Modelling and optimal control of the market of a telecommunications operator

by

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SUMMARY – ENGLISH

Modelling and optimal control of the market of a telecommunications operator

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A South African GSM telecommunications market consisting of two incumbents and an entering third player, is modelled utilising a non-linear, system dynamics approach. The model calculates subscriber choice based on a calculated utility. The utility is used to obtain a probability which is fed into a Bass type differential equation relating the different states in the model to their time derivatives. The model encapsulates all the prominent postpaid price plans in the market, as well as five different demographic market segments. Model Predictive Control is used to synthesise a linear feedback controller which uses the observed market state to optimally determine a price time series for one of the operators' products. The series will maximise Average Revenue Per User (ARPU) for the operator over the simulation time interval. Besides ARPU, the controller is also able to increase total revenue and minimise churn over the simulated interval for the optimising operator, and thus provides valuable decision support to the marketing management of such an operator.

Keywords: Model Based Control, Predictive Control, Optimal Control, Decision Support Systems, Management Systems, Telecommunication, Forecasts, GSM, ARPU.

SUMMARY – AFRIKAANS

Modelering en optimale beheer van die mark van 'n telekommunikasie operateur

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'n GSM telekommunikasiemark bestaande uit twee huidige diensverskaffers en 'n derde rolspeler wat besig is om die mark te betree, word gemodelleer deur gebruik te maak van 'n nie-lineêre, stelseldinamika benadering. Die model bereken 'n kliënt keuse gegrond op 'n berekende utilitieit. Die utiliteit word gebruik om 'n waarskynlikheid te bepaal wat in 'n Bass-tipe differensiaal vergelyking ingevoer word wat die verskillende toestande in die model in terme van hul tyd afgeleides uitdruk. Die model omsluit al die prominente kontrak tariefplanne in die mark, asook vyf verskillende demografiese marksegmente. Model Voorspellende Beheer word gebruik om 'n lineêre terugvoerbeheerder te ontwerp wat die waargenome marktoestand gebruik om 'n tydreeks vir die pryse van een van die diensverskaffers optimaal te bepaal, wat Gemiddelde Inkomste Per Gebruiker (GIPG) maksimeer oor die simulasie interval. Behalwe vir GIPG, slaag die beheerder ook daarin om die totale inkomste van die diensverskaffer te vermeerder, en die verlies aan kliënte te minimeer oor die simulasie interval vir die optimiserende rolspeler. Op die manier kan die geslotelusstelsel waardevolle besluitnemings ondersteuning aan die bemarkingsbestuur van so 'n diensverskaffer verskaf.

Sleutelwoorde: Model Gebaseerde Beheer, Voorspellende Beheer, Optimale Beheer, Besluitnemings Ondersteuningstelsels, Bestuurs Stelsels, Telekommunikasie, Vooruitskatting, GSM, GIPG.

ACKNOWLEDGEMENTS

LIST OF ABBREVIATIONS

ARPU   Average Revenue Per User.

GSM   Global System for Mobile telecommunications.

ISE   Integral Squared Error.

LTI   Linear Time Invariant.

MIMO   Multiple Input Multiple Output.

MPC   Model Predictive Control.

OO   Optimising Operator.

O2   Operator 2.

O3   Operator 3.

PSTN   Public Switched Telephone Network.

VAS   Value Added Services.

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1   INTRODUCTION

The parameters of a telecommunications network operator are often not optimally determined, but rather fixed by making use of legacy methods. Amongst the most critical of such parameters are the rates per minute at which the various products and price plans in the market are marketed by the operator. By researching the approach adopted in this dissertation, it is attempted to near optimally determine these parameters, which will result in driving the operator towards its strategic objectives.

The market needs to be modelled and simulated before the market can be optimised from the point of view of the operator being studied. The problem addressed in this dissertation is therefore the modelling of the market of a telecommunications operator, the simulation of the model, and the optimisation of strategic outputs by manipulating model inputs in a particular way.

1.1   Motivation

A significant amount has been published with regards to the modelling of systems such as the market of a telecommunications operator. However, the models that have been built have often been static and therefore do not lend itself to dynamic optimisation ([1] and [2]). Models built from an econometric foundation, are useful for forecasting, and dominate the current literature [3]. A discussion of the literature and background to modelling the market of a telecommunications operator is given in chapter 2.

In the field of dynamic optimisation, work has been published focussing on market models making use of historical data, and also models of organisations in the start-up phase [4]. Little has been published in the field of telecommunications though, and in particular not in the area of real time optimisation and decision support systems ([5] and [6]). A discussion of the literature and background to the control strategy used in the research (Model Predictive Control) is found in chapter 5.

The motivation for this project is therefore the scarcely researched area of dynamic market modelling and decision support enabling optimisation for telecommunication network operators.

1.2   Aims and Contributions

The main contribution of this dissertation is a dynamic system dynamics model of a telecommunications market, and a solution to the dynamic optimisation problem of marketing strategy (pricing) of telecommunication services. As far as could be ascertained, this is not available in the open literature.

In approaching the goal of the project, the following objectives have been pursued:

1. A suitable market was identified to be used for the project. This market may have been real or of academic nature.

2. A thorough understanding of the market was acquired. Insight into the market would determine the success of the next objective of the project.

3. The market was modelled by applying a structured, modular, preferably state-space based modelling methodology.

4. The model which had been built was simulated by feeding it with realistic industry data and testing the model against the true empirical historical data.

5. A feedback methodology was designed, tested and implemented as part of a closed-loop simulation which would optimise one or more strategic outputs of the model built in point 3 above.

1.3   Organisation of this Dissertation

The dissertation has been structured according to the chapters given below. Additionally the chapters represent the steps followed in the research done:

1.   Introduction.

2.   Process background – literature study of market modelling.

3.   Development of market model.

4.   Open-loop simulation.

5.   Control Background – literature study of control literature.

6.   Controller design.

7.   Closed-loop simulation.

8.   Conclusion and recommendations for future research.

2.   Process background – literature study of market modelling.

2   PROCESS BACKGROUND

2.1   Demand Forecasting

The last decade has seen rapid advances in telecommunications technology in an increasingly deregulated and competitive industry worldwide ([33] and [34]). Companies operating in these various markets have relied and are relying on market forecasts (specifically forecasting the demand) to justify the considerable investment needed to ensure sufficient capacity at the right time [2].

The core concept at the heart of most telecommunication demand models [2] is simple – that expected usage determines access demand, which in turn determines access requirements [35].

Telecommunication forecasting has been hampered by the unavailability of data, which is partly the result of the continued deregulation of the industry worldwide [2] (this deregulation trend has been very positive as it increased competition and lowered prices). Another major contributing factor to the unavailability of data is the commercial sensitivity of market data, and the subsequent reluctance of operators to freely release such data.

Short data series results in model parameter estimates being unreliable [17]. What is needed in order to do good forecasting, is the careful capturing of available industry data. This data can then be used to build models and to test hypotheses. Despite the emphasis on proper modelling in the literature, very few studies elevate the out of sample (i.e. tested) performance of the proposed models. Their claims to be useful in forecasting must thus be questioned. Very few models that have been built have been validated using data that were not used to derive the model, and there is a great need for best practice econometric analysis to be done on the models that have been built [36].

Most of the telecommunications demand forecasting literature consists of standard econometric practice (which is not extensive) implemented in the field [2]. The bulk of work that has been done has made use of methods that are inherently static, classical

econometric, models which relate independent and dependant variables in the model, but do not capture dynamic or time series information [24].

2.2   Key Factors to Incorporate

Usage is modelled as a function of price, income as well as other variables that are dependant on access [3]. Access is modelled based on individual choices made between alternatives. The choice that is made depends on the consumer surplus – the benefits that accrue from calling minus the costs of those calls. The cost depends in turn on the demand function (which is dependant on the usage price and income, as well as demographic variables) and the price of the corresponding calls, including access [36]. See section 2.3 for more information in the modelling and definition of the choices which are involved.

Price elasticity may be an increasing function of price – that means that if this is true, sensitivity to price is more for larger price values. Detailed cross-sectional data also permit a sharper focus on the determinants of price elasticities [37]. This approach has been adopted and used successfully by [4]. Income elasticities affect the number of calls more than the duration of calls [3].

Market demand can be segmented by customer, product and carrier to derive revenue forecasts. The benefits of this approach include the ability to convert volume forecasts into revenue [3]. Income, age and education could also be factors that separate a particular market segment from another. Each individual market needs to be modelled as a separate entity because of the big difference between e.g. markets in Africa and the US [38]. Despite the long history of Public Switched Telephone Networks (PSTNs), we now see that other technologies are making inroads into the traditional market of PSTN [3]. Not the least of these is the mobile GSM industry in South Africa.

2.3   Modelling Subscriber Choice

More and more service providers are offering different products as bundled packages of services. The customers' choice of service can be modelled through the logit function

(2-2). In addition, churn – the movement of subscribers between providers - is a variable of particular interest because of the operational and cost consequences.   A study of duration of stay with a particular operator [23], show that as well as price, non-price elements such as quality are important. Higher spending segments of the market seem to be more prone to shift between carriers.

The choice of carrier and the usage demand is jointly determined (rather than sequential where usage depends on the chosen carrier) [39]. Like wise, the number of calls and their duration are jointly determined [40]. It is important that a distinction be made between actual usage benefits and perceived benefits in a market. The future as being modelled in a forecasting model will in its essence be different from the perceived reality before the choices between alternatives have been made [41].

Aggregate Access studies in isolation (see the diffusion modelling discussion below in section 2.4) have limitations for the telecommunications supplier desiring to understand the market and develop policies and forecasts. As mentioned above, in order to obtain such insights, modelling the choice of subscribers has to be attempted. Models have been built that attempt to explain how different customers choose between different service options and rate structures, for example. A general model of individual choice is of the form [3]:

$$choice = f\left(preferred\ uses, attributes\ of\ the\ service\ relative\ to\ others, price\ and\ demographics\right)$$

(2-1)

A standard model form of (2-1) is the multinomial logit [8] where the probability ($P_n(i)$) of the nth individual choosing alternative i from a choice set C_n is [3],

$$P_n\left(i\right) = \frac{e^{\beta \mathbf{x}_{in}}}{\sum_{j \in C_n} e^{\beta' \mathbf{x}_{jn}}} \qquad i \in C_n,$$

(2-2)

where x_{jnk} are the variables that explain the nth consumer's choice of alternative j, based on k personal and service characteristics, described by the column vector $\mathbf{x}_{jn} = (x_{jn1}, x_{jn2}, \ldots x_{jnK})$. β and β' are column vectors indicating the impact of the characteristics on the exponents (utility). The β vectors are transposed and matrix multiplied with the $\mathbf{x}$ vectors to form the exponents in

(2-2).

Individual choices and their determinants are of interest in their own right. However, policy consequences can only be evaluated through aggregate forecasts, which can be calculated from the individual probabilities by either estimating the probabilities for each market segment, or sampling from the population as a whole ([8]).

In the dynamic telecommunication markets of the current day, the functional form and the specification of the price variable and the impact of it on the market and uptake is critical. Current models in marketing literature demonstrate possible non-linear asymmetric effects ([44] and [45]). However, very few of the models that have been published in the literature do reflect the complexities of consumer response to price changes. With increased competition in these established markets, advertising variables (either directly or indirectly through a "value for money driver") are now likely to prove important drivers of market share. In this area published work is also very limited [3].

Structural changes in the marketplace caused by technology and changed regulatory regimes, affect the demand for telecommunication services [42]. Very little work has been done on the impact of any structural change on the market, such as the entry of another operator into the market [3]. Constant elasticities have been assumed in almost all work that has been done. When a new entrant enters a telecommunication market, some customers are price sensitive while others are loyal to the incumbent(s) [18].

Customers choose the service to minimise their costs [42]. Consumers are generally assumed to allocate limited income to the available goods and services so as to maximise their satisfaction [15]. Potential customers are differentially more likely to adopt a particular product or service compared to others offered in the market based on the comparative rating of the different options against particular criteria, including: Price, brand value, reliability of service and network coverage. These variables would then constitute an overall attractiveness rating for the particular product or service [3]. This attractiveness rating is calculated from an utility which depends on the criteria mentioned above.

Services are placed on a multi dimensional product map and consumers, making the pending decision, trade off their valuation of individual attributes to reach an overall preference. Different groups of customers may fall into distinct benefit segments leading to targeted strategies. Forecasting the demand for a new service or price plan could just be seen as calculating the desirability of its position compared to competitors by market segment. Conjoint analysis [8] is the best established of the approaches within this framework and has been used by [9] to determine pricing structures and their effect on demand. Wittink and Bergestuen [10] examine its strengths in forecasting applications and offer principles that enhance accuracy.

The quantity demanded depends on the following: The amount of money available to be spent as well as the relative price of the product under consideration; the level of stock and its consumption; the preferences or tastes which depends on the necessity, usefulness, convenience and technological competitive power of the product [15].

In summary, demand patterns result from choice behaviour, where customers choose a product to maximise their utility [16]. The choice based model is useful in that it enables the description of complicated environments, and provides the flexibility to include marketing mix variables such as price and advertising in the analysis. Sales patterns have become complicated because of the entry of many competing and advanced services and products [16]. Saturation models are not adequate for the description of such markets characterised by competition between products because they are derived for single product markets [16]. In the paragraphs below a potential model for this theory is documented.

Define the utility that the *i*-th potential customer (i.e. non-subscriber) would obtain by choosing to subscribe to a service or product at time *t* as [16]:

$$U_t^{(0,k)} = V_t^{(0,k)} + \varepsilon_t^{(0,k)}$$

(2-3)

where *k* indicates different types of service subscription. In the superscript the first term indicates the status of the individual before the choice is made (zero by default) and the

second term the choice that is made at that time. V and ε denote the deterministic and the error term of the utility U respectively.

The probability, P_i, that a subscriber subscribes to service k (where k is either 0, 1 or 2 – assuming there are only three alternative states) while currently on 0, can be represented by

$$P_t^{(0,k)} = \frac{\exp\left(V_t^{(0,k)}\right)}{\exp\left(V_t^{(0,0)}\right) + \exp\left(V_t^{(0,1)}\right) + \exp\left(V_t^{(0,2)}\right)},$$

(2-4)

where only the deterministic term in the utility is used because we are assuming that the error terms follow independently and identically the Gumbel distribution [8]. Define the total market potential of the choice based model at time t, M_t, as [16]:

$$M_t = M_k.$$

(2-5)

If we denote the total number of subscribers to a particular service or product at time $t - 1$ as Y_{t-1}, then the total number of non-subscribers at time t is $M_t - Y_{t-1}$. If the number of new subscribers for a different service, k, at time t is given by $Z_t^{(0,k)}$, and $P_t^{(0,k)}$ is the purchase probability among the non-subscribers at that time, then the expected number of new subscribers for service k is:

$$E\left(Z_t^{(0,k)}\right) = \left(M_t - Y_{t-1}\right)P_t^{(0,k)} \qquad k = 1,2,$$

(2-6)

where $P_t^{(0,k)}$ is given by equation (2-4) [16].

Where there are a certain number of existing services with a new, competing service envisaged there are certain econometric problems that arise. A key theoretical issue is that of structuring the choice set - two methods are used in the literature [3]:

1.   The relative choice probability $P(i\,|\,C_n)/P(j\,|\,C_n)$ remains unchanged if the choice set, C_n, is enlarged, i.e. it is independent of the existence and attributes of any other alternative. This is called the Independence of Irrelevant Alternatives (IIA) property.

2.   While the IIA property will often hold in market analysis, it generally does not hold if competitive role-players enter the market and change the fundamental relationships between alternatives [3].

2.4   Diffusion Modelling

Typically when a new technology enters a market, the uptake of subscribers on the technology follows an S-shaped diffusion path [15] at an aggregate level (the choice analysis in section 2.3 has been at the sub-aggregate level). The best known model of this process has been introduced by Bass [11]. If m is the market potential and $N(t)$ the number of consumers who have adopted a new product, process or service at time t, then Bass proposed that:

$$n(t) = \frac{dN(t)}{dt} = g(t)(m - N(t)).$$

(2-7)

Different forms of g(t) result in different forms of the S-shaped curve that is assumed to characterise the adoption process [14].

Other models than Bass' that are used in the literature for growth curves, are the following [14]:

1.   The modified exponential curve:

$$X_t = a - c\,e^{-bt} + \varepsilon_t \qquad a, b \in R$$
$$a, b > 0$$

(2-8)

where X_t is the market development at time t (i.e. cumulative sales up to time t), a is the saturation level, b and c (both positive) are scaling coefficients, and ε_t is an error

term. In all the model definitions that are given in points 2. to 5. below, the interpretation of a, X_t and t will be the same. The definitions of b and c will differ between the different models (although these numbers are subject to the same constraints as b and c defined in (2-8)).

2.   The simple logistic curve with the same notation as above:

$$X_t = \frac{a}{1 + c\,e^{-bt}} + \varepsilon_t.$$

(2-9)

The coefficient b is called the coefficient of imitation or the coefficient of internal influence.

3.   The Gompertz curve:

$$X_t = a\,e^{(-c\exp(-bt))} + \varepsilon_t.$$

(2-10)

4.   The cumulative normal curve:

$$X_t = a\int_{-x}^{t} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(y-\mu)^2}{2\sigma^2}\right) dy + \varepsilon_t$$

(2-11)

where μ and σ are scaling coefficients.

5.   The cumulative log-normal curve:

$$X_t = a\int_{0}^{t} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(\ln(y)-\mu)^2}{2\sigma^2}\right) dy + \varepsilon_t.$$

(2-12)

The aggregate diffusion rate in the telecommunication industry (or any growth industry for that matter) tends to show a steady increase pattern annually. However, individual cumulative diffusion rate for a particular product may represent a typical product life cycle pattern with a potentially decreasing phase after a saturation point. The potential decrease of cumulative diffusion rate is not well explained in the existing diffusion forecasting models [15].

Genetic algorithms have been successfully used [27] to estimate the diffusion parameters in diffusion models, such as the Bass model. This research has focused on the growth phase in particular, which has proven to be a challenge for product managers in general.

Evidence of the success of diffusion type models is very limited indeed. Proper testing and evaluation of these models have been performed to a very limited extent in the literature [12].

In many telecommunications firms the following process has been implemented and used for forecasting [3]:

1.   Establish homogeneous market segments.

2.   Estimate the market potential for the new product or service.

3.   Estimate and forecast the relative prices of the new product (either through curve fitting or judgemental estimation).

4.   Estimate and forecast the switching probabilities as they depend on price and other variables.

5.   Using diffusion style curves, calculate the uptake of the products or the churn that it imposes on its alternatives.

2.5   System Dynamics

The above approach has been formalised in the literature. The modelling language of system dynamics which is based on a set of non-linear difference equations which can be easily programmed and simulated ([3] and [13]) (see section 5.2 for more information on applications of system dynamics) has most frequently been used.

System dynamics type models has long been considered focussed on obtaining insight into the internal relationships of a telecommunication market, and was thought to not be intended to do accurate forecasting as such. For this reason, it has not been evaluated to the extent that it needs to have been. In order to apply the methodology to its full potential, however, the need for such evaluation to be done as well has arisen [3].

2.6   Other Industries

Insights that have been obtained from forecasting work that has been done in industries other than Telecommunications, have proved that little value is added when trying to compare and implement similar approaches across the industry boundary [3]. The more deregulated an industry becomes, the more complex the task of forecasting becomes, for new products and services in particular [2].

2.6   Other Industries

3   MARKET MODEL DEVELOPMENT

3.1   Introduction

A market model has been built and simulated modelling the various attributes of a telecommunication market. The model is discussed in this section by essentially following the development process that was used. Frequent reference will be made to chapter 2 for concepts implemented. For each element in the model, the motivations for the design decisions taken are also documented.

The actual data which were used to test the developed model, were provided by an independent South African telecommunication consulting firm. Nine months of actual data form the basis of the model. The data were extracted in the form of a snapshot figure for the various fields per calendar month.

The market that has been modelled is a GSM telecommunications market, and specifically the postpaid, contract voice market where customers have contracts with the operator, and pay their bills at the end of the month. The balance of the market, occupied by the prepaid subscribers where airtime is bought upfront, was not included in the model.

3.2   Basic Premises for the Model

The following basic premises constitute the foundation of the model:

1. Two types of tariffs are used as prices in the market: Peak and Off-Peak rates. This is consistent with how typical telecommunications operators classify tariffs.

2. The postpaid market is divided into the following 5 market segments according to income and amount of minutes spent phoning per month (starting from low income and low usage at the one end, and going to high income and high usage at the other end): Consumer Low, Consumer High, Medium, Business Low, Business High (see sections 2.2 and 2.3 for a discussion of the impact of income on choice and usage, as well as segments in the market).

3. Three operators (telecom companies) are included in the model. For the one operator 6 price plans or tariff plans are modelled, and 5 each for the other two. A price plan is defined as a standalone market offering of an operator with its own rates and other benefits. Subscribers on a price plan of one of the operators may migrate to other price plans of the same operator, or to a price plan of a different operator (the modelling of this movement is discussed at great length later in this chapter). Furthermore, new subscribers joining the market can choose between any of the available price plan options (this input will also be discussed in more depth later in this chapter). In Figure 3-1 the potential movement of subscribers, as modelled, is graphically depicted.
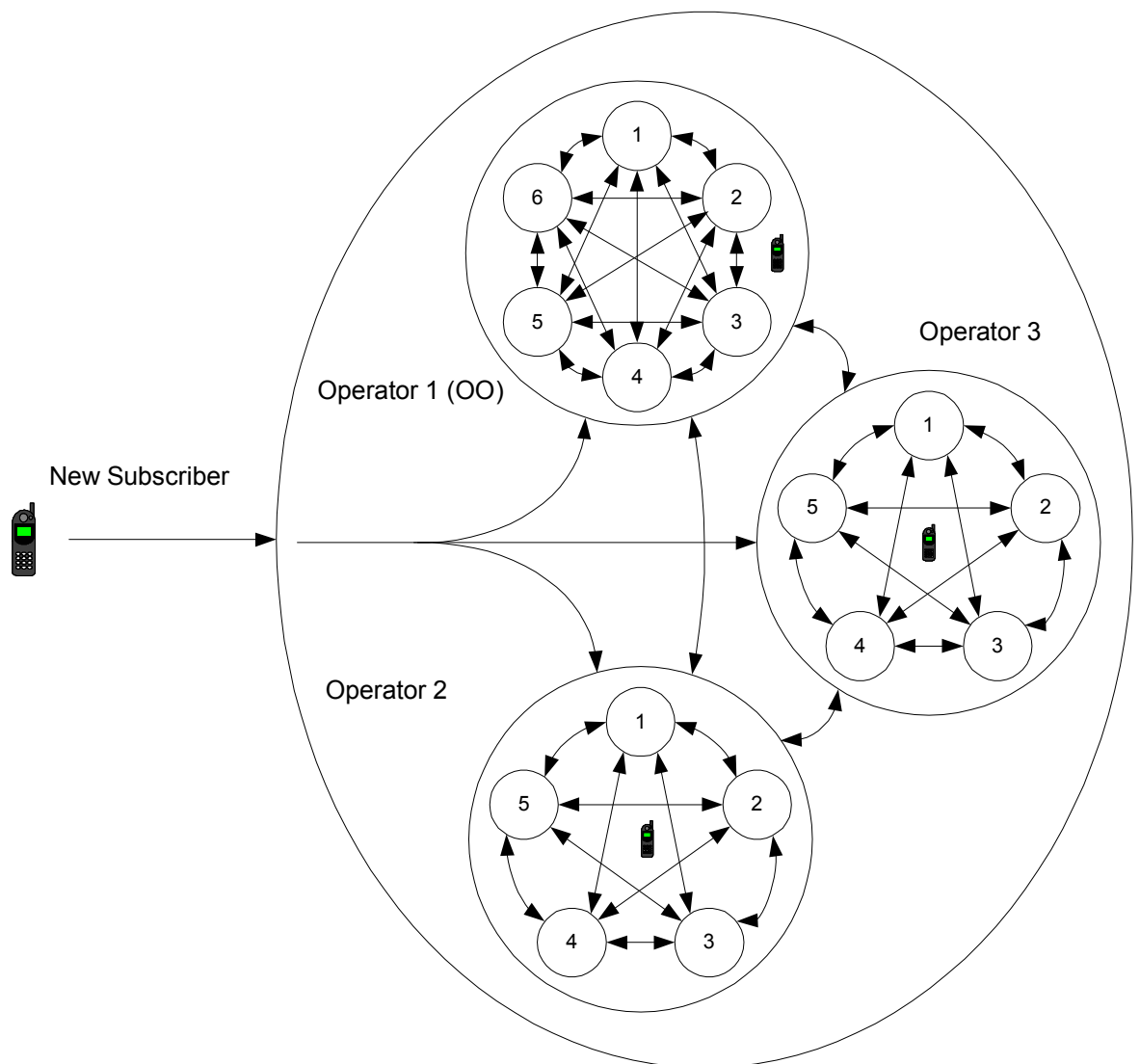
Figure 3-1: The Different Operators, Price Plans and Migrations in the Market Model

4. All calls made in the market can terminate on one of three different types of destinations: On-net (on the network of the operator that the phoning customer has a contract with), off-net (on the network of one of the other GSM operators operating in the market), or on a fixed line network.

5. Matlab was chosen for the implementation and simulation of the developed model. This choice was motivated by Matlab's ability to solve many technical and computation problems, especially those with matrix and vector formulations in a fraction of the time it would take to write a program in a scalar non-interactive language like C++ [25].

3.3   Factors Excluded from the Model

The market attributes, variables and inputs discussed in the following paragraphs are either not included in the model, or are included to a lesser degree.   The reason(s) for not including these attributes are also discussed.   A few initial assumptions were made (see section 3.2 above) and the analysis was conducted using these as a starting point.

1. Geographical distribution and its impact on the market are ignored. The reason for this is the need to simplify the model to a sufficient extent in order to reach the goal of the project i.e. to significantly contribute to the area of optimisation via decision support in the industry (see section 1.2).   Likewise the geographical length of calls made is ignored. This is however intrinsic to the market that has been studied - being a cellular market, the distance between subscribers in the same country does not impact the finances involved, as it would have done for a fixed line network.

2. The granularity of the analysis and all aggregation on the time dimension is restricted to a calendar month. For instance, the duration of calls made per time period (total amount of minutes called) as well as all other variables modelled are assigned a time value of the beginning of a month. The reasons for this restriction is the format of the industry data used (the sample length of these data is per month), the Nyquist analysis performed on the data (see section 6.1), as well as simplicity in the model.

3. The purpose of calls made as well as the impact of technological changes on the market are not taken into account. The reason being simplicity of the model, and it not being necessary for reaching the goal of the project (see section 1.2).   The intent of the

model, and the closed-loop system (see chapter 6) subsequently designed, is to simulate and forecast the market only over the horizon in which the variables included in the model change. This can be derived by looking at the shortcomings of the model. All other variables as listed above which are not included are assumed to be constant over this period of time. If any of these variables should change, the intent is that the model should be updated to include the structural impact of the change on the market, and the simulation rerun.

4. As all the customers in the model are using in essence the same GSM technology, they can all communicate with one another. Therefore the impact of different technologies that are mutually exclusive as far as usage is concerned and the impact of that on the choices made by customers are not modelled.

5. It is assumed that customers would not change the market segment they are on. This is viable if the typical horizon over which the simulation would be critically evaluated is considered. It is assumed that customers will not gain a substantial raise in income and make a subsequent jump to the next market segment within that time period on average.

6. It is assumed that no customer will decide to stop using his cell phone or stop the contract with his operator, but that all current and future customers would stay on a price plan of one of the three operators at all times. See point 5. above for a motivation for this assumption.

7. Operational issues (e.g. failures in the physical network, the requirements for call centres and other network externalities) are not included in the model. The chief reason for this exclusion is to minimise complexity of the model, in order to maximise the achievement of the strategic intent of the research (see section 1.2).

8. Marketing effort (the amount of effort or capital invested in marketing by the firm) data were not available and thus historical input for that input variable set ($\mathbf{d}_3$ as documented in section 3.4 and (3-10)) could not be used in the simulation. Appropriate values have been assumed, see section 3.4 for more information.

9. Only 9 months worth of data were available (see section 2.1 for the impact that this has on the model accuracy). As documented in section 2.1, this is a trend that is commonly observed in the literature ([2], [17] and [36]).

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3.4   Market Model Development

A system dynamics model was chosen as the appropriate model for the market (see section 2.5). A state-space model has been developed to represent the market because of the inherent differential equation nature of a system dynamics model. Furthermore, a state-space model is very flexible, and useful for Multiple Input Multiple Output (MIMO) problems in particular [19]. In addition, a state-space representation lends itself to the implementation of a feedback control system and subsequent optimisation. The standard way of representing a non-linear, time invariant, continuous, state-space model (the model falls into this category) is [28]

$$\dot{\mathbf{x}}(t) = f(\mathbf{x}(t), \mathbf{u}(t)) \quad \mathbf{x}(0) = \mathbf{x}_0$$
$$\mathbf{y}(t) = g(\mathbf{x}(t), \mathbf{u}(t))$$

,

(3-1)

where $\mathbf{x} \in R^n, \mathbf{u} \in R^m, \mathbf{y} \in R^l$, f and g are real-valued functions, t is the continuous time variable, n is the number of states in the model, m is the number of inputs into the model and p is the number of outputs of the model.

Although a continuous-time model was developed, the model is sampled at discrete intervals for all simulations performed. Therefore, the model is referenced with the discrete-time variable k all through the dissertation.

For each price plan in each market segment the amount of subscribers at a particular discrete time is represented by a state in the model. This accounts for $(6 + 5 + 5)*5 = 80$ states in the model. The rest of the states in the model represent key monthly variables for each market segment that is used to calculate or represent key output variables. The model was built with the objective of optimising key output variables for one of the operators in the market. Therefore most of these additional states are specific to that operator and are given below in vector form as the scalar variables are aggregated across the different market segments (this operator will from now on be referred to as the optimising operator or OO):

1. The amount of subscribers that have joined the market during the month and will be moving to a price plan offered by one of the incumbent operators (this variable is the only one in this list that is not specific to the optimising operator).

2. The net gross connections (the amount of new subscribers). These states will be referred to as the Migrations Vector.

3. The Average Revenue Per User (ARPU) for the optimising operator in the particular market segment. These states will be referred to as the ARPU Vector.

4. The revenue from usage. The sources of the revenue are monthly subscriptions and the usage of airtime by subscribers phoning a particular amount of minutes each month on the network. Note that this figure does not include connection fees that new subscribers would pay, nor interconnect revenue – funds received from other operators payable to the optimising operator when they terminate calls on the optimising operator's network. The reason being that the intent of the modelling and optimisation is the usage of services by the subscribers joining or being on the optimising network. These states will be referred to as the Revenue Vector.

5. The average amount of minutes phoned by all subscribers. These states will be referred to as the Average Usage Vector.

6. The total amount of minutes phoned by all subscribers. These states will be referred to as the Usage Vector.

The above variables comprise the rest of the states of the model: Six variables per market segment times the amount of market segments (5), i.e. 30. Therefore the total amount of states on the model is $n = 80 + 30 = 110$.

The inputs to the model consist of financial and addressable market information. The first set of financial inputs is the tariffs per minute that are asked of subscribers on each price plan (of all the operators) for the 2 times (peak and off-peak) with 3 types of destinations possible. This means that each price plan has $2 \times 3 = 6$ tariffs that can change, and since the 3 operators have 16 price plans among them (6, 5 and 5 respectively) for which the tariffs are changeable respectively, the total amount of tariff input variables is $16 \times 6 = 96$ (this figure is designated m_u in the rest of the dissertation). The tariffs of 5 of the 6 price

plans belonging to the optimising operator are the manipulated variables in the closed-loop system – $6 \times 5 = 30$ inputs (as designed in chapter 6).

The distribution for the destination of calls per market segment has been obtained by processing data from the company that sourced the subscriber data. The following distribution was obtained and used in the model (note the sum of the non-rounded elements of each row is 1, in the table the values have been rounded to two decimals):

Table 3-1. Call Destination Distribution per Market Segment

Market Segment	Fraction of calls from each GSM operator that terminates on-net.	Fraction of calls from each GSM operator that terminates off-net on a mobile network.	Fraction of calls from each GSM operator that terminates on a fixed line network.
Consumer Low	0.24	0.38	0.34
Consumer High	0.25	0.36	0.36
Medium	0.24	0.38	0.35
Business Low	0.13	0.46	0.39
Business High	0.05	0.51	0.43

The distribution for the time that calls are made (peak or off-peak) has been similarly derived. Table 3-2 describes this distribution fed into the model:

Table 3-2. Time-of-call Distribution per Market Segment

Market Segment	Fraction of Calls Made During Peak Time	Fraction of Calls Made During Off-peak Time
Consumer Low	0.44	0.56
Consumer High	0.69	0.31
Medium	0.71	0.29
Business Low	0.82	0.18
Business High	0.87	0.13

The actual data provided included values for the total amount of subscribers on each price plan for the optimising operator, as well as values for the tariffs on those price plans in the form of time series. During the time interval for which data was provided, the tariffs were

increased/decreased once, which is an annual practice for most operators. Actual figures for the subscriber numbers for the other operators were only available for the end of the simulation time period. These were used and initial values for the states ($\mathbf{x}[0]$ – where the square brackets indicate a discrete-time sample of the continuous-time model) were calculated using the growth in subscriber numbers for the optimising operator over the same time period. For graphs of actual data see the comparison with the model in section 4.1. The Integral Square Error (ISE) between the model output for state variables and that of the actual data was 3.6848×10^{10}, with ISE defined as [29]

$$\sum_{k=1}^{N} \left(\mathbf{e}[k]^{\mathrm{T}} \mathbf{e}[k] \right) \Delta k ,$$

(3-2)

with $\mathbf{e}[k]$ the discrete error column vector between the output signal and the benchmark it is compared to sampled at a period of Δk, $\mathbf{e}[k]^{\mathrm{T}}$ its transpose, k the discrete time index, and N the amount of samples in the series.

The relative size of the different market segments was inferred from the distribution of subscribers on the price plans of the optimising operator at the end of the simulation period. This relative distribution will be constant over time as a result of the assumptions made with regards to subscribers and their market segments in section 3.2.

Additional competition (above that offered by the second operator in the market) is accommodated by the model in the form of a third operator entering the market and the impact of the entry on the existing customer base of the incumbent role-players. Other than this entry, as well as one new product launched during the simulation period by the optimising operator, and the steady increase in the addressable market (see paragraph below), the market which was modelled is a fairly stable one. These changes in the market impacted the model in the form of structural change (see sections 2.2 and 2.3).

Another input variable into the model, is the number of new subscribers that are entering the market in a particular month (d_1 is the symbol used in this chapter). This is the amount of new (to the total market) customers that will be choosing and subscribing to a price plan of one of the operators in that month. For the period of time for which the open-loop

analysis has been performed, and the model and open-loop simulation developed (simulation period), this variable was inferred from the actual growth in the total subscriber numbers of the three operators.

The cost of an initial connection to a price plan (d_2), the monthly subscription (d_5) as well as the connection incentive (d_4 - the value of the handset which is subsidised and provided with a new contract) for each price plan for each operator was also obtained as a time series over the simulation period, and included as three additional sets of input variables. These fees and values have been raised with the annual tariff increase that occurred over the period as discussed above. Each input set was fed into the model as a time series per price plan. Therefore the total amount of distinct input variables contributed to the model by these sets (d_2, d_4 and d_5) is $3 \times 16 = 48$.

Advertising variables (the value for money driver mentioned in section 2.3) were modelled by including its influence into a soft issues utility set of input variables (designated d_3) in the form of a time series over the simulation period per price plan offered by all the operators. The variables in the market that were included into this utility variable are:

1. Network coverage and quality.

2. Brand value.

3. Reliability of service.

4. Additional Value Added Services (VAS), such as voice mail.

5. Customer service effectiveness.

Arbitrary, constant values expressed as a percentage have been assumed for d_3. The values listed in Table 3-3 were used for the price plans of the respective operators.

Table 3-3. Soft Issues Utility (d3) Values for Each Operator per Market Segment

Price Plan	Soft Issues Utility for Optimising Operator	Soft Issues Utility for Operator 2	Soft Issues Utility for Operator 3
Price Plan 1	85	95	100
Price Plan 2	80	90	95
Price Plan 3	75	85	90
Price Plan 4	70	80	85
Price Plan 5	65	75	80
Price Plan 6	60	Not applicable	Not applicable

When considering the above paragraphs it is apparent that the total amount of input variables into the model is 96 (tariffs per price plan) + 1 (d_1) + 48 ($\mathbf{d}_2$, $\mathbf{d}_4$ and $\mathbf{d}_5$) + 16 ($\mathbf{d}_3$) = 161.

For each price plan in the model, a particular amount of free minutes is allocated, ranging from 100 per month for the lowest usage price plans to 1000 per month for the highest.

The influence of each of the sets of inputs into the model ($\mathbf{u}$ vector at time k in this chapter) has been adjusted and fine tuned by 5 weights applicable to each market segment. These will be highlighted again in the paragraphs below:

Table 3-4. Utility Weights Used for Tuning of Model

Market Segment	Influence of Tariffs and Subscription	Influence of Connection Fee	Influence of Soft Issues	Influence of Connection Incentive
Consumer Low	0.6	0.15	0.10	0.15
Consumer High	0.6	0.10	0.10	0.20
Medium	0.6	0.10	0.05	0.25
Business Low	0.6	0.05	0.05	0.30
Business High	0.6	0.0	0.05	0.40

After processing the actual data provided (aggregating scattered data in the appropriate manner), the following truncated and scaled normal distributions for the monthly usage of airtime (measured in minutes per month) of subscribers in each market segment were obtained. The industry trend is that the more wealthy segments spend more time making calls on average per month, but with a higher standard deviation as well, and the converse is true for the less wealthy segments. The line style key used for the market segments is included in the figure.
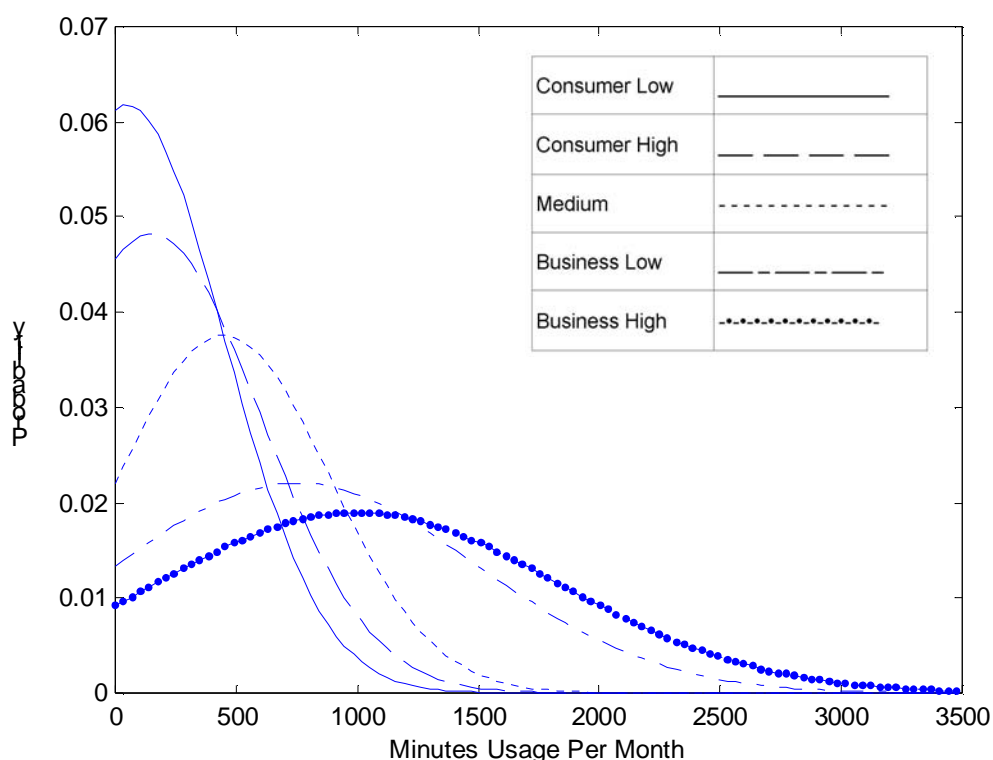
Figure 3-2: Subscriber Usage of Airtime per Month per Market Segment

The influence of tariff change on the usage of subscribers on a particular price plan in a certain market segment is captured in the following way: The more that is charged per minute the less a subscriber on that price plan and market segment would phone, and vice versa. The change in usage is calculated in direct proportion to the change in tariff. Actual data were used here to ascertain the relationship.

A choice based model has been built (see chapter 2). The model essentially consists of a subscriber number matrix (this 16 × 5 matrix consists of the first 80 states in the model),

where the 16 rows represent the different price plans of the different operators available in the postpaid contract, GSM, cellular market, and the 5 columns represent the different market segments. The model operates from the assumption that the customer would prefer the product that offers the best quality of service at the least cost. The choices that customers will make are modelled by calculating a probability that a particular choice will be made from a utility for the option (see section 2.3). The developed probability functions do not hold to the IIA property (see the end of section 2.3). Before the utility for a particular price plan in a certain market segment can be calculated, however, the perceived expense that the customer would have per month on that price plan needs to be obtained. For the expense to be calculated, a relationship between expense and the amount of minutes that a customer in the market segment uses per month, needs to be obtained. The expense at $t = 0$ minutes of usage per month equals the subscription fee per month applicable, and stays at that value for as long as the subscriber still has free minutes left. After that the curve is a linear function of minutes usage per month, of which the slope is the weighted sum of the products of the fractions shown in Table 3-1 and Table 3-2, and the tariffs applicable to each destination and time of call combination. An example of such a function for one of the price plans modelled in the market, is shown in Figure 3-3:
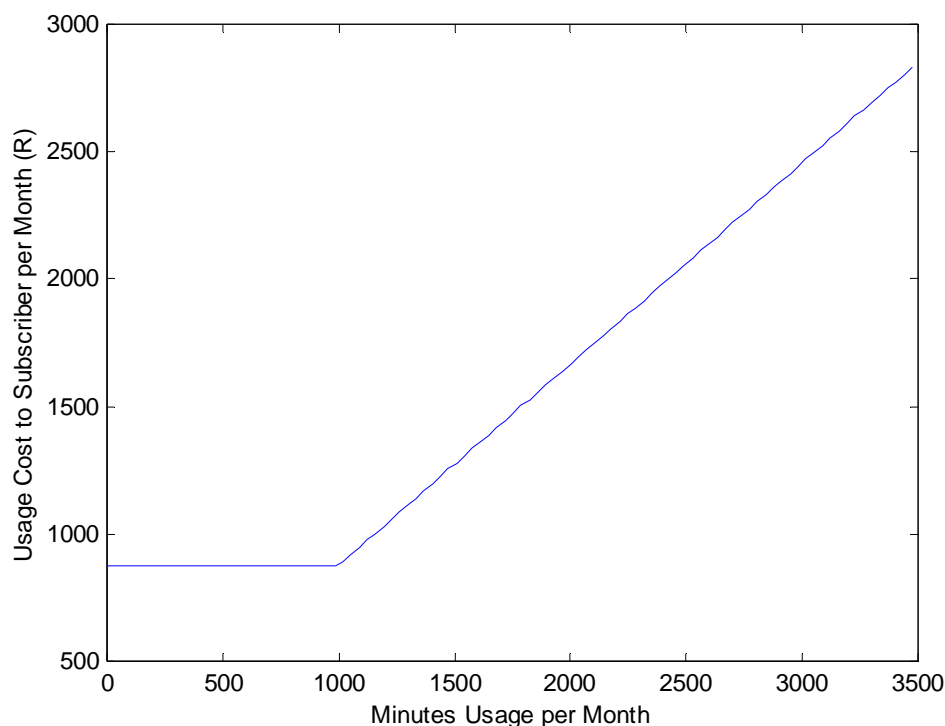
Figure 3-3: Cost to Subscriber per Month as a Function of Usage per Price Plan

The perceived expense $b_{i,j}$ for a subscriber for month k in the simulation ($b_{i,j}[k]$) for the market segment j and price plan i can now be obtained by mapping the distribution (Figure 3-2) for the segment to the above function (Figure 3-3). The resulting equation is:

$$b_{i,j}[k] = \sum_{t=0}^{T_{Max}} c(i,t) \cdot d(j,t),$$

(3-3)

where the functions $c(i,t)$ and $d(j,t)$ are the applicable cost to subscriber for the price plan i (Figure 3-3) and distribution for the market segment j (Figure 3-2) functions. T_{Max} is the maximum amount of minutes included in the distribution function – in Figure 3-2 this is close to 3500 minutes per month.

