5th International Conference on Heat Transfer, Fluid Mechanics, and Thermodynamics
1-4 July 2007, Sun City, South Africa
BR2

Heat transfer intensification using a cellular automaton

BOICHOT Raphaël, LUO Lingai*

*Corresponding author.
Laboratoire d'Optimisation de la Conception et Ingénierie de l'Environnement.
Polytech'Savoie, Site de Savoie Technolac
73376 Le Bourget du Lac, France
Tel : +33479758193 Fax : +33479758144
E-mail : Lingai.luo@univ-savoie.fr

Abstract

The aim of this study is to propose an optimisation method for heat transfer in a complex configuration inspired from well known constructal theory. Two cases of optimisation will be presented here: the case of a square heater cooled by a single-sided heat sink and the case of a disc-shaped area to be cooled with a central heat sink. The method used for optimisation is a cellular automaton which can progressively modify the shape of a conductive path linked to the heat sink with very simple rules. The rules imitate natural morphogenesis: as motionless organisms can grow along nutrient or water gradients with roots and pseudopodia, the cellular automaton created here initiates growth around the conductive path where thermal gradients are high and initiates destruction of conductive matter where gradients are low, so that the amount of conductive matter remains constant. The actual limitations of this procedure are exposed.

Keywords: Cellular automaton, constructal, heat conduction, auto optimisation.

Nomenclature

C_p	Heat capacity	$J.kg^{-1}K^{-1}$
K	Conductance	$J.s^{-1}m^{-1}K^{-1}$
k_p / k_0	Ratio of conductivities	$[-]$
p	Volume heating	$J.s^{-1}m^{-3}$
T	Temperature	K
λ	Thermal conductivity	$J.s^{-1}m^{-1}K^{-1}$
ρ	Density	$kg.m^{-3}$

1. Introduction

Observation of natural structures like trees, roots, or river basins provides good examples of total optimisation under multiple constraints, sometimes on several length scales. Shaped by time, natural selection and environmental conditions, these structures evolve to ease the flow through them and consequently minimise entropy production. These constructing laws (summarized by Bejan and Lorente [1]) are now well understood and have been applied to physical problems since the arrival of constructal theory in 1997 (Bejan [2], [3] and Bejan and Tondeur [4]). Bejan attempted to understand the reason a branching structure appeared, assuming and demonstrating analytically that it was the best way to relate a sink to a distributed source with a limited quantity of high-conductivity matter (e.g. metal, low-porosity material).

At this time, the main limitations of constructal theory was its purely analytical development leading to fixed angle elbows and constant path diameter tree shapes. Ledezma and al. [6] initiated a trend consisting in numerically adding freedom to the shapes obtained to improve their performances. Errera and Bejan [7] finally obtained an infinite freedom to morph by studying phenomena leading to the construction of river basins. But the approach of Errera and Bejan [7] has a limitation : even if the freedom to morph is infinite - the authors removed blocks from a wet terrain when pressure gradients are up to a critical value, by a step by step manner, to construct a tree network – it is unsure that the shapes obtained are not situated in a local minimum. Indeed, only the action consisting in removing a low conductivity block is possible, the shape can never reverse its evolution and destruct

eventual intermediary branches that become useless during morphogenesis.

This is why we propose a new manner to solve this ten years old problem. The domain of application here is heat conduction, but this is mathematically similar to Darcy flow. We choose to extend the method proposed by Errera and Bejan by starting from an initial conductive path that is progressively shaped by gradients equalization, both by removing and adding matter locally.

Indeed, for a given non-optimised shape of a conductive path, the presence of high thermal gradients at the outer frontier (at the low-conductive material side) means that areas with high temperatures (compared to the path local temperature) could be drained by adding conductive matter locally. On the other hand, the presence of low thermal gradients at the inner frontier (in the high-conductive material) means that conductive material drains little heat and could be used in other places of the area to cool. Consequently, little by little, using a cellular automaton, the distribution of conductive matter could be optimised according to a thermal gradients equalization principle to try to obtain the minimal thermal resistance.

