Universal problem for Kähler differentials in $\mathcal{A}$-modules: non-commutative and commutative cases

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Abstract

Let $\mathcal{A}$ be an associative and unital $\mathcal{K}$-algebra sheaf, where $\mathcal{K}$ is a commutative ring sheaf, and $\mathcal{E}$ an $(\mathcal{A}, \mathcal{A})$-*bimodule,* that is, a sheaf of $(\mathcal{A}, \mathcal{A})$-bimodules. We construct an $(\mathcal{A}, \mathcal{A})$-bimodule which is $\mathcal{K}$-isomorphic with the $\mathcal{K}$-module $\mathcal{D}_{\mathcal{K}}(\mathcal{A}, \mathcal{E})$ of germs of $\mathcal{K}$-derivations. A similar isomorphism is obtained, this time around with respect to $\mathcal{A}$, between the $\mathcal{K}$-module $\mathcal{D}_{\mathcal{K}}(\mathcal{A}, \mathcal{E})$ with the $\mathcal{A}$-module $\mathcal{H}om_{\mathcal{A}}(\Omega_{\mathcal{K}}(\mathcal{A}), \mathcal{E})$, where $\mathcal{A}$, in addition of being associative and unital, is assumed to be commutative, and $\Omega_{\mathcal{K}}(\mathcal{A})$ denotes the $\mathcal{A}$-*module of germs of Kähler differentials.* Finally, we expound on functoriality of Kähler differentials.

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1 Introduction

This paper is concerned with the generalization of modules of Kähler differentials by changing to the setting of sheaves of $\mathcal{A}$-modules, where, in general, $\mathcal{A}$ stands for a sheaf of associative and unital algebras on a topological space X. Differentials are amongst objects that play tremendous roles in commutative algebra and algebraic geometry similar to those of tangent and cotangent bundles in differentiable geometry (see [3, Chapter 16]). Kähler differentials of an R-algebra S over a ring R form an S-module, denoted $\Omega_{S/R}$, which is such that, for any S-module M, one has

$$\mathrm{Der}_R(S, M) \simeq \mathrm{H}om_S(\Omega_{S/R}, M);$$

in other words, the construction of $\Omega_{S/R}$, as D. Eisenbud puts it, "*linearizes*" the construction of derivations. The above isomorphism is usually used to compute $\mathrm{Der}_R(S, M)$ in terms of $\Omega_{S/R}$. This is the direction we intend to go along for the analogous version of derivations in the setting of sheaves of $\mathcal{A}$-modules.

The main result (Theorem 3.2) is derived from the classical theorem which states that *given a ring R, an R-algebra S and I the kernel of the multiplication map $\mu : S \otimes S \longrightarrow S$, if $e : S \longrightarrow I/I^2$ is the map defined by $b \longmapsto 1 \otimes b - b \otimes 1$, there is an isomorphism $\varphi : \Omega_{S/R} \simeq I/I^2$ such that the pair $(d, \Omega_{S/R})$ is isomorphic to $(e, I/I^2)$, i.e.,*

$$\mathrm{Der}_R(S, I/I^2) \simeq \mathrm{H}om_S(\Omega_{S/R}, I/I^2).$$

Now, in trying to enlarge the traditional framework of modules to sheaves of modules over sheaves of rings (or in particular, over sheaves of algebras), one is faced with the fact/reality that for non-complete generating presheaves, the sought counterparts of classical results are not straightforward and require a good measure of control over the sheafification functor. Every sheaf and presheaf considered within the bounds of this paper are assumed to be defined on a fixed topological space X.

2 Non-commutative case

Definition 2.1 Suppose that we have the following: a topological space (X, τ_X), a sheaf $\mathcal{K}$ of rings, a sheaf $\mathcal{A}$ of $\mathcal{K}$-algebras, and a sheaf $\mathcal{E}$ of $\mathcal{A}$-modules. A $\mathcal{K}$-*derivation* of $\mathcal{A}$ into $\mathcal{E}$ is a sheaf morphism

$$\partial : \mathcal{A} \longrightarrow \mathcal{E} \tag{1}$$

such that the following conditions are satisfied:

(1) ∂ is a $\mathcal{K}$-morphism.

(2) For every $U \in \tau_X$ and sections $s, t \in \mathcal{A}(U)$, ∂_U satisfies the Leibniz condition, that is,

$$\partial_U(s \cdot t) = \partial_U(s) \cdot t + s \cdot \partial_U(t). \tag{2}$$

The triple $(\mathcal{A}, \partial, \mathcal{E})$, thus obtained, is called a $\mathcal{K}$-*differential triad* on X (see [6, pp. 2, 3]).

Of course, concerning (2), we have used the morphism of the (complete) presheaves of sections of $\mathcal{A}$ and $\mathcal{E}$ induced by ∂; for the sake of simplicity, the latter is still denoted ∂ (a practice that we shall often apply in the sequel without specific mention).

Now, let us consider the generating presheaf of the tensor product sheaf $\mathcal{A} \otimes_{\mathcal{K}} \mathcal{A}$, that is, the presheaf given by the correspondence

$$U \longmapsto \Gamma(U, \mathcal{A}) \otimes_{\Gamma(U,\mathcal{K})} \Gamma(U, \mathcal{A}) \equiv \mathcal{A}(U) \otimes_{\mathcal{K}(U)} \mathcal{A}(U),$$

where U runs over the open sets in X, along with the obvious restriction maps (that is, $\{\rho^U_V \otimes \rho^U_V;\ U, V \in \tau_X \text{ and } V \subseteq U\}$ if $\{\rho^U_V;\ U, V \in \tau_X,\ V \subseteq U\}$ is the family of restriction maps of $\Gamma(\mathcal{A})$). For every $U \in \tau_X$, the tensor product $\mathcal{A}(U) \otimes_{\mathcal{K}(U)} \mathcal{A}(U)$ has a canonical $(\mathcal{A}(U), \mathcal{A}(U))$-bimodule structure, given by the following prescription

$$x \cdot (s \otimes_{\mathcal{K}(U)} t) \cdot y \equiv x(s \otimes t)y := (xs) \otimes (ty) \equiv (xs) \otimes_{\mathcal{K}(U)} (ty),$$

where $s, t, x, y \in \mathcal{A}(U)$. On the other hand, let us also consider, for every $U \in \tau_X$, the $\mathcal{K}(U)$-linear map $m_U : \mathcal{A}(U) \otimes_{\mathcal{K}(U)} \mathcal{A}(U) \longmapsto \mathcal{A}(U)$ determined by the $\mathcal{K}(U)$-bilinear map $\mathcal{A}(U) \times \mathcal{A}(U) \longrightarrow \mathcal{A}(U)$; $(s,t) \longmapsto st$, so

