Convergence of Ishikawa Iterations on Noncompact Sets.

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Abstract

Recall that Ishikawa's theorem [4] provides an iterative procedure that yields a sequence which converges to a fixed point of a Lipschitz pseudocontrative map $T : C \to C$, where C is a compact convex subset of a Hilbert space X. The conditions on T and C, as well as the fact that X has to be a Hilbert space, are clearly very restrictive. Modifications of the Ishikawa's iterative scheme have been suggested to take care of, for example, the case where C is no longer compact or where T is only continuous. The purpose of this paper is to explore those cases where the *unmodified* Ishikawa iterative procedure still yields a sequence that converges to a fixed point of T, with C no longer compact. We show that, if T has a fixed point, then every Ishikawa iteration sequence converges in norm to a fixed point of T if C is boundedly compact or if the set of fixed points of T is "suitably large". In the process, we also prove a convexity result for the fixed points of continuous pseudocontractions.

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1 Preliminaries and notation:

Throughout this paper, X is a real Hilbert space. The default topology on X will be the norm topology. For any map $T : M \to M$, where M is a subset of X, we denote the set of all fixed points of T by $\mathcal{F}(T)$. If u and v are any distinct vectors of X, we will use $\Gamma_+(u, v)$ to denote the set of all $x \in X$ for which $\langle x - u | v - u \rangle > ||v - u||^2$ and $\Gamma_-(u, v)$ to denote the set of all $x \in X$ for which $\langle x - u | v - u \rangle < ||v - u||^2$.

Definition 1.1 *Let M be a nonempty subset of X and let $T : M \to M$ be any map. Then*

(a) T *is* pseudocontractive, or a pseudocontraction, *if* $\langle Tx - Ty, x - y \rangle \leq ||x - y||^2$ *for every* x *and* y *in* M.

(b) T *is* hemicontractive, or a hemicontraction, *if* $\mathcal{F}(T) \neq \emptyset$ *and* $Tx \in \overline{\Gamma_-(u, x)}$ *for every* x *in* M *and* u *in* $\mathcal{F}(T)$.

In the above definitions, it is clear that, if $\mathcal{F}(T) \neq \emptyset$, then T is a pseudocontraction implies that T is a hemicontraction

Definition 1.2 *Let C be a nonempty closed convex subset of X and $T : C \to C$ be any map. An Ishikawa sequence x_n for T, with starting point $x_0 \in C$, is recursively defined as follows:*

$$x_{n+1} = x_n + \alpha_n(Ty_n - x_n) \qquad y_n = x_n + \beta_n(Tx_n - x_n)$$

where α_n and β_n are real sequences for which $0 \leq \alpha_n \leq \beta_n < 1$ for all $n \in \mathbb{N}$, $\lim \beta_n = 0$, and $\sum_{n\in\mathbb{N}} \alpha_n\beta_n$ is divergent.

It is worth noting that the proof of Lemma 2.1 in [2] can be slightly modified to show that any convergent Ishikawa sequence for a *continuous* T, in the setting of Definition 1.2, converges to a fixed point of T. This convergence is guaranteed under certain conditions on C and T. A typical result in this regard, and the main focus of this paper, is the following original result of Ishikawa, [4]:

Theorem 1.3 *Let C be a nonempty compact convex subset of X and let $T : C \to C$ be a Lipschitz pseudocontraction of C. Then every Ishikawa sequence for T converges to a point of $\mathcal{F}(T)$.* $\square$

We remark that Liu Qihou [8], shows that this result remains true if T is a continuous hemicontraction, providing $\mathcal{F}(T)$ has only finitely many points. As we see from the convexity Theorem 1.5 below, the latter condition is equivalent to "$\mathcal{F}(T)$ is a singleton".

Recall that a subset M of X is called *boundedly compact* if $M \bigcap A$ is compact for any closed bounded subset A of X. The purpose of this paper is to show that, provided $\mathcal{F}(T)$ remains nonempty, the conclusion in Theorem 1.3 remains valid if (a) the the condition "C is compact" is replaced with "C is boundedly compact" or (b) the compactness condition is relaxed but $\mathcal{F}(T)$ is 'large' in X, where 'large' is to be precisely defined later. In fact, we will prove these results for T a Lipschitz hemicontraction. Other extensions of Ishikawa's result have been obtained by imposing stronger conditions on T, see for example [3]. In similar spirit, Yuan Qing and Liu Qihou [9] have also extended a result of Borwein and Borwein [1] to noncompact intervals.