Two cases of application for cooling will be presented here: the case of a square heater for which Bejan [2] gave a solution and the case of a disc-shaped area for which Rocha et al. [5] found an approximated solution. The solutions of [2] and [5] permits to give an estimation of the performances of the shapes obtained in term of temperature decrease of the domain.

2. Theoretical background

Heat equation

The method used here for solving the heat equation in 2D is the centered finite difference scheme with five points. This can easily give the temperature distribution with square elements on the surface to optimise. The heat equation for a material in non-stationary conditions is defined as follows:

$$\rho C_p \frac{\partial T}{\partial t} = \lambda \Delta T + p \qquad (1)$$

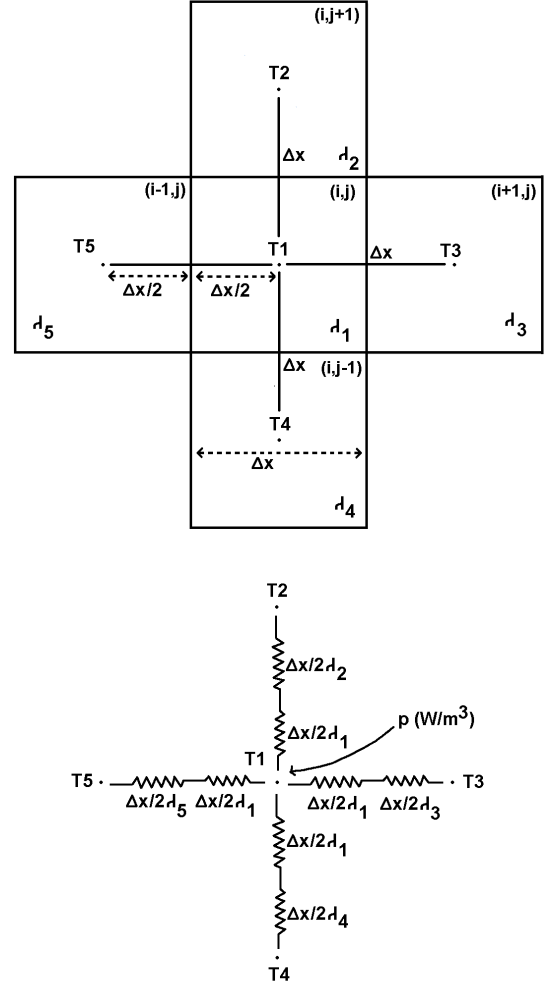The domain to study then is dicretised in small elemental squares as represented in Figure 1.

Figure 1: Five-cell discretisation for the finite difference method and thermal equivalent resistance net.

Either applying a finite difference discretisation on equation (1) or simply making a heat flux balance on the central node represented in Figure 1 (down) gives the following equation for the calculation of the temperature equilibrium in a stationary state (indices according to Figure 1):

$$T_1 = \frac{\sum_{n=2}^{5} K_{1 \to n} T_n + p_1(t)\Delta x^2}{\sum_{n=2}^{5} K_{1 \to n}} \qquad (2)$$

With $\qquad K_{1 \to n} = \dfrac{\Delta x}{\dfrac{\Delta x / 2}{\lambda_1} + \dfrac{\Delta x / 2}{\lambda_n}} \qquad (3)$

Equation (2) is applied in each cell of the calculation domain by iteration until convergence of the temperature solution. The norm of the thermal gradient at each node is evaluated at each temperature equilibrium by the following equations (indices according to Figure 1):

$$\left\| \overrightarrow{gradT} \right\| = \sqrt{\frac{(T_3 - T_5)^2}{2\Delta x} + \frac{(T_2 - T_4)^2}{2\Delta x}} \qquad (4)$$

Boundary conditions

With the finite difference method, the domain is divided into many square cells. At least four types of cells are needed to treat a large number of problems with this algorithm:

- High-conductivity isothermal cells (heat sink);
- Low-conductivity cells with a constant volume power generation rate (distributed source);
- High-conductivity cells without volume power generation (conductive path);
- Very-low-conductivity cells to close the domain (adiabatic borders).

