On the real representation of quaternion random variables

Mattheüs T. Loots[a], Andriëtte Bekker[a].[1], Mohammad Arashi[b] and Jacobus J.J. Roux[a]

[a] *Department of Statistics,Faculty of Natural and Agricultural Sciences, University of Pretoria, Pretoria, 0002, South Africa.*

[b] *Faculty of Mathematics,Shahrood University of Technology, P. O. Box 316-3619995161, Shahrood, Iran.*

Abstract

For the first time, the matrix-variate quaternion normal and quaternion Wishart distributions are derived from first principles, i.e. from their real counterparts, exposing the relations between their respective densities and characteristic functions. Applications of this theory in hypothesis testing are presented, and the density function of Wilks' statistic is derived for quaternion Wishart matrices.

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1 Introduction

The first appearance of quaternion distributions in the statistical literature from the *real representation approach* is in papers of Kabe ([10], [11], [12]). Rautenbach [19] and Rautenbach and Roux ([20], [21]) also utilised the representation theory in their papers, and more recently there is the refreshing contribution by Teng and Fang [22].

This paper is set apart from the other recent papers in this field, in that the *matrix-variate quaternion normal distribution* as well as the *quaternion Wishart distribution* are derived from first principles for the first time, i.e. from their **real counterparts**, exposing the relations between their respective densities and characteristic functions.

The paper is organized as follows: In Section 2 a collection of fundamental mathematical results are presented for use in later sections. This is included since the authors believe that it has didactic value. Section 3 is first devoted to a review on the derivation of the p-variate quaternion normal distribution, using the representation theory. Armed with these results, the matrix-variate quaternion normal distribution is derived by generalising this approach. The key idea in Section 4 is how the quaternion Wishart distribution relates to its real counterpart. We conclude in Section 5 by showing that a simple quaternion hypothesis may be represented with an associated real hypothesis, and thereafter derive an expression for the density function of Wilks' statistic in the case of quaternion Wishart matrices.

The results in this paper should therefore not be seen as corollaries of the work in real normed division algebras (see for example [5],[6], [7], [13] and [14]), but the significant contribution that is made in the statistical context, by using the real counterpart approach.

2 Mathematical Preliminaries

A number of useful theorems and other general results that are found in the literature will be discussed in this section, and will be referred to frequently in subsequent sections.

[1] Corresponding author .E-mail:andriette.bekker@up.ac.za

Let $\mathbb{R}$ denote the field of real numbers, and $\mathbb{Q}$ the quaternion (Hamiltonion) division algebra over $\mathbb{R}$, respectively. Hence, every $z \in \mathbb{Q}$ can be expressed as

$$z = x_1 + ix_2 + jx_3 + kx_4,$$

where i, j, and k satisfy the following relations:

$$i^2 = j^2 = k^2 = -1, ij = -ji = k, \; jk = -kj = i, ki = -ik = j$$

and where $x_1, x_2, x_3, x_4 \in \mathbb{R}$. The conjugate of a quaternion element is defined in a similar fashion to that of a complex number, and is given by:

$$\bar{z} = x_1 - ix_2 - jx_3 - kx_4$$

Now let $M_{n\times p}(\mathbb{R})$ and $M_{n\times p}(\mathbb{Q})$ denote the set of all $n \times p$ matrices over $\mathbb{R}$ and $\mathbb{Q}$, respectively. In the case of square matrices, say $p \times p$, this will be indecated by $M_p(\mathbb{R})$ and $M_p(\mathbb{Q})$ instead. Similar to the scalar form above, any $\mathbf{Z} \in M_{n\times p}(\mathbb{Q})$ may be rewritten as:

$$\mathbf{Z} = [z_{ij}]_{n\times p} = \mathbf{X}_1 + i\mathbf{X}_2 + j\mathbf{X}_3 + k\mathbf{X}_4,$$

where $z_{ij} \in \mathbb{Q}$, and $\mathbf{X}_1$, $\mathbf{X}_2$, $\mathbf{X}_3$, and $\mathbf{X}_4 \in M_{n\times p}(\mathbb{R})$. $\mathbf{X}_1$ is the real part of $\mathbf{Z}$, and will be denoted by $\operatorname{Re}\mathbf{Z}$. By setting $n = 1$, this reduces to the vector form in an obvious way.

We will denote the transpose of a matrix $\underset{n\times p}{\mathbf{Z}}$ as $\mathbf{Z}' = \underset{p\times n}{\mathbf{Z}}$. The conjugate transpose of $\mathbf{Z}$ is therefore given by

$$\bar{\mathbf{Z}}' = \left[\bar{z}'_{ij}\right]_{p\times n} = \mathbf{X}'_1 - i\mathbf{X}'_2 - j\mathbf{X}'_3 - k\mathbf{X}'_4.$$

and we say $\mathbf{Z}$ is Hermitian if $\bar{\mathbf{Z}}' = \mathbf{Z}$.

The vec operator is frequently used in expressions involving matrices of quaternions, see for instance Li and Xue [14], and is defined as

$$\text{vec}\mathbf{Z} = \left[\underline{Z}'_1, \ldots, \underline{Z}'_p\right]' \in M_{np\times 1}(\mathbb{Q}),$$

where $\underline{Z}_\alpha \in M_{n\times 1}(\mathbb{Q}), \alpha = 1, \ldots, p$ are the columns of $\mathbf{Z}$.

We will make use of the representation theory throughout this paper, and although quaternions may be represented by real matrices in various ways, see for instance Teng and Fang [22], we will use the representation employed by Kabe (for instance [10] and [12]) and Rautenbach [19]. Specifically suppose that
$z = x_1 + ix_2 + jx_3 + kx_4 \in \mathbb{Q}$ may be represented by $\mathbf{z}_0 \in M_4(\mathbb{R})$, as

$$\underset{4\times 4}{\mathbf{z}_0} = \begin{bmatrix} x_1 & -x_2 & -x_3 & -x_4 \\ x_2 & x_1 & -x_4 & x_3 \\ x_3 & x_4 & x_1 & -x_2 \\ x_4 & -x_3 & x_2 & x_1 \end{bmatrix}.$$

Now, if $\underset{n\times p}{\mathbf{Z}} \in M_{n\times p}(\mathbb{Q})$, i.e. the case where we have matrices with quaternion elements (or vectors by setting $n = 1$), we have $\underset{n\times p}{\mathbf{Z}} = [z_{st}]$, where $z_{st} = x_{1st} + ix_{2st} + jx_{3st} + kx_{4st} \in \mathbb{Q}, s = 1, \ldots, n$, and $t = 1, \ldots, p$. By an elementwise generalization of our representation of the scalar, to the matrix (or vector) form, we have

$$\underset{4\times 4}{\mathbf{z}_{0st}} = \begin{bmatrix} x_{1st} & -x_{2st} & -x_{3st} & -x_{4st} \\ x_{2st} & x_{1st} & -x_{4st} & x_{3st} \\ x_{3st} & x_{4st} & x_{1st} & -x_{2st} \\ x_{4st} & -x_{3st} & x_{2st} & x_{1st} \end{bmatrix}$$

in other words, $\underset{n\times p}{\mathbf{Z}}$ may be represented with the real matrix $\underset{4n\times 4p}{\mathbf{Z}_0}$ as: $\underset{4n\times 4p}{\mathbf{Z}_0} = [\mathbf{z}_{0st}]$.

Moreover, if we define the mapping

$$f\left(\underset{4p\times 4p}{\mathbf{R}}\right) = \underset{p\times p}{\mathbf{Q}} \ \forall \mathbf{R} \in M_{4p}(\mathbb{R}), \mathbf{Q} \in \mathbb{Q}.$$

it was shown in Rautenbach [19] that f is a faithful representation. When $\underset{4p\times 4p}{\mathbf{R}} \in M_{4p}(\mathbb{R})$ and $f(\mathbf{R}) = \mathbf{Q} \in M_p(\mathbb{Q})$ then it will be indicated as $\mathbf{R} \simeq \mathbf{Q}$.

For the proofs of the results in Lemma 1, see Rautenbach and Roux ([20],[21]) and Mehta [17].

Lemma 1 *Suppose* $\underset{4p\times 4p}{\mathbf{R}} \in M_{4p}(\mathbb{R})$, $\underset{p\times p}{\mathbf{Q}} \in M_p(\mathbb{Q})$ *and* $\mathbf{R} \simeq \mathbf{Q}$.

1. $\mathbf{R}$ *is symmetric if and only if* $\mathbf{Q}$ *is a quaternion Hermitian matrix.*

2. $\mathbf{R}$ *is orthogonal if and only if* $\mathbf{Q}$ *is symplectic.*

3. $\mathbf{R}$ *is nonsingular with inverse* $\underset{4p\times 4p}{\mathbf{R}}^{-1} = [r^{st}] = \left[\begin{bmatrix} a_1^{st} & -a_2^{st} & -a_3^{st} & -a_4^{st} \\ a_2^{st} & a_1^{st} & -a_4^{st} & a_3^{st} \\ a_3^{st} & a_4^{st} & a_1^{st} & -a_2^{st} \\ a_4^{st} & -a_3^{st} & a_2^{st} & a_1^{st} \end{bmatrix}\right]$ *if and only if* $\mathbf{Q}$ *is nonsingular with inverse* $\underset{p\times p}{\mathbf{Q}}^{-1} = [q^{st}] = [a_1^{st} + ia_2^{st} + ja_3^{st} + ka_4^{st}]$.

4. *There exists a symplectic matrix* $\mathbf{H}$ *such that:* $\mathbf{Q} = \mathbf{H}\mathbf{D}_\lambda\mathbf{H}^{-1} = \mathbf{H}\mathbf{D}_\lambda\bar{\mathbf{H}}'$ *where* $\mathbf{D}_\lambda = (\lambda_1, \ldots, \lambda_p)$ *with* $\lambda_s = \lambda_{s\mathbf{R}} + i0 + j0 + k0$, $s = 1, \ldots, p$ *the real characteristic roots of* $\mathbf{Q}$.

