A Generalized Multivariate Beta Distribution: control charting when the measurements are from an exponential distribution

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Abstract In Statistical Process Control (SPC) there exists a need to model the run-length distribution of a Q-chart that monitors the process mean when measurements are from an exponential distribution with an unknown parameter. To develop exact expressions for the probabilities of run-lengths the joint distribution of the charting statistics is needed. This gives rise to a new distribution that can be regarded as a generalized multivariate beta distribution. An overview of the problem statement as identified in the field of SPC is given and the newly developed generalized multivariate beta distribution is proposed. Statistical properties of this distribution are studied and the effect of the parameters of this generalized multivariate beta distribution on the correlation between two variables is also discussed.

Keywords Multivariate beta · Chi-squared · Statistical Process Control · Run-length · Shewhart · Hypergeometric functions · Correlation

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1. Introduction

Practitioners in the quality control environment require methods to monitor a process from the start of the production, whether or not prior (historical or past) information is available for estimating the parameters. Quesenberry (1991) presented Q-charts assuming that the observations from each sample are independent and identically distributed normal random variables. However Human and Chakraborti (2010) highlighted that there are cases where the assumption of normality is not valid. They proposed a Q-chart design for monitoring the process mean when the measurements are from an exponential distribution and the parameter of the distribution is unknown. To gain more insight into the performance of a control chart, one needs to consider the run-length distribution of the proposed chart. The run-length of a control chart is the number of samples collected until the shift is detected; this can also be viewed as the waiting time until a signal is observed following a shift or change in the process parameter. To develop exact expressions for the probabilities of the run-lengths the joint distribution of the charting statistics is needed. The following is an overview of the problem statement identified in the field of SPC; for more detail see Human and Chakraborti (2010).

To describe the proposed Q-charts in more detail, let $(X_{r1}, X_{r2}, \ldots, X_{rn})$, $r = 1, 2, \ldots$ represent successive and independent samples of $n \geq 1$ measurements made on a sequence of items and assume that these values are independent and identically distributed having been collected from an $EXP(\theta)$ distribution where the parameter θ denotes the unknown process mean.

Because we assume θ is unknown, the first sample is used to obtain an initial estimate of θ. This initial estimate is continuously updated using the new incoming samples as they are collected (as long as the value of θ does not change).

To this end, let

$$\overline{\overline{X}}_r = \frac{1}{r}\sum_{i=1}^{r}\overline{X}_i = \frac{1}{rn}\sum_{i=1}^{r}\sum_{j=1}^{n}X_{ij} \quad \text{for} \quad r = 1, 2, \ldots \tag{1}$$

where $\overline{\overline{X}}_r$ denote the overall mean of all the measurements up to and including sample r and $\overline{X}_i$ denote the i^{th} sample mean. The sequential sample quantity in (1) is computed as each new sample becomes available and can be calculated using the following updating formula

$$\overline{\overline{X}}_r = \frac{1}{r}\left[\overline{X}_r + (r-1)\,\overline{\overline{X}}_{r-1}\right] \quad \text{for} \quad r = 1, 2, \ldots \tag{2}$$

The updating formula in (2) allows the computation of $\overline{\overline{X}}_r$ from the latest (previous) overall mean $\overline{\overline{X}}_{r-1}$ and the most recent sample mean $\overline{X}_r$. Note that, when $r = 1$ we have that $\overline{\overline{X}}_1 = \overline{X}_1$. Also note that, as long as the process remains in-control $\overline{\overline{X}}_r$ is the MLE of θ when the samples $1, 2, \ldots, r$ are pooled/combined.

Therefore, the first sample is used to obtain an initial estimate of θ, that is, $\overline{\overline{X}}_1 = \overline{X}_1$ estimates θ at sample number one, and at sample number two $\overline{X}_2$ is compared to $\overline{\overline{X}}_1$ to check if the value of θ is still the same. If the value of θ is still the same at sample number two, a new updated estimate of θ is obtained. The updated estimate is $\overline{\overline{X}}_2 = \frac{1}{2}\left[\overline{X}_2 + \overline{\overline{X}}_1\right]$ and includes the information from samples one and two; the estimate $\overline{\overline{X}}_2$ is then used to check if the value of θ is still the same at sample $r = 3$ by comparing $\overline{X}_3$ to $\overline{\overline{X}}_2$. This sequential updating-and-testing procedure continues until a change is detected in the value of θ.

To describe the procedure in more general terms, suppose that there are two independent samples. The first sample consists of the measurements of the first $r-1$ samples combined and the second sample consists of the measurements of the r^{th} sample only, i.e.
Sample 1: $\{X_{11}, X_{12}, \ldots, X_{1n}; \ldots; X_{r-11}, X_{r-12}, \ldots, X_{r-1n}\}$ and **Sample 2:** $\{X_{r1}, X_{r2}, \ldots, X_{rn}\}$. Let $\overline{\overline{X}}_{r-1}$ in expression (1) denote the overall mean of Sample 1 and let $\overline{X}_r$ denote the mean of the observations from Sample 2. The well-known two-sample statistic for testing the hypothesis at time r that the two independent samples are from exponential distributions with the same unknown parameter, is based on the statistic

$$U_r^* = \frac{\overline{X}_r}{\overline{\overline{X}}_{r-1}} \quad \text{for} \quad r = 2, 3, \ldots \tag{3}$$

Note that, without loss of generality, it is assumed that Sample 1 is from an exponential distribution with parameter θ (i.e. $Y = \frac{2n(r-1)\overline{\overline{X}}_{r-1}}{\theta} \sim \chi^2_{2n(r-1)}$) and that Sample 2 is from an exponential distribution with parameter $\theta_1 = \lambda\theta$ where $\lambda > 0$ (i.e. $X = \frac{2n\overline{X}_r}{\theta_1} \sim \chi^2_{2n}$) so that U_r^* is in fact a test to check whether $\lambda = 1$ (i.e. the parameters are the same) versus $\lambda \neq 1$ (i.e. the parameter changed). The distribution of U_r^* when $\lambda = 1$ is an F distribution with numerator degrees of freedom $2n$ and denominator degrees of freedom $2n\,(r-1)$. This result is established by re-writing (3) as

$$U_r^* = \frac{\overline{X}_r}{\overline{\overline{X}}_{r-1}} = \lambda Z \;\; \text{with} \;\; \lambda = \frac{\theta_1}{\theta} = 1 \quad \text{for} \quad r = 2, 3, \ldots$$

where $Z = \frac{\frac{X}{2n}}{\frac{Y}{2n(r-1)}} \sim F_{2n,2n(r-1)}$, and the two random variables i.e. X and Y, are independent because it is assumed that successive samples are independent.