The perceived usage is different to the actual usage which will result from a choice made according to the probability calculated from the perceived utility (see section 2.3). This is because the usage that a customer expects to have based on his/her market segment (which is used to make choices between available price plans), is different to the actual usage the customer will have on a price plan when the rates applicable on the price plan is changed (as discussed above). Therefore the $ARPU_{i,j}$ for price plan i and market segment j will differ from $b_{i,j}[k]$ in the following manner:

$$ARPU_{i,j}[k] = \sum_{t=0}^{T_{Max}} c(i,t,r_i) \cdot d_{adjusted}(j,t,r_i)$$

(3-4)

where $d_{adjusted}$ is the adjusted distribution function as a function of r_i - the changed tariffs (or rates).

Customers (both new and old) choose a price plan based on a set of criteria and the rating they give to each option in the market (see section 2.1, 2.2 and 2.3), from the perspective of the particular market segment they are in. The weights reflected in Table 3-4 are applied to 4 utilities which are calculated from the inputs of the model in the following manner:

1.   The **b**[k] matrix, consisting of the different $b_{i,j}[k]$ scalars, is used to obtain the tariff utility matrix at month k (**ut**$_1$[k]). The **b**[k] matrix is fed it into a non-linear matrix function s(**M**) which processes (scaling of the elements of the columns by a common factor such that the sum of the elements in each column equals unity – function g(**M**)) the reciprocal of each element in the matrix. The function s(**M**) emphasises higher subsequent values in the columns of the matrix by increasing its value and decreasing lower values. This function is very similar to the logit function defined in section 2.3,

2.   (2-2), and often used by researchers ([3] and [8]) to convert utility into probability as is the case here. The reciprocal is taken for each bill-for-the-month value in the matrix in order to obtain a utility because of the inverse relationship between desirability of a price plan, and the bill that results from usage of that price plan. Higher resulting values of desirability are increased because of the human tendency to far rather choose the cheapest product than its desirability would indicate if it was not changed in relation to the total desirability for the whole market segment

3.   (2-2), [3] and [8]). The opposite applies to the lowest desirability. The whole matrix is then processed by g(**M**) again, and multiplied by a square matrix (**w**$_1$, size 5 × 5) of which all the elements are zero except the main diagonal. The main diagonal has as its elements the different tariff and subscription utility weights for each market segment. The dot notation before matrix powers is used in this dissertation to indicate that the power applies to each element of the matrix (as in Matlab). The above can then mathematically be expressed as follows:

$$\mathbf{ut}_1[k] = s\!\left(\left(s(b[k])\right)^{-1}\right)\!\cdot \mathbf{w}_1$$

(3-5)

The s function in (3-5) is subsequently defined in the next few paragraphs.

Given **M** (dimensions $\alpha \times \beta$), g(**M**) can be defined in terms of its elements $g_{\gamma,\delta}(\mathbf{M})$:

$$g_{\gamma,\delta}(\mathbf{M}) = \frac{M_{\gamma,\delta}}{\displaystyle\sum_{p=1}^{\beta} M_{p,\delta}}\,.$$

(3-6)

The function min($\mathbf{M}$) for the matrix $\mathbf{M}$ is defined as a row vector containing the minimum element of each column of $\mathbf{M}$ (as in Matlab).

The function max($\mathbf{M}$) for the matrix $\mathbf{M}$ is defined as a row vector containing the maximum element of each column of $\mathbf{M}$ (as in Matlab).

Given $\mathbf{M}$ (dimensions $\alpha \times \beta$), h($\mathbf{M}$) can be defined in terms of its elements $h_{\gamma,\delta}(\mathbf{M})$:

$$h_{\gamma,\delta}(\mathbf{M}) = M_{\gamma,\delta} - (\min(\mathbf{M}))_{\delta} + 0.01 \max(\mathbf{M})_{\delta}.$$

(3-7)

Given g($\mathbf{M}$), min($\mathbf{M}$) and max($\mathbf{M}$), s($\mathbf{M}$) is defined as:

$$s(\mathbf{M}) = g(h(\mathbf{M})).$$

(3-8)

4. The connection fee utility matrix calculation at month k ($\mathbf{ut}_2[k]$ based on the connection fee - cost of connecting to) is done by feeding g($\mathbf{M}$) with the following: A unity matrix ($\mathbf{U}$), minus a matrix formed by applying g to a matrix with the connection fee 16×1 input column vector for all its columns. The dimension of the matrix being referred to here is 16×5 (the amount of price plans in the market by the number of market segments). $\mathbf{w}_2$ is a square matrix (5×5) of which all the elements are zero, except the main diagonal which has as its elements the different connection fee utility weights for each market segment If g($\mathbf{M}$) is as defined in (3-6), and $\mathbf{U}$ is a matrix of the required dimension with each element being equal to 1 and $\mathbf{d}_2[k]$ is the connection fee input column vector, then

$$\mathbf{ut}_2[k] = g\left(\mathbf{U}_{6x15} - g\left(\mathbf{d}_2[k]\mathbf{U}_{1x5}\right)\right) \cdot \mathbf{w}_2.$$

(3-9)

5.   Using the notation defined in point 4 above, defining the soft issues input column vector at month k as $\mathbf{d}_3[k]$, and using the soft issues utility weights as the elements of the diagonal of $\mathbf{w}_3$ (5×5), the utility for the soft issues is calculated as follows:

$$\mathbf{ut}_3[k] = s\big(g\big(\mathbf{d}_3[k] \cdot \mathbf{U_{1x5}}\big)\big) \cdot \mathbf{w}_3 .$$

(3-10)

6.   Using the notation defined in point 4. above, defining the connection incentive input column vector as $\mathbf{d}_4[k]$, and using the connection incentive utility weights as the elements of the diagonal of $\mathbf{w}_4$ (5×5), the utility for the connection incentive is calculated as follows:

$$\mathbf{ut}_4[k] = g\big(\mathbf{d}_4[k] \cdot \mathbf{U_{1x5}}\big) \cdot \mathbf{w}_4 .$$

(3-11)

The dynamic model operates from the fundamental precept that the rate of change of the number of subscribers on a certain price plan in a particular market segment is proportional to a calculated probability multiplied by the total amount of subscribers in the market segment. This is a valid starting point because it is an alternative expression of (2-6) in section 2.3 as can be seen in (3-12). The probability is the normalised sum of the utilities calculated in the above paragraphs in (3-5) and (3-9) to (3-11) (this is an alternative application of the principle behind (2-4) in section 2.3). Customers are not only driven by utility in deciding which price plan to choose or move to. Customers that are currently on an existing price plan are less likely to move to a different one because of brand loyalty, resistance in the movement of marketing information to the customer, and other factors (see sections 2.2 and 2.3). The result of this phenomenon is that less customers than are expected do move and change price plans, and or operators. A limiting factor has thus been introduced that would limit the amount of subscribers that do effectively migrate. With this limiting factor represented by l_f ($l_f = 0.0078$ used in the model), and the amount of new customers from outside the market joining the market in the current month k by $d_{1,j}[k]$, the price plan under scope by underscore i, the market segment under scope by underscore j, the amount of subscribers on the price plan in the market segment under scope by $x_{i,j}[k]$ and q, p and v index variables, then

$$\Delta x_{i,j}[k] = l_f \cdot \left\{ \left(\sum_{v=1}^{v=16} x_{p,j}[k] + d_{1,j}[k] \right) \cdot \frac{\sum_{q=1}^{4} ut_{q,i,j}[k]}{\sum_{v=1}^{16} \left(\sum_{q=1}^{4} ut_{q,p,j}[k] \right)} - x_{i,j}[k] \right\}$$

$$= l_f \cdot \left\{ \frac{\left(\sum_{v=1}^{v=16} x_{p,j}[k] + d_{1,j}[k] \right) \cdot}{\left[s\left((s(\mathbf{b}[k]))^{-1} \right) \cdot \mathbf{w}_1 + g\left(\mathbf{U}_{6 \times 15} - g\left(\mathbf{d}_2[k]\mathbf{U}_{1 \times 5} \right) \right) \cdot \mathbf{w}_2 + s\left(g\left(\mathbf{d}_3[k]\mathbf{U}_{1 \times 5} \right) \right) \cdot \mathbf{w}_3 + g\left(\mathbf{d}_4[k]\mathbf{U}_{1 \times 5} \right) \cdot \mathbf{w}_4 \right]_{i,j}}{\sum_{p=1}^{16} \left(\left[s\left((s(\mathbf{b}[k]))^{-1} \right) \cdot \mathbf{w}_1 + g\left(\mathbf{U}_{6 \times 15} - g\left(\mathbf{d}_2[k]\mathbf{U}_{1 \times 5} \right) \right) \cdot \mathbf{w}_2 + s\left(g\left(\mathbf{d}_3[k]\mathbf{U}_{1 \times 5} \right) \right) \cdot \mathbf{w}_3 + g\left(\mathbf{d}_4[k]\mathbf{U}_{1 \times 5} \right) \cdot \mathbf{w}_4 \right]_{p,j} \right)} - x_{i,j}[k] \right\}$$

$$= l_f \cdot \left\{ \frac{\left(\sum_{v=1}^{v=16} x_{p,j}[k] + d_{1,j}[k] \right) \cdot}{\left[g\left(h\left(\left(g\left(h(\mathbf{b}[k]) \right) \right)^{-1} \right) \right) \cdot \mathbf{w}_1 + g\left(\mathbf{U}_{6 \times 15} - g\left(\mathbf{d}_2[k]\mathbf{U}_{1 \times 5} \right) \right) \cdot \mathbf{w}_2 + g\left(h\left(g\left(\mathbf{d}_3[k]\mathbf{U}_{1 \times 5} \right) \right) \right) \cdot \mathbf{w}_3 + g\left(\mathbf{d}_4[k]\mathbf{U}_{1 \times 5} \right) \cdot \mathbf{w}_4 \right]_{i,j}}{\sum_{p=1}^{16} \left(\left[g\left(h\left(\left(g\left(h(\mathbf{b}[k]) \right) \right)^{-1} \right) \right) \cdot \mathbf{w}_1 + g\left(\mathbf{U}_{6 \times 15} - g\left(\mathbf{d}_2[k]\mathbf{U}_{1 \times 5} \right) \right) \cdot \mathbf{w}_2 + g\left(h\left(g\left(\mathbf{d}_3[k]\mathbf{U}_{1 \times 5} \right) \right) \right) \cdot \mathbf{w}_3 + g\left(\mathbf{d}_4[k]\mathbf{U}_{1 \times 5} \right) \cdot \mathbf{w}_4 \right]_{p,j} \right)} - x_{i,j}[k] \right\}$$

(3-12)

In (3-12) the state vectors $\mathbf{x}$ and $\Delta \mathbf{x}$ are represented in the form of matrices which are internally reshaped in the code of the model so as to become vectors. In the text here they are represented as matrices to make the price plan and market segment coordinates clear. Equation (3-12) forms the heart of the state-space model, since it relates the state variables and inputs to the delta change of the state variables from month k to $k + 1$. It is apparent that the structure of the equation resembles that of the Bass model with the current state being subtracted (see (2-7) in section 2.4). From the relationship in the equation above between the probability of adoption of a particular price plan and the tariffs involved, it can be readily seen that the price elasticities in the model are not constant (as is often assumed – see section 2.2) but highly non-linear and dynamic.

In the paragraphs that follow, the equations for the output variables of the model are discussed. The output equations have been assigned to specific states in the model (except for the first set of outputs), which have then in turn been assigned to the output variables. All variables are evaluated at month k, unless otherwise indicted.

The amount of subscribers of the optimising operator in each market segment, or $y_1[k]$ to $y_5[k]$ at month k is calculated as follows:

$$y_j[k] = \sum_{i=1}^{NOP} x_{i,j}[k]$$

(3-13)

where *NOP* is the Number of Optimising Price Plans (the amount of price plans that the optimising operator has in the market for which the optimisation will be attempted in the chapters to follow, and for which the tariffs will become manipulated variables). In (3-13) the same notation for the state variables is used as has been used in (3-12), with *i* the price plan and *j* the market segment.

The Migrations Vector, $y_6[k]$ to $y_{10}[k]$ or $x_{18,1}[k]$ to $x_{18,5}[k]$ is calculated as follows with the same notation used as in (3-13):

$$x_{18,j}[k] = \sum_{i=1}^{NOP} \Delta x_{i,j}[k],$$

(3-14)

The revenue vector, $y_{16}[k]$ to $y_{20}[k]$ or $x_{20,1}[k]$ to $x_{20,5}[k]$ is calculated as follows with the same notation used as in (3-13):

$$x_{20,j}[k] = \sum_{i=1}^{NOP} ARPU_{i,j}[k] \cdot x_{i,j}[k]$$

(3-15)

The ARPU vector, $y_{11}[k]$ to $y_{15}[k]$ or $x_{19,1}[k]$ to $x_{19,5}[k]$ is calculated as the quotient of the Revenue and the Sum Vectors as follows with the same notation used as in (3-13):

$$x_{19,j}[k] = \frac{x_{20,j}[k]}{y_j[k]}$$

(3-16)

The average usage per user vector, $y_{21}[k]$ to $y_{25}[k]$ or $x_{21,1}[k]$ to $x_{21,5}[k]$ is calculated as the quotient of the total usage and the sum vectors as follows with the same notation used as in (3-13):

$$x_{21,j}[k] = \frac{x_{22,j}[k]}{y_j[k]}$$

$$\text{(3-17)}$$

The usage vector, $y_{26}[k]$ to $y_{30}[k]$ or $x_{22,1}[k]$ to $x_{22,5}[k]$ is calculated using the usage per subscriber on each price plan per market segment, $use_{i,j}[k]$, as follows with the same notation used as (3-13):

$$x_{22,j}[k] = \sum_{i=1}^{NOP} use_{i,j}[k] \cdot x_{i,j}[k]$$

$$\text{(3-18)}$$

From the above equations it is apparent that the amount of outputs of the model is $p = 6 \times 5 = 30$. Therefore, the whole non-linear state-space representation can be expressed as a 161 $\times$ 30 system with 110 internal states.

In Figure 3-4 below the different inputs and outputs of the model are depicted graphically and added to the representation of the model states and interaction as found in Figure 3-1. Figure 3-4 can also be seen as a representation of the control problem at hand because of the input and output identification (see chapter 6).
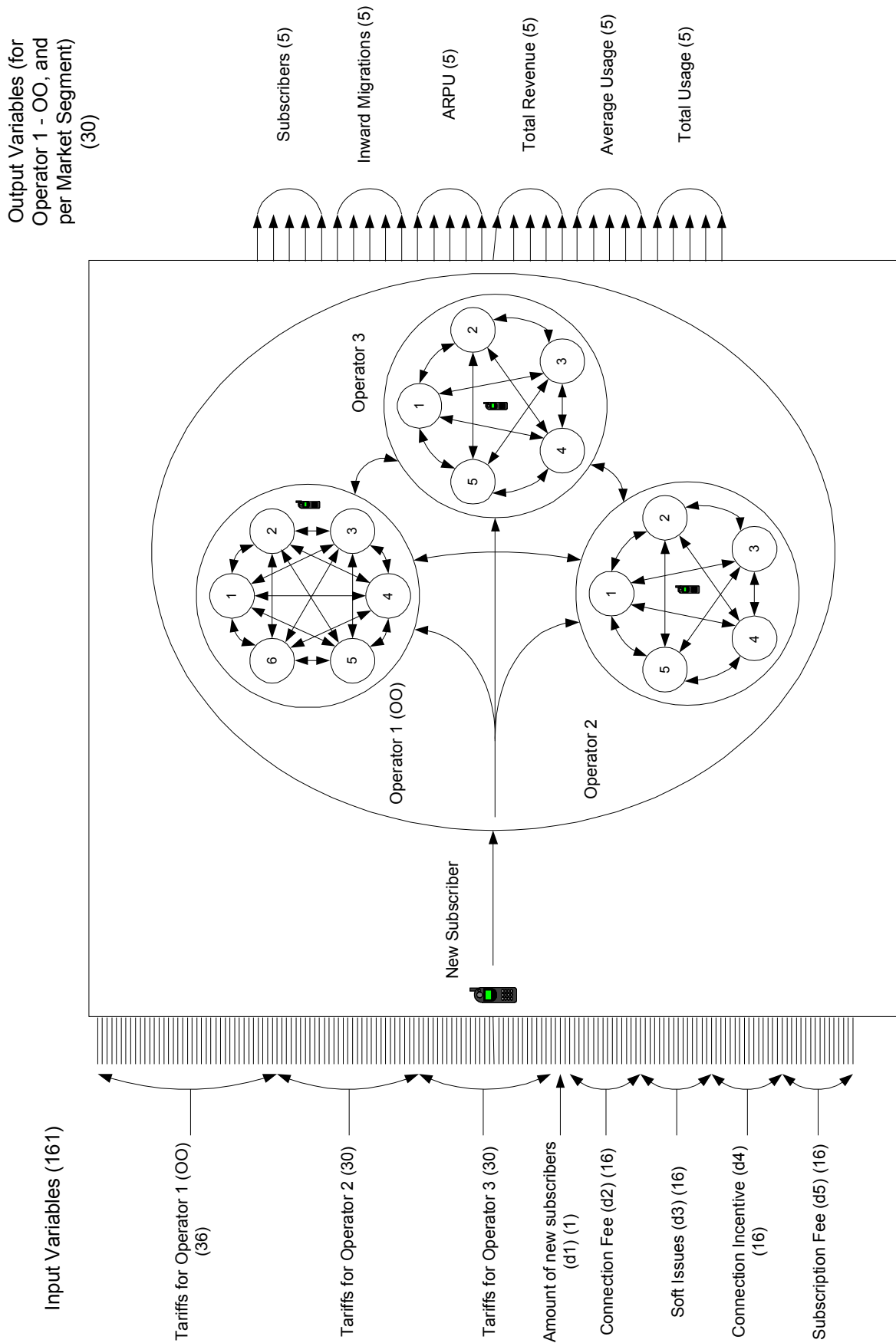
Figure 3-4: Schematic Representation of the Model with Inputs and Outputs

3.5   Conclusion

The model as documented in this section contains all parameters as used in the actual simulations in the chapters following this one, with the exception of tariffs and subscriber numbers. The figures used for these variables are not documented in this thesis in detail due to commercial sensitivity. All the graphs in the thesis contain the numbers as used in the simulations though.

Principles from all the sections in chapter 2 have been used in the derivation of the model as represented by (3-1) to (3-18) in this chapter. The model is sufficiently accurate for the purposes of the project as the states follow the actual data trends, especially in terms of the sign (positive or negative) of the influence of the inputs into the model (see chapter 4 and sections 4.1 and 4.2 for a comparison of the model and the industry data).

4   OPEN-LOOP SIMULATION

In this chapter the results of the simulation of the model discussed in chapter 3 are documented. Results are discussed (section 4.2) and conclusions are drawn (section 4.3) after the graphical results (section 4.1). Please note that the optimising operator is referred to as OO, and the other two operators as O2 and O3.

4.1   Results

The non-linear model is simulated by solving the differential equations with Euler integration. The simulation results of the open-loop non-linear model as developed in chapter 3, are displayed in this section in the form of trend graphs of key variables in the model over the 9 month simulation period. The first 6 graphs are that of the 6 price plans of the optimising operator. The 6 output graphs of the model will then be discussed followed by some comparison graphs between the model output and the actual data obtained. The trends of states and outputs of the model that are not given in this section, are shown in graphs in Appendix B. However, these trends will be discussed in this chapter.

In the following 6 graphs, the number of subscribers on each of the 6 price plans of the optimising operator are displayed. Each graph has 5 trends on it, one representing each market segment in the model, with the line style key as indicated on the graphs. Some of the graphs do not have a key on them, in such cases the key of the graph directly prior to the applicable graph applies.
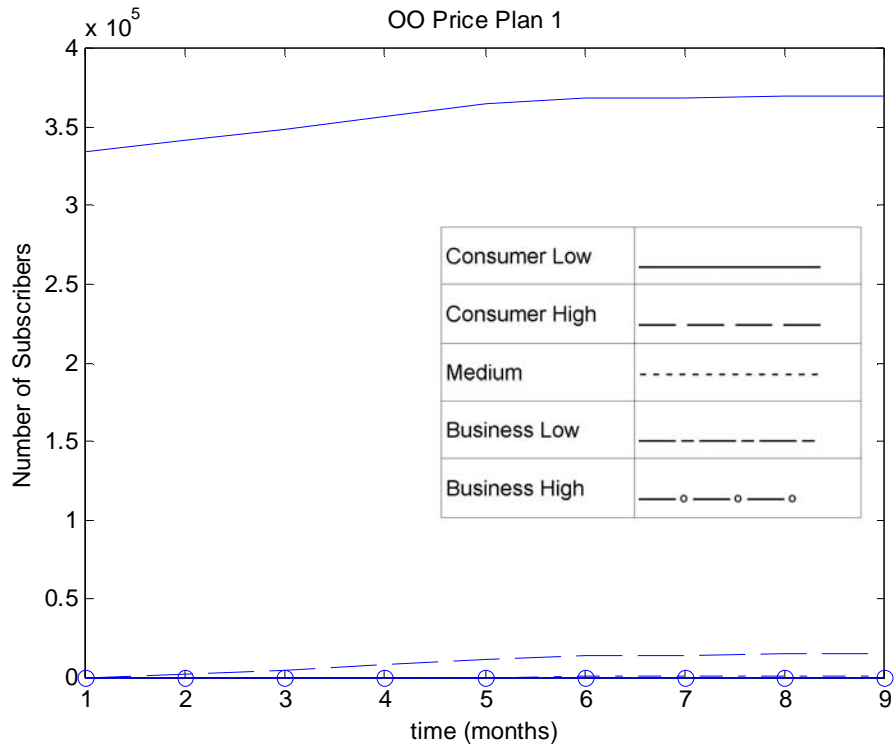
Figure 4-1: The Number of Subscribers for Price Plan 1 of OO
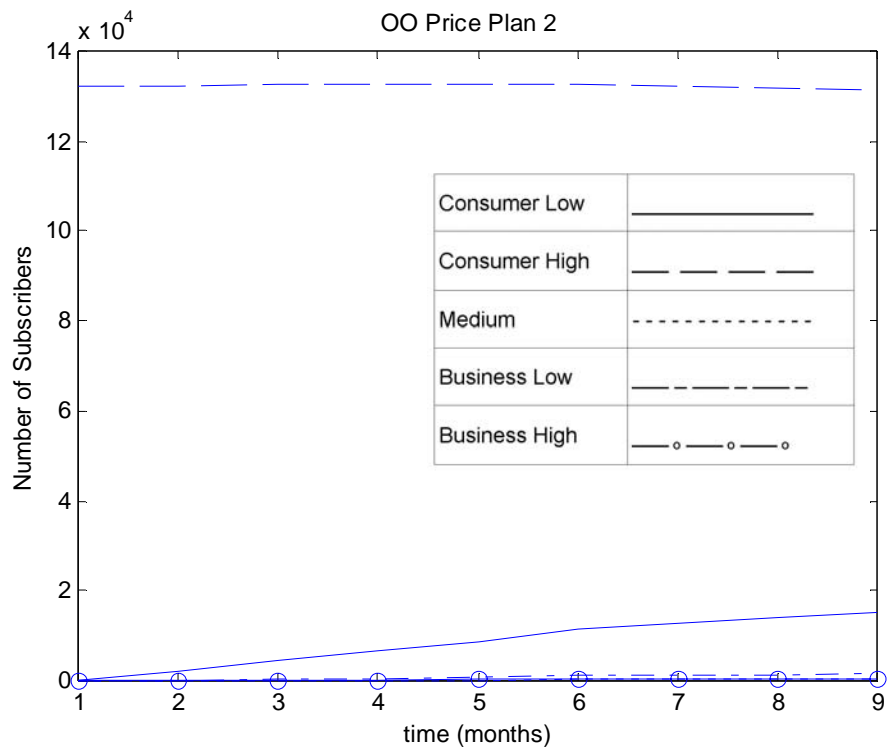
Figure 4-2: The Number of Subscribers for Price Plan 2 of OO

OO Price Plan 3

Consumer Low	————————
Consumer High	— — — —
Medium	- - - - - - - - -
Business Low	— – — – —
Business High	—o—o—o
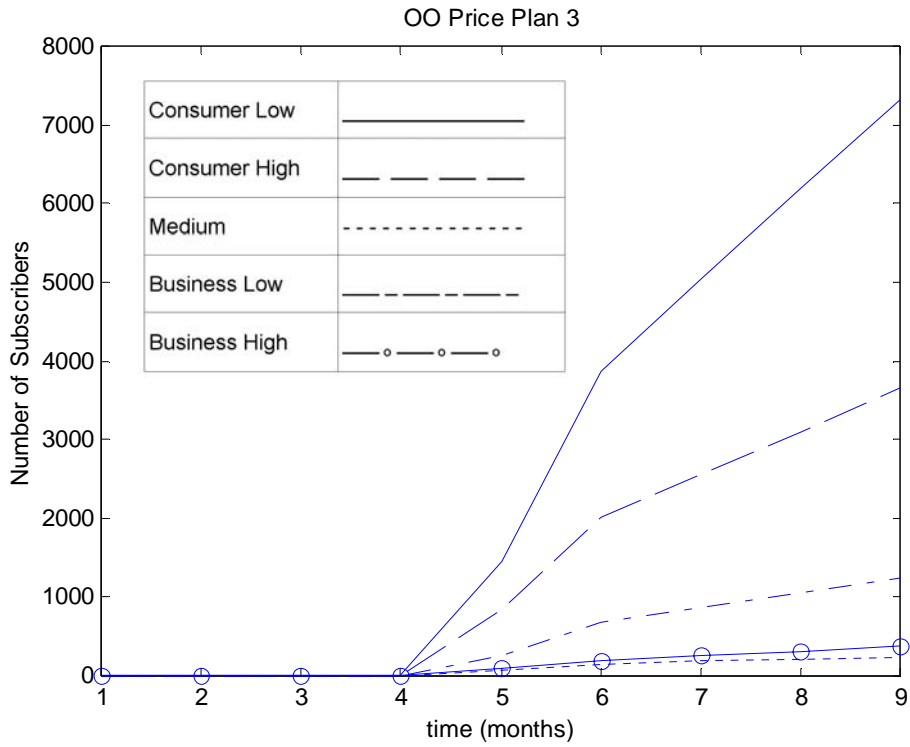
Figure 4-3: The Number of Subscribers for Price Plan 3 of OO

OO Price Plan 4

Consumer Low	————————
Consumer High	— — — —
Medium	- - - - - - - - -
Business Low	— – — – —
Business High	—o—o—o
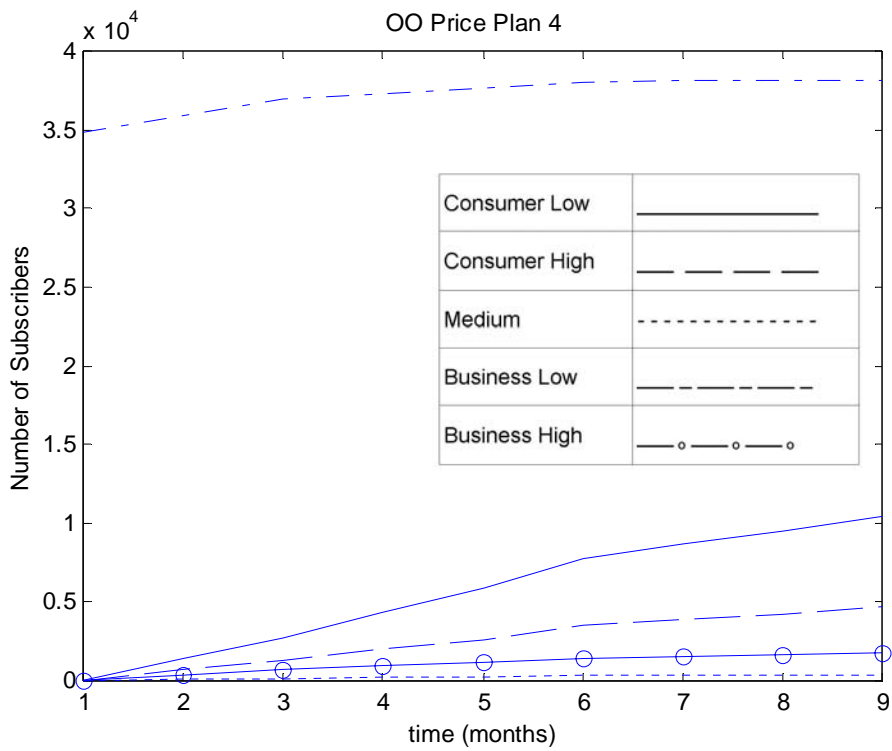
Figure 4-4: The Number of Subscribers for Price Plan 4 of OO
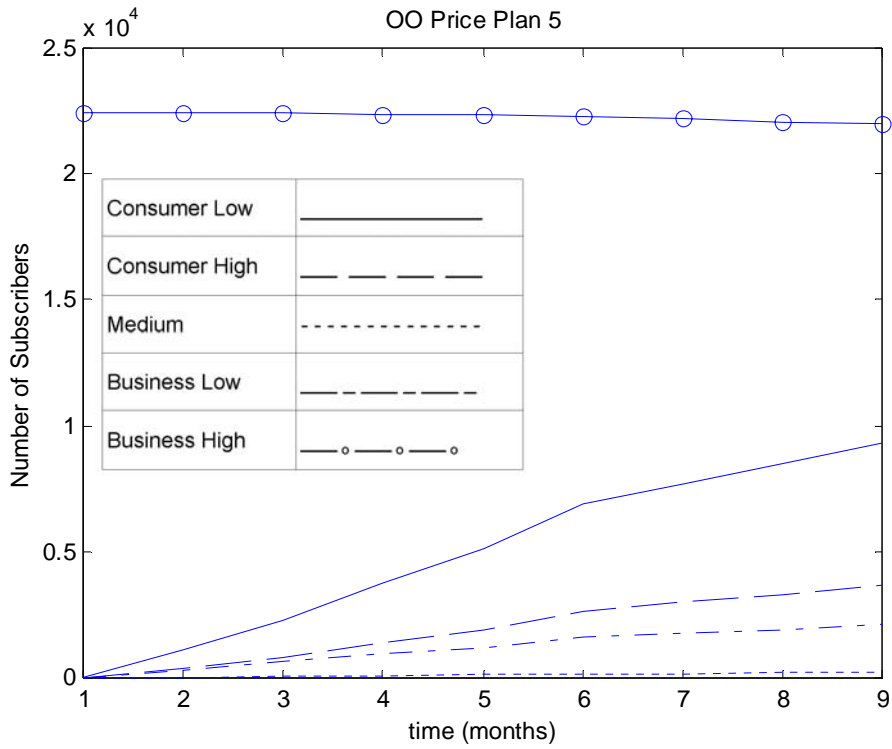
Figure 4-5: The Number of Subscribers for Price Plan 5 of OO
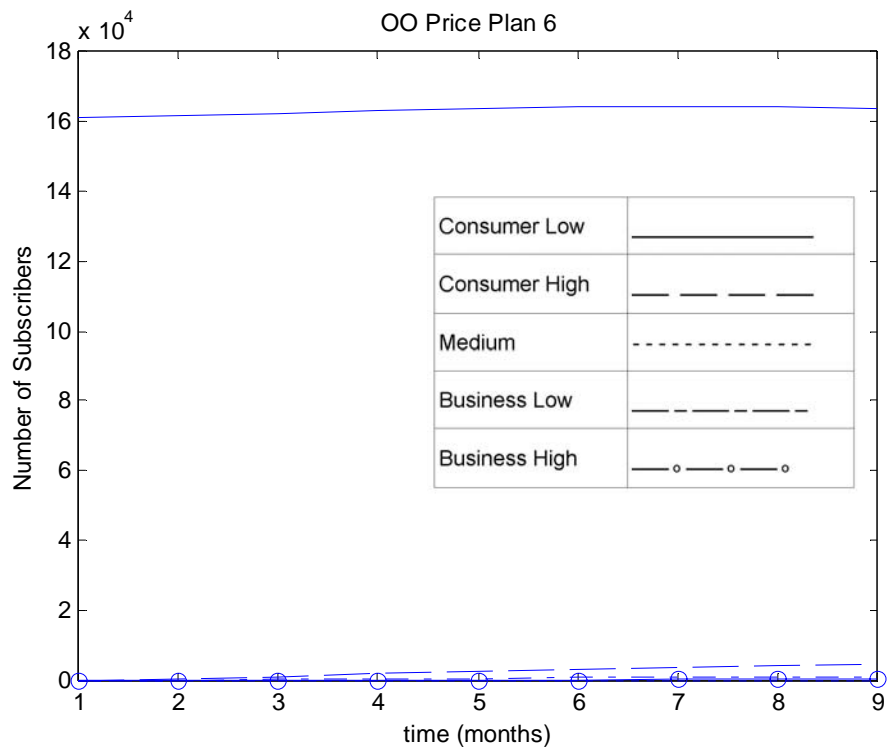
Figure 4-6: The Number of Subscribers for Price Plan 6 of OO

The next 6 graphs show the output variables of the model. Similar to the previous graphs, the different market segments are differentiated by different linestyles.
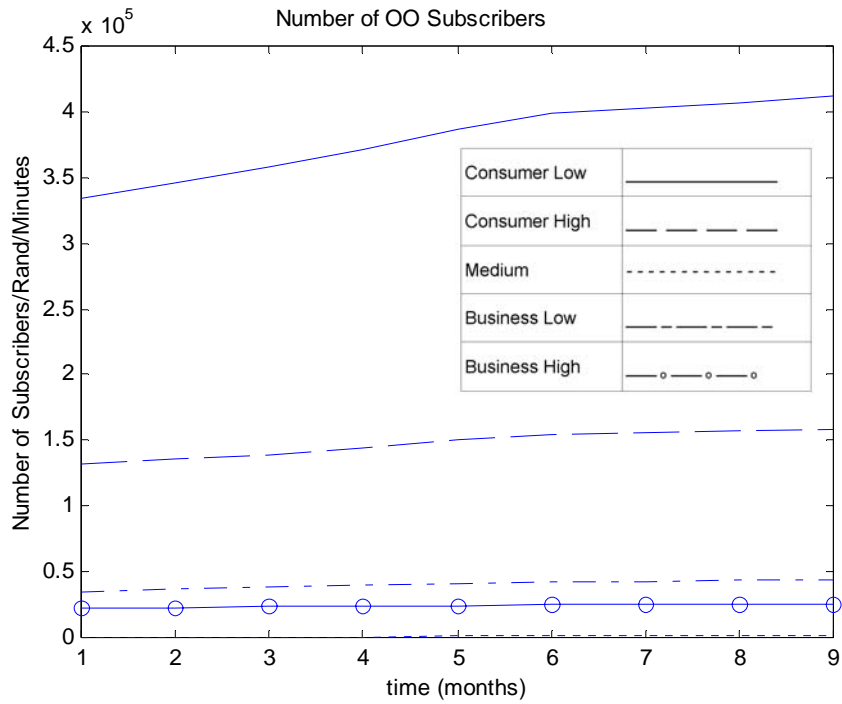
Figure 4-7: The Number of Subscribers per Market Segment for OO
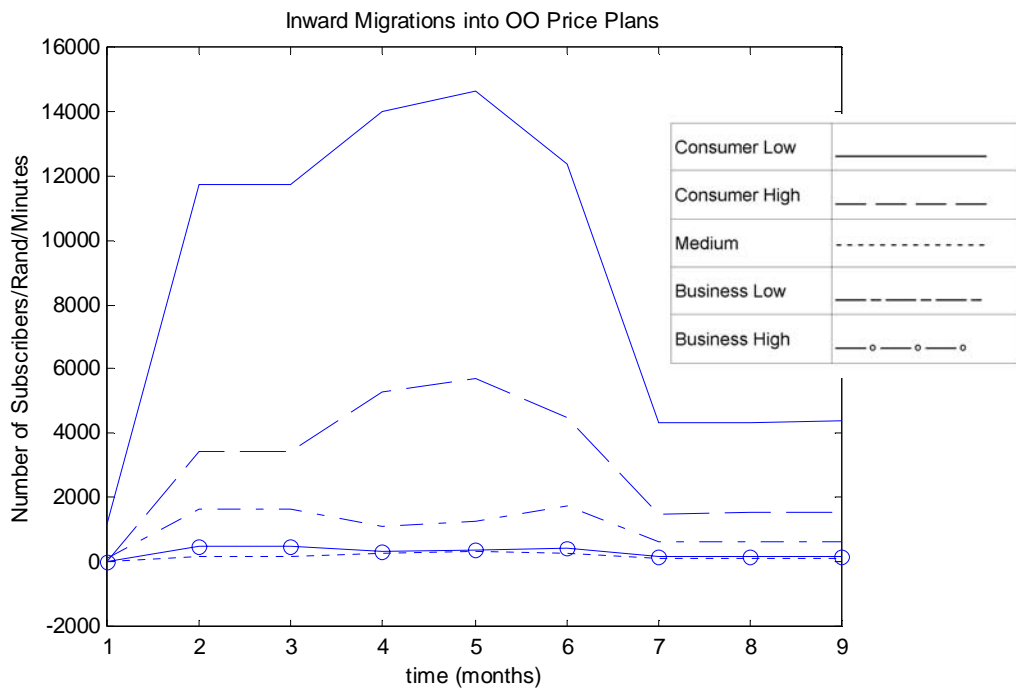
Figure 4-8: Gross Connections per Market Segment for OO

OO ARPU
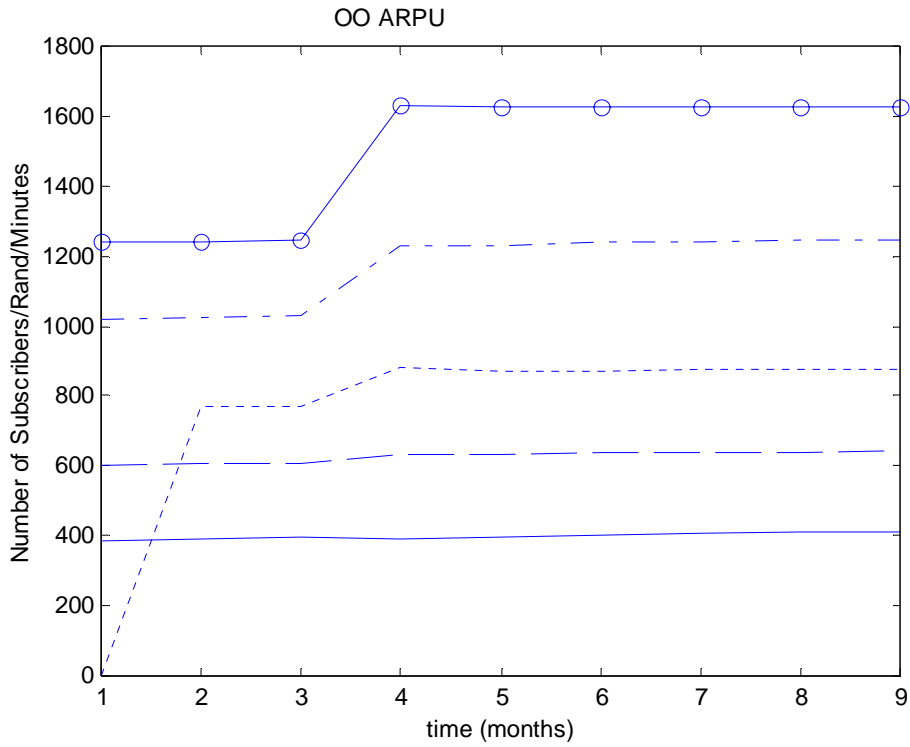
Figure 4-9: ARPU per Market Segment for OO