If one replaces the condition "compact" on the domain C of Theorem 1.3 by a "closed and bounded", then the Lipschitz pseudocontraction condition on T is still sufficient to guarantee the existence of a fixed point [6]. Early indications of this result can be found in, inter alia, the work of Martin ([5], Proposition 4). If boundedness is removed, then of course a fixed point can no longer be guaranteed. In this case, we naturally turn to hemicontractive maps for which the existence of a fixed point is part of the definition. As pointed out earlier, such maps are generalizations of pseudocontractive maps with fixed points.

We now show that, under conditions that are weaker than those of Theorem 1.3, a nonempty $\mathcal{F}(T)$ is always convex; and note that mere continuity of T is sufficient for the result. But first, we prove the following geometric lemma.

Lemma 1.4 *Let $w = \frac{1}{2}(u + v)$ for some distinct vectors u, $v \in X$ and let $z \in X$ be such that $z \neq w$ and such that $\langle z - w | u - v\rangle = 0$. Then $\overline{\Gamma_-(u, z)} \bigcap \overline{\Gamma_-(v, z)} \subset \overline{\Gamma_-(w, z)}$.*

Proof:
Let x be in $\overline{\Gamma_-(u, z)}$. Then, by definition, $\langle x - u|z - u\rangle \leq ||z - u||^2$. Hence

$$\langle(x - w) + (w - u)|(z - w) + (w - u)\rangle \leq ||(z - w) + (w - u)||^2,$$

which immediately yields

$$\langle x-w|z-w\rangle + \langle x-w|w-u\rangle \leq ||z-w||^2.$$

If x is also in $\overline{\Gamma_-(v,z)}$, then similarly,

$$\langle x-w|z-w\rangle + \langle x-w|w-v\rangle \leq ||z-w||^2$$

is also true. Adding the last two inequalities, and noting that $2w-u-v=0$, immediately leads to the required result. □

Theorem 1.5 *Let C be a nonempty closed convex subset of X and let $T: C \to C$ be a continuous hemicontraction. Then $\mathcal{F}(T)$ is convex.*

Proof:
Let distinct vectors u and v be both in $\mathcal{F}(T)$ and and let $w = \frac{1}{2}(u+v)$. Because T is continuous, it is sufficient to show that w is in $\mathcal{F}(T)$. The hemicontraction condition yields $Tw \in \overline{\Gamma_-(u,w)} \bigcap \overline{\Gamma_-(v,w)} = \{x \in X : \langle x-u|v-u\rangle = ||v-u||^2\} = \Gamma$, say. Suppose now, for the purpose of establishing a contradiction, that $Tw \neq w$. For each $\lambda \in (0,1)$, let $w_\lambda = w + \lambda(Tw - w)$. Then, for each $\lambda \in (0,1)$, we have $w_\lambda \neq w$ and, because Tw is in Γ, we also have that $\langle w_\lambda - w|u-v\rangle = 0$. By Lemma 1.4, the hemicontraction condition $Tw_\lambda \in \overline{\Gamma_-(u,w_\lambda)} \bigcap \overline{\Gamma_-(v,w_\lambda)}$ implies that $Tw_\lambda \in \overline{\Gamma_-(w,w_\lambda)} = \overline{\Gamma_+(Tw,w_\lambda)}$, for all $\lambda \in (0,1)$. If we put $\Gamma_\lambda = \overline{\Gamma_+(Tw,w_\lambda)}$, then it is clear that w_λ is the (unique) vector in Γ_λ that is nearest to Tw. Hence $||Tw - Tw_\lambda|| \geq (1-\lambda)||w - Tw||$ for all $\lambda \in (0,1)$. By considering the limit $\lambda \to 0$, it is clear that there is a contradiction with the continuity of T at w, completing the proof. □

