Existence result for a class of stochastic quasilinear partial differential equations with non-standard growth

by

Ali Zakaria Idriss

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DECLARATION

I, the undersigned, hereby declare that the thesis submitted herewith for the degree Magister Scientiae to the University of Pretoria contains my own, independent work and has not been submitted for any degree at any other university.

Signature:

Name: Ali zakaria Idriss

Date: December 2010

To my parents

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Abstract

In this dissertation, we investigate a very interesting class of quasi-linear stochastic partial differential equations. The main purpose of this article is to prove an existence result for such type of stochastic differential equations with non-standard growth conditions. The main difficulty in the present problem is that the existence cannot be easily retrieved from the well known results under Lipschitz type of growth conditions [42].

Contents

Chapter 1

Introduction

1.1 Motivation and Preliminaries

The study of quasi-linear partial differential equations with $p(x)$-growth conditions, in the deterministic case, has received much attention recently see for instance [18], [36], [37], [50], [66], and reference given therein. Different approaches are taken in the above papers to establish the existence of solutions for the problems in question. The present work extends a result in Samokin [66] to a class of stochastic partial differential equations. The passage to the result relies heavily on the variable exponent theory of generalized Lebesgue-Sobolev spaces. We study the notion of probabilistic weak solutions of the initial-boundary value problem for equations that generalize the equations of polytropic elastic filtration with random perturbations.

Let D be an open and bounded domain of the Euclidean space $\mathbb{R}^n$, $n \geq 1$ with C^2 boundary ∂D. We consider the cylindrical domain $Q_T = (0, T) \times D$ with some given final time $T > 0$ and denote by Q_t the cylinder $(0, t) \times D$ for $t \leqslant T$. We investigate the initial boundary value problem (I-BVP) for the stochastic parabolic equation

$$du - \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p(x)-2}\frac{\partial u}{\partial x_i}\right) dt =$$

$$f(t,u(t,x))dt + G(t,u(t,x))\; dW(t) \text{ in } Q_T = (0,T)\times D, \tag{1.1}$$

$$u(t,x) = \quad 0 \text{ on } \Gamma = (0,T)\times \partial D, \tag{1.2}$$

$$u(0,x) = \quad u(x) \text{ in } D, \tag{1.3}$$

where u is unknown function, the nonlinear terms $f(t,u)$ and $G(t,u)$ are known functions depending on u, $u_0(x)$ is a given function in $L^2(D)$, W is a $\mathbb{R}^d$-standard Wiener process, d a positive integer and p is a measurable function over D with values in the interval $[1,\infty]$ and is independent of t. Equations of this type appear in the mathematical modeling of various physical phenomena. They model processes ranging from the theory of non-Newtonian fluids to Continuum Mechanics. For more detailed information about the physical applications of these models we refer to [4]-[12], [43], [45], [65] and the bibliography therein. Equation (1.1) is degenerate when the gradient vanishes $(\nabla u = 0)$. If $p(x) = 2$ then we obtain the Laplacian equation. Polytropic filtration describes a large class of non-Newtonian fluids such as natural gas, extraction processes of crude oil, etc.... For more physical background we refer to [4]- [12], [43], [51] and [65]. They have properties such as global existence, global nonexistence, existence and uniqueness, blow-up, qualitative behavior, localization properties of solutions, etc.... We set

$$Au(t) = -\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\left|\frac{\partial u(t,x)}{\partial x_i}\right|^{p(x)-2}\frac{\partial u(t,x)}{\partial x_i}\right), \tag{1.4}$$

with $1 < p(x) < \infty$.

We can motivate the spaces $W_0^{1,p(x)}(D)$ through the following example of transmission problems for nonlinear elliptic equations.

Let D be bounded domain in $\mathbb{R}^n$ with boundary Γ. Let D_1 be a proper sub-domain of D with the boundary Γ_1, and D_2 the domain $D \setminus D_1$. Then D_2 is bounded by Γ_1 and Γ_2. We assume that Γ_1 and Γ_2 are sufficiently smooth. Let $\overrightarrow{n}_1$ (resp. $\overrightarrow{n_2}$) be the field of unit normal vectors to Γ_1(resp. Γ_2) oriented toward the interior of D_2, and $p_1, p_2 \in (1,\infty)$ are constants.

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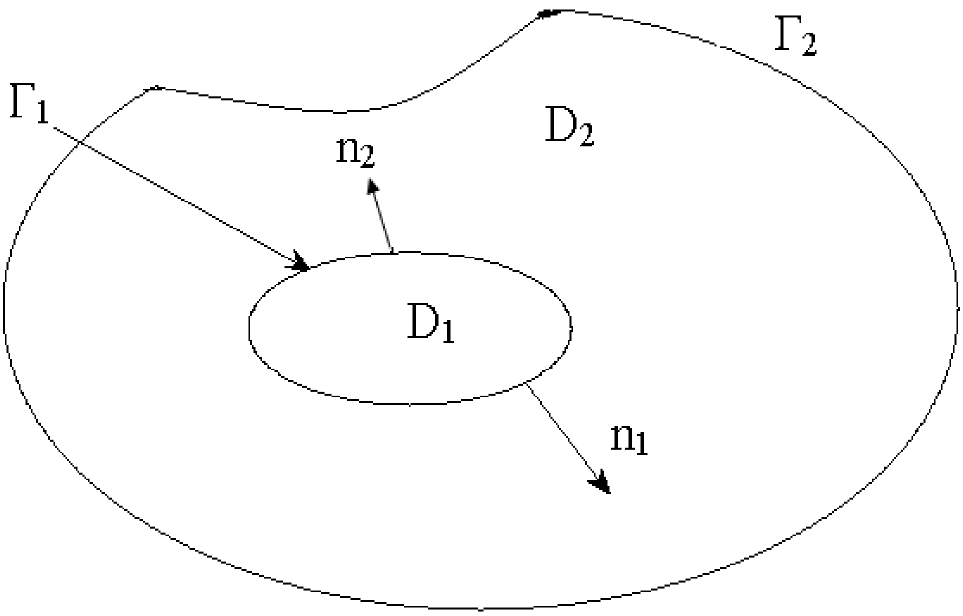

Fig

We consider the operator

$$\Delta_{p_k}\varphi = \operatorname{div}\left(\left|\frac{\partial\varphi}{\partial x}\right|^{p_k-2}\nabla\varphi\right), \tag{1.5}$$

∇ denotes the gradient.

We consider the following transmission problem

$$\Delta_{p_k}u_k = f_k \quad \text{on} \quad D_k, \quad k = 1, 2, \tag{1.6}$$

$$u_1(x) = u_2(x) \quad \text{on } \Gamma_1 \tag{1.7}$$

$$\frac{\partial_{\Delta_{p_2}}u_2}{\partial n_1}(x) = \frac{\partial_{\Delta_{p_1}}u_1}{\partial n_1}(x) \text{ on } \Gamma_1 \tag{1.8}$$

$$u_2 = 0 \quad \text{on} \quad \Gamma_2, \tag{1.9}$$

where,

$$\frac{\partial_{\Delta_{p_k}}\varphi}{\partial n} = \left|\frac{\partial\varphi}{\partial x}\right|^{p_k-2}\nabla\varphi.n.$$

Integrating by parts with test functions φ_k on D_k we get

$$\sum_{k=1}^{2}\int_{D_k}\text{div}\left(\left|\frac{\partial u_k}{\partial x}\right|^{p_k-2}\nabla u_k\right)\varphi_k\,dx$$
$$=-\sum_{k=1}^{2}\int_{D_k}\left|\frac{\partial u_k}{\partial x}\right|^{p_k-2}\nabla u_k\nabla\varphi_k\,dx+\sum_{k=1}^{2}\int_{\partial D_k}\left|\frac{\partial u_k}{\partial x}\right|^{p_k-2}\varphi\nabla u_k n_k\,dx, \tag{1.10}$$

by using the transmission conditions (1.7) and (1.8) in (1.10) with $\varphi_k = u_k$ we end up with

$$\sum_{k=1}^{2}\int_{D_k}\text{div}\left(\left|\frac{\partial u_k}{\partial x}\right|^{p_k-2}\nabla u_k\right)u_k dx=\sum_{k=1}^{2}\int_{D_k}|\nabla u_k|^{p_k}\,dx. \tag{1.11}$$

Defining

$$p(x)=\begin{cases} p_1 & \text{if } x\in\overline{D}_1,\\ p_2 & \text{if } x\in D_2\cup\Gamma_2.\end{cases}$$

We can rewrite (1.11) as

$$\|u\|_{W_0^{1,p(x)}}(D)=\int_D|\nabla u|^{p(x)}\,dx, \tag{1.12}$$

where,

$$u=\begin{cases} u_1 & \text{in } D_1,\\ u_2 & \text{in } D_2.\end{cases}$$

The relations (1.10)-(1.12) suggests that it may be reasonable to seek for an appropriate weak solution u of problem (1.6)-(1.9) in the functional setting of Sobolev space $W_0^{1,p(x)}(D)$ for a.e $t\in[0,T]$. Our aim is to establish an existence result of a probabilistic solution u to the stochastic parabolic problem (1.1)-(1.3).

In the deterministic version of problem (1.1)-(1.3) Samokin gave a detailed investigation of the weak solvability of the problem (1.1)-(1.3) in [66] with $G(t,u)=0$; further references can be found therein. A special interest in the study of such equations is motivated by their application in Science. They appear in the mathematical modeling of non-Newtonian fluids and Continuum Mechanics such as the processes of electro-rheological(ER) fluids (see [43], [65]), filtration through inhomogeneous anisotropic porous media (see [6], [7]). Actually, they are frequently used in optical application such as the processing of digital image, image recovery, etc.... For more information on the possible applications of the real world processes we refer to [4]-[12], [43], [65], [51], [89] where further references can be found. In the last decade, several authors have studied and obtained many results on problems with non-standard growth conditions. For more details we refer to [1], [2], [12], [32].

More recently, similar problems were discussed by many authors; among them, we refer to [85]. The authors of [85] gave a detailed investigation of nonlinear parabolic initial-boundary value problem with $p(x)$-growth conditions with respect to u and ∇u by introducing a compactness method combined with the Galerkin approximation.

The purpose of this dissertation is to prove an existence result for the initial-boundary value problem (1.1)-(1.3) in the functional setting of generalized Sobolev spaces $W^{1,p(x)}(D)$. For the proof we use a Galerkin approximation scheme combined with some deep analytic and probabilistic compactness results.

They are variety of closely related problems that we could not include all the references in this dissertation, we can only cite few [45], [73], [78] and [89]. The case of doubly degenerate parabolic equations with non-standard growth was initially studied by Sango in [70].

For fundamental properties of the generalized Lebesgue space $L^{p(x)}(D)$ and the corresponding Sobolev space $W^{k,p(x)}(D)$ we refer to the work of Samokin [66] and the work of Andrej Kováčik, Žilina and Rákosník [48], where some examples and counter examples on the sobolev embedding theorems can be found. We also note more recently, the works of L. Diening [25], D.E. edmunds and J. Rákosník [30, 31] and X.L Fan [34]. It has been shown in [65] that the crucial difference between the spaces $L^{p(x)}(D)$ and the usual Lebesgue spaces $L^{p}(D)$ is the fact that the elements of the generalized Lebesgue spaces $L^{p(x)}(D)$ are not in general $p(x)$-mean continuous (see definition 3). For applications of the spaces $L^{p(\cdot)}$ in mathematical modeling of electro-rheological fluids we refer to [65]. In the case when $p(x) = p$ is a constant function, many results have been obtained on the existence and regularity properties of the solutions, for instance we refer to the works in the bibliography [53] and [71]. For a localization property of weak solutions for parabolic equations with nonstandard growth conditions, we refer to [41].

The framework followed in this dissertation has proved successful in [14], [21], [23], [63],

[69]-[72] and [90].

Stochastic quasi-linear parabolic equations with non-standard growth are being investigated for the first time in the present dissertation.

The non-standard growth introduces new difficulties in the derivation of a priori estimates which were absent in the standard growth case. The work provides a new framework for applications of generalized sobolev spaces to stochastic partial differential equations.

1.2 Organization of the thesis

The present dissertation consists of three chapters.

In chapter one, we motivate the reason for the study of the present class of quasi-linear stochastic partial differential equations. This is basically motivated by an application arising in mathematical physics (see [43] and [66], for instance). Chapter two of the dissertation concerns a characterization of various type of functions spaces which are essential in the proof of the main result of the present work. Such a characterization can be found in Kufner [50]. The main result of this work in contained in chapter 3, see **Theorem 24: Existence theorem**. In this chapter, as a passage to the existence result, several results concerning estimates of the weak solution of the quasi-linear SPDEs are given under certain conditions. The derivation of these estimates is non-trivial.

1.3 Notation

Throughout this dissertation we denote by $|A|$ the Lebesgue measure of any subset A of D and by χ_A its characteristic function. By $\mathcal{P}(D)$ we denote the set of all measurable functions p on D that range in the interval $[1, \infty]$. We denote here by $\|f\|_X$ the norm of a measurable function on a space X.

For $p \in \mathcal{P}(D)$ we write

$$D_1^p = \{x \in D : p(x) = 1\},$$

$$D_\infty^p = \{x \in D : p(x) = \infty\},$$

$$D_0^p = D \setminus (D_1^p \cup D_\infty^p),$$

$$p_* = ess \inf_{D_0^p} p(x), \text{and} \quad p^* = ess \sup_{D_0^p} p(x) \quad \text{if} \quad |D_0^p| > 0$$

$$\text{and} \quad p_* = p^* = 1 \quad \text{if} \quad |D_0^p| = 0,$$

$$C_p = \|\chi_{D_0^p}\|_{L^\infty(D)} + \|\chi_{D_1^p}\|_{L^\infty(D)} + \|\chi_{D_\infty}^p\|_{L^\infty(D)}$$

and $r_p = C_p + 1/p_* - 1/p^*$. Here we consider the use of the convention $\dfrac{1}{\infty} = 0$.

Chapter 2

Function spaces

2.1 Introduction

Following [66], we give some definitions and establish some basic facts of properties of the theory of the generalized Lebesgue and Sobolev spaces. Moreover the generalized Lebesgue spaces $L^{p(x)}(D)$ and the usual Lebesgue space $L^p(D)$ have many common properties. In contrast to the classical Lebesgue space $L^p(D)$ and Orlicz spaces the generalized Lebesgue spaces $L^{p(x)}(D)$ are in general invariant with regard to the translation operator (see [48] for $p(x)$-mean continuity of their elements). By this, we mean the elements of $L^{p(x)}(D)$ are not in general $p(x)$-mean continuous (see Definition 3 below). For this reason, many problems can arise with regard to convolutions, Sobolev embeddings, denseness of smooth functions in $W_0^{1,p(x)}(D)$ and boundedness of integral operators. For further details we refer to [24]-[27], [34] and [48].

2.2 Generalized Lebesgue spaces

Let $p \in \mathcal{P}(D)$. On the set of functions (all functions here are considered to be measurable), we define the mapping ϱ_p by

$$\varrho_p(f) = \int_{D \setminus D_\infty} |f(x)|^{p(x)} \; dx + \operatorname{ess} \sup_{x \in D_\infty} |f(x)|. \tag{2.1}$$

and the norm

$$\|f\|_{L^{p(x)}(D)} = \inf\{\lambda > 0; \varrho_p(f/\lambda) \leqslant 1\}. \tag{2.2}$$

It has been shown in [66] and [48] that the functional ϱ_p has the following properties:

$$\varrho_p(f) \geqslant 0. \tag{2.3}$$

$$\varrho_p(f) = 0 \quad \text{if and only if} \quad f = 0. \tag{2.4}$$

$$\varrho_p(f) \quad \text{is a convex functional.} \tag{2.5}$$

$$\varrho_p(-f) = \varrho_p(f) \quad \text{for every function} \quad f. \tag{2.6}$$

The modular space $L^{p(x)}(D)$ generated by the functional ϱ_p can be defined as follows:

$$L^{p(x)}(D) \;=\; \{f : D \longrightarrow \mathbb{R}, \lim_{\lambda \to 0^+} \varrho_p(\lambda f) = 0\}$$

The functional ϱ_p preserves ordering, i.e.

$$\text{If} \, |f(x)| \geqslant |g(x)| \, \forall \, x \in D \;\; \text{and if} \, \varrho_p(f) \, < \infty, \; \text{then} \, \varrho_p(f) \geqslant \varrho_p(g); \tag{2.7}$$

the last inequality is strict if $|f(x)| \neq |g(x)|$.
Thus the space $L^{p(x)}(D)$ is a special and particular case of Musielak-Orlicz space called sometime Nakano space.

Definition 1. *The functional ϱ_p is*

(i) left-continuous, if $\lim_{\lambda \to 1^-} \varrho_p(\lambda f) = \varrho_p(f), \; \forall \; f \in L^{p(x)}(D)$,

(ii) right-continuous, if $\lim_{\lambda \to 1^+} \varrho_p(\lambda f) = \varrho_p(f), \; \forall \, f \in L^{p(x)}(D)$,

(iii) continuous, if it is both, left-continuous and right-continuous.

If $0 < \varrho_p(f) < \infty$, then the function $\lambda \mapsto \varrho_p(f/\lambda)$ is continuous and nonincreasing on the interval $[1,\infty)$. By continuous here we mean that the function $\varrho_p(\frac{f}{\lambda})$ is both left continuous and right continuous in the space $L^{p(x)}(D)$. We have the following properties

$$\varrho_p(f/\|f\|_{L^{p(x)}(D)}) \leqslant 1 \quad \forall\ f \in L^{p(x)}(D), \text{ with } \quad 0 < \|f\|_{L^{p(x)}(D)} < \infty. \tag{2.8}$$

$$\text{If } p^* < \infty,\ \varrho_p(f/\alpha) = 1 \Leftrightarrow 0 < \ \|f\|_{L^{p(x)}(D)} = \alpha \ < \ \infty, \tag{2.9}$$

$$\forall\, f \in L^{p(x)}(D) \text{ with } 0 < \|f\|_{L^{p(x)}(D)} < \infty.$$

The following property is a straight forward consequence of (2.5), (2.4) and (2.8).

$$\text{If } \|f\|_{L^{p(x)}(D)} \leqslant 1, \quad \text{then} \quad \varrho_p(f) \leqslant \|f\|_{L^{p(x)}(D)}. \tag{2.10}$$

We can summarize all the above properties by

Lemma 1. *Consider $f \in L^{p(x)}(D)$, then*

(i) $\|f\|_{L^{p(x)}(D)} \leqslant 1(> 1)$ *if and only if* $\varrho_p(f) \leqslant 1(> 1)$

(ii) *If* $\|f\|_{L^{p(x)}(D)} < 1$, *then* $\|f\|^{p^*}_{L^{p(x)}(D)} \leqslant \varrho_p(f) \leqslant \|f\|^{p_*}_{L^{p(x)}(D)}$

(iii) *If* $\|f\|_{L^{p(x)}(D)} > 1$, *then* $\|f\|^{p_*}_{L^{p(x)}(D)} \leqslant \varrho_p(f) \leqslant \|f\|^{p^*}_{L^{p(x)}(D)}$

We give here the definition of the generalized Lebesgue space $L^{p(x)}(D)$

Definition 2. *The generalized Lebesgue space is the class of functions f defined on D such that $\varrho_p(\lambda f)$ is a positive finite number for some strictly positive λ depending on f. Shortly*

$$L^{p(x)}(D) = \{f,\ f : D \longrightarrow \mathbb{R}^+ :\ \varrho_p(\lambda) < \infty \ \text{ for some } \lambda = \lambda(f) > 0\}.$$

When the Lebesgue measure of D^p_∞ vanishes i.e. $|D^p_\infty| = 0$, then the space $L^{p(x)}(D)$ endowed with the norm

$$\|f\|_{L^{p(x)}(D)} = \inf\{\lambda > 0 : \int_{D \setminus D_\infty} |f(x)|^{p(x)} \ \ dx \leqslant 1$$

becomes a Banach space. When $|D^p_\infty| > 0$, then the space $L^{p(x)}(D)$ can be introduced as

$$L^{p(x)}(D) = \left\{ f : D \longrightarrow \mathbb{R}, \varrho_p(f/\lambda) < \infty, \|f\|_{L^{p(x)}(D)} \leqslant 1 \right\}.$$

This space is a Banach space.

If $p(x) \equiv p$ is a constant function, then the generalized Lebesgue space coincide with the usual L^p space and we also have

$$\|f\|_{L^{p(x)}(D)} = \|f\|_{L^p(D)} = \left(\int_D |f(x)|^p dx \right)^{\frac{1}{p}},$$

and there is no confusion of notations.

Definition 3. *$f \in L^{p(x)}(D)$ is $p(x)$-mean continuous if $\forall \varepsilon > 0$ there exists a positive number $\delta = \delta(\varepsilon) > 0$ such that $\varrho_p(f_h - f) < \varepsilon$ for $h \in \mathbb{R}^n$, $|h|_{\mathbb{R}^n} < \delta$, where $f_h(x) = f(x+h)$, $x \in \mathbb{R}^n$ and the symbol $|h|_{\mathbb{R}^n}$ stands for the Euclidean norm in the Euclidean space $\mathbb{R}^n$.*

For more examples, theorems and details about the above definition we refer to [48].

For a given measurable function $p \in \mathcal{P}(D)$, we shall define the dual function pointwise known as the conjugate exponent function $q \in \mathcal{P}(D)$ to $p \in \mathcal{P}(D)$ is defined as

$$q(x) = \begin{cases} \infty & \text{for } x \in D^p_1, \\ 1 & \text{for } x \in D^p_\infty, \\ \frac{p(x)}{p(x)-1} & \text{for } x \in D^p_0. \end{cases}$$

The following result is the generalized Hölder's inequality:

Theorem 1. *Let $p \in \mathcal{P}(D)$. Then the inequality*

$$\int_D |f(x)\, g(x)| \; dx \;\; \leqslant \;\; r_p \;\; \|f\|_{L^{p(x)}(D)} \, \|g\|_{L^{q(x)}(D)} \tag{2.11}$$

is valid for each function $f \in L^{p(x)}(D)$ and $g \in L^{q(x)}(D)$ with the constant r_p defined in section 1.3 above.

For a given measurable function f on D, we shall introduce the generalized Lebesgue norm $|||\cdot|||_p$ on the space $L^{p(x)}(D)$ generated by the functional ϱ_p defined in (2.1).