Total OO Revenue

Consumer Low	————————
Consumer High	— — — — —
Medium	················
Business Low	— · — · — · —
Business High	—o—o—o—
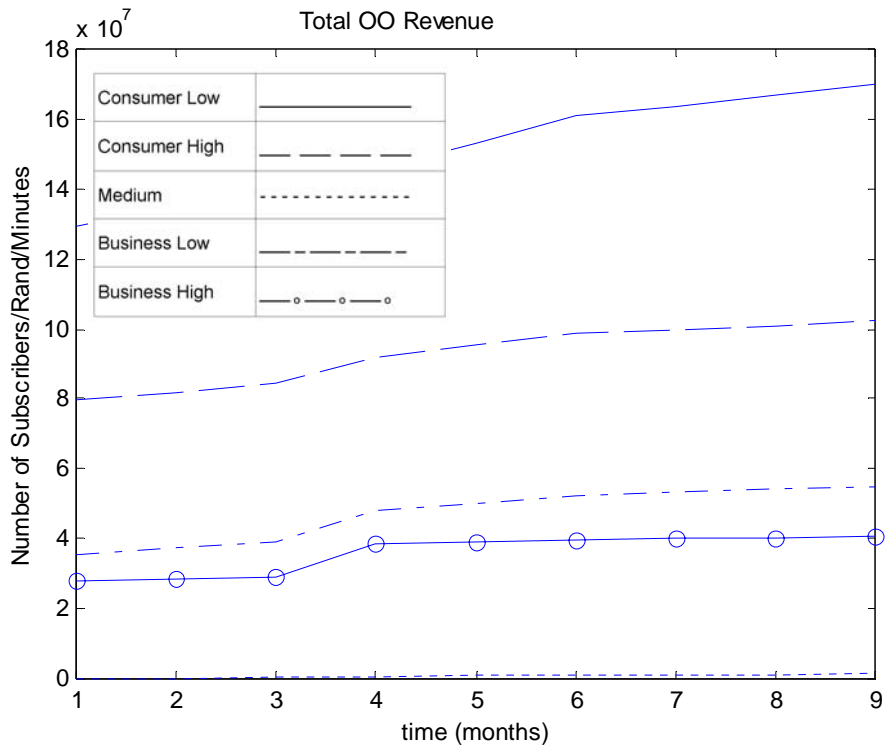
Figure 4-10: Total Revenue per Market Segment for OO
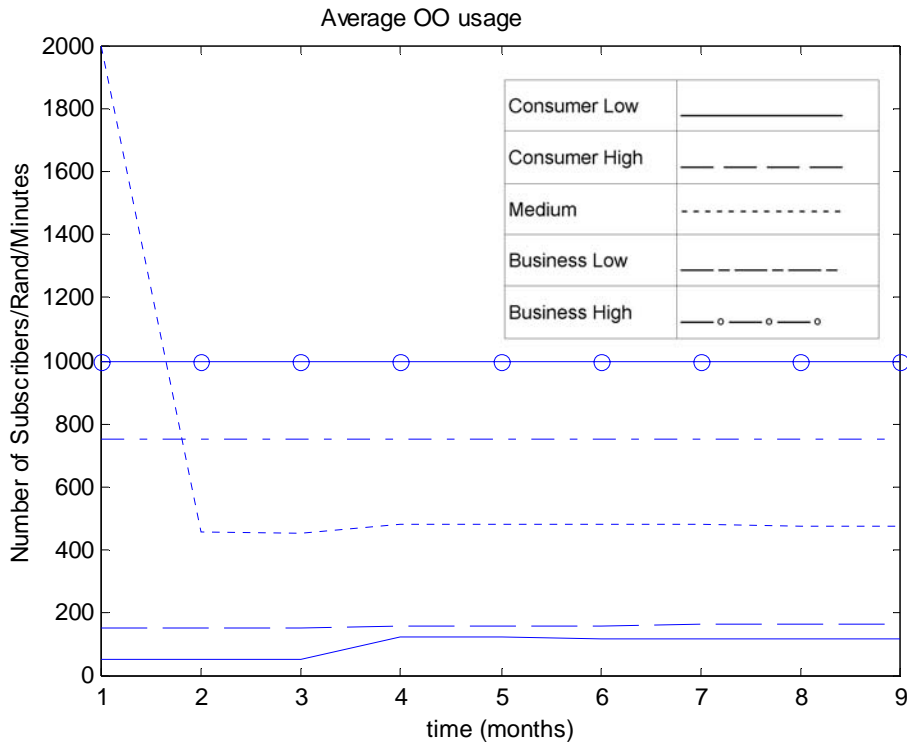
Figure 4-11: Average Usage per Market Segment for OO
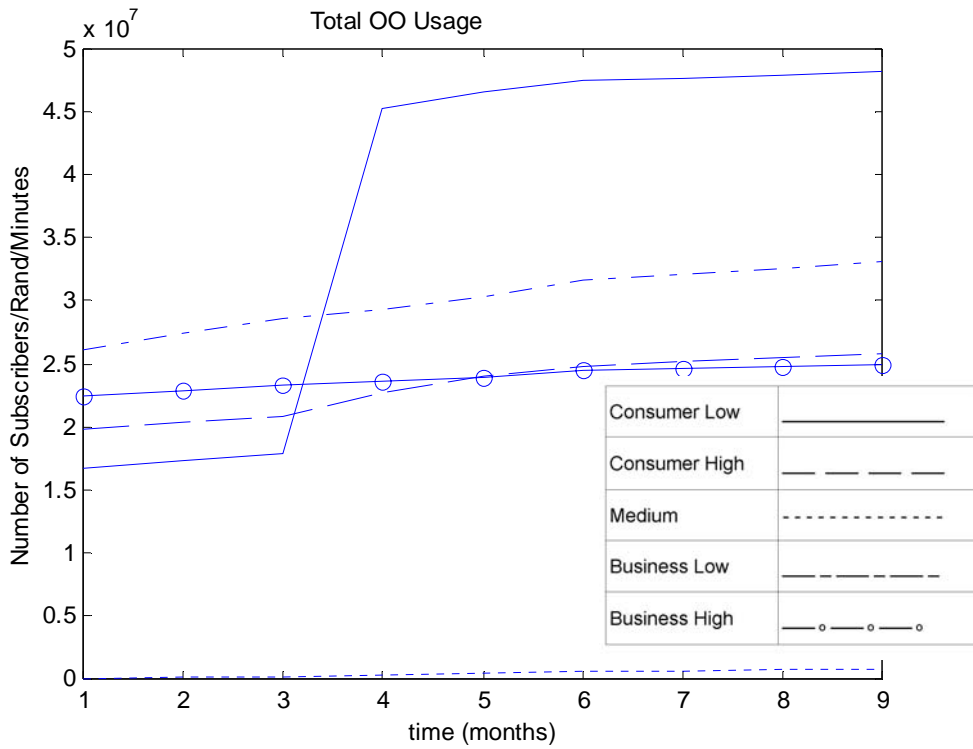
Figure 4-12: Total Usage per Market Segment for OO

In the next few graphs the actual data obtained for the simulation period (in the dotted line) are plotted against the model output for the total subscribers (across all market segments) on each price plan offered by the optimising operator.
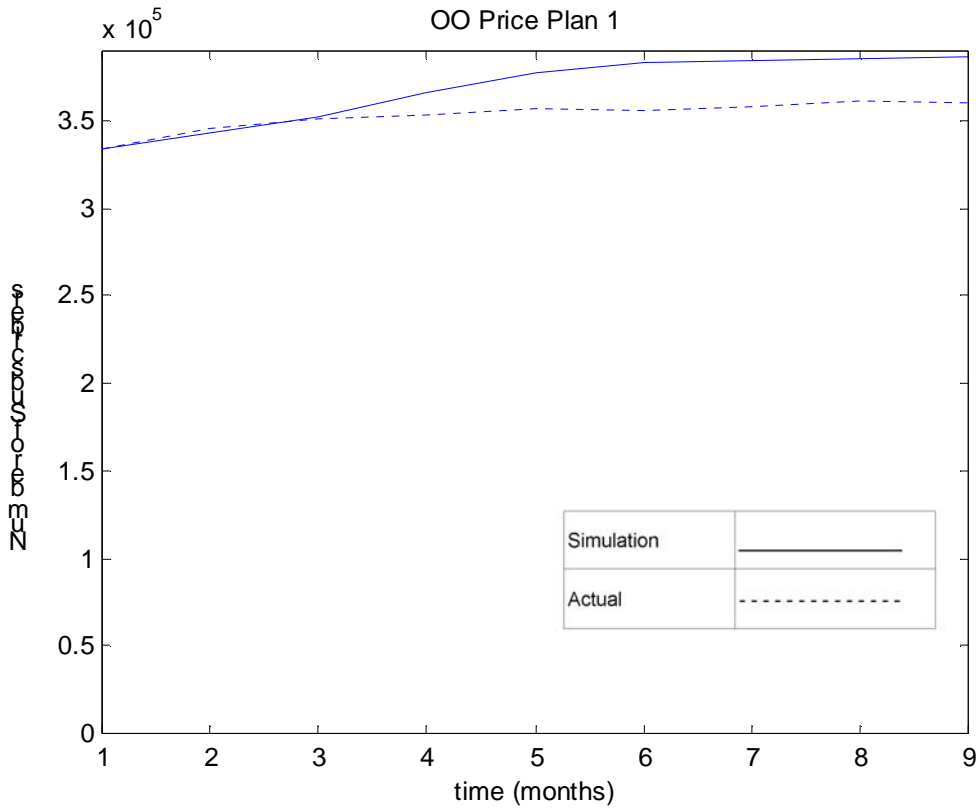
Figure 4-13: Total Subscribers on OO Price Plan 1: Model vs. Actual
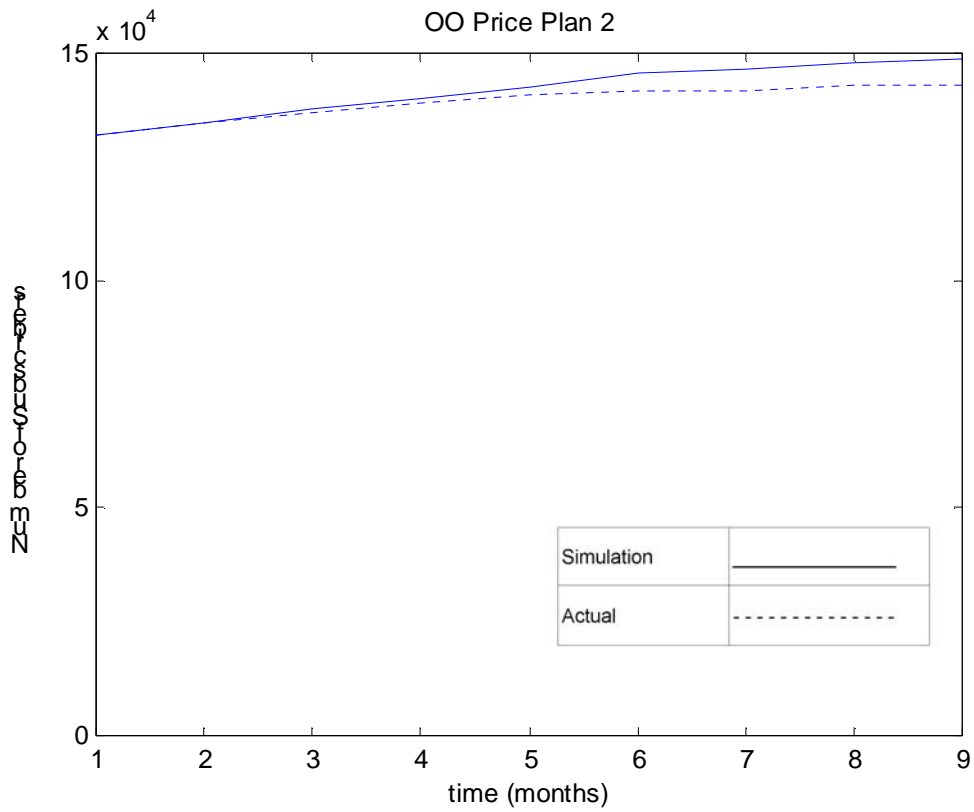
Figure 4-14: Total Subscribers on OO Price Plan 2: Model vs. Actual
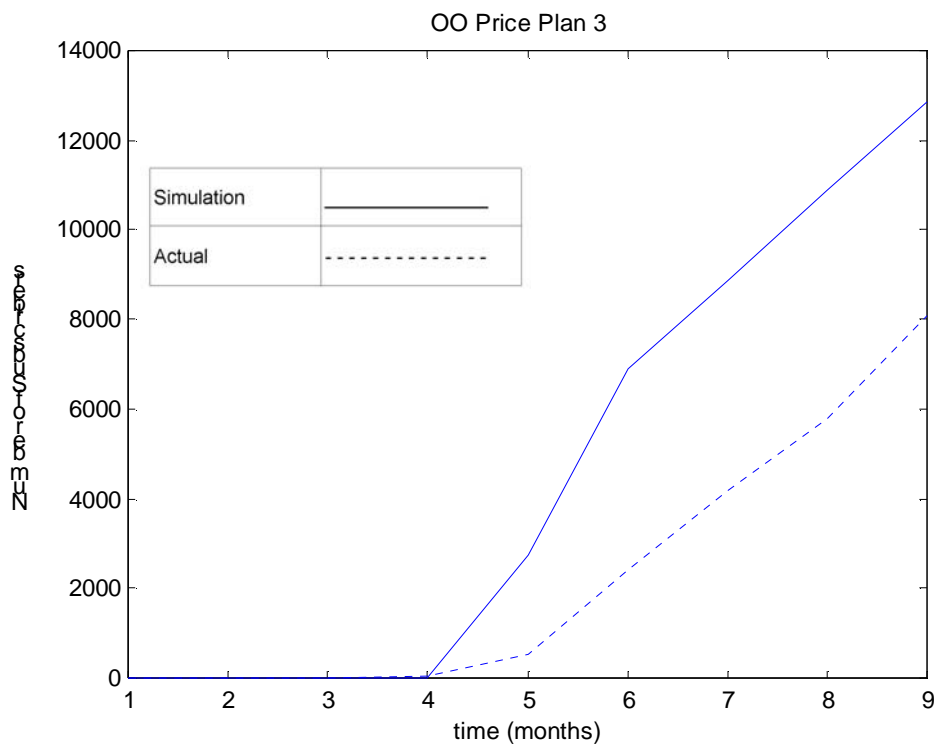
Figure 4-15: Total Subscribers on OO Price Plan 3: Model vs. Actual
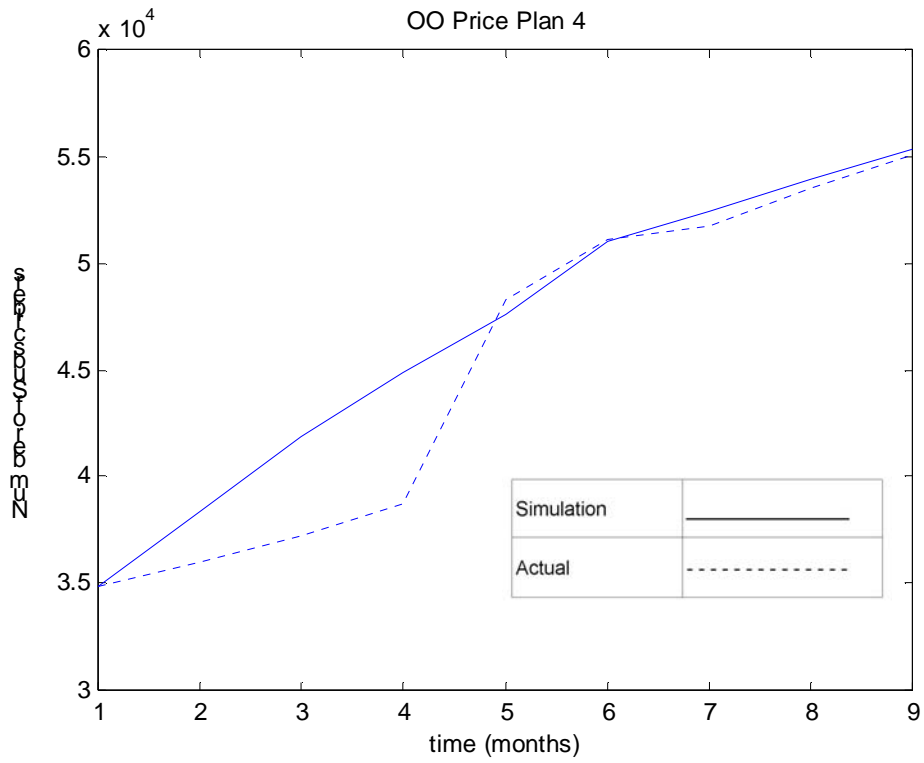
Figure 4-16: Total Subscribers on OO Price Plan 4: Model vs. Actual
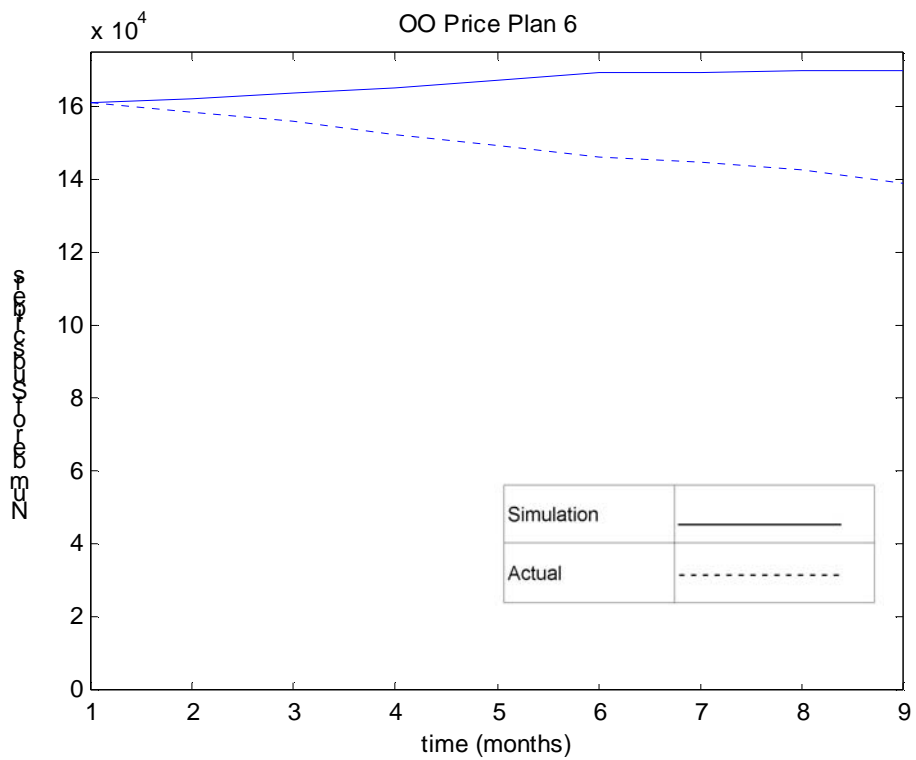
Figure 4-17: Total subscribers on OO price plan 6: Model vs. Actual

4.2 Discussion

In this section various observations concerning the results portrayed in the figures in section 4.1 are made:

1. In Figure 4-1 the number of subscribers on the price plan is steadily increasing mainly due to subscribers joining the market and opting for this price plan. The price plan was not adversely influenced by a tariff increase which occurred in this period, but the subscriber growth rate for the price plan did decrease and approach zero due the entry of the third operator towards the second half of the simulation period. Some of the trends of the market segments are not visible on the graph since those trends are clustered with the bulk of the trends at the bottom of the graph right on the x-axis.

2. In Figure 4-2 the different market segments were attracted differently to the price plan.

3. In Figure 4-3 above the launch of a new price plan is apparent.

4. In Figure 4-4 the influence of the price increase from the 3rd to the 4th month (of the simulation period) on the market segment with the most subscribers can be clearly seen. A significant amount of churn occurred because of the price increase.

5. In Figure 4-5 as well as in the previous 4 graphs the trend that one market segment is targeted by a particular price plan can be clearly seen from the amount of subscribers joining that price plan from a specific market segment when compared to the other segments. In Figure 4-5 the Business High market segment is very prominent in the subscriber numbers on the price plan – which is aimed at the high end of the income spectrum.

6. In Figure 4-6 it can be observed that the growth in the subscriber numbers on the particular price plan is much lower than the other price plans belonging to the OO. This is due to the fact that the operator is busy phasing this price plan out of the market by giving it a higher tariff than the others.

7. From Figure 4-7 it is clear that the low end of the market is the majority.

8. The impact of a tariff increase on the start of the 4th month and the 3rd operator entering in the 6th month is clearly discernable on this very strategic inward migrations (gross connections) output in Figure 4-8.

9. The very strategic output ARPU trends in Figure 4-9 clearly show the difference in income that differentiates the market segments. The Medium segment started of with zero as there were no subscribers in the market segment before the start of the simulation. The price plan that OO launched during the simulation period at the start of the 4th month was designed to target this segment in particular.

10. The positive impact of the tariff increases at the start of the 4th month is clearly discernable in the total revenue per market segment in Figure 4-10.

11. In Figure 4-11 above the medium segment starts off with a very high average usage because there are no subscribers in this segment at the start of the simulation (under the line in the calculation). The positive effect of the tariff increase on some segments can be seen in the graph as well.

12. The stimulating effect of the tariff increase is apparent on the total usage in Figure 4-12 above as well.

13. Similar observations can be made on the graphs for the other two operators as shown in section in Appendix B.

14. Figure 4-15 shows the greatest difference between the actual data and the open-loop simulation figures of all the actual vs. simulation graphs. The fact that the model over estimates the growth in the subscriber numbers is due to the complex non-linearities of the start-up of a price plan not being captured that well in the model.

4.3   Conclusion

The following conclusions can be drawn from the results of section 4.1 and 4.2:

1. The model correlates well with the actual data supplied as is evident in Figure 4-13 to Figure 4-17. It appears as if the signs of all the gains in the model are correct when looking at the tracking of the actual signal by the model output signal.

2. As can be seen from the line styles of the different market segments in the graphs above, each price plan is aimed at a particular market segment and attracts the majority of its subscribers from that market segment.

3. The model responds adequately to the main inputs, i.e. the tariff values (which were changed once in the open-loop simulation) as well as the growth in industry size.

4. The interaction between the different states in the model gives sound results since the entrance of the third operator into the market had the desired influence on the other states.

5. The good forecasting results produced by the model shows that a system dynamics approach to the forecasting problem in telecommunications markets is a viable and probably very good approach when applied and implemented in the correct manner. This has been an area of research to be addressed and given more priority according to the literature (see section 2.5), as it has not been proven to be viable in past research due to insufficient work done in the field.

6. The model succeeds in forecasting the uptake of 6 new price plans in the market with good accuracy – that of the first price plan in the Medium market segment for the optimising operator, and the 5 new price plans launched by the third operator when it entered into the market. The effect of new products and services are very difficult to forecast according to the literature (see chapter 2 above). The results obtained here are therefore potentially invaluable.

7. The non-linear effects of the start-up of new price plans are not modelled very accurately.

5   CONTROL BACKGROUND

In this section background to Model Predictive Control (MPC) and the implementation of optimisation techniques to problems similar to the one attempted in this research, as found in the literature, are reported on.

5.1   Model Predictive Control Background

The typical MIMO model predictive control feedback control system in its simplest configuration is shown in Figure 5-1 ([30] and [46]).
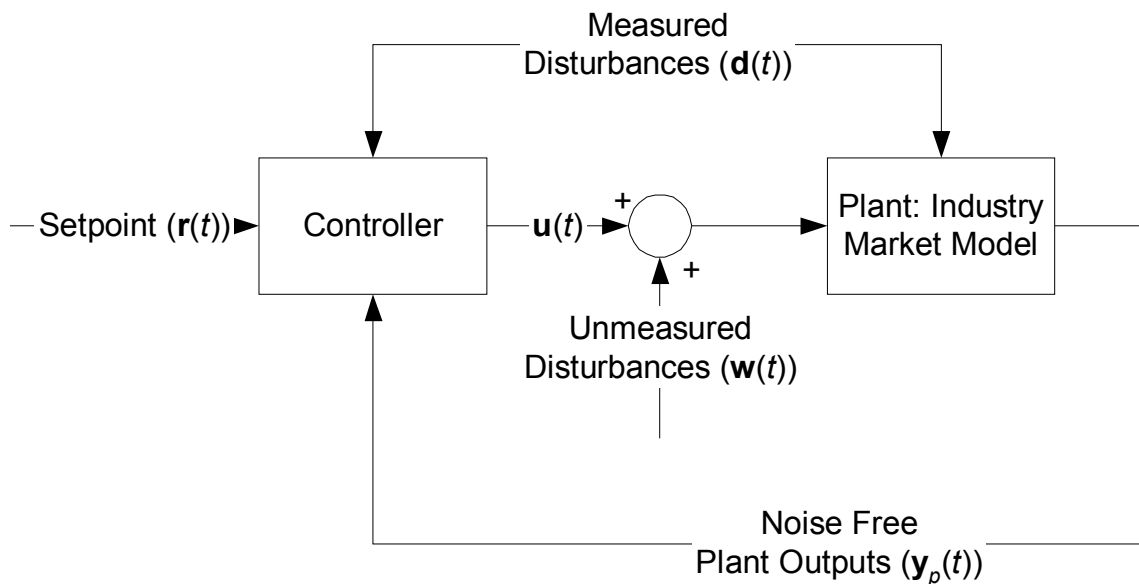
Figure 5-1: The Typical MPC Feedback Control System

In Figure 5-1 the vector signals $\mathbf{r}(t)$, $\mathbf{u}(t)$, $\mathbf{d}(t)$ and $\mathbf{y}_P(t)$ are the setpoint, control signal, disturbance inputs and the plant output signals respectively. For the purposes of this research, attention will be focused on an MPC control approach.

MPC yields the optimisation of a performance index with respect to some future control sequence, using predictions of the output signal based on some process model [47]. MPC has been very successful as a control strategy in modern control. Its main advantages include the following:

1. MPC provides high performance controllers that can easily be applied to difficult high–order and multivariable processes [47].

2. State control in the presence of constraints can be handled in a simple and systematic way, and controllers for such applications are easy to tune [47].

3. A large number of industrial examples and applications have been documented ([19] and [47]).

MPC can be expressed mathematically in different notations (e.g. state-space or frequency domain). However, since the model (see chapter 3) is expressed in state-space format, a state-space formulation has been pursued in this chapter as well.

MPC is a control strategy based on solving an online optimal control problem. The receding horizon approach is used. The latter can be summarised in the following steps [19]:

1. At time k and for the current state $\mathbf{x}(k)$, solve, on-line, an open-loop optimal control problem over some future interval, taking into account the current and future constraints.

2. Implement the first step in the optimal control sequence in the plant.

3. Repeat 1. and 2. above at time $(k+1)$, using the current state $\mathbf{x}(k+1)$.

The solution is converted into a closed-loop strategy by using the measured value of the state (obtained by direct measurement or by means of an observer) as the current state. Assume that $\mathbf{x}(k)$ is known, then in a general, discrete (MPC is as applicable to continuous systems) non-linear setting [19]:

Given a model

$$\mathbf{x}(l+1) = \mathrm{f}(\mathbf{x}(l), \mathbf{u}(l)), \qquad \mathbf{x}(k) = \mathbf{x},$$

(5-1)

where $\mathbf{x}$ is the initial state vector, and k the initial time, the MPC control output at event $(\mathbf{x}, k)$ is computed by solving a constrained optimal control problem:

$$P_N(\mathbf{x}): \quad V_N^0(\mathbf{x}) = \min_{\mathbf{U} \in \mathbf{U}_N} V_N(\mathbf{x}, \mathbf{U}),$$

where

$$\mathbf{U} = \{\mathbf{u}(k), \mathbf{u}(k+1), ..., \mathbf{u}(k+N-1)\}$$

$$V_N(\mathbf{x}, \mathbf{U}) = \sum_{l=k}^{k+N-1} L(\mathbf{x}(l), \mathbf{u}(l)) + F(\mathbf{x}(k+N))$$

(5-2)

and $\mathbf{U}_N$ is the set of $\mathbf{U}$ that satisfies the constraints over the entire interval $[k, k+N-1]$:

$$\begin{array}{ll} \mathbf{u}(l) \in U & l = k, k+1, ..., k+N-1 \\ \mathbf{x}(l) \in X & l = k, k+1, ..., k+N \end{array}$$

(5-3)

together with the terminal constraint

$$\mathbf{x}(k+N) \in \mathbf{W}.$$

(5-4)

Usually, $\mathbf{U} \subset \mathbf{R}^m$ is convex and compact, $\mathbf{X} \subset \mathbf{R}^n$ is convex and closed, and $\mathbf{W}$ is a set that can be selected appropriately to achieve stability.

In the above formulation the model and cost function (or performance index [47]) are time invariant. Hence, one obtains a time-invariant feedback control law. In particular, we can set $k = 0$ in the open-loop control problem without loss of generality. Then, at event $(\mathbf{x}, k)$, we solve [19]:

$$P_N(\mathbf{x}): \quad V_N^0(\mathbf{x}) = \min_{\mathbf{U} \in \mathbf{U}_N} V_N(\mathbf{x}, \mathbf{U}),$$

where

$$\mathbf{U} = \{\mathbf{u}(0), \mathbf{u}(1), ..., \mathbf{u}(N-1)\}$$

$$V_N(\mathbf{x}, \mathbf{U}) = \sum_{l=0}^{k+N-1} L(\mathbf{x}(l), \mathbf{u}(l)) + F(\mathbf{x}(N))'$$

(5-5)

subject to the appropriate constraints. Standard optimisation methods (such as Quadratic Programming - QP [19]) are used to solve the above problem.

Let the minimising control sequence be [19]

$$\mathbf{U}_{\mathbf{x}}^0 = \{\mathbf{u}_{\mathbf{x}}^0(0), \mathbf{u}_{\mathbf{x}}^0(1), ..., \mathbf{u}_{\mathbf{x}}^0(N-1)\}.$$

(5-6)

Then the actual control applied at time k is the first element of this sequence, i.e.

$$\mathbf{u} = \mathbf{u}_{\mathbf{x}}^o(0)$$

(5-7)

The above steps are then repeated for all time instants in the time interval for which control is applied. The amount of control moves that the controller plans into the future during each iteration is called the control horizon (N_C), and the amount of time intervals that the system output is predicted into the future in response to the control moves, is called the prediction horizon (N_P) ([21] and [47]).

As an example of the above, Figure 5-2 explains the idea of receding horizon. At time k the future control sequence {$\mathbf{u}(k)$, $\mathbf{u}(k + N - 1)$} is optimised such that the future output signal $\mathbf{y}(k + j)$ is as close as possible to reference signal $\mathbf{r}(k+j)$, measured in the performance-index $V_N(\mathbf{x}, \mathbf{U})$ and subject to constraints. At time k only the first element of the optimal sequence is applied to the real process, and the horizon is shifted and a new optimisation at time $k + 1$ is done [47].
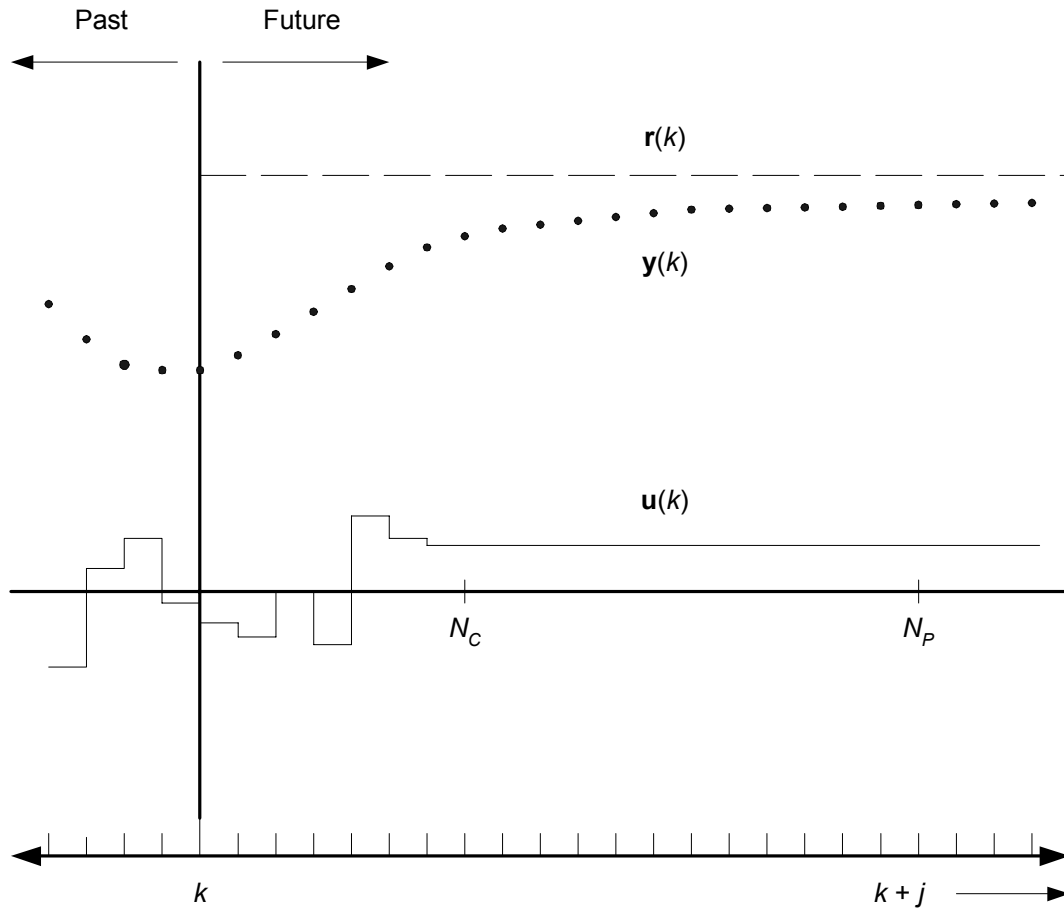
Figure 5-2: The Receding Horizon Concept in Predictive Control [47]

A remarkable property of MPC is that one can establish stability of the resultant feedback system, at least with full state information. This is made possible by the fact that the value function of the optimal control problem acts as a Lyapunov function for the system [19].

In the case of a quadratic cost function, the optimal control problem can be expressed in the following way [19]:

$$J_0 = [\mathbf{x}(N) - \mathbf{x}_s]^T \boldsymbol{\Psi}_f [\mathbf{x}(N) - \mathbf{x}_s] + \sum_{l=0}^{N-1} \mathbf{e}^T(l) \boldsymbol{\Psi} \mathbf{e}(l) + \sum_{l=0}^{M-1} [\mathbf{u}(l) - \mathbf{u}_s]^T \boldsymbol{\Phi} [\mathbf{u}(l) - \mathbf{u}_s]$$

$$(5-8)$$

where $\mathbf{e}(l)$ is the difference between the output and the setpoint, and the constant weighting matrices $\boldsymbol{\Psi} \geq 0, \boldsymbol{\Phi} > 0, \boldsymbol{\Psi}_f \geq 0$.

5.2   Optimisation Theory applied to Management Problems

In this section a very short survey of some of the literature on the area of dynamic modelling applied to price optimisation will be reported on.

Knowledge support systems have been successfully applied to management problems, especially to pricing [5].   Typically expert systems are applied at the lower levels of a company structure, where decision support systems are used at the top strategic levels. Senior and middle management levels in between would use a hybrid of the two types of systems [5].   Prices are typically determined by senior management in the marketing departments of companies if they are not set by the buyers or regulators.   Management is becoming more willing to use "smart" pricing techniques to enable better performance in a competitive environment.

Decision makers who lack an adequate model of the dynamics of a problem cannot apprehend the shortcomings of their actions and run the risk of propagating and perpetuating flawed decisions [26].   Any rational analysis for price setting in a company needs to take into account the demand, cost structures, competitor prices and offers and the company's own strategic objectives [5].   It is also important for companies in the economy of today to not only look at the pricing of one of their products in isolation, but to do effective portfolio pricing where a whole portfolio is considered.   Sing [5] has successfully developed a control theory based system called Price Strat which solves an optimisation problem for companies and advises them on the prices to implement based on the above-mentioned ideas.   He has implemented the system in the petrochemical, banking and automobile industries.

In [4] on the other hand, a combination of dynamic optimisation and decision calculus techniques have been applied to the problem of finding optimal marketing strategies in the telecommunications industry.   The models generated in this study are very robust and also model the market to a considerable level of detail (including elasticity).   The models seem to have a limited view of the market, though, and the whole portfolio of products when compared to the multivariable control approach in [5].

MPC has proven itself as a viable tool to use for a robust, flexible decision framework for dynamic management problems in areas and industries other than telecommunications. For example MPC has been proven to manage semiconductor supply chains in a robust manner [31].

The field of system dynamics has been mentioned in section 2.5. The field essentially has its roots in classical feedback control, and the techniques used are very successful in analysing and simulating industrial systems, including markets. Trends such as cyclicality can be uncovered easily. The strength of the systems dynamics approach in essence lies in the fact that real markets are not static (as many assume) but rather dynamic and complex; and that such complexities can only be accurately addressed by dynamic modelling of the systems under study [13]. Such modelling forms the basis for the implementation of feedback control.

As stated in the chapter 1, there is a considerable lack of literature on the topic of real-time dynamic optimisation in industrial markets. This is especially true of the telecommunication industry. Much use has been made of consumer surveys, as in [6], but tools such as those presented by Sing [5] are very scarce.

6   CONTROLLER DESIGN

6.1   Model Linearisation

In order to apply MPC, a linear time invariant (LTI) model was defined for use as internal model in the controller. The model (as developed in chapter 3) was linearised by making use of standard linear control theory [20]. A generic linear representation of the model is given in (6-1):

$$\dot{\mathbf{x}}(t) = \mathbf{A}\,\mathbf{x}(t) + \mathbf{B}\,\mathbf{u}(t) \quad \mathbf{x}(0) = \mathbf{x}_0$$
$$\mathbf{y}(t) = \mathbf{C}\,\mathbf{x}(t) + \mathbf{D}\,\mathbf{u}(t)$$

(6-1)

where $\mathbf{x} \in \mathrm{R}^n, \mathbf{u} \in \mathrm{R}^m, \mathbf{y} \in \mathrm{R}^l$, $\mathbf{A} \in \mathrm{R}^{n \times n}, \mathbf{B} \in \mathrm{R}^{n \times m}, \mathbf{C} \in \mathrm{R}^{p \times n}, \mathbf{D} \in \mathrm{R}^{p \times m}$, t is the continuous time variable, n is the number of states in the model, m is the amount of inputs into the model and p is the amount of outputs of the model.

As in section 3.4 the continuous linear model is sampled at discrete intervals for all simulations performed. Therefore, the model is referenced with the discrete time variable k. The continuous-time model was discretised using the matrix exponential method in Matlab. The [·] notation is used to denote discrete time.

The amount of inputs into the model for which it was linearised was considerably decreased from the total in the non-linear model (see Figure 3-4). Only the tariffs of 5 of the optimising operator's price plans, i.e. 30 variables, were included as manipulated variables in the linear model. The balance of the inputs were decreased to 5 measured disturbances only – the amount of new subscribers joining the addressable market per month per market segment. The reason for all the other market inputs not being included in the linear model is because of the fact that these variables stayed constant over the simulation period to be used for the controller simulation (control simulation period). These variables could thus be excluded from the linear model. Therefore the linear model has 35 inputs in total, of which 30 are manipulated and 5 are measured disturbances. The

non-linear plant model is however still used in the closed-loop simulation, with all the inputs still fed with their proper time series.

A model is usually linearised around an operating point where the resulting linear model is considered to be a good approximation of the non-linear model [20]. This operating point was calculated from the average value of all the states, outputs and inputs, as obtained over the control simulation period of the non-linear open-loop simulation.

The model was linearised by making use of a numeric approach. The same methodology described in [20] was used: All the states and inputs (both disturbances and manipulated variables) were increased by a small negligible factor (0.00001) one by one, and the subsequent change in state derivatives and outputs logged. The change in state derivative or output was then divided by the change in input or state, and the resulting quotient used for the appropriate element of the **A**, **B** or **C** matrix. The **D** matrix was set to zero (30 × 35).

In Appendix A the linear model as linearised, is displayed. Since all the matrices (**A**, **B** and **C**) have large dimensions, only parts of it could be fitted onto one page. Therefore the matrices are presented in broken down fashion in Appendix A. The **A** matrix is block diagonal, due to the exclusivity of the different market segments. States in different market segments do not influence one another because of subscribers not migrating between different segments. The states are clustered in blocks according to market segment.

The **B** matrix is of dimension 110 × 35.

The **C** matrix (30 × 110) is very sparsely populated because of the majority of the outputs being states, and thus resulting in only a few elements of the **C** matrix being non-zero and equal to 1. The first output of each market segment, however, is the sum of the states of the number of subscribers belonging to the price plans in that market segment for the optimising operator. The corresponding row for the output matrix **C** therefore has a series of 5 ones in it.

The **D** matrix is a 30×35 matrix of zeros.

6.2   Model Analysis

The linear model discussed above has the majority of its poles in the Open Left Half-Plane (OLHP), but there are 10 poles at zero. This implies that the open-loop linear model is unstable. These poles are zero mainly due to states that are totally insensitive to change in, or the value of, other states. The states being referred to are the amount of new subscribers that have joined each market segment from outside the total market (the 17th state in each block in Table A-1 in Appendix A). These states are only influenced by the disturbance variable d_1 (see section 3.4). Furthermore, these states do not influence any other state in the system, but are just used as place holders in the model. Therefore the instability caused by these states are not serious due to the reason for the poles being zero being the structure of the model, rather than a more fundamental characteristic of the system. Furthermore, the non-negative poles are zero and not positive [28]. All the poles of the model are listed in Table A-49 in Appendix A.

The model has only two zeros, both finite and at –0.0078. Therefore, there are no non-minimum phase zeros in the model.

Controllability and observability analysis gave the following results:

1.   Rank of the controllability matrix: 49.

2.   Rank of observability matrix: 51.

The model is therefore neither controllable nor observable. MPC, though, has the ability to move unstable poles to the LHP and does not require full controllability or observability.