In SPC, once the process encountered a permanent / sustained upward or downward step shift, one is interested in determining the probability of detecting the change in the parameter θ as soon as possible. To this end, suppose that from sample κ the process parameter has changed from θ to $\theta_1 = \lambda\theta$ where $\lambda \neq 1$ and $\lambda > 0$. This can be summarised as follows:

$$\begin{array}{rcc} \text{Sample mean:} & \underbrace{\overline{X}_1, \overline{X}_2, \ldots, \overline{X}_{\kappa-1}} & \underbrace{\overline{X}_\kappa, \overline{X}_{\kappa+1}, \ldots, \overline{X}_{\kappa+t-1}, \overline{X}_{\kappa+t}, \overline{X}_{\kappa+t+1}, \ldots} \\ \text{Distribution of the sample:} & EXP(\theta) & EXP(\theta_1) \end{array}$$

Following a change in the process parameter at sample κ the random variable $U^*_{\kappa+t}$, $\kappa = 2, 3, \ldots$ and $t = 0, 1, 2, \ldots$ can be written as

$$\begin{aligned} U^*_{\kappa+t} &= \frac{\overline{X}_{\kappa+t}}{\overline{\overline{X}}_{\kappa+t-1}} \\ &= \frac{\overline{X}_{\kappa+t}}{\frac{1}{\kappa+t-1}\left[(\kappa-1)\overline{\overline{X}}_{\kappa-1} + t\overline{\overline{X}}_{[\kappa:\kappa+t-1]}\right]} \\ &= \frac{\frac{\theta_1}{\theta}\frac{\overline{X}_{\kappa+t}}{\theta_1}}{\frac{1}{\kappa+t-1}\left[(\kappa-1)\frac{\overline{\overline{X}}_{\kappa-1}}{\theta} + t\frac{\theta_1}{\theta}\frac{\overline{\overline{X}}_{[\kappa:\kappa+t-1]}}{\theta_1}\right]} \\ &= \frac{\lambda\left\{\frac{W_{\kappa+t}}{2n}\right\}}{\frac{1}{\kappa+t-1}\left[(\kappa-1)\left\{\frac{W_{[1:\kappa-1]}}{2n(\kappa-1)}\right\} + t\lambda\left\{\frac{W_{[\kappa:\kappa+t-1]}}{2nt}\right\}\right]} \\ &= (\kappa+t-1)\frac{\lambda W_{\kappa+t}}{W_{[1:\kappa-1]} + \lambda W_{[\kappa:\kappa+t-1]}} \end{aligned} \quad (4)$$

where $\overline{\overline{X}}_{[\kappa:\kappa+t-1]} = \frac{1}{t}\sum_{j=\kappa}^{\kappa+t-1}\overline{X}_j$ is the mean of all the observations from sample κ to sample $\kappa+t-1$, and the random variables $W_{\kappa+t} = \frac{2n\overline{X}_{\kappa+t}}{\theta_1} \sim \chi^2_{2n}$, $W_{[1:\kappa-1]} = \frac{2n(\kappa-1)\overline{\overline{X}}_{\kappa-1}}{\theta} \sim \chi^2_{2n(\kappa-1)}$ and $W_{[\kappa:\kappa+t-1]} = \frac{2nt\overline{\overline{X}}_{[\kappa:\kappa+t-1]}}{\theta_1} \sim \chi^2_{2nt}$ are independent and $\lambda = \frac{\theta_1}{\theta}$. Take note, when $t = 0$, the term $W_{[\kappa:\kappa+t-1]}$ is undefined and therefore the denominator will only consist of $W_{[1:\kappa-1]}$.

Q-charts are constructed by plotting the charting statistic on a Shewhart type chart with lower- and upper control limits (LCL and UCL) and a center line over time. The charting statistic for the Q-chart is a function of the random variable (4). The process is declared out of control if the charting statistic plots on or outside the control limits.

As described above the process is regarded to be in control when the charting statistic plots between the lower- and upper control limits. In terms of the random variable (4), this translates to,

$$LCL^*_{\kappa+t} < U^*_{\kappa+t} = (\kappa+t-1)\frac{\lambda W_{\kappa+t}}{W_{[1:\kappa-1]} + \lambda W_{[\kappa:\kappa+t-1]}} < UCL^*_{\kappa+t}$$

which can be rewritten as,

$$LCL_{\kappa+t} < U_{\kappa+t} = \frac{\lambda W_{\kappa+t}}{W_{[1:\kappa-1]} + \lambda W_{[\kappa:\kappa+t-1]}} < UCL_{\kappa+t}$$

Take note that, for example, $UCL_{\kappa+t} = \frac{UCL^*_{\kappa+t}}{\kappa+t-1}$, which is a function of UCL and time. This is also true for $LCL_{\kappa+t}$.

This paper focuses on the random variable

$$U_{\kappa+t} = \frac{\lambda W_{\kappa+t}}{W_{[1:\kappa-1]} + \lambda W_{[\kappa:\kappa+t-1]}} \quad \text{for } \kappa = 2, 3, \ldots \text{ and } t = 0, 1, 2, \ldots \tag{5}$$

because the distribution of this random variable is unknown and is needed to determine the probabilities of the run-lengths as explained below.

Once a shift in the process parameter occurred, the run-length is the number of samples collected from time κ (i.e. first sample after the change) until an out-of-control signal is observed (i.e. charting statistic plots on or outside the control limits). The discrete random variable defining the run-length is called the run-length random variable and typically denoted by N. The distribution of N is called the run-length distribution. The probability of detecting a shift immediately, in other words, the probability of a run-length of one, is the likelihood that a signal is obtained at time κ and to calculate this probability the marginal distribution of the random variable U_κ is needed. The probability that the run-length is one, is one minus the probability that the random variable, U_κ, plots between the control limits,

$$\Pr(N = 1) = 1 - \int_{LCL_\kappa}^{UCL_\kappa} f(u_\kappa)\, du_\kappa$$

To develop exact expressions for the probabilities of run-lengths greater than one the joint distribution of the charting statistics is needed because after a change occurred, the charting statistics are no longer independent. Consider as an example a run-length of two and define the following two events, $A = \{LCL_\kappa < U_\kappa < UCL_\kappa\}$ and $B = \{LCL_{\kappa+1} < U_{\kappa+1} < UCL_{\kappa+1}\}$, then the probability of a run-length of two is the probability of having no signal in the first sample and having a signal in the second sample,

$$\begin{aligned} \Pr(N = 2) &= \Pr(A \cap B^c) \\ &= \Pr(A) - \Pr(A \cap B) \\ &= \int_{LCL_\kappa}^{UCL_\kappa} f(u_\kappa)\, du_\kappa - \int_{LCL_{\kappa+1}}^{UCL_{\kappa+1}} \int_{LCL_\kappa}^{UCL_\kappa} f(u_\kappa, u_{\kappa+1})\, du_\kappa du_{\kappa+1} \end{aligned}$$

Thus for a run-length of two, the bivariate distribution of U_κ and $U_{\kappa+1}$ is required and for a run-length of three, the trivariate distribution is required, etc.