$$m_U(s \otimes t) = st.$$

It is easily seen that every m_U is an $(\mathcal{A}(U), \mathcal{A}(U))$-bimodule morphism; whence, the family

$$m \equiv (m_U)_{U \in \tau_X} : \Gamma(\mathcal{A}) \otimes_{\Gamma(\mathcal{K})} \Gamma(\mathcal{A}) \longrightarrow \Gamma(\mathcal{A}) \tag{3}$$

is a $\Gamma(\mathcal{K})$-morphism. By sheafifying (3), we obtain a $\mathcal{K}$-morphism of $\mathcal{A}$-modules

$$\mathbf{S}(m) \equiv m : \mathcal{A} \otimes_{\mathcal{K}} \mathcal{A} \longrightarrow \mathcal{A}, \tag{4}$$

where $\mathbf{S}$ is the sheafifying functor, (see [5, pp. 33- 37]). Since every m_U is an $(\mathcal{A}(U), \mathcal{A}(U))$-bimodule morphism, it follows that (4) is an $(\mathcal{A}, \mathcal{A})$-bimodule morphism (in this bimodule morphism, we are considering $\mathcal{A}$ as an $(\mathcal{A}, \mathcal{A})$-bimodule). By [5, p. 108, (2.10)], the kernel of m, $\mathcal{I} \equiv (\ker m, \pi|_{\ker m}, X)$, is a sub-$(\mathcal{A}, \mathcal{A})$-bimodule of $\mathcal{A} \otimes_{\mathcal{K}} \mathcal{A}$.

For the purpose of Lemma 2.1, we recall the following notion: *If $S \equiv (S(U), \rho^U_V)$ is a presheaf (of sets), then $\rho_U : S(U) \longrightarrow S(U, \pi)$ defines the (canonical) map sending an element of $S(U)$ to a section of the sheaf generated by the presheaf S over U*; cf [5, pp. 30, (7.8), (7.9), (7.11)]. π is the *local homeomorphism* obtained through sheafification.

Lemma 2.1 *Let X be a topological space, $\mathcal{K}$ a sheaf of commutative rings, $\mathcal{A}$ a sheaf of associative and unital $\mathcal{K}$-algebras, and $m : \mathcal{A} \otimes_{\mathcal{K}} \mathcal{A} \longrightarrow \mathcal{A}$ the $\mathcal{A}$-morphism which corresponds to the usual multiplication $\mathcal{A}$-bilinear morphism on $\mathcal{A}$. Then, the family $\partial \equiv (\partial_U)_{U \in \tau_X}$ such that*

$$\partial_U(s) = s \otimes 1 - 1 \otimes s \in \mathcal{A}(U) \otimes_{\mathcal{K}(U)} \mathcal{A}(U) \tag{5}$$

for every $U \in \tau_X$ and section $s \in \mathcal{A}(U)$, yields a $\mathcal{K}$-derivation of $\mathcal{A}$ into $\mathcal{I} :=$ $\ker m$. *Moreover, for every $U \in \tau_X$, the left $\mathcal{A}(U)$-module $\mathcal{I}(U) = \ker m_U$ is generated by $\{\widetilde{\partial_U(s)} \equiv \rho_U(\partial_U(s));\ s \in \mathcal{A}(U)\}$.*

Proof. The first assertion is straightforward. Indeed, let's consider the collection $I \equiv (I(U), (\rho^U_V \otimes \rho^U_V)|_I)$, where, for every $U \in \tau_X$, $I(U)$ is the $\mathcal{A}(U)$-module generated by the set

$$\{s \otimes 1 - 1 \otimes s;\ s \in \mathcal{A}(U)\} \subseteq \mathcal{A}(U) \otimes_{\mathcal{K}(U)} \mathcal{A}(U).$$

It is clear that I is a subpresheaf of $\Gamma(\mathcal{A}) \otimes_{\Gamma(\mathcal{K})} \Gamma(\mathcal{A})$, a generating presheaf of the sheaf $\mathcal{A} \otimes_{\mathcal{K}} \mathcal{A}$. Since, for every $U \in \tau_X$, $\partial_U : \mathcal{A}(U) \longrightarrow I(U)$ (given by that $\partial_U(s) = s \otimes 1 - 1 \otimes s$) is a $\mathcal{K}(U)$-derivation, it follows that the family $\partial \equiv (\partial_U)_{U \in \tau_X}$ yields a $\mathcal{K}$-derivation of $\mathcal{A}$ into the sheafification $\mathcal{I}$ of the presheaf I.

On the other hand, let $J \subseteq \mathbb{N}$ be finite, $s_i,\ t_i \in \mathcal{A}(U)$, where U is open in X. If $\widetilde{\sum_{i \in J} s_i \otimes t_i} \equiv \rho_U(\sum_{i \in J} s_i \otimes t_i)$ belongs to $\mathcal{I}(U)$, then $\sum_{i \in J} s_i t_i = 0$ and thus

$$\rho_U\Big(\sum_{i \in J} s_i \otimes t_i\Big) = \rho_U\Big(\sum_{i \in J} s_i (1 \otimes t_i - t_i \otimes 1)\Big) = \sum_{i \in J} s_i \rho_U(\partial_U(t_i)).$$

■

We should draw our attention to the fact that, in Lemma 2.1, we have taken ∂ as the sheaf morphism obtained after sheafification of the presheaf morphism $\overline{\partial} \equiv \partial : \Gamma(\mathcal{A}) \longrightarrow \Gamma(\mathcal{A}) \otimes_{\Gamma(\mathcal{K})} \Gamma(\mathcal{A})$, rather than the sheaf morphism induced by the presheaf morphism $\Gamma(\partial) : \Gamma(\mathcal{A}) \longrightarrow \Gamma(\mathcal{A} \otimes_{\mathcal{K}} \mathcal{A})$.