The frequency response analysis of the linear model revealed that the majority of input-output pairings (30 outputs, 35 inputs) followed a response similar to that of the 1st output (number of the optimising operator's subscribers per Subscriber Low Market Segment) and the 1st input (tariffs for the first price plan of the optimising operator, during peak time for on-net termination) pairing displayed in Figure 6-1 below. Note that the time unit used in

the denominator of the frequency measurement is per month for the model.   That is because the state-space differential equations were developed with the assumption that the time rate of change will be measured per month due to the availability of actual data in the time series at that frequency, but also the inherent characteristic of the industry to measure all market parameters per month.   Also note that the vertical axis for Figure 6-1 to Figure 6-3 are expressed in linear units.
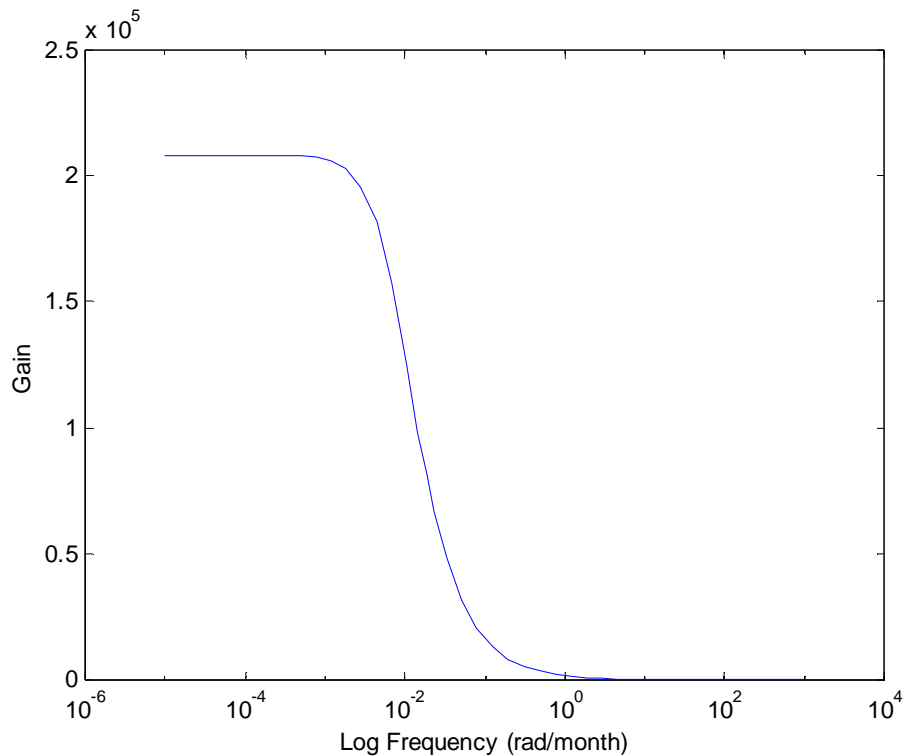
Figure 6-1: The Frequency Response of the 1-1 Input-output Pairing

The highest frequencies were found in a couple of input-output channels where a more complex response manifested. All these responses looked similar to that of the 6th output (the inward migrations into the optimising operator's price plans in the consumer low market segment) and the first input as shown in the Figure 6-2 below.
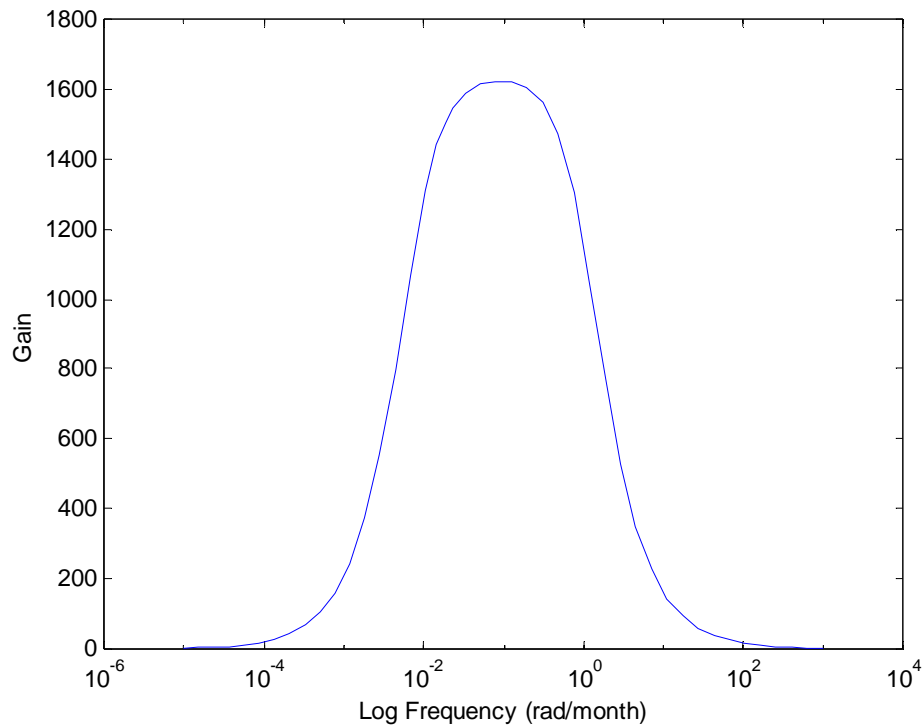
Figure 6-2: The Frequency Response of the 1-6 Input-Output pairing

For a band pass system (Figure 6-2) the bandwidth is defined as the range of frequencies over which the amplitude response remains within $1/\sqrt{2}$ times its value at the mid-band frequency [22]. From Figure 6-2 it is apparent that the cut-off frequency and bandwidth in this case is $10^0 = 1$ rad/month. This was the case for all the responses – the maximum bandwidth in the model is 1 rad/month. That being the case, the Nyquist frequency was calculated to be $1/\pi$ per month [32]. Therefore the minimum sampling rate, and subsequently the minimum simulation time increment as well as controller simulation time increment is π months. Because of the inherent nature of the model it was decided to make the subsequent simulation time increment 1 month. This was the rate that was used in the open-loop non-linear simulation as well. The chosen sampling rate is therefore π times greater than the Nyquist frequency, but smaller than the standard control practice of sampling at not less than 10 times the Nyquist frequency. It is therefore possible that some dynamics might be lost.

It was only the migration vector outputs that showed the band pass frequency response of Figure 6-2 in the input channels of the tariffs. A few other channels showed a response

similar to the one in Figure 6-2, but with less aggressive attenuation of low frequency inputs:
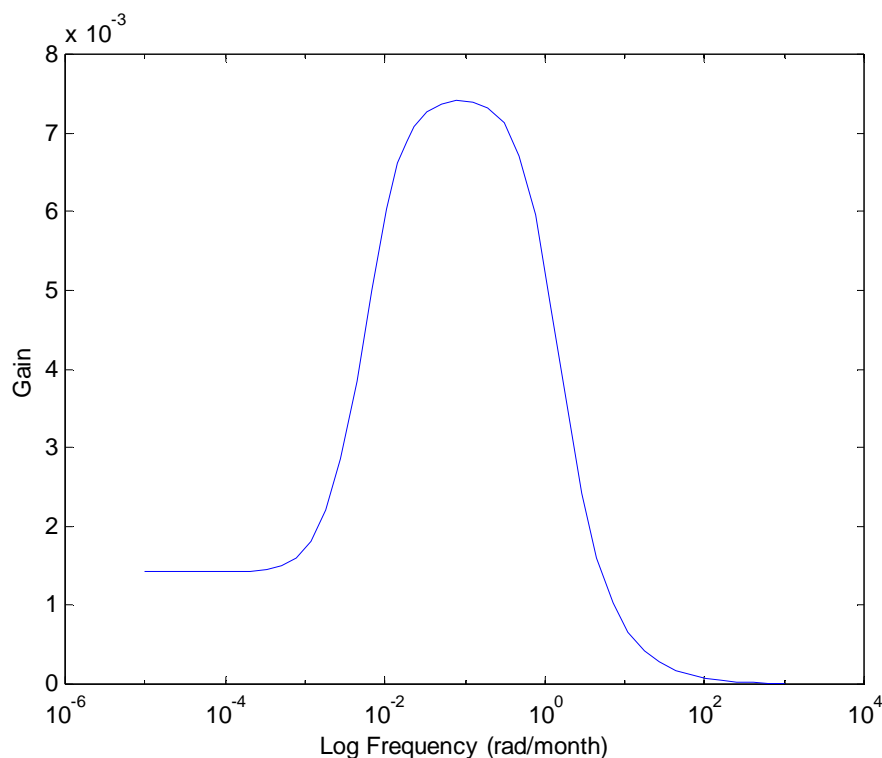
Figure 6-3: The Frequency Response of the 11-10 Input-Output Channel

Figure 6-3 is the input-output channel response for the 11th output (ARPU for the Consumer Low Market Segment) with the 10th input (peak rate for off-net calls from the 2nd optimising operator tariff). The greatest contributing reason to the lowpass phenomena in Figure 6-2 and Figure 6-3 is the operating point at which the market has been linearised. In those pairings where the different frequency responses resulted, the system proved to be a low/band pass filter for the applicable inputs. It should not be a rule for the system though, since similar pairs (types of variables involved) showed normal behaviour.

6.3   Controller Design

The objectives of the controller which was designed are in line with the over arching objectives of the research as documented in section 1.2, and are the following:

1.   To enable the optimising operator to implement a price strategy aimed at particular outputs, and particular market segments, with a specific strategic intent. The price strategy implies obtaining the best tariff trend for the intent of the operator. The strategic intent of the operator would be defined as a weighed mix of the different outputs to be optimised per market segment.

2.   To optimise the response of the optimising operator to the price moves and other changes that the competition introduces into the market.

In order to meet the above objectives a feedback control methodology had to be implemented for the model which had been described in chapter 3.
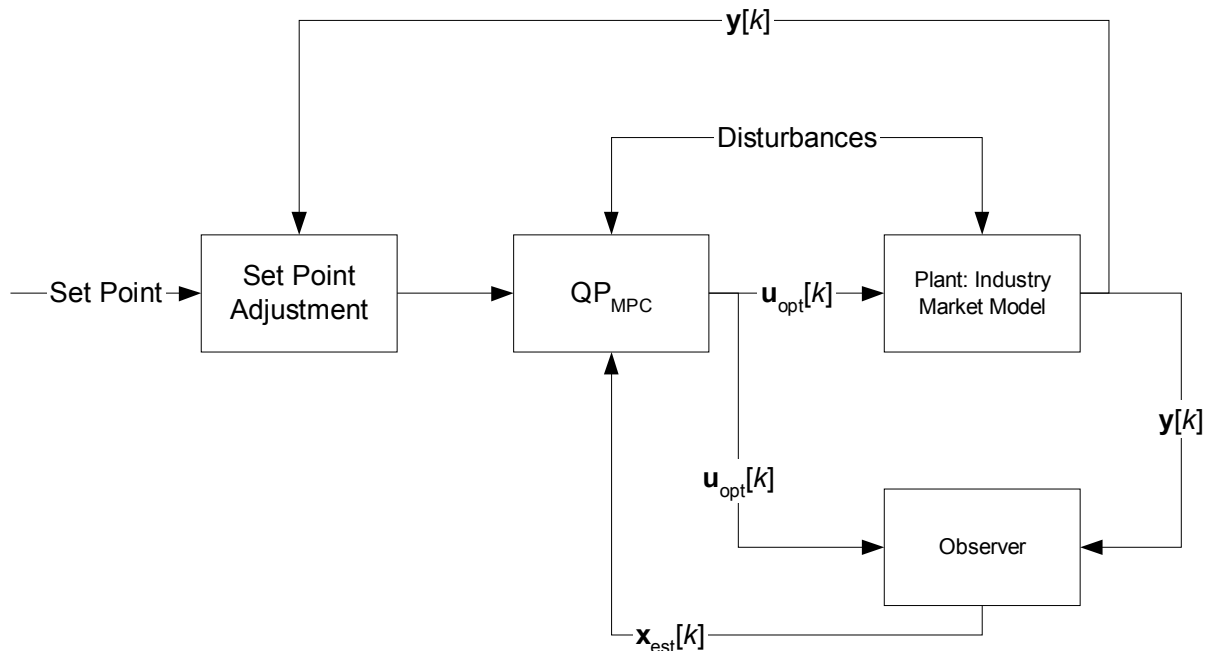
Figure 6-4: The Implemented MPC Controller [46]

In Figure 6-4 the designed and implemented MPC control loop is portrayed. The MPC algorithm uses dynamic quadratic programming (QP) to solve for the optimum input to the market (see section 5.1). The observer is linear, and estimates the current states from the model outputs and inputs which are used internally in the optimisation function. In the following paragraphs the design decisions that led to the controller as shown will be discussed in more detail.

The constraints of the market are the maximum and minimum values of the tariffs that may be charged per minute. These have been set at R0.00 (absolutely), and R50.00 above the initial value of each tariff on each price plan, respectively.

The time interval over which the closed-loop was simulated was chosen as the latter part of the interval used for the open-loop simulation. This part of the simulation period was decided upon because of the relative stability of the structure of the market over this period. In the months before this period, one additional price plan was launched into the market and a third cellular operator entered the market. These changes in the structure of the market resulted in severe model non-linearities which would typically not be addressed in the designed controller. The market did however still change during the period over which the simulation was done. The new price plans in the market drew new subscribers which resulted in an increased total addressable market. Such disturbances (new price plans and new subscribers in the market) are a reality of the industry, and therefore the controller was designed to deal with them.

The control problem is not a trivial one. Various approaches were experimented with. It was found that if the controller focuses on all the outputs of the model with the same priority, that a meaningful solution to the problem, resulting from bounded changes in the manipulated variables (tariffs) could not be obtained. A higher priority thus had to be assigned to some outputs. It was decided to focus the attention of the controller on the ARPU set of outputs of the model. The reason being that it is one of the most strategic outputs of any telecommunications operator, frequently looked at by investors and shareholders as a key (if not *the* key) parameter indicating performance for an operator.

If one output is given more priority than others, marketing management would not (and will not want to) dismiss the rest. They would be carefully trended, and kept within close proximity to budgeted values. The controller was designed to assist management in this regard by still assigning priority to the other variables, only much less than that of the key output.

The priority assigned to each output variable manifests itself in a weight that is assigned to each output. In this case the ARPU output of each market segment (i.e. 5 ARPU outputs)

was assigned a relative weight of 10000 divided by the average of the output over the control simulation period.   All the other outputs were assigned a relative weight of 1 divided by the average of the output over the control simulation period.

In the 1990's a generation of MPC developed that dealt with several levels of constraints (soft, hard and ranked according to priority) mechanisms to recover from infeasible solutions [19].   A form of this strategy has been applied in this design.   Setpoints were assigned to each output. The setpoint of the ARPU outputs were made a certain percentage higher than the value of the respective ARPU outputs per month obtained from the open-loop non-linear simulation of the model (see chapter 4). All the other outputs were given setpoints equal to the values of those outputs per month obtained from the open-loop simulation. If any output is lower than the desired setpoint for the month, the setpoint is kept constant and the controller would attempt to get the output closer to the setpoint (if not equal to it) for the next month by applying optimal tariff inputs. If an output is higher than the desired setpoint for a month, however, the controller sets this setpoint equal to the output for that month so as to zero the error signal for that output, to enable the controller to focus on outputs that were lower than setpoint (see the setpoint adjustment block in Figure 6-4 and how it links to the other blocks in the design).

The percentage increase in the ARPU setpoint is 5%. This value was obtained empirically by increasing the ARPU setpoint in the closed-loop simulation until it became apparent that the controller could not obtain a particular increase in ARPU without totally losing control of the other outputs.   The controller thus successfully optimises ARPU by increasing it with 5 % while maintaining all the other outputs as close as possible to their respective setpoints (see chapter 7 for the results with the controller implemented).

The relative weights applied to the manipulated variables in the objective function of the controller, were made 1000 divided by the average of those tariffs over the control simulation period in the open-loop case. A relatively large value was chosen in order to effectively punish excessive control action, as management would desire a smooth change in tariff as decision support.

The optimal values for the prediction and control horizon parameters (see section 5.1) were determined empirically. The 3-dimensional diagram in Figure 6-5 shows the value of the ISE on the z-axis with the prediction- and control horizons on the x- and y-axis respectively.
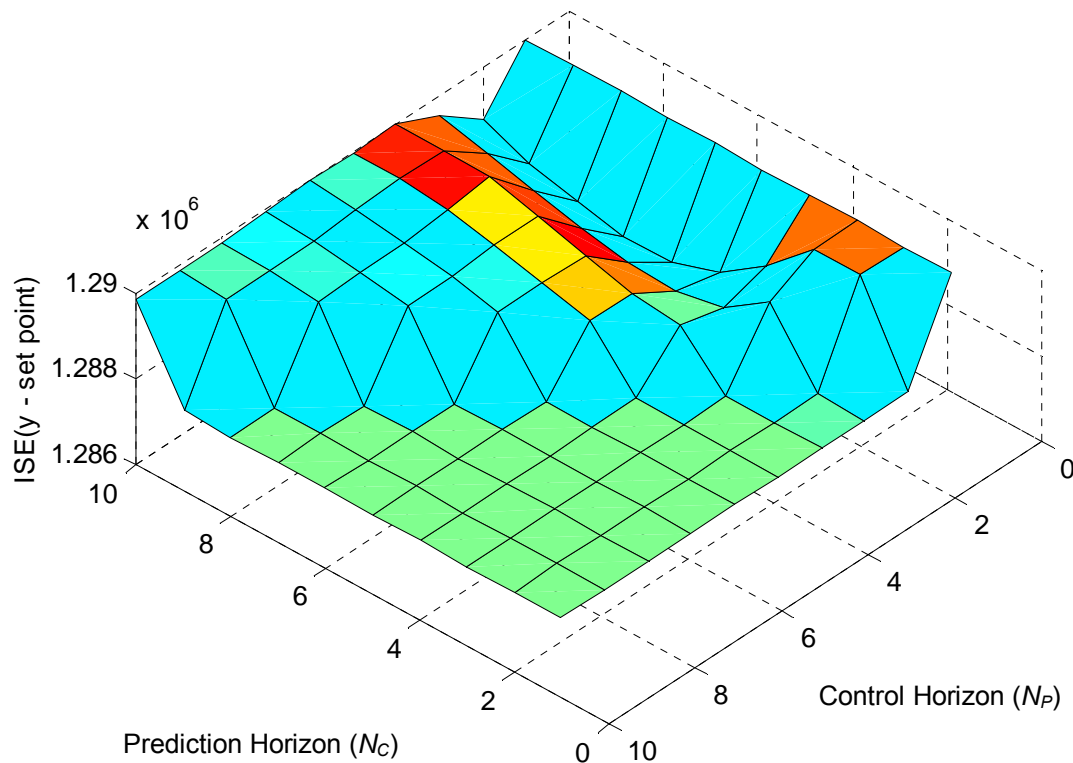
Figure 6-5: The ISE of the Output Variables Plotted vs. N_C and N_P

The lowest value for the ISE occurs in a "valley" between two "rifts" at the crossing of a control horizon of 2 and a prediction horizon of 7. The numerical values shown in Figure 6-5 can be found in Table A-50.

7   CLOSED-LOOP SIMULATION

In this chapter the results of the simulation of the feedback system described in chapter 6 (non-linear market model with linear feedback control implemented) are documented. Results are discussed (section 7.2) and conclusions are drawn (section 7.3) after the graphical results (section 7.1). Please note that the optimising operator is referred to as OO, and the other two operators as O2 and O3.

7.1   Results

The closed-loop simulation results (utilising Euler integration for the solving of the differential equations) are first given from an output perspective – the graph of each controlled variable over the control simulation interval is shown. On many of the graphs the setpoint and the output cannot be distinguished from one another – that is because the controller managed to have the output track the setpoint perfectly. In cases where the setpoint has been increased based on values recorded during the open-loop simulation (see section 4.1), the open-loop simulation values are shown as dashed lines.
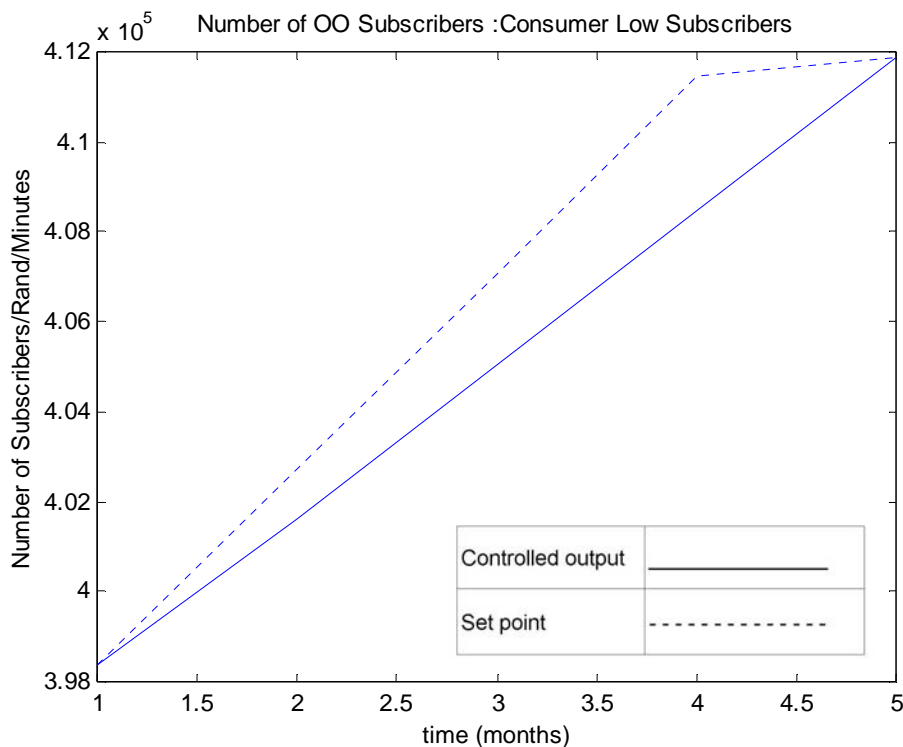
Figure 7-1: The Number of OO Subscribers in the Consumer Low Market Segment

The setpoint (in dotted line) as well as the controlled output (in solid line) are drawn on the same graph.
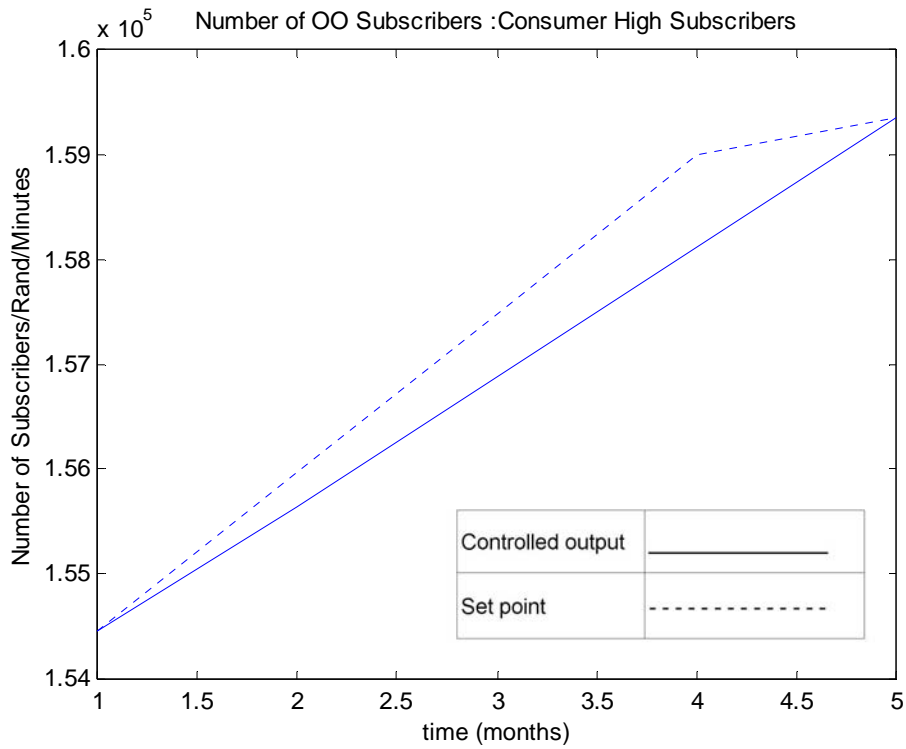
Figure 7-2: The Number of OO Subscribers in the Consumer High Market Segment

Number of OO Subscribers :Medium Subscribers
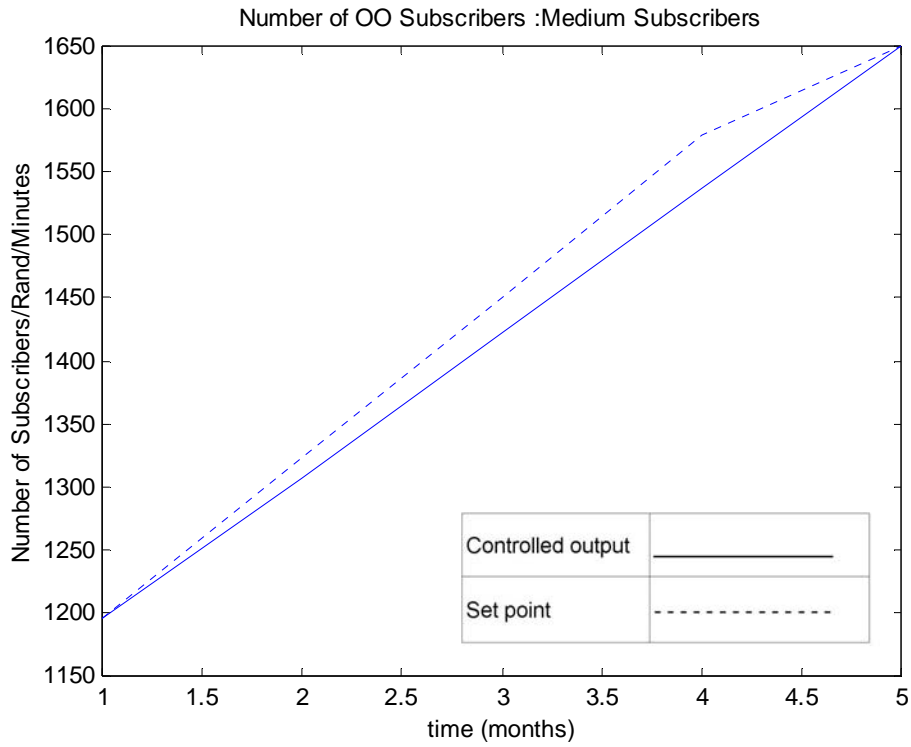
Figure 7-3: The Number of OO Subscribers in the Medium Market Segment