The distribution of the random variable (5) is derived in Section 2. To simplify the notation, define

$$\begin{aligned} U_0 &= \tfrac{\lambda W_0}{X} \\ U_j &= \frac{\lambda W_j}{X + \lambda \sum\limits_{k=0}^{j-1} W_k} \quad \text{where } j = 1, 2, \ldots \text{ and } \lambda > 0 \end{aligned} \tag{6}$$

where X, W_j with $j = 0, 1, 2, \ldots$ are independent chi-squared random variables with degrees of freedom a and v_j with $j = 0, 1, 2, \ldots$ respectively. The random variables defined in expression (6) are equivalent to (5) where j has the role of t, i.e. it indicates the number of samples after the parameter θ changed. For example, U_κ and $U_{\kappa+1}$ in (5) is the same as U_0 and U_1 in (6). The information regarding κ, the sample number when the parameter θ changed to $\theta_1 = \lambda\theta$, in (5) is contained in the degrees of freedom of the random variable X in (6), i.e. $a = 2n(\kappa - 1)$ where n represents the sample size, therefore a represents the total number of observations (for all samples) before the shift occurred. The degrees of freedom for W_j depends also on the sample size, i.e. $v_j = 2n$.

Note that, the random variables in (6) are constructed from independent chi-squared random variables using the variables-in-common (or trivariate reduction) technique. Other bivariate distributions that are also constructed in this way that is defined on the positive domain includes the usual bivariate F (Balakrishnan and Lai, 2009) and the extended bivariate F proposed by El-Bassiouny and Jones (2008). Gupta et al. (2009) derived a non-central bivariate beta type 1 distribution that is defined on the unit square; applying the appropriate transformation will yield a distribution defined on the positive domain.

In this paper the distribution of $(U_0, U_1, \ldots, U_p)$ i.e. a new generalized multivariate beta distribution is derived. In terms of the problem identified from the SPC context, this distribution is used to develop exact expressions for the probability of detecting a change in the parameter θ as explained above. This new generalized multivariate beta distribution is explored in detail in Sections 2, 3 and 4. Section 2 focuses on the joint and marginal distributions. Because the moments are used to investigate the correlation structure between the random variables, this statistical property of the distribution is studied in this paper. In Section 3 a general expression is derived for the product moments of the distribution, while the correlation of the charting statistics for the bivariate case together with a shape analysis of the univariate and bivariate distributions are considered in Section 4.

2. The Generalized Multivariate Beta Distribution

In this section the joint distribution of the random variables in (6) is derived. This gives rise to a new distribution that can be regarded as a generalized multivariate beta distribution. The construction of these random variables (see (6)) and their dependence structure originated from the problem identified in SPC. In Section 2.1 the joint density of the new multivariate beta distribution is derived. The marginal distributions including the univariate distribution, the bivariate distribution and the distribution of a subset of $(U_0, U_1, \ldots, U_p)$ are presented in Section 2.2.

2.1. Multivariate Distribution Function

In this section the joint density of $(U_0, U_1, \ldots, U_p)$ is derived.

Theorem 1 *Let X, W_j with $j = 0, 1, 2, \ldots, p$ be independent chi-squared random variables with degrees of freedom a and v_j with $j = 0, 1, 2, \ldots, p$ respectively. Let $U_0 = \frac{\lambda W_0}{X}$, and $U_j = \frac{\lambda W_j}{X + \lambda \sum_{k=0}^{j-1} W_k}$ where $j = 1, 2, \ldots, p$ and $\lambda > 0$. The joint density of $(U_0, U_1, \ldots, U_p)$ is given by*

$$f(u_0, u_1, \ldots, u_p)$$

$$= \frac{\Gamma\left(\frac{a}{2} + \sum_{j=0}^{p} \frac{v_j}{2}\right) \lambda^{\frac{a}{2}}}{\Gamma\left(\frac{a}{2}\right) \prod_{j=0}^{p} \Gamma\left(\frac{v_j}{2}\right)} \left(\prod_{j=0}^{p} u_j^{\frac{v_j}{2} - 1} \right) \left(\prod_{k=0}^{p-1} (1 + u_k)^{\sum_{j=k+1}^{p} \frac{v_j}{2}} \right) \left(\lambda + u_0 + \sum_{j=1}^{p} u_j \prod_{k=0}^{j-1} (1 + u_k) \right)^{-\left(\frac{a}{2} + \sum_{j=0}^{p} \frac{v_j}{2} \right)}, \tag{7}$$

$$u_j > 0,\ j = 0, 1, \ldots, p$$

Proof The joint density of $X, W_0, W_1, \ldots, W_p$ is

$$f(x, w_0, w_1, \ldots, w_p) = \frac{1}{2^{\frac{a}{2} + \sum_{j=0}^{p} \frac{v_j}{2}} \Gamma\left(\frac{a}{2}\right) \prod_{j=0}^{p} \Gamma\left(\frac{v_j}{2}\right)} x^{\frac{a}{2} - 1} e^{-\frac{x}{2}} \prod_{j=0}^{p} w_j^{\frac{v_j}{2} - 1} e^{-\frac{w_j}{2}}$$

Let $U = X$, $U_0 = \frac{\lambda W_0}{X}$ and $U_j = \frac{\lambda W_j}{X + \lambda \sum_{k=0}^{j-1} W_k}$ where $j = 1, 2, \ldots, p$. This gives the inverse transformation, $X = U, W_0 = \frac{1}{\lambda} U_0 U$ and $W_j = \frac{1}{\lambda} U_j U \prod_{k=0}^{j-1} (1 + U_k)$ where $j = 1, 2, \ldots, p$ with Jacobian $J(x, w_0, w_1, \ldots, w_p \to u, u_0, u_1, \ldots, u_p) = \left(\frac{u}{\lambda}\right)^{p+1} \prod_{k=0}^{p-1} (1 + u_k)^{p-k}$