Theorem 2.1 *Let $\mathcal{A}$ be an associative and unital $\mathcal{K}$-algebra sheaf, where $\mathcal{K}$ is a commutative algebra sheaf, and $\partial : \mathcal{A} \longrightarrow \mathcal{I} \subseteq \mathcal{A} \otimes_{\mathcal{K}} \mathcal{A}$ the $\mathcal{K}$-derivation, given by* (5). *Then, ∂ has the following universal property: For every $(\mathcal{A}, \mathcal{A})$-bimodule $\mathcal{E}$ and every $\mathcal{K}$-derivation $\delta : \mathcal{A} \longrightarrow \mathcal{E}$, there exists a unique $(\mathcal{A}, \mathcal{A})$-morphism $\varphi : \mathcal{I} \longrightarrow \mathcal{E}$ such that*

$$\delta = \varphi \circ \partial. \tag{6}$$

Proof. It is clear that for every $(\mathcal{A}, \mathcal{A})$-morphism $\varphi : \mathcal{I} \longrightarrow \mathcal{E}$, $\varphi \circ \partial$ is a $\mathcal{K}$-derivation. Conversely, let $\delta : \mathcal{A} \longrightarrow \mathcal{E}$ be a $\mathcal{K}$-derivation; then we first show that if there exists an $(\mathcal{A}, \mathcal{A})$-morphism $\varphi : \mathcal{I} \longrightarrow \mathcal{E}$ such that $\delta = \varphi \circ \partial$, such a φ must be unique. To this end, let's consider the presheaf morphism $\overline{\varphi} : I \longrightarrow \Gamma(\mathcal{E})$, where I is the presheaf such that $I(U)$ is generated by the set in (5), such that $\Gamma(\delta) \equiv \delta = \overline{\varphi} \circ \partial \equiv \overline{\varphi} \circ \Gamma(\partial)$; so, for any $U \in \tau_X$ and section $s \in \mathcal{A}(U)$,

$$\overline{\varphi}_U(s \otimes 1 - 1 \otimes s) = \delta_U(s).$$

Since $I(U)$ is generated by $\partial_U(\mathcal{A}(U))$, it follows that

$$\overline{\varphi}_U\Big(\sum_{i\in I} s_i \otimes t_i\Big) = \sum_{i\in I} s_i \overline{\varphi}_U(1 \otimes t_i - t_i \otimes 1) = \sum_{i\in I} s_i \delta_U(t_i);$$

wherefore, $\overline{\varphi}$ is unique. Hence, the sheafification $\varphi := \mathbf{S}(\overline{\varphi})$ of $\overline{\varphi}$ is unique and satisfies (23).

Now, as every mapping $(s,t) \longmapsto -s\delta_U(t)$ of $\mathcal{A}(U) \times \mathcal{A}(U)$ into $\mathcal{E}(U)$ is $\mathcal{K}(U)$-bilinear, there exists a unique $\Gamma(\mathcal{K})$-morphism

$$\psi \equiv (\psi_U)_{U\in\tau_X} : \Gamma(\mathcal{A}) \otimes_{\Gamma(\mathcal{K})} \Gamma(\mathcal{A}) \longrightarrow \Gamma(\mathcal{E})$$

such that

$$\psi_U(s \otimes t) = -s\delta_U(t);$$

it suffices to verify that the restriction $\overline{\varphi}$ of ψ to I is $\Gamma(\mathcal{A})$-linear for the left and right $\mathcal{A}$-module structures. The first assertion is obvious since, for every $U \in \tau_X$ and sections $r, s, t \in \mathcal{A}(U)$,

$$\psi_U(r(s \otimes t)) = -(rs)\delta_U(t) = r\psi_U(s \otimes t).$$

To prove linearity on the right-hand side, note that, if $\sum_{i\in I} s_i \otimes t_i \in I(U)$ and $t \in \mathcal{A}(U)$, then

$$\sum_{i\in I} s_i\delta_U(t_it) = \sum_{i\in I} s_i\delta_U(t_i)t + \sum_{i\in I}(s_it_i)\delta_U(t);$$

but $\sum_{i\in I} s_it_i = 0$, therefore

$$\sum_{i\in I} s_i\delta_U(t_it) = (\sum_{i\in I} s_i\delta_U(t_i))t.$$

■

Thus, for any $U \in \tau_X$, we have a *canonical* $\mathcal{K}|_U$*-isomorphism* $\varphi \longmapsto \varphi \circ \partial$

$$\mathrm{H}om_{(\mathcal{A}|_U, \mathcal{A}|_U)}(\mathcal{I}|_U, \mathcal{E}|_U) \longrightarrow \mathrm{D}_{\mathcal{K}|_U}(\mathcal{A}|_U, \mathcal{E}|_U), \tag{7}$$

where the right-hand side is the $\mathcal{K}(U)$-module of $\mathcal{K}|_U$-derivations of $\mathcal{A}|_U$ into $\mathcal{E}|_U$.

Hence, under the hypotheses of Theorem 2.1, we have

Corollary 2.1 *If* $\mathcal{D}_{\mathcal{K}}(\mathcal{A}, \mathcal{E})$ *denotes the* $\mathcal{K}$*-module of germs of* $\mathcal{K}$*-derivations of* $\mathcal{A}$ *into* $\mathcal{E}$*, then*

$$\mathcal{H}om_{(\mathcal{A},\mathcal{A})}(\mathcal{I}, \mathcal{E}) = \mathcal{D}_{\mathcal{K}}(\mathcal{A}, \mathcal{E}), \tag{8}$$

within a $\mathcal{K}$*-isomorphism.*

3 Commutative case

We will assume, throughout this section, that whenever one considers tensor products of $\mathcal{A}$-modules, the "*coefficient sheaf*" $\mathcal{A}$ is a *sheaf of commutative and unital* $\mathbb{C}$*-algebras.* Keeping with the rule of the previous section, we assume that all sheaves involved in this section are defined over the same topological space X.

For the purpose of the main results of this section, we recall the following results (cf. [5, pp. 130- 132, (5.11) and Theorem 5.1]).

Lemma 3.1 *If* $\mathcal{E}$ *is an* $\mathcal{A}$*-module, then the functor*

$$\otimes_{\mathcal{A}} \mathcal{E} : \mathcal{A}\text{-}\mathcal{M}od_X \longrightarrow \mathcal{A}\text{-}\mathcal{M}od_X \tag{9}$$

is covariant and right exact.

Lemma 3.2 *If* $\mathcal{E}$ *is a vector sheaf, then the tensor product functor* $\otimes_{\mathcal{A}} \mathcal{E}$ *is exact.*

A result more general than the one of Lemma 3.1 goes as follows.

Lemma 3.3 *Let $\mathcal{R}, \mathcal{R}', \mathcal{R}''$ be right $\mathcal{A}$-modules, $\mathcal{E}$ a left $\mathcal{A}$-module and*

$$\mathcal{R}' \xrightarrow{\varphi} \mathcal{R} \xrightarrow{\psi} \mathcal{R}'' \longrightarrow 0 \tag{10}$$

an exact $\mathcal{A}$-sequence. Then, the $\mathcal{A}$-sequence

$$\mathcal{R}' \otimes_{\mathcal{A}} \mathcal{E} \xrightarrow{\varphi \otimes 1_{\mathcal{E}}} \mathcal{R} \otimes_{\mathcal{A}} \mathcal{E} \xrightarrow{\psi \otimes 1_{\mathcal{E}}} \mathcal{R}'' \otimes_{\mathcal{A}} \mathcal{E} \longrightarrow 0 \tag{11}$$

is exact.