Let $T: C \to C$ be a Lipschitz hemicontraction with Lipschitz constant $L > 0$. As is well known, T remains a hemicontraction under a u translation for any u in X. Indeed, consider $\tilde{C} = C - u = \{c - u : c \in C\}$, the u translate of C. Then $\tilde{T} : \tilde{C} \to \tilde{C}$, the corresponding u translate of T, defined by $\tilde{T}(\tilde{x}) = Tx - u$ for each $\tilde{x} \in C$, where $\tilde{x} = x - u$ is the u translate of x, is also a Lipschitz hemicontraction of $\tilde{C}$ with Lipschitz constant L. Clearly, $x \in \mathcal{F}(T)$ if and only if $\tilde{x} \in \mathcal{F}(\tilde{T})$. Furthermore, if we let $\tilde{x}_n = x_n - u$ and $\tilde{y}_n = y_n - u$, then $x_{n+1} = x_n + \alpha_n(Ty_n - x_n)$ and $y_n = x_n + \beta_n(Tx_n - x_n)$ if and only if $\tilde{x}_{n+1} = \tilde{x}_n + \alpha_n(\tilde{T}\tilde{y}_n - \tilde{x}_n)$ and $\tilde{y}_n = \tilde{x}_n + \beta_n(\tilde{T}\tilde{x}_n - \tilde{x}_n)$. We also note that, if $\tilde{p}$ and $\tilde{q}$ are the respective u translates of $p \in C$ and $q \in C$, then $Tp - q = \tilde{T}\tilde{p} - \tilde{q}$. When considering an Ishikawa sequence for a hemicontraction $T: C \to C$, it is clear from the preceding discussion that there is no loss in generality in assuming that $0 \in C$ and that $0 \in \mathcal{F}(T)$.

Now we show that, if all the β_n are chosen small enough, then the corresponding Ishikawa sequence has some useful monotonicity properties. We first prove the following lemma:

Lemma 1.6 *Let C be a nonempty closed convex subset of X and let $T: C \to C$ be a Lipschitz hemicontraction of C, Lipschitz constant $L > 0$, such that $0 \in \mathcal{F}(T)$. Let x_n be an Ishikawa sequence for T as set out in Definition 1.2. Given any $\epsilon \in (0,1)$, there exists $N \in \mathbb{N}$ such that, if $n \geq N$ and $u \in \mathcal{F}(T)$, then*

$$\langle -\tilde{x}_n|\tilde{T}\tilde{y}_n - \tilde{x}_n\rangle \geq (1-\epsilon)\beta_n||\tilde{T}\tilde{y}_n - \tilde{x}_n||^2,$$

where $\tilde{T}$, $\tilde{x}_n$ and $\tilde{y}_n$ are the u translates of T, x_n and y_n respectively.

Proof:
Assume the hypothesis of the lemma. As already discussed, we may replace T, x_n and y_n in Definition 1.2, by $\tilde{T}$, $\tilde{x}_n$ and $\tilde{y}_n$ respectively. Hence, for each $n \in \mathbb{N}$, we have

$$|\; ||\tilde{T}\tilde{y}_n - \tilde{x}_n|| - ||\tilde{T}\tilde{x}_n - \tilde{x}_n||\; | \leq ||\tilde{T}\tilde{y}_n - \tilde{T}\tilde{x}_n|| \leq L\beta_n||\tilde{T}\tilde{x}_n - \tilde{x}_n||.$$

Pick $N' \in \mathbb{N}$ such that $L\beta_n < \frac{1}{3}$ for all $n \geq N'$. Then

$$\tfrac{2}{3}||\tilde{T}\tilde{x}_n - \tilde{x}_n|| \leq ||\tilde{T}\tilde{y}_n - \tilde{x}_n|| \leq \tfrac{3}{2}||\tilde{T}\tilde{x}_n - \tilde{x}_n||$$

for all $n \geq N'$. Again, from Definition 1.2, we have that

$$\tilde{y}_n - \tilde{x}_{n+1} = \beta_n(\tilde{T}\tilde{x}_n - \tilde{T}\tilde{y}_n) + (\beta_n - \alpha_n)(\tilde{T}\tilde{y}_n - \tilde{x}_n).$$

Hence, if $n \geq N'$,

$$\langle \tilde{y}_n - \tilde{x}_{n+1} | \tilde{T}\tilde{y}_n - \tilde{x}_n\rangle \geq (\beta_n - \alpha_n)||\tilde{T}\tilde{y}_n - \tilde{x}_n||^2 - \beta_n||\tilde{T}\tilde{x}_n - \tilde{T}\tilde{y}_n||\,||\tilde{T}\tilde{y}_n - \tilde{x}_n|| \geq$$

$$(\beta_n - \alpha_n - \tfrac{3}{2}\beta_n^2 L)||\tilde{T}\tilde{y}_n - \tilde{x}_n||^2.$$

We note here that N' is not depended upon u since *all* the terms in the above double inequality are invariant under any translation.