More details about the connection between the two norms $|||\cdot|||_p$ and $\|\cdot\|_{L^{p(x)}(D)}$ can be found in [56]. From the generalized Hölder inequality established in theorem 1, we can characterize the norm $|||\cdot|||_p$ by the following theorem.

Theorem 2. *Let f and g be measurable functions such that*

$$\varrho_q(g) = \int_{D\setminus D^q_\infty} |g(x)|^{q(x)}\ dx\ +\ ess \sup_{x\in D^q_\infty} |g(x)|\ \leqslant\ 1.$$

For functions f we define

$$|||f|||_p = \sup\left\{\int_D f(x)\, g(x)\ dx\, ;\, \varrho_q(g)\ \leqslant\ 1\right\}. \tag{2.12}$$

Following [66] and [48], $|||\cdot|||_p$ is a norm on the class of functions f with $|||f|||_p < \infty$. This is an analogue of the Orlicz norm defined in [56]. The next result shows the connection between the norms $\|\cdot\|_{L^{p(x)}(D)}$ and $|||\cdot|||_p$.

Theorem 3. *With the new norm defined by (2.12) in theorem 2, One can have:*

$$L^{p(x)}(D)\ =\ \{\ f : D \longrightarrow \mathbb{R}\ :\ |||f|||_p \leqslant \infty\},$$

the following inequalities

$$C_p^{-1}\|f\|_{L^{p(x)}(D)} \leqslant |||f|||_p \leqslant r_p\|f\|_{L^{p(x)}(D)}\,,$$

hold, where C_p and r_p are constants form subsection 1.3.

Proof 1. *For the detailed proof of this theorem we refer to [48].*

In particular, for the norm $|||\,.\,|||_p$, we have the H$\ddot{o}$lder inequality

$$\left|\int_D f(x)\, g(x)\, dx\right| \leqslant r_p\, \|f\|_{L^{p(x)}(D)}|||g|||_p, \quad \forall f\in L^{p(x)}(D), g\in L^{q(x)}(D), \tag{2.13}$$

where $\dfrac{1}{p(x)} + \dfrac{1}{q(x)} = 1.$

In order to investigate any kind of norms convergence for sequences in the generalized Lebesgue spaces $L^{p(x)}(D)$, we first have to provide those spaces with a suitable notion of convergence. Since, spaces $L^{p(x)}(D)$ can be seen as modular spaces, it is more convenient to replace the notion of norms convergences by those of modular convergences.

We describe the modular convergence as follows:

we consider a sequence of functions $(f_n) \subset L^{p(x)}(D)$, we shall say that f_n converge modularly to a function $f \in L^{p(x)}(D)$, if

$$\lim_{n \longrightarrow \infty} \varrho_p(f_n - f) = 0.$$

In [48] and [56], it has been shown that in the generalized Lebesgue space $L^{p(x)}(D)$ and Orlicz spaces there is a substantial difference between the norm convergence and the modular convergence. In [48] it has been shown that the norm convergence is stronger than the modular convergence.

The dual space $(L^{p(x)}(D))^*$ of the space $L^{p(x)}(D)$ is the space of all continuous linear functionals over $L^{p(x)}(D)$. Letting

$$L^{q(x)}(D) := \{f : D \longrightarrow \mathbb{R} \text{ such that } \int_D |g(x)|^{q(x)}\, dx < \infty\}.$$

The following equivalence result characterizes the dual $L^{q(x)}(D)$ of the space $L^{p(x)}(D)$.

Theorem 4. *The following statements are equivalent*

i) $p \in L^\infty(D)$;

ii) *for any functional* $J \in \left(L^{p(x)}(D)\right)^*$ *there exists a unique function* $g \in L^{q(x)}(D)$ *such that*

$$J(g)(x) = \int_D f(x)\, g(x)\ dx, \quad f \in L^{p(x)}(D);$$

and

$$C_p^{-1}\|g\|_{L^{q(x)}(D)} \leqslant \|J\|_{\left(L^{p(x)}(D)\right)^*} \leqslant r_p\|g\|_{L^{q(x)}(D)}.$$

Proof 2. *See [48] for the proof of this theorem.*

The following results can be found in [48].

Corollary 1. *(***cf. [48]***)*

(i) The dual space of $L^{p(x)}(D)$ is the space $L^{q(x)}(D)$ if and only if $p \in L^{\infty}(D)$;

(ii) The space $L^{p(x)}(D)$ is reflexive if and only if

$$1 \quad < \quad ess\inf_{D} p(x) \quad \leqslant \quad ess\sup_{D} p(x) \quad < \quad \infty; \tag{2.14}$$

Given two Banach spaces X and Y, we denote the continuous embedding of the space X into the space Y by the symbol $X \circlearrowleft Y$.

Theorem 5. *Let $0 < |\Omega| < \infty$ and consider two measurable functions p and q on Ω . Then the following conditions are equivalent*

(i)

$$L^{q(x)}(D) \circlearrowleft L^{p(x)}(D) \quad \text{for } 0 < |D| < \infty. \tag{2.15}$$

(ii)

$$p(x) \quad \leqslant \quad q(x) \quad for\ a.e \quad x \in \Omega. \tag{2.16}$$

Theorem 6. *If $p \in \mathcal{P}(D) \cap L^{\infty}(D)$, then the space $L^{p(x)}(D)$ is separable.*

2.3 Generalized Sobolev Spaces

We shall consider D a bounded open domain of $\mathbb{R}^n$ with smooth boundary ∂D, $n \geq 1$ and let k and m be natural numbers.

Consider a multi-index $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$ of order $|\alpha| = \alpha_1 +, \ldots, +\alpha_n \leqslant k$ and set $D^{\alpha} = D_1^{\alpha_1} \ldots D_m^{\alpha_m}$, where $D^i = \partial/\partial x_i$ is the generalized derivative operator.

Definition 4. *Let $p \in \mathcal{P}(D)$ and $k \in \mathbb{N}$. The generalized Sobolev space $W^{k,p(x)}(D)$ is the class of all measurable functions $f : D \longrightarrow \mathbb{R}$ such that for each multi-index α with order $|\alpha| \leqslant k$, the generalized derivatives of f of order α, $D^{\alpha} f(x)$ exists and belongs to $L^{p(x)}(D)$. Shortly*

$$W^{k,p(x)}(D) = \{f,\ f : D \longrightarrow \mathbb{R}, D^{\alpha} f \in L^{p(x)}(D)\}.$$

Definition 5. *We define the norm in $W^{k,p(x)}(D)$ by*

$$\|f\|_{L^{p(x)}(D)} = \sum_{|\alpha|\leqslant k} \|D^{\alpha} f(x)\|_{L^{p(x)}(D)},\ f \in W^{k,p(x)}(D). \tag{2.17}$$

We define $C_0^{\infty}(D)$ to be the space of infinitely differentiable functions, with compact support in D.

Definition 6. *We define*

$$W_0^{k,p(x)}(D) = \overline{C_0^{\infty}}^{W^{k,p(x)}(D)},$$

that is the space $W_0^{k,p(x)}(D)$ is the closure of the space C_0^{∞} of infinitely differentiable functions with respect to the norm of $W^{k,p(x)}(D)$.

Next, we shall give the definition of the norm on the space $W_0^{1,p(x)}(D)$.

Definition 7. *If $f \in W_0^{1,p(x)}(D)$, we define its norm by*

$$\|f\|_{W_0^{1,p(x)}(D)} = \sum_{i=1}^{n} \left\| \frac{\partial f}{\partial x_i} \right\|_{L^{p(x)}(D)}. \tag{2.18}$$

The norm (2.18) is equivalent to the norm (2.17) (with $k = 1$).

The space $W_0^{k,p(x)}(D)$ is a proper subspace of $W^{k,p(x)}(D)$, provided that D is a proper subset of $\mathbb{R}^n$. If $k = 0$, since D is open bounded domain of $\mathbb{R}^n$, we have the density result.

Theorem 7. *Let $p \in \mathcal{P}(D) \cap L^{\infty}(D)$. Then the set $C_0^{\infty}(D)$ is dense in $L^{p(x)}(D)$.*

Theorem 8. *Let $p \in \mathcal{P}(D)$. The spaces $W^{k,p(x)}(D)$ and $W_0^{k,p(x)}(D)$ are Banach spaces under their respecttive norms (2.17) and (2.18). They are separable if $p \in L^{\infty}(D)$ and reflexive if $1 < ess\inf_{x\in D} p(x) \leqslant ess\sup_{x\in D} p(x) < \infty$.*

Consider two Banach spaces X and Y. We denote their continuous embedding by $\circlearrowleft$. As a consequence of Theorem 5 we have the following trivial embedding:

$$\text{If } q(x) \leqslant p(x) \text{ for a.e. } x \in D, \text{ then } W^{k,p(x)}(D) \circlearrowleft W^{k,q(x)}(D). \tag{2.19}$$

The compact embedding of X into Y will be symboled $X \circlearrowleft\circlearrowleft Y$.

Theorem 9. *Let D and p satisfy one of the following conditions:*

(i) p is continuous on $\bar{D}$;

(ii) there exist numbers p_i and r_i and subsets $G_i \subset D$, $i = 1, 2, \ldots, m$, that contain finitely many components with Lipschitzian boundaries such that $|D \setminus \cup_{i=1}^{m} G_i| = 0$, the interiors of G_i are mutually disjoint, $1 = p_1 < p_2 < r_1 < p_3 < r_2 < \cdots < p_{m-1} < r_{m-2} < n < p_m < r_{m-1} < r_m = \infty$, $r_i < np_i/(n - p_i)$, $i = 1, 2, \ldots, m - 1$ and $p_i \leqslant p(x) \leqslant r_i$ for $i = 1, 2, \ldots, m$ and for all $x \in G_i$.

Then the space $W_0^{1,p(x)}(D)$ is compactly embedded in the space $L^{p(x)}(D)$:

$$W_0^{1,p(x)}(D) \circlearrowleft\circlearrowleft L^{p(x)}(D).$$

We shall write $(\ ,\ )$ to denote the inner product in $L^2(D)$.

Using the generalized Holder inequality and the characterization of the dual $\left(L^{p(x)}(D)\right)^*$, we have the following characterization of the dual space $\left(W_0^{k,p(x)}(D)\right)^*$ of $W_0^{1,p(x)}(D)$:

Theorem 10. *Let $p \in \mathcal{P}(D) \cap L^\infty(D)$, then for any functional $J \in \left(W_0^{k,p(x)}(D)\right)^*$ there exists a unique system of functions $\{g_\alpha \in L^{q(x)}(D) : |\alpha| \leqslant k\}$ such that*

$$J(f) = \sum_{|\alpha| \leqslant k} \int_D D^\alpha f(x)\, g_\alpha(x)\; dx,$$

for every function $f \in W_0^{1,p(x)}(D)$.

Next, we introduce some other sorts of Lebesgue and Sobolev spaces of measurable functions defined on the closed interval $[0, T]$, with values in various Banach spaces. They are essential in order to construct generalized solutions to the initial boundary value problem (I-BVP).

Definition 8. *The space*

$$L^r\left(0,\ T\ ;\ W_0^{1,p(x)}(D)\right)$$

consists of all measurable functions u that are defined on the closed interval $[0,T]$, taking values in the space $W_0^{1,p(x)}(D)$, satisfying $\|u(t,x)\|_{W_0^{1,p(x)}(D)} \in L^r([0,T])$. We endow this space with the following norm

$$\|u(t,x)\|_{L^r\left(0,T;W_0^{1,p(x)}(D)\right)} = \left(\int_0^T \|u(t,x)\|^r_{W_0^{1,p(x)}(D)}\, dt\right)^{\frac{1}{r}}.$$

Next, we introduce an intermediary space which is important for the construction of our probabilistic weak solution.

Definition 9. *The space*

$$\overset{\circ}{V}(Q_T)$$

consists of all measurable functions u defined on $[0,T]$ and taking values in $W_0^{1,p(x)}(D)$. We endow $\overset{\circ}{V}(Q_T)$ with the finite norm

$$\begin{aligned}\|u(t,x)\|_{\overset{\circ}{V}(Q_T)} &= \sum_{i=1}^{n}\left\|\frac{\partial u(t,x)}{\partial x_i}\right\|_{L^{p(x)}(Q_T)} \\ &= \sum_{i=1}^{n}\inf\left\{\lambda_i{>}0\ :\ \int_{Q_T}\left(\left|\frac{\partial u(t,x)}{\partial x_i}\right|/\lambda_i\right)^{p(x)}\ dt\,dx\ \leqslant 1\right\},\end{aligned} \tag{2.20}$$

where $Q_T = (0,T)\times D$.

The space $\overset{\circ}{V}(Q_t)$ is introduced similarly, where $Q_t = (0,t)\times D$, $0 \leqslant t \leqslant T$.
Following [66] and [85], from previous theorems 1, 4, 8 and the completeness of the spaces $L^{p(x)}(D)$, $W^{1,p(x)}(D)$ and $W_0^{1,p(x)}(D)$ we can derive the following result.

Theorem 11. *The space $\overset{\circ}{V}(Q_T)$ with the norm $\|\ .\ \|_{\overset{\circ}{V}(Q_T)}$ defined above is a Banach space.*
If $p \in \mathcal{P}(D)\cap L^\infty(D)$, then $\overset{\circ}{V}(Q_T)$ is separable.
If $p(x) \geq 2$, then $\overset{\circ}{V}(Q_T)$ is reflexive.

From the characterization of the generalized Lebesgue and Sobolev spaces we see that it is necessary to give the characterization of the dual space $(\overset{\circ}{V}(Q_T))^*$ of the space

$\mathring{V}(Q_T)$. We equip the dual space $(\mathring{V}(Q_T))^*$ of $\mathring{V}(Q_T)$ with the norm

$$\|f\|_{(\mathring{V}(Q_T))^*} = \sup_{\|u\|_{\mathring{V}(Q_T)} \leqslant 1} |\langle f\,,\,u\rangle| = \inf \sum_{|\alpha|\leqslant 1} \int_0^T \|f_\alpha(t)\|_{L^{q(x)}(Q_T)} dt,$$

where the infimum is taken over all possible decompositions

$$f(t) = \sum_\alpha D_x^\alpha f_\alpha(t), \qquad f_\alpha(t) \in L^{q(x)}(Q_T).$$

We have the following result. For the proof we follow the lines of [85] (Lemma 2.10 page 316), but with appropriate changes.

Lemma 2. *Let $p \in \mathcal{P}(D)$ such that $p \in C(\bar{D})$ or $p \in L^\infty(D)$, and moreover, let $p(x) \geq 2$. Then the following continuous embedding holds: $\mathring{V}(Q_T) \circlearrowleft L^2(0,T;W_0^{1,p(x)}(D))$*

Proof 3. *For every number $\lambda > 1$, we have $\lambda\varrho_p(u) \leqslant \varrho_p(\lambda u)$.*
Therefore $\lambda\varrho_p(\frac{\partial u}{\partial x_i}/\lambda) \leqslant \varrho_p(\frac{\partial u}{\partial x_i})$. We use the fact that

$$\left\| \frac{\frac{\partial u}{\partial x_i}}{\|\frac{\partial u}{\partial x_i}\|_{L^{p(x)}(D)}} \right\|_{L^{p(x)}(D)} = 1$$

and Lemma 1 to get

$$\int_D \left| \frac{\frac{\partial u}{\partial x_i}}{\|\frac{\partial u}{\partial x_i}\|_{L^{p(x)}(D)}} \right|^{p(x)} dx = 1.$$

If $\dfrac{\|\frac{\partial u}{\partial x_i}\|_{L^{p(x)}(D)}}{\|\frac{\partial u}{\partial x_i}\|_{L^{p(x)}(Q_T)}} \geq 1$,

$$\int_D \left| \frac{\left|\frac{\partial u}{\partial x_i}\right|}{\left\|\frac{\partial u}{\partial x_i}\right\|_{L^{p(x)}(Q_T)}} \right|^{p(x)} dx = \int_D \left| \frac{\left\|\frac{\partial u}{\partial x_i}\right\|_{L^{p(x)}(D)}}{\left\|\frac{\partial u}{\partial x_i}\right\|_{L^{p(x)}(Q_T)}} \frac{\left|\frac{\partial u}{\partial x_i}\right|}{\left\|\frac{\partial u}{\partial x_i}\right\|_{L^{p(x)}(D)}} \right|^{p(x)} dx$$
$$= \int_D \left(\frac{\left\|\frac{\partial u}{\partial x_i}\right\|_{L^{p(x)}(D)}}{\left\|\frac{\partial u}{\partial x_i}\right\|_{L^{p(x)}(Q_T)}} \right)^{p(x)} \left| \frac{\left|\frac{\partial u}{\partial x_i}\right|}{\left\|\frac{\partial u}{\partial x_i}\right\|_{L^{p(x)}(D)}} \right|^{p(x)} dx,$$

$p(x) \geq 2$, *since* $\dfrac{\left\|\frac{\partial u}{\partial x_i}\right\|_{L^{p(x)}(D)}}{\left\|\frac{\partial u}{\partial x_i}\right\|_{L^{p(x)}(Q_T)}} \geq 1$, *then*

$$\left(\frac{\left\|\frac{\partial u}{\partial x_i}\right\|_{L^{p(x)}(D)}}{\left\|\frac{\partial u}{\partial x_i}\right\|_{L^{p(x)}(Q_T)}} \right)^{p(x)} \geq \left(\frac{\left\|\frac{\partial u}{\partial x_i}\right\|_{L^{p(x)}(D)}}{\left\|\frac{\partial u}{\partial x_i}\right\|_{L^{p(x)}(Q_T)}} \right)^2.$$

Therefore

$$\int_D \left| \frac{\left|\frac{\partial u}{\partial x_i}\right|}{\left\|\frac{\partial u}{\partial x_i}\right\|_{L^{p(x)}(Q_T)}} \right|^{p(x)} dx \geq \frac{\left\|\frac{\partial u}{\partial x_i}\right\|^2_{L^{p(x)}(D)}}{\left\|\frac{\partial u}{\partial x_i}\right\|^2_{L^{p(x)}(Q_T)}} \int_D \left| \frac{\left|\frac{\partial u}{\partial x_i}\right|}{\left\|\frac{\partial u}{\partial x_i}\right\|_{L^{p(x)}(D)}} \right|^{p(x)} dx$$
$$\geq \frac{\left\|\frac{\partial u}{\partial x_i}\right\|^2_{L^{p(x)}(D)}}{\left\|\frac{\partial u}{\partial x_i}\right\|^2_{L^{p(x)}(Q_T)}}.$$

Thus we obtain

$$\frac{\left\|\frac{\partial u}{\partial x_i}\right\|^2_{L^{p(x)}(D)}}{\left\|\frac{\partial u}{\partial x_i}\right\|^2_{L^{p(x)}(Q_T)}} \leqslant \int_D \left| \frac{\left|\frac{\partial u}{\partial x_i}\right|}{\left\|\frac{\partial u}{\partial x_i}\right\|_{L^{p(x)}(Q_T)}} \right|^{p(x)} dx + 1.$$

The intergration of this inequality with respect to t from 0 to T and use of Lemma 1 yields

$$\int_0^T \frac{\left\|\frac{\partial u}{\partial x_i}\right\|^2_{L^{p(x)}(D)}}{\left\|\frac{\partial u}{\partial x_i}\right\|^2_{L^{p(x)}(Q_T)}} dt \leqslant \int_0^T \int_D \left| \frac{\left|\frac{\partial u}{\partial x_i}\right|}{\left\|\frac{\partial u}{\partial x_i}\right\|_{L^{p(x)}(Q_T)}} \right|^{p(x)} dx\, dt + T$$
$$= \int_{Q_T} \left| \frac{\left|\frac{\partial u}{\partial x_i}\right|}{\left\|\frac{\partial u}{\partial x_i}\right\|_{L^{p(x)}(Q_T)}} \right|^{p(x)} dx\, dt + T = 1 + T.$$

Hence, we obtain the inequality

$$\int_0^T \left\|\frac{\partial u}{\partial x_i}\right\|^2_{L^{p(x)}(D)} dt \leqslant (1+T) \left\|\frac{\partial u}{\partial x_i}\right\|^2_{L^{p(x)}(Q_T)}.$$

This implies

$$\left(\sum_{i=1}^n \int_0^T \left\|\frac{\partial u}{\partial x_i}\right\|^2_{L^{p(x)}(D)} dt\right)^{1/2} \leqslant C \sum_{i=1}^n \left\|\frac{\partial u}{\partial x_i}\right\|_{L^{p(x)}(Q_T)}$$
$$= C\|u\|_{\mathring{V}(Q_T)}.$$

Since,

$$\int_0^T \|u\|^2_{W_0^{1,p(x)}(D)} dt = \int_0^T \left(\sum_{i=1}^n \left\|\frac{\partial u}{\partial x_i}\right\|_{L^{p(x)}(D)}\right)^2 dt \leqslant C \sum_{i=1}^n \int_0^T \left\|\frac{\partial u}{\partial x_i}\right\|^2_{L^{p(x)}(D)} dt,$$

It follows with an appropriate choice of the constant C that

$$\left(\int_0^T \|u\|^2_{W_0^{1,p(x)}(D)}\,dt\right)^{1/2} \leqslant C\,\|u\|_{\mathring{V}(Q_T)}\,.$$

It follows that $\mathring{V}(Q_T) \subset L^2(0,T;W_0^{1,p(x)}(D))$; thus we can get the embedding

$$\mathring{V}(Q_T) \circlearrowleft L^2(0,T;W_0^{1,p(x)}(D)).$$

From the continuous embedding of the generalized Lebesgue spaces we easily have the following continuous embedding

$$L^s\left(0,T;W_0^{1,s}(D)\right) \circlearrowleft \mathring{V}(Q_T) \circlearrowleft L^r\left(0,T;W_0^{1,r}(D)\right),$$

where $2 \leqslant r = \operatorname{ess\,inf}_{x\in D} p(x) \leqslant s = \operatorname{ess\,sup}_{x\in D} p(x) < \infty$. And then clearly the embedding of their duals follows:

$$L^{r/(r-1)}\left(0,T;W_0^{-1,r/(r-1)}(D)\right) \circlearrowleft \left(\mathring{V}(Q_T)\right)^* \circlearrowleft L^{s/(s-1)}\left(0,T;W_0^{-1,s/(s-1)}(D)\right).$$

2.4 Functional-Analytic Statement of the Problem

We consider the family of operators $A(t) : W_0^{1,p(x)}(D) \longrightarrow \left(W_0^{1,p(x)}(D)\right)^*$, for $t \in [0,T]$ such that

$$(\,Au(t),v) \;=\; \sum_{i=1}^{n}\int_D \left|\frac{\partial u(t,x)}{\partial x_i}\right|^{p(x)-2} \frac{\partial u(t,x)}{\partial x_i}\,\frac{\partial v}{\partial x_i}\,dx \tag{2.21}$$

holds for all $u,v \in W_0^{1,p(x)}(D)$ and for $s \in [0,t]$.
If $u \in \mathring{V}(Q_T)$ then $Au(s) \in \left(\mathring{V}(Q_T)\right)^*$.
We next introduce some probabilistic evolutions spaces.