Proof. The proof is similar to the proof of Lemma 3.1. ■

As for a dual statement, we have

Lemma 3.4 *Let $\mathcal{S}, \mathcal{S}', \mathcal{S}''$ be left $\mathcal{A}$-modules, $\mathcal{E}$ a right $\mathcal{A}$-module and*

$$\mathcal{S}' \xrightarrow{\delta} \mathcal{S} \xrightarrow{\lambda} \mathcal{S}'' \longrightarrow 0 \tag{12}$$

an exact $\mathcal{A}$-sequence. Then, the $\mathcal{A}$-sequence

$$\mathcal{E} \otimes_{\mathcal{A}} \mathcal{S}' \xrightarrow{1_{\mathcal{E}} \otimes \delta} \mathcal{E} \otimes_{\mathcal{A}} \mathcal{S} \xrightarrow{1_{\mathcal{E}} \otimes \lambda} \mathcal{E} \otimes_{\mathcal{A}} \mathcal{S}'' \longrightarrow 0 \tag{13}$$

is also exact.

By applying mutatis mutandis the proof of [2, p. 252, Proposition 6], one has

Theorem 3.1 *Assuming the hypotheses of* Lemmas 3.3 *and* 3.4, *the $\mathcal{A}$-morphism $\psi \otimes \lambda : \mathcal{R} \otimes_{\mathcal{A}} \mathcal{S} \longrightarrow \mathcal{R}'' \otimes_{\mathcal{A}} \mathcal{S}''$ is surjective. Furthermore,*

$$\ker(\psi \otimes \lambda) = \mathrm{Im}(\varphi \otimes 1_{\mathcal{S}}) + \mathrm{Im}(1_{\mathcal{R}} \otimes \lambda) \tag{14}$$

within an $\mathcal{A}$-isomorphism.

In other words,

Corollary 3.1 *Assuming the hypotheses of* Lemmas 3.3 *and* 3.4, *with* $\mathcal{R}'$ *and* $\mathcal{S}'$ *sub-*$\mathcal{A}$*-modules of* $\mathcal{R}$ *and* $\mathcal{S}$*, respectively, one has*

$$(\mathcal{R} \otimes_{\mathcal{A}} \mathcal{S})/[\mathrm{Im}(\mathcal{R}' \otimes_{\mathcal{A}} \mathcal{S}) + \mathrm{Im}(\mathcal{R} \otimes_{\mathcal{A}} \mathcal{S}')] = (\mathcal{R}/\mathcal{R}') \otimes_{\mathcal{A}} (\mathcal{S}/\mathcal{S}') \tag{15}$$

within an $\mathcal{A}$*-isomorphism.*

If the $\mathcal{A}$-modules of Corollary 3.1 are sheaves of $\mathcal{A}$-algebras ; more precisely, suppose that $\mathcal{R}$ is the sheaf $\mathcal{A}$, $\mathcal{R}' \equiv \mathcal{I}$ an *ideal subsheaf* of $\mathcal{A}$ and $\mathcal{S}'$ the *trivial subsheaf* 0 of $\mathcal{S}$, we have

$$(\mathcal{A}/\mathcal{I}) \otimes_{\mathcal{A}} \mathcal{S} = \mathcal{S}/\mathcal{I}\mathcal{S} \tag{16}$$

within an $(\mathcal{A}/\mathcal{I})$-isomorphism.

Now, let $\mathcal{K}$ be a *sheaf of commutative rings with an identity element*, $\mathcal{A}$ a sheaf of commutative $\mathcal{K}$-algebras and $\mathcal{E}$ an $\mathcal{A}$-module. *Since* $\mathcal{A}$ *is commutative,* $\mathcal{E}$ *can be considered as an* $(\mathcal{A}, \mathcal{A})$*-bimodule.* Indeed, for every open $U \in \tau_X$ and sections $\alpha, \beta \in \mathcal{A}(U)$ and $s \in \mathcal{E}(U)$, since $\mathcal{A}(U)$ is commutative, there is a one-to-one correspondence $\mathcal{A}(U) \times \mathcal{E}(U) \longrightarrow \mathcal{E}(U) \times \mathcal{A}(U)$, given by

$$(\beta\alpha)s \longmapsto s(\alpha\beta) = s(\beta\alpha).$$

We now show that the left and right $\mathcal{A}$-module sheaf structures of $\mathcal{E}$ are compatible. To this end, we note the following:

$$(\alpha s)\beta \equiv (s\alpha)\beta = s(\alpha\beta) = s(\beta\alpha) = (s\beta)\alpha \equiv \alpha(s\beta),$$

which corroborates the claim that $\mathcal{E}$ may be regarded as an $(\mathcal{A}, \mathcal{A})$-bimodule. On the other hand, since, for every $U \in \tau_X$ and sections $r, s, t, u \in \mathcal{A}(U)$,

$$r(s \otimes t)u = (rs) \otimes (tu) = (rs) \otimes (ut) = (r \otimes u)(s \otimes t), \tag{17}$$

the $(\Gamma(\mathcal{A}), \Gamma(\mathcal{A}))$-*bimodule presheaf structure* on $\Gamma(\mathcal{A}) \otimes_{\Gamma(\mathcal{K})} \Gamma(\mathcal{A})$ is identical with its $(\Gamma(\mathcal{A}) \otimes_{\Gamma(\mathcal{K})} \Gamma(\mathcal{A}))$-*module presheaf structure. Sheaf-wise*, we have that the $(\mathcal{A}, \mathcal{A})$-bimodule sheaf structure of the $\mathcal{K}$-module $\mathcal{A} \otimes_{\mathcal{K}} \mathcal{A}$ is equivalent to its $(\mathcal{A} \otimes_{\mathcal{K}} \mathcal{A})$-module sheaf structure. By virtue of (17), it follows that the *kernel* (*sheaf*) $\mathcal{I}$ of the $\mathcal{K}$-morphism (4) is an ideal sheaf of the $\mathcal{K}$-algebra sheaf $\mathcal{A}\otimes_{\mathcal{K}}\mathcal{A}$. But m is surjective, therefore

$$(\mathcal{A}\otimes_{\mathcal{K}}\mathcal{A})/\mathcal{I}=\mathcal{A} \tag{18}$$