The combination of all these cells easily defines unspecified geometries. Practically speaking, the geometry is drawn under a classical graphical tool, each color for a pixel representing a type of cell (a pixel has a square shape). Then the image file is interpreted by the procedure to store the characteristics of all cells in several matrices that can be treated by the numerical procedure.

The cellular automaton

The aim of the calculation procedure is to provide the best arrangement of conductive cells to decrease the difference between the maximal temperature of the domain and the temperature of the heat sink. The principle is to attempt to equalize the thermal gradients at the frontier between materials. It is possible to approach this configuration iteratively by displacing elements of the high-conductive material linked to the heat sink in a step-by-step manner. The procedure chosen is the following: for each calculation step (calculation of a temperature-stable solution for a given configuration of cells), a small amount of conductive material against the internal frontier of conductive path is displaced from the area where gradients are the lowest to the area against the external frontier where thermal gradients are the highest. Intuitively adding conductive matter where temperature gradients are high will support the growth of branches toward high temperatures and will facilitate heat conduction around this area. Similarly, if gradients are locally low in high-conductivity material, the heat transfer is locally poor, and removing matter here has less importance. The total number of cells remains the same during calculations.

This calculation procedure has three major advantages. The first is that the rate of cells moving at each step can be relatively high (typically 6%).

The second is that the localisation of the highest and lowest gradients can be rapidly evaluated, without waiting for the complete stable solution of the heat equation. Concretely, this means that the computation can be substantially accelerated. The third advantage is that only good solutions are tested: the configuration at one step is always better than the preceding one.

3. Optimisation results

Square heater cooled by a sided heat sink

The case of a square heater with a sided heat sink was widely studied by Bejan [2]. The aim here is to try to compare the result of cellular automaton directly to the analytical constructal result. The problem encountered here is that the minimum size that can be taken into account in our case is one pixel (equal to one cell) and the maximum size of the domain to obtain convergence in a reasonable amount of time is 200×200 pixels (cells). Moreover, numerical convergence is difficult to achieve if the ratio between high and low conductivity is very high, because the slenderness of branches to obtain is too high compared to the size of the cells (the procedure converges with difficulty). Consequently, the constraint of discretisation by numerical simulation imposes testing at best a ratio of conductivity between materials of 200. Taking a minimal width of the initial path of one cell, a square elemental area of 50×50 cells is found to be optimal. Applying four levels of optimisation as proposed by Bejan [2] leads to the geometry represented in Figure 2 (first picture). The size of cells is 1×1 mm. As the temperatures found will be normalised, the power generation rate has no importance here. Four other conductivity ratios between the conductive path and the heat-generating material are tested with the same initial configuration: $k_p/k_0 = 50$, 10, 5 and 2. Indeed constructal theory gives no solution for these four other ratios and our discretisation constraint: the constructal solution for $k_p/k_0 = 200$ will be used as a probable best shape in all the other cases tested. Now the cellular automaton algorithm is applied. The results in terms of shapes are given in Figure 2.

In this figure, the ratio of conductivity is the same in the two one-line images and varies from the highest value on the top to the lowest value on the bottom. The color scale refers to the initial configuration equilibrium in term of temperature and thermal gradients, which is not represented here.