2.4.1 Some results from probabilistic evolutions spaces

Given a Banach space B, for $1 \leqslant q \leqslant \infty$, we denote by $L^q(0,T;B)$ the set of functions defined on $[0,T]$ and taking values in B. We endow $L^q(0,T;B)$ with the norm

$$\|u\|_{L^q(0,T;B)} = \left(\int_0^T \|u(t)\|_B^q\,dt\right)^{1/q} \quad \text{if } 1 \leqslant q < \infty.$$

When $q = \infty$, the space $L^\infty(0,T;B)$ is the space of all essentially bounded functions on the closed interval $[0,T]$ with values in B with the norm

$$\|u\|_{L^\infty(0,T;B)} = \operatorname{ess}\sup_{[0,T]} \|u\|_B \ < \ \infty.$$

Let B be a Banach space, we assume that $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space and let $(\mathcal{F}_t)_{0\leqslant t\leqslant T}$ be a filtration i.e. an increasing and right continuous family of sub σ-algebra of $\mathcal{F}$ with $\mathcal{F}_0$ contains all the $\mathbb{P}$-null sets of the filtration. Throughout we denote by $\mathbb{E}$ the mathematical expectation with respect to the probability measure $\mathbb{P}$.

Let $1 \leqslant p \ \leqslant \infty$. The space

$$L^p\left(\Omega, \mathcal{F}, \mathbb{P}, L^q\left(0,T;B\right)\right) \tag{2.22}$$

consists of all random functions $u(t,x,\omega)$ defined on $[0,T]\times\Omega$ and taking values in B such that the function u is measurable w.r.t. (t,ω) and for almost all t, u is measurable w.r.t. the filtration $\mathcal{F}$. We furthermore endow this space with the norm

$$\|u\|_{L^p(\Omega,\mathcal{F},\mathbb{P},L^q(0,T;B))} = \left(\mathbb{E}\|u\|^p_{L^q(0,T;B)}\right)^{1/p}. \tag{2.23}$$

When $q = \infty$, then the norm in the space $L^p(\Omega, \mathcal{F}, \mathbb{P}, L^\infty(0,T;B))$ is given by

$$\|u\|_{L^p(\Omega,\mathcal{F},\mathbb{P},L^\infty(0,T;B))} = \left(\mathbb{E}\ \|u\|^p_{L^\infty(0,T;B)}\right)^{1/p}.$$

Theorem 12. *$L^p(\Omega, \mathcal{F}, \mathbb{P}, L^q(0,T;B))$ with the norm defined in (2.23) is a Banach space.*

We shall give some useful compactness results.

Lemma 3. *Let $(g_\kappa)_{\kappa=1,2,\ldots}$ and g be some functions in the space $L^q(0,T;L^q(D))$ with $1 < q < \infty$ such that*

$$\|g_\kappa\|_{L^q(0,T;L^q(D))} \leqslant C, \quad \forall\, \kappa$$

and as $\kappa \longrightarrow \infty \quad g_\kappa \longrightarrow g$ for almost all $(t,x) \in Q_T$. Then g_κ converges weakly to g in $L^q(0,T;L^q(D))$.

Proof 4. *For a detailed proof see [53, Chap. 1, Lemma 1.3].*

Remark 1. *The above lemma is still valid if instead of $L^q(0,T;L^q(D))$ we have $L^p(\Omega,\mathcal{F},P,L^q(Q_T))$ for almost all (t,x,ω).*

In order to sharpen our result, we collect the next lemma from [75, sect. 8, Theorem 5].

Lemma 4. *Given some Banach spaces B, F and H with F a subset of H such that B is compactly embedded into F. For any p, $q \in [1,\infty]$, let V be a bounded set in $L^q(0,T;B)$ such that*

$$\lim_{\theta\longrightarrow\infty}\int_0^{T-\theta}\|v(t+\theta)-v(t)\|_H^p\,dt=0,\text{ uniformly for all } v\in V.$$

Then V is relatively compact in $L^p(0,T,F)$.

2.4.2 Some Facts and definitions from Stochastic Calculus

In this section, we give some fundamental definitions of probabilistic concepts and provide some well-known prerequisites from probability theories and stochastic calculus, which will be used throughout the dissertation. For more details about these basic results we refer for example to [16], [17], [20], [44], [46], [60], [76], [77], [86], [87], [74].

A probability space is a triple $(\Omega,\mathcal{F},\mathbb{P})$, where

- Ω is the set of all elements ω (ω are called sometimes elementary events),
- $\mathcal{F}$ is the Borel σ-field of subsets of Ω
- $\mathbb{P}$ is a probability measure.

If $(\Omega,\mathcal{F},\mathbb{P})$ is a given probability space, and $(E,\mathcal{G})$ a measurable space, then a function $f:(\Omega,\mathcal{F})\longrightarrow(E,\mathcal{G})$ is called $\mathcal{F}$-measurable if

$$f^{-1}(B)=\{\omega\in\Omega; f(\omega)\in B\}\in\mathcal{F},$$

for all sets $B\in\mathcal{G}$(or, equivalently, for all subset B of E); f is also called E-valued random variable.

Expectation

Next, we shall define the expectation $\mathbb{E}X$ of an arbitrary random variable without giving any of its properties.

Definition 10. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space.*
We define the expectation of the random variable X by

$$\mathbb{E}X = \int_{\Omega} X(\omega)\,\mathbb{P}(d\omega) \text{ or } \int_{\Omega} X\,d\mathbb{P}.$$

Conditional Expectation

In this section we shall define the conditional expectations with respect to a σ-algebra $\mathcal{G}$ (sub-σ-algebra of $\mathcal{F}$). The concept of conditional expectation is of major importance in the definition of martingale. Suppose again $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space and that $\mathcal{G}$ is a sub-σ-field of $\mathcal{F}$.

Definition 11. *Suppose that X is an E-valued integrable random variable defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Then the conditional expectation of X with respect to the σ-field $\mathcal{G}$ (conditional expectation of X given $\mathcal{G}$) is the (a.s. unique) integrable random variable $\mathbb{E}[X|\mathcal{G}]$ satisfying*

1 $\mathbb{E}[X|\mathcal{G}]$ is $\mathcal{G}$-measurable;

2 for every $B \in \mathcal{G}$

$$\int_H X\,d\mathbb{P} = \int_H \mathbb{E}[X|\mathcal{G}]\,d\mathbb{P}, \text{ or } \mathbb{E}[XI_B] = \mathbb{E}[\mathbb{E}[X|\mathcal{G}]I_B], \text{ for all } H \in \mathcal{G}. \quad (2.24)$$

Note that the existence and uniqueness of $\mathbb{E}[X|\mathcal{G}]$ follows from the following result known as the Radon-Nikodym theorem.

Theorem 13. *Let μ be the measure on $\mathcal{G}$ defined by*

$$\mu(H) = \int_H X\,d\mathbb{P};\ H \in \mathcal{G}.$$

Then μ is absolutely continuous with respect to $\mathbb{P}|\mathcal{G}$, so there exists a $\mathbb{P}|\mathcal{G}$-unique $\mathcal{G}$-measurable random variable Y on Ω such that

$$\mu(H) = \int_H Y\, d\mathbb{P} \text{ for all } H \in \mathcal{G}.$$

Thus the conditional expectation $\mathbb{E}[X|\mathcal{G}]$ is a modification (see definition below) of Y. The random variable $Y =: \mathbb{E}[X|\mathcal{G}]$ is indeed unique a.s. with respect to the measure $\mathbb{P}|\mathcal{G}$.

Filtration

Our basic structure is a measurable space $(\Omega, \mathcal{F})$.

Definition 12. *A filtration is an increasing sequence of σ-algebras on a measurable space. That is, given a measurable space $(\Omega, \mathcal{F})$, a filtration is a sequence of σ-algebras $\{\mathcal{F}_t\}_{0\leqslant t\leqslant T}$ with $\mathcal{F}_t \subseteq \mathcal{F}$ for each $t \in [0,T]$ and satisfying*

$$s \leqslant t \Longrightarrow \mathcal{F}_s \subset \mathcal{F}_t.$$

Similarly, a filtered probability space (known as a stochastic basis) is a probability space equipped with a filtration of its σ-algebras. When $T = \infty$ we define $\mathcal{F}_\infty$ as the σ-algebra generated by the infinite union of the $\mathcal{F}_t$'s, which is also contained in $\mathcal{F}$:

$$\mathcal{F}_\infty = \sigma\left(\bigcup_{t\in[0,T)} \mathcal{F}_t\right) \subseteq \mathcal{F}.$$

As a convention we write $\mathcal{F}_\infty = \bigvee_t \mathcal{F}_t$.
We define $\mathcal{F}_{t^+} = \bigcap_{s>t} \mathcal{F}_s$ and, for $t > 0$, we define $\mathcal{F}_{t^-} = \bigvee_{s<t} \mathcal{F}_s$.
The filtration is said to be right continuous if $\mathcal{F}_t = \mathcal{F}_{t^+}$.

Stopping Time

We pass next to the definition of stopping times.

Definition 13. *Suppose that we are given a measurable space* $(\Omega, \mathcal{F})$ *equipped with a filtration* $\mathcal{F} = \{\mathcal{F}_t\}_{0 \leqslant t \leqslant T}$. *A random variable* $\tau : \Omega \longrightarrow [0, \infty]$ *is called a* **stopping time** *with respect to the filtration* $\mathcal{F}$ *(or an* $\mathcal{F}$*-stopping time or simply a stopping time if there is no confusion) if the event* $\{\tau \leqslant t\} = \{\omega : \tau(\omega) \leqslant t\} \in \mathcal{F}_t$, *for every* t, $t \geq 0$.

Stochastic Process

We assume as given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Definition 14. *A* **stochastic process** X *defined on a measurable space* $(\Omega, \mathcal{F})$ *with values in a measurable space* $(E, \mathcal{G})$ *is a family of random variables* $(X(t))_{0 \leqslant t \leqslant T}$ *with values in* E, *indexed by* $t \in [0, T]$.

(1) For a fixed sample point $\omega \in \Omega$, the function $t \mapsto X(t, \omega)$; $0 \leqslant t \leqslant T$ is the sample path of the process X associated with ω.

(2) X is continuous if its sample paths $X(t, \omega)$ is a continuous funtions of t, for almost all (almost everywhere) $\omega \in \Omega$.

Definition 15. *Suppose that* $\mathcal{F} = \{\mathcal{F}_t\}$, $0 \leqslant t \leqslant T$ *is a filtration of the measurable space* $(\Omega, \mathcal{F})$, *and* X *is a stochastic process defined on* $(\Omega, \mathcal{F})$ *with values in* $(E, \mathcal{G})$. *Then* X *is said to be* **adapted** *to the filtration* $(\mathcal{F}_t)_{0 \leqslant t \leqslant T}$ *(or* $\mathcal{F}_t$*-adapted) if* $X(t) \in \mathcal{F}_t$ *that is* $\mathcal{F}_t$*-measurable random variable, for each* $t \in [0, T]$.

Definition 16. *Suppose that* $X = \{X(t)\}_{0 \leqslant t \leqslant T}$ *is a stochastic process defined on a measurable space* $(\Omega, \mathcal{F})$, *and taking values in the measurable space* $(E, \mathcal{G})$. *Then* X *is said to be* **measurable process** *if the map* $(t, \omega) \longmapsto X(t, \omega)$ *is measurable provided that* $[0, T] \times \Omega$ *is given the product* σ*-field* $\mathcal{B}([0, T]) \otimes \mathcal{F}$, *where* $\mathcal{B}([0, T])$ *denotes the Borel* σ*-fields.*

We shall introduce one of the most important theorems for constructing Wiener process. For further proofs and more information about these theorems and definitions we refer to [58].

Theorem 14. *(***Kolmogorov Extension Theorem***)*

For all $t_1, t_2, \ldots, t_k \in [0,T]$, $k \in \mathbb{N}$ let $\nu_{t_1}, \ldots, \nu_{t_k}$ be probability measures on $\mathbb{R}^{nk}$ such that

$$\nu_{t_{\sigma(1)},\ldots,\ldots t_{\sigma(k)}}\left(F_1 \times \cdots \times F_k\right) = \nu_{t_1,\ldots t_k}\left(F_{\sigma^{-1}(1)} \times \cdots \times F_{\sigma^{-1}(k)}\right) \tag{2.25}$$

for all permutations σ on $\{1, 2, \ldots, k\}$ and

$$\nu_{t_1,\ldots,\ldots t_k}\left(F_1 \times \cdots \times F_k\right) = \nu_{t_1,\ldots t_k,t_{k+1},\ldots,t_{k+m}}\left(F_1 \times \cdots \times F_k \times \mathbb{R}^n \times \cdots \times \mathbb{R}^n\right) \tag{2.26}$$

for all $m \in \mathbb{N}$, where the set on the right hand side has a total of $m+n$ factors. Then there exists a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a stochastic process $X = \{X(t)\}$ on Ω, $X(t) : \Omega \longrightarrow \mathbb{R}^n$, such that

$$\nu_{t_1,\ldots,\ldots t_k}\left(F_1 \times \cdots \times F_k\right) = \mathbb{P}\left[X(t_1) \in F_1, \cdots, X(t_k) \in F_k\right],$$

for all $t_i \in [0,T]$, $k \in \mathbb{N}$ and all Boral sets F_i.

In order to contruct Wiener process we need the following.
Fix $x \in \mathbb{R}^n$ and define

$$p(t,x,y) = (2\pi t)^{-n/2} \exp\left(-\frac{|x-y|_{\mathbb{R}^n}}{2t}\right) \text{ for } y \in \mathbb{R}^n,\ t > 0.$$

If $0 \leqslant t_1 \leqslant t_2 \leqslant \cdots \leqslant t_k$ define a measure $\nu_{t_1,\ldots,t_k}$ on $\mathbb{R}^n$ by

$$\begin{aligned} &\nu_{t_1,\ldots,t_k}\left(F_1 \times \cdots \times F_k\right) = \\ &\int_{F_1 \times \cdots \times F_k} p(t_1,x,x_1)p(t_2-t_1,x_1,x_2)\cdots p(t_k-t_{k-1},x_{k-1},x_k)\,dx_1\cdots dx_k, \end{aligned} \tag{2.27}$$

where $dx = dx_1 \cdots dx_k$ stands for the Lebesgue measure and the $p(0,x,y)dy = \delta_x(y)$, is the unit point mass at x. The extension of this definition rest on (2.25). It is clear that $p(t,x,y)$ satisfies $\int_{\mathbb{R}^n} p(t,x,y)\,dy = 1$ for all $t \in [0,T]$, hence property (2.26) is valid. Then we apply Theorem 14 to find a probability space $(\Omega, \mathcal{F}, \mathbb{P}^x)$ and a stochastic process $W = \{W(t)\}_{0 \leqslant t \leqslant T}$ on Ω such that the finite dimensional distributions on $W(t)$ are given by

$$\begin{aligned} &\mathbb{P}^x\left(W_{t_1} \in F_1, \cdots, W_{t_k} \in F_k\right) = \\ &\int_{F_1 \times \cdots \times F_k} p(t_1,x,x_1)p(t_2-t_1,x_1,x_2)\cdots p(t_k-t_{k-1},x_{k-1},x_k)\,dx_1\cdots dx_k. \end{aligned} \tag{2.28}$$

Definition 17. *A process $W = \{W(t)\}_{t\in[0,T]}$ satisfying properties (2.27) and (2.28) above is called Wiener process on a measurable space $(\Omega, \mathcal{F})$ with a family of probability measures $\mathbb{P}^x$, i.e, $\mathbb{P}^x(W_0 = x) = 1$, and W is a Wiener process starting at x under $\mathbb{P}^x$.*

A Wiener process is characterized by the following properties:

(i) $W(t)$ is a Gaussian process, i.e. for all $0 \leqslant t_1 \leqslant \cdots \leqslant t_k$ the random variable $W = \{W(t)\}_{t\in[t_1,t_k]}$ has a normal distribution.

(ii) $W(t)$ has independent increments, i.e.

$$W_{t_1}, W_{t_2} - W_{t_1}, \cdots, W_{t_k} - W_{t_{k-1}}$$

are independent for all $0 \leqslant t_1 < t_2 < \cdots < t_k$.

Definition 18. *Two stochastic processes $X = \{X(t)\}$ and $Y = \{Y(t)\}$ defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with values in $(E, \mathcal{G})$ are said to be a* **modification** *of (or a* **version** *of) each other if*

$$\mathbb{P}\left(\omega : X(t,\omega) = Y(t,\omega)\right) = 1, \ \ \forall t \in [0,T].$$

Note that if $X(t)$ is a modification of $Y(t)$, then $X(t)$ and $Y(t)$ have the same finite-dimensional distributions.

Next, we introduce another famous theorem of Kolmogorov which can help to justify the existence of a continuous version of Wiener process:

Theorem 15. *(***Kolmogorov's continuity theorem***). Suppose that the process $X = \{X(t)\}_{t\in[0\leqslant t\leqslant T}$ satisfies the following condition: for all $T > 0$ there exist positives constant α, β, C such that*

$$\mathbb{E}\left[|X(t+h) - X(t)|^{\alpha}\right] \leqslant C|h|^{1+\beta}; \ 0 \leqslant t, \ h \leqslant T.$$

Then there exists a continuous version of X.

Theorem 16. *Let W be a Wiener process. Then there exists a modification of W which has continuos paths a.s.*

To check whether or not a given proess $W(t)$ is a Wiener process, we need the following necessary and sufficient condition: for and arbitrary n, $0 = t_0 < t_1 < \cdots < t_n = T$, and $z_0, z_1, \ldots, z_n$

$$\mathbb{E} \exp \left\{ i \sum_{k=1}^{n} z_k \left[W(t_k) - W(t_{k-1}) \right] + i z_0 W(t_0) \right\} = \exp \left\{ -\frac{1}{2} \sum_{k=1}^{n} z_k^2 (t_k - t_{k-1}) \right\}.$$

Martingale

Definition 19. *A stochastic process $X = (X(t))_{0 \leqslant t \leqslant T}$ taking values in E, adapted to the filtration $\mathcal{F} = \{\mathcal{F}_t\}$ is called a* **martingale***(or martingale with respect to the filtration) if*

$$\begin{cases} i) \ X(t) \in L^1(d\mathbb{P}); \text{that is, } \mathbb{E}\left(|X(t)|\right) < \infty, \ \forall t \in [0, T], \\ \quad ii) \ \mathbb{E}\left(X(t)|\mathcal{F}_s\right) = X_s, \ \mathbb{P} - \text{a.s., for any} \quad t > s \geq 0. \end{cases}$$

The main ingredient in the theory of integration is the concept of square integrable martingales.

Definition 20. *A random variable X is said to be* **square integrable** *if it has a finite second moment (or mean square), that is $\mathbb{E}[X^2] < \infty$. A process $X = \{X(t)\}_{t \in [0,T]}$ is square integrable if $\sup_{t \in [0,T]} \mathbb{E}[X(t)^2] < \infty$. If the process X satisfies the following:*

a) X is a martingale,

b) X is square integrable,

then X is called square integrable martingale.

Let $1 \leqslant p < \infty$. We denote by

$$L^p(\Omega, \mathcal{F}, \mathbb{P})$$

the space of all stochastic processes (resp. martingales) $X = \{X(t)\}_{0 \leqslant t \leqslant T}$ with values in E, that satisfy the following two properties

i) $\|X(t)\|_E$ is measurable,

ii) $\mathbb{E}[\|X(t)\|_E^p] < \infty$.

Any stochastic process X for which the above two properties hold is called p-th integrable stochastic (resp. p-th integrable martingale) process.

Definition 21. *(cf. [47])*
An adapted process $X = \{X(t)\}_{0 \leqslant t \leqslant T}$ with values in E is said to be a local martingale if there exists an increasing sequence of of stopping times τ_n, such that

(i) $\tau_n \longrightarrow \infty$ almost surely as $n \longrightarrow \infty$,

(ii) for each n the stopped processes $X(t \wedge \tau_n)$ is uniformly integrable (the definition will follow) martingale in t.

Stochastic Integrals

In this section we shall introduce the definition of the stochastic integral

$$\int_0^T X(t)\,dW(t) \tag{2.29}$$

of a process $X = X(t)$ for any $t \in [0, T]$ with respect to a standard one dimensional Wiener process W.
Let X be an $\mathcal{F}_t$-measurable process for each t, for which

$$\int_0^T X^2(t)\,dt < \infty.$$

Then we can define the Itô integral (2.29) for the process X as follows:

$$\int_0^T X(t)\,dW(t) = \lim \sum_{i=0}^{n-1} X(t_i)\left(W(t_{i+1}^n) - W(t_i^n)\right), \tag{2.30}$$

as $|\delta_n| \longrightarrow 0$ and $n \longrightarrow \infty$, where for each n, $\{t_i^n\}$, is a partition of the interval $[0, T]$, and the limit is taken over all partitions with $\delta_n = \max_{1 \leqslant i \leqslant n-1}(t_{i+1}^n - t_i^n)$ is the mesh of the partition $t_i^n = \{t_0^n < t_1^n < \cdots < t_n^n = T\}$ of the interval $[0, T]$.

Theorem 17. *For a process $X(t)$ possessing the above properties, the stochastic integrals*

$$\int_0^t X(s)\,dW(s)$$

are continuous martingales in t with zero mean, that is

$$\mathbb{E}\int_0^{\cdot} X(t)\,dW(t) = 0.$$

Suppose that h is an $\mathcal{F}_t$-adapted process such that

$$\int_0^T h^2(t)\,dt < \infty \text{ almost surely,}$$

now consider $Y(t)$ to be an $\mathbb{R}$-valued Itô integral with respect to a $\mathbb{R}^m$ standard Brownian motion defined by

$$Y(t) = \int_0^t h(s)\,dW(s).$$

Then, the corresponding Itô integrals are defined for any $t \leqslant T$ and they are local martingales.