Thus, the joint density of $U, U_0, U_1, \ldots, U_p$ is given by

$$f\left(u, u_0, u_1, \ldots, u_p\right) = \frac{\lambda^{\left(-\sum_{j=0}^{p}\frac{v_j}{2}\right)}}{2^{\frac{a}{2}+\sum_{j=0}^{p}\frac{v_j}{2}}\Gamma\left(\frac{a}{2}\right)\prod_{j=0}^{p}\Gamma\left(\frac{v_j}{2}\right)} u^{\frac{a}{2}+\sum_{j=0}^{p}\frac{v_j}{2}-1} u_0^{\frac{v_0}{2}-1}\left(\prod_{j=1}^{p}u_j^{\frac{v_j}{2}-1}\right)\left(\prod_{k=0}^{p-1}\left(1+u_k\right)^{\sum_{j=k+1}^{p}\frac{v_j}{2}}\right)e^{-\frac{u}{2}\left(1+\frac{u_0}{\lambda}+\sum_{j=1}^{p}\frac{u_j}{\lambda}\prod_{k=0}^{j-1}(1+u_k)\right)}$$

(Note that $\prod_{j=1}^{p}\left[\prod_{k=0}^{j-1}\left(1+u_k\right)\right]^{\frac{v_j}{2}-1} = \prod_{k=0}^{p-1}\left(1+u_k\right)^{\sum_{j=k+1}^{p}\frac{v_j}{2}-(p-k)}$)

Now, integrating this expression with respect to u using the definition of the gamma integral function (see Prudnikov 1986, Eq. 2.3.3(1), p322), yields the desired result. □

Remarks

(i) An alternative expression for the joint probability density function in (7) in terms of the hypergeometric function, ${}_1F_0(\cdot)$, can be obtained as follows,

$$f\left(u_0, u_1, \ldots, u_p\right) = \frac{\Gamma\left(\frac{a}{2}+\sum_{j=0}^{p}\frac{v_j}{2}\right)\lambda^{\frac{a}{2}}}{\Gamma\left(\frac{a}{2}\right)\prod_{j=0}^{p}\Gamma\left(\frac{v_j}{2}\right)}\left(\prod_{j=0}^{p}u_j^{\frac{v_j}{2}-1}\right)\left(\prod_{k=0}^{p}\left(1+u_k\right)^{-\left(\frac{a}{2}+\sum_{j=0}^{k}\frac{v_j}{2}\right)}\right)\left(\frac{\lambda+\prod_{k=0}^{p}(1+u_k)-1}{\prod_{k=0}^{p}(1+u_k)}\right)^{-\left(\frac{a}{2}+\sum_{j=0}^{p}\frac{v_j}{2}\right)}$$

using the fact that $\lambda + u_0 + \sum_{j=1}^{p} u_j \prod_{k=0}^{j-1}\left(1+u_k\right) = \lambda + \prod_{k=0}^{p}\left(1+u_k\right) - 1.$

This can be simplified using the binomial series ${}_1F_0(\alpha; z) = (1-z)^{-\alpha}$ for $|z| < 1$ (Mathai, 1993, p25) with $1 - z = \frac{\lambda+\prod_{k=0}^{p}(1+u_k)-1}{\prod_{k=0}^{p}(1+u_k)}$ and $\alpha = \frac{a}{2} + \sum_{j=0}^{p}\frac{v_j}{2}$, then

$$f\left(u_0, u_1, \ldots, u_p\right) = \frac{\Gamma\left(\frac{a}{2}+\sum_{j=0}^{p}\frac{v_j}{2}\right)\lambda^{\frac{a}{2}}}{\Gamma\left(\frac{a}{2}\right)\prod_{j=0}^{p}\Gamma\left(\frac{v_j}{2}\right)}\left(\prod_{j=0}^{p}u_j^{\frac{v_j}{2}-1}\right)\left(\prod_{k=0}^{p}\left(1+u_k\right)^{-\left(\frac{a}{2}+\sum_{j=0}^{k}\frac{v_j}{2}\right)}\right){}_1F_0\left(\frac{a}{2}+\sum_{j=0}^{p}\frac{v_j}{2}; \frac{1-\lambda}{\prod_{k=0}^{p}\left(1+u_k\right)}\right) \tag{8}$$

(ii) The joint density of $(U_0, U_1, \ldots, U_p)$ can also be expressed in terms of the product of beta type II densities by expanding ${}_1F_0(\cdot)$ in expression (8) in series form. Therefore,

$$f(u_0, u_1, \ldots, u_p) = \lambda^{\frac{a}{2}}\sum_{b=0}^{\infty}\frac{\left(\frac{a}{2}\right)_b}{b!}\left(1-\lambda\right)^b\prod_{j=0}^{p}Beta^{II}\left(\frac{v_j}{2}, \frac{a}{2}+\sum_{k=0}^{j-1}\frac{v_k}{2}+b\right) \tag{9}$$

where $(\alpha)_i$ is the Pochhammer coefficient defined as $(\alpha)_i = \alpha\left(\alpha+1\right)\ldots\left(\alpha+i-1\right) = \frac{\Gamma(\alpha+i)}{\Gamma(\alpha)}$ and $Beta^{II}\left(\cdot\right)$ denotes the known beta type II distribution. This alternative representation of the joint density is used in some of the subsequent derivations.

(iii) The joint density of $(U_0, U_1, \ldots, U_p)$ can be used to calculate the probability of a run-length of $p+1$, for $p = 1, 2, \ldots$. (See also Human and Chakraborti (2010) expressions (19) to (21).)

(iv) If $\lambda = 1$, i.e. the process is in control, the joint density (7) simplifies to

$$f(u_0, u_1, \ldots, u_p) = \frac{\Gamma\left(\frac{a}{2}+\sum_{j=0}^{p}\frac{v_j}{2}\right)}{\Gamma(\frac{a}{2})\prod_{j=0}^{p}\Gamma(\frac{v_j}{2})}\left(\prod_{j=0}^{p} u_j^{\frac{v_j}{2}-1}\right)\left(\prod_{k=0}^{p}(1+u_k)^{-\left(\frac{a}{2}+\sum_{j=0}^{k}\frac{v_j}{2}\right)}\right), \ u_j > 0$$

This confirms the independency of the random variables when $\lambda = 1$.

2.2. Marginal Distributions

This section focuses on the density of any subset of random variables of the generalized multivariate beta distribution. The marginal density of U_j, $j = 0, 1, \ldots p$, in general is derived in Section 2.2.1. In Section 2.2.2 the bivariate density of (U_j, U_{j+m}) is derived which will be used to investigate the correlation structure which is discussed in Section 4. This section is concluded with the derivation of the density of a subset of $(U_0, U_1, \ldots, U_p)$. The marginal distributions derived in this section will also be used to determine the moments in Section 3.