5. *If* $\mathbf{R}$ *is symmetric, then* $\det \mathbf{R} = (\det \mathbf{Q})^4$.

6. *Let* $\underset{4p\times 1}{\underline{q}_0} = [b_{11}, b_{21}, b_{31}, b_{41}, \ldots, b_{1p}, b_{2p}, b_{3p}, b_{4p}]'$ *and* $\underset{p\times 1}{\underline{q}} = [q_1, \ldots, q_p]'$ *with* $q_s = b_{1s} + ib_{2s} + jb_{3s} + kb_{4s}$, $s = 1, \ldots, p$. *Let* $\mathbf{R}$ *be symmetric, such that* $\mathbf{Q}$ *is quaternion Hermitian. It now follows that* $\underline{q}_0'\mathbf{R}\underline{q}_0 = \bar{\underline{q}}'\mathbf{Q}\underline{q}$.

7. $\mathbf{R}$ *is symmetric positive definite if and only if* $\mathbf{Q}$ *is a positive definite quaternion Hermitian matrix.*

From this it is clear that the eigenvalues of a Hermitian matrix are all real since they correspond to that of the real associated, symmetric, matrix (for which they are repeated with multiplicity 4). Moreover, if the eigenvalues are all positive, then the real associated, symmetric matrix are positive definite, and we may conclude that the Hermitian quaternion matrix is a positive definite quaternion matrix. In other words, If $\mathbf{Q} \in M_p(\mathbb{Q})$ is a Hermitian positive definite matrix, then its eigenvalues $\lambda_s, s = 1, \ldots, p$ are real and positive and there exists a Hermitian positive matrix, written as $\mathbf{Q}^{\frac{1}{2}}$ such that $\mathbf{Q} = \mathbf{Q}^{\frac{1}{2}}\mathbf{Q}^{\frac{1}{2}}$, see Teng and Fang [22]. For more technical results and a review on quaternion matrix algebra, see Zhang [23].

The trace operator is frequently used in the symplification of expressions, and although the multiplication of quaternions is noncommutative, we may go about it as follows, see Zhang [23].

Let Re tr($\mathbf{A}$) =tr(Re $\mathbf{A}$) for $\mathbf{A} \in M_p(\mathbb{Q})$, we have

$$\begin{aligned} \operatorname{Re}\operatorname{tr}(\mathbf{A}) &= \tfrac{1}{2}\operatorname{tr}(\mathbf{A} + \bar{\mathbf{A}}') \\ \operatorname{Re}\operatorname{tr}(\mathbf{AB}) &= \operatorname{Re}\operatorname{tr}(\mathbf{BA}) \quad \forall \mathbf{A}, \mathbf{B} \in M_p(\mathbb{Q}) \end{aligned} \tag{1}$$

Moreover, if $\mathbf{A} = \bar{\mathbf{A}}' \in M_p(\mathbb{Q})$, i.e. a Hermitian matrix, and using Lemma 1(4) then this becomes

$$\operatorname{Re}\operatorname{tr}(\mathbf{A}) \quad = \operatorname{tr}(\mathbf{A}) = \sum_{\alpha=1}^{p} \lambda_\alpha,$$

where $\lambda_1, \ldots, \lambda_p$ are the eigenvalues of $\mathbf{A}$.

We now define some concepts specifically pertaining to the development of the quaternion distribution theory, and show how they interact with their real associated counterparts.

Definition 2 *Let*

$$\underset{p\times 1}{\underline{Z}} = \underline{X}_1 + i\underline{X}_2 + j\underline{X}_3 + k\underline{X}_4$$

be a quaternion probability vector with real associated probability vector given by

$$\underset{4p\times 1}{\underline{Z}_0} = \left[\underline{X}_1', \underline{X}_2', \underline{X}_3', \underline{X}_4'\right]'.$$

A function $f_{\underline{Z}}(\underline{z})$ of the p quaternion variables $z_s = x_{1s} + ix_{2s} + jx_{3s} + kx_{4s},\ s = 1, \ldots, p$ is called the probability density function (pdf) of $\underline{Z}$ if

$$f_{\underline{Z}}(\underline{z}) = f_{\underline{Z}_0}(\underline{z}_0)$$

where $f_{\underline{Z}_0}(\underline{z}_0)$ is the $4p$-variate real pdf of $\underset{4p\times 1}{\underline{Z}_0}$.

Note, here we require the associated counterpart of $\underset{p\times 1}{\underline{Z}}$ to be $\underset{4p\times 1}{\underline{Z}_0}$ and not necessarily $\underset{4p\times 4}{\mathbf{Z}_0}$ as defined earlier. This is due to the derivation of the quaternion normal distribution, which is discussed in Section 3.

The notion of an associated real probability matrix may be introduced in a similar fashion. Let $\underset{n\times p}{\mathbf{Z}} = \mathbf{X}_1 + i\mathbf{X}_2 + j\mathbf{X}_3 + k\mathbf{X}_4$ be a quaternion probability matrix. The associated real probability matrix is given by

$$\underset{n\times 4p}{\mathbf{Z}_0} = \left[\mathbf{X}_1', \mathbf{X}_2', \mathbf{X}_3', \mathbf{X}_4'\right].$$

Definition 3 *Let $\underset{n\times p}{\mathbf{Z}} = \mathbf{X}_1 + i\mathbf{X}_2 + j\mathbf{X}_3 + k\mathbf{X}_4$ be a quaternion probability matrix. The characteristic function of $\mathbf{Z}$ is defined as*

$$\phi_{\mathbf{Z}}(\mathbf{T}) = E\left[\exp \tfrac{\iota}{2} tr\left(\bar{\mathbf{Z}}'\mathbf{T} + \bar{\mathbf{T}}'\mathbf{Z}\right)\right], \tag{2}$$

where $\underset{n\times p}{\mathbf{T}} = \mathbf{T}_1 + i\mathbf{T}_2 + j\mathbf{T}_3 + k\mathbf{T}_4$ is a quaternion matrix and ι is the usual imaginary complex unit.

If we let $\mathbf{V} = \bar{\mathbf{Z}}'\mathbf{T} + \bar{\mathbf{T}}'\mathbf{Z}$, say, then it follows that $\bar{\mathbf{V}}' = \bar{\mathbf{T}}'\mathbf{Z} + \bar{\mathbf{Z}}'\mathbf{T} = \mathbf{V}$ implying that $\mathbf{V}$ is Hermitian and hence, from (1), the characteristic function may be written as

$$\phi_{\mathbf{Z}}(\mathbf{T}) = E\left[\exp \tfrac{\iota}{2} \operatorname{Re}\operatorname{tr}\left(\bar{\mathbf{Z}}'\mathbf{T} + \bar{\mathbf{T}}'\mathbf{Z}\right)\right] = E\left[\exp \iota \operatorname{Re}\operatorname{tr}\left(\bar{\mathbf{Z}}'\mathbf{T}\right)\right] \tag{3}$$

In the case of a quaternion probability vector, $\underset{p\times 1}{\underline{Z}} = \underline{X}_1 + i\underline{X}_2 + j\underline{X}_3 + k\underline{X}_4$, (2) reduces to

$$\phi_{\underline{Z}}(\underline{t}) = E\left[\exp \frac{\iota}{2}\left(\bar{\underline{Z}}'\underline{t} + \bar{\underline{t}}'\underline{Z}\right)\right], \tag{4}$$

where $\underset{p\times 1}{\underline{t}} = \underline{t}_1 + i\underline{t}_2 + j\underline{t}_3 + k\underline{t}_4$ is a quaternion vector and ι is the usual imaginary complex unit. An expression for the characteristic function of $\underset{p\times 1}{\underline{Z}}$ in terms of Re tr can be derived in a similar fashion as in (3) and is given by

$$\phi_{\underline{Z}}(\underline{t}) = E\left[\exp \iota \operatorname{Re}\operatorname{tr}\left(\bar{\underline{Z}}'\underline{t}\right)\right].$$

The nature of the problem will determine the form of the characteristic function used in its derivation.

It now follows that

$$\phi_{\mathbf{Z}}(\mathbf{T}) = E\left[\exp \iota \operatorname{Re}\operatorname{tr}\left(\mathbf{X}_1'\mathbf{T}_1 + \mathbf{X}_2'\mathbf{T}_2 + \mathbf{X}_3'\mathbf{T}_3 + \mathbf{X}_4'\mathbf{T}_4\right)\right]$$

such that

$$\phi_{\mathbf{Z}}(\mathbf{T}) = \phi_{\mathbf{Z}_0}(\mathbf{T}_0). \tag{5}$$

$\phi_{\mathbf{Z}_0}(\mathbf{T}_0)$ is the characteristic function of the associated real probability matrix, $\underset{n\times 4p}{\mathbf{Z}_0} = [\mathbf{X}_1', \mathbf{X}_2', \mathbf{X}_3', \mathbf{X}_4']$ and further is $\underset{n\times 4p}{\mathbf{T}_0} = [\mathbf{T}_1', \mathbf{T}_2', \mathbf{T}_3', \mathbf{T}_4']$ a real matrix. It is therefore clear that the characteristic function of a quaternion probability matrix is equivalent to the characteristic function of a $n \times 4p$-variate real probability matrix. A similar result holds for the special case in which $n = 1$, yielding a quaternion probability vector.

3 The quaternion normal distribution

The p-variate quaternion normal distribution forms the basis from which the quaternion distribution theory is further developed, and will be discussed in Section 3.1. We briefly note the univariate as a special case of the p-variate quaternion normal distribution.

In Section 3.2 we derive the matrix-variate quaternion normal distribution using the real representation thereof. We derive the probability density and characteristic function, and emphasise the relationship between the quaternion and associated real cases.

3.1 The p-variate quaternion normal distribution

In this section the approach of Kabe ([10], [11], [12]), Rautenbach [19] and Rautenbach and Roux ([20], [21]) are followed in deriving the p-variate quaternion normal distribution. Although the results in this section 3.1 are in general not new, it is shown how they relate to those given by Teng and Fang [22], and with particular emphasis on the quaternion and related real characteristic functions.