Stochastic differential and Itô processes

Consider the Itô integral

$$Y(t) = \int_0^t X(s)dW(s).$$

A process $Y = \{Y_t\}_{0 \leqslant t \leqslant T}$ is said to be an Itô process if for any $0 \leqslant t \leqslant T$ it can be expressed as follows:

$$Y(t) = Y(0) + \int_0^t \mu(s)\,ds + \int_0^t \sigma(s)\,dW(s), \tag{2.31}$$

where processes $\mu(t)$ and $\sigma(t)$ satisfy the properties:

(i) $\mu(t)$ is adapted and $\int_0^T |\mu(t)|\,dt < \infty$ almost surely

(ii) $\sigma(t)$ is predictible and $\int_0^T \sigma^2(t)\,dt < \infty$ almost surely.

We now introduce the notion of stochastic differential with respect to a standard brownian motion. The differential relation

$$dY(t) = X(t)\,dW(t)$$

is called by convention stochastic differential with respect to a standard brownian motion. If Y is an Itô process given by (2.31), we will then say that the process $Y(t)$ has a stochastic differential on the interval $[0,T]$

$$dY(t) = \mu(t)\,dt + \sigma(t)\,dW(t), \quad 0 \leqslant t \leqslant T. \tag{2.32}$$

If μ and σ depend on t through the process $Y(t)$, we then write

$$dY(t) = \mu(Y(t))\,dt + \sigma(Y(t))\,dW(t), \quad 0 \leqslant t \leqslant T. \tag{2.33}$$

Itô's formula

In the following result we introduce the Itô's formula for $\varphi(X(t))$.

Theorem 18. *Let $X(t)$ have a stochastic differential for $0 \leqslant t \leqslant T$*

$$dX(t) = b(s)\,ds + h(s)\,dW(s),$$

where $b(t)$ is an $\mathbb{R}$-valued, $\mathcal{F}_t$-measurable and integrable process over $[0,T]$.
Suppose that $\varphi(x)$ is once continuously differentiable in t and twice continuously differentiable in x. Then the the process $Y(t) = \varphi(X(t))$ also possesses a stochastic differential and is given by

$$d\varphi(X(t)) = \varphi'(X(t))\,dX(t) + \frac{1}{2}\varphi''(X(t))h^2(t)\,dt \tag{2.34}$$

$$= \left[\varphi'(X(t))b(t) + \frac{1}{2}\varphi''(X(t))h^2(t)\right]\,dt + \varphi'(X(t))h(t)\,dW(t). \tag{2.35}$$

In integral notations

$$\varphi(X(t)) = \varphi(X(0)) + \int_0^t \varphi'(X(s))\,dX(s) + \frac{1}{2}\varphi''(X(s))h^2(s)\,ds. \tag{2.36}$$

Formula (2.36) is called Itô's formula for $\varphi(X(t))$.

We shall introduce a useful result which is known as the Burkholder-Davis-Gundy inequality. This gives bounds for the maximum of a martingale in terms of the quadratic variations. The proof of the Burkholder-Davis-Gundy inequality can be found in [61] and [64].

Theorem 19. *Suppose that $Y(t) = \int_0^t X(s)\, dW(s)$ is the Itô's integral process such that*

$$\mathbb{E}\left(\int_0^\tau X^2(t)\, dt\right)^{p/2} < \infty.$$

Then, for any real number $p > 0$ there are constants $c_p > 0$ and $0 < C_p < \infty$ depending only on p, such that for any stopping time τ

$$c_p \mathbb{E}\left[\int_0^\tau X^2(t)\,(dW(t))^2\right]^{p/2} \leqslant \mathbb{E} \sup_{0\leqslant t\leqslant \tau}\left|\int_0^t X(s)\, dW(s)\right|^p \leqslant C_p \mathbb{E}\left[X^2(s)\, ds\right]^{p/2}. \tag{2.37}$$

We collect some powerful theorems from Prokhorov [60] and Skorohod [76] which are compactness results . A detailed proof of these results can be found in [20].

We firstly introduce the tightness of probability measures.

We shall consider E to be a separable complete metric space and consider its Borel σ-field $\mathcal{B}(E)$. We have the following definitions of relative compactness and of tightness of probability measure.

Definition 22. *A family of probability measures Π_n on $(E, \mathcal{B}(E))$ is said to be relatively compact if from every sequence of elements of Π_n we can extract a subsequence Π_{n_j} such that Π_{n_j} converges weakly to the measure Π. This can also be formulated as follows:*

For any continuous and bounded function ϕ on E

$$\lim_{j\longrightarrow\infty}\int_E \phi(s)\, d\Pi_{n_j} \longrightarrow \int_E \phi(s)\, d\Pi.$$

We define the tightness of Π_n by

Definition 23. *A family of probability measures Π_n on $(E, \mathcal{B}(E))$ is tight if for any $\varepsilon > 0$, we can find a compact subset K_ε of E such that*

$$\mathbb{P}(K_\varepsilon) \geq 1 - \varepsilon \text{ for every } \mathbb{P} \in \Pi_n.$$

Theorem 20 (Prokhorov)**.** *The family of probability measures Π_n is relatively compact if and only if it is tight.*

The weak convergence of probability measures can also be related to the almost everywhere convergence of random variables by the following theorem from [76].

Theorem 21 (Skorokhod)**.** *For any sequence of probability measures Π_n on $(E, \mathcal{B}(E))$ which converges weakly to a probability measure Π, there exists a probability space $(\Omega', \mathcal{F}', \mathbb{P}')$ and random variables $X, X_1, \ldots, X_n, \ldots$ with values in E such that the probability law of X_n is Π_n and that of X is Π and*

$$\lim_{n \to \infty} \Pi_n = \Pi, \;\; \mathbb{P}' - \text{a.s.}$$

Uniform Integrability

The concept of uniform integrability is very important in Probability Theory. Assume that we are given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $L^1(\Omega, \mathcal{F}, \mathbb{P})$ is the space of (equivalence classes) of real random variables X such that $\|X\|_{L^1(\Omega,\mathcal{F},\mathbb{P})} = \mathbb{E}[|X|] < \infty$.

Definition 24. *A family of random variables is said to be* **uniformly integrable** *provided that*

$$\sup_n \int_{\{|X_n|>c\}} |X_n(\omega)| \, d\mathbb{P}(\omega) \tag{2.38}$$

converges to 0 uniformly as $c \longrightarrow \infty$, or in a different notation,

$$\sup_n \mathbb{E}\left[|X_n(\omega)| I_{\{|X_n|>c\}}\right] \tag{2.39}$$

converges to 0 uniformly, as $c \longrightarrow \infty$.

The definition of uniformly integrable random variables is very useful in limit theorems, as the generalization of dominated Lebesgue's convergence theorem. That is known as Vitali's theorem (see definition below). The following result is very useful test for uniform integrability test and is known as de la Vallée-Poussin theorem.

Theorem 22. *Let $\{X_n\}_{n\in\mathbb{N}}$ be a sequence of integrable random variables. That is $\mathbb{E}[|X_n|] < \infty$. Let $\varphi = \varphi(t) : [0,T] \longrightarrow [0,\infty)$ be a nonnegative increasing, convex function ($\varphi(\lambda x + (1-\lambda)y) \leqslant \lambda\varphi(x) + (1-\lambda)\varphi(y)$) for all $x, y \in [0,T]$, $\lambda \in [0,1]$ such that*

$$\lim_{t\longrightarrow\infty} \frac{\varphi(t)}{t} = \infty; \tag{2.40}$$

and

$$\sup_{n\in\mathbb{N}} \int_\Omega \varphi(|X_n(\omega)|)\, d\mathbb{P}(\omega) < \infty. \tag{2.41}$$

Then the family $\{X_n\}_{n\in\mathbb{N}}$ is uniformly integrable if and only if (2.40) and (2.41) hold.

Note: the function φ defined above is called a u.i. (uniform integrability) test function.

Corollary 2. *If the process $X = \{X_t\}_{0\leqslant t\leqslant T}$ is square integrable, that is, $\sup_{0\leqslant t\leqslant T} \mathbb{E}[X_t^2] < \infty$, then X is uniformly integrable.*

We introduce a necessary and sufficient condition for a family $\{X_n\}_{n\in\mathbb{N}}$ of random variables to be uniformly integrable. That will help us to relate the two, i.e., uniform integrability and the convergence to random variables. For the proof of the following results, we refer to [74, page 186-189 and 191] and the bibliography therein.

Theorem 23 (Vitali's Theorem). *Suppose that $\{X_n\}_{n\in\mathbb{N}}$ is sequence of intergrable random variables, that is $X_n \in L^1(\Omega, \mathcal{F}, \mathbb{P})$ with $\mathbb{E}(|X_n|) < \infty$. Suppose as well that X_n converges in probability to the random variable X with $\mathbb{E}(|X|) < \infty$. Then, the following properties are equivalent*

i) the sequence of random variables $(|X_n|)_{n\in\mathbb{N}}$ are uniformly integrable, *ii) In probabilistic language, a sequence of integrable random variables* $(X_n)_{n\in\mathbb{N}}$ *converges to* X *in mean. That is,* $\mathbb{E}[|X_n|] \longrightarrow \mathbb{E}[|X|]$,

iii) a sequence $\{X_n\}_{n\in\mathbb{N}}$ *converges to* X *in the* L^1*-norm. That is,* $X_n \longrightarrow X$ *in* $L^1(\Omega, \mathcal{F}, \mathbb{P})$.

Hölder's inequality. Let $1 < p, q < \infty$, and $\dfrac{1}{p} + \dfrac{1}{q} = 1$. If $\mathbb{E}[|X|^p] < \infty$ and $\mathbb{E}[|Y|^q] < \infty$, Then $\mathbb{E}[|XY|] < \infty$ and

$$\mathbb{E}[XY] \leqslant \left(\mathbb{E}[|X|^p]\right)^{1/p} \left(\mathbb{E}[|Y|^q]\right)^{1/q}. \tag{2.42}$$

Chapter 3

SETTING AND MAIN RESULT

In this section we shall give the definition of the weak probabilistic solution to our problem, formulate our main result and prove the existence theorem of the weak probabilistic solutions by using the monotonicity of the operator A introduced in (2.21) subsection 2.4.

3.1 Assumptions

We now introduce the conditions on the nonlinear operators in equation (1.1).
We assume that

$$f : (0,T) \times L^2(D) \longrightarrow L^2(0,T; (W_0^{1,p(x)}(D))^*), \ \text{measurable}$$

$$\text{a.e.} \quad (t,u) \longrightarrow f(t,u) : \text{continuous w.r.t the second variable}$$

$$\|f(t,u)\|_{L^2(0,T;(W_0^{1,p(x)}(D))^*)} \leqslant C\left(1 + \|u\|_{L^2(Q_T)}\right) \tag{3.1}$$

$$G : (0,T) \times L^2(D) \longrightarrow \left(L^2(D)\right)^d, \text{measurable}$$

$$\text{a.e.} \quad (t,u) \longrightarrow G(t,u) : \text{continuous from w.r.t the second variable}$$

$$\|G(t,u)\|_{(L^2(D))^d} \leqslant C(1 + \|u(t)\|_{L^2(D)}). \tag{3.2}$$

Suppose that $p(x) \geq 2$ and satisfies the inequalities

$$2 \leqslant r = \text{ess} \inf_{x \in D_0^p} p(x) \leqslant s = \text{ess} \sup_{x \in D_0^p} p(x) < \infty. \tag{3.3}$$

Next, we shall define the concept of probabilistic weak solution of the I-BVP (1.1)-(1.3) as follows:

Definition 25. A probabilistic weak solution of the *I-BVP (1.1-1.3)* is a probabilistic system

$$\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leqslant t \leqslant T}, \mathbb{P}, W, u\right), \tag{3.4}$$

where

(1) $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space, $\mathcal{F}_t$ is a filtration on it,

(2) W is an d-dimensional $\mathcal{F}_t$-standard Wiener process,

(3) $u(t)$ is $\mathcal{F}_t$-measurable,

(4) $u(t)$ is an element of

$$L^q(\Omega, \mathcal{F}, \mathbb{P}, L^\infty(0, T; L^2(D))) \cap \mathring{V}(Q) \cap L^{q/2}(\Omega, \mathcal{F}, \mathbb{P}, L^r(0, T; W_0^{1,p(x)}(D)))$$

for any $q \in [2, \frac{2r'}{r'-1}]$,

(5) for a.e $t \in [0, T]$, $u(t)$ satisfies the integral identity

$$\begin{aligned}(u(t), v) - \int_0^t (A(s)u(s), v)ds = (u_0, v) + \int_0^t (f(s, u(s), v)\ ds + \\ + \left(\int_0^t G(s, u(s))\, dW(s), v \right),\end{aligned} \tag{3.5}$$

for all $v \in W_0^{1,p(x)}(D)$.

Our main result is

3.2 Main Result

Theorem 24 (Existence theorem)**.** *Let $p(x) \geq 2$ be a measurable function such that $p \in C(\bar{D})$ and moreover $p(x)$ satisfies the second condition in Theorem 9. In addition, assume that (3.1), (3.2), (3.3) hold and $u_0 \in L^2(D)$. Then there exists a probabilistic weak solution $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leqslant t\leqslant T}, \mathbb{P}, W, u)$ in the sense of the above definition.*

The equivalent stochastic parabolic problem to (3.5) is written as follows:

$$du - Au\,dt = f\,dt + G\,dW \ \text{ in } (W^{1,p(x)}(D))^*. \tag{3.6}$$

Remark 2. *By virtue of Theorems 9 and 10, $W_0^{1,p(x)}(D) \circlearrowleft\circlearrowleft L^2(D) \subset (W_0^{1,p(x)}(D))^*$, for $p(x) \geq 2$. Identity (3.5) with the inclusion $\mathring{V}(Q) \cap L^r(0,T; W_0^{1,r}(D)) \subset L^2(Q)$, implies that u is weakly continuous with values in the dual space $(W_0^{1,p(x)}(D))^*$. Following [79, subsect 1.4, page 263], we argue that, since the function $u(\omega)$ belongs to the space $L^\infty(0,T; L^2(D))$, then u is weakly continuous with values in $L^2(D)$; therefore the initial condition for u for $t = 0$ is meaningful.*

3.3 Proof of the Existence Theorem

In order to prove our main theorem, as an essential auxiliary tool, we use the Galerkin method.

Our existence proof will proceed in several steps and it follows the scheme of BENSOUSSAN [14] for the case of stochastic nonlinear parabolic equation, but with appropriate changes. We shall prove the existence by firstly constructing approximate solutions to the I-BVP (1.1)-(1.3) through a Galerkin scheme of the problem (1.1)-(1.3). At the second step we derive a "$priori$" estimates for the approximating solutions of these Galerkin systems. At the third step we pass to the limit in the finite dimensional equation by choosing from the sequence of solutions (u_m) a subsequence (u_{m_μ}) which converges weakly in appropriate topologies. Then at the final step, we shall prove that the limit u

of u_m is a solution of the I-BVP (1.1)-(1.3) by using the monotonicity of the operator A.

3.3.1 Construction of an approximating sequence

Let $w_1(x), w_2(x), \ldots, w_m(x), \ldots$ be a basis in the space $W_0^{1,p(x)}(D)$. For each $m \in \mathbb{N}$ let us denote the span of $\{w_1(x), \ldots, w_m\}$ by S_m.

Consider the probabilistic system

$$(\bar{\Omega}, \bar{\mathcal{F}}, \{\bar{\mathcal{F}}_t\}_{0\leqslant t\leqslant T}, \bar{\mathbb{P}}, \bar{W}).$$

We seek approximate solutions $u_m(t)$ of the problem (1.1)-(1.3) in the form

$$u_m(t,x) = \sum_{j=1}^{m} C_{jm}(t) w_j(x). \tag{3.7}$$

The functions $C_{1m}(t)$, $C_{2m}(t), \ldots, C_{mm}(t)$ in the expansion (3.7) are found from the system of stochastic ordinary differential equations

$$\begin{aligned}
&(du_m(t)\,,\, w_j) + (Au_m(t)\,,\, w_j)\; dt \\
&= (f(t, u_m(t)), w_j)\, dt + \int_D G(t, u(t)) w_j \, dx \; d\bar{W}(t), \quad j = 1, 2, \ldots, m,\; t \in [0,T],
\end{aligned} \tag{3.8}$$

with the initial conditions $C_{1m}(0) = C_1$, $C_{2m}(0) = C_2 \ldots$, $C_{mm}(0) = C_m$, where the constants C_k , $k = 1, 2, \ldots, m, \ldots$, are the coefficients in the expansion of $u_0(x)$ with respect to the basis $w_1(x), w_2(x), \ldots, w_m(x), \ldots$ in $L^2(D)$. Hence

$$u_m(0,x) = u_{0m}(x) = \sum_{j=1}^{m} C_{jm}(0) w_j(x) \longrightarrow u_0(x) \,\text{strongly in}\; L^2(D),\; \text{as}\, m \longrightarrow \infty.$$

Then we can rewrite explicitly system (3.8) in the form

$$\begin{aligned}
&\sum_{k=1}^{m} C'_{jm}(t)(w_k, w_j) + \sum_{i=1}^{n} \int_D \left| \sum_{k=1}^{m} C_{km}(t) \right|^{p(x)-2} \left(\sum_{k=1}^{m} C_{km}(t) \frac{\partial w_k(x)}{\partial x_i} \right) \frac{\partial w_j(x)}{\partial x_i} \, dx \\
&= (f(t, u_m(t)), w_j)\, dt + \int_D G(t, u_m(t)) w_j \, dx \; d\bar{W}(t),\; j \in [1,m], t \in [0,T].
\end{aligned} \tag{3.9}$$

Under our conditions on f and G, the system (3.9) satisfies the conditions of existence (see [77, page 121, Theorem 2], [76], and [81]). The solution u_m exists on some interval $[0, t_m]$, $t_m \leqslant T$. The a $priori$ estimates of the functions $u_m(t)$ obtained below implies that u_m exists on the interval $[0, T]$.

3.3.2 A "priori" estimates for the approximate solutions

We shall now establish a $priori$ estimates for the Galerkin approximate solutions u_m. To do that, we shall introduce a stopping time argument.

For each natural number $k \geq 1$, consider the following $\mathcal{F}_t$-stopping times:

$$\tau_k^m = \begin{cases} \inf\left\{t \in [0, T] : \quad \|u_m(t)\|_{L^2(D)} \geq k\right\}, \\ \qquad\qquad \text{if } \left\{\bar{\omega} \in \bar{\Omega} : \|u_m(t)\|_{L^2(D)} \geq k\right\} \neq \emptyset, \\ T, \quad \text{if} \left\{\bar{\omega} \in \bar{\Omega} : \|u_m(t)\|_{L^2(D)} \geq k\right\} = \emptyset. \end{cases}$$

We shall give the first main result in the following lemma from which follows the existence of u_m over the entire closed interval $[0, T]$.

Lemma 5. *There exists a positive constant K independent of m such that the following a priori estimates hold*

$$\bar{\mathbb{E}} \sup_{0 \leqslant t \leqslant T} \|u_m(s)\|^2_{L^2(D)} \leqslant K, \tag{3.10}$$

$$\bar{\mathbb{E}} \, \|u_m\|_{\mathring{V}(Q)} \leqslant K. \tag{3.11}$$

Here $\bar{\mathbb{E}}$ is the mathematical expectation on the probability space $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}})$.

Proof 5. *Let $u_m(t) \in S_m = \mathrm{Span}\{w_j(x)\}_{j=1}^m$. By multiplying equation (3.8) by $C_{jm}(t, \omega)$ and by summing the resulting relations over j from 1 to m we obtain*

$$\begin{aligned} &(du_m, u_m) + \int_D \sum_{i=1}^n \left|\frac{\partial u_m(t,x)}{\partial x_i}\right|^{p(x)-2} \frac{\partial u_m(t,x)}{\partial x_i} \frac{\partial u_m(t,x)}{\partial x_i} dx dt \\ &= (f(t, u_m(t)), u_m(t))\, dt + \int_D G(t, u_m(t)) u_m(t) dx dW(t) \quad \textit{for } j = 1, 2, \ldots, m. \end{aligned} \tag{3.12}$$

By applying Ito's formula to equation (3.12) we get

$$d\|u_m(t)\|^2_{L^2(D)} = 2\left[-\sum_{i=1}^{n}\int_D \left|\frac{\partial u(t,x)}{\partial x_i}\right|^{p(x)} dx + (f(t,u_m(t)),u_m(t))\right] dt+$$
$$+\int_D \|G(t,u_m(t))\|^2_{(L^2(D))^d}\,dx\ dt + 2\int_D G(t,u_m(t,x)u_m(t,x)\,dx\,dW(t).$$

The integration of this relation over the interval $[0,t]$ *yields the following*

$$I \equiv \|u_m(t)\|^2_{L^2(D)} + 2\int_0^t \sum_{i=1}^{n}\int_D \left|\frac{\partial u(s,x)}{\partial x_i}\right|^{p(x)} dx\ ds = 2\int_0^t (f(s,u_m(s)),u_m(s))\ ds+$$
$$+2\int_0^t \frac{1}{2}\int_D \|G(s,u_m(s,x))\|^2_{(L^2(D))^d}\ dx\,ds + 2\int_0^t\int_D G(s,u_m(s,x))u_m(s,x)\,dx\,dW(s)+$$
$$\|u_{0m}\|^2_{L^2(D)}. \tag{3.13}$$

Hence, we obtain the inequality

$$I \leqslant 2\,K_1\|u_m\|_{L^2(0,T;W_0^{1,p(x)}(D))}\|f(t,u_m(t))\|_{L^2\left(0,T;\left(W_0^{1,p(x)}(D)\right)^*\right)} +$$
$$+\int_0^t \|G(s,u_m(s,x))\|^2_{(L^2(D))^d}\,ds + 2\int_0^t (G(s,u_m(s,x)),u_m(s))\,dW(s) + \|u_{0m}\|^2_{L^2(D)}, \tag{3.14}$$

where the positive constant K_1 *is independent of* m.