2.2.1. Univariate distribution

Theorem 2 *Let X, W_j with $j = 0, 1, 2, \ldots, p$ be independent chi-squared random variables with degrees of freedom a and v_j with $j = 0, 1, 2, \ldots, p$ respectively. Let $U_0 = \frac{\lambda W_0}{X}$ and $U_j = \frac{\lambda W_j}{X+\lambda\sum_{k=0}^{j-1}W_k}$ where $j = 1, 2, \ldots, p$ and $\lambda > 0$. If the joint density of $U_0, U_1, \ldots, U_p$ is given by (7), then the marginal density of*

(a) U_j, $j = 1, 2, \ldots p$ is given by

$$f(u_j) = \frac{\Gamma\left(\frac{a}{2}+\sum_{k=0}^{j}\frac{v_k}{2}\right)}{\Gamma(\frac{v_j}{2})\Gamma\left(\frac{a}{2}+\sum_{k=0}^{j-1}\frac{v_k}{2}\right)}\lambda^{\frac{a}{2}} u_j^{\frac{v_j}{2}-1}(1+u_j)^{-\left(\frac{a}{2}+\sum_{k=0}^{j}\frac{v_k}{2}\right)} {}_2F_1\left(\frac{a}{2}+\sum_{k=0}^{j}\frac{v_k}{2}, \frac{a}{2}; \frac{a}{2}+\sum_{k=0}^{j-1}\frac{v_k}{2}; \frac{1-\lambda}{1+u_j}\right), u_j > 0 \tag{10}$$

and of

(b) U_0 is given by

$$f(u_0) = \frac{\Gamma(\frac{a}{2}+\frac{v_0}{2})\lambda^{\frac{a}{2}}}{\Gamma(\frac{a}{2})\Gamma(\frac{v_0}{2})} u_0^{\frac{v_0}{2}-1}(\lambda+u_0)^{-\left(\frac{a}{2}+\frac{v_0}{2}\right)}, u_0 > 0 \tag{11}$$

where ${}_2F_1(\cdot)$ denotes the Gauss hypergeometric function.

Proof (a) Let $T = \sum_{k=0}^{j-1} W_k$, therefore $T \sim GAM(2, b)$ with $b = \sum_{k=0}^{j-1}\frac{v_k}{2}$. The joint density of W_j, X and T is given by

$$f(w_j, x, t) = \frac{1}{2^{\frac{a}{2}+\frac{v_j}{2}+b}\Gamma\left(\frac{v_j}{2}\right)\Gamma\left(\frac{a}{2}\right)\Gamma(b)} w_j^{\frac{v_j}{2}-1} x^{\frac{a}{2}-1} t^{b-1} e^{-\frac{w_j}{2}-\frac{x}{2}-\frac{t}{2}}$$

By applying the transformation $U_j = \frac{\lambda W_j}{X+\lambda T}$ and $Y = X + \lambda T$ with Jacobian $J\left(w_j, x, t \rightarrow u_j, y, t\right) = \frac{y}{\lambda}$ the joint density of U_j, Y and T is

$$f\left(u_j, y, t\right) = \frac{\lambda^{\frac{a}{2}-\frac{v_j}{2}-1}}{2^{\frac{a}{2}+\frac{v_j}{2}+b}\Gamma\left(\frac{v_j}{2}\right)\Gamma\left(\frac{a}{2}\right)\Gamma\left(b\right)} u_j^{\frac{v_j}{2}-1} y^{\frac{v_j}{2}} \left(\frac{y}{\lambda}-t\right)^{\frac{a}{2}-1} t^{b-1} e^{-\frac{y}{2}\left(1+\frac{u_j}{\lambda}\right)-\frac{t}{2}(1-\lambda)}$$

Therefore, the marginal density of U_j is,

$$f\left(u_j\right) = \frac{\lambda^{\frac{a}{2}-\frac{v_j}{2}-1}}{2^{\frac{a}{2}+\frac{v_j}{2}+b}\Gamma\left(\frac{v_j}{2}\right)\Gamma\left(\frac{a}{2}\right)\Gamma\left(b\right)} u_j^{\frac{v_j}{2}-1} \int_0^\infty y^{\frac{v_j}{2}} e^{-\frac{y}{2}\left(1+\frac{u_j}{\lambda}\right)} \int_0^{\frac{y}{\lambda}} t^{b-1} \left(\frac{y}{\lambda}-t\right)^{\frac{a}{2}-1} e^{-\frac{t}{2}(1-\lambda)} dtdy$$

Using Gradshteyn and Ryzhik (2007) Eq. 3.383(1), p347, and Eq. 7.522(9), p815 and relation Eq. 9.131(1), p1008, the desired result (10) follows after simplification.

(b) For $j = 0$, the marginal density of U_0 can be obtained using a similar approach as in (a). □

Remarks

(i) The marginal density, $f(u_0)$, can be used to calculate the probability of a run-length of one as explained in the introduction. (See also Human and Chakraborti (2010) expression (18).)

(ii) If $\lambda = 1$, i.e. when the process is in control, the marginal density (10,11) simplifies to a beta type II density with parameters $\frac{v_j}{2}$ and $\frac{a}{2} + \sum_{k=0}^{j-1} \frac{v_k}{2}$.

$$f\left(u_j\right) = \frac{\Gamma\left(\frac{a}{2}+\sum_{k=0}^{j}\frac{v_k}{2}\right)}{\Gamma(\frac{v_j}{2})\Gamma\left(\frac{a}{2}+\sum_{k=0}^{j-1}\frac{v_k}{2}\right)} u_j^{\frac{v_j}{2}-1} \left(1+u_j\right)^{-\left(\frac{a}{2}+\sum_{k=0}^{j}\frac{v_k}{2}\right)} \quad , u_j > 0$$

(See also remark (iv) of Section 2.1.)