Definition 4 *Let*

$$\underset{p\times 1}{\underline{Z}} = [Z_1, \ldots, Z_p]' = \begin{bmatrix} X_{11} + iX_{21} + jX_{31} + kX_{41} \\ X_{12} + iX_{22} + jX_{32} + kX_{42} \\ \vdots \\ X_{1p} + iX_{2p} + jX_{3p} + kX_{4p} \end{bmatrix}$$

be a quaternion probability vector with real associated probability vector

$$\underset{4p\times 1}{\underline{Z}_0} = [X_{11}, \ldots, X_{1p}, X_{21}, \ldots, X_{2p}, X_{31}, \ldots, X_{3p}, X_{41}, \ldots, X_{4p}]' = \left[\underset{1\times p}{\underline{X}_1'}, \underset{1\times p}{\underline{X}_2'}, \underset{1\times p}{\underline{X}_3'}, \underset{1\times p}{\underline{X}_4'}\right]'.$$

Then, $\underline{Z}$ has a p-variate quaternion normal distribution if $\underline{Z}_0$ has a $4p$-variate real normal distribution.

Teng and Fang [22] used a different matrix structure for representing quaternions by matrices. They supposed that $\underset{p\times 1}{\underline{Z}} = \underline{X}_1 + i\underline{X}_2 + j\underline{X}_3 + k\underline{X}_4$ may be represented by

$$\underset{4p\times 4}{\mathbf{Z}_{00}} = \begin{bmatrix} \underset{p\times 1}{\underline{X}_1} & \underset{p\times 1}{\underline{X}_2} & \underset{p\times 1}{\underline{X}_3} & \underset{p\times 1}{\underline{X}_4} \\ -\underset{p\times 1}{\underline{X}_2} & \underset{p\times 1}{\underline{X}_1} & -\underset{p\times 1}{\underline{X}_4} & \underset{p\times 1}{\underline{X}_3} \\ -\underset{p\times 1}{\underline{X}_3} & \underset{p\times 1}{\underline{X}_4} & \underset{p\times 1}{\underline{X}_1} & -\underset{p\times 1}{\underline{X}_2} \\ -\underset{p\times 1}{\underline{X}_4} & -\underset{p\times 1}{\underline{X}_3} & \underset{p\times 1}{\underline{X}_2} & \underset{p\times 1}{\underline{X}_1} \end{bmatrix} = \left[\underset{4p\times 1}{\underline{Y}_1}, \underset{4p\times 1}{\underline{Y}_2}, \underset{4p\times 1}{\underline{Y}_3}, \underset{4p\times 1}{\underline{Y}_4}\right].$$

Thus the conjugate $\underset{p\times 1}{\bar{\underline{Z}}} = \underline{X}_1 - i\underline{X}_2 - j\underline{X}_3 - k\underline{X}_4$ of $\underset{p\times 1}{\underline{Z}}$ may be represented as

$$\underset{4p\times 4}{\bar{\mathbf{Z}}_{00}} = \begin{bmatrix} \underset{p\times 1}{\underline{X}_1} & -\underset{p\times 1}{\underline{X}_2} & -\underset{p\times 1}{\underline{X}_3} & -\underset{p\times 1}{\underline{X}_4} \\ \underset{p\times 1}{\underline{X}_2} & \underset{p\times 1}{\underline{X}_1} & \underset{p\times 1}{\underline{X}_4} & -\underset{p\times 1}{\underline{X}_3} \\ \underset{p\times 1}{\underline{X}_3} & -\underset{p\times 1}{\underline{X}_4} & \underset{p\times 1}{\underline{X}_1} & \underset{p\times 1}{\underline{X}_2} \\ \underset{p\times 1}{\underline{X}_4} & \underset{p\times 1}{\underline{X}_3} & -\underset{p\times 1}{\underline{X}_2} & \underset{p\times 1}{\underline{X}_1} \end{bmatrix} = \left[\underset{4p\times 1}{\underline{Y}_1^*}, \underset{4p\times 1}{\underline{Y}_2^*}, \underset{4p\times 1}{\underline{Y}_3^*}, \underset{4p\times 1}{\underline{Y}_4^*} \right].$$

From this it is clear that $\underline{Z}_0 = \underline{Y}_1^*$. Teng and Fang [22] showed that any of the $\underline{Y}_s$, $s = 1, 2, 3, 4$ may be used to arive at the same form of the probability density function for the p-variate quaternion normal distribution.

The covariance matrix $\mathbf{\Sigma}$ of $\underline{Z}$ relates to its real associated covariance matrix $\mathbf{\Sigma}_0$, i.e. the covariance matrix of $\underline{Z}_0$, in the following way:

$$\begin{aligned} \underset{4p\times 4p}{\mathbf{\Sigma}_0} = \operatorname{cov}\left(\underline{Z}_0, \underline{Z}_0'\right) = E\left[\left(\underline{Z}_0 - \underline{\mu}_{\underline{Z}_0}\right)\left(\underline{Z}_0 - \underline{\mu}_{\underline{Z}_0}\right)'\right] &= \frac{1}{4}\begin{bmatrix} \mathbf{\Sigma}_1 & -\mathbf{\Sigma}_2 & -\mathbf{\Sigma}_3 & -\mathbf{\Sigma}_4 \\ \mathbf{\Sigma}_2 & \mathbf{\Sigma}_1 & -\mathbf{\Sigma}_4 & \mathbf{\Sigma}_3 \\ \mathbf{\Sigma}_3 & \mathbf{\Sigma}_4 & \mathbf{\Sigma}_1 & -\mathbf{\Sigma}_2 \\ \mathbf{\Sigma}_4 & -\mathbf{\Sigma}_3 & \mathbf{\Sigma}_2 & \mathbf{\Sigma}_1 \end{bmatrix} \\ &\simeq \frac{1}{4}\left(\mathbf{\Sigma}_1 + i\mathbf{\Sigma}_2 + j\mathbf{\Sigma}_3 + k\mathbf{\Sigma}_4\right) \\ &= \frac{1}{4}\operatorname{cov}\left(\underline{Z}, \underline{Z}'\right) \end{aligned} \tag{6}$$

where $\underset{p\times p}{\mathbf{\Sigma}_1}$ is a real symmetric matrix, and $\underset{p\times p}{\mathbf{\Sigma}_2}$, $\underset{p\times p}{\mathbf{\Sigma}_3}$ and $\underset{p\times p}{\mathbf{\Sigma}_4}$ are skew symmetric matrices. Returning to the $\underline{Y}_s$, $s = 1, 2, 3, 4$ defined by Teng and Fang [22], they showed that $\frac{1}{4}\operatorname{cov}\left(\underline{Z}, \underline{Z}'\right) \simeq \operatorname{cov}\left(\underline{Y}_s, \underline{Y}_s'\right)$, $s = 1, 2, 3, 4$.

In order to apply the representation theory discussed in Section 2, i.e. by an elementwise expansion of the quaternion probability vector, we now rearrange the components of the real associated probability vector as follows:

$$\underset{4p\times 1}{\underline{Z}_0^*} = \left[X_{11}, X_{21}, X_{31}, X_{41}, \ldots, X_{1p}, X_{2p}, X_{3p}, X_{4p}\right]'.$$

The components of $\underline{\mu}_0$ and $\mathbf{\Sigma}_0$ are rearranged accordingly in forming $\underset{4p\times 1}{\underline{\mu}_0^*}$ and $\underset{4p\times 4p}{\mathbf{\Sigma}_0^*}$ respectively, and now yield the pdf of $\underline{Z}_0^*$ as

$$f_{\underline{Z}_0^*}(\underline{z}_0^*) = (2\pi)^{-\frac{1}{2}(4p)} \left\{\det \mathbf{\Sigma}_0^*\right\}^{-\frac{1}{2}} \exp\left\{-\frac{1}{2}\left(\underline{z}_0^* - \underline{\mu}_0^*\right)' \mathbf{\Sigma}_0^{*-1}\left(\underline{z}_0^* - \underline{\mu}_0^*\right)\right\} \tag{7}$$

for $\underline{z}_0^* \in B_0^* = \left\{\underline{z}_0^* = \left[x_{11}, x_{21}, x_{31}, x_{41}, \ldots, x_{1p}, x_{2p}, x_{3p}, x_{4p}\right]', -\infty < x_{1s}, x_{2s}, x_{3s}, x_{4s} < \infty, s = 1, \ldots, p\right\}$.
Since $f_{\underline{Z}_0}(\underline{z}_0) = f_{\underline{Z}_0^*}(\underline{z}_0^*)$ for all $\underline{z}_0$ and corresponding $\underline{z}_0^*$, we may use $\underline{Z}_0^*$ as real associated probability vector when deriving the pdf of $\underline{Z}$.

Theorem 5 *Let $\underset{p\times 1}{\underline{Z}}$ be a probability vector that has a p-variate quaternion normal distribution, as given in Definition 4, with $E[\underline{Z}] \equiv \underline{\mu}$ and $\mathbf{\Sigma} \equiv [\sigma_{st}]$ as given in (6). The pdf of $\underset{p\times 1}{\underline{Z}} \sim \mathbb{Q}N(p; \underline{\mu}, \mathbf{\Sigma})$, is given by:*

$$f_{\underline{Z}}(\underline{z}) = 2^{2p}\pi^{-2p}\left(\det \mathbf{\Sigma}\right)^{-2} \exp\left\{-2\left(\overline{\underline{z} - \underline{\mu}}\right)' \mathbf{\Sigma}^{-1}\left(\underline{z} - \underline{\mu}\right)\right\} \tag{8}$$

where $\underline{z} \in B = \left\{\underline{z} = [z_1, \ldots, z_p]' : z_s = x_{1s} + ix_{2s} + jx_{3s} + kx_{4s}, -\infty < x_{1s}, x_{2s}, x_{3s}, x_{4s} < \infty, s = 1, \ldots, p.\right\}$

Proof See Rautenbach and Roux [21].