By applying Young's inequality to the first term on the right-hand side of relation (3.14) and using the continuous embedding established in Lemma 2, we obtain

$$\begin{aligned} I &\leqslant C(\varepsilon)\|f(t,u_m(t))\|^{\frac{r}{r-1}}_{L^2\left(0,T;\left(W_0^{1,p(x)}(D)\right)^*\right)} + C\varepsilon^r\|u_m\|^r_{\mathring{V}(Q_t)} + \|u_{0m}\|^2_{L^2(D)} \\ &+\int_0^t \|G(s,u_m(s))\|^2_{(L^2(D))^d}\,ds + 2\int_0^t (G(s,u_m(s)),u_m(s))\,dW(s), \end{aligned} \tag{3.15}$$

where ε *is a positive constant to be chosen later.*

We consider the following alternatives. Either $\|u_m\|_{\mathring{V}(Q_t)} \geq 1$, *or* $\|u_m\|_{\mathring{V}(Q_t)} \leqslant 1$.

If $\|u_m\|_{\mathring{V}(Q_t)} \geq 1$, *we set*

$$\lambda = \sum_{i=1}^{n}\int_0^t\int_D \left|\frac{\partial u_m}{\partial x_i}\right|^{p(x)} dx\,dt,$$

by definition

$$\|u_m\|_{\overset{\circ}{V}(Q_t)} = \sum_{i=1}^{n} \inf \left\{ \lambda_i \ > \ 0 : \int_0^t \int_D \left(\left| \frac{\partial u_m}{\partial x_i} \right| / \lambda_i \right)^{p(x)} \ dx\, dt \leqslant 1 \right\},$$

and from Lemma 1 we obtain $\lambda \geq 1$. *Since,* $2 \leqslant r \leqslant p(x)$, *we have* $\frac{1}{p(x)} \leqslant \frac{1}{r}$. *This implies* $\dfrac{1}{\lambda^{1/r}} \leqslant \dfrac{1}{\lambda^{1/p(x)}}$. *We have*

$$1 = \frac{\sum_{i=1}^{n} \int_0^t \int_D \left| \dfrac{\partial u_m}{\partial x_i} \right|^{p(x)} \ dx\, dt}{\sum_{i=1}^{n} \int_0^t \int_D \left| \dfrac{\partial u_m}{\partial x_i} \right|^{p(x)} \ dx\, dt} = \sum_{i=1}^{n} \int_0^t \int_D \left| \frac{\partial u_m}{\partial x_i} / \lambda^{1/p(x)} \right|^{p(x)} \ dx\, dt;$$

which implies that

$$\sum_{i=1}^{n} \int_0^t \int_D \left| \frac{\partial u_m}{\partial x_i} / \lambda^{1/r} \right|^{p(x)} \ dx\, dt \leqslant \sum_{i=1}^{n} \int_0^t \int_D \left| \frac{\partial u_m}{\partial x_i} / \lambda^{1/p(x)} \right|^{p(x)} \ dx\, dt = 1.$$

This inequality, together with the definition of the norm in the space $\overset{\circ}{V}(Q_t)$ *enable us to say that*

$$\|u_m\|_{\overset{\circ}{V}(Q_t)} = \sum_{i=1}^{n} \left\| \frac{\partial u_m}{\partial x_i} \right\|_{L^{p(x)}(Q_t)} \leqslant \lambda^{1/r},$$

hence, $\|u_m\|^r_{\overset{\circ}{V}(Q_t)} \leqslant \lambda$. *Thus chosing* ε *sufficiently small in (3.15), we get*

$$\begin{aligned} &\|u_m(t)\|^2_{L^2(D)} + \|u_m\|^r_{\overset{\circ}{V}(Q_t)} \leqslant \\ &+ C(\varepsilon) \|f(t, u_m(t))\|^{r/(r-1)}_{L^2\left(0,T;\left(W_0^{1,p(x)}(D)\right)^*\right)} + \|u_{0m}\|^2_{L^2(D)} + \\ &\int_0^t \|G(s, u_m(s,x))\|^2_{(L^2(D))^d}\, ds + 2 \int_0^t \int_D G(s, u_m(s)) u_m(s)\, dx\, dW(s). \end{aligned} \tag{3.16}$$

Let now $\|u_m\|_{\overset{\circ}{V}(Q_t)} \leqslant 1$. *Then*

$$\|u_m\|^r_{\overset{\circ}{V}(Q_t)} \leqslant 1. \tag{3.17}$$

By (3.15) and (3.17) we have

$$\begin{aligned} &\|u_m(t)\|^2_{L^2(D)} \leqslant \\ &C(\varepsilon) \|f(t, u_m(t))\|^{r/(r-1)}_{L^2\left(0,T;\left(W_0^{1,p(x)}(D)\right)^*\right)} + \tilde{C}_\varepsilon + \|u_{0m}\|^2_{L^2(D)} + \\ &\int_0^t \|G(s, u_m(s,x))\|^2_{(L^2(D))^d}\, ds + 2 \int_0^t \int_D G(s, u_m(s)) u_m(s) dx\, dW(s). \end{aligned} \tag{3.18}$$

Conbining (3.16) and (3.18) we come up with

$$\|u_m(t)\|^2_{L^2(D)} + \|u_m\|^r_{\overset{\circ}{V}(Q_t)} \leqslant$$
$$C + C(\varepsilon)\|f(t,u_m(t))\|^{r/(r-1)}_{L^2\left(0,T;\left(W_0^{1,p(x)}(D)\right)^*\right)} + \|u_{0m}\|^2_{L^2(D)} +$$
$$\int_0^t \|G(s,u_m(s,x))\|^2_{(L^2(D))^d}\,ds + 2\int_0^t\int_D G(s,u_m(s))u_m(s)\,dx\,dW(s). \tag{3.19}$$

Since $r \geq 2$ *then its conjugate* $\dfrac{r}{r-1} \leqslant 2$. *We have by (3.1) and the result of Young's inequality*

$$\begin{aligned}\|f(t,u_m(t))\|^{r/(r-1)}_{L^2\left(0,T;\left(W_0^{1,p(x)}(D)\right)^*\right)} &\leqslant C_1(\varepsilon) + C_2(\varepsilon)\|f(t,u_m(t))\|^2_{L^2\left(0,T;\left(W_0^{1,p(x)}(D)\right)^*\right)}\\ &\leqslant C\left(1+\|u_m\|_{L^2(Q_t)}\right)^2.\end{aligned} \tag{3.20}$$

Let us take the supremum over the interval $[0, t\wedge\tau_k^m]$ *and pass to the expectation in (3.19). Then we can estimate the terms in the resulting right-hand side.*
We have

$$\bar{\mathbb{E}}\sup_{s\in[0,\,t\wedge\tau_k^m]}\|u_m(s)\|^2_{L^2(D)} + \bar{\mathbb{E}}\|u_m\|^r_{\overset{\circ}{V}(Q_{t\wedge\tau_k^m})} \leqslant$$
$$C + C(\varepsilon)\bar{\mathbb{E}}\|u_m(s)\|^2_{L^2(Q_{t\wedge\tau_k^m})} + \bar{\mathbb{E}}\|u_{0m}\|^2_{L^2(D)} +$$
$$\bar{\mathbb{E}}\int_0^{t\wedge\tau_k^m}\|G(s,u_m(s,x))\|^2_{(L^2(D))^d}\,ds + 2\bar{\mathbb{E}}\sup_{s\in[0,t\wedge\tau_k^m]}\int_0^s\int_D G(s,u_m(s))u_m(s)dx\,dW(s). \tag{3.21}$$

Here we have used the notation $t\wedge\tau_k^m = \min\{t\,,\,\tau_k^m\}$ *and* $t\in[0,\tau_k^m]$.
By assumption (3.2) we have

$$\int_0^{t\wedge\tau_k^m}\|G(s,u_m(s))\|^2_{(L^2(D))^d}\;ds \leqslant C\int_0^{t\wedge\tau_k^m}\left(1+\|u_m(s)\|^2_{L^2(D)}\right)ds. \tag{3.22}$$

We proceed to estimate the stochastic integral term in the right-hand side of (3.21).

We use the Burkholder-Gundy-Davis inequality. We have for any $\eta > 0$,

$$\begin{aligned}
&\bar{\mathbb{E}} \sup_{s\in[0,t\wedge\tau_k^m]} \left| \int_0^s \int_D G(l,u_m(l))u_m(l)\,dx\,dW(l) \right| \\
&\leqslant C\bar{\mathbb{E}} \left(\int_0^{t\wedge\tau_k^m} (\ G(s,u_m(s))\ ,\ u_m(s)\ )^2\ ds \right)^{\frac{1}{2}} \\
&\leqslant C\bar{\mathbb{E}}\ \left(\int_0^{t\wedge\tau_k^m} \|G(s,u_m(s))\|^2_{(L^2(D))^d}\, \|u_m(s)\|^2_{L^2(D)}\ ds \right)^{\frac{1}{2}} \\
&\leqslant C\ \bar{\mathbb{E}} \left(\int_0^{t\wedge\tau_k^m} [1+\|u_m(s)\|_{L^2(D)}]^2 \|u_m(s)\|^2_{L^2(D)}\ ds \right)^{1/2} \\
&\leqslant C\ \bar{\mathbb{E}} \sup_{s\in[0,t\wedge\tau_k^m]} \|u_m(s)\|_{L^2(D)} \left(\int_0^{t\wedge\tau_k^m} [1+\|u_m(s)\|_{L^2(D)}]^2\ ds \right)^{1/2} \\
&\leqslant \eta\ \bar{\mathbb{E}} \sup_{s\in[0,t\wedge\tau_k^m]} \|u_m(s)\|^2_{L^2(D)} + C_\eta\, \bar{\mathbb{E}} \int_0^{t\wedge\tau_k^m} [1+\|u_m(s)\|_{L^2(D)}]^2\ ds. \qquad (3.23)
\end{aligned}$$

Here we have used (3.2), Hölder's and Young's inequalities.

Combining the above inequalities with an appropriate choice of the parameter η, we obtain

$$\begin{aligned}
&\bar{\mathbb{E}} \sup_{s\in[0,\,t\wedge\tau_k^m]} \|u_m(s)\|^2_{L^2(D)} + \bar{\mathbb{E}}\|u_m\|^r_{\overset{\circ}{V}(Q_{t\wedge\tau_k^m})} \leqslant \\
&\|u_{0m}\|^2_{L^2(D)} + C + C\,\bar{\mathbb{E}} \int_0^{t\wedge\tau_k^m} [1+\|u_m(s)\|_{L^2(D)}]^2\ ds. \qquad (3.24)
\end{aligned}$$

This implies that for all $s \in [0, t\wedge t_k^m]$ and for all $m,\, k \geq 1$

$$\bar{\mathbb{E}} \sup_{s\in[0,t\wedge\tau_k^m]} \|u_m(s)\|^2_{L^2(D)} + \bar{\mathbb{E}}\|u_m\|^r_{\overset{\circ}{V}(Q_{t\wedge\tau_k^m})} \leqslant K, \qquad (3.25)$$

where K is a positive constant independent of m. As $k \longrightarrow \infty$, the sequence $t\wedge\tau_k^m$ converges to t. Then passing to the limit in (3.25) as $k \longrightarrow \infty$, we obtain the key estimates

$$\bar{\mathbb{E}} \sup_{t\in[0,T]} \|u_m(t)\|^2_{L^2(D)} \leqslant K, \qquad (3.26)$$

$$\bar{\mathbb{E}}\|u_m\|^r_{\overset{\circ}{V}(Q_T)} \leqslant K, \qquad (3.27)$$

for all $m \geq 1$ and hence the proof of the lemma is complete.

Lemma 6. *Let $q \in [2, \frac{2r'}{r'-1}]$, then u_m satisfies the estimate*

$$E \sup_{0 \leqslant t \leqslant T} \|u_m(s)\|^q_{L^2(D)} \leqslant C, \tag{3.28}$$

where C is positive constant independent of m and r' is the conjugate of r.

Proof 6. *Let $q \geq 4$. Applying Ito's formula to (3.13), we get*

$$\begin{aligned}
&d\|u_m\|^{q/2}_{L^2(D)} = \frac{q}{2}\|u_m\|^{(q-4)/2}_{L^2(D)} \times (-2)\left(Au_m(t), u_m(t)\right) dt + \\
&q\|u_m\|^{(q-4)/2}_{L^2(D)} \left[\left(f(t, u_m)\,,\, u_m(t)\right)\; dt + \int_D G(t, u_m) u_m dx\, dW(t) \right] \\
&+ \frac{q}{2}\frac{(q-4)}{4}\|u_m(t)\|^{(q/2)-4}_{L^2(D)}\|G(t, u_m)\|^2_{(L^2(D))^d} dt.
\end{aligned}$$

After integrating this relation over the closed interval $[0, t]$, we obtain the inequality

$$\begin{aligned}
&\|u_m(t)\|^{q/2}_{L^2(D)} + q\int_0^t \|u_m(s)\|^{(q/2)-2}_{L^2(D)} \int_D \sum_{i=1}^n \left|\frac{\partial u_m}{\partial x_i}\right|^{p(x)} dx\, ds \\
&\leqslant \|u_{0m}\|^{q/2}_{L^2(D)} + q\int_0^t \|u_m(s)\|^{(q/2)-2}_{L^2(D)} \int_D f(s, u_m(s)) u_m(s) dx\, ds \\
&+ \frac{q}{2}\int_0^t \|u_m(s)\|^{(q/2)-2}_{L^2(D)} \left(G(s, u_m(s)), u_m(s)\right)\; dW(s) + \\
&+ \frac{q}{2}\frac{q-4}{4}\int_0^t \|u_m(s)\|^{(q/2)-4}_{L^2(D)}\|G(s, u_m(s))\|^2_{(L^2(D))^d} ds.
\end{aligned}$$

It follows that

$$\begin{aligned}
&J \equiv \sup_{0 \leqslant s \leqslant t} \|u_m(s)\|^{q/2}_{L^2(D)} + \frac{q}{2}\sum_{i=1}^n \int_0^t \|u_m(s)\|^{(q/2)-2}_{L^2(D)} \int_D \left|\frac{\partial u_m}{\partial x_i}\right|^{p(x)} dx\; ds \\
&\leqslant \|u_{0m}\|^{q/2}_{L^2(D)} + q\int_0^t \|u_m(s)\|^{(q/2)-2}_{L^2(D)} \left(f(s, u_m(s))\,,\, u_m(s)\right)\; ds + \\
&+ \frac{q}{2}\sup_{0 \leqslant s \leqslant t}\int_0^t \|u_m(s)\|^{(q/2)-2}_{L^2(D)}\|G(s, u_m(s))\|_{(L^2(D))^d}\|u_m(s)\|_{L^2(D)} dW(s) + \\
&+ \int_0^t \|u_m(s)\|^{(q/2)-4}_{L^2(D)}\|G(s, u_m(s))\|^2_{(L^2(D))^d}\; ds.
\end{aligned} \tag{3.29}$$

To estimate the terms in the right-hand side of (3.29), we take the square and the mathematical expectation in both sides of this inequality. By using assumption (3.1), *Hölder's and Young's inequalities we have*

$$
\begin{aligned}
&\bar{\mathbb{E}}\left(\int_0^t \|u_m(s)\|_{L^2(D)}^{(q/2)-2}(f(s,u_m(s))\,,\,u_m(s))\;ds\right)^2\\
&\leqslant C\bar{\mathbb{E}}\left[\sup_{s\in[0,t]}\|u_m(s)\|_{L^2(D)}^{q-4}\left(\int_0^t (f(s,u_m(s))\,,\,u_m(s))\;ds\right)^2\right]\\
&\leqslant C\bar{\mathbb{E}}\left[\sup_{s\in[0,t]}\|u_m(s)\|_{L^2(D)}^{q-4}\|f(s,u_m(s))\|_{L^2\left(0,t;\left(W_0^{1,p(x)}(D)\right)^*\right)}^2\|u_m\|_{L^2\left(0,t;W_0^{1,p(x)}(D)\right)}^2\right]\\
&\leqslant C\bar{\mathbb{E}}\left[\|u_m\|_{\mathring{V}(Q_t)}^2\sup_{s\in[0,t]}\|u_m(s)\|_{L^2(D)}^{q-4}\left(1+\|u_m\|_{L^2(Q_t)}\right)^2\right]\\
&\leqslant C(\varepsilon)\bar{\mathbb{E}}\|u_m\|_{\mathring{V}(Q_t)}^r+\varepsilon\bar{\mathbb{E}}\sup_{s\in[0,T]}\|u_m(t)\|_{L^2(D)}^{r'(q-4)}+\varepsilon\bar{\mathbb{E}}\|u_m(t)\|_{L^2(D)}^{r'(q-2)}\\
&\leqslant C(\varepsilon)\bar{\mathbb{E}}\|u_m\|_{\mathring{V}(Q_t)}^r+\varepsilon'\bar{\mathbb{E}}\sup_{s\in[0,T]}\|u_m(t)\|_{L^2(D)}^{r'(q-4)}+C(\varepsilon').
\end{aligned}
$$

Since $\dfrac{2r'}{r'-1}\geq q$ *then* $r'(q-2)\leqslant q$. *Therefore by Young's inequality*

$$
\begin{aligned}
&\bar{\mathbb{E}}\left(\int_0^t \|u_m(s)\|_{L^2(D)}^{(q/2)-2}(f(s,u_m(s))\,,\,u_m(s))\;ds\right)^2\\
&\leqslant C\bar{\mathbb{E}}\|u_m\|_{\mathring{V}(Q_t)}^r+\varepsilon\bar{\mathbb{E}}\sup_{t\in[0,T]}\|u_m(t)\|_{L^2(D)}^q+C.
\end{aligned}
$$

Similarly, we have by assumption (3.2),

$$
\begin{aligned}
&\bar{\mathbb{E}}\left(\int_0^t \|u_m(s)\|_{L^2(D)}^{(q-4)/2}\|G(s,u_m(s))\|_{(L^2(D))^d}^2\;ds\right)^2\\
&\leqslant C\bar{\mathbb{E}}\left(\int_0^t \|u_m(s)\|_{L^2(D)}^{(q/2)-2}\left(1+\|u_m(s)\|_{L^2(D)}\right)^2\;ds\right)^2\\
&\leqslant C\bar{\mathbb{E}}\left(\int_0^t \left[\|u_m(s)\|_{L^2(D)}^{(q/2)-2}+\|u_m(s)\|_{L^2(D)}^{q/2}\right]\;ds\right)^2\\
&\leqslant C\bar{\mathbb{E}}\left(\int_0^t \left[\|u_m(s)\|_{L^2(D)}^{q/2}+1\right]\;ds\right)^2\\
&\leqslant C\,T\,\bar{\mathbb{E}}\int_0^t\left(\|u_m(s)\|_{L^2(D)}^q+1\right)ds.
\end{aligned}
$$

Let us estimate the stochastic integral in (3.29). We use martingale inequality

$$\begin{aligned}
&\bar{\mathbb{E}}\left(\sup_{s\in[0,\,t]}\int_0^s \|u_m(l)\|_{L^2(D)}^{(q/2)-2}\left(G(l,u_m(l))\,,u_m(l)\right)\,dW(l)\right)^2\\
&\leqslant C\bar{\mathbb{E}}\int_0^t \|u_m(s)\|_{L^2(D)}^{q-4}\left(G(s,u_m(s))\,,u_m(s)\right)^2\,ds\\
&\leqslant C\bar{\mathbb{E}}\int_0^t \|u_m(s)\|_{L^2(D)}^{q-4}\|G(s,u_m(s))\|_{(L^2(D))^d}\|u_m(s)\|_{L^2(D)}ds\\
&\leqslant C\bar{\mathbb{E}}\int_0^t \|u_m(s)\|_{L^2(D)}^{q-4}\|u_m(s)\|_{L^2(D)}^{2}\left(1+\|u_m(s)\|_{L^2(D)}\right)^2\,ds\\
&\leqslant C\bar{\mathbb{E}}\int_0^t \left(\|u_m(s)\|_{L^2(D)}^{q-2}+\|u_m(s)\|_{L^2(D)}^{q}\right)\,ds\\
&\leqslant C\,T\,\bar{\mathbb{E}}\int_0^t \left(\|u_m(s)\|_{L^2(D)}^{q}+1\right)\,ds. \qquad (3.30)
\end{aligned}$$

Combining these estimates for sufficiently small ε and ε' with an appropriate choice of the constants $C(\varepsilon)$ and $C(\varepsilon')$, we obtain

$$\begin{aligned}
&\bar{\mathbb{E}}\sup_{s\in[0,\,t]}\|u_m(s)\|_{L^2(D)}^{q}+\bar{\mathbb{E}}\left(\sum_{i=1}^n\int_0^t \|u_m(s)\|_{L^2(D)}^{(q/2)-2}\int_D\left|\frac{\partial u_m}{\partial x_i}\right|^{p(x)}dx\,ds\right)^2\\
&\leqslant C\,T\,\bar{\mathbb{E}}\int_0^t \left(\|u_m(s)\|_{L^2(D)}^{q}+1\right)\,ds. \qquad (3.31)
\end{aligned}$$

This implies that

$$\bar{\mathbb{E}}\sup_{s\in[0,\,t]}\|u_m(s)\|_{L^2(D)}^{q}\leqslant C\,T\,\bar{\mathbb{E}}\int_0^t \left(\|u_m(s)\|_{L^2(D)}^{q}+1\right)\,ds.$$

Hence, we obtain the inequality

$$\bar{\mathbb{E}}\left(\sup_{0\leqslant t\leqslant T}\|u_m(t)\|_{L^2(D)}^{q}\right)\leqslant C. \qquad (3.32)$$

3.3.3 Estimates involving the dual space $\left(W_0^{1,p(x)}(D)\right)^*$

In this subsection, we shall derive an estimate on the norm of the difference $u_m(t+\theta)-u_m(t)$ in the dual space $(W_0^{1,p(x)}(D))^*$.