2.2.2 Bivariate distribution

Theorem 3 *Let X, W_j with $j = 0, 1, 2, \ldots, p$ be independent chi-squared random variables with degrees of freedom a and v_j with $j = 0, 1, 2, \ldots, p$ respectively. Let $U_0 = \frac{\lambda W_0}{X}$ and $U_j = \frac{\lambda W_j}{X+\lambda\sum_{k=0}^{j-1} W_k}$ where $j = 1, 2, \ldots, p$ and $\lambda > 0$. If the joint density of $U_0, U_1, \ldots, U_p$ is given by (7), then the bivariate density of U_j and U_{j+m} is given by*

$$f(u_j, u_{j+m}) = \lambda^{\frac{a}{2}} \sum_{b=0}^{\infty} \frac{\left(\frac{a}{2}\right)_b}{b!} (1-\lambda)^b \times Beta^{II}\left(\frac{v_j}{2}, \frac{a}{2} + \sum_{k=0}^{j-1} \frac{v_k}{2} + b\right) \times Beta^{II}\left(\frac{v_{j+m}}{2}, \frac{a}{2} + \sum_{k=0}^{j+m-1} \frac{v_k}{2} + b\right) \qquad (12)$$

$$= \frac{\lambda^{\frac{a}{2}}\Gamma\left(\frac{a}{2}+\sum_{k=0}^{j}\frac{v_k}{2}\right)\Gamma\left(\frac{a}{2}+\sum_{k=0}^{j+m}\frac{v_k}{2}\right)}{\Gamma(\frac{v_j}{2})\Gamma(\frac{v_{j+m}}{2})\Gamma\left(\frac{a}{2}+\sum_{k=0}^{j-1}\frac{v_k}{2}\right)\Gamma\left(\frac{a}{2}+\sum_{k=0}^{j+m-1}\frac{v_k}{2}\right)} u_j^{\frac{v_j}{2}-1}(1+u_j)^{-\left(\frac{a}{2}+\sum_{k=0}^{j}\frac{v_k}{2}\right)} u_{j+m}^{\frac{v_{j+m}}{2}-1}\times$$

$$(1+u_{j+m})^{-\left(\frac{a}{2}+\sum_{k=0}^{j+m}\frac{v_k}{2}\right)} {}_3F_2\left(\frac{a}{2}+\sum_{k=0}^{j+m}\frac{v_k}{2},\frac{a}{2}+\sum_{k=0}^{j}\frac{v_k}{2},\frac{a}{2};\frac{a}{2}+\sum_{k=0}^{j+m-1}\frac{v_k}{2},\frac{a}{2}+\sum_{k=0}^{j-1}\frac{v_k}{2};\frac{1-\lambda}{(1+u_j)(1+u_{j+m})}\right) \tag{13}$$

where $u_j, u_{j+m} > 0$, $j = 0, 1, 2, \dots, p$, $m = 1, 2, 3, \dots$ and ${}_3F_2(\cdot)$ denotes the hypergeometric function.

Proof Equation (12) follows from integrating the appropriate variables from the joint distribution in the form given in (9). Expression (13) in terms of the hypergeometric function, ${}_3F_2(\cdot)$, follows from expanding the product of the beta type II densities, rearranging the terms and simplifying

$$f(u_j, u_{j+m})$$

$$= \frac{\lambda^{\frac{a}{2}}}{\Gamma(\frac{v_j}{2})\Gamma(\frac{v_{j+m}}{2})} u_j^{\frac{v_j}{2}-1}(1+u_j)^{-\left(\frac{a}{2}+\sum_{k=0}^{j}\frac{v_k}{2}\right)} u_{j+m}^{\frac{v_{j+m}}{2}-1}(1+u_{j+m})^{-\left(\frac{a}{2}+\sum_{k=0}^{j+m}\frac{v_k}{2}\right)}\times$$

$$\sum_{b=0}^{\infty}\frac{\Gamma\left(\frac{a}{2}+\sum_{k=0}^{j+m}\frac{v_k}{2}+b\right)\Gamma\left(\frac{a}{2}+\sum_{k=0}^{j}\frac{v_k}{2}+b\right)\Gamma\left(\frac{a}{2}+b\right)}{\Gamma\left(\frac{a}{2}+\sum_{k=0}^{j+m-1}\frac{v_k}{2}+b\right)\Gamma\left(\frac{a}{2}+\sum_{k=0}^{j-1}\frac{v_k}{2}+b\right)\Gamma\left(\frac{a}{2}\right)b!}\left(\frac{1-\lambda}{(1+u_j)(1+u_{j+m})}\right)^b$$

$$= \frac{\lambda^{\frac{a}{2}}\Gamma\left(\frac{a}{2}+\sum_{k=0}^{j}\frac{v_k}{2}\right)\Gamma\left(\frac{a}{2}+\sum_{k=0}^{j+m}\frac{v_k}{2}\right)}{\Gamma(\frac{v_j}{2})\Gamma(\frac{v_{j+m}}{2})\Gamma\left(\frac{a}{2}+\sum_{k=0}^{j-1}\frac{v_k}{2}\right)\Gamma\left(\frac{a}{2}+\sum_{k=0}^{j+m-1}\frac{v_k}{2}\right)} u_j^{\frac{v_j}{2}-1}(1+u_j)^{-\left(\frac{a}{2}+\sum_{k=0}^{j}\frac{v_k}{2}\right)} u_{j+m}^{\frac{v_{j+m}}{2}-1}(1+u_{j+m})^{-\left(\frac{a}{2}+\sum_{k=0}^{j+m}\frac{v_k}{2}\right)}\times$$

$$\sum_{b=0}^{\infty}\frac{\left(\frac{a}{2}+\sum_{k=0}^{j+m}\frac{v_k}{2}\right)_b\left(\frac{a}{2}+\sum_{k=0}^{j}\frac{v_k}{2}\right)_b\left(\frac{a}{2}\right)_b}{\left(\frac{a}{2}+\sum_{k=0}^{j+m-1}\frac{v_k}{2}\right)_b\left(\frac{a}{2}+\sum_{k=0}^{j-1}\frac{v_k}{2}\right)_b b!}\left(\frac{1-\lambda}{(1+u_j)(1+u_{j+m})}\right)^b \qquad \square$$

Remarks

In the following two remarks the cases $m = 1$ and $j = 0$ in expression (13) is considered respectively, since the bivariate density will be illustrated for these cases in Section 4.

(i) If $m = 1$,

$$f(u_j, u_{j+1}) = \frac{\lambda^{\frac{a}{2}}\Gamma\left(\frac{a}{2}+\sum_{k=0}^{j}\frac{v_k}{2}\right)\Gamma\left(\frac{a}{2}+\sum_{k=0}^{j+1}\frac{v_k}{2}\right)}{\Gamma(\frac{v_j}{2})\Gamma(\frac{v_{j+1}}{2})\Gamma\left(\frac{a}{2}+\sum_{k=0}^{j-1}\frac{v_k}{2}\right)\Gamma\left(\frac{a}{2}+\sum_{k=0}^{j}\frac{v_k}{2}\right)} u_j^{\frac{v_j}{2}-1}(1+u_j)^{-\left(\frac{a}{2}+\sum_{k=0}^{j}\frac{v_k}{2}\right)} u_{j+1}^{\frac{v_{j+1}}{2}-1}\times$$

$$(1+u_{j+1})^{-\left(\frac{a}{2}+\sum_{k=0}^{j+1}\frac{v_k}{2}\right)} {}_2F_1\left(\frac{a}{2}+\sum_{k=0}^{j+1}\frac{v_k}{2},\frac{a}{2};\frac{a}{2}+\sum_{k=0}^{j-1}\frac{v_k}{2};\frac{1-\lambda}{(1+u_j)(1+u_{j+1})}\right)$$