Initial configuration

- ☐ Heat generating material
- ■ Heat sink
- ■ High conductivity material
- ■ Adiabatic material

External thermal gradients
■ Min ■ Max

Temperature field
■ Min ■ Max
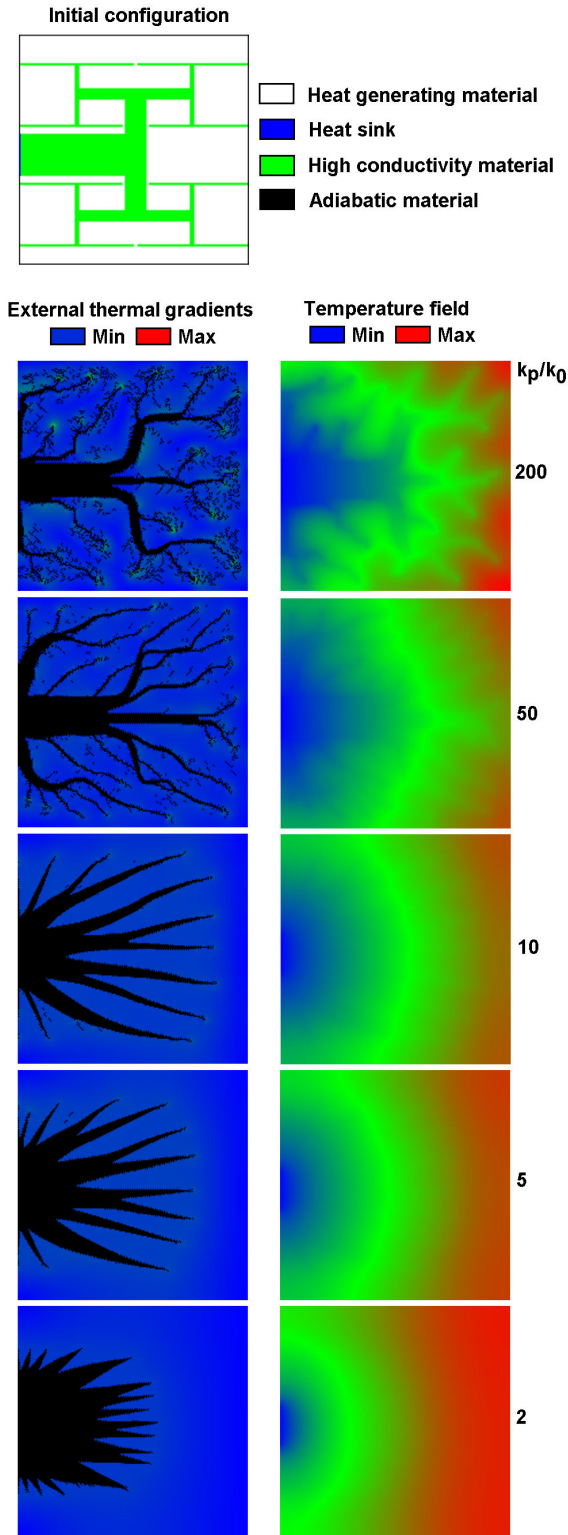

k_p/k_0

200

50

10

5

2

Figure 2: Initial geometry and cellular automaton geometry found (left column: thermal gradients in the low-conductive material, right column: temperature field). Conductive path is coloured in black in the left column images.

The average and maximum temperatures at equilibrium are divided by the value obtained in initial configuration for the same parameters and are presented in Table 1 (in percent).

Ratio of conductivity k_p/k_o	200	50	10	5	2
Average T / Average T initial	79.3%	70.8%	75.4%	83.2%	95.9%
Maximal T / Maximal T initial	98.1%	87.8%	85.3%	89.3%	95.7%