Now, because $0 \in \mathcal{F}(T)$ and T is a hemicontraction, we have that $\langle x_n | y_n - x_n\rangle \leq 0$ and $\langle y_n | Ty_n - y_n\rangle \leq 0$ for all $n \in \mathbb{N}$. These inequalities remain true under all u translations, providing u remains in $\mathcal{F}(T)$. Therefore, for any $n \in \mathbb{N}$, we have

$$\langle \tilde{y}_n | \tilde{T}\tilde{y}_n - \tilde{x}_n\rangle = \langle \tilde{y}_n | \tilde{T}\tilde{y}_n - \tilde{y}_n\rangle + \langle \tilde{y}_n | \tilde{y}_n - \tilde{x}_n\rangle \leq \langle \tilde{y}_n | \tilde{y}_n - \tilde{x}_n\rangle =$$

$$\langle \tilde{y}_n - \tilde{x}_n | \tilde{y}_n - \tilde{x}_n\rangle + \langle \tilde{x}_n | \tilde{y}_n - \tilde{x}_n\rangle \leq ||\tilde{y}_n - \tilde{x}_n||^2 \leq \tfrac{9}{4}\beta_n^2||\tilde{T}\tilde{y}_n - \tilde{x}_n||^2.$$

Now let $\epsilon > 0$ be arbitrary, and pick $N \geq N'$ such that $(\frac{9}{4} + \frac{3}{2}L)\beta_j \leq \epsilon$ for all $j \geq N$. Then, if $n \geq N$,

$$\langle -\tilde{x}_n | \tilde{T}\tilde{y}_n - \tilde{x}_n\rangle = \langle \tilde{x}_{n+1} - \tilde{x}_n | \tilde{T}\tilde{y}_n - \tilde{x}_n\rangle + \langle \tilde{y}_n - \tilde{x}_{n+1} | \tilde{T}\tilde{y}_n - \tilde{x}_n\rangle - \langle \tilde{y}_n | \tilde{T}\tilde{y}_n - \tilde{x}_n\rangle \geq$$

$$[\beta_n - \tfrac{3}{2}\beta_n^2 L - \tfrac{9}{4}\beta_n^2]\,||\tilde{T}\tilde{y}_n - \tilde{x}_n||^2 \geq (1-\epsilon)\beta_n||\tilde{T}\tilde{y}_n - \tilde{x}_n||^2,$$

as required. Once again, we note that the integers N' and N depend only on ϵ, L and the sequence β_n. Thus N is independent of the translation vector $u \in \mathcal{F}(T)$. This completes the proof. □

Proposition 1.7 *Let C be a nonempty closed convex subset of X and let $T : C \to C$ be a Lipschitz hemicontraction of C. Let x_n be an Ishikawa sequence for T as set out in Definition 1.2. Then*

(a) *There exists $N \in \mathbb{N}$ such that, if u is in $\mathcal{F}(T)$, then $||x_{n+1} - u|| \leq ||x_n - u||$ for all $n \geq N$.*

(b) *There exists a subsequence x_{n_j} of x_n for which $Tx_{n_j} - x_{n_j}$ converges to 0.*

Proof:
Let $u \in \mathcal{F}(T)$ and let $\tilde{x}_n$, $\tilde{y}_n$, $\tilde{T}$ be the u translates of x_n, y_n, T respectively. Then, from Definition 1.2, we have

$$||\tilde{x}_{n+1}||^2 = ||\tilde{x}_n||^2 + \alpha_n^2||\tilde{T}\tilde{y}_n - \tilde{x}_n||^2 + 2\alpha_n\langle\tilde{x}_n|\tilde{T}\tilde{y}_n - \tilde{x}_n\rangle.$$

By Lemma 1.6, there exists $N \in \mathbb{N}$, independent of u, such that $||\tilde{T}\tilde{x}_n - \tilde{x}_n|| \leq \frac{3}{2}||\tilde{T}\tilde{y}_n - \tilde{x}_n||$ and $\langle -\tilde{x}_n|\tilde{T}\tilde{y}_n - \tilde{x}_n\rangle \geq \frac{3}{4}\beta_n||\tilde{T}\tilde{y}_n - \tilde{x}_n||^2$ for all $n \geq N$. Hence, for such n,

$$||\tilde{x}_n||^2 - ||\tilde{x}_{n+1}||^2 = -\alpha_n^2||T\tilde{y}_n - \tilde{x}_n||^2 - 2\alpha_n\langle\tilde{x}_n|\tilde{T}\tilde{y}_n - \tilde{x}_n\rangle \geq \tfrac{1}{2}\alpha_n\beta_n||\tilde{T}\tilde{y}_n - \tilde{x}_n||^2 \geq 0,$$

showing that, for each $u \in \mathcal{F}(T)$, the N-tail of the sequence $||x_n - u||$ is decreasing. This completes the proof of the first assertion.