Remark 6 *1. Let $tr(\cdot)$ be the trace on $M_p(\mathbb{Q})$. Rautenbach and Roux [21] argued that the trace, as used by Kabe ([10], [11], [12]) in simplifying the exponent of $f_{\underline{Z}}(\underline{z})$, is not allowed, because of the noncommutativity* *property of quaternions, i.e.* $tr(\mathbf{AB}) \neq tr(\mathbf{BA})$. *However, from (1), and the fact that for a quaternion random vector* $\underset{p\times 1}{\underline{Z}}$, *we have* $\bar{\underline{Z}}'\underline{Z} \in \mathbb{R}$, *and from Lemma 1 (6) we may write*

$$\exp\left\{\bar{\underline{Z}}'\mathbf{\Sigma}^{-1}\underline{Z}\right\} = \exp\left\{tr\,\mathrm{Re}(\bar{\underline{Z}}'\mathbf{\Sigma}^{-1}\underline{Z})\right\} = \exp\left\{\mathrm{Re}\,tr\left(\bar{\underline{Z}}'\mathbf{\Sigma}^{-1}\underline{Z}\right)\right\} = \exp\left\{\mathrm{Re}\,tr\left(\mathbf{\Sigma}^{-1}\underline{Z}\bar{\underline{Z}}'\right)\right\}. \tag{9}$$

2. *Rautenbach [19] showed that*

$$f_{\underline{Z}}(\underline{z}) = 2^{2p}\pi^{-2p}\left(\det\mathbf{\Sigma}\right)^{-2}\exp\left\{-2\left(\overline{\underline{z}-\underline{\mu}}\right)'\mathbf{\Sigma}^{-1}\left(\underline{z}-\underline{\mu}\right)\right\}$$

with $\mathbf{\Sigma}_0$ *as associated covariance matrix or*

$$f_{\underline{Z}}(\underline{z}) = 2^{2p}\pi^{-2p}\left(\det\mathbf{\Sigma}\right)^{-2}\exp\left\{-2\left(\underline{z}-\underline{\mu}\right)'\mathbf{\Sigma}^{-1}\left(\overline{\underline{z}-\underline{\mu}}\right)\right\}$$

with $\mathbf{\Sigma}_{00}$ *as the associated covariance matrix, which is obtained by using the representation used by Teng and Fang [22] as discussed above.*

Example 7 *The pdf of the univariate quaternion normal distribution is obtained by setting* $p = 1$ *in (8), and is given by*

$$f_Z(z) = 4\pi^{-2}\sigma_1^{-4}\exp\left\{-\frac{2}{\sigma_1^2}\left(\overline{z-\mu}\right)(z-\mu)\right\} \tag{10}$$

for $z \in B = \{z = x_1 + ix_2 + jx_3 + kx_4, -\infty < x_1, x_2, x_3, x_4 < \infty\}, \sigma_1^2 \equiv var(Z) = var(X_1) + var(X_2) + var(X_3) + var(X_4)$ *and* $\mu \equiv E[Z] = E[X_1] + iE[X_2] + jE[X_3] + kE[X_4]$ *are the variance and expected value of the quaternion probability variable* $Z = X_1 + iX_2 + jX_3 + kX_4 \sim \mathbb{Q}N(\mu, \sigma_1^2)$, *respectively. Note that*

$$\mathbf{\Sigma}_0^* = \frac{1}{4}\begin{bmatrix} \sigma_1^2 & 0 & 0 & 0 \\ 0 & \sigma_1^2 & 0 & 0 \\ 0 & 0 & \sigma_1^2 & 0 \\ 0 & 0 & 0 & \sigma_1^2 \end{bmatrix} \simeq \sigma_1^2.$$

Theorem 8 *The characteristic function of* $\underset{p\times 1}{\underline{Z}} \sim \mathbb{Q}N(p; \underline{\mu}, \mathbf{\Sigma})$, *is given by:*

$$\phi_{\underline{Z}}(\underline{t}) = \exp\left\{\frac{\iota}{2}\left(\bar{\underline{\mu}}'\underline{t} + \bar{\underline{t}}'\underline{\mu}\right) - \frac{1}{8}\bar{\underline{t}}'\mathbf{\Sigma}\underline{t}\right\} \tag{11}$$

for every quaternion vector $\underset{p\times 1}{\underline{t}}$ *and* ι *the usual imaginary complex unit.*

Proof. See Rautenbach and Roux [21]. ■

If we let $v = \bar{\underline{\mu}}'\underline{t} + \bar{\underline{t}}'\underline{\mu}$ then $\bar{v}' = \bar{\underline{t}}'\underline{\mu} + \bar{\underline{\mu}}'\underline{t} = v$, i.e. $v \in \mathbb{R}$, so that we can rewrite (11) in the form

$$\phi_{\underline{Z}}(\underline{t}) = \exp\left\{\iota\,\mathrm{Re}\left(\bar{\underline{\mu}}'\underline{t}\right) - \frac{1}{8}\bar{\underline{t}}'\mathbf{\Sigma}\underline{t}\right\} = \exp\left\{\iota\underline{\mu}_0^{*\prime}\underline{t}_0^* - \frac{1}{2}\underline{t}_0^{*\prime}\mathbf{\Sigma}_0^*\underline{t}_0^*\right\} = \phi_{\underline{Z}_0^*}(\underline{t}_0^*),$$

so that (5) holds in the multivariate case.

3.2 The matrix-variate quaternion normal distribution

We now turn our attention to the derivation of the matrix-variate quaternion normal distribution by expanding our results of the previous section. Once again our goal is to emphasise the relationship between the *real associated form* and its resultant counterpart. In Theorem 10 the relationship between characteristic function of the matrix-variate quaternion normal distribution and its real associated counterpart will be established.

Theorem 9 *Let* $\underset{p\times 1}{\underline{Z}_\alpha}, \alpha = 1,\ldots,n$ *be n probability vectors each having a p-variate quaternion normal distribution, as given in Definition 4. Now, suppose (for* $\alpha = 1,\ldots,n,\ \beta = 1,\ldots,p$*) that*

$$\underset{n\times p}{\mathbf{Z}} = [Z_{\alpha\beta}] = \begin{bmatrix} Z_{11} & \ldots & Z_{1p} \\ \vdots & \ddots & \vdots \\ Z_{n1} & \ldots & Z_{np} \end{bmatrix} = \begin{bmatrix} \underset{1\times p}{\underline{Z}'_1} \\ \vdots \\ \underset{1\times p}{\underline{Z}'_n} \end{bmatrix} = \left[\underset{n\times 1}{\underline{Z}_{(1)}}, \ldots, \underset{n\times 1}{\underline{Z}_{(p)}} \right]$$

i.e. the rows of $\mathbf{Z}$ *are* $\mathbb{Q}N(p; \underline{\mu}_\alpha, \mathbf{\Sigma})$ *distributed,* $\alpha = 1,\ldots,n$ *with dependence structure given by* $\underset{n\times n}{\mathbf{R}}$ *not necessarily equal to* $\mathbf{I}_n$*. It may be assumed without loss of generality that* $\mathbf{R}$ *is real-valued. Similarly, define*

$$\underset{n\times p}{\boldsymbol{\mu}} = \left[\mu_{\alpha\beta}\right],\ \alpha = 1,\ldots,n,\ \beta = 1,\ldots,p.$$

Then

$$\underset{np\times 1}{vec\mathbf{Z}} = \begin{bmatrix} \underset{n\times 1}{\underline{Z}_{(1)}} \\ \vdots \\ \underset{n\times 1}{\underline{Z}_{(p)}} \end{bmatrix} \sim \mathbb{Q}N\left(np; vec\boldsymbol{\mu}, \mathbf{\Sigma}\otimes\mathbf{R}\right)$$

i.e. a matrix-variate quaternion normal distribution where $\underset{np\times 1}{vec\boldsymbol{\mu}} = \begin{bmatrix} \underset{n\times 1}{\underline{\mu}_{(1)}} \\ \vdots \\ \underset{n\times 1}{\underline{\mu}_{(p)}} \end{bmatrix}$

(denote $\underset{n\times p}{\mathbf{Z}} \sim \mathbb{Q}N\left(n\times p; \boldsymbol{\mu}, \mathbf{\Sigma}\otimes\mathbf{R}\right)$*).*

Proof.

1. In order to apply the methodology set out in Section 3.1, we define the real associated probability vector of $\underline{Z}_{(\beta)}$ as

$$\underset{n\times 1}{\underline{Z}_{(\beta)}} = \begin{bmatrix} Z_{1\beta} \\ \vdots \\ Z_{n\beta} \end{bmatrix} = \begin{bmatrix} X_{11\beta} + iX_{21\beta} + jX_{31\beta} + kX_{41\beta} \\ \vdots \\ X_{1n\beta} + iX_{2n\beta} + jX_{3n\beta} + kX_{4n\beta} \end{bmatrix}$$
$$= \underset{n\times 1}{\underline{X}_{(1\beta)}} + i\underset{n\times 1}{\underline{X}_{(2\beta)}} + j\underset{n\times 1}{\underline{X}_{(3\beta)}} + k\underset{n\times 1}{\underline{X}_{(4\beta)}}$$

with associated real counterpart given by

$$\underset{n\times 4}{\underline{Z}^*_{(0\beta)}} = \left[\underset{n\times 1}{\underline{X}_{(1\beta)}}, \underset{n\times 1}{\underline{X}_{(2\beta)}}, \underset{n\times 1}{\underline{X}_{(3\beta)}}, \underset{n\times 1}{\underline{X}_{(4\beta)}} \right].$$