Table 1: Ratio of temperatures for optimised shape evaluation

Only the results with a conductivity ratio of 200 can be directly compared to constructal initial shape. In this configuration, we can obtain a slight decrease in the maximum temperature and a significant decrease in the average temperature of the domain. For all the other cases, the cellular automaton gives better results than initial shape. Two major trends can be observed: for k_p/k_0 = 10, 5 and 2, the structure found at convergence has a radiating shape from the heat sink to the surface. This is intuitively the best shape: the thermal resistances along streamlines from the surface to the heat sink are at best decreased by this configuration. For k_p/k_0 = 200 and 50, the shape obtained seems to derive from the initial shape. We observed this phenomenon for many different initial shapes. We explain this trend by the fact that the width of the potential branches becomes smaller than the cells used: approaching the tip, the numerical code rapidly becomes unstable and the shape freezes in the incompletely optimised configuration. This trend has a direct implication in results: the results for k_p/k_0 = 200 and 50 should be significantly better. This problem could be avoided by using a finer mesh and a higher cell moving rate. For: k_p/k_0 < 10, the temperature equilibrium only has to be evaluated approximately to achieve a rapid convergence of shape, which is independent of the initial shape used. We verified this assumption with several other tests.

Disc area cooled by a central heat sink

The case of a disc area cooled by conduction through a central heat sink has no obvious solution. Following Rocha et al. [5], an optimal configuration could be found after a process that requires a large number of approximations, in particular at the branching stage: the surface after the first level of blades is not completely covered because the optimal fixed triangular shapes do not precisely intercept the surface to cool. Rather than trying to simulate an optimal shape according to Rocha, the choice is made here to start from an non-optimised circular conductive area and to compare the performance of the cellular automaton with Rocha's analytical course. The maximal conductivity ratio that can be simulated is 200. The size of the cells is 1×1 mm, so the circle radius is 0.1 m (200×200 cells). With a 31% proportion of conductive matter and a conductivity ratio of 200, Rocha's analytical course gives an optimal dimensionless thermal resistance of 0.022. Four other conductivity ratios are tested with the same initial shape: k_p/k_0 = 50, 10, 5 and 2. The results in

terms of shapes are given in Figure 3, and the temperature equilibrium compared to that of the initial shape in Table 2. The color scale refers to the initial configuration.

Initial configuration

- □ Heat generating material
- ■ Heat sink
- ■ High conductivity material
- ■ Adiabatic material

Ratio of conductivity k_p/k_0	200	50	10	5	2
Average T / Average T initial	16,4%	33,1%	74,2%	89,7%	98,9%
Maximal T / Maximal T initial	17,6%	31,7%	72,7%	89,2%	98,7%

Table 2: Ratio of temperatures for optimised shape evaluation

In figure 3, the ratio of conductivity is the same in the two one-line images and varies from the highest value on the top to the lowest value on the bottom.

Table 2 gives the average and maximal temperature at equilibrium divided by the value obtained with the initial configuration for the same parameters. In each case, the optimised configuration provides better cooling than the initial one. The dimensionless thermal resistance found for the comparable case ($k_p/k_0 = 200$, analytical constructal theory becomes invalid for low k_p/k_0) is 0.010, which is two times less the analytical constructal dimensionless thermal resistance.

Figure 3 shows that the higher the ratio of conductivity is, the fewer first branches there are (only eight for $k_p/k_0 = 200$), but the higher the tendency to branch is. On the contrary, the branches are numerous for $k_p/k_0 = 2$ (28), but there are no secondary constructs. As for the preceding case, the branches tend to be thinner and longer if the ratio of conductivity increases, and the temperatures fall compared to the initial circular conductive path. Compared to Rocha's analytical geometries, the solutions presented here also show that the optimal branches could be shorter than the circle radius, which is not evocated by this author.

4. Discussion

It is necessary to explain in what the gradient equalization method can be similar to the maximal temperature minimization. In a separated study we demonstrated that this two methods are similar under certain conditions. First with this kind of problem we observed that the maximal temperature minimization is achieved only if the temperatures at the adiabatic boundary of the domain are equalized. We also observed that the gradients equalization at the frontier between materials and the temperature equalization at the external adiabatic boundary coincide when the distance to travel for heat from the boundary to the heat sink is similar in all directions (approximated case of a centered heat sink for example), or if the k_p/k_0 ratio is high. So the optimisation method proposed here is a convenient approximation, easy to implement, that can reach the maximal temperature minimization under this two conditions. In all other cases, the less this two conditions are respected, the less optimised the shape obtained is.