Lemma 7. *The followins holds*

$$\bar{\mathbb{E}}\sup_{0\leqslant|\theta|\leqslant\delta\leqslant 1}\int_0^T\|u_m(t+\theta)-u_m(t)\|^2_{(W_0^{1,p(x)}(D))^*}\,dt\leqslant C\delta \qquad (3.33)$$

Proof 7. *Noting that the functions* $\{C_{jm}(t)w_j(x)\}_{j=1,2,\ldots}$ *form an orthonormal basis of the dual* $\left(W_0^{1,p(x)}(D)\right)^*$ *of* $W_0^{1,p(x)}(D)$*, we introduce the orthogonal projection of* $\left(W_0^{1,p(x)}(D)\right)^*$ *onto the span of* $S_m = \{C_{1m}(t)w_1(x),\ldots,C_{jm}(t)w_j(x),\ldots,C_{mm}(t)w_m(x)\}$*;*

$$P_m : (W_0^{1,p(x)}(D))^* \longrightarrow S_m; \tag{3.34}$$

$$P_m\xi = \sum_{i=1}^{m} C_{jm}(t)\langle \xi, w_j\rangle_{W_0^{1,p(x)}(D)\times\left(W_0^{1,p(x)}(D)\right)^*} w_j(x), \tag{3.35}$$

where $\langle\cdot\,,\,\cdot\rangle_{W_0^{1,p(x)}(D)\times\left(W_0^{1,p(x)}(D)\right)^*}$ *denotes the duality pairing between the space* $W_0^{1,p(x)}(D)$ *and its dual* $\left(W_0^{1,p(x)}(D)\right)^*$*.*

Then writing (3.12) in integrated form as an equality between random variables with values in the space $\left(W_0^{1,p(x)}(D)\right)^*$*,*

$$u_m(t,x) + \int_0^t P_m\left[A(s)u_m(s) - f(s,u_m)\right] ds = u_{0m} + \int_0^t P_m(G(s,u_m))dW(s). \tag{3.36}$$

For positive θ *such that* $u_m(t+\theta)$ *is defined for* $t \in [0,T]$*, we have*

$$\begin{aligned} &u_m(t+\theta) - u_m(t) \\ &= -\int_t^{t+\theta} P_m A u_m(s) ds + P_m\left(\int_t^{t+\theta} f(s,u_m(s))ds + \int_t^{t+\theta} G(s,u_m(s))d\bar{W}(s)\right); \end{aligned} \tag{3.37}$$

and then by definition

$$\|u_m(t+\theta)-u_m(t)\|_{(W_0^{1,p(x)}(D))^*} = \sup_{\varphi\in W_0^{1,p(x)}(D):\|\varphi\|_{W_0^{1,p(x)}(D)}=1} \int_D (u_m(t+\theta)-u_m(t))\varphi(x)dx.$$

We set

$$y_t(\theta) = \left\| -\int_t^{t+\theta} P_m\left[Au_m(s) - f(s,u_m)\right] ds + \int_t^{t+\theta} P_m G(s,u_m)\, d\bar{W}(s)\right\|_{\left(W_0^{1,p(x)}(D)\right)^*}.$$

From this we have the following

$$
\begin{aligned}
\|u_m(t+\theta)-u_m(t)\|_{(W_0^{1,p(x)}(D))^*} &= y_t(\theta)\\
&\leqslant \left\|\int_t^{t+\theta} P_m A u_m(s)\,ds\right\|_{\left(W_0^{1,p(x)}(D)\right)^*}+\\
&+\left\|P_m\int_t^{t+\theta} f(s,u_m(s))\,ds\right\|_{\left(W_0^{1,p(x)}(D)\right)^*}+\\
&+\left\|\int_t^{t+\theta} P_m G(s,u_m(s))\,d\bar{W}(s)\right\|_{\left(W_0^{1,p(x)}(D)\right)^*}.
\end{aligned}
\tag{3.38}
$$

Since A is bounded from $W_0^{1,p(x)}(D)\longrightarrow(W_0^{1,p(x)}(D))^*$, we have by Fubini's theorem and Hölder's inequality

$$
\begin{aligned}
&\left\|\int_t^{t+\theta} P_m A u_m(s)ds\right\|_{(W_0^{1,p(x)}(D))^*}\\
&=\sup_{\varphi\in W_0^{1,p(x)}(D):\|\varphi\|_{W_0^{1,p(x)}(D)}=1}\int_D\left(\int_t^{t+\theta}P_mAu_m(s)\varphi(x)ds\right)dx\\
&\leqslant\int_t^{t+\theta}\|P_mAu_m(s)\|_{(W_0^{1,p(x)}(D))^*}ds\\
&\leqslant C\int_t^{t+\theta}\|u_m(s)\|_{W_0^{1,p(x)}(D)}ds\\
&\leqslant C\theta^{1/2}\left(\int_t^{t+\theta}\|u_m(s)\|^2_{W_0^{1,p(x)}(D)}\right)^{1/2}.
\end{aligned}
\tag{3.39}
$$

Similarly we also have

$$
\begin{aligned}
\left\|P_m\int_t^{t+\theta}f(s,u_m(s))ds\right\|_{(W_0^{1,p(x)}(D))^*} &\leqslant \int_t^{t+\theta}\|f(s,u_m(s))\|_{(W_0^{1,p(x)}(D))^*}ds\\
&\leqslant C\theta^{1/2}\left(\int_t^{t+\theta}\|f(s,u_m(s))\|^2_{(W_0^{1,p(x)}(D))^*}ds\right)^{1/2}.
\end{aligned}
\tag{3.40}
$$

According to (3.37), (3.39) and (3.40) and taking into account the inequality (3.38), we *have*

$$\begin{aligned}
&\|u_m(t+\theta)-u_m(t)\|_{(W_0^{1,p(x)}(D))^*}\\
&\leqslant C\theta^{1/2}\left[\left(\int_t^{t+\theta}\|u_m(s)\|^2_{W_0^{1,p(x)}(D)}ds\right)^{1/2}+\left(\int_t^{t+\theta}\|f(s,u_m(s))\|^2_{(W_0^{1,p(x)}(D))^*}ds\right)^{1/2}\right]\\
&+\left\|\int_t^{t+\theta}P_mG(s,u_m(s))d\bar{W}(s)\right\|_{(W_0^{1,p(x)}(D))^*}.
\end{aligned}\tag{3.41}$$

Taking the square in (3.41), we get

$$\begin{aligned}
&\|u_m(t+\theta)-u_m(t)\|^2_{\left(W_0^{1,p(x)}(D)\right)^*}\\
&\leqslant C\theta\int_t^{t+\theta}\|u_m(s)\|^2_{W_0^{1,p(x)}(D)}ds+\\
&+C\theta\int_t^{t+\theta}\|f(s,u_m(s))\|^2_{(W_0^{1,p(x)}(D))^*}ds\\
&+\left\|\int_t^{t+\theta}G(s,u_m(s))d\bar{W}(s)\right\|^2_{(W_0^{1,p(x)}(D))^*}.
\end{aligned}\tag{3.42}$$

We fix $\delta<1$ and we take the supremum over $\theta\leqslant\delta$, then we obtain

$$\begin{aligned}
&\sup_{0\leqslant\theta\leqslant\delta<1}[y_t(\theta)]^2\\
&\leqslant CT\delta^2\sup_{0\leqslant t\leqslant T}\|u_m(t)\|^2_{W_0^{1,p(x)}(D)}+C\delta\sup_{0\leqslant\theta\leqslant\delta<1}\int_t^{t+\delta}\|f(s,u_m(s))\|^2_{(W_0^{1,p(x)}(D))^*}ds\\
&+\sup_{0\leqslant\theta\leqslant\delta<1}\left\|\int_t^{t+\theta}G(s,u_m(s))d\bar{W}(s)\right\|^2_{(W_0^{1,p(x)}(D))^*}.
\end{aligned}\tag{3.43}$$

Integrating the inequality (3.43) with respect to t from δ to $T-\delta$, we obtain

$$\begin{aligned}
&\bar{\mathbb{E}}\sup_{0\leqslant\theta\leqslant\delta<1}\int_\delta^{T-\delta}\|u_m(t+\theta)-u_m(t)\|^2_{\left(W_0^{1,p(x)}(D)\right)^*}dt\\
&\leqslant CT\delta^2\bar{\mathbb{E}}\sup_{0\leqslant t\leqslant T}\|u_m(t)\|^2_{W_0^{1,p(x)}(D)}\\
&+C\delta\bar{\mathbb{E}}\int_\delta^{T-\delta}\left(\sup_{0\leqslant\theta\leqslant\delta<1}\int_t^{t+\delta}\|f(s,u_m(s))\|^2_{(W_0^{1,p(x)}(D))^*}ds\right)dt\\
&+\bar{\mathbb{E}}\int_\delta^{T-\delta}\sup_{0\leqslant\theta\leqslant\delta<1}\left\|\int_t^{t+\theta}G(s,u_m(s))d\bar{W}(s)\right\|^2_{(W_0^{1,p(x)}(D))^*}.
\end{aligned}\tag{3.44}$$

We first estimate the second term in the right-hand side of (3.44). We have

$$
\begin{aligned}
\bar{\mathbb{E}}\int_\delta^{T-\delta}\left(\int_t^{t+\delta}\|f(s,u_m(s))\|^2_{(W_0^{1,p(x)}(D))^*}ds\right)dt &\leqslant C\bar{\mathbb{E}}\int_0^T\|f(t,u_m(t))\|^2_{(W_0^{1,p(x)}(D))^*}dt\\
&= C\bar{\mathbb{E}}\|f(t,u_m(t))\|^2_{L^2(0,T;(W_0^{1,p(x)}(D))^*)}\\
&\leqslant C\bar{\mathbb{E}}\left(1+\|u_m\|_{L^2(Q_t)}\right)^2\\
&\leqslant C\bar{\mathbb{E}}\|u_m\|^2_{L^2(Q_t)}+C\\
&\leqslant C\bar{\mathbb{E}}\sup_{0\leqslant t\leqslant T}\|u_m\|^2_{L^2(D)}+C; \qquad (3.45)
\end{aligned}
$$

and we also note that

$$
\begin{aligned}
\bar{\mathbb{E}}\int_\delta^{T-\delta}C\delta\left(\int_t^{t+\delta}\|u_m(s)\|^2_{W_0^{1,p(x)}(D)}\right)dt &\leqslant C\delta\bar{\mathbb{E}}\int_0^T\|u_m(t)\|^2_{W_0^{1,p(x)}(D)}dt\\
&= C\delta\bar{\mathbb{E}}\|u_m\|^2_{L^2(0,T;W_0^{1,p(x)}(D))}\\
&\leqslant C\delta\bar{\mathbb{E}}\|u_m\|^2_{\mathring{V}(Q_T)}\\
&\leqslant C\delta\left(\bar{\mathbb{E}}\|u_m\|^r_{\mathring{V}(Q_T)}\right)^{\frac{2}{r}}. \qquad (3.46)
\end{aligned}
$$

For the estimate of the stochastic integral term we use martingale's inequality. We have by using assumption (3.2) and Fubini's Theorem

$$
\begin{aligned}
&\bar{\mathbb{E}}\int_\delta^{T-\delta}\sup_{0\leqslant\theta\leqslant\delta<1}\left\|\int_t^{t+\theta}G(s,u_m(s))d\bar{W}(s)\right\|^2_{(W_0^{1,p(x)}(D))^*}dt\\
&\leqslant\bar{\mathbb{E}}\int_\delta^{T-\delta}\left(\int_t^{t+\delta}[G(s,u_m(s))]^2\,ds\right)dt\\
&\leqslant\int_\delta^{T-\delta}\bar{\mathbb{E}}\left(\int_t^{t+\delta}\|G(s,u_m(s))\|^2_{(L^2(D))^d}\,ds\right)dt\\
&\leqslant C\int_0^{T-\delta}\bar{\mathbb{E}}\left(\int_t^{t+\delta}\left(1+\|u_m(s)\|_{L^2(D)}\right)^2ds\right)dt\\
&\leqslant TC\delta+C\delta\int_0^T\bar{\mathbb{E}}\left(\int_t^{t+\delta}\|u_m(s)\|^2_{L^2(D)}\,ds\right)dt\\
&\leqslant TC\delta+C\delta\bar{\mathbb{E}}\sup_{t\in[0,T]}\|u_m(t)\|^2_{L^2(D)}. \qquad (3.47)
\end{aligned}
$$

Taking into account the estimates of the previous Lemmas we have

$$
\bar{\mathbb{E}}\int_\delta^{T-\delta}\sup_{0\leqslant\theta\leqslant\delta<1}\left\|\int_t^{t+\theta}G(s,u_m(s))d\bar{W}(s)\right\|^2_{(W_0^{1,p(x)}(D))^*}dt\leqslant C\delta. \qquad (3.48)
$$

Combining the inequalities (3.38)-(3.48), with the key estimates (3.26), (3.27) and (3.28) of Lemmas 5 and 6, we obtain

$$\bar{\mathbb{E}} \sup_{0\leqslant\theta\leqslant\delta<1} \int_0^T \|u_m(t+\theta) - u_m(t)\|^2_{\left(W_0^{1,p(x)}(D)\right)^*} dt \leqslant C\delta,$$

where the constant C is independent of δ and m. Finally, collecting all the estimates and making a similar reasoning with $\theta < 0$, we obtain

$$\bar{\mathbb{E}} \sup_{|\theta|\leqslant\delta<1} \int_0^T \|u_m(t+\theta) - u_m(t)\|^2_{\left(W_0^{1,p(x)}(D)\right)^*} dt \leqslant C\delta. \tag{3.49}$$

This proves Lemma 8.

3.3.4 A variation of the compactness result

Following Bensoussan [13], we reformulate the Lemma 5 as;

Proposition 1. *Let μ_n and ν_n be both sequences of positive real numbers such that both sequences tend to 0 as n tends to ∞. Then, we have the following compact embedding*

$$\mathcal{W}_{\mu_n,\nu_n} = \left\{ z \in \left| \begin{array}{c} L^2\left(0,T; W_0^{p(x)}(D)\right) \cap L^\infty(0,T;L^2(D)) \\ \sup_n \dfrac{1}{\nu_n} \sup_{|\theta|\leqslant\mu_n} \left(\int_0^T \|z(t+\theta) - z(t)\|^2_{(W_0^{p(x)}(D))^*}\, dt\right)^{1/2} < \infty \end{array} \right. \right\}$$

is compactly embedded in $L^2(0,T;L^2(D))$.

We define the norm in the space $\mathcal{W}_{\mu_n,\nu_n}$ by

$$\begin{aligned} \|z\|_{\mathcal{W}_{\mu_n,\nu_n}} = \sup_{0\leqslant t\leqslant T} \|z(t)\|_{L^2(D)} + \left(\int_0^T \|z(t)\|^2_{W_0^{1,p(x)}(D)}\, dt\right)^{1/2} + \\ + \sup_n \frac{1}{\nu_n} \sup_{|\theta|\leqslant\mu_n} \left(\int_0^T \|z(t+\theta) - z(t)\|^2_{(W_0^{p(x)}(D))^*}\, dt\right)^{1/2}. \end{aligned} \tag{3.50}$$

The space $\mathcal{W}_{\mu_n,\nu_n}$ with the norm (3.50) is a Banach space.

We also consider the probabilistic evolution space

$$\mathcal{Z}_{\mu_n,\nu_n}$$

of random variables z such that

$$E\left(\int_0^T \|z\|^2_{W_0^{1,p(x)}(D)}\,dt\right)^{1/2} < \infty, \quad \left(E\sup_{0\leqslant t\leqslant T}\|z\|^q_{L^2(D)}\right)^{1/q} < \infty;$$

$$\text{and } E\sup_n \frac{1}{\nu_n}\sup_{|\theta|\leqslant\mu_n}\left(\int_0^T \|z(t+\theta)-z(t)\|^2_{(W_0^{p(x)}(D))^*}\,dt\right)^{1/2} < \infty.$$

The space $\mathcal{Z}_{\mu_n,\nu_n}$ endowed with the norm

$$\|z\|_{\mathcal{Z}_{\mu_n,\nu_n}} = E\left(\int_0^T \|z\|^2_{W_0^{1,p(x)}(D)}\,dt\right)^{1/2} + \left(E\sup_{0\leqslant t\leqslant T}\|z\|^q_{L^2(D)}\right)^{1/q} + \\ + E\sup_n \frac{1}{\nu_n}\sup_{|\theta|\leqslant\mu_n}\left(\int_0^T \|z(t+\theta)-z(t)\|^2_{(W_0^{p(x)}(D))^*}\,dt\right)^{1/2},$$

is a Banach space. The a priori estimates established in the previous Lemmas allow us to assert that for any $q \in \left[2, \frac{2r'}{r'-1}\right]$, and for μ_n, ν_n such that the series $\sum_{n=1}^{\infty}\dfrac{\sqrt{\mu_n}}{\nu_n} < \infty$, the Galerkin solutions $\{u_m : m \in \mathbb{N}\}$ remain in a bounded subset of $\mathcal{Z}_{\mu_n,\nu_n}$ since $\mathring{V}(Q_T) \circlearrowleft L^2\left(0,T;W_0^{1,p(x)}(D)\right)$.

Next, we shall prove the tightness property of the Galerkin solutions. Similar proof with more details can be found in [14], [23], [63], [68], [71], [69] and [72].

3.3.5 Tightness property of Galerkin approximating solutions

We consider the set

$$S = C(0,T;\mathbb{R}^d) \times L^2(0,T;L^2(D)).$$

We equip S with its Borel σ-algebra denoted by $\mathcal{B}(S)$: the σ-algebra of the Borel sets (subsets of S) of S.

For each m, we consider the following mapping

$$\phi : \bar{\Omega} \longrightarrow S : \bar{\omega} \rightsquigarrow \left(\bar{W}(.,\bar{\omega}), u_m(.,\bar{\omega})\right).$$

For each m, we consider Π_m to be the probability measure on $(S,\mathcal{B}(S))$ given by

$$\Pi_m(A) = \bar{\mathbb{P}}(\phi^{-1}(A)), \tag{3.51}$$

for all borel set $A \subset S$. We have the following main result concerning the tightness of the family of probability measures Π_m on S.

Theorem 25. *The family of probability measures* $\{\Pi_m \ m = 1, 2, \ldots\}$ *is tight.*

Proof 8. *(cf. [14], [23], [62], [63], [68], [71], [72])*
We shall find for any $\varepsilon > 0$ *compact subsets*

$$W_\varepsilon \subset C(0, T; \mathbb{R}^d), \ Z_\varepsilon \subset L^2(0, T; L^2(D)),$$

such that

$$\Pi_m \left((\bar{W}, u_m) \in W_\varepsilon \times Z_\varepsilon \right) \geq 1 - \varepsilon.$$

This can also be proved by the following inequality

$$\bar{\mathbb{P}} \left(\omega : \bar{W}(., \bar{\omega}) \in W_\varepsilon ; u_m(., \bar{\omega}) \in Z_\varepsilon \right) \geq 1 - \varepsilon \tag{3.52}$$

which in its turn can be proved by the following

$$\bar{\mathbb{P}} \left(\bar{\omega} : \bar{W}(., \bar{\omega}) \notin W_\varepsilon \right) \leqslant \frac{\varepsilon}{2}, \tag{3.53}$$

$$\bar{\mathbb{P}} \left(\bar{\omega} : u_m(., \bar{\omega}) \notin Z_\varepsilon \right) \leqslant \frac{\varepsilon}{2}. \tag{3.54}$$

For as a constant L_ε *depending on* ε *to be chosen later on and* N *a natural number so that as* $N \longrightarrow \infty$; $\sum \frac{\sqrt{\mu_N}}{\nu_N} < \infty$. *We consider a subdivision* $\{\frac{j}{N^6}\}$ *of length* $\frac{T}{N^6}$ *of the interval* $[0, T]$. *We next consider the following set*

$$W_\varepsilon = \left\{ W(.) \in C(0, T; \mathbb{R}^m), \text{ such that } \sup_{\substack{s,t \in [0,T] \\ |t-s| < T/N^6}} N \left| W(t) - W(s) \right| \leqslant L_\varepsilon \right\}$$

In view of Arzela-Ascoli's Theorem, the subset W_ε *is compact in the space* $C(0, T; \mathbb{R}^d)$. *For the rest of the proof, we need the following.*

Theorem 26. *Let* ξ *be a random variable on the probability space* $(\bar{\Omega}, \bar{F}, \bar{\mathbb{P}})$. *For any positive constant* $C > 0$, *and for any* $k > 0$ *we have*

$$\bar{\mathbb{P}} \left(\omega : \xi(\omega) \geq C \right) \leqslant \frac{1}{C^k} \bar{\mathbb{E}} \left(|\xi(\omega)|^k \right). \tag{3.55}$$

(3.55) is known as the Markov inequality.

We have

$$\bar{\mathbb{P}}\left(\bar{\omega} : \bar{W}(.,\bar{\omega}) \notin W_\varepsilon\right) \leqslant \bar{\mathbb{P}}\left(\bigcup_{N=1}^{\infty}\left\{W(.) : \sup_{\substack{s,t\in[0,T]\\ |t-s|<T/N^6}} |W(t)-W(s)| > \frac{L_\varepsilon}{N}\right\}\right)$$

$$\leqslant \sum_{N=1}^{\infty}\bar{\mathbb{P}}\left(\left\{W(.) : \sup_{\substack{s,t\in[0,T]\\ |t-s|<T/N^6}} |W(t)-W(s)| > \frac{L_\varepsilon}{N}\right\}\right) \tag{3.56}$$

Furthermore we get

$$\begin{aligned}\bar{\mathbb{P}}\left(\bar{\omega} : \bar{W}(.,\bar{\omega}) \notin W_\varepsilon\right) &\leqslant \sum_{N=1}^{\infty}\bar{\mathbb{P}}\left(\bigcup_{j=1}^{N^6}\left(\sup_{\frac{(j-1)T}{N^6}\leqslant t\leqslant\frac{jT}{N^6}} |W(t)-W(s)| > \frac{L_\varepsilon}{N}\right)\right)\\ &\leqslant \sum_{N=1}^{\infty}\sum_{j=1}^{N^6}\bar{\mathbb{P}}\left(\sup_{\frac{(j-1)T}{N^6}\leqslant t\leqslant\frac{jT}{N^6}} |W(t)-W(s)| > \frac{L_\varepsilon}{N}\right)\end{aligned} \tag{3.57}$$

Next recall the following well known fact about the Wiener Process $W(t)$;

$$\bar{\mathbb{E}}\left|\bar{W}(t)-\bar{W}(s)\right|^n \leqslant (n-1)!\,(t-s)^{n/2},\ n=2,3,\ldots \tag{3.58}$$

Combining this with Markov's inequality (see Theorem (26)) we obtain

$$\begin{aligned}&\bar{\mathbb{P}}\left(\bar{\omega} : \bar{W}(.,\bar{\omega}) \notin W_\varepsilon\right) \leqslant\\ &\leqslant \sum_{N=1}^{\infty}\sum_{j=1}^{N^6}\left(\frac{N}{L_\varepsilon}\right)^4\bar{\mathbb{E}}\left(\sup_{\frac{(j-1)T}{N^6}\leqslant t\leqslant\frac{jT}{N^6}}\left|W(t)-W\left(\frac{(j-1)T}{N^6}\right)\right|^4\right)\\ &\leqslant \sum_{N=1}^{\infty}C\left(\frac{L_\varepsilon}{N}\right)^{-4}(TN^{-6})^2N^6 = \frac{CT^2}{L_\varepsilon^4}\sum_{N=1}^{\infty}N^{-2}.\end{aligned} \tag{3.59}$$

We take as a function of ε,

$$L_\varepsilon^4 = 2CT^2\varepsilon^{-1}\sum_{N=1}^{\infty}\frac{1}{N^2}$$

and, thus, the required inequality (3.53) is thereby proved.