From this we now have

$$\underset{n\times p}{\mathbf{Z}} = \left[\underset{n\times 1}{\underline{Z}_{(1)}}, \ldots, \underset{n\times 1}{\underline{Z}_{(p)}}\right]$$
$$= \left[\underset{n\times 1}{\underline{X}_{(11)}} + i\underset{n\times 1}{\underline{X}_{(21)}} + j\underset{n\times 1}{\underline{X}_{(31)}} + k\underset{n\times 1}{\underline{X}_{(41)}}, \ldots, \underset{n\times 1}{\underline{X}_{(1p)}} + i\underset{n\times 1}{\underline{X}_{(2p)}} + j\underset{n\times 1}{\underline{X}_{(3p)}} + k\underset{n\times 1}{\underline{X}_{(4p)}}\right]$$

from which the real associated matrix $\underset{n\times 4p}{\mathbf{Z}_0^*}$ of $\underset{n\times p}{\mathbf{Z}}$ immediately follows as

$$\underset{n\times 4p}{\mathbf{Z}_0^*} = [X_{\gamma\alpha\beta}] = \left[\underset{n\times 1}{\underline{X}_{(11)}}, \underset{n\times 1}{\underline{X}_{(21)}}, \underset{n\times 1}{\underline{X}_{(31)}}, \underset{n\times 1}{\underline{X}_{(41)}}, \ldots, \underset{n\times 1}{\underline{X}_{(1p)}}, \underset{n\times 1}{\underline{X}_{(2p)}}, \underset{n\times 1}{\underline{X}_{(3p)}}, \underset{n\times 1}{\underline{X}_{(4p)}}\right]$$
$$= \left[\underset{n\times 4}{\underline{Z}^*_{(01)}}, \ldots, \underset{n\times 4}{\underline{Z}^*_{(0p)}}\right].$$

where $\gamma = 1, \ldots, 4,\ \alpha = 1, \ldots, n,\ \beta = 1, \ldots, p$. If we note that

$$\underset{4n\times 1}{\mathrm{vec}\underline{Z}^*_{(0\beta)}} = \begin{bmatrix} \underset{n\times 1}{\underline{X}_{(1\beta)}} \\ \underset{n\times 1}{\underline{X}_{(2\beta)}} \\ \underset{n\times 1}{\underline{X}_{(3\beta)}} \\ \underset{n\times 1}{\underline{X}_{(4\beta)}} \end{bmatrix},$$

then we have

$$\underset{4np\times 1}{\mathrm{vec}\mathbf{Z}_0^*} = \left[\underset{1\times n}{\underline{X}'_{(11)}}, \underset{1\times n}{\underline{X}'_{(21)}}, \underset{1\times n}{\underline{X}'_{(31)}}, \underset{1\times n}{\underline{X}'_{(41)}}, \ldots, \underset{1\times n}{\underline{X}'_{(1p)}}, \underset{1\times n}{\underline{X}'_{(2p)}}, \underset{1\times n}{\underline{X}'_{(3p)}}, \underset{1\times n}{\underline{X}'_{(4p)}}\right]' = \begin{bmatrix} \underset{4n\times 1}{\mathrm{vec}\underline{Z}^*_{(01)}} \\ \vdots \\ \underset{4n\times 1}{\mathrm{vec}\underline{Z}^*_{(0p)}} \end{bmatrix}$$
$$\simeq \mathrm{vec} \underset{np\times 1}{\mathbf{Z}}.$$

In a similar fashion, we can show that $\underset{4np\times 1}{\mathrm{vec}\boldsymbol{\mu}_0^*} \simeq \underset{np\times 1}{\mathrm{vec}\boldsymbol{\mu}}$.

2. We can now rewrite the problem in terms of n real associated probability vectors as $\underset{4p\times 1}{\underline{Z}^*_{0\alpha}},\ \alpha = 1, \ldots, n$ each having a $4p$-variate real normal distribution, as given in (7). The real associated quantity $\underset{4np\times 1}{\mathrm{vec}\mathbf{Z}_0^*}$ now has a density given by $N\left(\mathrm{vec}\boldsymbol{\mu}_0^*, \boldsymbol{\Sigma}_0^* \otimes \mathbf{R}\right)$, i.e. a real matrix-variate normal distribution, where $\underset{n\times n}{\mathbf{R}}$ denotes the dependence structure of the rows of $\mathbf{Z}_0^*$ and is equal to that of $\mathbf{Z}$ and since $\mathbf{R}$ is real-valued.

3. Thus, the pdf of $\mathbf{Z}_0^*$ is given by:

$$f_{\mathbf{Z}_0^*}\left(\underset{n\times 4p}{\boldsymbol{\mu}_0^*}, \underset{4p\times 4p}{\boldsymbol{\Sigma}_0^*}, \underset{n\times n}{\mathbf{R}}\right) = (2\pi)^{-2np} \det(\mathbf{R})^{-2p} \det(\boldsymbol{\Sigma}_0^*)^{-\frac{n}{2}} \exp\left\{-\frac{1}{2}\mathrm{tr}\left[\boldsymbol{\Sigma}_0^{*-1}\left(\mathbf{Z}_0^* - \boldsymbol{\mu}_0^*\right)' \mathbf{R}^{-1}\left(\mathbf{Z}_0^* - \boldsymbol{\mu}_0^*\right)\right]\right\}$$

and by the isomorphic relations already established above the pdf of $\mathbf{Z}$ follows as

$$f_{\mathbf{Z}}\left(\underset{n\times p}{\boldsymbol{\mu}}, \underset{p\times p}{\boldsymbol{\Sigma}}, \underset{n\times n}{\mathbf{R}}\right) = \frac{2^{2np}}{\pi^{2np}\left(\det\mathbf{R}\right)^{2p}\left(\det\boldsymbol{\Sigma}\right)^{2n}} \exp\left\{-2\,\mathrm{Re}\,\mathrm{tr}\left[\boldsymbol{\Sigma}^{-1}\left(\overline{\mathbf{Z}-\boldsymbol{\mu}}\right)' \mathbf{R}^{-1}\left(\mathbf{Z}-\boldsymbol{\mu}\right)\right]\right\} \qquad (12)$$

from which the desired result follows.

■

We now establish the relationship between the characteristic function of the matrix-variate quaternion normal distribution and that of its real associated matrix-variate normal distribution.

Theorem 10 *Let* $\underset{n\times p}{\mathbf{Z}} \sim \mathbb{Q}N\left(n\times p;\boldsymbol{\mu},\boldsymbol{\Sigma}\otimes\mathbf{R}\right)$ *as defined in Theorem 9 above. The characteristic function of* $\mathbf{Z}$ *is given by*

$$\phi_{\mathbf{Z}}(\mathbf{T}) = \exp \operatorname{Re} tr\left\{\iota\bar{\boldsymbol{\mu}}'\mathbf{T} - \frac{1}{8}\boldsymbol{\Sigma}\bar{\mathbf{T}}'\mathbf{R}\mathbf{T}\right\} \tag{13}$$

where $\underset{n\times p}{\mathbf{T}} \in M_{n\times p}(\mathbb{Q})$ *and where* ι *is the usual complex unit.*

Proof. From (3) and (12) we have

$$\begin{aligned}\phi_{\mathbf{Z}}(\mathbf{T}) &= E\left[\exp\iota\operatorname{Re}\operatorname{tr}\left(\bar{\mathbf{Z}}'\mathbf{T}\right)\right] = \int_B \frac{2^{2np}}{\pi^{2np}\left(\det\mathbf{R}\right)^{2p}\left(\det\boldsymbol{\Sigma}\right)^{2n}} \\ &\quad\times\exp\left\{-2\operatorname{Re}\operatorname{tr}\left[\boldsymbol{\Sigma}^{-1}\left(\overline{\mathbf{Z}-\boldsymbol{\mu}}\right)'\mathbf{R}^{-1}\left(\mathbf{Z}-\boldsymbol{\mu}\right)-\tfrac{\iota}{2}\bar{\mathbf{Z}}'\mathbf{T}\right]\right\}\mathrm{d}\mathbf{Z}\end{aligned} \tag{14}$$

where
$B=\left\{\mathbf{Z}=\left[z_{st}\right]' : z_{st}=x_{1st}+ix_{2st}+jx_{3st}+kx_{4st}; -\infty < x_{1st},x_{2st},x_{3st},x_{4st}<\infty, s=1,\ldots,n, t=1,\ldots,p\right\}$ is the region of variability for $\mathbf{Z}$. We can rewrite the argument of Re tr in the form

$$\begin{aligned}\boldsymbol{\Sigma}^{-1}\left(\overline{\mathbf{Z}-\boldsymbol{\mu}}\right)'\mathbf{R}^{-1}\left(\mathbf{Z}-\boldsymbol{\mu}\right)-\frac{\iota}{2}\bar{\mathbf{Z}}'\mathbf{T} &= \boldsymbol{\Sigma}^{-1}\left[\left(\overline{\mathbf{Z}-\left(\boldsymbol{\mu}+\frac{\iota}{4}\mathbf{RT\Sigma}\right)}\right)\mathbf{R}^{-1}\left(\mathbf{Z}-\left(\boldsymbol{\mu}+\frac{\iota}{4}\mathbf{RT\Sigma}\right)\right)\right] \\ &\quad+\boldsymbol{\Sigma}^{-1}\left[\bar{\boldsymbol{\mu}}'\mathbf{R}^{-1}\boldsymbol{\mu}-\left(\overline{\boldsymbol{\mu}+\frac{\iota}{4}\mathbf{RT\Sigma}}\right)'\mathbf{R}^{-1}\left(\boldsymbol{\mu}+\frac{\iota}{4}\mathbf{RT\Sigma}\right)\right].\end{aligned}$$

Noting the fact that $\boldsymbol{\Sigma}$ and $\mathbf{R}$ are Hermitian, that the conjugate of Re tr is again Re tr,and that according to (1) the arguments of Re tr may be rearranged, (14) reduces to

$$\begin{aligned}\phi_{\mathbf{Z}}(\mathbf{T}) &= \exp\operatorname{Re}\operatorname{tr}\left\{-2\boldsymbol{\Sigma}^{-1}\left[\bar{\boldsymbol{\mu}}'\mathbf{R}^{-1}\boldsymbol{\mu}-\left(\overline{\boldsymbol{\mu}+\frac{\iota}{4}\mathbf{RT\Sigma}}\right)'\mathbf{R}^{-1}\left(\boldsymbol{\mu}+\frac{\iota}{4}\mathbf{RT\Sigma}\right)\right]\right\} \\ &= \exp\operatorname{Re}\operatorname{tr}\left\{\iota\bar{\boldsymbol{\mu}}'\mathbf{T}-\frac{1}{8}\boldsymbol{\Sigma}\bar{\mathbf{T}}'\mathbf{RT}\right\}, \quad \text{as required.}\end{aligned}$$

■

Note that $\phi_{\mathbf{Z}}(\mathbf{T}) = \exp\operatorname{Retr}\left\{\iota\bar{\boldsymbol{\mu}}'\mathbf{T}-\frac{1}{8}\boldsymbol{\Sigma}\bar{\mathbf{T}}'\mathbf{RT}\right\} = \exp tr\left\{\iota\boldsymbol{\mu}_0^{*\prime}\mathbf{T}_0^*-\frac{1}{2}\boldsymbol{\Sigma}_0^*\mathbf{T}_0^{*\prime}\mathbf{RT}_0^*\right\} = \phi_{\mathbf{Z}_0^*}(\mathbf{T}_0^*)$, satisfying (5).