within a $\mathcal{K}$-isomorphism. In addition, let's consider $\mathcal{E}$ as an $(\mathcal{A}\otimes_{\mathcal{K}}\mathcal{A})$-module by means of the $\mathcal{K}$-morphism m; in fact, for any $U\in\tau_X$ and sections $\alpha,\beta\in\mathcal{A}(U)$, $s\in\mathcal{E}(U)$, we set

$$(\alpha\otimes\beta)s:=m(\alpha\otimes\beta)s=(\alpha\beta)s.$$

Then,

$$\mathcal{H}om_{(\mathcal{A},\mathcal{A})}(\mathcal{I},\mathcal{E})=\mathcal{H}om_{\mathcal{A}\otimes_{\mathcal{K}}\mathcal{A}}(\mathcal{I},\mathcal{E}) \tag{19}$$

within a $\mathcal{K}$-isomorphism, for, given an $U\in\tau_X$ and sections $\alpha,\beta\in\mathcal{A}(U)$, $s\in\mathcal{I}(U)$, using the fact that the $(\mathcal{A},\mathcal{A})$-bimodule sheaf structure of $\mathcal{A}\otimes_{\mathcal{K}}\mathcal{A}$ is identical with its $(\mathcal{A}\otimes_{\mathcal{K}}\mathcal{A})$-module sheaf structure and $\mathcal{I}$ is an ideal sheaf in $\mathcal{A}\otimes_{\mathcal{K}}\mathcal{A}$, we have: $\varphi\equiv(\varphi_V)_{U\supseteq V,\ open}\in\mathcal{H}om_{(\mathcal{A},\mathcal{A})}(\mathcal{I},\mathcal{E})(U)$ implies that

$$(\alpha|_V\otimes\beta|_V)\varphi_V(s)=\alpha|_V\varphi_V(s)\beta|_V=\varphi_V(\alpha|_V s\beta|_V)=\varphi_V((\alpha|_V\otimes\beta|_V)s),$$

for any subopen V of U, which means that $\varphi\in\mathcal{H}om_{\mathcal{A}\otimes_{\mathcal{K}}\mathcal{A}}(\mathcal{I},\mathcal{E})(U)$. In the same way, one shows that $\mathcal{H}om_{\mathcal{A}\otimes_{\mathcal{K}}\mathcal{A}}(\mathcal{I},\mathcal{E})(U)\subseteq\mathcal{H}om_{(\mathcal{A},\mathcal{A})}(\mathcal{I},\mathcal{E})(U)$. Hence, (19) is proved.

On the other hand, for any $U\in\tau_X$ and sections $s\in\mathcal{A}(U)$, $t\in\mathcal{E}(U)$

$$(s\otimes 1-1\otimes s)t=0;$$

so,

$$\mathcal{I}\mathcal{E}=0;$$

that is, the ideal sheaf $\mathcal{I}$ is contained in the annihilator of the $(\mathcal{A}\otimes_{\mathcal{K}}\mathcal{A})$-module $\mathcal{E}$. On applying the sheaf isomorphism (16) to $\mathcal{K}$-algebra sheaves $\mathcal{I}$ and $(\mathcal{A}\otimes_{\mathcal{K}}\mathcal{A})/\mathcal{I}$, one has the canonical $(\mathcal{A}\otimes_{\mathcal{K}}\mathcal{A})/\mathcal{I}$-isomorphism

$$((\mathcal{A}\otimes_{\mathcal{K}}\mathcal{A})/\mathcal{I})\otimes_{\mathcal{A}}\mathcal{I}=\mathcal{I}/\mathcal{I}^2;$$

hence, by virtue of (18) and [5, p. 304, (6.11)], we have the following sheaf isomorphisms

$$\begin{aligned}\mathcal{H}om_{\mathcal{A}}(\mathcal{I}/\mathcal{I}^2,\mathcal{E}) &= \mathcal{H}om_{\mathcal{A}}(((\mathcal{A}\otimes_{\mathcal{K}}\mathcal{A})/\mathcal{I})\otimes_{\mathcal{A}}\mathcal{I},\mathcal{E})\\ &= \mathcal{H}om_{\mathcal{A}}((\mathcal{A}\otimes_{\mathcal{K}}\mathcal{A})/\mathcal{I},\mathcal{H}om_{\mathcal{A}}(\mathcal{I},\mathcal{E}))\\ &= \mathcal{H}om_{\mathcal{A}}(\mathcal{A},\mathcal{H}om_{\mathcal{A}}(\mathcal{I},\mathcal{E}))\\ &= \mathcal{H}om_{\mathcal{A}}(\mathcal{I},\mathcal{E}).\end{aligned}$$

But $\mathcal{A}$ is commutative, therefore

$$\mathcal{H}om_{\mathcal{A}}(\mathcal{I},\mathcal{E}) = \mathcal{H}om_{(\mathcal{A},\mathcal{A})}(\mathcal{I},\mathcal{E});$$

so

$$\mathcal{H}om_{\mathcal{A}\otimes_{\mathcal{K}}\mathcal{A}}(\mathcal{I},\mathcal{E}) = \mathcal{H}om_{\mathcal{A}}(\mathcal{I}/\mathcal{I}^2,\mathcal{E}) \tag{20}$$

within an $\mathcal{A}$-isomorphism.

It follows, from the discussion above and Theorem 2.1, that we have proved the analogous of [2, p. 569, Proposition 18]. Also see [3, pp. 410, 411, Theorem 16.24] and [4, pp. 746- 748, Theorem 3.1].

Theorem 3.2 *Let $\mathcal{K}$ be a sheaf of commutative rings, $\mathcal{A}$ a sheaf of commutative and unital $\mathcal{K}$-algebras and $\mathcal{I}$ the ideal kernel sheaf of the surjective $\mathcal{K}$-morphism* (4)*. Moreover, let $\partial_{\mathcal{A}/\mathcal{K}} : \mathcal{A} \longrightarrow \mathcal{I}/\mathcal{I}^2$ be the $\mathcal{K}$-morphism such that $\partial_{\mathcal{A}/\mathcal{K}} = q\circ\partial$, where $q : \mathcal{I} \longrightarrow \mathcal{I}/\mathcal{I}^2$ is the canonical $\mathcal{K}$-morphism and ∂ is the $\mathcal{K}$-morphism given component-wise by* (5)*. Then, the mapping $\partial_{\mathcal{A}/\mathcal{K}}$ is a $\mathcal{K}$-derivation and, for every $\mathcal{A}$-module $\mathcal{E}$ and every $\mathcal{K}$-derivation $\delta : \mathcal{A} \longrightarrow \mathcal{E}$, there exists a unique $\mathcal{A}$-morphism $\varphi : \mathcal{I}/\mathcal{I}^2 \longrightarrow \mathcal{E}$ such that $\delta = \varphi \circ \partial_{\mathcal{A}/\mathcal{K}}$.*