Number of OO Subscribers :Business Low Subscribers
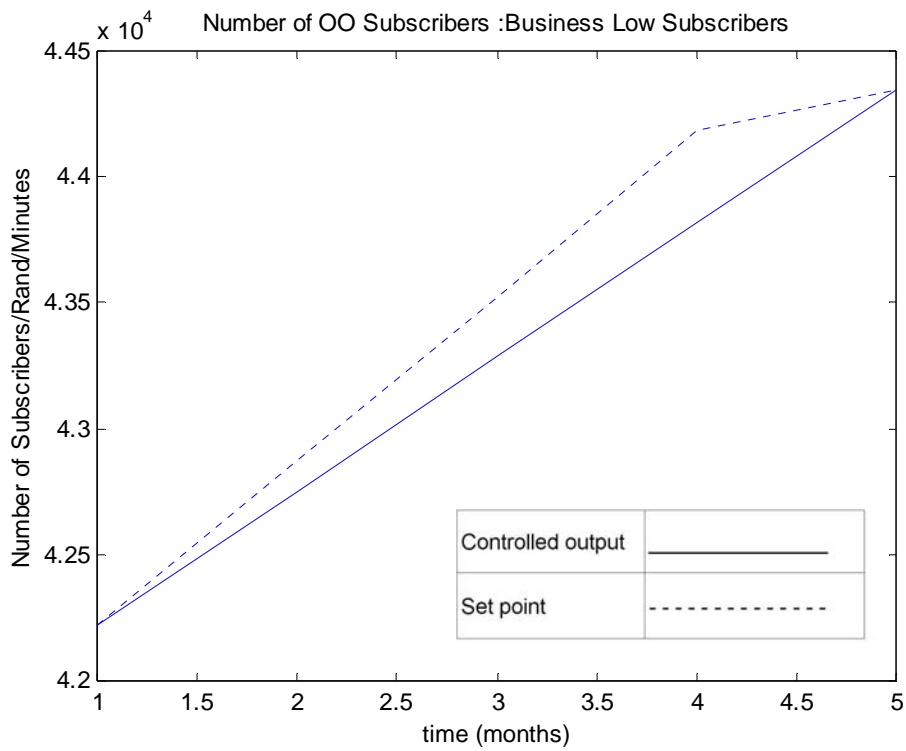
Figure 7-4: The Number of OO Subscribers in the Business Low Market Segment
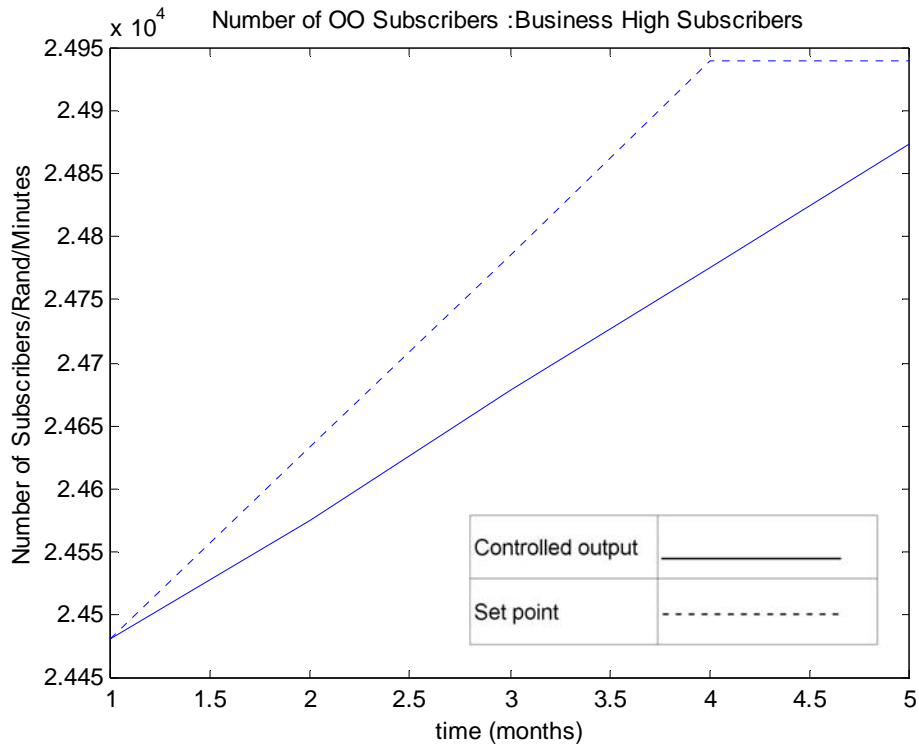
Figure 7-5: The Number of OO Subscribers in the Business High Market Segment
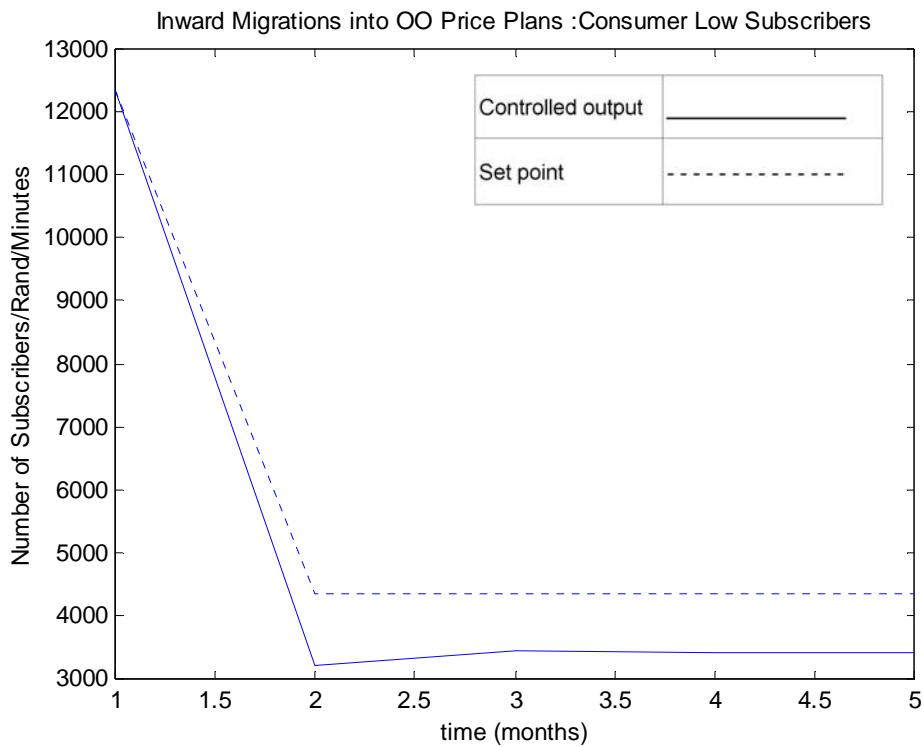
Figure 7-6: Migrations into OO Price Plans in the Consumer Low Market Segment
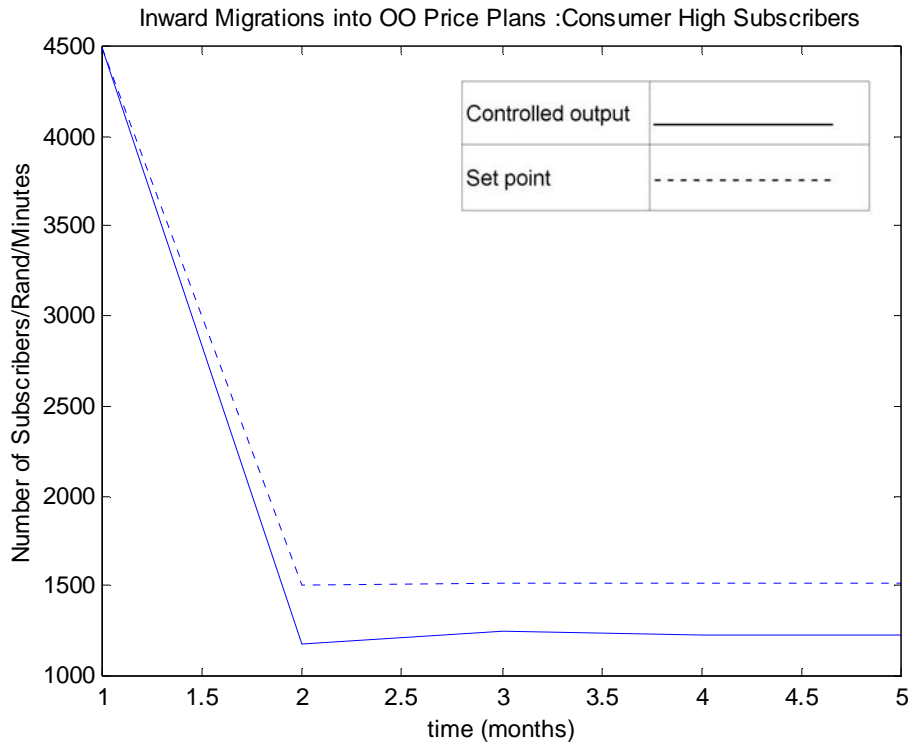
Figure 7-7: Migrations into OO Price Plans in the Consumer High Market Segment
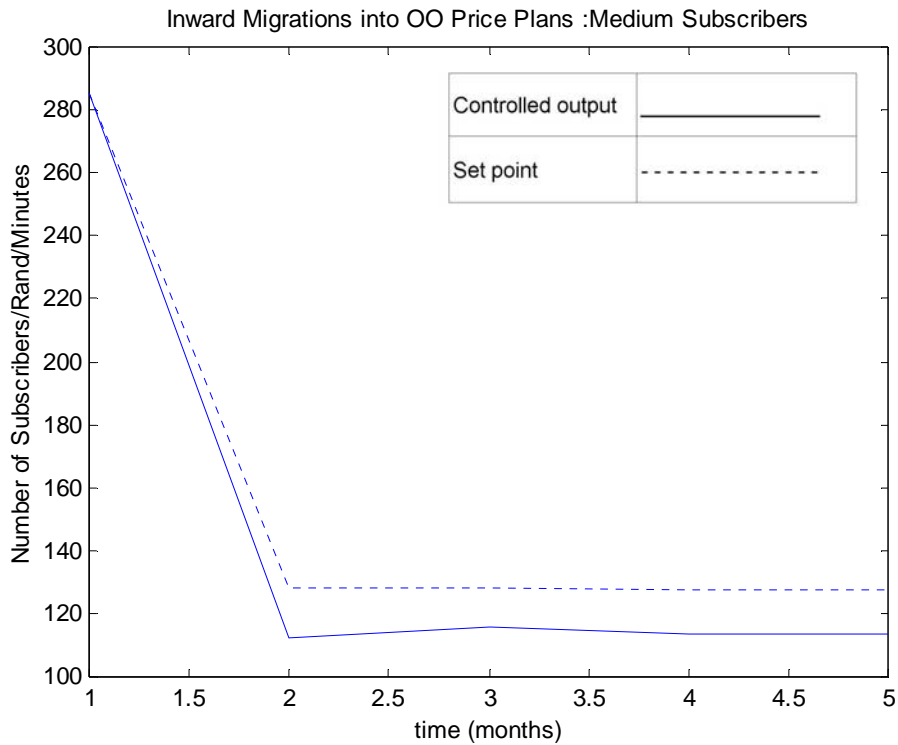
Figure 7-8: Migrations into OO Price Plans in the Medium Market Segment
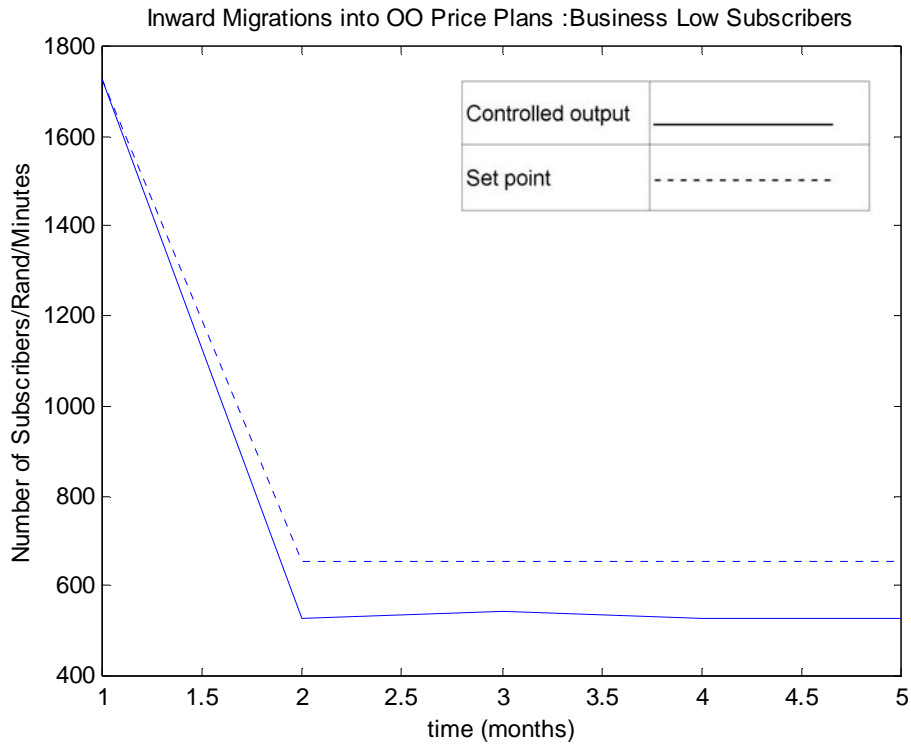
Figure 7-9: Migrations into OO Price Plans in the Business Low Market Segment
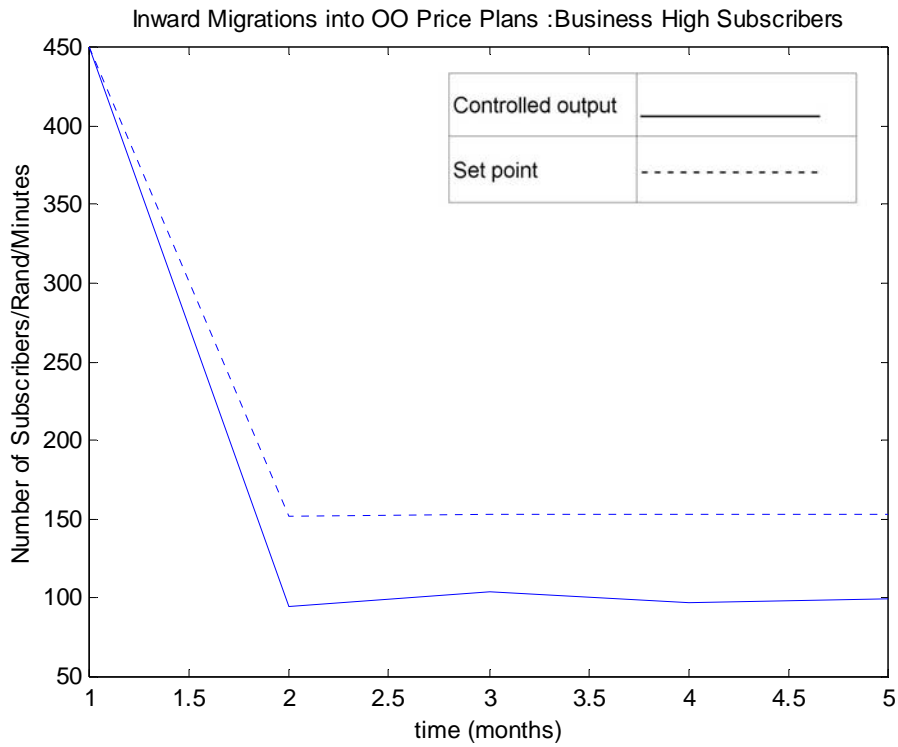
Figure 7-10: Migrations into OO Price Plans in the Business High Market Segment
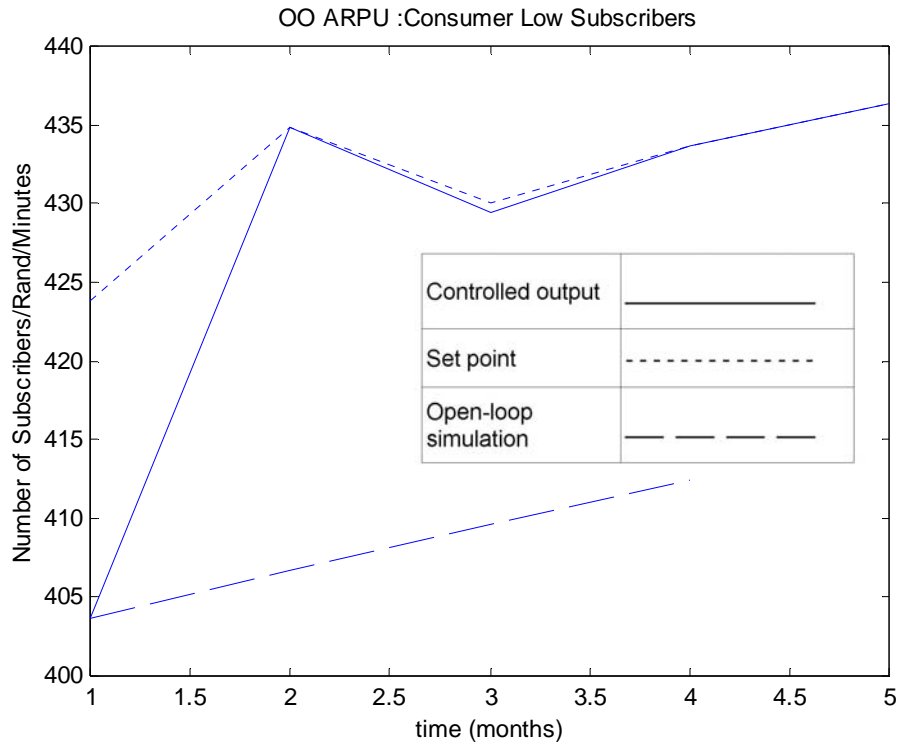
Figure 7-11: OO ARPU in the Consumer Low Market Segment
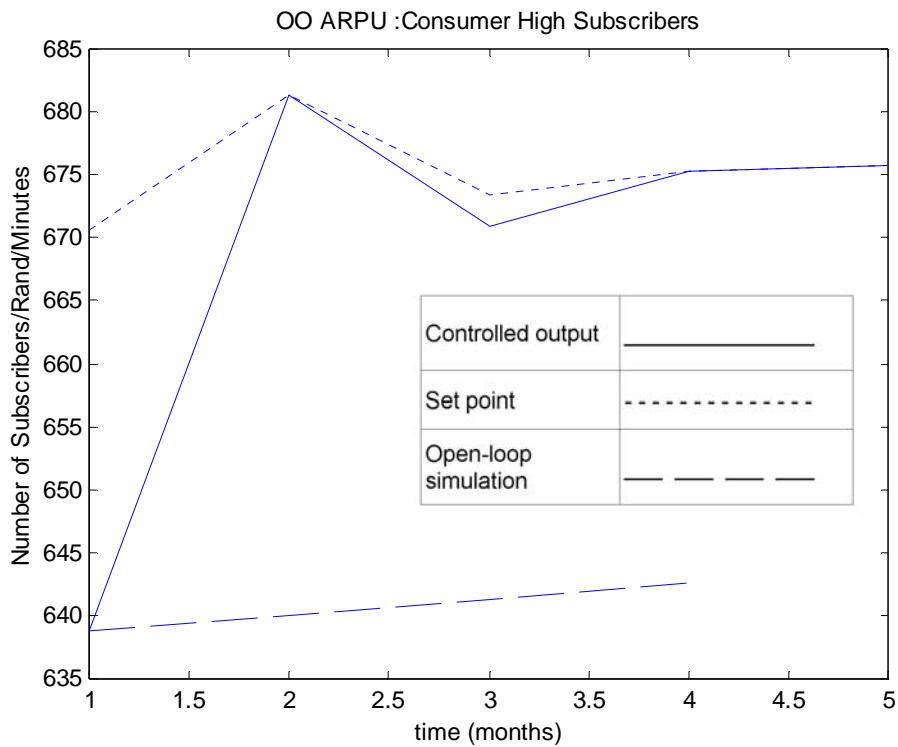
Figure 7-12: OO ARPU in the Consumer High Market Segment
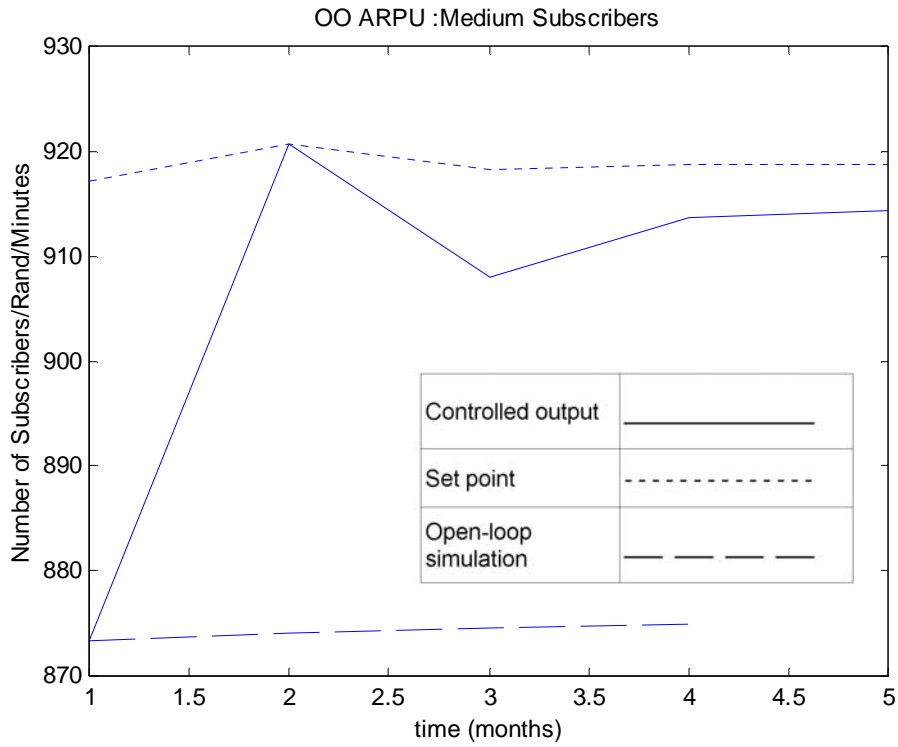
Figure 7-13: OO ARPU in the Medium Market Segment
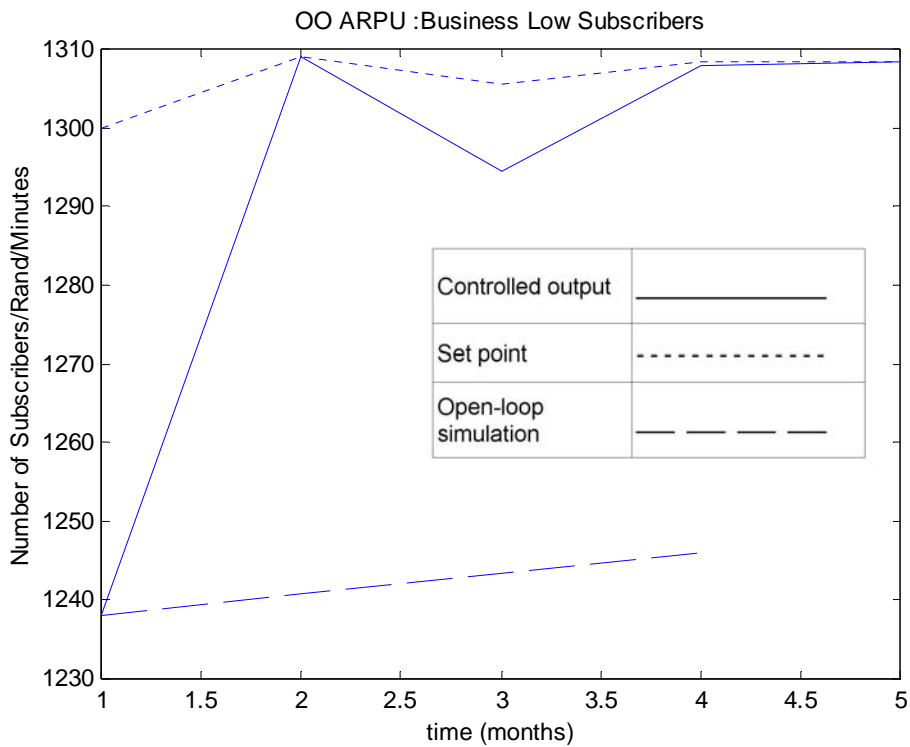
Figure 7-14: OO ARPU in the Business Low Market Segment
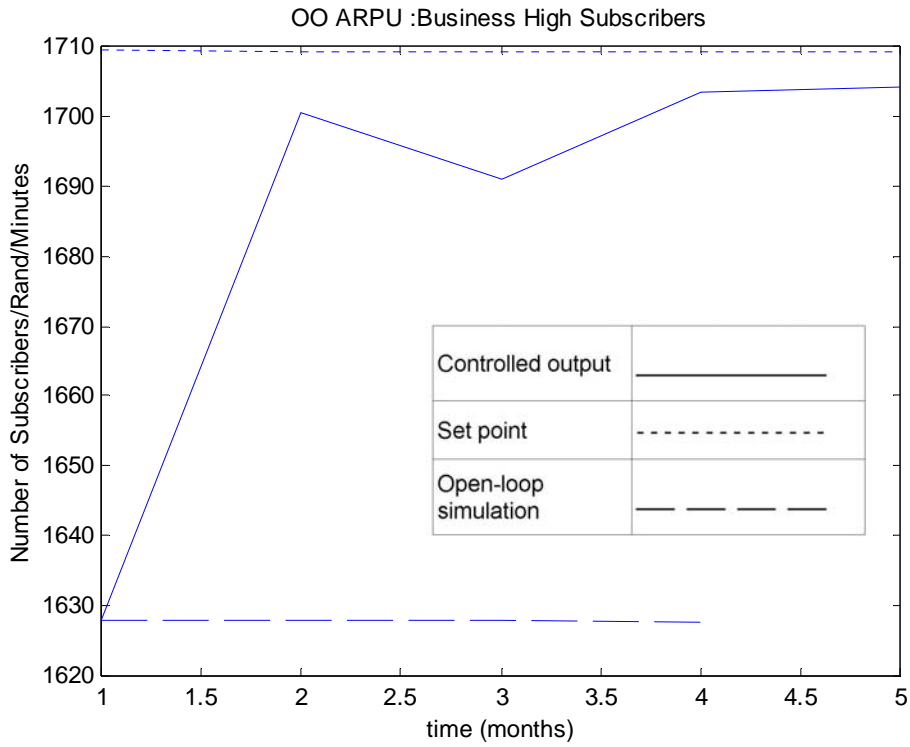
Figure 7-15: OO ARPU in the Business High Market Segment
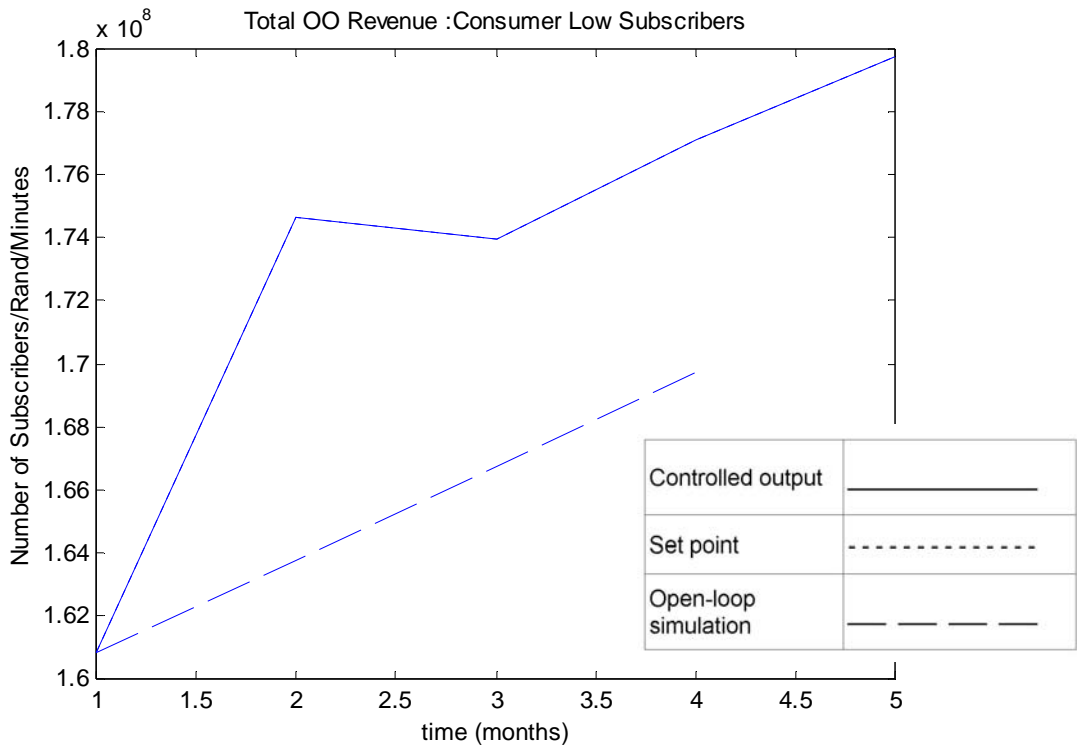
Figure 7-16: Total OO Revenue in the Consumer Low Market Segment
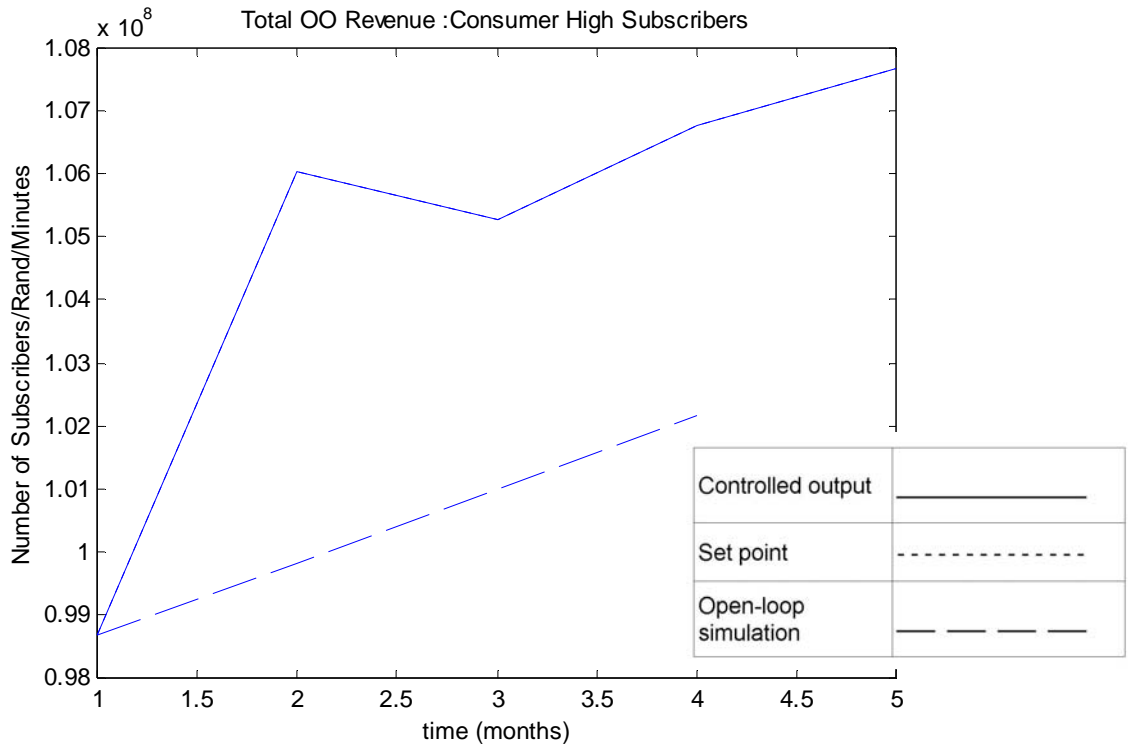
Figure 7-17: Total OO Revenue in the Consumer High Market Segment
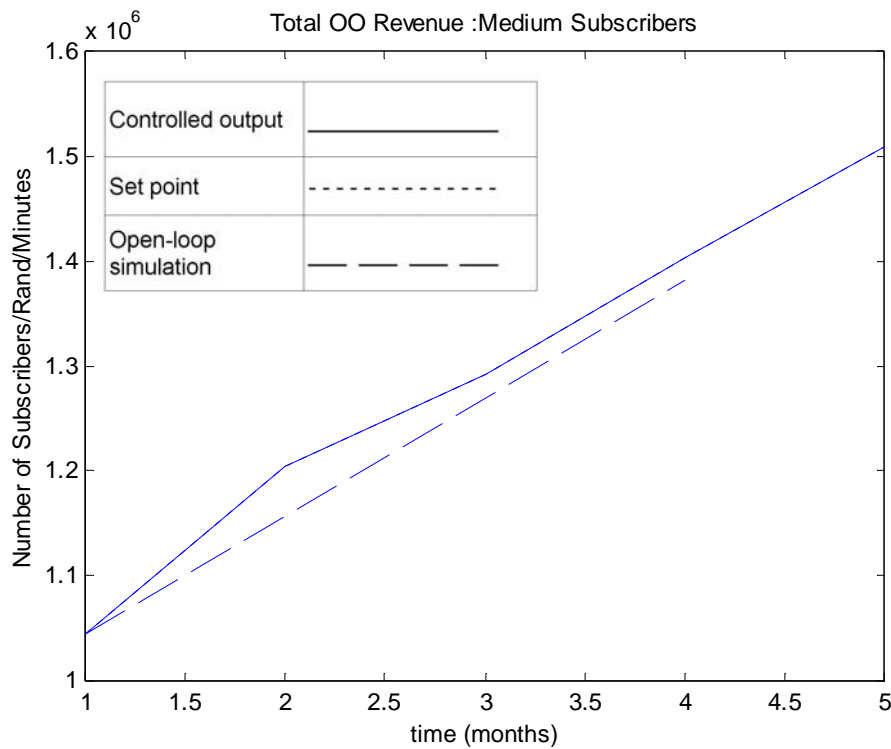
Figure 7-18: Total OO Revenue in the Medium Market Segment
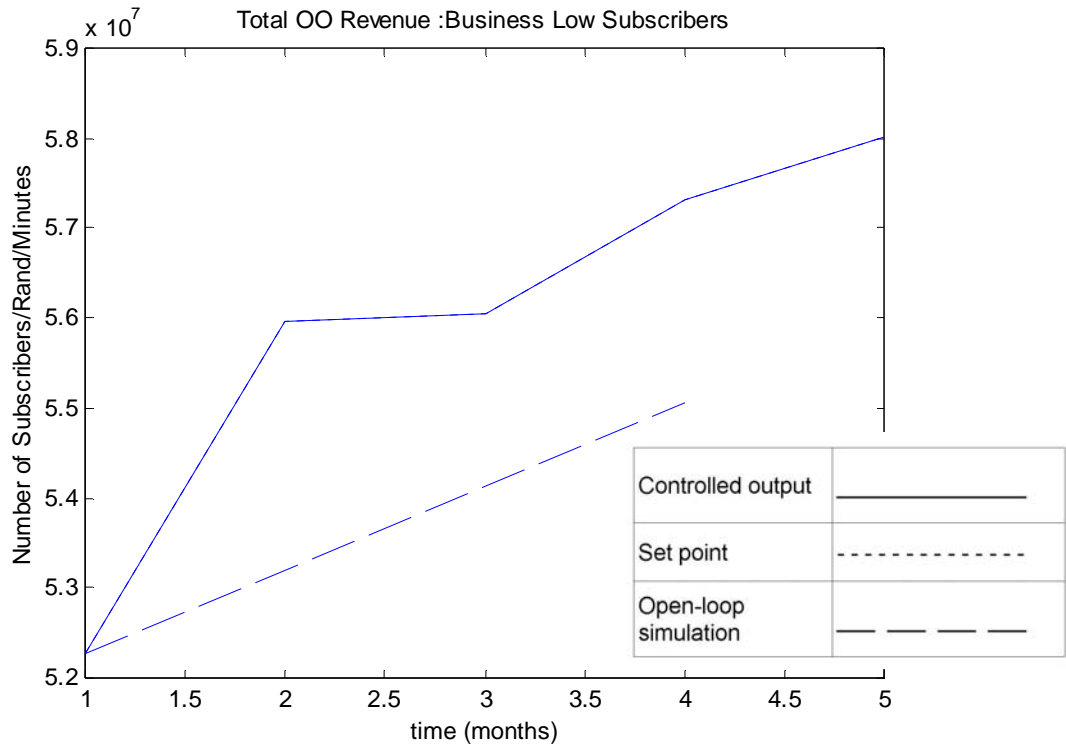
Figure 7-19: Total OO Revenue in the Business Low Market Segment
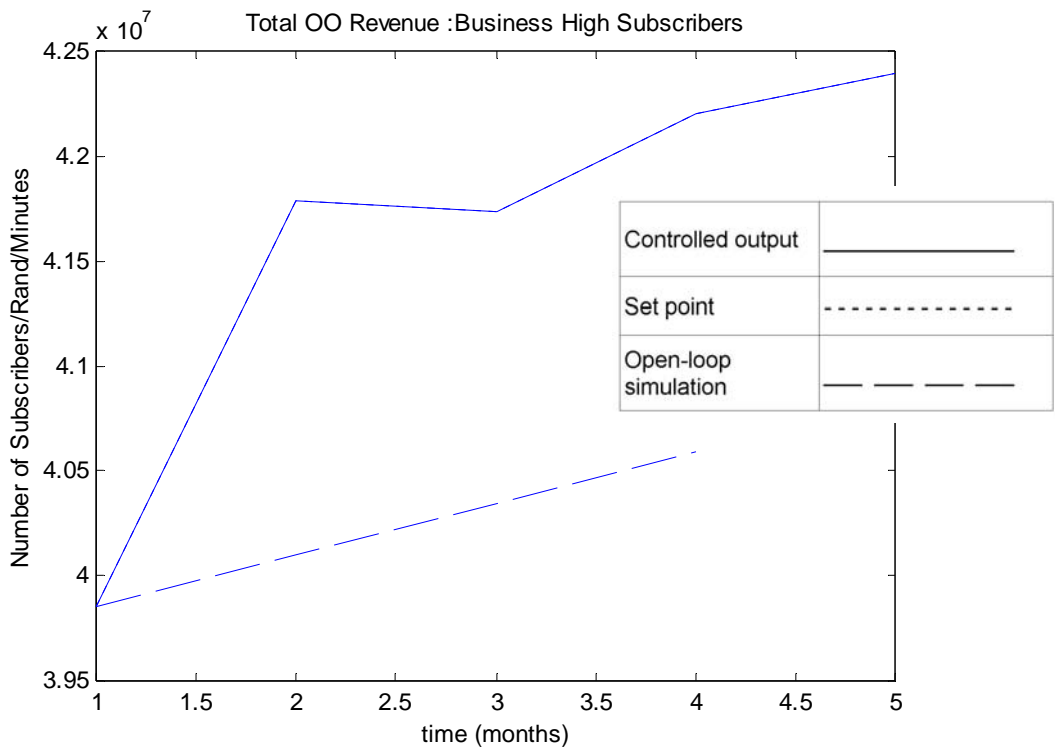
Figure 7-20: Total OO Revenue in the Business High Market Segment
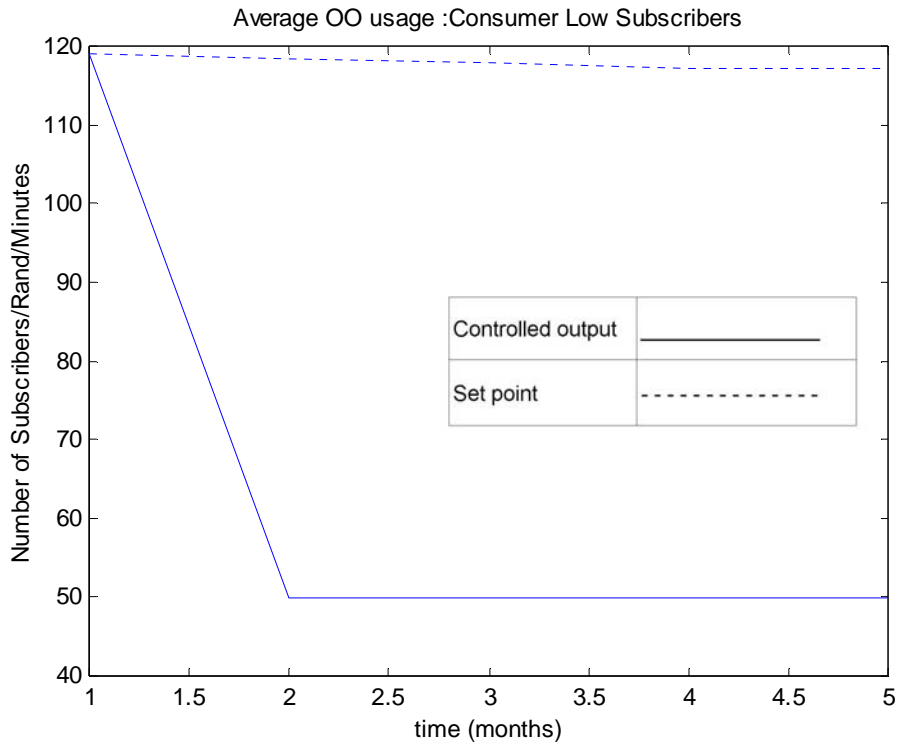
Figure 7-21: Average OO Usage in the Consumer Low Market Segment
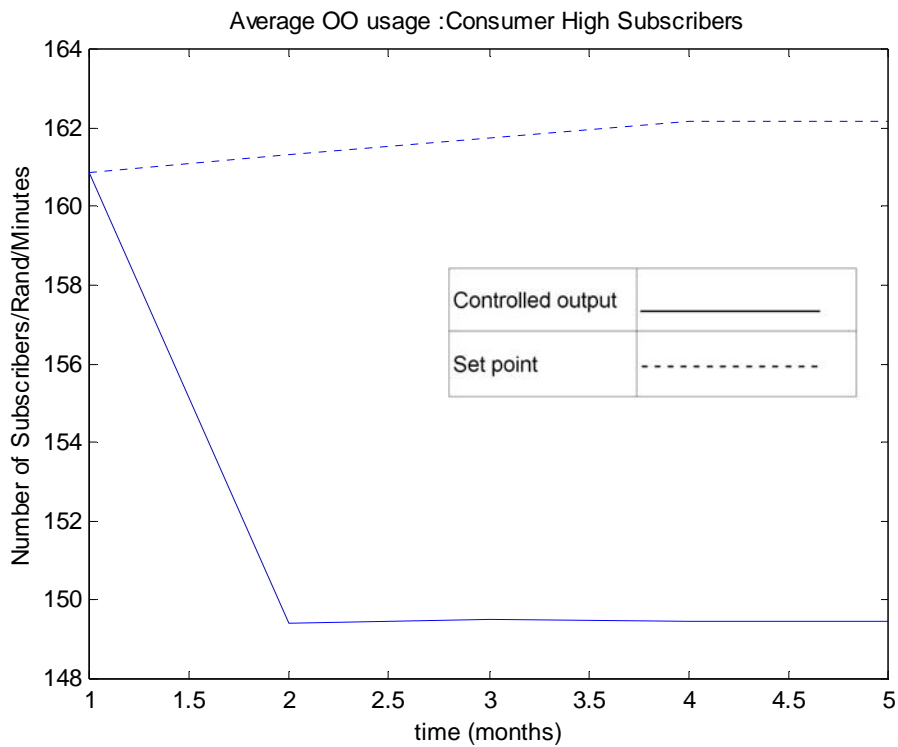
Figure 7-22: Average OO Usage in the Consumer High Market Segment

Average OO usage :Medium Subscribers
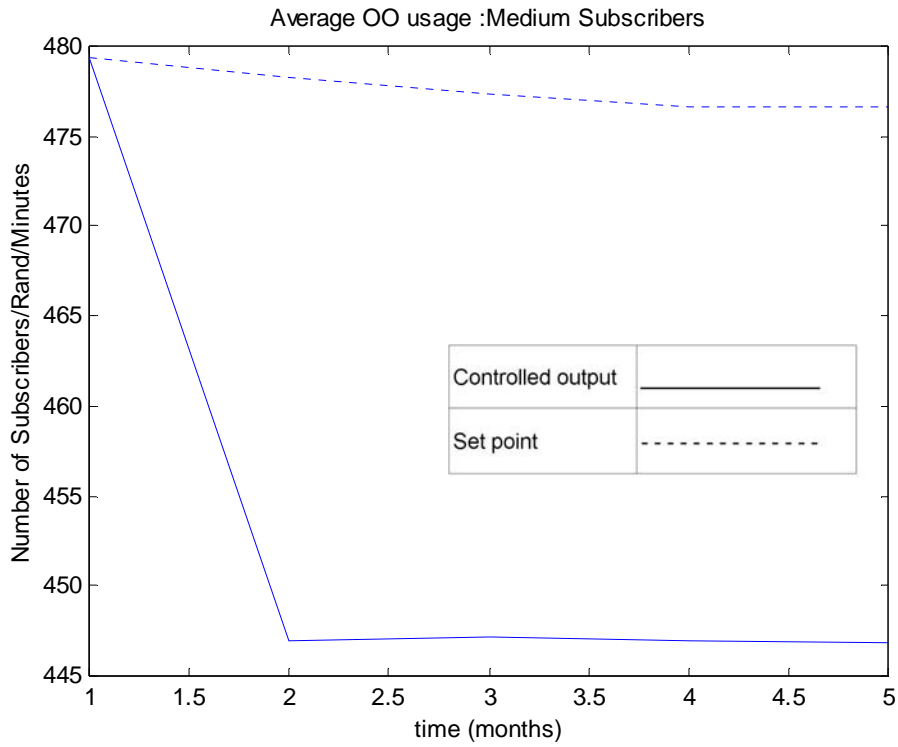
Figure 7-23: Average OO Usage in the Medium Market Segment

Average OO usage :Business Low Subscribers
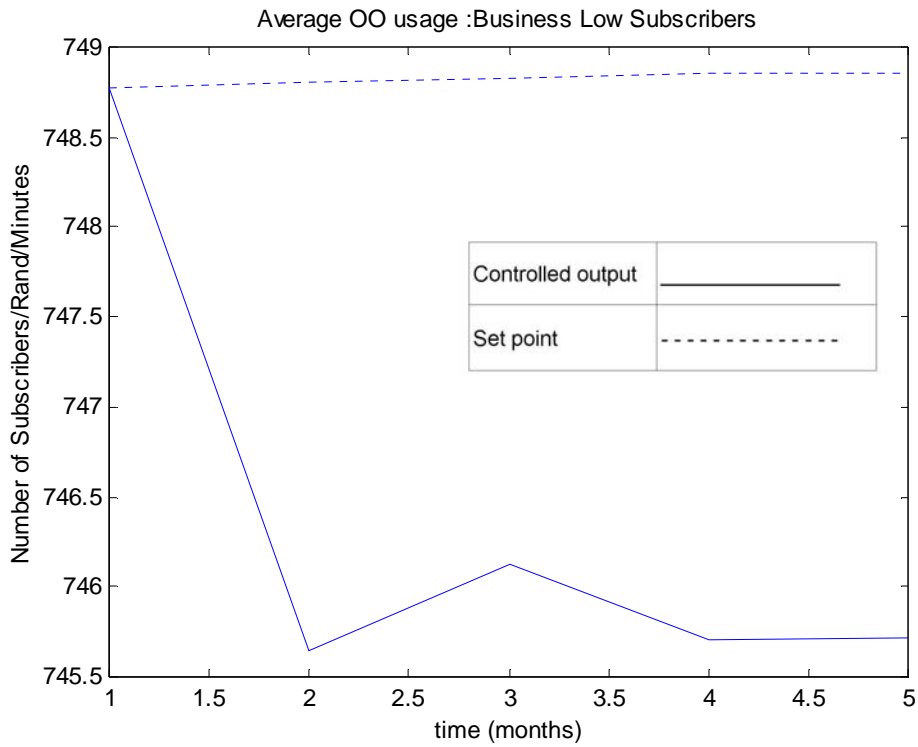
Figure 7-24: Average OO Usage in the Business Low Market Segment
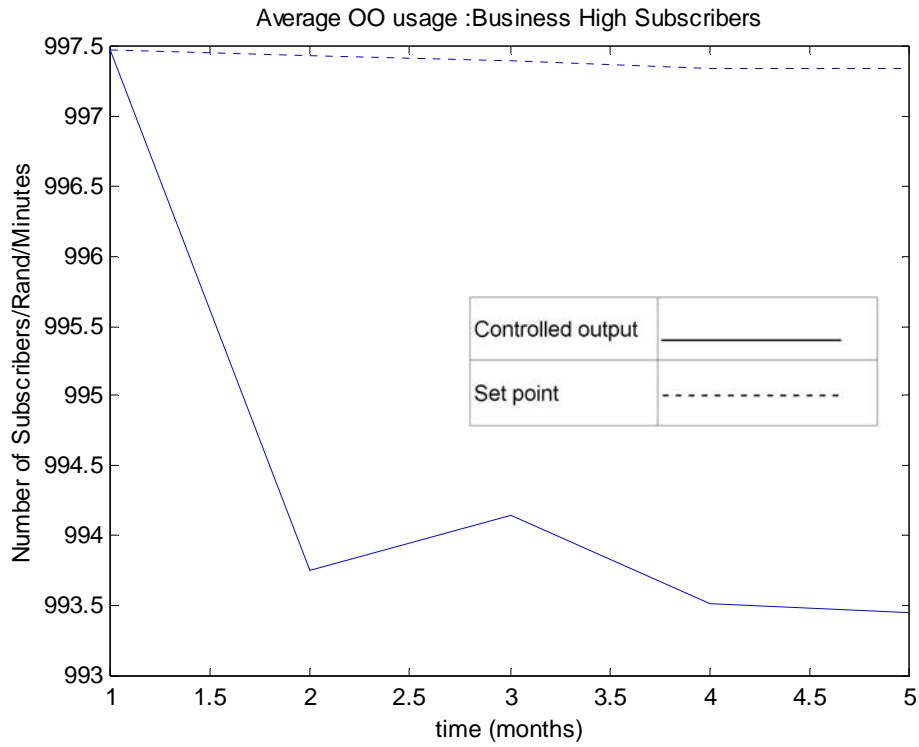
Figure 7-25: Average OO Usage in the Business High Market Segment
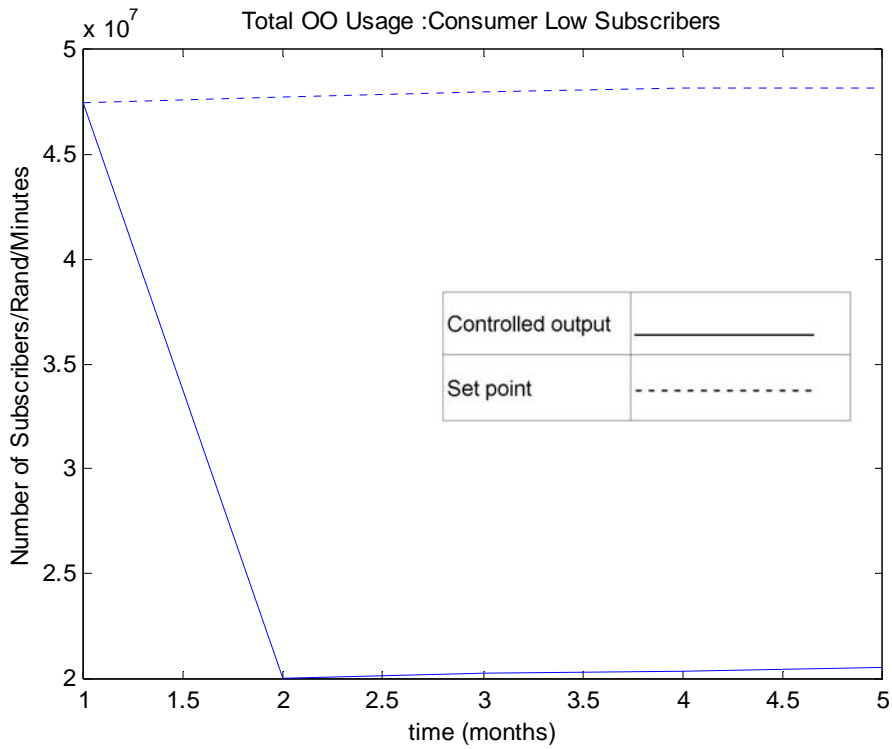
Figure 7-26: Total OO Usage in the Consumer Low Market Segment
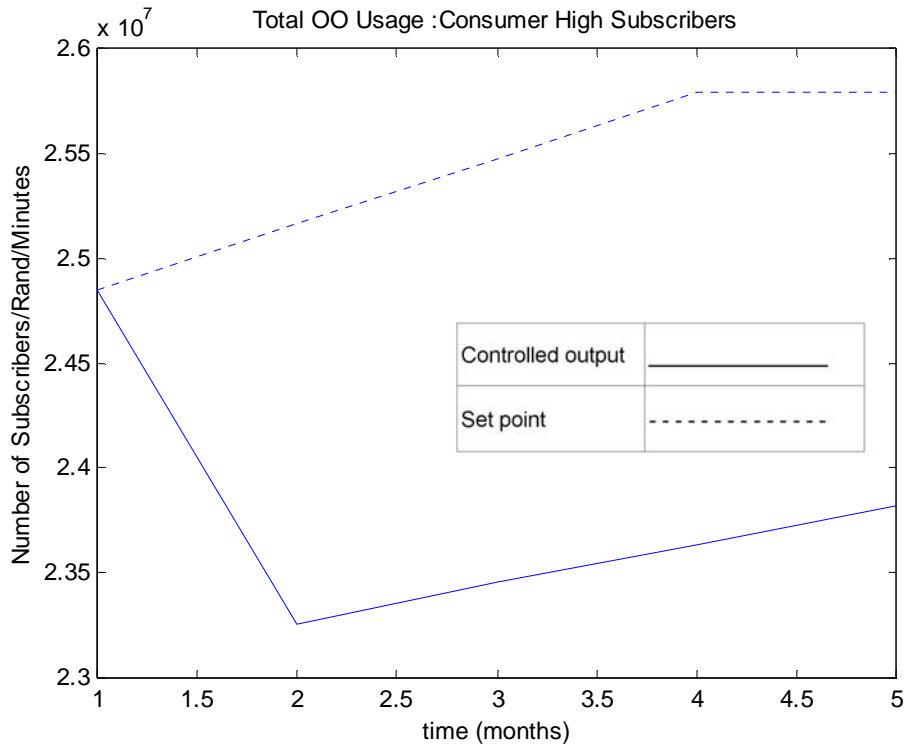
Figure 7-27: Total OO Usage in the Consumer High Market Segment
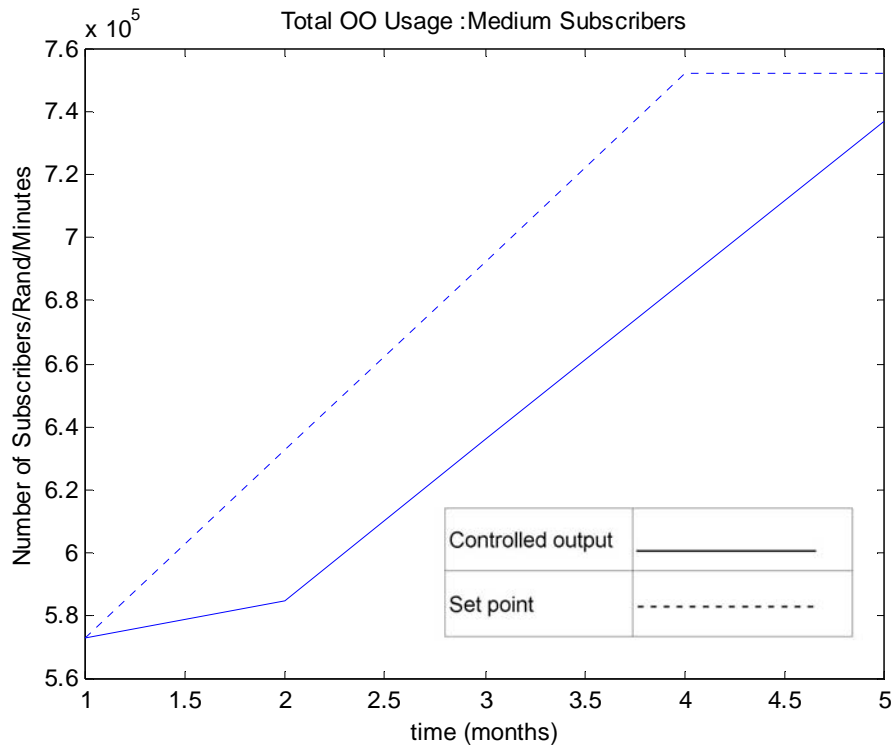
Figure 7-28: Total OO Usage in the Medium Market Segment
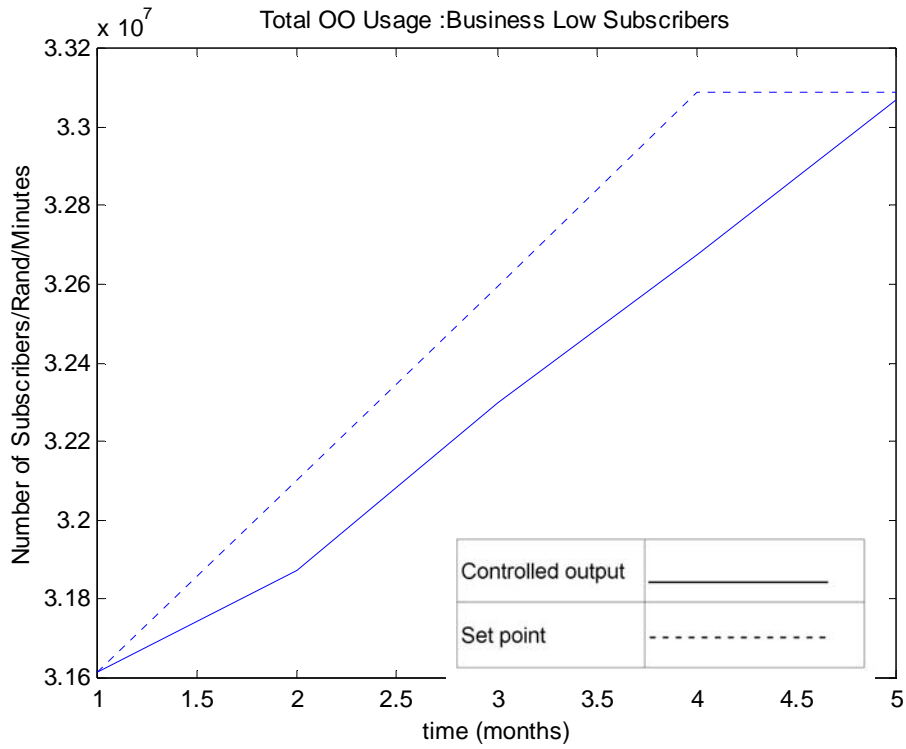
Figure 7-29: Total OO Usage in the Business Low Segment
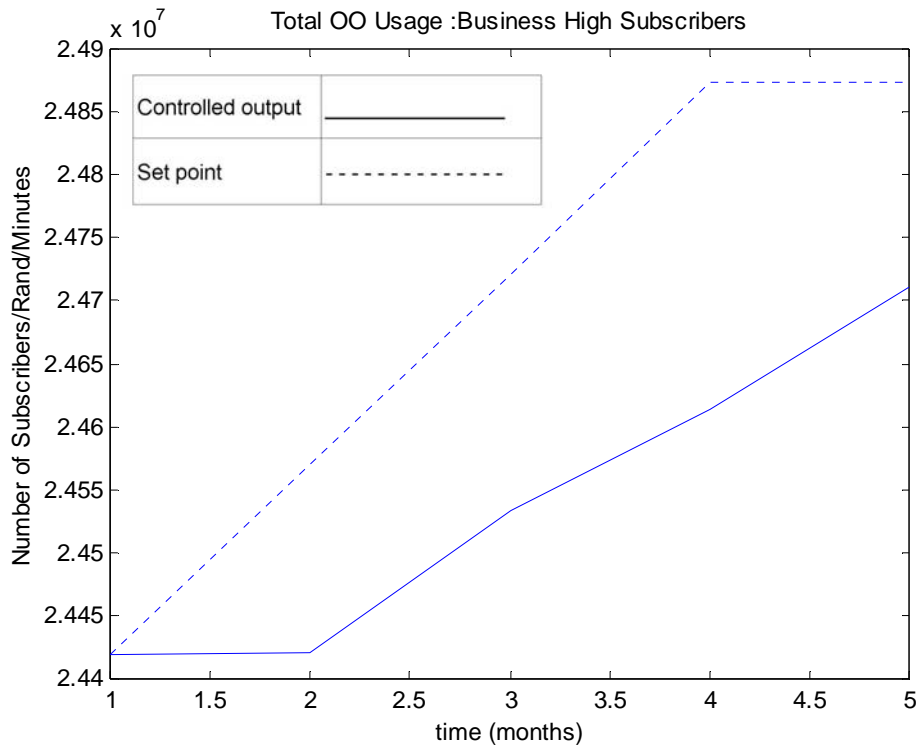
Figure 7-30: Total OO Usage in the Business High Segment