4 The quaternion Wishart distributions

Is it possible to find the density function of the quaternion Wishart matrix from the real associated Wishart matrix?

In this section, we derive the quaternion Wishart distribution from the real matrix-variate normal distribution associated with the matrix-variate quaternion normal distribution by which it is defined. We once again emphasise the link between the characteristic functions of the quaternion and real associated Wishart distributions.

Kabe ([10], [11], [12]) derived the hypercomplex Wishart distribution directly from the hypercomplex normal distribution using the Q generalized Sverdrup's lemma. Teng and Fang [22] showed that the maximum likelihood estimator $\hat{\boldsymbol{\Sigma}}$ of $\boldsymbol{\Sigma}$ followed a quaternion Wishart distribution. They used a Fourier transform on the results given by Andersson[1] to yield explicit expressions for the probability density and characteristic functions of the quaternion Wishart distribution. The non-central quaternion Wishart distribution was discussed by Kabe [12], while Li and Xue [14] derived the singular quaternion Wishart distribution. More technical results, specifically regarding Selberg-type squared matrices, gamma and beta integrals are found in the paper by Gupta and Kabe [9].

Theorem 11 *Let* $\underset{n\times p}{\mathbf{Z}} \sim \mathbb{Q}N\left(n\times p;\mathbf{0},\mathbf{\Sigma}\otimes\mathbf{I}_n\right)$ *(see Theorem 9). Then for* $n\geq p$, $\mathbf{W}=\bar{\mathbf{Z}}'\mathbf{Z}$ *is said to have the quaternion Wishart distribution with* n *degrees of freedom, i.e.* $\mathbf{W}\sim\mathbb{Q}W_p\left(n,\mathbf{\Sigma}\right)$, *with density function given by*

$$f(\mathbf{W})=\frac{2^{2np}}{\mathbb{Q}\Gamma_p(2n)\left(\det\mathbf{\Sigma}\right)^{2n}}\exp\left\{-2\,\mathrm{Re}\;\; tr\left(\mathbf{\Sigma}^{-1}\mathbf{W}\right)\right\}\det\left(\mathbf{W}\right)^{2n-2p+1}, \tag{15}$$

with $\mathbf{W}=\bar{\mathbf{W}}'>\mathbf{0}$ *and where* $\mathbb{Q}\Gamma_p\left(\cdot\right)$ *is the multivariate quaternion gamma function, as given in [8].*

Proof

Let $\underset{4p\times 4p}{\mathbf{W}_0^*}=\underset{4p\times n}{\mathbf{Z}_0^{*\prime}}\underset{n\times 4p}{\mathbf{Z}_0^*}$ where $\underset{n\times p}{\mathbf{Z}_0^*}\sim N\left(n\times 4p;\mathbf{0},\mathbf{\Sigma}_0^*\otimes\mathbf{I}_n\right)$ is the real associated matrix of $\mathbf{Z}$ as given in Theorem 9. Let $\mathbf{T}=(t_{ls})$, $l,s=1,2,\ldots,p$ where $t_{ls}=\bar{t}_{ls}$ and $t_{ls}=t_{1ls}+it_{2ls}+j_{3ls}+kt_{4ls}$. From (5) we have

$$\begin{aligned}\phi_{\mathbf{W}}(\mathbf{T}) &= \phi_{\mathbf{W_0}}(\mathbf{T}_0)\\ &= \det(\mathbf{I}_{4p}-2\iota\mathbf{\Sigma}_{\mathbf{0}}^*\mathbf{T}_0^*)^{-\frac{n}{2}},\end{aligned}$$

where $\underset{4p\times 4p}{\mathbf{T}_0^*}$ is the real associated symmetric vector of $\underset{p\times p}{\mathbf{T}}$. Let $\mathbf{Y}_0^*=(\mathbf{I}_{4p}-2\iota\mathbf{\Sigma}_{\mathbf{0}}^*\mathbf{T}_0^*)$, then $\mathbf{Y}_0^{*\prime}=(\mathbf{I}_{4p}-2\iota\mathbf{T}_0^*\mathbf{\Sigma}_{\mathbf{0}}^*)=\mathbf{Y}_0^*$ and from Lemma 1 (1) $\mathbf{Y}$ is Hermitian. Therefore,

$$\begin{aligned}\phi_{\mathbf{W}}(\mathbf{T}) &= \det(\mathbf{Y})^{-2n}\\ &= \det\left(\mathbf{I}_p-\tfrac{\iota}{2}\mathbf{\Sigma}\mathbf{T}\right)^{-2n}\end{aligned}$$

Let $\mathbf{S}=-\iota\mathbf{T}$, using the definition of the quaternion Gauss hypergeometric function of matrix argument (see [13]), we have

$$\begin{aligned}\det\left(\mathbf{I}_p-\frac{\iota}{2}\mathbf{\Sigma}\boldsymbol{T}\right)^{-2n} &= \det(\frac{1}{2}\mathbf{\Sigma})^{-2n}\det(\mathbf{S})^{-2n}\det\left(\mathbf{I}_p+2\mathbf{\Sigma}^{-1}\mathbf{S}^{-1}\right)^{-2n}\\ &= \det(\frac{1}{2}\mathbf{\Sigma})^{-2n}\det(\mathbf{S})^{-2n}\sum_{k=0}^{\infty}\sum_{\kappa}\frac{(2n)_\kappa\mathbb{Q}C_\kappa(-2\mathbf{\Sigma}^{-\frac{1}{2}}\mathbf{S}^{-1}\mathbf{\Sigma}^{-\frac{1}{2}})}{k!}\end{aligned}$$

where $\mathbb{Q}C_\kappa(\cdot)$ is the zonal polynomial of the quaternion Hermitian matrix (see [8] and [13]). Using the inverse Laplace transformation ([4], equation 4.13) and ${}_0\mathbb{Q}F_0(a;\mathbf{A})=etr(\mathbf{A})$, we have that

$$\begin{aligned}f(\mathbf{W}) &= \frac{2^{2p(p-1)}}{(2\pi\iota)^{2p(p-1)+p}}2^{2np}\det(\mathbf{\Sigma})^{-2n}\sum_{k=0}^{\infty}\sum_{\kappa}\frac{(2n)_\kappa}{k!}\int_{\mathbf{S}-\mathbf{S}_0\in\Phi}\mathrm{etr}(\mathbf{WS})\det\left(\mathbf{S}\right)^{-2n}\mathbb{Q}C_\kappa(-2\mathbf{\Sigma}^{-\frac{1}{2}}\mathbf{S}^{-1}\mathbf{\Sigma}^{-\frac{1}{2}})d\mathbf{S}\\ &= 2^{2np}\det(\mathbf{\Sigma})^{-2n}\sum_{k=0}^{\infty}\sum_{\kappa}\frac{(2n)_\kappa}{k!\mathbb{Q}\Gamma_p(2n,\kappa)}\det(\mathbf{W})^{2n-2(p-1)-1}\mathbb{Q}C_\kappa(-2\mathbf{W}\mathbf{\Sigma}^{-1})\\ &= \frac{2^{2np}}{\mathbb{Q}\Gamma_p(2n)}\det(\mathbf{\Sigma})^{-2n}\det(\mathbf{W})^{2n-2(p-1)-1}\;{}_0\mathbb{Q}F_0(-2\mathbf{\Sigma}^{-1}\mathbf{W})\end{aligned}$$

and the desired result follows. ■

Remark 12 *What is the distribution of* $\underline{\bar{Z}}'\underline{Z}$, *for* $\underset{p\times 1}{\underline{Z}}$ *a p-variate quaternion normal distributed probability vector, from the real representation perspective?*

Rautenbach ([19], p 161) answered this question:

Let $\underset{p\times 1}{\underline{Z}} \sim \mathbb{Q}N(p;\underline{0},\mathbf{I}_p)$. If $V = \underline{\bar{Z}}'\underline{Z}$, then $W = 4V \sim \chi^2_{4p}$ (chi-squared distribution with $4p$ degrees of freedom) distribution. Also, if $\underline{Z} \sim \mathbb{Q}N\left(p;\underline{\mu},\mathbf{\Sigma}\right)$ then $4\left(\overline{\underline{Z}-\underline{\mu}}\right)'\mathbf{\Sigma}^{-1}\left(\underline{Z}-\underline{\mu}\right) \sim \chi^2_{4p}$ (see [21]).

5 Applications illustrating the role of the quaternion normal distributions in hypothesis testing

The reader is referred to the valuable contribution of Bhavsar [3] where asymptotic distributions of likelihood ratio criteria for two testing problems are considered. In this section we show that a simple quaternion hypothesis may be represented with an associated real hypothesis, and thereafter derive an expression for the density function of Wilks' statistic in the case of quaternion Wishart matrices.