The $\mathcal{A}$-module $\mathcal{I}/\mathcal{I}^2$ is called the *$\mathcal{A}$-module of germs of Kähler differentials* (*or $\mathcal{K}$-differentials*) of $\mathcal{A}$ and is denoted by $\Omega_{\mathcal{K}}(\mathcal{A})$ (see [2, p. 569]). By virtue of (8), (19) and (20), there is a canonical $\mathcal{A}$-isomorphism

$$\mathcal{H}om_{\mathcal{A}}(\Omega_{\mathcal{K}}(\mathcal{A}),\mathcal{E}) = \mathcal{D}_{\mathcal{K}}(\mathcal{A},\mathcal{E}). \tag{21}$$

4 Functoriality of $\mathcal{K}$-differentials

Throughout this section, all the algebra sheaves are assumed to be associative and unital and all the algebra sheaf morphisms are assumed to be unital.

Lemma 4.1 *Let $\mathcal{K} \equiv (\mathcal{K},\sigma,X)$ be a sheaf of commutative rings and $\mathcal{A} \equiv (\mathcal{A},\pi,X)$ a sheaf of $\mathcal{K}$-algebras where the continuous* "exterior multiplication" *is given via a ring sheaf morphism $\eta : \mathcal{K} \longrightarrow \mathcal{A}$, that is, the map*

$$\mathcal{K}\circ\mathcal{A} \longrightarrow \mathcal{A} : (k,a) \longmapsto \eta_x(k)a \in \mathcal{A}_x \subseteq \mathcal{A},$$

with $\sigma(k) = \pi(a) = x \in X$*. (We have used the familiar notation as in* [5, p. 96, (1.37)]*, i.e.,* $\mathcal{K} \circ \mathcal{A} = \{(k,a) \in \mathcal{K} \times \mathcal{A} : \sigma(k) = \pi(a)\}$*.) In the same vein, let* $\mathcal{K}'$ *be another sheaf of commutative rings and* $\mathcal{A}'$ *a sheaf of* $\mathcal{K}'$*-algebras via a sheaf morphism* $\eta' : \mathcal{K}' \longmapsto \mathcal{A}'$*. Moreover, let* $\psi : \mathcal{K} \longrightarrow \mathcal{K}'$ *and* $\varphi : \mathcal{A} \longrightarrow \mathcal{A}'$ *be sheaf morphisms such that*

$$\varphi \circ \eta = \eta' \circ \psi.$$

Then, there exists a unique $\mathcal{A}$*-morphism* $\vartheta : \Omega_{\mathcal{K}}(\mathcal{A}) \longrightarrow \Omega_{\mathcal{K}'}(\mathcal{A}')$ *such that*

$$\vartheta \circ \partial_{\mathcal{A}/\mathcal{K}} = \partial_{\mathcal{A}'/\mathcal{K}'} \circ \varphi,$$

where $\partial_{\mathcal{A}'/\mathcal{K}'}$ *is the* $\mathcal{K}'$*-derivation of* $\mathcal{A}'$ *into the* $\mathcal{A}'$*-module* $\Omega_{\mathcal{K}'}(\mathcal{A}')$ *of germs of Kähler differentials of* $\mathcal{A}'$*.*

Proof. First let us note that $\Omega_{\mathcal{K}'}(\mathcal{A}')$ is an $\mathcal{A}$-module by composing φ with the $\mathcal{K}'$-derivation $\partial_{\mathcal{A}'/\mathcal{K}'}$. That $\partial_{\mathcal{A}'/\mathcal{K}'} \circ \varphi : \mathcal{A} \longrightarrow \Omega_{\mathcal{K}'}(\mathcal{A}')$ is a $\mathcal{K}$-derivation is clear; the existence and uniqueness of ϑ follow from Theorem 3.2 ■

We shall denote the $\mathcal{A}$-morphism ϑ of Lemma 4.1 by $\Omega(\varphi)$; clearly, if $\mathcal{K}''$ is a sheaf of commutative rings, $\mathcal{A}''$ a sheaf of $\mathcal{K}''$-algebras via a sheaf morphism η'', $\psi' : \mathcal{K}' \longrightarrow \mathcal{K}''$ and $\varphi'' : \mathcal{A}' \longrightarrow \mathcal{A}''$ are ring and algebra sheaf morphisms, respectively, such that

$$\eta' \circ \psi = \varphi \circ \eta, \quad \eta'' \circ \psi' = \varphi' \circ \eta',$$

then

$$\Omega(\varphi' \circ \varphi) = \Omega(\varphi') \circ \Omega(\varphi).$$

By considering the canonical $\mathcal{A}$-morphism

$$\iota_{\mathcal{A}} : \Omega_{\mathcal{K}}(\mathcal{A}) \longrightarrow \Omega_{\mathcal{K}}(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{A}',$$

which is the sheafification of the canonical $\Gamma(\mathcal{A})$-morphism

$$\Gamma(\iota_{\mathcal{A}}) : \Gamma(\Omega_{\mathcal{K}}(\mathcal{A})) \longrightarrow \Gamma(\Omega_{\mathcal{K}}(\mathcal{A})) \otimes_{\Gamma(\mathcal{A})} \Gamma(\mathcal{A}')$$

such that, for any $U \in \tau_X$ and section $s \in \Gamma(\Omega_{\mathcal{K}}(\mathcal{A}))(U) \equiv \Gamma(U, \Omega_{\mathcal{K}}(\mathcal{A}))$,

$$\Gamma(\iota_{\mathcal{A}})(s) = s \otimes 1,$$

where $1 \in \Gamma(\mathcal{A}')(U) \equiv \mathcal{A}'(U)$ is the identity section of $\mathcal{A}'$ over U, we obtain a $\Gamma(\mathcal{A}')$-morphism

$$\Gamma(\Omega_0(\varphi)) : \Gamma(\Omega_{\mathcal{K}}(\mathcal{A})) \otimes_{\Gamma(\mathcal{A})} \Gamma(\mathcal{A}') \longrightarrow \Gamma(\Omega_{\mathcal{K}'}(\mathcal{A}'))$$

such that

$$\Gamma(\Omega(\varphi)) = \Gamma(\Omega_0(\varphi)) \circ \Gamma(\iota_{\mathcal{A}}).$$