The changes that were made to the tariffs by the controller in order to achieve the output trends above, are now given. For each OO price plan, two graphs are shown, one graph for the peak rates to the three different destinations, followed by the off-peak rates. The key included in each graph describes the different line types used for the different destinations.
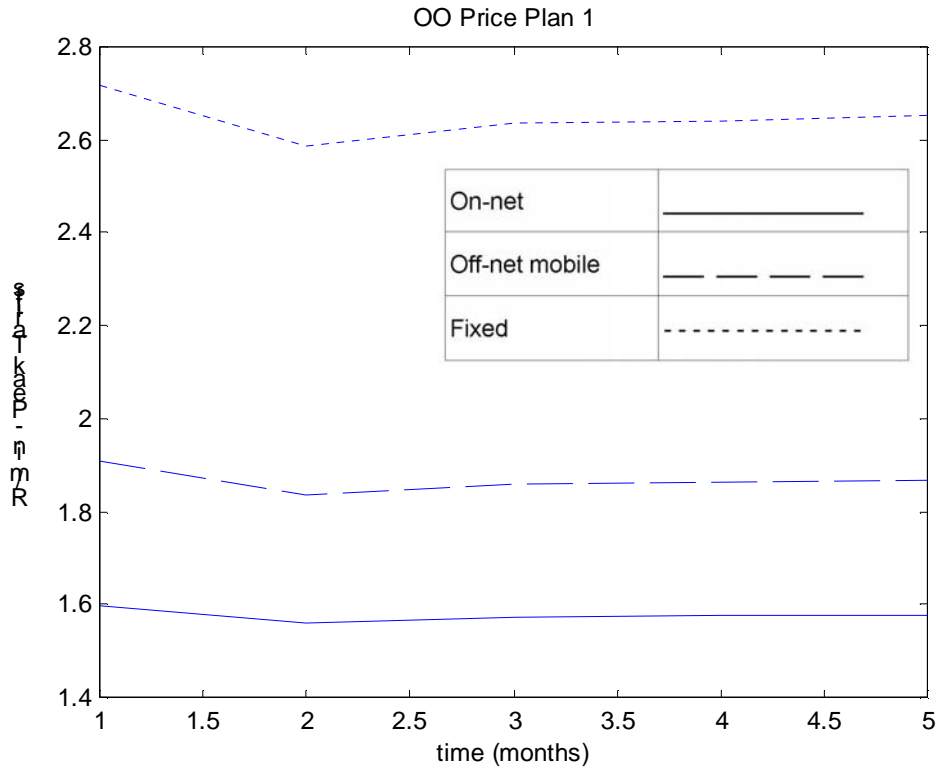
Figure 7-31: Peak Tariffs for OO Price Plan 1
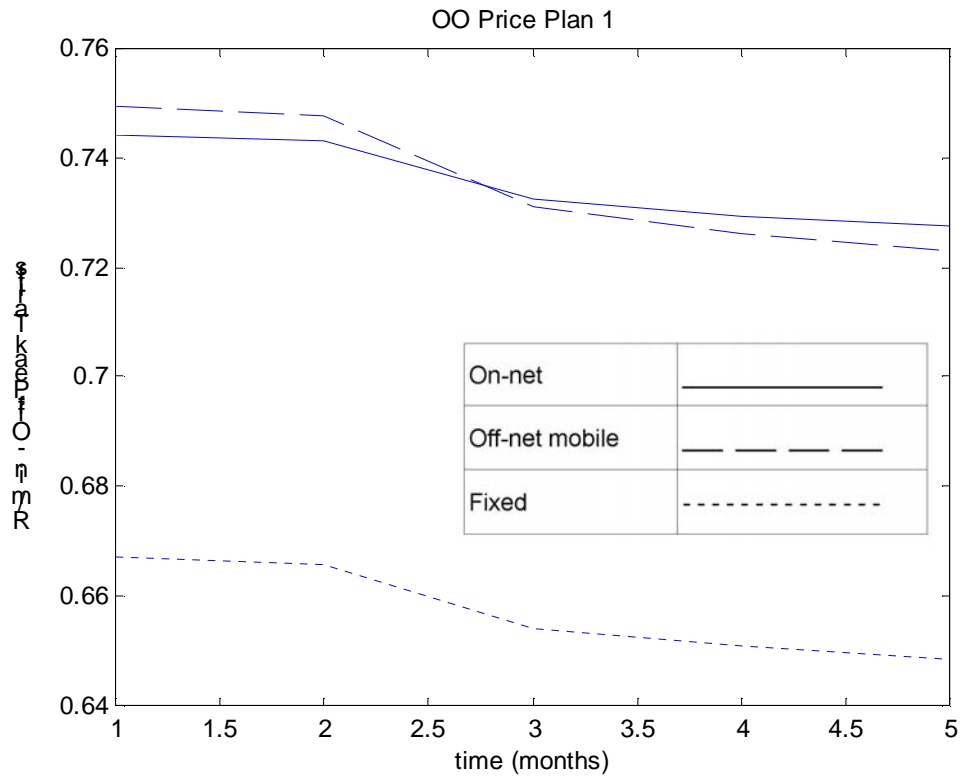
Figure 7-32: Off-Peak Tariffs for OO Price Plan 1
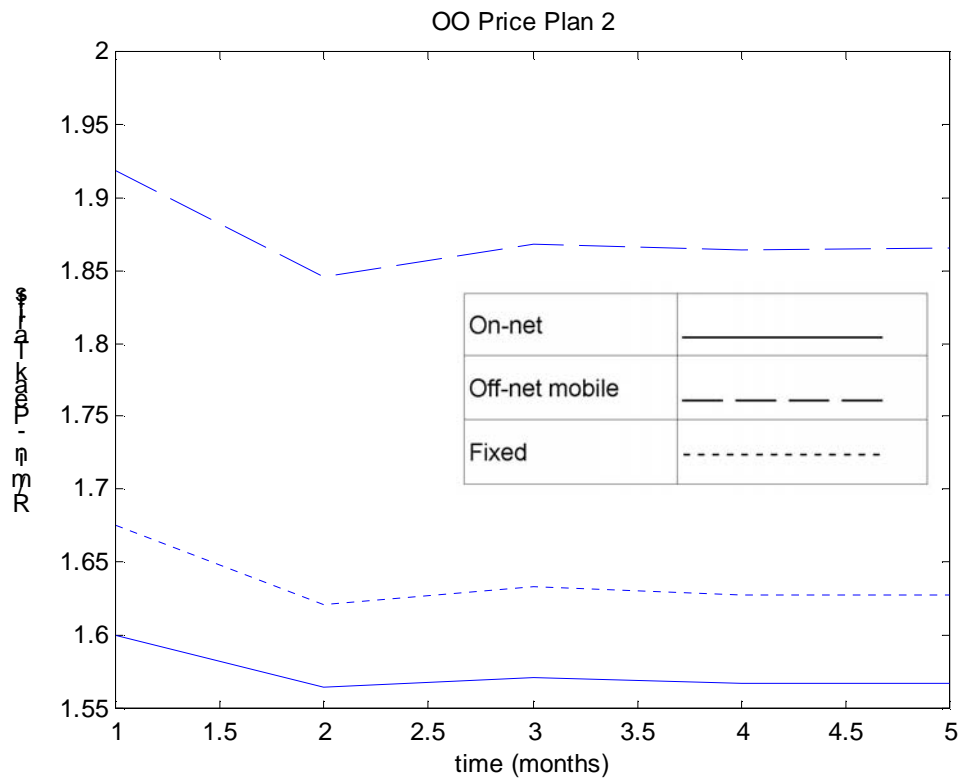
Figure 7-33: Peak Tariffs for OO Price Plan 2
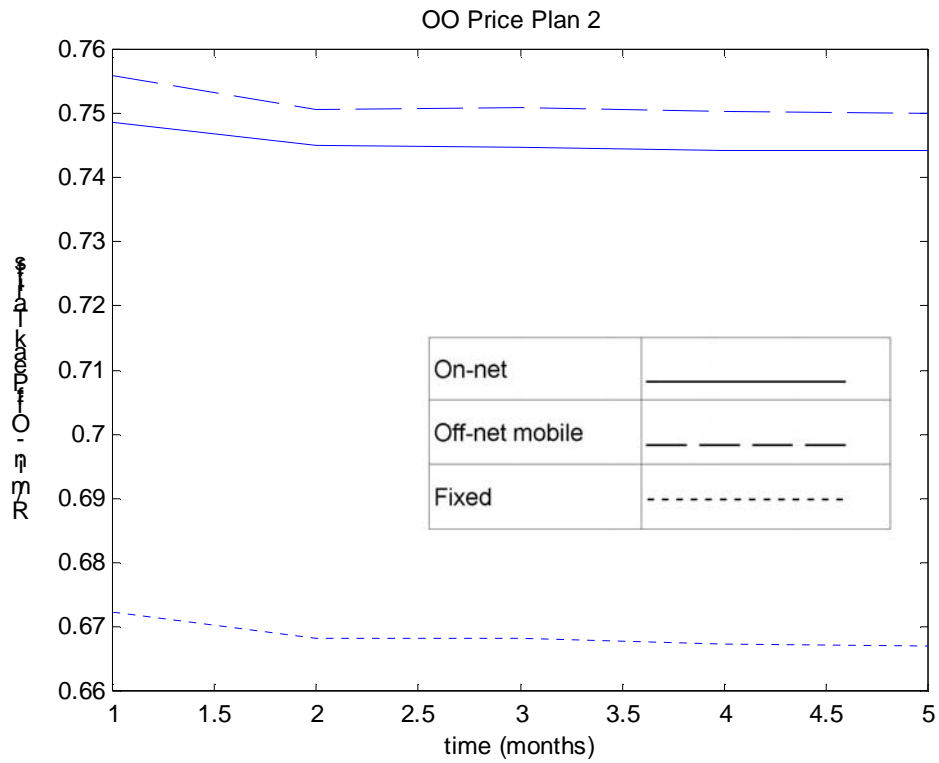
Figure 7-34: Off-Peak Tariffs for OO Price Plan 2
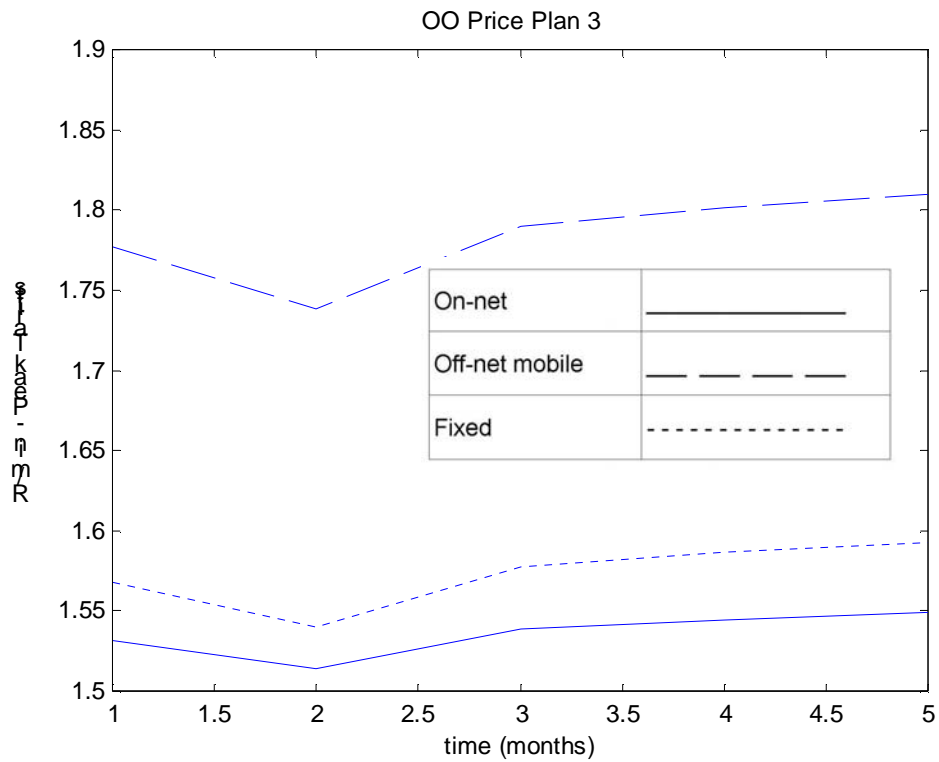
Figure 7-35: Peak Tariffs for OO Price Plan 3
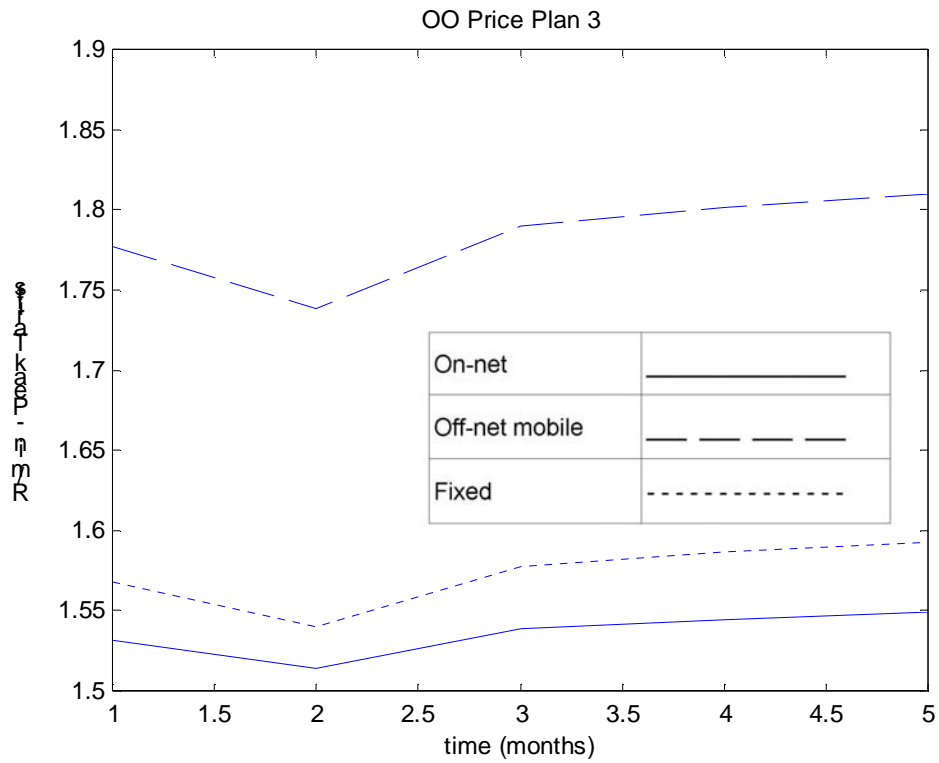
OO Price Plan 3

Figure 7-36: Off-Peak Tariffs for OO Price Plan 3
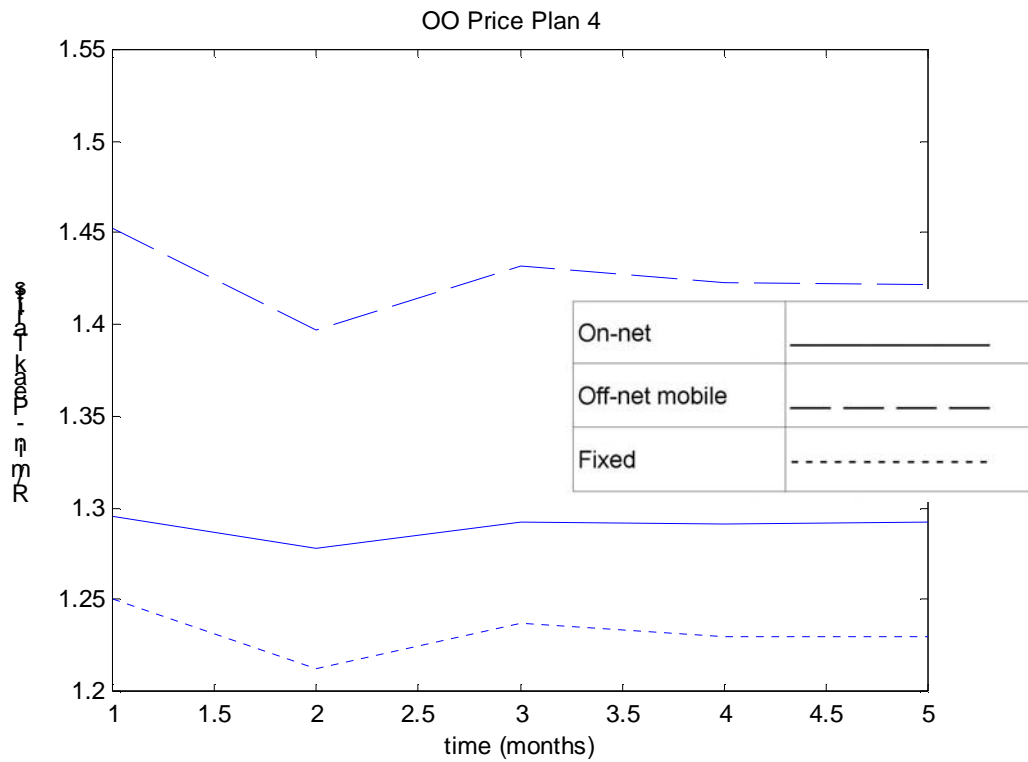
OO Price Plan 4

Figure 7-37: Peak Tariffs for OO Price Plan 4
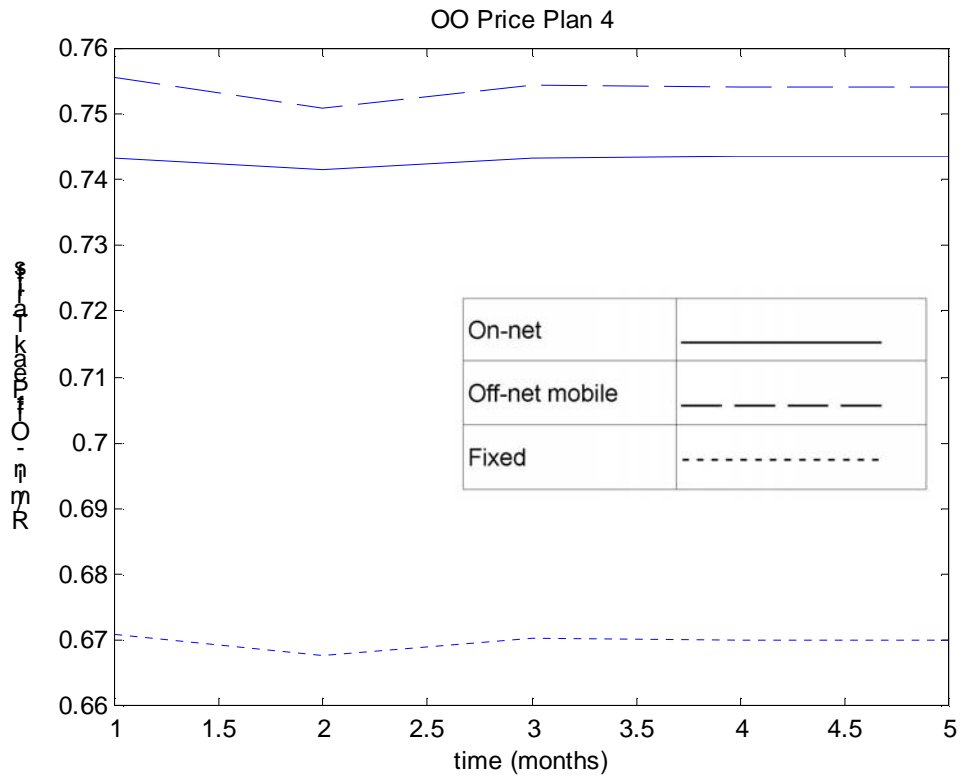
Figure 7-38: Off-Peak Tariffs for OO Price Plan 4
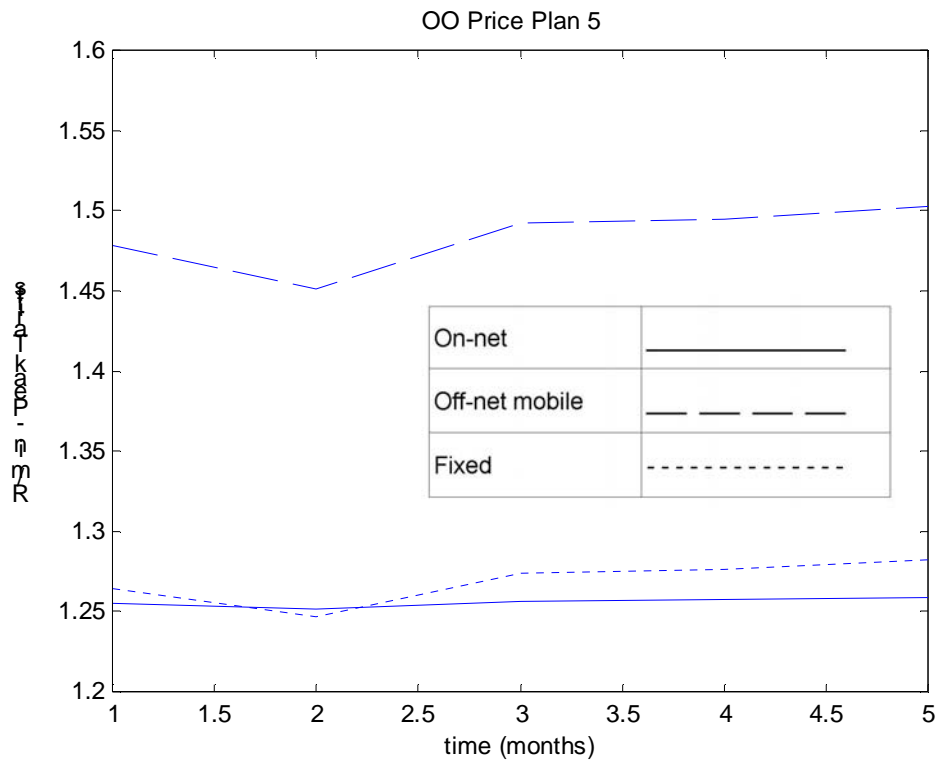
Figure 7-39: Peak Tariffs for OO Price Plan 5
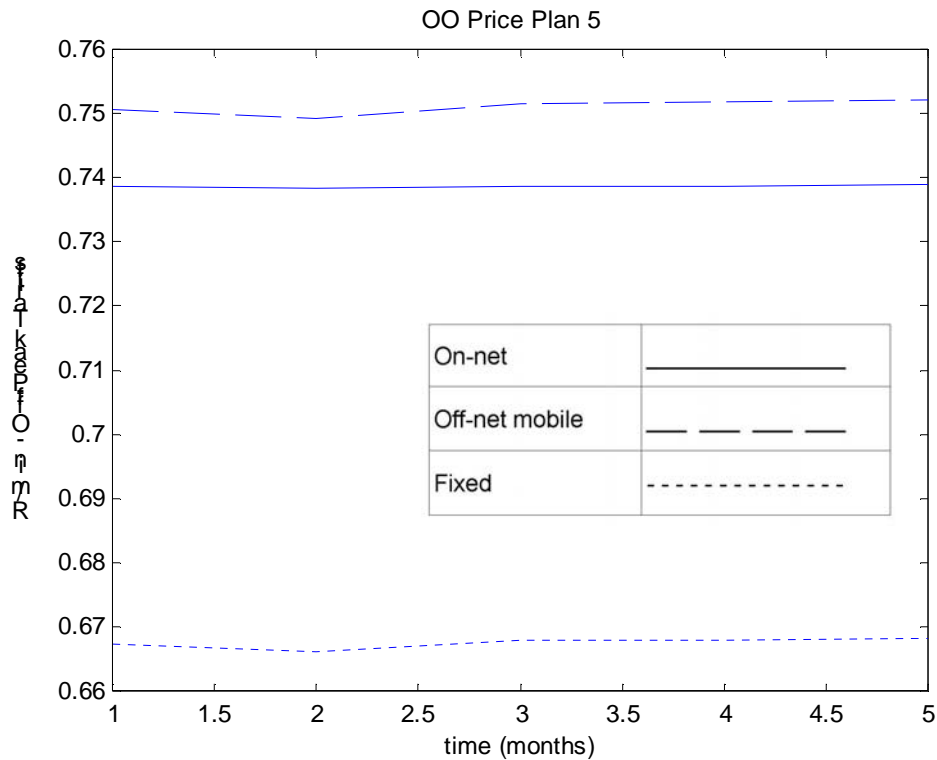
Figure 7-40: Off-Peak Tariffs for OO Price Plan 5

7.2   Discussion

In Figure 7-1 to Figure 7-5, it is apparent that the number of subscribers in each market segment did drop a little from the open-loop simulation over the same time interval, due to the focus on maximising the ARPU of the operator per market segment. The drop is not significant though, and the controller manages to minimise the impact by bringing the various output variables back to setpoint for most of the output variables at the end of the simulation period.

The inward migrations or gross connections (negative churn) for all the OO price plans fell from the case of the open-loop simulation over the same time period (setpoint), as can be seen in Figure 7-6 to Figure 7-10. As in the case of number of subscribes per market segment, the controller's emphasis on ARPU has a negative effect on this variable. The controller does manage to constrain these variables though and keep them at a fixed difference from the setpoint signals.

The controller manages to successfully maximise the ARPU signal (the ARPU signals approach the increased setpoints of 5% above the values recorded in the open-loop simulation) for all 5 market segments, as is apparent from Figure 7-11 to Figure 7-15. For 4 of the 5 segments the controller manages to reach the setpoint signals within the first 2 months of the simulation, and for 3 of the segments the ARPU values track the setpoints at the end of the simulation period. The other 2 segments also converge at that particular stage.

By increasing the ARPU for each market segment the controller manages to increase the revenue per market segment above the totals recorded per month in the open-loop simulation (although this was not the primary focus of the controller). If the controlled outputs shown in Figure 7-16 to Figure 7-20 are compared to the total revenue output of the open-loop simulation as shown in Figure 4-10 as well as in the dashed lines in Figure 7-16 to Figure 7-20, the increase in revenue is apparent. In Figure 7-16 to Figure 7-20 the open-loop simulation stops one month before the closed-loop simulation – this was done to increase closed-loop simulation time. On the revenue graphs in this section the difference between the setpoint and the controlled output is not apparent, that is due to the setpoint increase algorithm implemented with the controller (see Figure 6-4). The setpoint was raised because the actual revenue output was above setpoint in each month of the simulation. The revenue increase for each month, per market segment in total as well as aggregated in the control simulation period is as shown in percentage form in Table 7-1. The total additional revenue achieved by increasing the ARPU by 5 %, is 3.62 % over the 5 months of the open-loop simulation.

Table 7-1. The Revenue Increase (in Percentage) Achieved by the Controller by Focussing on an ARPU Increase of 5%

Month in Simulation	Consumer Low	Consumer High	Medium	Business Low	Business High
1	0.00%	0.00%	0.00%	0.00%	0.00%
2	6.64%	6.23%	4.08%	5.19%	4.22%
3	4.33%	4.22%	1.80%	3.54%	3.43%
4	4.38%	4.50%	1.61%	4.10%	3.96%
Total	3.86%	3.75%	1.90%	3.23%	2.91%
Grand Total					3.62%

The average usage per user as well as the total usage per market segment have fallen as a result of the effort of the controller to increase ARPU as well as revenue. This is apparent from Figure 7-21 to Figure 7-30. The decrease in these outputs is not significant, except in the case of the Consumer Low Market Segment.

The manipulated variables and the dynamic change made to them over time as calculated by the QP optimiser of the MPC controller are plotted over the control simulation interval in Figure 7-31 to Figure 7-40. Some tariffs were increased and others decreased with the strategic intent of maximising ARPU per market segment while at least maintaining the other output variables at their open-loop simulation values over time. The changes to the tariffs are smooth due to the large weighting applied to the manipulated variables in the MPC controller (see section 6.3 above).

7.3 Conclusion

Based on the results and discussion in the previous two sections, the non-linear model with a linear MPC controller implemented in a feedback control loop achieved all the objectives of the research (section 1.2 as well as 6.3 above). The following has been achieved in addition to the *a priori* objectives:

1. The ARPU output variables per market segment were chosen as the strategic output variables to be optimised. This was achieved with the spin-off that the total revenue per market segment was raised by a total of 3.62% over the control simulation interval. This phenomenon, that a focused increase of ARPU results in an increase in total revenue, will probably always occur because of the inherent structure of the market.

2. With the closed-loop system as developed, the optimising operator is able to see the impact of price changes on volume and margin in the future in all market segments across all products offered - thus avoiding costly mistakes.

3. Although the linear model is open-loop unstable, and neither controllable nor observable (see section 6.2), the linear controller applied to the non-linear model in the simulation produces a controlled output with no sign of instability.

8   CONCLUSIONS AND RECOMMENDATIONS

8.1   Conclusions

The main contribution of this dissertation is a dynamic system dynamics model of a telecommunications market, and a solution to the dynamic optimisation problem of marketing strategy (pricing) of telecommunication services.

The different conclusions documented in sections 3.5, 4.3, and 7.3 can be summarised in the following points:

1. The model as developed implements methodology from a wide spectrum of literature in the field of forecasting and in particular that of telecommunications forecasting.

2. The developed model adequately represents the industry modelled, based on comparisons with relevant industry data. The model incorporates an entering third operator in the market, adding to the scope and complexity of the model.

3. The model being dynamic and in a system dynamics format contributes to the field of telecommunication forecasting in a meaningful way.

4. A linear feedback, model based predictive controller implemented on the non-linear model successfully optimises the strategic ARPU output vector of the model for the optimising operator by manipulating the tariffs of the optimising operator.

5. In addition to the ARPU vector, the revenue vector is also increased by a substantial margin potentially resulting in large profit increase for the operator. Most of the other controlled variables are still controlled at or close to their respective setpoints.

6. The implemented controller contributes to the field of price optimisation in telecommunications in a unique way.

7. The goal of the project as stated in section 1.2 has been successfully achieved, and a foundation has been built for future research as detailed in the section 8.2.

8.2   Further Study

A number of areas have been identified for future research in the course of this research. These are documented under the headings below.

8.2.1   Market Modelling

The following possible improvements can be made to the model:

1. Different forecasting methods, for instance classical diffusion modelling versus system dynamics modelling, have not been tested and compared with one another in the literature to a significant enough degree to provide decision support to forecasting practitioners on which technique to apply [3].

2. The modelling of the growth of the telecommunication market in South Africa could be done with more accuracy and precision. Some of the diffusion modelling techniques discussed in section 2.4 could be applied for this purpose. Future work in the field could involve such an emphasis which will result in a more complete and holistic model of the entire market and all interactions present in it.

3. Interconnect revenue could be added to the revenue and ARPU modelling.

4. More of the market attributes, variables and inputs listed in section 3.3 above which have not included in this model could be included in future work. For example, the capturing and usage of actual marketing effort data as dynamic input into the model.

8.2.2   Control and Decision Support for Market Optimisation

Further research into the following areas is needed in order to bring this research to a state of greater maturity which might lead to commercialisation:

1. The observation that an increase in total revenue follows when ARPU is maximised will have to be verified.

2. Other outputs than the ARPU could be chosen as the primary strategic output to focus on as well.   Fluctuation between different priority outputs and parameters as key performance indicators often occur in the marketing departments of operators.

One particular output typically remains key for a season and then changes to a different one. The controller implemented in this research could easily be adjusted in such a scenario to focus on the new output variable. It might be more difficult to maximise the gross connections (migrations vector) output as the strategic output because if its inherent band-pass frequency response (see Figure 6-2) and subsequent rejection of low frequencies.

3. Additional analysis and investigation regarding the inherent instability of the linear model should be conducted. The stability of the non-linear model needs to be ascertained; and formal control design techniques for open-loop unstable systems should be experimented with.

4. Other control strategies could be applied to the problem and compared with MPC.

5. Adaptive control which makes it possible to run the system on-line and real time with the internal model "learning" and adjusting to changes in the external market, would add a significant dimension of decision support to management.

6. The suggestions made by the controller for price change should be implemented in real-time on an actual case study and the market response verified with empirical data.

9   REFERENCES

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A   APPENDIX A

In this appendix the different matrices of the linear model are documented.

The **A** matrix of the linear model can be summarised in the following manner:

Table A-1. The A-matrix (110 × 110) of the Linear Model Summarised

Matrix **A**1 (22 × 22)	0 (22 × 22)	0 (22 × 22)	0 (22 × 22)	0 (22 × 22)
0 (22 × 22)	Matrix **A**2 (22 × 22)	0 (22 × 22)	0 (22 × 22)	0 (22 × 22)
0 (22 × 22)	0 (22 × 22)	Matrix **A**3 (22 × 22)	0 (22 × 22)	0 (22 × 22)
0 (22 × 22)	0 (22 × 22)	0 (22 × 22)	Matrix **A**4 (22 × 22)	0 (22 × 22)
0 (22 × 22)	0 (22 × 22)	0 (22 × 22)	0 (22 × 22)	Matrix **A**5 (22 × 22)

The different sub-matrices in Table A-1 are defined in the following 5 tables:

Table A-2. The A1 Matrix (22 × 22) as per Table A-1

-0.007059	0.000784	0.000784	0.000784	0.000784	0.000784	0.000784	0.000784	0.000784	0.000784	0.000784	0.000784	0.000784	0.000784	0.000784	0.000784	0	0	0	0	0	0
0.000325	-0.00752	0.000325	0.000325	0.000325	0.000325	0.000325	0.000325	0.000325	0.000325	0.000325	0.000325	0.000325	0.000325	0.000325	0.000325	0	0	0	0	0	0
0.000293	0.000293	-0.00755	0.000293	0.000293	0.000293	0.000293	0.000293	0.000293	0.000293	0.000293	0.000293	0.000293	0.000293	0.000293	0.000293	0	0	0	0	0	0
0.000234	0.000234	0.000234	-0.00761	0.000234	0.000234	0.000234	0.000234	0.000234	0.000234	0.000234	0.000234	0.000234	0.000234	0.000234	0.000234	0	0	0	0	0	0
0.000213	0.000213	0.000213	0.000213	-0.00763	0.000213	0.000213	0.000213	0.000213	0.000213	0.000213	0.000213	0.000213	0.000213	0.000213	0.000213	0	0	0	0	0	0
0.000259	0.000259	0.000259	0.000259	0.000259	-0.00758	0.000259	0.000259	0.000259	0.000259	0.000259	0.000259	0.000259	0.000259	0.000259	0.000259	0	0	0	0	0	0
0.000552	0.000552	0.000552	0.000552	0.000552	0.000552	-0.00729	0.000552	0.000552	0.000552	0.000552	0.000552	0.000552	0.000552	0.000552	0.000552	0	0	0	0	0	0
0.000311	0.000311	0.000311	0.000311	0.000311	0.000311	0.000311	-0.00753	0.000311	0.000311	0.000311	0.000311	0.000311	0.000311	0.000311	0.000311	0	0	0	0	0	0
0.000332	0.000332	0.000332	0.000332	0.000332	0.000332	0.000332	0.000332	-0.00751	0.000332	0.000332	0.000332	0.000332	0.000332	0.000332	0.000332	0	0	0	0	0	0
0.00026	0.00026	0.00026	0.00026	0.00026	0.00026	0.00026	0.00026	0.00026	-0.00758	0.00026	0.00026	0.00026	0.00026	0.00026	0.00026	0	0	0	0	0	0
0.000227	0.000227	0.000227	0.000227	0.000227	0.000227	0.000227	0.000227	0.000227	0.000227	-0.00762	0.000227	0.000227	0.000227	0.000227	0.000227	0	0	0	0	0	0
0.001778	0.001778	0.001778	0.001778	0.001778	0.001778	0.001778	0.001778	0.001778	0.001778	0.001778	-0.00607	0.001778	0.001778	0.001778	0.001778	0	0	0	0	0	0
0.00133	0.00133	0.00133	0.00133	0.00133	0.00133	0.00133	0.00133	0.00133	0.00133	0.00133	0.00133	-0.00651	0.00133	0.00133	0.00133	0	0	0	0	0	0
0.000391	0.000391	0.000391	0.000391	0.000391	0.000391	0.000391	0.000391	0.000391	0.000391	0.000391	0.000391	0.000391	-0.00745	0.000391	0.000391	0	0	0	0	0	0
0.000301	0.000301	0.000301	0.000301	0.000301	0.000301	0.000301	0.000301	0.000301	0.000301	0.000301	0.000301	0.000301	0.000301	-0.00754	0.000301	0	0	0	0	0	0
0.000253	0.000253	0.000253	0.000253	0.000253	0.000253	0.000253	0.000253	0.000253	0.000253	0.000253	0.000253	0.000253	0.000253	0.000253	-0.00759	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
-0.005995	-0.00599	-0.00599	-0.00599	-0.00599	0.001849	0.001849	0.001849	0.001849	0.001849	0.001849	0.001849	0.001849	0.001849	0.001849	0.001849	0	-1	0	0	0	0
-8.22E-05	0.000182	0.000289	0.000847	0.001984	7.16E-07	7.16E-07	7.16E-07	7.16E-07	7.16E-07	7.16E-07	7.16E-07	7.16E-07	7.16E-07	7.16E-07	7.16E-07	0	0	-1	0	0	0
374.9311	483.0211	526.768	755.0776	1220.429	1.052	1.052	1.052	1.052	1.052	1.052	1.052	1.052	1.052	1.052	1.052	0	0	0	-1	0	0
1.76E-05	-0.00016	-0.00017	-0.00016	-0.00016	-1.61E-07	-1.61E-07	-1.61E-07	-1.61E-07	-1.61E-07	-1.61E-07	-1.61E-07	-1.61E-07	-1.61E-07	-1.61E-07	-1.61E-07	0	0	0	0	-1	0
123.968	49.7147	48.7335	49.6678	49.6678	0.15125	0.15125	0.15125	0.15125	0.15125	0.15125	0.15125	0.15125	0.15125	0.15125	0.15125	0	0	0	0	0	-1

Table A-3. The A2 Matrix (22 × 22) as per Table A-1

-0.0073	0.000531	0.000531	0.000531	0.000531	0.00053	0.00053	0.00053	0.00053	0.00053	0.00053	0.00053	0.00053	0.00053	0.00053	0.00053	0	0	0	0	0	0
0.00037	-0.00748	0.000366	0.000366	0.000366	0.00037	0.00037	0.00037	0.00037	0.00037	0.00037	0.00037	0.00037	0.00037	0.00037	0.00037	0	0	0	0	0	0
0.00037	0.000368	-0.00748	0.000368	0.000368	0.00037	0.00037	0.00037	0.00037	0.00037	0.00037	0.00037	0.00037	0.00037	0.00037	0.00037	0	0	0	0	0	0
0.00027	0.000272	0.000272	-0.00757	0.000272	0.00027	0.00027	0.00027	0.00027	0.00027	0.00027	0.00027	0.00027	0.00027	0.00027	0.00027	0	0	0	0	0	0
0.00023	0.00023	0.00023	0.00023	-0.00761	0.00023	0.00023	0.00023	0.00023	0.00023	0.00023	0.00023	0.00023	0.00023	0.00023	0.00023	0	0	0	0	0	0
0.00026	0.000262	0.000262	0.000262	0.000262	-0.0076	0.00026	0.00026	0.00026	0.00026	0.00026	0.00026	0.00026	0.00026	0.00026	0.00026	0	0	0	0	0	0
0.00041	0.000405	0.000405	0.000405	0.000405	0.00041	-0.00744	0.00041	0.00041	0.00041	0.00041	0.00041	0.00041	0.00041	0.00041	0.00041	0	0	0	0	0	0
0.00034	0.000338	0.000338	0.000338	0.000338	0.00034	0.00034	-0.00751	0.00034	0.00034	0.00034	0.00034	0.00034	0.00034	0.00034	0.00034	0	0	0	0	0	0
0.00041	0.000411	0.000411	0.000411	0.000411	0.00041	0.00041	0.00041	-0.00743	0.00041	0.00041	0.00041	0.00041	0.00041	0.00041	0.00041	0	0	0	0	0	0
0.00032	0.000316	0.000316	0.000316	0.000316	0.00032	0.00032	0.00032	0.00032	-0.00753	0.00032	0.00032	0.00032	0.00032	0.00032	0.00032	0	0	0	0	0	0
0.00024	0.000242	0.000242	0.000242	0.000242	0.00024	0.00024	0.00024	0.00024	0.00024	-0.0076	0.00024	0.00024	0.00024	0.00024	0.00024	0	0	0	0	0	0
0.00071	0.000708	0.000708	0.000708	0.000708	0.00071	0.00071	0.00071	0.00071	0.00071	0.00071	-0.00714	0.00071	0.00071	0.00071	0.00071	0	0	0	0	0	0
0.00225	0.002246	0.002246	0.002246	0.002246	0.00225	0.00225	0.00225	0.00225	0.00225	0.00225	0.00225	-0.0056	0.00225	0.00225	0.00225	0	0	0	0	0	0
0.00045	0.000449	0.000449	0.000449	0.000449	0.00045	0.00045	0.00045	0.00045	0.00045	0.00045	0.00045	0.00045	-0.0074	0.00045	0.00045	0	0	0	0	0	0
0.00042	0.000417	0.000417	0.000417	0.000417	0.00042	0.00042	0.00042	0.00042	0.00042	0.00042	0.00042	0.00042	0.00042	-0.0074	0.00042	0	0	0	0	0	0
0.00028	0.000281	0.000281	0.000281	0.000281	0.00028	0.00028	0.00028	0.00028	0.00028	0.00028	0.00028	0.00028	0.00028	0.00028	-0.00756	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
-0.0061	-0.00608	-0.00608	-0.00608	-0.00608	0.00177	0.00177	0.00177	0.00177	0.00177	0.00177	0.00177	0.00177	0.00177	0.00177	0.00177	0	-1	0	0	0	0
-0.0005	-6.93E-05	-5.63E-05	0.000955	0.00372	8.07E-07	8.07E-07	8.07E-07	8.07E-07	8.07E-07	8.07E-07	8.07E-07	8.07E-07	8.07E-07	8.07E-07	8.07E-07	0	0	-1	0	0	0
557.922	627.102	629.1673	789.1849	1226.726	1.2606	1.2606	1.2606	1.2606	1.2606	1.2606	1.2606	1.2606	1.2606	1.2606	1.2606	0	0	0	-1	0	0
0.00071	-7.56E-05	-9.82E-05	-7.70E-05	-7.70E-05	2.79E-07	2.79E-07	2.79E-07	2.79E-07	2.79E-07	2.79E-07	2.79E-07	2.79E-07	2.79E-07	2.79E-07	2.79E-07	0	0	0	0	-1	0
273.125	149.0209	145.4469	148.8067	148.8067	0.32969	0.32969	0.32969	0.32969	0.32969	0.32969	0.32969	0.32969	0.32969	0.32969	0.32969	0	0	0	0	0	-1