As for the second assertion, we note that the above monotonicity of the N-tail of $||x_n - u||$ implies that $\sum_{n\in\mathbb{N}} \alpha_n\beta_n||\tilde{T}\tilde{y}_n - \tilde{x}_n||^2 = \sum_{n\in\mathbb{N}} \alpha_n\beta_n||Ty_n - x_n||^2$ is convergent. Since $||\tilde{T}\tilde{x}_n - \tilde{x}_n|| \leq \frac{3}{2}||\tilde{T}\tilde{y}_n - \tilde{x}_n||$ for $n \geq N$ we must have that $\sum_{n\in\mathbb{N}} \alpha_n\beta_n||Tx_n - x_n||^2$ is also convergent. The condition $\sum_{n\in\mathbb{N}} \alpha_n\beta_n = \infty$ implies the existence of a subsequence $Tx_{n_j} - x_{n_j}$ of $Tx_n - x_n$ which converges to 0, as claimed. □

Because X has the fixed point property for continuous pseudocontractions, the following is an easy consequence of Proposition 1.7.

Corollary 1.8 *Let C be a nonempty closed bounded (unbounded) convex subset of X and let $T : C \to C$ be a Lipschitz pseudocontraction (hemicontraction) of C. Then every Ishikawa sequence for T weakly converges to a fixed point of T.* □

We now examine two special cases where C is not necessarily compact.

2 The case C is boundedly compact

In this section, we focus our attention to the situation where $T : C \to C$ is a Lipschitz hemicontraction with Lipschitz constant $L > 0$ and $C \subset X$ a convex, boundedly compact set.

Theorem 2.1 *Let C be a nonempty boundedly compact convex subset of X and let $T : C \to C$ be a Lipschitz hemicontraction of C. Then every Ishikawa sequence for T converges to a point of $\mathcal{F}(T)$.*

Proof:
Let x_n be an Ishikawa sequence for T as set out in Definition 1.2. We deduce from Proposition 1.7 that x_n is bounded and that there exists a subsequence x_{n_k} of x_n which converges to some $x \in C$ such that $Tx_{n_k} - x_{n_k}$ converges to 0. Then from

$$||Tx - x|| \leq ||Tx - Tx_{n_k}|| + ||Tx_{n_k} - x_{n_k}|| + ||x_{n_k} - x||,$$

and the continuity of T, we deduce that $x \in \mathcal{F}(T)$. Because $||x_n - x||$ is eventually decreasing, we conclude that x_n converges to x, completing the proof. □

Theorem 2.1 shows that the convergence of Ishikawa sequences may be used to characterize those Lipshitz pseudocontractions T on a boundedly compact closed convex set for which $\mathcal{F}(T)$ is not empty:

Corollary 2.2 *Let C be a nonempty boundedly compact convex subset of X and let $T : C \to C$ be a Lipschitz pseudocontraction of C. Then $\mathcal{F}(T) \neq \emptyset$ if and only if every Ishikawa sequence for T is convergent.* □

3 The case C is not boundedly compact.

In this section we are only interested in X infinite dimensional with the domain C of T no longer boundedly compact. Although we will not be able to prove general convergence for this case, we will show that there is convergence in the special case where $\mathcal{F}(T)$ is "suitably large" in relation to C. To this end, we start with a definition of our notion of "suitably large", as well as a lemma.

Definition 3.1 *We will say that a convex set $M \subset X$ is a large subset of X if there exist $u \in X$ and a closed subspace E of X of finite codimension in X such that M is a full subset of $u+E$, that is M is a subset of $u+E$ with a nonempty interior relative to $u+E$.* □

Lemma 3.2 *Let A be a closed, convex and large subset of X. Let x_n be a sequence in X which weakly converges to q, and for which $||x_n - a|| \geq ||x_{n+1} - a||$ for every $a \in A$ and $n \in \mathbb{N}$. Then x_n converges to q.*