The main observation from results is that the parameter that shapes the conductive path is the

External temperature gradients   **Temperature field**
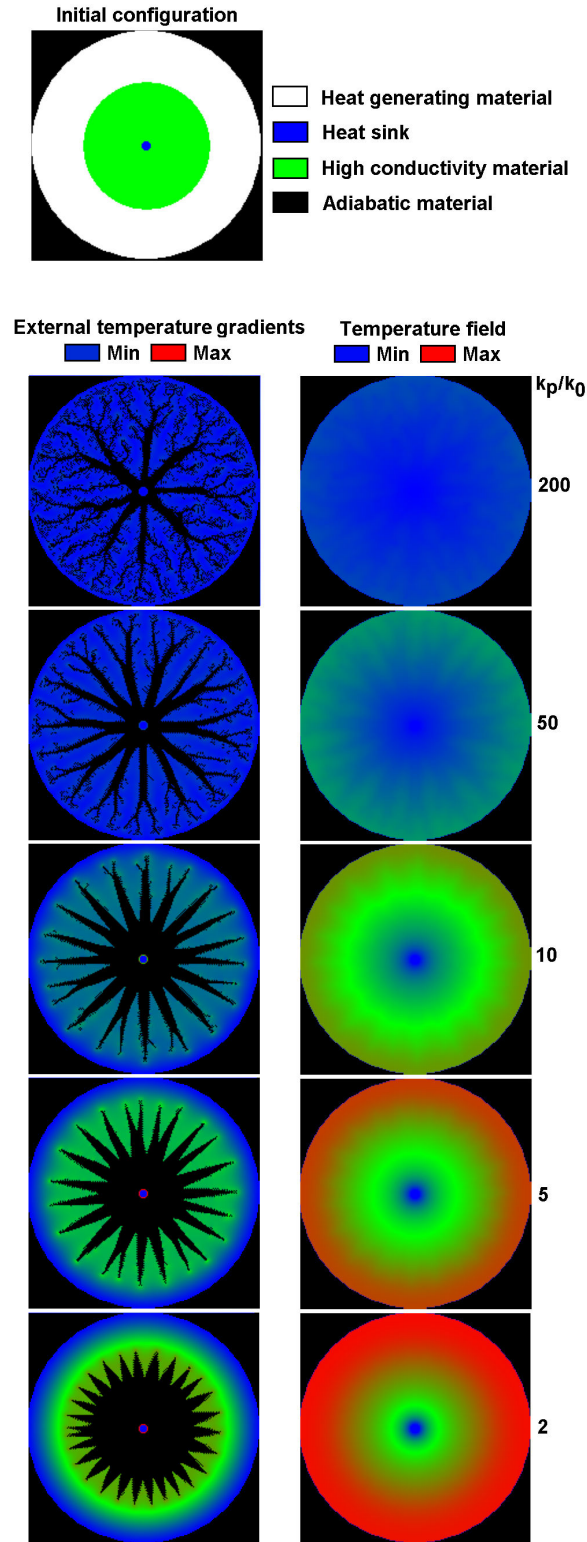■ Min ■ Max   ■ Min ■ Max

Figure 3: Non-optimised initial geometry (first image) and cellular automaton geometry found (left column: thermal gradients in the low-conductive material, right column: temperature field). Conductive path is coloured in black in the left column images.

conductivity ratio, as stated in [7]. Indeed, the higher the k_p/k_0 ratio is, the more slender the branches become, (and secondary constructs become numerous). This trend can be explained quite simply: if a branch is long and thin, its internal resistance increases and its ability to drain heat decreases. The procedure can support the growth of long branches or the generation of new branches only if their temperature stays low compared to the surrounding matter (in other words, the thermal gradient remains high compared to other areas of the domain). This is only possible while the resistance of the total branch has the same order of magnitude as the resistance of the local area that the branch drains. If this is not the case, the growth will continue towards this equipartition of resistance or the path will be dismantled little by little. Consequently, high-conductivity ratios produce better territory occupation by the conductive material. On the other hand, if this ratio is low, internal resistance increases rapidly during their growth and their extension stops early. Only thick conductive branches with a high angle at their tips can be sustained in this environment.

