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Discrete pulse transform of images and applications

by
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I, Inger Nicolette Fabris-Rotelli declare that the dissertation, which I hereby submit for the degree PhD Mathematical Sciences at the University of Pretoria, is my own work and has not previously been submitted by me for a degree at this or any other tertiary institution.

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Summary

The LULU operators L_n and U_n operate on neighbourhoods of size n. The Discrete Pulse Transform (DPT) of images is obtained via recursive peeling of so-called local maximum and minimum sets with the LULU operators as n increases from 1 to the maximum number of elements in the array. The DPT provides a new nonlinear decomposition of a multidimensional array. This thesis investigates the theoretical and practical soundness of the decomposition for image analysis. Properties for the theoretical justification of the DPT are provided as consistency of the decomposition (a pseudo-linear property), and its setting as a nonlinear scale-space, namely the LULU scale-space. A formal axiomatic theory for scale-space operators and scale-spaces is also presented. The practical soundness of the DPT is investigated in image sharpening, best approximation of an image, noise removal in signals and images, feature point detection with ideas to extending work to object tracking in videos, and image segmentation. LULU theory on multidimensional arrays and the DPT is now at a point where concrete signal, image and video analysis algorithms can be developed for a wide variety of applications.

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