(ii) If $j = 0$, $$f(u_0, u_m) = \frac{\lambda^{\frac{a}{2}}\Gamma(\frac{a}{2}+\frac{v_0}{2})\Gamma\left(\frac{a}{2}+\sum_{k=0}^{m}\frac{v_k}{2}\right)}{\Gamma(\frac{v_0}{2})\Gamma(\frac{v_m}{2})\Gamma(\frac{a}{2})\Gamma\left(\frac{a}{2}+\sum_{k=0}^{m-1}\frac{v_k}{2}\right)} u_0^{\frac{v_0}{2}-1}(1+u_0)^{-\left(\frac{a}{2}+\frac{v_0}{2}\right)} u_m^{\frac{v_m}{2}-1}\times$$

$$(1+u_m)^{-\left(\frac{a}{2}+\sum_{k=0}^{m}\frac{v_k}{2}\right)} {}_2F_1\left(\frac{a}{2}+\sum_{k=0}^{m}\frac{v_k}{2},\frac{a}{2}+\frac{v_0}{2};\frac{a}{2}+\sum_{k=0}^{m-1}\frac{v_k}{2};\frac{1-\lambda}{(1+u_j)(1+u_m)}\right)$$

Take note in this case $\sum_{k=0}^{j-1}\frac{v_k}{2} = 0$ for $j < 1$.

2.2.3 Distribution of a subset

Theorem 4 *Let X, W_j with $j = 0, 1, 2, \ldots, p$ be independent chi-squared random variables with degrees of freedom a and v_j with $j = 0, 1, 2, \ldots, p$ respectively. Let $U_0 = \frac{\lambda W_0}{X}$ and $U_j = \frac{\lambda W_j}{X + \lambda \sum_{k=0}^{j-1} W_k}$ where $j = 1, 2, \ldots, p$ and $\lambda > 0$. If the joint density of $U_0, U_1, \ldots, U_p$ is given by (7), then the joint density of the subset $U_r, U_{r+1}, \ldots, U_p$ where $r = 0, 1, \ldots, p$ is given by*

$$f(u_r, u_{r+1}, \ldots, u_p) = \lambda^{\frac{a}{2}} \sum_{b=0}^{\infty} \frac{\left(\frac{a}{2}\right)_b}{b!} (1-\lambda)^b \prod_{j=r}^{p} Beta^{II}\left(\frac{v_j}{2}, \frac{a}{2} + \sum_{k=0}^{j-1} \frac{v_k}{2} + b\right), \; u_j > 0, \; j = r, \ldots, p \quad (14)$$

$$= \frac{\lambda^{\frac{a}{2}} \Gamma\left(\frac{a}{2} + \sum_{k=0}^{p} \frac{v_k}{2}\right)}{\Gamma\left(\frac{a}{2} + \sum_{k=0}^{r-1} \frac{v_k}{2}\right) \prod_{k=r}^{p} \Gamma\left(\frac{v_k}{2}\right)} \left(\prod_{j=r}^{p} u_j^{\frac{v_j}{2}-1} (1+u_j)^{-\left(\frac{a}{2} + \sum_{k=0}^{j} \frac{v_k}{2}\right)}\right) \times \quad (15)$$

$${}_2F_1\left(\frac{a}{2} + \sum_{k=0}^{p} \frac{v_k}{2}, \frac{a}{2}; \frac{a}{2} + \sum_{k=0}^{r-1} \frac{v_k}{2}; \frac{1-\lambda}{\prod_{j=r}^{p}(1+u_j)}\right), \; u_j > 0, \; j = r, \ldots, p$$

Proof This proof is similar to theorem 3. □

3. Moments

Theorem 5 provides a derivation of the joint moments of $U_0, U_1, \ldots, U_p$. The product moments for the bivariate case, $E\left(u_j^r u_{j+m}^s\right)$ and a subset, $E(u_r^{h_r} u_{r+1}^{h_{r+1}} \ldots u_p^{h_p})$ are given in Theorem 6 and 7, respectively. The moments for the bivariate case will be used in Section 4 to investigate the correlation structure.

Theorem 5 *The joint moments of $U_0, U_1, \ldots, U_p$, where $(U_0, U_1, \ldots, U_p)$ has joint density* (8), *is given by*

$$E\left(u_0^{h_0} u_1^{h_1} \ldots u_p^{h_p}\right) = \frac{\lambda^{\frac{a}{2}} \Gamma\left(\frac{a}{2} + \sum_{k=0}^{p} \frac{v_k}{2}\right)}{\Gamma\left(\frac{a}{2}\right) \prod_{k=0}^{p} \Gamma\left(\frac{v_k}{2}\right)} \sum_{b=0}^{\infty} \frac{\left(\frac{a}{2} + \sum_{k=0}^{p} \frac{v_k}{2}\right)_b}{b!} (1-\lambda)^b \prod_{j=0}^{p} \frac{\Gamma\left(\frac{v_j}{2} + h_j\right) \Gamma\left(\frac{a}{2} + \sum_{k=0}^{j-1} \frac{v_k}{2} + b - h_j\right)}{\Gamma\left(\frac{a}{2} + \sum_{k=0}^{j} \frac{v_k}{2} + b\right)} \quad (16)$$

Take note $\sum_{k=0}^{j-1} \frac{v_k}{2} = 0$ if $j < 1$.

Proof By expanding the hypergeometric function in series form in expression (8) it follows that

$$E\left(u_0^{h_0} u_1^{h_1} \ldots u_p^{h_p}\right) = \frac{\lambda^{\frac{a}{2}} \Gamma\left(\frac{a}{2} + \sum_{k=0}^{p} \frac{v_k}{2}\right)}{\Gamma\left(\frac{a}{2}\right) \prod_{k=0}^{p} \Gamma\left(\frac{v_k}{2}\right)} \sum_{b=0}^{\infty} \frac{\left(\frac{a}{2} + \sum_{k=0}^{p} \frac{v_k}{2}\right)_b}{b!} (1-\lambda)^b \prod_{j=0}^{p} \left(\int_0^{\infty} u_j^{\frac{v_j}{2} + h_j - 1} (1+u_j)^{-\left(\frac{a}{2} + \sum_{k=0}^{j} \frac{v_k}{2} + b\right)} du_j\right)$$

Evaluation of the above integrals using the definition of the beta type II integral function (see Prudnikov 1986, Eq 2.2.4(24), p298), yields the desired expression (16). □