5.1 Example: Quaternion hypothesis test for $H_{01}:\underline{\mu}=\underline{0}$, $\mathbf{\Sigma}$ known

Suppose that $\underset{2\times 1}{\underline{Z}} = \underline{X}_1 + i\underline{X}_2 + j\underline{X}_3 + k\underline{X}_4 \sim \mathbb{Q}N\left(2;\underline{\mu},\mathbf{\Sigma}\right)$ where $\underset{2\times 1}{\underline{\mu}} = [\mu_1,\mu_2]' = \underline{\mu}_{X_1} + i\underline{\mu}_{X_2} + j\underline{\mu}_{X_3} + k\underline{\mu}_{X_4}$ and

$$\underset{2\times 2}{\mathbf{\Sigma}} = \begin{bmatrix} \sigma_1^2 & \xi_{11}+i\xi_{12}+j\xi_{13}+k\xi_{14} \\ \xi_{11}-i\xi_{12}-j\xi_{13}-k\xi_{14} & \sigma_2^2 \end{bmatrix}$$

is a positive definite Hermitian matrix. We are interested in testing the quaternion null hypothesis $H_{01}:\underline{\mu}=\underline{0}$, $\mathbf{\Sigma}$ known based upon a random sample, $\underline{Z}_1,\ldots,\underline{Z}_n$, against $H_{a1}:\underline{\mu}\neq\underline{0}$, $\mathbf{\Sigma}$ known. In order to derive a test criterion for such a test, consider the following likelihood function

$$\begin{aligned} L = &\left(\frac{2}{\pi}\right)^{4n}\left[\sigma_1^2\sigma_2^2-\left(\xi_{11}^2+\xi_{12}^2+\xi_{13}^2+\xi_{14}^2\right)\right]^{-2n} \\ &\times\exp\left\{-2\left(\sigma_1^2\sigma_2^2-\left(\xi_{11}^2+\xi_{12}^2+\xi_{13}^2+\xi_{14}^2\right)\right)^{-1}\left[\sigma_2^2\sum_{s=1}^n\left(\overline{z_{1s}-\mu_1}\right)\left(z_{1s}-\mu_1\right)\right.\right. \\ &\left.\left.-2\operatorname{Re}\sum_{s=1}^n\left(\overline{z_{1s}-\mu_1}\right)\left(\xi_{11}+i\xi_{12}+j\xi_{13}+k\xi_{14}\right)\left(z_{2s}-\mu_2\right)+\sigma_1^2\sum_{s=1}^n\left(\overline{z_{2s}-\mu_2}\right)\left(z_{2s}-\mu_2\right)\Big)\right]\right\}. \end{aligned}$$

The likelihood ratio criterion is given by $\Lambda^* = \frac{\max_{H_{01}} L\left(\underline{\mu},\mathbf{\Sigma}\right)}{\max_{H_{a1}} L\left(\underline{\mu},\mathbf{\Sigma}\right)}$. Under H_0 we have

$$\begin{aligned} \max_{H_{01}} L\left(\underline{\mu},\mathbf{\Sigma}\right) &= \left(\tfrac{2}{\pi}\right)^{4n}\left(\sigma_1^2\sigma_2^2-\left(\xi_{11}^2+\xi_{12}^2+\xi_{13}^2+\xi_{14}^2\right)\right)^{-2n} \\ &\times\exp\left\{-2\left(\sigma_1^2\sigma_2^2-\left(\xi_{11}^2+\xi_{12}^2+\xi_{13}^2+\xi_{14}^2\right)\right)^{-1}\right. \\ &\left.\times\left[\sigma_2^2\sum_{s=1}^n\bar{z}_{1s}z_{1s}-2\operatorname{Re}\sum_{s=1}^n\bar{z}_{1s}\left(\xi_{11}+i\xi_{12}+j\xi_{13}+k\xi_{14}\right)z_{2s}+\sigma_1^2\sum_{s=1}^n\bar{z}_{2s}z_{2s}\right]\right\} \end{aligned} \tag{16}$$

Rautenbach ([19], Theorem 6.3.2) showed that the maximum likelihood estimator of $\underline{\mu}$ is given by $\hat{\underline{\mu}} = \frac{1}{n}\sum_{s=1}^n\underline{z}_s = (\text{avg } z_1, \text{avg } z_2)'$ (where avg z_j =average value of $z_{j1},\ldots z_{jn}, j=1,2$)
such that

$$
\begin{aligned}
\max_{H_{a1}} L\left(\underline{\mu}, \mathbf{\Sigma}\right) &= L\left(\underline{\hat{\mu}}, \mathbf{\Sigma}\right) \\
&= \left(\tfrac{2}{\pi}\right)^{4n}\left(\sigma_1^2\sigma_2^2-\left(\xi_{11}^2+\xi_{12}^2+\xi_{13}^2+\xi_{14}^2\right)\right)^{-2n}\exp\Big\{-2\left(\sigma_1^2\sigma_2^2-\left(\xi_{11}^2+\xi_{12}^2+\xi_{13}^2+\xi_{14}^2\right)\right)^{-1} \\
&\times\Big[\sigma_2^2\sum_{s=1}^{n}\left(\overline{z_{1s}-\text{avg } z_1}\right)\left(z_{1s}-\text{avg } z_1\right) \\
&-2\,\text{Re}\sum_{s=1}^{n}\left(\overline{z_{1s}-\text{avg } z_1}\right)\left(\xi_{11}+i\xi_{12}+j\xi_{13}+k\xi_{14}\right)\left(z_{2s}-\text{avg } z_2\right) \\
&+\sigma_1^2\sum_{s=1}^{n}\left(\overline{z_{2s}-\text{avg } z_2}\right)\left(z_{2s}-\text{avg } z_2\right)\Big]\Big\}
\end{aligned}
\tag{17}
$$

From (16) and (17) it now follows that

$$
\begin{aligned}
\Lambda^* &= \exp\Big\{-2n\left(\sigma_1^2\sigma_2^2-\left(\xi_{11}^2+\xi_{12}^2+\xi_{13}^2+\xi_{14}^2\right)\right)^{-1}\left[\sigma_1^2\text{avg}\bar{z}_1\text{avg } z_1\right. \\
&\quad \left.-2\,\text{Re}\,\text{avg } \bar{z}_1\left(\xi_{11}+i\xi_{12}+j\xi_{13}+k\xi_{14}\right)\text{avg } z_2+\sigma_2^2\text{avg } \bar{z}_2\text{avg } z_2\right]\Big\} \\
&= \exp\left\{-2n\text{avg}\underline{\bar{z}}'\mathbf{\Sigma}^{-1}\text{avg } \underline{z}\right\}.
\end{aligned}
$$

The null hypothesis, H_{01}, is now rejected at the $100\alpha\%$ significance level, in favour of H_{a1}, if $\exp\left\{-2n\text{avg } \underline{\bar{z}}'\mathbf{\Sigma}^{-1}\text{avg } \underline{z}\right\} \leq \lambda_\alpha^*$ where the constant λ_α^* is such that $P\left[\Lambda^* \leq \lambda_\alpha^*|H_{01}\right]=\alpha$. Thus, H_{01} is rejected if $y = 4n\text{avg}\underline{\bar{z}}'\mathbf{\Sigma}^{-1}\text{avg } \underline{z} \geq -2\ln\lambda_\alpha^* = \lambda_\alpha'$. Under H_{01} avg $\underline{z} \sim \mathbb{Q}N\left(2;\underline{0},\frac{1}{n}\mathbf{\Sigma}\right)$ and from Remark 12 it follows that $Y = 4n\text{avg } \underline{\bar{Z}}'\mathbf{\Sigma}^{-1}\text{avg } \underline{Z} \sim \chi_8^2$. Since $P\left[y \geq \chi^2_{8,1-\alpha}\right]=\alpha$ the null hypothesis is rejected if $y \geq \chi^2_{8,1-\alpha}$ where $\chi^2_{8,1-\alpha}$ is the $100\left(1-\alpha\right)^{th}$ percentile of χ_8^2.

An alternative approach in deriving a test criterion in this case involves the use of the real associated probability vector of $\underline{z}$. We know that $\underset{8\times 1}{\underline{z}_0^*} \sim N\left(8;\underline{\mu}_0^*,\mathbf{\Sigma}_0^*\right)$, i.e. a real multivariate normal distribution, where $\underset{8\times 1}{\underline{\mu}_0^*} = \left[\mu_{11},\mu_{21},\mu_{31},\mu_{41},\mu_{12},\mu_{22},\mu_{23},\mu_{24}\right]'$ and

$$
\mathbf{\Sigma}_0^* = \frac{1}{4}\begin{bmatrix}
\sigma_1^2 & 0 & 0 & 0 & \xi_{11} & -\xi_{12} & -\xi_{13} & -\xi_{14} \\
0 & \sigma_1^2 & 0 & 0 & \xi_{12} & \xi_{11} & -\xi_{14} & \xi_{13} \\
0 & 0 & \sigma_1^2 & 0 & \xi_{13} & \xi_{14} & \xi_{11} & -\xi_{12} \\
0 & 0 & 0 & \sigma_1^2 & \xi_{14} & -\xi_{13} & \xi_{12} & \xi_{11} \\
\xi_{11} & \xi_{12} & \xi_{13} & \xi_{14} & \sigma_2^2 & 0 & 0 & 0 \\
-\xi_{12} & \xi_{11} & \xi_{14} & -\xi_{13} & 0 & \sigma_2^2 & 0 & 0 \\
-\xi_{13} & -\xi_{14} & \xi_{11} & \xi_{12} & 0 & 0 & \sigma_2^2 & 0 \\
-\xi_{14} & \xi_{13} & -\xi_{12} & \xi_{11} & 0 & 0 & 0 & \sigma_2^2
\end{bmatrix}.
$$

The test criterion for testing $H_{01}^*:\underline{\mu}_0^*=\underline{0}$, $\mathbf{\Sigma}_0^*$ known, against $H_{a1}^*:\underline{\mu}_0^*\neq\underline{0}$, $\mathbf{\Sigma}_0^*$ known, based upon a random sample $\underline{Z}_{01}^*,\ldots,\underline{Z}_{0n}^*$ of $\underline{Z}_0^*$ is given by

$$
Y_0 = n\left(\text{avg } \underline{Z}_0^{*\prime}\mathbf{\Sigma}_0^{*-1}\text{avg } \underline{Z}_0^*\right) \sim \chi_8^2
$$

where avg $\underline{Z}_0^* = \left[\text{avg } X_{11},\text{avg } X_{21},\text{avg } X_{31},\text{avg } X_{41},\text{avg } X_{12},\text{avg } X_{22},\text{avg } X_{32},\text{avg } X_{42}\right]'$ such that the null hypothesis is rejected if $y_0 \geq \chi^2_{8,1-\alpha}$ where $\chi^2_{8,1-\alpha}$ is the $100(1-\alpha)^{th}$ percentile of χ_8^2.