Once more arguing as in [14], we establish (3.54). We choose Z_ε to be a ball in $\mathcal{W}_{\mu_N,\nu_N}$ of center the point 0 and radius M_ε with the sequences $\mu_N,\ \nu_N \longrightarrow 0$ such that both sequences are taken to be independent of ε, and satisfy $\sum_{N=1}^{\infty} \frac{\sqrt{\mu_N}}{\nu_N} < \infty$. Now, applying Proposition 1 to the ball Z_ε, it follows that Z_ε is a compact subset of $L^2(0,T;L^2(D))$. Furthermore, by Bienaymé-Tchebitcheff's inequality, we have

$$\begin{aligned}
\bar{\mathbb{P}}\left(\bar{\omega} : u_m(.\bar{\omega}) \notin Z_\varepsilon\right) &\leqslant \bar{\mathbb{P}}\left(\omega : \|u_m\|_{\mathcal{W}_{\mu_N,\nu_N}} > M_\varepsilon\right) \\
&\leqslant \frac{\bar{\mathbb{E}}\|u_m\|_{\mathcal{W}_{\mu_N,\nu_N}}}{M_\varepsilon} \\
&\leqslant \frac{1}{M_\varepsilon}\|u_m\|_{\mathcal{W}_{\mu_N,\nu_N}} \\
&\leqslant \frac{C}{M_\varepsilon}.
\end{aligned} \tag{3.60}$$

We then choose $M_\varepsilon = \dfrac{2C}{\varepsilon}$, and get

$$\bar{\mathbb{P}}\left(\bar{\omega} : u_m \notin Z_\varepsilon\right) \leqslant \frac{C}{M_\varepsilon} = \frac{\varepsilon}{2}. \tag{3.61}$$

The proof of the theorem is complete.

3.3.6 Application of Prokhorov and Skorokhod Theorems

It follows from the tightness property of Π_m proved above and by Prokhorov's Theorem, we can extract a subsequence of probability measures Π_{m_ν} which is weakly convergent to a probability measure Π on S. It follows from Skorokhod's Theorem that we can also find a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and S-valued random variables W_{m_ν}, u_{m_ν} and W, u such that

$$(W_{m_\nu}, u_{m_\nu}) \longrightarrow (W, u) \text{ on } S,\ \mathbb{P} - a.s. \tag{3.62}$$

The probability law of (W_{m_ν}, u_{m_ν}) is Π_{m_ν} and the one of (W,u) is Π.

Next, we choose the filtration $\{\mathcal{F}_t\}_{0\leqslant t\leqslant T}$ by setting

$$\mathcal{F}_t = \sigma\{W(s), u(s)\}_{0\leqslant s\leqslant t}. \tag{3.63}$$

To check that the limiting process $W(t)$ is an $\mathcal{F}_t$-Wiener process we proceed as follows. We use the following characterization of Wiener processes through their characteric functions. For further details we refer to [68]. To fix the ideas, we show that for any $l \in \mathbb{N}$, $0 = t_0 < t_1 < \cdots < t_l$ and $\lambda_0, \lambda_1, \ldots, \lambda_l$ with $\lambda_k \in \mathbb{R}^l$, $k \in [0, l]$, the increments process $W(t_k) - W(t_{k-1})$ are independent with respect to $\mathcal{F}_{t_{k-1}}$, normally distributed with zero mean and variance $t_k - t_{k-1}$. It is sufficient to prove that

$$\mathbb{E} \exp \left\{ i \sum_{k=1}^{l} \lambda_k \cdot [W(t_k) - W(t_{k-1})] \right\} = \Pi_{k=1}^{l} \exp \left\{ -\frac{1}{2} |\lambda_k|^2_{\mathbb{R}^l} (t_k - t_{k-1}) \right\}; \quad (3.64)$$

where $i = \sqrt{-1}$ the imaginary unit.

This will follow if we show that

$$\mathbb{E} \exp \left\{ i \sum_{k=1}^{l} \lambda \cdot (W(t+\theta) - W(t)) \right\} = \exp \left\{ -\frac{1}{2} |\lambda|^2_{\mathbb{R}^l} \theta \right\}, \quad (3.65)$$

for all $\theta > 0$ and any $\lambda \in \mathbb{R}^l$.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let ξ and η be random variables for which $\mathbb{E}(\xi)$, $\mathbb{E}(\eta)$ are defined. Let ξ be $\mathcal{F}$-measurable such that $\mathbb{E}(\xi\,\eta)$ is defined. Then we have the following properties of conditional expectations:

(i) $\mathbb{E}(\mathbb{E}(\eta/\mathcal{F})) \;=\; \mathbb{E}(\eta)$

(ii) $\mathbb{E}(\xi\,\eta/\mathcal{F}) \;=\; \eta\,\mathbb{E}(\xi/\mathcal{F})$.

Then

$\mathbb{E}(X\,Y) = \mathbb{E}\,(X\,\mathbb{E}(Y/\mathcal{F}))$.

Let $J_t(W, u)$ be a bounded continuous functional depending on S which depends only on the values of $W(t)$ and $u(t)$ for $0 < t$.

To prove (3.65), it is sufficient to prove that

$$\begin{aligned} &\mathbb{E}\left[\exp\left(i\lambda \cdot (W(t+\theta) - W(t))\right) J_t(W(\cdot), u(\cdot))\right] \\ &= \exp\left(-\frac{1}{2}|\lambda|^2_{\mathbb{R}^l}\right) \mathbb{E}\left[J_t(W, u)\right]. \end{aligned} \quad (3.66)$$

Since W_{m_ν} is a Wiener process and the increments $W_{m_\nu}(t+\theta)-W_{m_\nu}(t)$ is independent of the functional $J_t(W_{m_\nu}, u_{m_\nu})$, it follows that, for any $\lambda \in \mathbb{R}^l$

$$\begin{aligned}
&\mathbb{E}\left(\exp\left(i\lambda(W_{m_\nu}(t+\theta)-W_{m_\nu}(t))\right) J_t(W_{m_\nu}, u_{m_\nu})\right) \\
&= \mathbb{E}\left(\exp\left(i\,\lambda(W_{m_\nu}(t+\theta)-W_{m_\nu}(t))\right)\right) \mathbb{E}\left(J_t(W_{m_\nu}, u_{m_\nu})\right) \\
&= \mathbb{E}(\exp(-|\lambda|^2_{\mathbb{R}^l}\,\theta/2))\,\mathbb{E}(J_t(W_{m_\nu}, u_{m_\nu})) \\
&= \exp(-|\lambda|^2_{\mathbb{R}^l}\theta^2/2)\,\mathbb{E}(J_t(W_{m_\nu}, u_{m_\nu})).
\end{aligned} \tag{3.67}$$

Using the fact that J_t is a continuous functional and taking into account the convergence (3.62), as $\nu \longrightarrow \infty$, it follows that (3.66) holds.

We set $m = m_\nu$, and integrate (3.8) with respect to s over the interval $[0,t]$. Then using the Wiener process W_{m_ν} instead of $\bar{W}$, we clearly see that W_{m_ν} and u_{m_ν} satisfy the following equation

$$\begin{aligned}
&\int_0^t (du_{m_\nu}(s), w_j(x)) - \int_0^t (Au_{m_\nu}(s)\,,\,w_j(x))\,ds \\
&= \int_0^t (f(s,u_{m_\nu}(s))\,,\,w_j(x))\,ds + \int_0^t (G(s,u_{m_\nu}(s))\,,\;w_j(x)\,)\,dW_{m_\nu}(s),
\end{aligned} \tag{3.68}$$

for $j = 1, 2, \ldots$. This equation is equivalent to the following

$$\begin{aligned}
&u_{m_\nu} + \int_0^t P_{m_\nu}\left(A(s)u_{m_\nu}(s)\right)\,ds \\
&= u_{0m_\nu} + \int_0^t P_{m_\nu}\left(f(s,u_{m_\nu}(s))\right)\,ds + \int_0^t P_{m_\nu}\left(G(s,u_{m_\nu})\right)\,dW_{m_\nu}(s).
\end{aligned} \tag{3.69}$$

By setting

$$\begin{aligned}
\sigma_m(t) = u_m(t) &- \int_0^t P_m\left(A(s)u_m(s) - f(s,u_m(s))\right)\,ds \\
&- u_{0m} - \int_0^t P_m\left(G(s,u_m(s))\right)d\bar{W}(s),
\end{aligned}$$

and we define

$$X_m = \int_0^T \|\sigma_m(t)\|^2_{(W_0^{1,p(x)}(D))^*}dt.$$

We trivially have $X_m = 0\,\bar{\mathbb{P}} -$ a.s, hence

$$\bar{E}\frac{X_m}{1+X_m} = 0.$$

We set

$$\vartheta_m(t) = u_{m_\nu}(t) - \int_0^t P_m\left(A(s)u_{m_\nu}(s) - f(s, u_{m_\nu}(s))\right) ds \\ - u_{0m_\nu} - \int_0^t P_m\left(G(s, u_{m_\nu}(s))\right) dW_{m_\nu}(s)$$

and define Y_{m_ν} the analogue of X_m with u_m replaced for u_{m_ν} and $\bar{W}$ replaced for W_{m_ν}. Then we have the expression

$$Y_{m_\nu} = \int_0^T \|\vartheta_m(t)\|^2_{(W_0^{1,p(x)}(D))^*} dt.$$

We shall show that

$$\mathbb{E}\frac{Y_{m_\nu}}{1 + Y_{m_\nu}} = 0. \tag{3.70}$$

The difficulty we are encountering here in the expression of X_m is that X_m is not a deterministic functional of the pair u_m, W for the reason that there is presence of a stochastic integral term in the expression of X_m. By using a mollification of G in t, we can cope with this obstacle. We define the regularizing function G^ε as follows:

$$G^\varepsilon(t, u) = \frac{1}{\varepsilon}\int_0^T \rho\left(\frac{t-s}{\varepsilon}\right) G(s, u(s))\, ds, \tag{3.71}$$

where ϱ is a standard mollifier. Noting that by the above definition of G^ε in (3.71), we obviously have the uniform estimate

$$\mathbb{E}\int_0^T \|G^\varepsilon(t, u(t))\|^2_{(L^2(D))^m}\, ds \leqslant C\mathbb{E}\int_0^T \|G(t, u(t))\|^2_{(L^2(D))^m}\, dt \tag{3.72}$$

and

$$G^\varepsilon(\,\cdot, u(\cdot)\,) \longrightarrow G(\,\cdot, u(\cdot)) \text{ in } L^2(\Omega, \mathcal{F}, \mathbb{P}, L^2(0,T; (L^2(D))^m)). \tag{3.73}$$

Let us denote by $X_{m,\varepsilon}$ and $Y_{m_\nu,\varepsilon}$ the analogue of X_m and Y_{m_ν} with G^ε instead of G. We define the mapping

$$\phi_{m,\varepsilon} : C\left(0, T; \mathbb{R}^d\right) \times L^2(0,T; L^2(D)) \longrightarrow (\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}}),$$

by

$$\phi_{m,\varepsilon}\left(\bar{W},\, u_m\right) = \frac{X_{m,\varepsilon}}{1 + X_{m,\varepsilon}}.$$

$\phi_{m,\varepsilon}$ is a bounded continuous functional on $C\left(0,T;\mathbb{R}^d\right) \times L^2(0,T;L^2(D))$. In a similar way, we introduce the mapping $\phi_{m_\nu,\varepsilon}\left(W_{m_\nu},\, u_{m_\nu}\right)$ given by

$$\phi_{m_\nu,\varepsilon}\left(W_{m_\nu},\, u_{m_\nu}\right) = \frac{Y_{m_\nu,\varepsilon}}{1 + Y_{m_\nu,\varepsilon}}.$$

We note that by applying Prokhorov's Theorem, we have

$$\begin{aligned}\mathbb{E}\frac{Y_{m_\nu,\varepsilon}}{1+Y_{m_\nu,\varepsilon}} &= \mathbb{E}\,\phi_{m_\nu,\varepsilon}\left(W_{m_\nu},\, u_{m_\nu}\right) = \int_S \phi_{m_\nu,\varepsilon}(W,\, x)\, d\,\Pi_{m_\nu} \\ &= \bar{\mathbb{E}}\,\phi_{m_\nu,\varepsilon}\left(\bar{W},\, u_{m_\nu}\right) = \bar{\mathbb{E}}\,\frac{X_{m_\nu,\varepsilon}}{1+X_{m_\nu,\varepsilon}}. \end{aligned} \tag{3.74}$$

By adding and substracting same terms

$$\begin{aligned}\mathbb{E}\frac{Y_{m_\nu}}{1+Y_{m_\nu}} - \bar{\mathbb{E}}\frac{X_{m_\nu}}{1+X_{m_\nu}} &= \mathbb{E}\left(\frac{Y_{m_\nu}}{1+Y_{m_\nu}} - \frac{Y_{m_\nu,\varepsilon}}{1+Y_{m_\nu,\varepsilon}}\right) + \\ &\quad + \bar{\mathbb{E}}\left(\frac{X_{m_\nu,\varepsilon}}{1+X_{m_\nu,\varepsilon}} - \frac{X_{m_\nu}}{1+X_{m_\nu}}\right). \end{aligned} \tag{3.75}$$

Moreover, the first term in the right-hand side of this equality can be estimated as follows:

$$\begin{aligned}\mathbb{E}\left|\frac{Y_{m_\nu}}{1+Y_{m_\nu}} - \frac{Y_{m_\nu,\varepsilon}}{1+Y_{m_\nu,\varepsilon}}\right| &= \mathbb{E}\left|\frac{Y_{m_\nu} - Y_{m_\nu,\varepsilon}}{(1+Y_{m_\nu})(1+Y_{m_\nu,\varepsilon})}\right| \\ &\leqslant \mathbb{E}\left|Y_{m_\nu} - Y_{m_\nu,\varepsilon}\right| \\ &\leqslant C\left(\mathbb{E}\int_0^T \|G^\varepsilon(t,\, u_{m_\nu}(t)) - G(t,\, u_{m_\nu})\|^2_{(L^2(D))^d}\; dt\right)^{1/2}. \end{aligned} \tag{3.76}$$

Here we have used Burkholder-Davis-Gundy's inequality.

In a similar way, the last term in the right-hand side of equation (3.75) can be estimated as follows:

$$\bar{\mathbb{E}}\left|\frac{X_{m_\nu,\varepsilon}}{1+X_{m_\nu,\varepsilon}} - \frac{X_{m_\nu}}{1+X_{m_\nu}}\right| \leqslant C\left(\mathbb{E}\int_0^T \|G^\varepsilon(t,\, u_{m_\nu}(t)) - G(t,\, u_{m_\nu}(t))\|^2_{(L^2(D))^d}\; dt\right)^{1/2}. \tag{3.77}$$

It follows from (3.74), (3.76) and (3.77) that

$$\left|\left|\mathbb{E}\frac{Y_{m_\nu}}{1+Y_{m_\nu}}\right| - \left|\bar{\mathbb{E}}\frac{X_{m_\nu}}{1+X_{m_\nu}}\right|\right| \leqslant \left|\mathbb{E}\frac{Y_{m_\nu}}{1+Y_{m_\nu}} - \bar{\mathbb{E}}\frac{X_{m_\nu}}{1+X_{m_\nu}}\right|$$
$$\leqslant C\left(\mathbb{E}\int_0^T \|G^\varepsilon(t,\, u_{m_\nu}(t)) - G(t,\, u_{m_\nu}(t))\|^2_{(L^2(D))^d}\, dt\right)^{1/2}. \tag{3.78}$$

Letting ε tends to 0 in (3.78) and taking into account (3.73), we obtain

$$\left|\mathbb{E}\frac{Y_{m_\nu}}{1+Y_{m_\nu}}\right| = \left|\bar{\mathbb{E}}\frac{X_{m_\nu}}{1+X_{m_\nu}}\right| = 0.$$

From this we deduce (3.70). Thus, the relation (3.69) is thereby proved.

We can assert that there exists a positive constant K_4 independent of m such that

$$\|Au_m\|_{L^2\left(0,T;\left(W_0^{1,p(x)}(D)\right)^*\right)} \leqslant K_4. \tag{3.79}$$

And $G(t, u_m(t))$ remains in a bounded subset of the space $L^2(\Omega, \mathcal{F}, P, L^2(0, T; (L^2(D))^m))$.

3.3.7 Passage to the limit and Monotonicity Method

In this subsection we establish some convergence properties of the sequence (u_{m_ν}) obtained in the previous section.

From (3.69) and the estimates on u_m, it follows that u_{m_ν} satisfies the a priori estimates

$$\mathbb{E}\sup_{t\in[0,T]}\|u_{m_\nu}(t)\|^q_{L^2(D)} \leqslant C,\; q \in [2, \frac{2r'}{r'-1}]; \tag{3.80}$$

$$\mathbb{E}\|u_{m_\nu}\|^r_{\mathring{V}(Q_T)} \leqslant C, \tag{3.81}$$

$$\mathbb{E}\int_0^T \|u_{m_\nu}(t)\|^2_{W_0^{1,p(x)}(D)}\, dt \leqslant C, \tag{3.82}$$

$$\mathbb{E}\sup_{0\leqslant\theta\leqslant\delta<1}\int_0^T \|u_{m_\nu}(t+\theta) - u_{m_\nu}(t)\|^2_{(W_0^{1,p(x)}(D))^*}\, dt \leqslant C\,\delta. \tag{3.83}$$

(3.82) is a consequence of the embedding result in Lemma 2.

From the last estimates, we can extract a new subsequence from $\{u_m\}$ still denoted u_{m_ν}

such that

$$u_{m_\nu} \longrightarrow u \text{ weakly } * \text{ in } L^q(\Omega, \mathcal{F}, \mathbb{P}, L^\infty(0,T; L^2(D))); \tag{3.84}$$

$$u_{m_\nu} \longrightarrow u \text{ weakly in } \quad L^r\left(\Omega, \mathcal{F}, \mathbb{P}; \mathring{V}(Q_T)\right); \tag{3.85}$$

$$u_{m_\nu} \longrightarrow u \text{ weakly in } L^2(\Omega, \mathcal{F}, \mathbb{P}, L^2(0,T; W_0^{1,p(x)}(D))); \tag{3.86}$$

$$A\, u_{m_\nu}(\omega) \longrightarrow \chi(\omega) \text{ weakly in } L^2\left(0,T; \left(W_0^{1,p(x)}(D)\right)^*\right) \tag{3.87}$$

$$u_{m_\nu}(T) \longrightarrow \xi \text{ weakly in } L^2(\Omega, \mathcal{F}, \mathbb{P}, L^2(D)), \tag{3.88}$$

and the subsequence satisfies the followings estimates

$$\mathbb{E} \sup_{t\in[0,T]} \|u(t)\|^q_{L^2(D)} \leqslant C,\ q \in [2, \frac{2r'}{r'-1}]; \tag{3.89}$$

$$\left(\mathbb{E}\|u\|^r_{\mathring{V}(Q_T)}\right)^{1/r} \leqslant C, \tag{3.90}$$

$$\mathbb{E}\int_0^T \|u(t)\|^2_{W_0^{1,p(x)}(D)}\, dt \leqslant C, \tag{3.91}$$

$$\mathbb{E} \sup_{0\leqslant\theta\leqslant\delta<1} \int_0^T \|u(t+\theta)-u(t)\|^2_{(W_0^{1,p(x)}(D))^*}\, dt \leqslant C\,\delta. \tag{3.92}$$

According to (3.62), (3.80) and Vitali's Theorem, we get

$$u_{m_\nu} \longrightarrow u \text{ in } \quad L^2(\Omega, \mathcal{F}, \mathbb{P}, L^2(0,T; L^2(D))). \tag{3.93}$$

Then

$$u_{m_\nu} \longrightarrow u \text{ for almost all } (t,\,\omega) \text{ w. r. t. the measure } d\mathbb{P}\,\times\,dt. \tag{3.94}$$

The convergence (3.94), the estimate (3.80) combined with the condition on f, and Vitali's Theorem imply that as $\nu \longrightarrow \infty$

$$f(\,.\,,\,u_{m_\nu}(\,.\,)) \longrightarrow f(\,.\,,u(\,.\,)) \text{ in } L^2\left(\Omega, \mathcal{F}, \mathbb{P}, L^2\left(0,T, \left(W_0^{1,p(x)}(D)\right)^*\right)\right). \tag{3.95}$$

The convergence (3.95) implies that for fixed j we have in particular the convergence

$$(f(\,.\,,\,u_{m_\nu}(\,.\,))\,,\,w_j(x)) \longrightarrow (f(\,.\,,u(\,.\,))\,,\,w_j(x)) \text{ in } L^2(\Omega, \mathcal{F}, \mathbb{P}, L^2(0,T)), \tag{3.96}$$

since $w_j \in W_0^{1,p(x)}(D)$.