Proof:
We need only consider the case X is infinite dimensional. We suppose that A is a full, closed and convex subset of $u+E$ for some $u \in X$ and some closed subspace E of X of finite codimension $j \geq 0$ in X. We will assume with no loss of generality that $q = 0$. We may further assume that u is a nonzero vector in the interior of A relative to $u+E$ and that u is also in $E^{\perp}$, the orthogonal complement of E in X. We justify this further assumption because one can always make the following "reduction" to match the assumptions: Pick a nonzero u' in the interior of A relative to $u+E$. Then $u'+E = u+E$. Put $E' = \{x \in X : \langle x|u\rangle = 0\} \bigcap E$. Then u' is in $E'^{\perp}$ and the closed subspace E' is of finite codimension $j' \leq j+1$ in X. Furthermore, if we put $A' = (u'+E') \bigcap A$, then A' is clearly a closed, convex and full subset of $u'+E'$ and u' is in the interior of A' relative to $u'+E'$.

For each $n \in \mathbb{N}$, let $x_n = a_n + b_n$, where a_n is in E and b_n is in $E^{\perp}$. It is clear that both a_n and b_n weakly converge to 0, and because $E^{\perp}$ is finite dimensional, b_n converges to 0. It is therefore sufficient to show that a_n converges to 0. Suppose this was not the case. Then we would be able find a subsequence x'_n of x_n for which $\gamma = \lim_{k \in \mathbb{N}} ||a'_n|| > 0$, where $x'_n = a'_n + b'_n$, with a'_n in E and b'_n in $E^{\perp}$. Let $\alpha \in (0, \gamma)$ be such that $(u+E) \bigcap \mathrm{B}(u, \alpha) \subset A$ and put $\mu = \sqrt{||u||^2 + (\gamma - \alpha)^2}$, and $\kappa = \sqrt{||u||^2 + (\gamma - \alpha/2)^2}$. Fix $m_0 \in \mathbb{N}$ such that, if $n \geq m_0$, then $||a'_n|| > 0$, $||x'_n - a'_n|| < \frac{\kappa - \mu}{3}$ and $|\,||a'_n|| - \gamma| < \min\{\alpha/2\,,\, \frac{\kappa-\mu}{3}\}$. Put $z = \frac{\alpha}{||a'_{m_0}||} a'_{m_0}$, $w = \frac{\gamma}{\alpha} z$ and $v = u + z$. Then $||w - a'_{m_0}|| < \frac{\kappa - \mu}{3}$. Because a'_n weakly converges to 0, there exists $k_0 > m_0$ such that $\langle a'_{k_0}|z\rangle \leq \alpha^2/2 = ||z||^2/2$. Hence $||a'_{k_0} - z|| \geq ||a'_{k_0}|| \geq \gamma - \alpha/2 > \gamma - \alpha = ||w - z||$, so that

$$||w - v|| = ||(w-z) - u|| = \mu < \kappa \leq ||(a'_{k_0} - z) - u|| = ||a'_{k_0} - v||.$$

Because $||z|| = \alpha$, we must have that v is in A. We then have

$$||x'_{m_0} - v|| \leq ||x'_{m_0} - a'_{m_0}|| + ||a'_{m_0} - w|| + ||w - v|| \leq$$

$$||x'_{m_0} - a'_{m_0}|| + ||a'_{m_0} - w|| + ||a'_{k_0} - v|| - (\kappa - \mu) <$$

$$||a'_{k_0} - v|| - \tfrac{1}{3}(\kappa - \mu) < ||a'_{k_0} - v|| - ||a'_{k_0} - x'_{k_0}|| \leq ||x'_{k_0} - v||,$$

in contradiction with $||x'_{m_0} - v|| \geq ||x'_{k_0} - v||$. This completes the proof. □

Our last result is an easy corollary of Lemma 3.2.

Theorem 3.3 *Let C be a closed convex subset of X and let $Y = \overline{\mathrm{span}(C)}$. Let $T : C \to C$ be a Lipschitz hemicontraction of C for which $\mathcal{F}(T)$ is large in Y, then every Ishikawa sequence for T converges to a point of $\mathcal{F}(T)$.*

Proof:
Let x_n be an Ishikawa sequence for T as set out in Definition 1.2. Because of Proposition 1.7, we may assume without losing generality that, if u in $\mathcal{F}(T)$, then $||x_{n+1} - u|| \leq ||x_n - u||$ for all $n \in \mathbb{N}$. By Corollary 1.8, x_n weakly converges to some $q \in \mathcal{F}(T)$. Lemma 3.2, implies that x_n converges to q as desired. □

Remark: In the hypothesis of Theorem 3.3, it is clearly necessary to work inside Y in case Y is of infinite codimension in X.

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