Direct comparison with analytical method for constructal theory is difficult: cellular automaton offers a very limited convergence if ratio $k_p/k_0 > 100$, due to the size of elemental cells (i.e. time to converge). On the contrary, analytical method must use a high ratio k_p/k_0 to facilitate the course. So the only solution is to compare the upper limit of cellular automaton (in term of conductivity ratio) which is the lower limit of analytical method. This is why for each configuration only one ratio can be used for comparison. By this aspect, the two methods can in fact be regarded as complementary.

However, severe limitations are encountered with this procedure. The first limitation (as stated before) is the convergence of the code, which is limited by the size of the elemental cell and the time allowed for the calculation. It can be observed for the convergence of Figures 2 and 3 that for $k_p/k_0 = 200$ and 50, the convergence is not at all satisfactory (residual mal-distribution of external gradients) because the procedure rapidly falls into a situation where the cells changing at each iteration are the same, since the conductive domain becomes dispersed and discontinuous. We tested different cases to find that the main reason is that the cells are too coarse for this ratio of conductivity and the cells moving rate too low. Moreover, it seems that in this same case it is necessary to wait for a better temperature equilibrium (compared with the rough equilibrium needed between each iteration with low-conductivity ratios). The parameters for a systematic good and fast convergence of the shape have still to be found.

5. Conclusion

The cellular automaton method can provide an interesting tool for constructal theory by using an infinite freedom to morph and a reverse evolution that avoid local minima. This is an improvement compared to the work of Errera and Bejan [7]. With two examples, we show that cellular automaton method can give better performances than analytical method for constructal theory. So the equipartition of gradients along the frontier of the conductor is a convenient method to optimise the tree shape. However we observe that this is only an approximation.

The geometry is very easily defined and treated by the procedure (as a pixel matrix), so even non-symmetrical or unspecified distributed sources can be optimised with exactly the same computational cost as the most simple problems. The main actual limitation that remains is to found the good parameters (cells moving rate, number of cells, number of iterations) to achieve a reliable convergence with the minimum time consumption.

References

[1] A. Bejan, S. Lorente. Conctructal theory of generation of configuration in nature and engineering. Journal of applied Physics 100, 041301, 2006.

[2] A. Bejan. Conctructal theory network of conducting paths for cooling a heat generating volume. International Journal of Heat and Mass Transfer Vol 40, No. 4, pp. 799-816, 1997.

[3] A. Bejan. Constructal network for fluid flow between a finite-size volume and one source or sink. Rev. Gen. Therm. Vol. 36, pp. 592-604, 1997.

[4] A. Bejan, D. Tondeur. Equipartition, optimal allocation, and the constructal approach to predicting organization in nature. Rev. Gen. Therm. Vol. 37, pp.165-180, 1998.

[5] LOA. Rocha, S. Lorente, A. Bejan. Constructal design for cooling a disc-shaped area by conduction. International Journal of Heat and Mass Transfer Vol 45, pp. 1643-1652, 2002.

[6] G.A. Ledezma, A Bejan, M. R. Errera. Constructal tree networks for Heat transfer. Journal of Applied Physics, Vol. 82, pp. 89-100, 1997.

[7] M. R. Errera, A. Bejan. Deterministic tree network for river drainage basins. Fractals, Vol. 6, 1998, pp. 245-261.