If

$$\Omega(\varphi) := \mathbf{S}\Gamma(\Omega(\varphi)), \quad \Omega_0(\varphi) := \mathbf{S}\Gamma(\Omega_0(\varphi)),$$

we have that

$$\Omega_0(\varphi) : \Gamma(\Omega_{\mathcal{K}}(\mathcal{A})) \otimes_{\Gamma(\mathcal{A})} \Gamma(\mathcal{A}') \longrightarrow \Omega_{\mathcal{K}'}(\mathcal{A}')$$

is the $\mathcal{A}'$-morphism such that

$$\Omega(\varphi) = \Omega_0(\varphi) \circ \iota_{\mathcal{A}}.$$

Theorem 4.1 below allows us to construct a commutative diagram of sheaves involving sheaves of germs of $\mathcal{K}$-derivations and sheaves of morphisms on $\mathcal{A}$-modules of germs of $\mathcal{K}$-differentials. In order to achieve this, we first examine the theorem in question. We note hereby that algebra sheaves $\mathcal{A}$ and $\mathcal{B}$ of Theorem 4.1 are arbitrary, that is, not necessarily associative and unital.

Theorem 4.1 *Let $\mathcal{A}$, $\mathcal{B}$ be algebra sheaves, $\mathcal{E}$ a locally free left $\mathcal{A}$-module (in other words, a vector sheaf) of rank m, and $\mathcal{G}$ a left $\mathcal{B}$-module. Moreover, let $\mathcal{F}$ be a $(\mathcal{B}, \mathcal{A})$-bimodule such that, as a left $\mathcal{B}$-module, $\mathcal{F}$ is locally free and of rank n. Then,*

$$\mathcal{H}om_{\mathcal{A}}(\mathcal{E}, \mathcal{H}om_{\mathcal{B}}(\mathcal{F}, \mathcal{G})) = \mathcal{H}om_{\mathcal{B}}(\mathcal{E} \otimes_{\mathcal{A}} \mathcal{F}, \mathcal{G}) \tag{22}$$

within an isomorphism of group sheaves.

Proof. Let $\mathcal{U}$ and $\mathcal{V}$ be local frames (see [5, p. 126, Definition 4.2]) of $\mathcal{E}$ and $\mathcal{F}$, respectively. That $\mathcal{W} \equiv \mathcal{U} \cap \mathcal{V} := \{U \cap V : U \in \mathcal{U}, V \in \mathcal{V}\}$ is a common local frame of $\mathcal{E}$ and $\mathcal{F}$ is clear. So, if $U \in \mathcal{W}$, then, applying [5, p. 137, (6.22), (6.23), (6.24')] and the fact that $\mathcal{H}om_{\mathcal{B}}(\mathcal{F},\mathcal{G})$ is a left $\mathcal{A}$-module, one has the following $\mathcal{A}|_U$-isomorphisms:

$$\begin{aligned}\mathcal{H}om_{\mathcal{A}}(\mathcal{E},\mathcal{H}om_{\mathcal{B}}(\mathcal{F},\mathcal{G}))|_U &= \mathcal{H}om_{\mathcal{A}|_U}(\mathcal{A}^m|_U,\mathcal{H}om_{\mathcal{B}|_U}(\mathcal{B}^n|_U,\mathcal{G}|_U))\\ &= \mathcal{H}om_{\mathcal{A}|_U}(\mathcal{A}^m|_U,\mathcal{H}om_{\mathcal{B}|_U}((\mathcal{B}|_U)^n,\mathcal{G}|_U)),\end{aligned}$$

that is,

$$\mathcal{H}om_{\mathcal{A}}(\mathcal{E},\mathcal{H}om_{\mathcal{B}}(\mathcal{F},\mathcal{G}))|_U = \mathcal{G}^{mn}|_U. \tag{23}$$

In the same way, one shows that

$$\mathcal{H}om_{\mathcal{B}}(\mathcal{E}\otimes_{\mathcal{A}}\mathcal{F},\mathcal{G})|_U = \mathcal{G}^{mn}|_U \tag{24}$$

within a $\mathcal{B}|_U$-isomorphism. On the other hand, for any open subset W of X, one has the following morphism

$$\mathrm{H}om_{\mathcal{A}|_W}(\mathcal{E}|_W,\mathcal{H}om_{\mathcal{B}}(\mathcal{F},\mathcal{G})|_W) \xrightarrow{\varphi_W} \mathrm{H}om_{\mathcal{B}|_W}(\mathcal{E}|_W\otimes_{\mathcal{A}|_W}\mathcal{F}|_W,\mathcal{G}|_W),$$

which is given by

$$\varphi_W(\alpha)(s\otimes t) := (\alpha_Z(s))_Z(t) \equiv \alpha(s)(t),$$

where $\alpha\in \mathrm{H}om_{\mathcal{A}|_W}(\mathcal{E}|_W,\mathcal{H}om_{\mathcal{B}}(\mathcal{F},\mathcal{G})|_W)$, $s\in(\mathcal{E}|_W)(Z)=\mathcal{E}(Z)$, $t\in\mathcal{F}(Z)$, with Z a subopen of W. Clearly, the family $\varphi\equiv(\varphi_W)_{W\in\tau_X}$ yields an $\mathcal{A}$-morphism. We shall indeed show that the sheafification $\mathbf{S}(\varphi)\equiv\widetilde{\varphi}$ of φ is an $\mathcal{A}$-isomorphism. For this purpose, we notice that, by virtue of (23) and (24),

$$\mathcal{H}om_{\mathcal{A}}(\mathcal{E},\mathcal{H}om_{\mathcal{B}}(\mathcal{F},\mathcal{G}))_x = \mathcal{G}^{mn}_x = \mathcal{H}om_{\mathcal{B}}(\mathcal{E}\otimes_{\mathcal{A}}\mathcal{F},\mathcal{G})_x, \tag{25}$$

for any $x\in X$. The equalities in (25) are valid up to group isomorphisms. Furthermore, as

$$\mathcal{H}om_{\mathcal{A}}(\mathcal{E},\mathcal{H}om_{\mathcal{B}}(\mathcal{F},\mathcal{G}))_x = \mathcal{H}om_{\mathcal{A}_x}(\mathcal{E}_x,\mathcal{H}om_{\mathcal{B}}(\mathcal{F},\mathcal{G})_x)$$

and

$$\mathcal{H}om_{\mathcal{B}}(\mathcal{E}\otimes_{\mathcal{A}}\mathcal{F},\mathcal{G})_x = \mathcal{H}om_{\mathcal{B}_x}(\mathcal{E}_x\otimes_{\mathcal{A}_x}\mathcal{F}_x,\mathcal{G}_x),$$

for any $x\in X$, φ_x is an $\mathcal{A}_x$-isomorphism (see [1, p. 198, Theorem 15.6]). Whence, by [5, p. 68, Theorem 12.1], φ is an $\mathcal{A}$-isomorphism, and the proof is complete. ∎

In particular, we have.