From the above discussion it is once again clear that two different approaches exist in order to test $H_{01}:\underline{\mu}=\underline{0}$ against $H_{a1}:\underline{\mu}\neq\underline{0}$ with $\mathbf{\Sigma}$ known. We may either conduct an analysis using quaternion quantities directly with y or, by utilising the real associated quantity, y_0, as test criterion respectively. From this it is clear that the quaternion hypothesis $H_{01}:\underline{\mu}=\underline{0}$ against $H_{a1}:\underline{\mu}\neq\underline{0}$ may also be expressed in terms of real quantities, i.e. $H_{01}^{*} : \underline{\mu}_0^{*} = \underline{0}$ against $H_{a1}^{*} : \underline{\mu}_0^{*} \neq \underline{0}$.

5.2 Example: Wilks' statistic

Let the rows of $\underset{n_1 \times p}{\mathbf{X}}$ and $\underset{n_2 \times p}{\mathbf{Y}}$ be independently $\mathbb{Q}N(p; \underline{0}, \mathbf{\Sigma})$ and $\mathbb{Q}N(p; \underline{\mu}, \mathbf{\Sigma})$ distributed, respectively. From Theorem 9 we know that $\mathbf{X} \sim \mathbb{Q}N(n_1 \times p, \mathbf{0}, \mathbf{\Sigma} \otimes \mathbf{I}_{n_1})$ and $\mathbf{Y} \sim \mathbb{Q}N(n_2 \times p; \boldsymbol{\mu}, \mathbf{\Sigma} \otimes \mathbf{I}_{n_2})$, and from Theorem 11 it furthermore follows that $\mathbf{A} = \mathbf{X}'\mathbf{X} \sim \mathbb{Q}W_p(n_1, \mathbf{\Sigma})$ and $\mathbf{B} = \mathbf{Y}'\mathbf{Y} \sim \mathbb{Q}W_p(n_2, \mathbf{\Sigma}, \mathbf{\Omega})$ (the *non-central quaternion Wishart distribution* with $\mathbf{\Omega} = \mathbf{\Sigma}^{-1}\boldsymbol{\mu}\bar{\boldsymbol{\mu}}'$, see [12]).
Wilks' statistic

$$\Lambda = \frac{\det(\mathbf{A})}{\det(\mathbf{A} + \mathbf{B})}$$

(with $\mathbf{A}$ and $\mathbf{B}$ as defined above) can be used as a likelihood ratio criterion for testing whether the matrix mean $\boldsymbol{\mu}$ is equal to zero or not.

What is the corresponding density function expression for Wilks' statistic in the case of quaternion Wishart matrices?

Mehta [18] provides many useful results for quaternion random matrices, for instance, on p 282 the probability density function for the determinant of a $n \times n$ random Hermitian matrix taken from the Gaussian unitary ensemble is calculated. The present work proposes the distribution of Wilks' statistic based on Meijer's G-function (see [16]) in a numerical feasible form. Since Λ is real according to Lemma 1(5) (also see Mehta [18], p 284), the result follows similarly as that given in Bekker, Roux and Arashi [2].

From the definition of the non-central quaternion matrix-variate beta type I distribution (see [15]), it follows that

$$\begin{aligned}
&E\left[\det(\mathbf{U})^{h-1}\right] \\
&= \frac{\exp\{\mathrm{Re}\,\mathrm{tr}(-2\mathbf{\Omega})\}}{\mathbb{Q}B_p(2n_1, 2n_2)} \int\limits_{\mathbf{0}<\mathbf{U}=\bar{\mathbf{U}}'<\mathbf{I}_p} \det(\mathbf{U})^{h+2n_1-2p} \det(\mathbf{I}_p - \mathbf{U})^{2n_2-2p+1} {}_1\mathbb{Q}F_1\left(2(n_1+n_2); 2n_2; 2\mathbf{\Omega}(\mathbf{I}_p - \mathbf{U})\right) d\mathbf{U} \\
&= \frac{\exp\{\mathrm{Re}\,\mathrm{tr}(-2\mathbf{\Omega})\}}{\mathbb{Q}B_p(2n_1, 2n_2)} \sum_{k=0}^{\infty}\sum_{\kappa} \frac{(2(n_1+n_2))_\kappa}{k!\,(2n_2)_\kappa} \int\limits_{\mathbf{0}<\mathbf{U}=\bar{\mathbf{U}}'<\mathbf{I}_p} \det(\mathbf{U})^{h+2n_1-2p} \det(\mathbf{I}_p - \mathbf{U})^{2n_2-2p+1} \mathbb{Q}C_\kappa\left(2\mathbf{\Omega}(\mathbf{I}_p - \mathbf{U})\right) d\mathbf{U}
\end{aligned}$$

where $\mathbb{Q}B_p(\cdot)$ is the quaternion multivariate beta function (see [12]) and ${}_1\mathbb{Q}F_1(\cdot)$ is the quaternion confluent hypergeometric function with a matrix argument (see [15]).
Let $\mathbf{T} = (\mathbf{I}_p - \mathbf{U})$, after applying equation (3.13) of [4] to the above expression, and then simplifying, we obtain

$$E\left[\det(\mathbf{U})^{h-1}\right] = \frac{\exp\{\mathrm{Re}\,\mathrm{tr}(-2\mathbf{\Omega})\}}{\mathbb{Q}\Gamma_p(2n_1)} \sum_{k=0}^{\infty}\sum_{\kappa} \frac{\mathbb{Q}\Gamma_p(2(n_1+n_2), \kappa)\mathbb{Q}\Gamma_p(2n_1+h-1)}{k!\mathbb{Q}\Gamma_p(2(n_1+n_2)+h-1, \kappa)} \mathbb{Q}C_\kappa(2\mathbf{\Omega})$$

From $\mathbb{Q}\Gamma_p(a) = \pi^{p(p-1)} \prod_{\alpha=1}^{p} \Gamma(a - 2(\alpha-1))$, $\mathrm{Re}(a) > 2(p-1)$,(see [8]) the density function of $\Lambda = \det(\mathbf{U})$ is uniquely determined by the inverse Mellin transform as given in ([16], Definition 1.8, p 23). Therefore,

$$f_\Lambda(\lambda) = \frac{\exp\{\mathrm{Re}\,\mathrm{tr}(-2\mathbf{\Omega})\}}{\mathbb{Q}\Gamma_p(2n_1)} \sum_{k=0}^{\infty}\sum_{\kappa} \mathbb{Q}C_\kappa(2\mathbf{\Omega}) \frac{\mathbb{Q}\Gamma_p(2(n_1+n_2), \kappa)}{k!} G_{p,p}^{p,0}\left(\lambda \,\middle|\, \begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_p \end{matrix}\right)$$

where $a_l = 2(n_1+n_2-l)+k_l+1$, $l = 1, \ldots, p$ and $b_l = 2n_1 - 2l + 1$, $l = 1, \ldots, p$, and $G(\cdot)$ is Meijer's G-function (see Mathai, [16], p 60).

For $\mathbf{\Omega} = \mathbf{0}$, using the definition of the quaternion matrix-variate beta type I distribution (see [15]), the distribution of Wilks' statistic, under the null hypothesis, is given by

$$f_{\Lambda}(\lambda) = \frac{\mathbb{Q}\Gamma_p(2(n_1+n_2))}{\mathbb{Q}\Gamma_p(2n_1)} G^{p,0}_{p,p}\left(\lambda \middle| \begin{array}{c} a_1,\ldots,a_p \\ b_1,\ldots,b_p \end{array}\right)$$

where $a_l = 2(n_1+n_2-l)+1$, $l=1,\ldots,p$ and $b_l = 2n_1 - 2l + 1$, $l = 1,\ldots,p$.

Remark 13 *Hotelling's T^2 statistic is given by $T^2 = n\underline{Y}'\mathbf{A}^{-1}\underline{Y}$ where $\underset{p\times 1}{\underline{Y}} \sim \mathbb{Q}N(p;\underline{0},\mathbf{\Sigma})$ and which is independently distributed of $\underset{p\times p}{\mathbf{A}} \sim \mathbb{Q}W_p(n,\mathbf{\Sigma})$. From Lemma 1(6) this is equal to $n\underline{Y}_0^{*\prime}\mathbf{A}_0^{-1}\underline{Y}_0$ where $\underset{4p\times 1}{\underline{Y}_0^*} \sim N(4p;\underline{0},\mathbf{\Sigma}_0)$ (from $Definition$ 4)and which is independently distributed of $\underset{4p\times 4p}{\mathbf{A}_0} \sim W_{4p}(n,\mathbf{\Sigma}_0)$ (from Theorem 11). Once again, this problem reduces to a problem in the real space, and familiar techniques and inference procedures in the real distribution theory may be applied.*

6 Concluding remarks

In this paper we focussed on the real representation of the matrix-variate quaternion normal distribution in distribution theory. For the first time the quaternion Wishart distribution was derived from the *real associated* Wishart distribution via the characteristic function. A simple quaternion hypothesis was represented with an associated real hypothesis, and thereafter an expression for the density function of Wilks' statistic in the case of quaternion Wishart matrices was derived.

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