Table A-4. The A3 Matrix (22 × 22) as per Table A-1

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22
-0.00747	0.00037	0.00037	0.00037	0.00037	0.00037	0.00037	0.00037	0.00037	0.00037	0.00037	0.00037	0.00037	0.00037	0.00037	0.00037	0	0	0	0	0	0
0.00035	-0.00749	0.00035	0.00035	0.00035	0.00035	0.00035	0.00035	0.00035	0.00035	0.00035	0.00035	0.00035	0.00035	0.00035	0.00035	0	0	0	0	0	0
0.000415	0.000415	-0.00743	0.000415	0.000415	0.000415	0.000415	0.000415	0.000415	0.000415	0.000415	0.000415	0.000415	0.000415	0.000415	0.000415	0	0	0	0	0	0
0.000335	0.000335	0.000335	-0.00751	0.000335	0.000335	0.000335	0.000335	0.000335	0.000335	0.000335	0.000335	0.000335	0.000335	0.000335	0.000335	0	0	0	0	0	0
0.000259	0.000259	0.000259	0.000259	-0.00758	0.000259	0.000259	0.000259	0.000259	0.000259	0.000259	0.000259	0.000259	0.000259	0.000259	0.000259	0	0	0	0	0	0
0.000255	0.000255	0.000255	0.000255	0.000255	-0.00759	0.000255	0.000255	0.000255	0.000255	0.000255	0.000255	0.000255	0.000255	0.000255	0.000255	0	0	0	0	0	0
0.000268	0.000268	0.000268	0.000268	0.000268	0.000268	-0.00758	0.000268	0.000268	0.000268	0.000268	0.000268	0.000268	0.000268	0.000268	0.000268	0	0	0	0	0	0
0.000303	0.000303	0.000303	0.000303	0.000303	0.000303	0.000303	-0.00754	0.000303	0.000303	0.000303	0.000303	0.000303	0.000303	0.000303	0.000303	0	0	0	0	0	0
0.00043	0.00043	0.00043	0.00043	0.00043	0.00043	0.00043	0.00043	-0.00741	0.00043	0.00043	0.00043	0.00043	0.00043	0.00043	0.00043	0	0	0	0	0	0
0.000423	0.000423	0.000423	0.000423	0.000423	0.000423	0.000423	0.000423	0.000423	-0.00742	0.000423	0.000423	0.000423	0.000423	0.000423	0.000423	0	0	0	0	0	0
0.000258	0.000258	0.000258	0.000258	0.000258	0.000258	0.000258	0.000258	0.000258	0.000258	-0.00759	0.000258	0.000258	0.000258	0.000258	0.000258	0	0	0	0	0	0
0.000414	0.000414	0.000414	0.000414	0.000414	0.000414	0.000414	0.000414	0.000414	0.000414	0.000414	-0.00743	0.000414	0.000414	0.000414	0.000414	0	0	0	0	0	0
0.002088	0.002088	0.002088	0.002088	0.002088	0.002088	0.002088	0.002088	0.002088	0.002088	0.002088	0.002088	-0.00575	0.002088	0.002088	0.002088	0	0	0	0	0	0
0.000422	0.000422	0.000422	0.000422	0.000422	0.000422	0.000422	0.000422	0.000422	0.000422	0.000422	0.000422	0.000422	-0.00742	0.000422	0.000422	0	0	0	0	0	0
0.000941	0.000941	0.000941	0.000941	0.000941	0.000941	0.000941	0.000941	0.000941	0.000941	0.000941	0.000941	0.000941	0.000941	-0.0069	0.000941	0	0	0	0	0	0
0.000313	0.000313	0.000313	0.000313	0.000313	0.000313	0.000313	0.000313	0.000313	0.000313	0.000313	0.000313	0.000313	0.000313	0.000313	-0.00753	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
-0.00611	-0.00611	-0.00611	-0.00611	-0.00611	0.001729	0.001729	0.001729	0.001729	0.001729	0.001729	0.001729	0.001729	0.001729	0.001729	0.001729	0	-1	0	0	0	0
-0.05918	-0.03437	-0.05415	-0.00752	0.24708	5.16E-06	5.16E-06	5.16E-06	5.16E-06	5.16E-06	5.16E-06	5.16E-06	5.16E-06	5.16E-06	5.16E-06	5.16E-06	0	0	-1	0	0	0
779.7197	817.2999	787.3403	857.9874	1243.707	1.5201	1.5201	1.5201	1.5201	1.5201	1.5201	1.5201	1.5201	1.5201	1.5201	1.5201	0	0	0	-1	0	0
0.04937	-0.01791	-0.0251	-0.01835	-0.01835	-9.38E-06	-9.38E-06	-9.38E-06	-9.38E-06	-9.38E-06	-9.38E-06	-9.38E-06	-9.38E-06	-9.38E-06	-9.38E-06	-9.38E-06	0	0	0	0	-1	0
548.8115	446.8884	435.9935	446.2151	446.2151	0.81033	0.81033	0.81033	0.81033	0.81033	0.81033	0.81033	0.81033	0.81033	0.81033	0.81033	0	0	0	0	0	-1

Table A-5. The A4 Matrix (22 × 22) as per Table A-1

0.000192	0.000192	0.000192	0.000192	0.000192	0.000192	0.000192	0.000192	0.000192	0.000192	0.000192	0.000192	0	0	0	0	0	0
0.000284	0.000284	0.000284	0.000284	0.000284	0.000284	0.000284	0.000284	0.000284	0.000284	0.000284	0.000284	0	0	0	0	0	0
0.000338	0.000338	0.000338	0.000338	0.000338	0.000338	0.000338	0.000338	0.000338	0.000338	0.000338	0.000338	0	0	0	0	0	0
0.000568	0.000568	0.000568	0.000568	0.000568	0.000568	0.000568	0.000568	0.000568	0.000568	0.000568	0.000568	0	0	0	0	0	0
-0.00752	0.000323	0.000323	0.000323	0.000323	0.000323	0.000323	0.000323	0.000323	0.000323	0.000323	0.000323	0	0	0	0	0	0
0.000204	-0.007639	0.000204	0.000204	0.000204	0.000204	0.000204	0.000204	0.000204	0.000204	0.000204	0.000204	0	0	0	0	0	0
0.000148	0.000148	-0.007695	0.000148	0.000148	0.000148	0.000148	0.000148	0.000148	0.000148	0.000148	0.000148	0	0	0	0	0	0
0.000244	0.000244	0.000244	-0.007599	0.000244	0.000244	0.000244	0.000244	0.000244	0.000244	0.000244	0.000244	0	0	0	0	0	0
0.000337	0.000337	0.000337	0.000337	-0.007506	0.000337	0.000337	0.000337	0.000337	0.000337	0.000337	0.000337	0	0	0	0	0	0
0.000798	0.000798	0.000798	0.000798	0.000798	-0.007045	0.000798	0.000798	0.000798	0.000798	0.000798	0.000798	0	0	0	0	0	0
0.000318	0.000318	0.000318	0.000318	0.000318	0.000318	-0.007525	0.000318	0.000318	0.000318	0.000318	0.000318	0	0	0	0	0	0
0.000184	0.000184	0.000184	0.000184	0.000184	0.000184	0.000184	-0.007659	0.000184	0.000184	0.000184	0.000184	0	0	0	0	0	0
0.000353	0.000353	0.000353	0.000353	0.000353	0.000353	0.000353	0.000353	-0.00749	0.000353	0.000353	0.000353	0	0	0	0	0	0
0.00033	0.00033	0.00033	0.00033	0.00033	0.00033	0.00033	0.00033	0.00033	-0.007513	0.00033	0.00033	0	0	0	0	0	0
0.002745	0.002745	0.002745	0.002745	0.002745	0.002745	0.002745	0.002745	0.002745	0.002745	-0.005099	0.002745	0	0	0	0	0	0
0.000477	0.000477	0.000477	0.000477	0.000477	0.000477	0.000477	0.000477	0.000477	0.000477	0.000477	-0.007366	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
-0.006138	0.001705	0.001705	0.001705	0.001705	0.001705	0.001705	0.001705	0.001705	0.001705	0.001705	0.001705	0	-1	0	0	0	0
0.005045	4.37E-06	4.37E-06	4.37E-06	4.37E-06	4.37E-06	4.37E-06	4.37E-06	4.37E-06	4.37E-06	4.37E-06	4.37E-06	0	0	-1	0	0	0
1458.351	2.3134	2.3134	2.3134	2.3134	2.3134	2.3134	2.3134	2.3134	2.3134	2.3134	2.3134	0	0	0	-1	0	0
-1.56E-05	4.55E-08	4.55E-08	4.55E-08	4.55E-08	4.55E-08	4.55E-08	4.55E-08	4.55E-08	4.55E-08	4.55E-08	4.55E-08	0	0	0	0	-1	0
743.5571	1.2789	1.2789	1.2789	1.2789	1.2789	1.2789	1.2789	1.2789	1.2789	1.2789	1.2789	0	0	0	0	0	-1

Table A-6. The A5 Matrix (22 × 22) as per Table A-1

-0.00767	0.000172	0.000172	0.000172	0.000172	0.000172	0.000172	0.000172	0.000172	0.000172	0.000172	0.000172	0.000172	0.000172	0.000172	0.000172	0	0	0	0	0	0
0.000284	-0.00756	0.000284	0.000284	0.000284	0.000284	0.000284	0.000284	0.000284	0.000284	0.000284	0.000284	0.000284	0.000284	0.000284	0.000284	0	0	0	0	0	0
0.000332	0.000332	-0.00751	0.000332	0.000332	0.000332	0.000332	0.000332	0.000332	0.000332	0.000332	0.000332	0.000332	0.000332	0.000332	0.000332	0	0	0	0	0	0
0.00072	0.00072	0.00072	-0.00712	0.00072	0.00072	0.00072	0.00072	0.00072	0.00072	0.00072	0.00072	0.00072	0.00072	0.00072	0.00072	0	0	0	0	0	0
0.000415	0.000415	0.000415	0.000415	-0.00743	0.000415	0.000415	0.000415	0.000415	0.000415	0.000415	0.000415	0.000415	0.000415	0.000415	0.000415	0	0	0	0	0	0
0.000202	0.000202	0.000202	0.000202	0.000202	-0.00764	0.000202	0.000202	0.000202	0.000202	0.000202	0.000202	0.000202	0.000202	0.000202	0.000202	0	0	0	0	0	0
0.000132	0.000132	0.000132	0.000132	0.000132	0.000132	-0.00771	0.000132	0.000132	0.000132	0.000132	0.000132	0.000132	0.000132	0.000132	0.000132	0	0	0	0	0	0
0.000237	0.000237	0.000237	0.000237	0.000237	0.000237	0.000237	-0.00761	0.000237	0.000237	0.000237	0.000237	0.000237	0.000237	0.000237	0.000237	0	0	0	0	0	0
0.000331	0.000331	0.000331	0.000331	0.000331	0.000331	0.000331	0.000331	-0.00751	0.000331	0.000331	0.000331	0.000331	0.000331	0.000331	0.000331	0	0	0	0	0	0
0.000976	0.000976	0.000976	0.000976	0.000976	0.000976	0.000976	0.000976	0.000976	-0.00687	0.000976	0.000976	0.000976	0.000976	0.000976	0.000976	0	0	0	0	0	0
0.000405	0.000405	0.000405	0.000405	0.000405	0.000405	0.000405	0.000405	0.000405	0.000405	-0.00744	0.000405	0.000405	0.000405	0.000405	0.000405	0	0	0	0	0	0
0.000156	0.000156	0.000156	0.000156	0.000156	0.000156	0.000156	0.000156	0.000156	0.000156	0.000156	-0.00769	0.000156	0.000156	0.000156	0.000156	0	0	0	0	0	0
0.000293	0.000293	0.000293	0.000293	0.000293	0.000293	0.000293	0.000293	0.000293	0.000293	0.000293	0.000293	-0.00755	0.000293	0.000293	0.000293	0	0	0	0	0	0
0.000329	0.000329	0.000329	0.000329	0.000329	0.000329	0.000329	0.000329	0.000329	0.000329	0.000329	0.000329	0.000329	-0.00751	0.000329	0.000329	0	0	0	0	0	0
0.002254	0.002254	0.002254	0.002254	0.002254	0.002254	0.002254	0.002254	0.002254	0.002254	0.002254	0.002254	0.002254	0.002254	-0.00559	0.002254	0	0	0	0	0	0
0.000604	0.000604	0.000604	0.000604	0.000604	0.000604	0.000604	0.000604	0.000604	0.000604	0.000604	0.000604	0.000604	0.000604	0.000604	-0.00724	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
-0.00592	-0.00592	-0.00592	-0.00592	-0.00592	0.001923	0.001923	0.001923	0.001923	0.001923	0.001923	0.001923	0.001923	0.001923	0.001923	0.001923	0	-1	0	0	0	0
0.013857	0.006059	0.002694	-0.00764	0.000254	-4.09E-07	-4.09E-07	-4.09E-07	-4.09E-07	-4.09E-07	-4.09E-07	-4.09E-07	-4.09E-07	-4.09E-07	-4.09E-07	-4.09E-07	0	0	-1	0	0	0
1962.655	1768.769	1685.088	1428.179	1624.443	3.1203	3.1203	3.1203	3.1203	3.1203	3.1203	3.1203	3.1203	3.1203	3.1203	3.1203	0	0	0	-1	0	0
0.000505	6.76E-05	-0.00105	7.19E-06	7.17E-06	-2.27E-07	-2.27E-07	-2.27E-07	-2.27E-07	-2.27E-07	-2.27E-07	-2.27E-07	-2.27E-07	-2.27E-07	-2.27E-07	-2.27E-07	0	0	0	0	-1	0
1004.027	993.1445	965.4701	991.6432	991.6428	1.9122	1.9122	1.9122	1.9122	1.9122	1.9122	1.9122	1.9122	1.9122	1.9122	1.9122	0	0	0	0	0	-1

Table A-7. The B-matrix (110 × 35) of the Linear Model Summarised

B11 (22x5)	**B**12 (22x5)	**B**13 (22x5)	**B**14 (22x5)	**B**15 (22x5)	**B**16 (22x5)	**Bd**1 (22x1)	0 (22x1)	0 (22x1)	0 (22x1)	0 (22x1)
B21 (22x5)	**B**22 (22x5)	**B**23 (22x5)	**B**24 (22x5)	**B**25 (22x5)	**B**26 (22x5)	0 (22x1)	**Bd**2 (22x1)	0 (22x1)	0 (22x1)	0 (22x1)
B31 (22x5)	**B**32 (22x5)	**B**33 (22x5)	**B**34 (22x5)	**B**35 (22x5)	**B**36 (22x5)	0 (22x1)	0 (22x1)	**Bd**3 (22x1)	0 (22x1)	0 (22x1)
B41 (22x5)	**B**42 (22x5)	**B**43 (22x5)	**B**44 (22x5)	**B**45 (22x5)	**B**46 (22x5)	0 (22x1)	0 (22x1)	0 (22x1)	**Bd**4 (22x1)	0 (22x1)
B51 (22x5)	**B**52 (22x5)	**B**53 (22x5)	**B**54 (22x5)	**B**55 (22x5)	**B**56 (22x5)	0 (22x1)	0 (22x1)	0 (22x1)	0 (22x1)	**Bd**5 (22x1)

Table A-8. The B11 Matrix (22 × 5) as per Table A-7

-1759.36	12.4827	3.6889	0.17454	0.002922
57.2734	-92.9656	0.80033	0.037867	0.000634
42.9565	2.0312	-27.6736	0.028401	0.000475
17.6395	0.83408	0.24649	-1.3261	0.000195
6.8593	0.32434	0.095851	0.004535	-0.02232
54.9286	2.5973	0.76756	0.036317	0.000608
160.0692	7.5688	2.2368	0.10583	0.001772
49.9746	2.363	0.69833	0.033041	0.000553
50.7993	2.402	0.70986	0.033586	0.000562
24.5255	1.1597	0.34271	0.016215	0.000271
6.8496	0.32388	0.095715	0.004529	7.58E-05
682.2913	32.262	9.5342	0.4511	0.007551
483.0982	22.8432	6.7507	0.31941	0.005346
71.3338	3.373	0.9968	0.047163	0.000789
37.472	1.7719	0.52363	0.024775	0.000415
13.2921	0.62851	0.18574	0.008788	0.000147
0	0	0	0	0
-1634.63	-77.2933	-22.842	-1.0808	-0.01809
23.0141	0.81711	0.1914	0.068998	0.004717
8747196	302652.7	68947.46	27794.89	1923.001
-154.596	0.01124	0.002769	-0.0024	-0.00232
-6.3E+07	-4478.56	-1549.59	-1107.31	-953.428

Table A-9. The B12 Matrix (22 × 5) as per Table A-7

-2785.65	19.7642	5.8408	0.27635	0.004626
90.6825	-147.195	1.2672	0.059956	0.001004
68.0142	3.216	-43.8166	0.044968	0.000753
27.9291	1.3206	0.39028	-2.0997	0.000309
10.8606	0.51354	0.15176	0.007181	-0.03533
86.9699	4.1124	1.2153	0.057501	0.000962
253.4419	11.984	3.5416	0.16757	0.002805
79.1261	3.7415	1.1057	0.052315	0.000876
80.4318	3.8032	1.1239	0.053179	0.00089
38.8318	1.8362	0.54263	0.025674	0.00043
10.8452	0.51282	0.15155	0.00717	0.00012
1080.29	51.0814	15.0958	0.71425	0.011955
764.9024	36.1683	10.6886	0.50572	0.008465
112.9447	5.3406	1.5783	0.074675	0.00125
59.3304	2.8054	0.82907	0.039227	0.000657
21.0457	0.99515	0.29409	0.013915	0.000233
0	0	0	0	0
-2588.16	-122.381	-36.1665	-1.7112	-0.02864
36.4389	1.2938	0.30305	0.10925	0.007468
13849730	479200.2	109166.8	44008.56	3044.748
-244.777	0.017796	0.004385	-0.00379	-0.00368
-1E+08	-7091.05	-2453.52	-1753.24	-1509.59

Table A-10. The B13 Matrix (22 × 5) as per Table A-7

-2492.42	17.6838	5.226	0.24726	0.004139
81.1371	-131.701	1.1338	0.053645	0.000898
60.8549	2.8775	-39.2043	0.040235	0.000673
24.9892	1.1816	0.34919	-1.8786	0.000277
9.7173	0.45948	0.13579	0.006425	-0.03161
77.8153	3.6795	1.0874	0.051448	0.000861
226.7641	10.7225	3.1688	0.14993	0.00251
70.7971	3.3476	0.98931	0.046808	0.000783
71.9654	3.4029	1.0056	0.047581	0.000796
34.7443	1.6429	0.48551	0.022972	0.000384
9.7036	0.45883	0.1356	0.006416	0.000107
966.5766	45.7044	13.5068	0.63906	0.010697
684.3872	32.3612	9.5635	0.45249	0.007574
101.0559	4.7784	1.4121	0.066814	0.001118
53.0852	2.5101	0.7418	0.035098	0.000587
18.8304	0.89039	0.26313	0.01245	0.000208
0	0	0	0	0
-2315.73	-109.499	-32.3595	-1.5311	-0.02563
32.6032	1.1576	0.27115	0.097747	0.006682
12391863	428758.1	97675.57	39376.09	2724.251
-219.011	0.015923	0.003923	-0.00339	-0.00329
-9E+07	-6344.63	-2195.25	-1568.69	-1350.69

Table A-11. The B14 Matrix (22 × 5) as per Table A-7

-2239.18	15.887	4.695	0.22214	0.003718
72.8933	-118.32	1.0186	0.048194	0.000807
54.6718	2.5851	-35.221	0.036147	0.000605
22.4502	1.0616	0.31371	-1.6878	0.000248
8.73	0.4128	0.12199	0.005772	-0.0284
69.909	3.3056	0.9769	0.046221	0.000774
203.7241	9.6331	2.8468	0.13469	0.002255
63.6039	3.0075	0.88879	0.042052	0.000704
64.6535	3.0571	0.90346	0.042746	0.000716
31.2142	1.476	0.43618	0.020638	0.000345
8.7177	0.41222	0.12182	0.005764	9.65E-05
868.3691	41.0607	12.1344	0.57413	0.00961
614.8511	29.0731	8.5918	0.40652	0.006804
90.7883	4.2929	1.2687	0.060026	0.001005
47.6916	2.2551	0.66643	0.031532	0.000528
16.9172	0.79993	0.2364	0.011185	0.000187
0	0	0	0	0
-2080.44	-98.3732	-29.0717	-1.3755	-0.02302
29.2906	1.04	0.2436	0.087815	0.006003
11132796	385194.4	87751.31	35375.31	2447.459
-196.758	0.014305	0.003525	-0.00305	-0.00296
-8.1E+07	-5699.99	-1972.21	-1409.3	-1213.45

Table A-12. The B15 Matrix (22 × 5) as per Table A-7

-3545.36	25.1544	7.4337	0.35172	0.005887
115.4137	-187.339	1.6128	0.076307	0.001277
86.5633	4.0931	-55.7665	0.057233	0.000958
35.546	1.6808	0.49671	-2.6723	0.000393
13.8225	0.6536	0.19315	0.009139	-0.04497
110.6887	5.2339	1.5468	0.073183	0.001225
322.5614	15.2523	4.5074	0.21327	0.00357
100.7056	4.7619	1.4072	0.066583	0.001115
102.3675	4.8404	1.4305	0.067682	0.001133
49.4222	2.3369	0.69062	0.032676	0.000547
13.803	0.65267	0.19288	0.009126	0.000153
1374.911	65.0127	19.2128	0.90904	0.015216
973.5093	46.0324	13.6037	0.64365	0.010774
143.7473	6.7971	2.0087	0.095041	0.001591
75.5112	3.5705	1.0552	0.049925	0.000836
26.7854	1.2665	0.3743	0.01771	0.000296
0	0	0	0	0
-3294.01	-155.757	-46.0301	-2.1779	-0.03645

46.3768	1.6466	0.38571	0.13904	0.009505
17626932	609891.2	138939.6	56010.9	3875.13
-311.534	0.02265	0.005581	-0.00483	-0.00468
-1.3E+08	-9024.97	-3122.66	-2231.4	-1921.3

Table A-13. The B16 Matrix (22 × 5) as per Table A-7

-3172.17	22.5066	6.6512	0.3147	0.005267
103.2651	-167.62	1.443	0.068275	0.001143
77.4515	3.6623	-49.8964	0.051208	0.000857
31.8043	1.5039	0.44443	-2.391	0.000352
12.3675	0.5848	0.17282	0.008177	-0.04024
99.0374	4.683	1.3839	0.06548	0.001096
288.608	13.6468	4.033	0.19082	0.003194
90.1052	4.2606	1.2591	0.059574	0.000997
91.5921	4.3309	1.2799	0.060557	0.001014
44.2199	2.0909	0.61792	0.029237	0.000489
12.35	0.58397	0.17258	0.008165	0.000137
1230.185	58.1692	17.1904	0.81335	0.013614
871.0359	41.1869	12.1717	0.5759	0.009639
128.6162	6.0816	1.7973	0.085036	0.001423
67.5628	3.1947	0.94411	0.04467	0.000748
23.9659	1.1332	0.3349	0.015845	0.000265
0	0	0	0	0
-2947.28	-139.362	-41.1848	-1.9486	-0.03262
41.495	1.4733	0.3451	0.1244	0.008504
15771464	545692.1	124314.4	50115.02	3467.22
-278.741	0.020266	0.004993	-0.00432	-0.00419
-1.1E+08	-8074.98	-2793.96	-1996.51	-1719.06

Table A-14. The B21 Matrix (22 × 5) as per Table A-7

-460.762	8.8687	5.0499	0.35061	0.007833
18.4992	-107.848	2.3574	0.16368	0.003657
18.1974	4.0726	-61.4473	0.16101	0.003597
7.591	1.6989	0.96735	-4.3601	0.001501
2.1947	0.49118	0.27968	0.019418	-0.09848
14.6207	3.2721	1.8632	0.12936	0.00289
25.4843	5.7034	3.2475	0.22548	0.005038
15.7866	3.5331	2.0117	0.13968	0.003121
20.58	4.6058	2.6226	0.18209	0.004068
11.6224	2.6011	1.4811	0.10283	0.002297
2.1901	0.49014	0.27909	0.019377	0.000433
56.5336	12.6523	7.2043	0.50019	0.011175
217.5421	48.6863	27.7221	1.9247	0.043002

23.406	5.2383	2.9827	0.20709	0.004627
21.2241	4.75	2.7047	0.18778	0.004195
5.2894	1.1838	0.67404	0.046799	0.001046
0	0	0	0	0
-414.279	-92.7164	-52.793	-3.6654	-0.08189
5.607	43.3175	0.65635	0.27801	0.029869
621291.4	6795009	69968.61	41639.07	4673.927
-20.3806	-0.42571	-0.0016	-0.01426	-0.01159
-3292115	-82381.8	-8804.89	-2850.27	-1846.69

Table A-15. The B22 Matrix (22 × 5) as per Table A-7

-663.494	12.7709	7.2718	0.50488	0.01128
26.6388	-155.301	3.3947	0.23569	0.005266
26.2042	5.8645	-88.4841	0.23185	0.00518
10.9311	2.4464	1.393	-6.2786	0.002161
3.1604	0.70729	0.40274	0.027962	-0.14181
21.0537	4.7119	2.6829	0.18628	0.004162
36.6973	8.2129	4.6765	0.32469	0.007254
22.7326	5.0876	2.8969	0.20113	0.004494
29.6351	6.6324	3.7765	0.2622	0.005858
16.7361	3.7456	2.1327	0.14808	0.003308
3.1537	0.7058	0.40188	0.027903	0.000623
81.4082	18.2193	10.3741	0.72028	0.016092
313.2596	70.1081	39.9198	2.7716	0.061923
33.7045	7.5431	4.2951	0.29821	0.006663
30.5626	6.84	3.8947	0.27041	0.006041
7.6167	1.7046	0.97062	0.06739	0.001506
0	0	0	0	0
-596.56	-133.511	-76.0218	-5.2782	-0.11792
8.0741	62.3772	0.94514	0.40034	0.043012
894660.5	9784813	100754.8	59960.26	6730.455
-29.3481	-0.61303	-0.00231	-0.02054	-0.01668
-4740644	-118630	-12679	-4104.39	-2659.23

Table A-16. The B23 Matrix (22 × 5) as per Table A-7

-663.494	12.7709	7.2718	0.50488	0.01128
26.6388	-155.301	3.3947	0.23569	0.005266
26.2042	5.8645	-88.4841	0.23185	0.00518
10.9311	2.4464	1.393	-6.2786	0.002161
3.1604	0.70729	0.40274	0.027962	-0.14181
21.0537	4.7119	2.6829	0.18628	0.004162
36.6973	8.2129	4.6765	0.32469	0.007254
22.7326	5.0876	2.8969	0.20113	0.004494

29.6351	6.6324	3.7765	0.2622	0.005858
16.7361	3.7456	2.1327	0.14808	0.003308
3.1537	0.7058	0.40188	0.027903	0.000623
81.4082	18.2193	10.3741	0.72028	0.016092
313.2596	70.1081	39.9198	2.7716	0.061923
33.7045	7.5431	4.2951	0.29821	0.006663
30.5626	6.84	3.8947	0.27041	0.006041
7.6167	1.7046	0.97062	0.06739	0.001506
0	0	0	0	0
-596.56	-133.511	-76.0218	-5.2782	-0.11792
8.0741	62.3772	0.94514	0.40034	0.043012
894660.5	9784813	100754.8	59960.26	6730.455
-29.3481	-0.61303	-0.00231	-0.02054	-0.01668
-4740644	-118630	-12679	-4104.39	-2659.23

Table A-17. The B24 Matrix (22 × 5) as per Table A-7

-207.01	3.9845	2.2688	0.15752	0.003519
8.3113	-48.4534	1.0591	0.073536	0.001643
8.1757	1.8297	-27.6068	0.072336	0.001616
3.4105	0.76327	0.43461	-1.9589	0.000674
0.98603	0.22067	0.12565	0.008724	-0.04424
6.5687	1.4701	0.83707	0.058118	0.001298
11.4495	2.5624	1.459	0.1013	0.002263
7.0926	1.5873	0.90383	0.062753	0.001402
9.2461	2.0693	1.1783	0.081807	0.001828
5.2217	1.1686	0.66541	0.0462	0.001032
0.98394	0.22021	0.12539	0.008706	0.000195
25.3993	5.6844	3.2367	0.22472	0.005021
97.7367	21.8736	12.4549	0.86474	0.01932
10.5158	2.3534	1.3401	0.09304	0.002079
9.5355	2.1341	1.2151	0.084367	0.001885
2.3764	0.53184	0.30283	0.021026	0.00047
0	0	0	0	0
-186.126	-41.6553	-23.7186	-1.6468	-0.03679
2.5191	19.4615	0.29488	0.1249	0.01342
279130.5	3052830	31435.16	18707.41	2099.881
-9.1565	-0.19126	-0.00072	-0.00641	-0.00521
-1479066	-37012.1	-3955.83	-1280.56	-829.672

Table A-18. The B25 Matrix (22 × 5) as per Table A-7

-298.094	5.7377	3.2671	0.22683	0.005068
11.9682	-69.7729	1.5252	0.10589	0.002366
11.773	2.6348	-39.7538	0.10416	0.002327

4.9111	1.0991	0.62584	-2.8208	0.000971
1.4199	0.31777	0.18094	0.012563	-0.06371
9.459	2.1169	1.2054	0.08369	0.00187
16.4873	3.6899	2.101	0.14587	0.003259
10.2133	2.2857	1.3015	0.090364	0.002019
13.3144	2.9798	1.6967	0.1178	0.002632
7.5192	1.6828	0.95819	0.066527	0.001486
1.4169	0.3171	0.18056	0.012536	0.00028
36.5749	8.1855	4.6609	0.3236	0.00723
140.7407	31.4979	17.935	1.2452	0.02782
15.1427	3.389	1.9297	0.13398	0.002993
13.7311	3.073	1.7498	0.12149	0.002714
3.422	0.76585	0.43608	0.030277	0.000676
0	0	0	0	0
-268.021	-59.9835	-34.1548	-2.3714	-0.05298
3.6275	28.0245	0.42463	0.17986	0.019324
401948.2	4396075	45266.64	26938.67	3023.83
-13.1854	-0.27542	-0.00104	-0.00923	-0.0075
-2129856	-53297.5	-5696.39	-1844	-1194.73

Table A-19. The B26 Matrix (22 × 5) as per Table A-7

-298.094	5.7377	3.2671	0.22683	0.005068
11.9682	-69.7729	1.5252	0.10589	0.002366
11.773	2.6348	-39.7538	0.10416	0.002327
4.9111	1.0991	0.62584	-2.8208	0.000971
1.4199	0.31777	0.18094	0.012563	-0.06371
9.459	2.1169	1.2054	0.08369	0.00187
16.4873	3.6899	2.101	0.14587	0.003259
10.2133	2.2857	1.3015	0.090364	0.002019
13.3144	2.9798	1.6967	0.1178	0.002632
7.5192	1.6828	0.95819	0.066527	0.001486
1.4169	0.3171	0.18056	0.012536	0.00028
36.5749	8.1855	4.6609	0.3236	0.00723
140.7407	31.4979	17.935	1.2452	0.02782
15.1427	3.389	1.9297	0.13398	0.002993
13.7311	3.073	1.7498	0.12149	0.002714
3.422	0.76585	0.43608	0.030277	0.000676
0	0	0	0	0
-268.021	-59.9835	-34.1548	-2.3714	-0.05298
3.6275	28.0245	0.42463	0.17986	0.019324
401948.2	4396075	45266.64	26938.67	3023.83
-13.1854	-0.27542	-0.00104	-0.00923	-0.0075
-2129856	-53297.5	-5696.39	-1844	-1194.73

Table A-20. The B31 Matrix (22 × 5) as per Table A-7

-15.5641	0.38995	0.43285	0.064305	0.001219
0.54096	-7.8676	0.29888	0.044402	0.000842
0.72012	0.35843	-8.634	0.059107	0.00112
0.39603	0.19712	0.2188	-1.3093	0.000616
0.067396	0.033546	0.037236	0.005532	-0.02533
0.3918	0.19502	0.21647	0.032159	0.00061
0.45522	0.22658	0.25151	0.037364	0.000708
0.4459	0.22195	0.24636	0.0366	0.000694
0.73422	0.36545	0.40565	0.060265	0.001142
0.72883	0.36277	0.40267	0.059822	0.001134
0.067113	0.033405	0.03708	0.005509	0.000104
0.94287	0.46931	0.52093	0.077391	0.001467
6.6298	3.3	3.6629	0.54418	0.010314
0.68991	0.3434	0.38117	0.056628	0.001073
2.5113	1.25	1.3875	0.20612	0.003907
0.24261	0.12076	0.13404	0.019913	0.000377
0	0	0	0	0
-13.8396	-6.8886	-7.6463	-1.1359	-0.02153
23.5256	14.9324	8.1725	5.0765	0.55847
23535.71	16597.12	5692.969	6697.25	827.2573
-48.7065	-0.15358	-0.00148	-0.34409	-0.21029
-80391.4	-3518.05	-3648.99	-1063.07	-328.862

Table A-21. The B32 Matrix (22 × 5) as per Table A-7

-24.643	0.61742	0.68534	0.10182	0.00193
0.85652	-12.457	0.47322	0.070304	0.001333
1.1402	0.56752	-13.6705	0.093586	0.001774
0.62704	0.31211	0.34644	-2.073	0.000976
0.10671	0.053114	0.058957	0.008759	-0.0401
0.62035	0.30878	0.34274	0.050919	0.000965
0.72076	0.35875	0.39822	0.05916	0.001121
0.70601	0.35141	0.39007	0.057949	0.001098
1.1625	0.57863	0.64228	0.095419	0.001809
1.154	0.57439	0.63757	0.094719	0.001795
0.10626	0.052891	0.058709	0.008722	0.000165
1.4929	0.74307	0.8248	0.12253	0.002323
10.4972	5.2249	5.7996	0.86161	0.016331
1.0923	0.54371	0.60352	0.08966	0.0017
3.9761	1.9791	2.1968	0.32636	0.006186
0.38412	0.1912	0.21223	0.031529	0.000598
0	0	0	0	0
-21.9125	-10.9069	-12.1066	-1.7986	-0.03409
37.2488	23.6429	12.9397	8.0377	0.88425
37264.94	26278.8	9013.889	10603.98	1309.824
-77.1186	-0.24316	-0.00234	-0.54481	-0.33296
-127286	-5570.22	-5777.55	-1683.2	-520.698

Table A-22. The B33 Matrix (22 × 5) as per Table A-7

-22.6975	0.56868	0.63123	0.093778	0.001778
0.7889	-11.4736	0.43586	0.064753	0.001227
1.0502	0.52272	-12.5913	0.086198	0.001634
0.57754	0.28747	0.31909	-1.9094	0.000899
0.098286	0.048921	0.054302	0.008067	-0.03694
0.57138	0.2844	0.31568	0.046899	0.000889
0.66386	0.33043	0.36678	0.05449	0.001033
0.65027	0.32367	0.35927	0.053374	0.001012
1.0707	0.53295	0.59157	0.087886	0.001666
1.0629	0.52904	0.58723	0.087241	0.001654
0.097873	0.048716	0.054074	0.008033	0.000152
1.375	0.68441	0.75969	0.11286	0.002139
9.6685	4.8124	5.3418	0.79359	0.015042
1.0061	0.50079	0.55587	0.082582	0.001565
3.6622	1.8229	2.0234	0.3006	0.005698
0.3538	0.1761	0.19547	0.02904	0.00055
0	0	0	0	0
-20.1826	-10.0458	-11.1508	-1.6566	-0.0314
34.3081	21.7764	11.9182	7.4032	0.81444
34322.96	24204.15	8302.262	9766.824	1206.417
-71.0303	-0.22397	-0.00216	-0.5018	-0.30667
-117237	-5130.47	-5321.43	-1550.31	-479.59

Table A-23. The B34 Matrix (22 × 5) as per Table A-7

-6.3572	0.15928	0.1768	0.026265	0.000498
0.22096	-3.2135	0.12208	0.018136	0.000344
0.29413	0.1464	-3.5266	0.024142	0.000458
0.16176	0.080515	0.089371	-0.53478	0.000252
0.027528	0.013702	0.015209	0.00226	-0.01035
0.16003	0.079656	0.088418	0.013135	0.000249
0.18594	0.092548	0.10273	0.015261	0.000289
0.18213	0.090654	0.10063	0.014949	0.000283
0.29989	0.14927	0.16569	0.024615	0.000467
0.29769	0.14817	0.16447	0.024434	0.000463
0.027412	0.013644	0.015145	0.00225	4.26E-05
0.38512	0.19169	0.21277	0.03161	0.000599
2.708	1.3479	1.4961	0.22227	0.004213
0.28179	0.14026	0.15569	0.02313	0.000438
1.0257	0.51055	0.56671	0.084191	0.001596
0.099093	0.049323	0.054748	0.008134	0.000154
0	0	0	0	0
-5.6528	-2.8136	-3.1231	-0.46398	-0.00879

9.609	6.0991	3.338	2.0735	0.22811
9613.16	6779.1	2325.292	2735.496	337.8939
-19.8942	-0.06273	-0.0006	-0.14054	-0.08589
-32835.9	-1436.95	-1490.44	-434.213	-134.324

Table A-24. The B35 Matrix (22 × 5) as per Table A-7

-10.0655	0.25219	0.27993	0.041587	0.000788
0.34985	-5.0881	0.19329	0.028716	0.000544
0.46571	0.23181	-5.5838	0.038225	0.000725
0.25612	0.12748	0.1415	-0.84674	0.000398
0.043586	0.021695	0.024081	0.003578	-0.01638
0.25339	0.12612	0.13999	0.020798	0.000394
0.2944	0.14653	0.16265	0.024164	0.000458
0.28837	0.14354	0.15932	0.023669	0.000449
0.47483	0.23634	0.26234	0.038974	0.000739
0.47135	0.23461	0.26042	0.038688	0.000733
0.043403	0.021604	0.02398	0.003563	6.75E-05
0.60977	0.30351	0.33689	0.05005	0.000949
4.2876	2.1341	2.3689	0.35193	0.006671
0.44617	0.22208	0.24651	0.036622	0.000694
1.6241	0.80837	0.89729	0.1333	0.002527
0.1569	0.078095	0.086684	0.012878	0.000244
0	0	0	0	0
-8.9503	-4.4549	-4.945	-0.73463	-0.01393
15.2143	9.657	5.2852	3.283	0.36117
15220.85	10733.58	3681.715	4331.203	534.9986
-31.4991	-0.09932	-0.00096	-0.22253	-0.136
-51990.2	-2275.17	-2359.85	-687.504	-212.68

Table A-25. The B36 Matrix (22 × 5) as per Table A-7

-9.2709	0.23228	0.25783	0.038304	0.000726
0.32223	-4.6864	0.17803	0.026449	0.000501
0.42894	0.2135	-5.1429	0.035208	0.000667
0.2359	0.11742	0.13033	-0.77989	0.000367
0.040145	0.019982	0.02218	0.003295	-0.01509
0.23338	0.11616	0.12894	0.019156	0.000363
0.27116	0.13497	0.14981	0.022256	0.000422
0.26561	0.1322	0.14675	0.021801	0.000413
0.43734	0.21769	0.24163	0.035897	0.00068
0.43413	0.21609	0.23986	0.035634	0.000675
0.039977	0.019898	0.022087	0.003281	6.22E-05
0.56163	0.27955	0.3103	0.046098	0.000874
3.9491	1.9657	2.1819	0.32414	0.006144

0.41095	0.20455	0.22705	0.033731	0.000639
1.4959	0.74455	0.82645	0.12278	0.002327
0.14451	0.071929	0.079841	0.011861	0.000225
0	0	0	0	0
-8.2437	-4.1032	-4.5546	-0.67664	-0.01283
14.0132	8.8946	4.868	3.0238	0.33266
14019.2	9886.19	3391.053	3989.266	492.7618
-29.0124	-0.09148	-0.00088	-0.20496	-0.12526
-47885.7	-2095.55	-2173.55	-633.227	-195.889

Table A-26. The B41 Matrix (22 × 5) as per Table A-7

-13.8774	0.28476	0.36971	1.2001	0.05665
0.26763	-22.5991	0.5697	1.8493	0.087293
0.37826	0.62019	-29.1051	2.6137	0.12338
1.0162	1.6661	2.1632	-90.0685	0.33145
0.21131	0.34646	0.44981	1.4601	-4.5141
0.20223	0.33157	0.43048	1.3974	0.065962
0.076769	0.12587	0.16342	0.53046	0.02504
0.22715	0.37244	0.48354	1.5696	0.074091
0.34316	0.56264	0.73048	2.3712	0.11193
1.7356	2.8457	3.6945	11.9927	0.5661
0.20926	0.34311	0.44546	1.446	0.068256
0.16934	0.27765	0.36047	1.1701	0.055234
0.53507	0.8773	1.139	3.6973	0.17453
0.30646	0.50246	0.65235	2.1176	0.099958
7.5319	12.3492	16.0331	52.0442	2.4567
0.66702	1.0936	1.4199	4.609	0.21756
0	0	0	0	0
-12.004	-19.6816	-25.5527	-82.9453	-3.9154
2.1479	2.9975	1.8301	41.6385	1.0819
79254.68	106960.5	48454.28	1722844	42575.6
-1.9192	-0.05653	-0.02841	-1.4123	-0.07487
-93155.6	-17217.7	-20380.8	-124049	-6215.63

Table A-27. The B42 Matrix (22 × 5) as per Table A-7

-49.1042	1.0076	1.3082	4.2465	0.20045
0.94699	-79.9656	2.0158	6.5435	0.30888
1.3384	2.1945	-102.987	9.2483	0.43656
3.5957	5.8955	7.6542	-318.701	1.1728
0.74771	1.2259	1.5916	5.1665	-15.9731
0.71557	1.1732	1.5232	4.9445	0.2334
0.27164	0.44538	0.57824	1.877	0.088603
0.80376	1.3178	1.711	5.5538	0.26217