Corollary 4.1 *Let $\mathcal{K}$ be a sheaf of commutative rings, $\mathcal{A}$ a $\mathcal{K}$-algebra sheaf such that its corresponding $\mathcal{A}$-module $\Omega_{\mathcal{K}}(\mathcal{A})$ of germs of $\mathcal{K}$-differentials is a vector sheaf, $\mathcal{B}$ an arbitrary $\mathcal{K}$-algebra sheaf and $\mathcal{E}$ a $\mathcal{B}$-module. Then,*

$$\mathcal{H}om_{\mathcal{B}}(\Omega_{\mathcal{K}}(\mathcal{A})\otimes\mathcal{B},\mathcal{E}) = \mathcal{H}om_{\mathcal{A}}(\Omega_{\mathcal{K}}(\mathcal{A}),\mathcal{E}) \tag{26}$$

within an isomorphism of group sheaves.

Lemma 4.2 *Suppose the conditions of Lemma 4.1, where the $\mathcal{K}$-algebra sheaf $\mathcal{A}$ is such that its $\mathcal{A}$-module $\Omega_{\mathcal{K}}(\mathcal{A})$ of germs of $\mathcal{K}$-differentials is a vector sheaf. Then, for every $\mathcal{A}'$-module $\mathcal{E}'$, the diagram*

$$\begin{array}{ccc} \mathcal{H}om_{\mathcal{A}'}(\Omega_{\mathcal{K}'}(\mathcal{A}'),\mathcal{E}') & \xrightarrow{\mathcal{H}om(\Omega_0(\varphi),1_{\mathcal{E}'})} & \mathcal{H}om_{\mathcal{A}'}(\Omega_{\mathcal{K}}(\mathcal{A})\otimes_{\mathcal{A}}\mathcal{A}',\mathcal{E}') \\ \sigma_{\mathcal{A}'}\downarrow & & \downarrow\sigma_{\mathcal{A}}\circ r_{\mathcal{A}} \\ \mathcal{D}_{\mathcal{K}'}(\mathcal{A}',\mathcal{E}') & \xrightarrow[C(\varphi)]{} & \mathcal{D}_{\mathcal{K}}(\mathcal{A},\mathcal{E}'), \end{array}$$

where, for any open $U\subseteq X$, $C(\varphi)_U:\mathcal{D}_{\mathcal{K}'}(\mathcal{A}',\mathcal{E}')(U)\longrightarrow\mathcal{D}_{\mathcal{K}}(\mathcal{A},\mathcal{E}')(U)$ is given by

$$C(\varphi)_U(D) := D\circ(\varphi|_U),$$

$\sigma_{\mathcal{A}'}:\mathcal{H}om_{\mathcal{A}'}(\Omega_{\mathcal{K}'}(\mathcal{A}'),\mathcal{E}')\simeq\mathcal{D}_{\mathcal{K}'}(\mathcal{A}',\mathcal{E}')$, $\sigma_{\mathcal{A}}:\mathcal{H}om_{\mathcal{A}}(\Omega_{\mathcal{K}}(\mathcal{A}),\mathcal{E}')\simeq\mathcal{D}_{\mathcal{K}}(\mathcal{A},\mathcal{E}')$ and $r_{\mathcal{A}}:\mathcal{H}om_{\mathcal{A}'}(\Omega_{\mathcal{K}}(\mathcal{A})\otimes_{\mathcal{A}}\mathcal{A}',\mathcal{E}')\simeq\mathcal{H}om_{\mathcal{A}}(\Omega_{\mathcal{K}}(\mathcal{A}),\mathcal{E}')$ are the canonical sheaf isomorphisms, commutes.

Proof. Fix an open set U in X, and let $\vartheta\in\mathcal{H}om_{\mathcal{A}'}(\Omega_{\mathcal{K}'}(\mathcal{A}'),\mathcal{E}')(U) = \mathrm{H}om_{\mathcal{A}'|_U}(\Omega_{\mathcal{K}'}(\mathcal{A}')|_U,\mathcal{E}'|_U)$. For every $\vartheta\in\mathcal{H}om_{\mathcal{A}'}(\Omega_{\mathcal{K}'}(\mathcal{A}'),\mathcal{E}')(U)$, by definition, one has

$$(\mathcal{H}om(\Omega_0(\varphi),1_{\mathcal{E}'}))_U(\vartheta) := \vartheta\circ\Gamma(\Omega_0(\varphi))|_U \equiv \vartheta\circ\Omega_0(\vartheta)|_U.$$

One the other hand, for any $\alpha\in\mathcal{H}om_{\mathcal{A}'}(\Omega_{\mathcal{K}}(\mathcal{A})\otimes_{\mathcal{A}}\mathcal{A}',\mathcal{E}')(U)$,

$$(r_{\mathcal{A}})_U(\alpha) := \alpha\circ\Gamma(\iota_{\mathcal{A}})|_U \equiv \alpha\circ(\iota_{\mathcal{A}})|_U.$$

Thus, one has

$$\begin{aligned}(\sigma_{\mathcal{A}} \circ r_{\mathcal{A}} \circ \mathcal{H}om(\Omega_0(\varphi), 1_{\mathcal{E}'}))_U(\vartheta) &= (\sigma_{\mathcal{A}} \circ r_{\mathcal{A}})_U(\vartheta \circ \Omega_0(\varphi)|_U) \\ &= (\sigma_{\mathcal{A}})_U(\vartheta \circ \Omega_0(\varphi)|_U \circ (\iota_{\mathcal{A}})|_U) \\ &= \vartheta \circ \Omega_0(\varphi)|_U \circ (\iota_{\mathcal{A}})|_U \circ (\partial_{\mathcal{A}/\mathcal{K}})|_U \\ &= \vartheta \circ (\Omega(\varphi))|_U \circ (\partial_{\mathcal{A}/\mathcal{K}})|_U \\ &= \vartheta \circ (\partial_{\mathcal{A}'/\mathcal{K}'})|_U \circ \varphi|_U \\ &= C(\varphi)_U(\vartheta \circ (\partial_{\mathcal{A}'/\mathcal{K}'})|_U) \\ &= (C(\varphi)_U \circ (\sigma_{\mathcal{A}'})_U)(\vartheta).\end{aligned}$$

■

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