From (3.2), the estimate (3.80) and Vitali's theorem we also have

$$(G(t, u_{m_\nu}(t)), w_j) \longrightarrow (G(t,u(t)), w_j) \text{ in } L^2\left(\Omega, \mathcal{F}, \mathbb{P}, L^2\,(0,T)\right). \tag{3.97}$$

Arguing as in [14], [23], [63], [68], [69], [71] and [72], we shall prove that

$$\forall t \ \left(\int_0^t G(s, u_{m_\nu}(s))\, dW_{m_\nu} \,,\ w_j(x) \right) \longrightarrow \left(\int_0^t G(s, u(s))\, dW \,,\ w_j(x) \right) \text{ weakly}$$
$$\text{in } L^2\left(\Omega, \mathcal{F}, \mathbb{P}, L^2\left(0,t\right)\right). \tag{3.98}$$

However (3.98) holds, if we prove that

$$\int_0^T G(s, u_{m_\nu}(s))\, dW_{m_\nu} \longrightarrow \int_0^T G(s, u(s))\, dW \text{ weakly}$$
$$\text{in } L^2\left(\Omega, \mathcal{F}, \mathbb{P}, L^2\left(D\right)\right). \tag{3.99}$$

Let G^ε be the regularization of G as introduced in (3.71)

$$G^\varepsilon(t,u) = \frac{1}{\varepsilon} \int_0^T \rho\left(\frac{t-s}{\varepsilon}\right) G(s,u(s))\ ds,$$

then one can check that for fixed m_ν, as $\varepsilon \longrightarrow 0$ we have

$$G^\varepsilon(., u_{m_\nu}(.)) \longrightarrow G(., u(.)) \quad \text{in} \quad L^2(\Omega, \mathcal{F}, \mathbb{P}, L^2(0,T, L^2(D))). \tag{3.100}$$

Moreover, the mapping G^ε satisfies (3.72) and the uniform estimate (in ε)

$$\begin{aligned} &\int_0^T \|G^\varepsilon(t, u_{m_\nu}(t)) - G^\varepsilon(t, u(t))\|^2_{(L^2(D))^d}\, dt \\ &\leqslant \int_0^T \|G(t, u_{m_\nu}(t)) - G(t, u(t))\|^2_{(L^2(D))^d}\, dt. \end{aligned} \tag{3.101}$$

Next, integrating by parts in the stochastic term we obtain

$$\begin{aligned} &\int_0^T G^\varepsilon(t, u_{m_\nu}(t))\, dW_{m_\nu}(t) \\ &= W_{m_\nu}(T) G^\varepsilon(T, u_{m_\nu}(T)) - \int_0^T W_{m_\nu}(t) G'^\varepsilon(t, u_{m_\nu})\, dt. \end{aligned} \tag{3.102}$$

Therefore using the convergence of the pairs

$$(W_{m_\nu}, u_{m_\nu}) \longrightarrow (W, u) \quad \text{in } S,\ \mathbb{P}-\text{a.s.},\ \text{as}\, \nu \longrightarrow \infty,$$

for fixed ε, we have

$$\begin{aligned} &\int_0^T G^\varepsilon(t, u_{m_\nu}(t))\, dW_{m_\nu}(t) \\ &\longrightarrow G^\varepsilon(T, u(T))\, W(T) - \int_0^T W(t)\, G'^\varepsilon(t, u(t)) dt \end{aligned} \tag{3.103}$$

for almost all ω, x.

we have

$$G^{\varepsilon}(T, u(T))\, W(T) - \int_0^T W(t)\, G'^{\varepsilon}(t, u(t))dt = \int_0^T G^{\varepsilon}(t, u(t))\, dW(t).$$

From (3.72), (3.2) and (3.89) we have

$$\mathbb{E}\left|\int_0^T G^{\varepsilon}(t, u_{m_\nu}(t))\, dW_{m_\nu}(t)\right|^2 \leqslant C\mathbb{E}\int_0^T \|G(t, u(t))\|^2_{(L^2(D))^d}\, dt \leqslant C. \qquad (3.104)$$

From Remark 1, (3.103) and (3.104) we deduce

$$\int_0^T G^{\varepsilon}(t, u_{m_\nu}(t))\, dW_{m_\nu}(t) \longrightarrow \int_0^T G^{\varepsilon}(t, u(t))\, dW(t) \text{ weakly in } L^2\left(\Omega, \mathcal{F}, \mathbb{P}, L^2(D)\right). \qquad (3.105)$$

Therefore, for any $\gamma \in L^2\left(\Omega, \mathcal{F}, \mathbb{P}, L^2(D)\right)$

$$\mathbb{E}\left(\gamma\,, \int_0^T G^{\varepsilon}(t, u_{m_\nu}(t))\, dW_{m_\nu}(t)\right) \longrightarrow \mathbb{E}\left(\gamma\,, \int_0^T G^{\varepsilon}(t, u(t))\, dW(t)\right), \qquad (3.106)$$

and it is easy to check that G satisfies the estimate

$$\mathbb{E}\left|\int_0^T G(t, u_{m_\nu}(t))dW_{m_\nu}(t)\right|^2 \leqslant \mathbb{E}\int_0^T \|G(t, u_{m_\nu}(t))\|^2_{(L^2(D))^d}\, dt \leqslant C. \qquad (3.107)$$

Taking a function $\zeta \in L^2(\Omega, \mathcal{F}, \mathbb{P}, L^2(D))$, we claim that for any $\gamma \in L^2(\Omega, \mathcal{F}, \mathbb{P}, L^2(D))$ and for fixed ε, as $\nu \longrightarrow \infty$

$$\mathbb{E}\left(\gamma\,, \int_0^T G(t, u_{m_\nu}(t))\, dW_{m_\nu}(t)\right) \longrightarrow \mathbb{E}\left(\gamma\,,\, \zeta\right). \qquad (3.108)$$

We next look for an identification of ζ. For that purpose, we proceed as follows.

$$\begin{aligned}
I_{\varepsilon,\nu} &\equiv \mathbb{E}\left(\gamma\,, \int_0^T G(t, u_{m_\nu})dW_{m_\nu}(t)\right) - \mathbb{E}\left(\gamma\,, \int_0^T G(t, u)dW(t)\right) \\
&= \mathbb{E}\left(\gamma\,, \int_0^T \left[G(t, u_{m_\nu}) - G^{\varepsilon}(t, u_{m_\nu}) + G^{\varepsilon}(t, u_{m_\nu})\right] dW_{m_\nu}(t)\right) \\
&- \mathbb{E}\left(\gamma\,, \int_0^T \left[G(t, u) - G^{\varepsilon}(t, u) + G^{\varepsilon}(t, u)\right] dW(t)\right).
\end{aligned} \qquad (3.109)$$

Hence

$$|I_{\varepsilon,\nu}| \leqslant C\left(|I_{1,\varepsilon,m_\nu}| + |I_{\varepsilon}| + |I_{2,\varepsilon,m_\nu}|\right), \qquad (3.110)$$

where

$$I_{1,\varepsilon,m_\nu} = \mathbb{E}\left(\gamma\,,\,\int_0^T [G(t,u_{m_\nu}) - G^\varepsilon(t,u_{m_\nu})]\,dW_{m_\nu}(t)\right)$$
$$I_\varepsilon = \mathbb{E}\left(\gamma\,,\,\int_0^T [G^\varepsilon(t,u) - G(t,u)]\,dW(t)\right)$$
$$I_{2,\varepsilon,m_\nu} = \mathbb{E}\left(\gamma\,,\,\left[\int_0^T G^\varepsilon(t,u_{m_\nu})dW_{m_\nu}(t) - \int_0^T G^\varepsilon(t,u)dw(t)\right]\right).$$

We write

$$I_{2,\varepsilon,m_\nu} = \mathbb{E}\left(\gamma\,,\,\int_0^T G^\varepsilon(t,u_{m_\nu})dW_{m_\nu}(t)\right) - \mathbb{E}\left(\gamma\,,\,\int_0^T G^\varepsilon(t,u)\,dW(t)\right). \tag{3.111}$$

Therefore, it follows according to (3.100) and (3.101) that $I_{2,\varepsilon,m_\nu} \longrightarrow 0$.

From the definition of G^ε we readily get that I_ε and I_{1,ε,m_ν} converge to zero as $\varepsilon \longrightarrow 0$.

Hence $I_{\varepsilon,\nu} \longrightarrow 0$ as $\varepsilon \longrightarrow 0$, $\nu \longrightarrow \infty$. Then we deduce that

$$\zeta = \int_0^T G(t,u(t))\,dW(t). \tag{3.112}$$

Therefore (3.98) follows.

Let us set $m = m_\nu$ in relation (3.8). Then integrating the first term in (3.8), we get

$$\begin{aligned}
&\int_0^T (Au_{m_\nu}\,,\,w_j(x))\,dt \\
&= \int_0^T (f(t,u_{m_\nu}(t))\,,\,w_j(x))\,dt + \int_0^T (G(t,u_{m_\nu}(t))\,,\,w_j(x))\,dW(t) \\
&+ (u_{m_\nu}(0,x)\,,\,w_j(x)) - (u_{m_\nu}(T,x)\,,\,w_j(x)), \quad j = 1,2,\ldots,m_\nu.
\end{aligned} \tag{3.113}$$

Recall that S_m denotes the span of the functions $w_j(x)$.

Passing to the limit in (3.113) as $\nu \longrightarrow \infty$, and making use of all the convergence results, we obtain

$$\begin{aligned}
&\int_0^T (\chi\,,\,w_j(x))\,dt \\
&= \int_0^T (f(t,u(t))\,,\,w_j(x))\,dt + \int_0^T (G(t,u(t))\,,\,w_j(x))\,dW(t) \\
&+ (u_0(x,\omega)\,,\,w_j(x)) - (\xi\,,\,w_j(x)), \quad \text{for any} \quad j = 1,2,\ldots.
\end{aligned} \tag{3.114}$$

This holds, if we replace $w_j(x)$ by any of their linear combinaitions. Hence since S_m is dense in $W_0^{1,p(x)}(D)$, we have

$$\begin{aligned}
&\int_0^T (\chi\,,\,v(x))\,dt \\
&= \int_0^T (f(t,u(t))\,,\,v(x))\,dt + \int_0^T (G(t,u(t))\,,\,v(x))\,dW(t) \\
&+ (u_0(x,\omega)\,,\,v(x)) - (u(T,x)\,,\,v(x)), \quad \text{for any} \quad j = 1,2,\ldots,
\end{aligned} \tag{3.115}$$

$\forall t \in [0,T]$, for any function $v \in W_0^{1,p(x)}(D)$.

Arguing similarly as in [66, subsect. 3.3, page 655], and taking into account the inclusion

$$\overset{\circ}{V}(Q_T) \cap L^2(0,T;W_0^{1,p(x)}(D)) \subset L^2(Q_T),$$

we see that $u(.,.,\omega) \in C([0,T];L^2(D))$ for a. e. $\omega \in \Omega$ and hence the initial condition is meaningful.

Following well known arguments from [66, page 655] and [51, page 1665] we obtain that

$$u(T,x) = \xi, \tag{3.116}$$

$$u_0(x) = u(0,x). \tag{3.117}$$

3.3.8 Monotonicity Method

It remains to identify the limit of

$$\int_0^t Au_{m_\nu}(s)\,ds,$$

which requires arguments of monotone operators. A similar approach can be found in [51], [66] and [59] and the bibliography therein. We shall prove that

$$\int_0^t \chi(\omega)\,ds = \int_0^t Au(s)(\omega)\,ds,$$

for any $t \in [0,T]$. Let v be an arbitrary function in $\overset{\circ}{V}(Q_T)$. Let us set

$$X_\nu = 2\mathbb{E}\int_0^t (Au_{m_\nu}(s) - Av(s)\,,\,u_{m_\nu}(s) - v(s))\;ds \;+\; \mathbb{E}\|u_{m_\nu}(s) - v(s)\|^2_{L^2(D)}.$$

Passing to the limit in (3.69) with $m = m_\nu$, as $\nu \longrightarrow \infty$, we obtain

$$u(t) + \int_0^t \chi(s)\, ds = u_0 + \int_0^t f(s, u(s))\, ds + \int_0^t G(s, u(s))\, dW(s), \tag{3.118}$$

and by Ito's formula, we have

$$\begin{aligned}
&\mathbb{E}\|u(t)\|^2_{L^2(D)} + 2\mathbb{E}\int_0^t (\chi(s)\,,\, u(s))\; ds \\
&= \|u_0\|^2_{L^2(D)} + 2\mathbb{E}\int_0^t (f(s, u(s)\,,\, u(s)))\; ds + \\
&+ \mathbb{E}\int_0^t \|G(s, u(s))\|^2_{(L^2(D))^d}\, ds + 2\mathbb{E}\int_0^t (G(s, u(s))\,,\, u(s))\, dW(s).
\end{aligned} \tag{3.119}$$

For any function $v \in W_0^{1,p(x)}(D)$ we have

$$\begin{aligned}
&X_\nu \\
&= 2\sum_{i=1}^n \iint_{Q_t} \left(\left|\frac{\partial u_{m_\nu}}{\partial x_i}\right|^{p(x)-2} \frac{\partial u_{m_\nu}}{\partial x_i} - \left|\frac{\partial v}{\partial x_i}\right|^{p(x)-2} \frac{\partial v}{\partial x_i} \right) \left(\frac{\partial u_{m_\nu}}{\partial x_i} - \frac{\partial v}{\partial x_i} \right) dx\, dt + \\
&+ \mathbb{E}\|u_{m_\nu}(s) - v(s)\|^2_{L^2(D)}.
\end{aligned} \tag{3.120}$$

In view of this and the monotonicity of the operator A, we obtain

$$X_\nu \geq 0. \tag{3.121}$$

It follows from the boundedness of the sequene u_{m_ν} established in (3.80), (3.81) and (3.116) that

$$\mathbb{E}\|u(T, x)\|^2_{L^2(D)} \leqslant \varliminf_{\nu \longrightarrow \infty} \mathbb{E}\|u_{m_\nu}(T, x)\|^2_{L^2(D)}. \tag{3.122}$$

On the other hand we have

$$\begin{aligned}
0 &\leqslant \int_0^t (Au_{m_\nu}(s) - Av(s)\,,\, u_{m_\nu}(s) - v(s))\; ds \\
&= \int_0^t (Au_{m_\nu}(s)\,,\, u_{m_\nu}(s))\; ds - \int_0^t (Av(s)\,,\, u_{m_\nu}(s))\; ds \\
&- \int_0^t (Au_{m_\nu}(s)\,,\, v(s))\; ds + \int_0^t (Av(s)\,,\, v(s))\; ds,
\end{aligned} \tag{3.123}$$

and

$$
\begin{aligned}
X_\nu = \mathbb{E}\|u_{m_\nu}(t)\|^2_{L^2(D)} &+ 2\mathbb{E}\int_0^t \left(Au_{m_\nu}(s)\,,\, u_{m_\nu}(s)\right)\,ds + \mathbb{E}\|v(s)\|^2_{L^2(D)} - \\
&- 2\mathbb{E}\int_0^t \left(Av(s)\,,\, u_{m_\nu}(s) - v(s)\right)\,ds - 2\mathbb{E}\int_0^t \left(Au_{m_\nu}(s)\,,\, v(s)\right)\,ds - \\
&- 2\mathbb{E}\left(u_{m_\nu}(s)\,,\, v(s)\right). \qquad (3.124)
\end{aligned}
$$

We write X_ν in the form: $X_\nu = Y_\nu + Z_\nu$, where,

$$
\begin{aligned}
Y_\nu =& 2\,\mathbb{E}\int_0^t \left(f(s,u_{m_\nu}(s))\,,\, u_{m_\nu}(s)\right)ds - 2\,\mathbb{E}\int_0^t \left(Au_{m_\nu}(s)\,,\, v(s)\right)ds - \\
&- 2\int_0^s \left(Av(s)\,,\, u_{m_\nu}(s) - v(s)\right)\,ds + \mathbb{E}\int_0^t \|G(s,u_{m_\nu}(s))\|^2_{(L^2(D))^d}\,ds, \qquad (3.125)
\end{aligned}
$$

and Z_ν is defined by the difference:

$$
\begin{aligned}
Z_\nu = X_\nu - Y_\nu = \mathbb{E}\|u_{0m_\nu}\|^2_{L^2(D)} &- \mathbb{E}\|v(t)\|^2_{L^2(D)} - 2\mathbb{E}\left(u_{m_\nu}(s)\,,\, v\right) \\
&+ 2\mathbb{E}\int_0^t \left(G(s,u_{m_\nu}(s))\,,\, u_{m_\nu}(s)\right)dW_{m_\nu}(s). \qquad (3.126)
\end{aligned}
$$

Passing to the limit in the expresion of X_ν as $\nu \longrightarrow \infty$, we obtain, in view of (3.88), (3.93), (3.121) and (3.123)

$$
\begin{aligned}
&\|u_0\|^2_{L^2(D)} + 2\mathbb{E}\int_0^t \left(f(s,u(s))\,,\, u(s)\right)\,ds + \mathbb{E}\int_0^t \|G(s,u(s))\|^2_{(L^2(D))^d}\,ds + \mathbb{E}\|v(t)\|^2_{L^2(D)} \\
&- 2\,\mathbb{E}\int_0^t \left(Av(s)\,,\, u(s) - v(s)\right)\,ds - 2\,\mathbb{E}\int_0^t \left(\chi(s)\,,\, v(s)\right)\,ds - 2\,\mathbb{E}\left(u(t)\,,\, v(t)\right) \\
&+ 2\,\mathbb{E}\int_0^t \left(G(s,u(s))\,,\, u(s)\right)dW(s) \\
&\geq \mathbb{E}\sum_{i=1}^n \int_{Q_t} \left(\left|\frac{\partial u}{\partial x_i}\right|^{p(x)-2}\frac{\partial u}{\partial x_i} - \left|\frac{\partial v}{\partial x_i}\right|^{p(x)-2}\frac{\partial v}{\partial x_i}\right)\left(\frac{\partial u}{\partial x_i} - \frac{\partial v}{\partial x_i}\right)dx\,dt. \qquad (3.127)
\end{aligned}
$$

Hence it follows from (3.125) and (3.127) that

$$
\begin{aligned}
&\mathbb{E}\|u(t) - v(t)\|^2_{L^2(D)} + 2\,\mathbb{E}\int_0^t \left(\chi(s) - Av(s)\,, u(s) - v(s)\,\right)\,ds \\
&\geq \mathbb{E}\sum_{i=1}^n \int_{Q_t} \left(\left|\frac{\partial u}{\partial x_i}\right|^{p(x)-2}\frac{\partial u}{\partial x_i} - \left|\frac{\partial v}{\partial x_i}\right|^{p(x)-2}\frac{\partial v}{\partial x_i}\right)\left(\frac{\partial u}{\partial x_i} - \frac{\partial v}{\partial x_i}\right)dx\,dt. \\
&\geq 0. \qquad (3.128)
\end{aligned}
$$

We have

$$\begin{aligned}\lim_{\nu\longrightarrow\infty}\sup X_\nu \leqslant\, & 2\mathbb{E}\int_0^t (f(s,u),u)\; ds + \mathbb{E}\|u_0\|^2_{L^2(D)} - 2\mathbb{E}\int_0^t (\chi(s)\,,\, v(s))\; ds \\ & - 2\mathbb{E}\int_0^t (Av(s)\,,\, u(s)-v(s))\; ds + \mathbb{E}\int_0^t (G(s,u)\,,\, u)\; dW(s) + \\ & + \mathbb{E}\int_0^t \|G(s,u)\|_{(L^2(D))^d}\, ds \\ & = 2\mathbb{E}\int_0^t (\chi\,,\, u)\, dt - 2\mathbb{E}\int_0^t (\chi\,,\, v)\, dt - 2\mathbb{E}\int_0^t (Av\,,\, u-v)\, dt. \end{aligned} \tag{3.129}$$

Hence, using (3.127)-(3.129), we have

$$2\mathbb{E}\int_0^t (\chi - Av\,,\, u-v)\; ds + \mathbb{E}\|u(t)-v(t)\|^2_{L^2(D)} \geq 0. \tag{3.130}$$

We pick $u, v, w \in \mathring{V}(Q_t)$ such that $v = u - \alpha w$, where α is a positive constant. Then passing to the limit as $\nu \longrightarrow \infty$ in (3.123) yields

$$\mathbb{E}\int_0^t (\chi\,,\, u)\, ds - \mathbb{E}\int_0^t (Av\,,\, u)\, ds - \mathbb{E}\int_0^t (\chi\,,\, v)\, ds + \mathbb{E}\int_0^t (Av\,,\, v)\, ds \geq 0. \tag{3.131}$$

Since the operator A is monotone, taking account of (3.123), using (3.130), we can write the term in the left-hand side of (3.131) as follows

$$\begin{aligned}\lim_{\nu\longrightarrow\infty} X_\nu &= \mathbb{E}\int_0^t (\chi\,,\, u)\, dt - \mathbb{E}\int_0^t (\chi\,,\, v)\, dt - \mathbb{E}\int_0^t (Av\,,\, u-v)\, dt \\ &= \mathbb{E}\int_0^t (\chi - Av\,,\, u-v)\, dt \\ &= \mathbb{E}\int_0^t (\chi - Av\,,\, \alpha w)\, dt \\ &= \mathbb{E}\int_0^t (\chi - A(u-\alpha w)\,,\, \alpha w)\; ds \geq 0; \end{aligned} \tag{3.132}$$

then we deduce after dividing the terms in (3.132) by α

$$\mathbb{E}\int_0^t (\chi - A(u-\alpha w)\,,\, w)\; ds \geq 0. \tag{3.133}$$

Letting α tend to 0 in (3.133) and using the fact that the operator A is semicontinuous we obtain the inequality

$$\mathbb{E}\int_0^t (\chi - Au\,,\, w)\; ds \geq 0.$$

Since w is an arbitrary function in $\mathring{V}(Q_t)$, the last inequality implies that $\chi = Au$. Therefore the integral identity (6.5) is satisfied.

Thus u is a solution of the problem (1.1)-(1.3). Thus the solution u belongs to the space $L^2(\Omega, \mathcal{F}, \mathbb{P}; C(0,T; L^2(D)))$ and hence $u(t)$ is an $L^2(D)$-valued measurable process. The proof of the existence Theorem is thereby complete.

Conclusion

In this dissertation we proved the existence of probabilistic weak solutions for a class of stochastic quasilinear parabolic partial differential equations with non-standard growth. We have used the Galerkin method to construct an approximation to the weak probabilistic solutions to our problem (1.1)-(1.3). In the proof, we combined the Galerkin methods with some analytic and probabilistic compactness results. To recover the main theorem of existence we used some probabilistic results from [60], [76] and [77]. The Galerkin method solves the weak formulation of the problem by converting it into a finite dimensional case. First, we proved that the Galerkin equations admit solutions. In the second step, we derived a "priori" estimates for the approximating solutions u_m. In the third step, we passed to the limit in the finite dimensional equation by choosing a subsequence $(u_{m_\nu}) \subset (u_m)$, which converges weakly in appropriate topologies. In the final step, we used the monotonicity of the operator A to prove that the limit u of u_m is a solution of the problem (1.1)-(1.3). At this final step, the analysis rests on two properties of the operator A which are the monotonicity and the semicontinuity of the operator A.

In the future, we hope to extend and explore

1. this type of models,
2. the work on electro-rheological fluids done by M. Růžička [65], in the framework of stochastic evolution problems.

and possibly we hope to explore the numerical analysis of models of this type.

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