1.2143	1.9909	2.5848	8.3902	0.39606
6.1413	10.0692	13.0729	42.4351	2.0031
0.74047	1.2141	1.5762	5.1165	0.24152
0.5992	0.98244	1.2755	4.1403	0.19544
1.8933	3.1043	4.0303	13.0825	0.61755
1.0844	1.7779	2.3083	7.4928	0.3537
26.6512	43.697	56.7321	184.1546	8.6929
2.3602	3.8698	5.0241	16.3086	0.76984
0	0	0	0	0
-42.4753	-69.6421	-90.4166	-293.496	-13.8544
7.6002	10.6066	6.4758	147.3362	3.8284
280439.9	378476.1	171454.3	6096220	150652.1
-6.7909	-0.20004	-0.10052	-4.9973	-0.26494
-329627	-60923.7	-72116.1	-438940	-21993.7

Table A-28. The B43 Matrix (22 × 5) as per Table A-7

-41.6319	0.85429	1.1091	3.6003	0.16995
0.80288	-67.797	1.7091	5.5477	0.26188
1.1348	1.8605	-87.3148	7.841	0.37013
3.0486	4.9984	6.4894	-270.203	0.99436
0.63393	1.0394	1.3494	4.3803	-13.5424
0.60668	0.99471	1.2914	4.1921	0.19788
0.23031	0.37761	0.49025	1.5914	0.07512
0.68145	1.1173	1.4506	4.7087	0.22227
1.0295	1.6879	2.1914	7.1135	0.33579
5.2068	8.5369	11.0836	35.9777	1.6983
0.62779	1.0293	1.3364	4.3379	0.20477
0.50802	0.83294	1.0814	3.5103	0.1657
1.6052	2.6319	3.417	11.0917	0.52358
0.91936	1.5074	1.957	6.3526	0.29987
22.5956	37.0475	48.099	156.1314	7.3701
2.0011	3.2809	4.2596	13.8269	0.65269
0	0	0	0	0
-36.0117	-59.0444	-76.6577	-248.834	-11.7461
6.4436	8.9926	5.4903	124.9154	3.2458
237764.2	320881.8	145363.3	5168534	127726.8
-5.7575	-0.1696	-0.08523	-4.2368	-0.22462
-279467	-51652.8	-61142	-372145	-18646.9

Table A-29. The B44 Matrix (22 × 5) as per Table A-7

-3.0463	0.062509	0.081156	0.26344	0.012435
0.058748	-4.9608	0.12506	0.40594	0.019162
0.083032	0.13614	-6.3889	0.57374	0.027083

0.22307	0.36574	0.47484	-19.7712	0.072758
0.046385	0.076053	0.09874	0.32051	-0.99091
0.044392	0.072784	0.094496	0.30674	0.014479
0.016852	0.02763	0.035872	0.11644	0.005497
0.049863	0.081754	0.10614	0.34454	0.016264
0.075328	0.12351	0.16035	0.5205	0.02457
0.38098	0.62466	0.811	2.6325	0.12427
0.045936	0.075316	0.097784	0.31741	0.014983
0.037172	0.060947	0.079128	0.25685	0.012125
0.11746	0.19258	0.25003	0.8116	0.038311
0.067271	0.1103	0.1432	0.46483	0.021942
1.6533	2.7108	3.5195	11.4244	0.53928
0.14642	0.24007	0.31168	1.0117	0.047758
0	0	0	0	0
-2.635	-4.3204	-5.6091	-18.2076	-0.85947
0.47148	0.65799	0.40173	9.1402	0.2375
17397.36	23479.12	10636.29	378185.2	9345.862
-0.42128	-0.01241	-0.00624	-0.31001	-0.01644
-20448.8	-3779.5	-4473.83	-27230.3	-1364.41

Table A-30. The B45 Matrix (22 × 5) as per Table A-7

-10.779	0.22119	0.28717	0.93216	0.044002
0.20788	-17.5536	0.4425	1.4364	0.067804
0.29381	0.48172	-22.607	2.0301	0.095831
0.78931	1.2942	1.6802	-69.9595	0.25745
0.16413	0.26911	0.34939	1.1341	-3.5063
0.15708	0.25754	0.33437	1.0854	0.051235
0.05963	0.097768	0.12693	0.41203	0.01945
0.17644	0.28928	0.37558	1.2192	0.057549
0.26654	0.43702	0.56739	1.8418	0.08694
1.3481	2.2103	2.8697	9.3151	0.43971
0.16254	0.2665	0.346	1.1231	0.053017
0.13153	0.21566	0.27999	0.90887	0.042902
0.41561	0.68143	0.8847	2.8718	0.13556
0.23804	0.39028	0.5067	1.6448	0.077641
5.8503	9.5921	12.4535	40.4246	1.9082
0.5181	0.84947	1.1029	3.58	0.16899
0	0	0	0	0
-9.3239	-15.2874	-19.8477	-64.4266	-3.0412
1.6683	2.3283	1.4215	32.3421	0.84038
61559.91	83080	37636.15	1338194	33069.97
-1.4907	-0.04391	-0.02207	-1.097	-0.05816
-72357.3	-13373.6	-15830.5	-96353.2	-4827.9

Table A-31. The B46 Matrix (22 × 5) as per Table A-7

-9.1388	0.18753	0.24347	0.79031	0.037306
0.17624	-14.8824	0.37517	1.2178	0.057486
0.2491	0.40842	-19.1668	1.7212	0.081248
0.6692	1.0972	1.4245	-59.3135	0.21827
0.13916	0.22816	0.29622	0.96154	-2.9727
0.13318	0.21835	0.28349	0.92022	0.043438
0.050556	0.08289	0.10762	0.34933	0.01649
0.14959	0.24526	0.31843	1.0336	0.048792
0.22598	0.37052	0.48105	1.5615	0.07371
1.143	1.874	2.433	7.8976	0.3728
0.13781	0.22595	0.29335	0.95223	0.044949
0.11152	0.18284	0.23738	0.77056	0.036374
0.35237	0.57773	0.75008	2.4348	0.11493
0.20181	0.33089	0.4296	1.3945	0.065826
4.96	8.1324	10.5584	34.2731	1.6178
0.43926	0.7202	0.93504	3.0352	0.14327
0	0	0	0	0
-7.9051	-12.9611	-16.8274	-54.6226	-2.5784
1.4145	1.974	1.2052	27.4205	0.7125
52192.1	70437.38	31908.9	1134556	28037.59
-1.2638	-0.03723	-0.01871	-0.93003	-0.04931
-61346.4	-11338.5	-13421.5	-81690.8	-4093.22

Table A-32. The B51 Matrix (22 × 5) as per Table A-7

-0.96339	0.018934	0.024491	0.25345	0.014959
0.016266	-1.9702	0.043348	0.4486	0.026477
0.021751	0.044814	-2.5338	0.59987	0.035405
0.10106	0.20821	0.26932	-24.0351	0.1645
0.027718	0.057107	0.073866	0.76443	-1.5379
0.013175	0.027144	0.03511	0.36334	0.021445
0.004433	0.009132	0.011812	0.12224	0.007215
0.014169	0.029192	0.03776	0.39077	0.023063
0.0194	0.039969	0.051698	0.53502	0.031577
0.16039	0.33044	0.42742	4.4233	0.26106
0.027501	0.05666	0.073288	0.75844	0.044764
0.008605	0.017729	0.022932	0.23732	0.014007
0.025161	0.051837	0.06705	0.69389	0.040954
0.017692	0.03645	0.047148	0.48792	0.028797
0.43666	0.89963	1.1637	12.0424	0.71075
0.069404	0.14299	0.18496	1.9141	0.11297
0	0	0	0	0
-0.79659	-1.6412	-2.1228	-21.9687	-1.2966
0.53744	0.81258	0.51928	2.1342	14.2799
12065.58	17531.45	9455.247	17301.61	352927.2
-0.29169	-0.01623	-0.00917	-0.06265	-0.78515
-8046.79	-2040.32	-2345.14	-23468.5	-20814.1

Table A-33. The B52 Matrix (22 × 5) as per Table A-7

-9.8265	0.19312	0.2498	2.5851	0.15258
0.16592	-20.0963	0.44215	4.5756	0.27006
0.22186	0.4571	-25.845	6.1186	0.36113
1.0308	2.1238	2.747	-245.152	1.6779
0.28272	0.58249	0.75343	7.797	-15.6869
0.13438	0.27686	0.35812	3.706	0.21874
0.045211	0.093147	0.12048	1.2468	0.073591
0.14453	0.29776	0.38515	3.9857	0.23525
0.19788	0.40768	0.52732	5.457	0.32208
1.6359	3.3705	4.3596	45.1161	2.6628
0.28051	0.57792	0.74753	7.7359	0.45659
0.087774	0.18084	0.23391	2.4206	0.14287
0.25664	0.52874	0.68391	7.0775	0.41773
0.18046	0.37179	0.4809	4.9767	0.29373
4.4539	9.1762	11.8691	122.8297	7.2496
0.70792	1.4585	1.8865	19.523	1.1523
0	0	0	0	0
-8.1251	-16.7399	-21.6526	-224.075	-13.2253
5.4819	8.2883	5.2966	21.7685	145.6554
123068.9	178821	96443.79	176483.5	3599858
-2.9753	-0.16552	-0.0935	-0.63903	-8.0085
-82077.2	-20811.1	-23920.2	-239374	-212304

Table A-34. The B53 Matrix (22 × 5) as per Table A-7

-8.2851	0.16283	0.21062	2.1796	0.12864
0.13989	-16.944	0.37279	3.8579	0.2277
0.18706	0.38539	-21.7909	5.1588	0.30448
0.86913	1.7906	2.3161	-206.697	1.4147
0.23838	0.49112	0.63525	6.574	-13.2263
0.1133	0.23343	0.30194	3.1247	0.18442
0.038119	0.078536	0.10158	1.0513	0.062047
0.12186	0.25105	0.32473	3.3605	0.19834
0.16684	0.34373	0.4446	4.6011	0.27156
1.3793	2.8418	3.6758	38.0392	2.2451
0.23651	0.48727	0.63027	6.5225	0.38497
0.074006	0.15247	0.19722	2.0409	0.12046
0.21638	0.4458	0.57663	5.9673	0.3522
0.15215	0.31347	0.40547	4.196	0.24766
3.7553	7.7368	10.0073	103.5628	6.1124
0.59687	1.2297	1.5906	16.4606	0.97153
0	0	0	0	0
-6.8506	-14.114	-18.2561	-188.927	-11.1508

4.622	6.9882	4.4658	18.3538	122.8075
103764	150770.6	81315.31	148798.8	3035174
-2.5086	-0.13956	-0.07883	-0.53879	-6.7523
-69202.4	-17546.6	-20168.1	-201826	-179001

Table A-35. The B54 Matrix (22 × 5) as per Table A-7

-0.14395	0.002829	0.00366	0.037872	0.002235
0.002431	-0.2944	0.006477	0.067033	0.003956
0.00325	0.006696	-0.37862	0.089636	0.00529
0.015101	0.031112	0.040243	-3.5915	0.02458
0.004142	0.008533	0.011038	0.11423	-0.22981
0.001969	0.004056	0.005246	0.054293	0.003204
0.000662	0.001365	0.001765	0.018266	0.001078
0.002117	0.004362	0.005642	0.058391	0.003446
0.002899	0.005972	0.007725	0.079945	0.004718
0.023966	0.049376	0.063867	0.66095	0.039009
0.004109	0.008466	0.010951	0.11333	0.006689
0.001286	0.002649	0.003427	0.035462	0.002093
0.00376	0.007746	0.010019	0.10369	0.00612
0.002644	0.005447	0.007045	0.072908	0.004303
0.065248	0.13443	0.17388	1.7994	0.1062
0.010371	0.021366	0.027637	0.28601	0.01688
0	0	0	0	0
-0.11903	-0.24523	-0.3172	-3.2827	-0.19375
0.080307	0.12142	0.077593	0.3189	2.1338
1802.902	2619.642	1412.852	2585.288	52736.26
-0.04359	-0.00242	-0.00137	-0.00936	-0.11732
-1202.39	-304.876	-350.424	-3506.8	-3110.15

Table A-36. The B55 Matrix (22 × 5) as per Table A-7

-1.4683	0.028858	0.037327	0.38629	0.022799
0.024792	-3.0029	0.066068	0.68373	0.040354
0.033152	0.068302	-3.8619	0.91429	0.053962
0.15403	0.31735	0.41048	-36.6327	0.25072
0.042246	0.087039	0.11258	1.1651	-2.344
0.02008	0.041371	0.053512	0.55379	0.032685
0.006756	0.013919	0.018003	0.18631	0.010996
0.021596	0.044493	0.057551	0.59558	0.035152
0.029568	0.060918	0.078795	0.81544	0.048128
0.24445	0.50364	0.65144	6.7416	0.3979
0.041916	0.086357	0.1117	1.156	0.068226
0.013116	0.027022	0.034952	0.36171	0.021349
0.038348	0.079007	0.10219	1.0576	0.062419
0.026965	0.055555	0.071859	0.74366	0.043891
0.66553	1.3712	1.7736	18.3543	1.0833
0.10578	0.21794	0.2819	2.9173	0.17218
0	0	0	0	0

-1.2141	-2.5014	-3.2355	-33.4833	-1.9762
0.81913	1.2385	0.79145	3.2528	21.7646
18389.6	26720.35	14411.1	26370.1	537909.8
-0.44458	-0.02473	-0.01397	-0.09549	-1.1967
-12264.4	-3109.73	-3574.32	-35769.3	-31723.6

Table A-37. The B56 Matrix (22 × 5) as per Table A-7

-1.238	0.024331	0.031472	0.32569	0.019223
0.020903	-2.5319	0.055705	0.57648	0.034024
0.027952	0.057588	-3.2561	0.77087	0.045497
0.12987	0.26757	0.34609	-30.8864	0.21139
0.03562	0.073386	0.094923	0.98233	-1.9763
0.016931	0.034881	0.045118	0.46692	0.027558
0.005696	0.011735	0.015179	0.15709	0.009271
0.018208	0.037514	0.048523	0.50216	0.029638
0.02493	0.051362	0.066435	0.68753	0.040578
0.20611	0.42464	0.54925	5.6841	0.33548
0.035341	0.072811	0.094179	0.97464	0.057524
0.011058	0.022783	0.02947	0.30497	0.018
0.032333	0.066614	0.086163	0.89169	0.052628
0.022735	0.046841	0.060587	0.62701	0.037006
0.56113	1.1561	1.4954	15.4752	0.91336
0.089188	0.18375	0.23768	2.4597	0.14517
0	0	0	0	0
-1.0237	-2.109	-2.7279	-28.231	-1.6662
0.69064	1.0442	0.6673	2.7425	18.3505
15504.96	22528.92	12150.54	22233.59	453531.8
-0.37484	-0.02085	-0.01178	-0.08051	-1.009
-10340.6	-2621.93	-3013.64	-30158.4	-26747.3

Table A-38. The Bd1 Matrix (22 × 1) as per Table A-7

0.099931
0.041498
0.037297
0.029834
0.027152
0.032987
0.070438
0.039716
0.042356
0.033199
0.028942
0.22667
0.16953
0.049818
0.038411
0.032225
0
0.23571
8.98E-05

133.6217
-2.07E-05
19.215

Table A-39. The Bd2 Matrix (22 × 1) as per Table A-7

0.067765
0.046707
0.04686
0.034657
0.029328
0.033457
0.051658
0.043052
0.05245
0.040343
0.030801
0.090259
0.28634
0.057209
0.053226
0.035888
0
0.22532
0.000103
160.8849
3.52E-05
42.0611

Table A-40. The Bd3 Matrix (22 × 1) as per Table A-7

0.047175
0.044646
0.052889
0.042662
0.033052
0.032521
0.034144
0.038605
0.054774
0.053893
0.032842
0.05273
0.26626
0.053835
0.12
0.03997
0
0.22043
0.00066
193.8107
-0.00119

103.3182

Table A-41. The Bd4 Matrix (22 × 1) as per Table A-7

0.024437
0.03623
0.043092
0.072471
0.04118
0.026003
0.018844
0.031166
0.04296
0.10174
0.040578
0.023482
0.045
0.042078
0.34993
0.060811
0
0.21741
0.000557
295.0415
5.32E-06
163.0388

Table A-42. The Bd5 Matrix (22 × 1) as per Table A-7

0.021919
0.036174
0.042328
0.091859
0.052893
0.025784
0.016842
0.030271
0.042248
0.12443
0.051678
0.019944
0.03738
0.041895
0.28741
0.076948
0
0.24517
-5.17E-05
397.7947
-2.99E-05
243.7837

Table A-43. The C Matrix (30 × 110) in the linear model

C1 (30 × 22)	C2 (30 × 22)	C3 (30 × 22)	C4 (30 × 22)	C5 (30 × 22)

The individual matrices referenced in Table A-43 will be defined in the tables below.

Table A-44. The C1 Matrix (30 × 22) as per Table A-43

1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Table A-45. The C2 Matrix (30 × 22) as per Table A-43

0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	**1**	**1**	**1**	**1**	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	**1**	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	**1**	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	**1**	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	**1**	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	**1**	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	**1**	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	**1**
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Table A-46. The C3 Matrix (30 × 22) as per Table A-43

0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	**1**	**1**	**1**	**1**	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	**1**	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	**1**	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	**1**	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	**1**	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	**1**
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Table A-47. The C4 Matrix (30 × 22) as per Table A-43

0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Table A-48. The C5 Matrix (30 × 22) as per Table A-43

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1

The poles of the linearised open-loop model are shown Table A-49:

Table A-49. The Poles of the Linear Model

-1.0014	-1	-0.9999	-0.0078	-0.0078	-0.0078	-0.0078	-0.0078	-0.0078	-0.0078	0
-1	-1	-0.9999	-0.0078	-0.0078	-0.0078	-0.0078	-0.0078	-0.0078	-0.0078	0
-1	-1	-0.9999	-0.0078	-0.0078	-0.0078	-0.0078	-0.0078	-0.0078	-0.0078	0
-1	-1	-0.9999	-0.0078	-0.0078	-0.0078	-0.0078	-0.0078	-0.0078	-0.0078	0
-1	-1	-0.9999	-0.0078	-0.0078	-0.0078	-0.0078	-0.0078	-0.0078	-0.0078	0
-1	-1	-0.0078	-0.0078	-0.0078	-0.0078	-0.0078	-0.0078	-0.0078	-0.0078	0
-1	-1	-0.0078	-0.0078	-0.0078	-0.0078	-0.0078	-0.0078	-0.0078	-0.0078	0
-1	-1	-0.0078	-0.0078	-0.0078	-0.0078	-0.0078	-0.0078	-0.0078	-0.0078	0
-1	-0.9999	-0.0078	-0.0078	-0.0078	-0.0078	-0.0078	-0.0078	-0.0078	0	-0.0078 - 0.0i
-1	-0.9999	-0.0078	-0.0078	-0.0078	-0.0078	-0.0078	-0.0078	-0.0078	0	-0.0078 + 0.0i

In Table A-50 the ISE for the output variables are shown with different prediction and control horizon values implemented. It is apparent that the lowest value of the output ISE of $1{,}287874 \times 10^6$ occurs at $N_C = 2$ and $N_P = 7$;

Table A-50. The ISE of the Output Variables Recorded vs. N_C and N_P

N_P	1	2	3	4	5	6	7	8	9	10
N_C										
1	1304483	1292529	1318335	1343186	1363151	1378941	1392055	1403357	1413415	1422609
2		1290085	1290117	1288987	1288222	1287893	*1287874*	1288060	1288380	1288788
3			1289423	1288687	1288479	1288534	1288720	1288972	1289256	1289552
4				1288963	1289074	1289346	1289660	1289980	1290292	1290587
5					1289099	1289365	1289636	1289889	1290121	1290330

6						1289355	1289584	1289774	1289935	1290070
7							1289573	1289731	1289849	1289940
8								1289724	1289822	1289888
9									1289817	1289873
10										1289870

B   APPENDIX B

In this appendix the results from the non-linear open-loop simulation for the other operators in the market is shown. Lines that are not clearly visible are ar or close to zero for each element of the series.
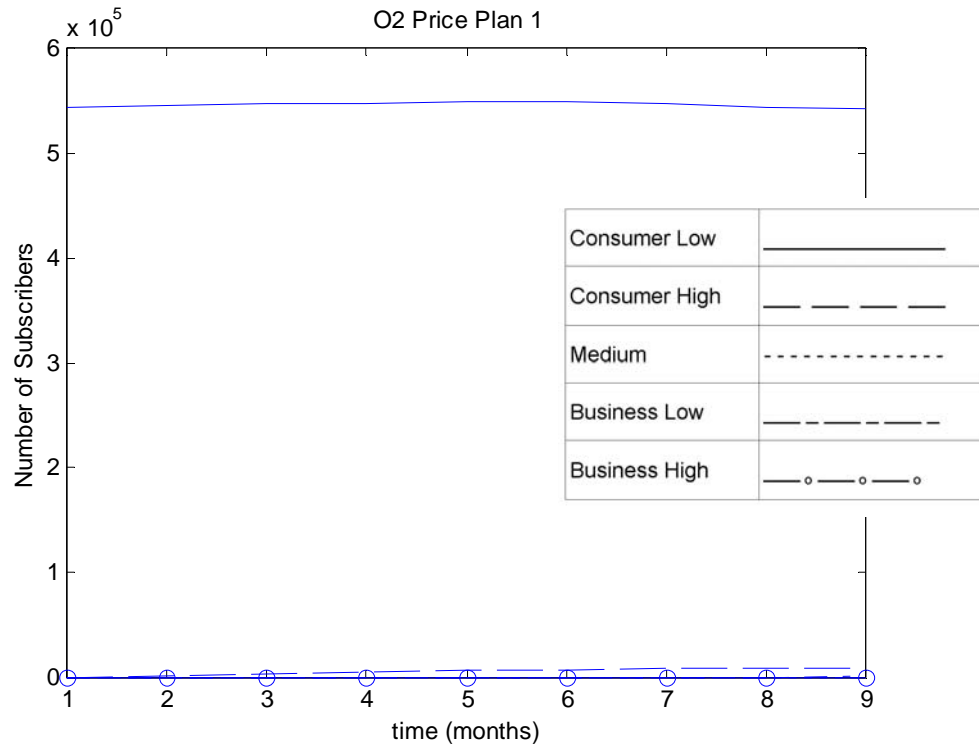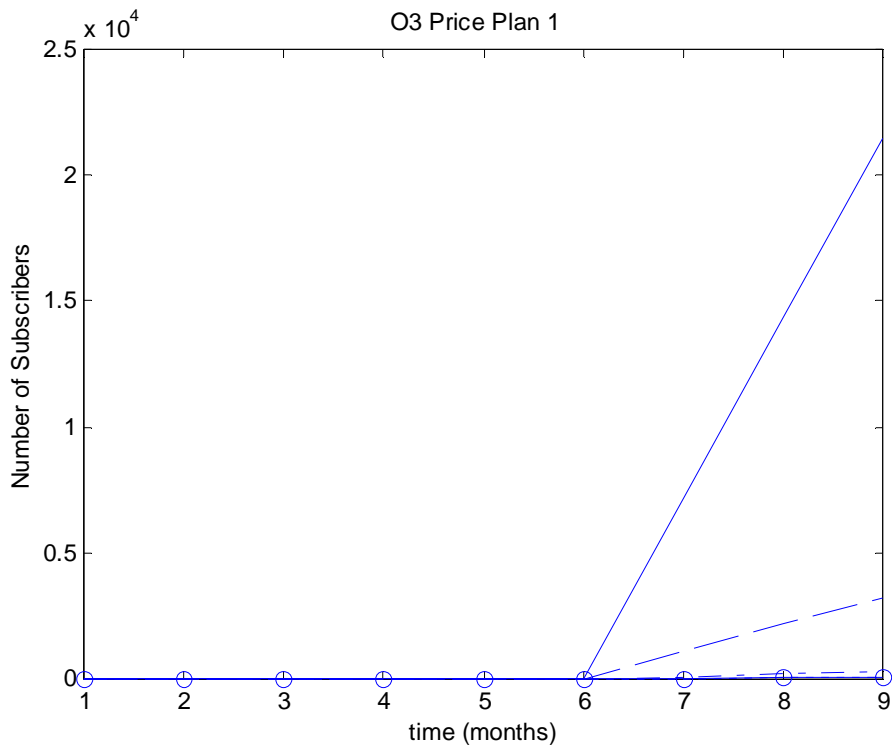

Figure B-1. The Number of Subscribers for Price Plan 1 of OP2
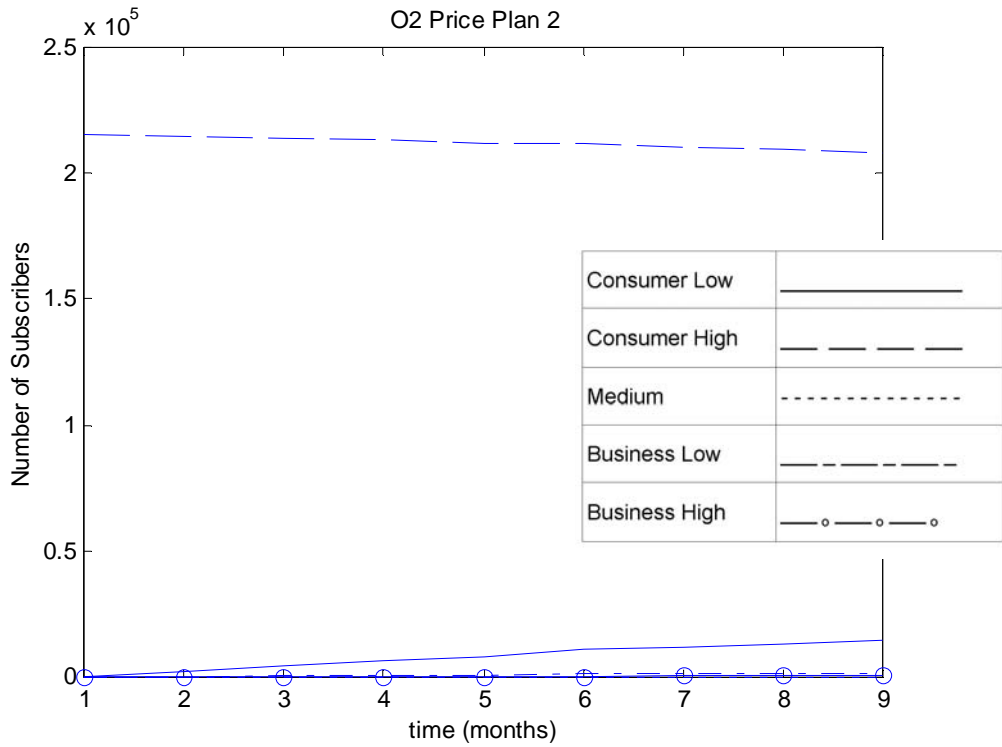
Figure B-2. The Number of Subscribers for Price Plan 2 of OP2
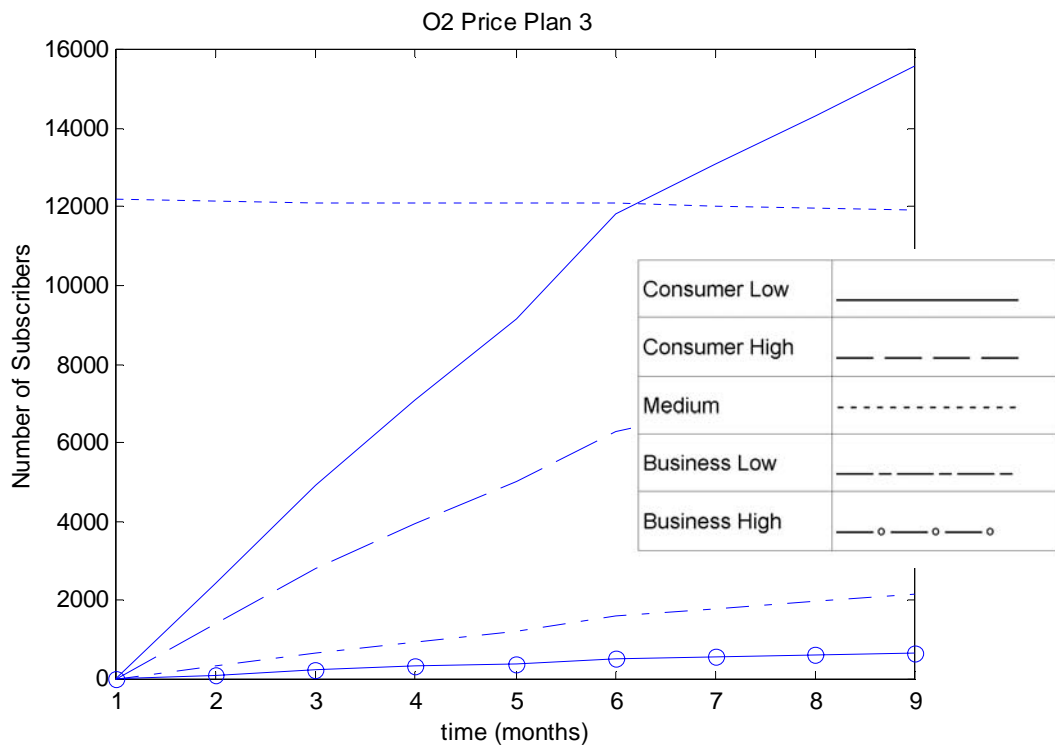
Figure B-3. The Number of Subscribers for Price Plan 3 of OP2
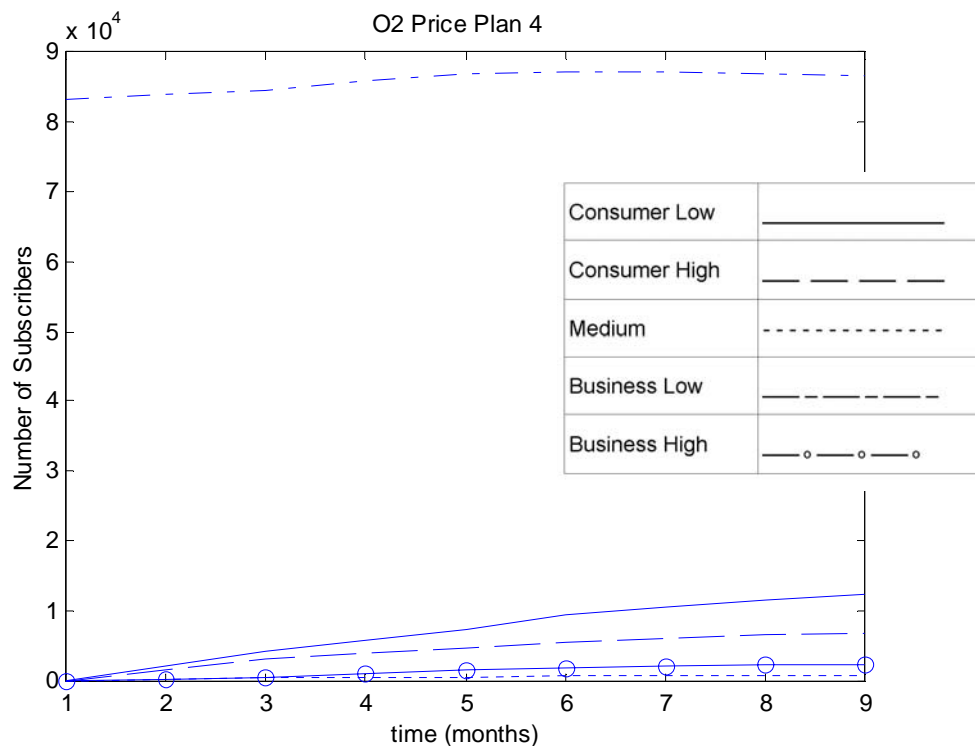
Figure B-4. The Number of Subscribers for Price Plan 4 of OP2

In Figure B-4 above the effect of the price increase can be seen in the 4th month on the top trend in the graph.
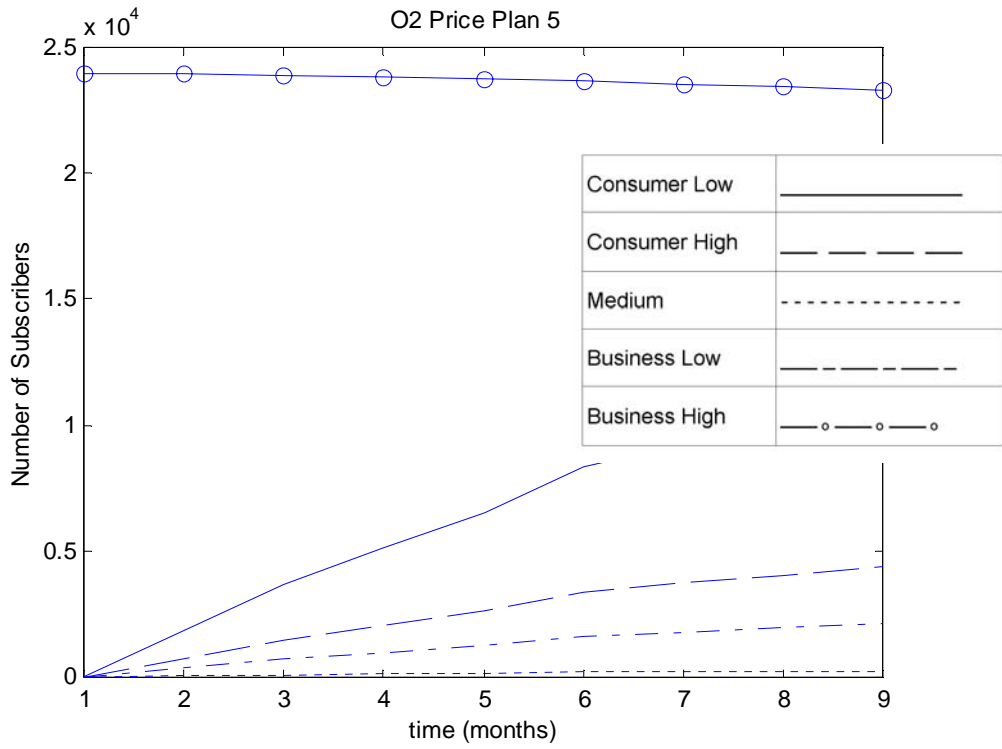
Figure B-5. The Number of Subscribers for Price Plan 5 of OP2

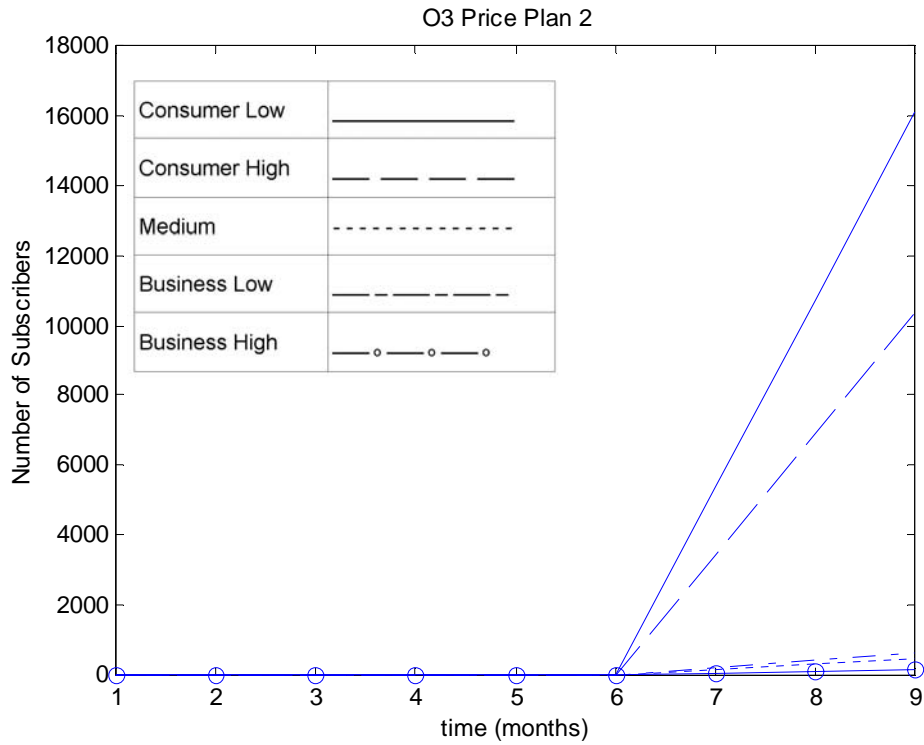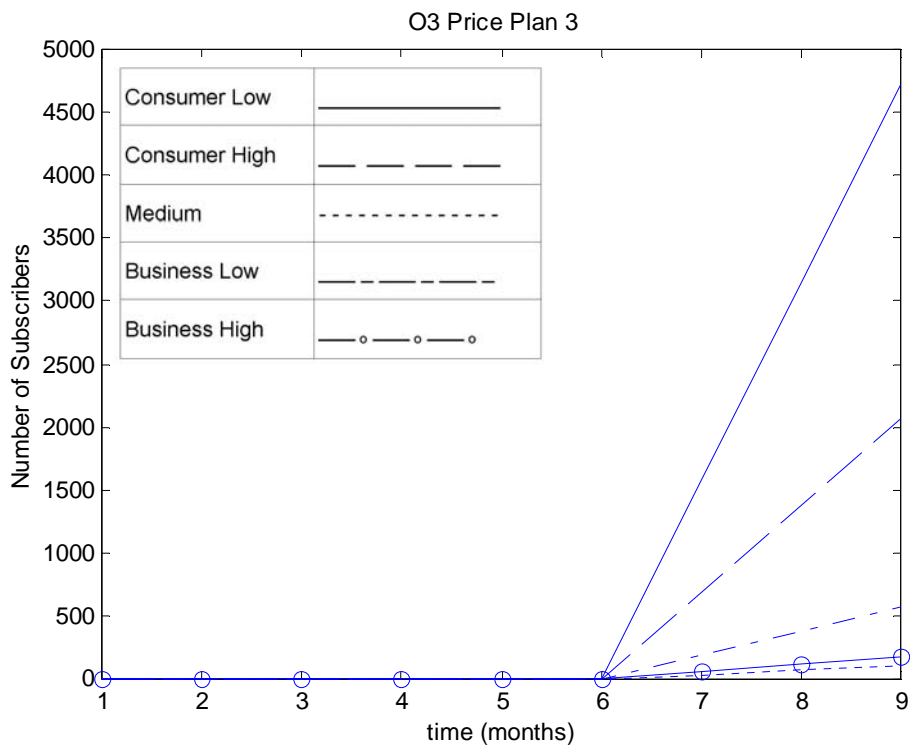
Figure B-6. The Number of Subscribers for Price Plan 1 of OP3

Figure B-7. The Number of Subscribers for Price Plan 2 of OP3

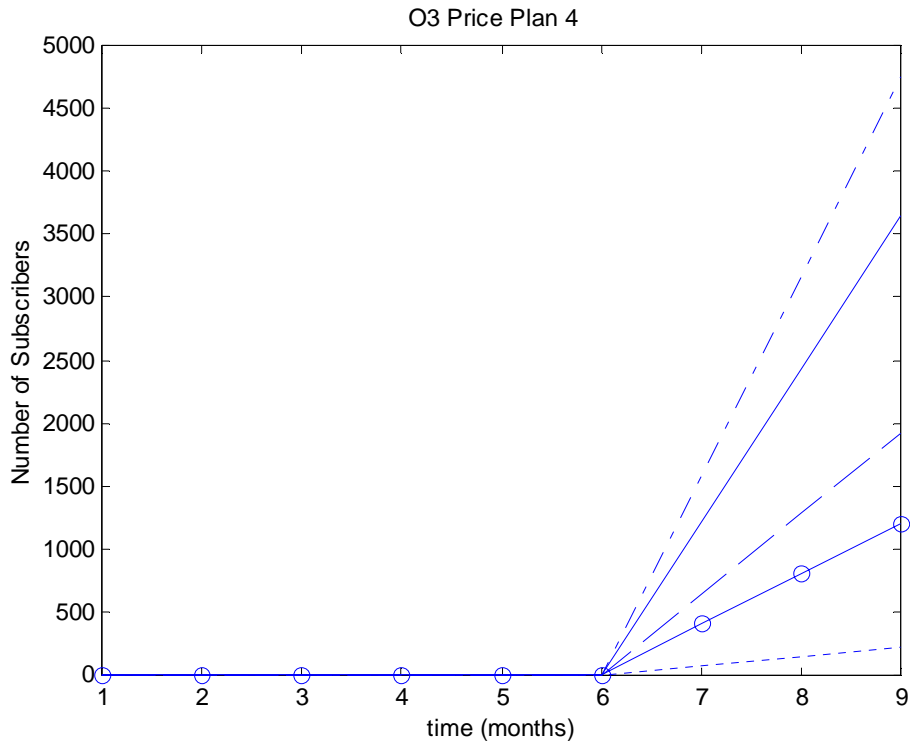**Figure B-8. The Number of Subscribers for Price Plan 3 of OP3**

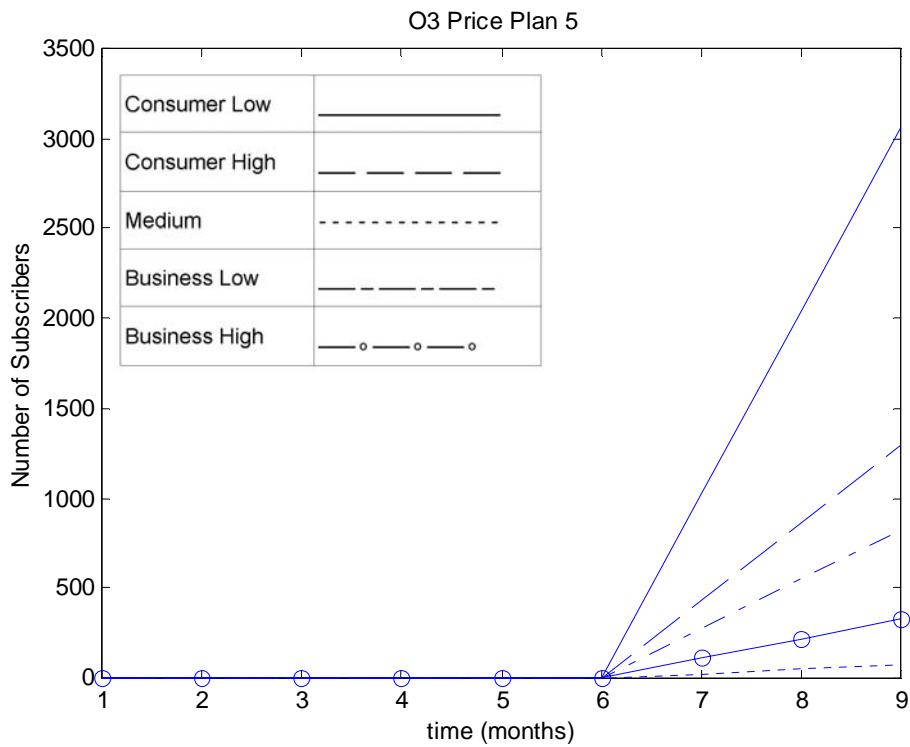**Figure B-9. The Number of Subscribers for Price Plan 4 of OP3**

Figure B-10. The Number of Subscribers for Price Plan 5 of OP3