Theorem 6 *The product moment of U_j, U_{j+m}, where (U_j, U_{j+m}) has bivariate density* (13), *is given by*

$$E\left(u_j^r u_{j+m}^s\right) = \frac{\lambda^{\frac{a}{2}}\Gamma(\frac{v_j}{2}+r)\Gamma(\frac{v_{j+m}}{2}+s)\Gamma\left(\frac{a}{2}+\sum_{k=0}^{j-1}\frac{v_k}{2}-r\right)\Gamma\left(\frac{a}{2}+\sum_{k=0}^{j+m-1}\frac{v_k}{2}-s\right)}{\Gamma(\frac{v_j}{2})\Gamma\left(\frac{a}{2}+\sum_{k=0}^{j-1}\frac{v_k}{2}\right)\Gamma(\frac{v_{j+m}}{2})\Gamma\left(\frac{a}{2}+\sum_{k=0}^{j+m-1}\frac{v_k}{2}\right)} \quad (17)$$
$${}_3F_2\left(\frac{a}{2}+\sum_{k=0}^{j-1}\frac{v_k}{2}-r, \frac{a}{2}+\sum_{k=0}^{j+m-1}\frac{v_k}{2}-s, \frac{a}{2}; \frac{a}{2}+\sum_{k=0}^{j+m-1}\frac{v_k}{2}, \frac{a}{2}+\sum_{k=0}^{j-1}\frac{v_k}{2}; 1-\lambda\right)$$

where $j = 0, 1, 2, \ldots$ and $m = 1, 2, 3, \ldots$

Proof This proof is similar to theorem 5. □

Remarks
In the following two remarks the cases $j = 1$ and $m = 1$ in expression (17) is considered respectively, since the correlation will be plotted for these cases in Section 4.

(i) If $j = 0$,

$$E\left(u_0^r u_m^s\right) = \frac{\lambda^{\frac{a}{2}}\Gamma(\frac{v_0}{2}+r)\Gamma(\frac{v_m}{2}+s)\Gamma(\frac{a}{2}-r)\Gamma\left(\frac{a}{2}+\sum_{k=0}^{m-1}\frac{v_k}{2}-s\right)}{\Gamma(\frac{a}{2})\Gamma(\frac{v_0}{2})\Gamma(\frac{v_m}{2})\Gamma\left(\frac{a}{2}+\sum_{k=0}^{m-1}\frac{v_k}{2}\right)}\,{}_2F_1\left(\frac{a}{2}-r, \frac{a}{2}+\sum_{k=0}^{m-1}\frac{v_k}{2}-s; \frac{a}{2}+\sum_{k=0}^{m-1}\frac{v_k}{2}; 1-\lambda\right)$$

Take note that $\sum_{k=0}^{j-1}\frac{v_k}{2} = 0$ if $j < 1$.

(ii) If $m = 1$,

$$E\left(u_j^r u_{j+1}^s\right) = \frac{\lambda^{\frac{a}{2}}\Gamma(\frac{v_{j+1}}{2}+s)\Gamma(\frac{v_j}{2}+r)\Gamma\left(\frac{a}{2}+\sum_{k=0}^{j}\frac{v_k}{2}-s\right)\Gamma\left(\frac{a}{2}+\sum_{k=0}^{j-1}\frac{v_k}{2}-r\right)}{\Gamma(\frac{v_j}{2})\Gamma\left(\frac{a}{2}+\sum_{k=0}^{j-1}\frac{v_k}{2}\right)\Gamma(\frac{v_{j+1}}{2})\Gamma\left(\frac{a}{2}+\sum_{k=0}^{j}\frac{v_k}{2}\right)}\times$$
$${}_3F_2\left(\frac{a}{2}+\sum_{k=0}^{j}\frac{v_k}{2}-s, \frac{a}{2}, \frac{a}{2}+\sum_{k=0}^{j-1}\frac{v_k}{2}-r; \frac{a}{2}+\sum_{k=0}^{j-1}\frac{v_k}{2}, \frac{a}{2}+\sum_{k=0}^{j}\frac{v_k}{2}; 1-\lambda\right)$$

Theorem 7 *The joint moments of a subset $(U_r, U_{r+1}, \ldots, U_p)$ where $r = 0, 1, \ldots, p$ with joint density* (15), *is given by*

$$E\left(u_r^{h_r}\ldots u_p^{h_p}\right)$$
$$= \frac{\lambda^{\frac{a}{2}}\Gamma\left(\frac{a}{2}+\sum_{k=0}^{p}\frac{v_k}{2}\right)}{\Gamma\left(\frac{a}{2}+\sum_{k=0}^{r-1}\frac{v_k}{2}\right)\prod_{k=r}^{p}\Gamma\left(\frac{v_k}{2}\right)}\sum_{b=0}^{\infty}\frac{\left(\frac{a}{2}+\sum_{k=0}^{p}\frac{v_k}{2}\right)_b\left(\frac{a}{2}\right)_b}{\left(\frac{a}{2}+\sum_{k=0}^{r-1}\frac{v_k}{2}\right)_b b!}(1-\lambda)^b\prod_{j=r}^{p}\frac{\Gamma\left(\frac{v_j}{2}+h_j\right)\Gamma\left(\frac{a}{2}+\sum_{k=0}^{j-1}\frac{v_k}{2}+b-h_j\right)}{\Gamma\left(\frac{a}{2}+\sum_{k=0}^{j}\frac{v_k}{2}+b\right)}$$

Proof This proof is similar to theorem 5. □

4. Shape analysis and computations

In this section the shape of the univariate and bivariate marginal densities of the generalized multivariate beta distribution will be illustrated for different values of the parameters λ, a and v_j. The effect of the different parameters on the correlation between U_j and U_{j+m} will also be investigated. In the SPC application discussed in the introduction $v_j = 2n$, therefore it depends on the sample size at each point in time. In this section it will be assumed that the sample sizes at each point in time are equal, in effect $v_j = v$.

Figure 1 illustrates the effect of the parameters λ, a and v on the univariate marginal density (see (10,11)). Panels (i) to (iii) focus on the random variable U_0. In panel (iv), the influence of j is investigated, where j represents the position of the random variable in the process. For larger values of j the density moves towards the vertical axis. In all four panels, the solid black line ($\lambda = 1.5$, $a = 20$ and $v = 10$) is the same and is used as a reference. In panel (i) the role of λ is investigated. Take note that when $\lambda = 1$, the density simplifies to that of a beta type II density. Panel (ii) shows that for larger values of $a = 2n(\kappa - 1) = v(\kappa - 1)$ (meaning the shift took place after a long time) the plot moves towards the vertical axis. Take note that a depends on the sample size at each point in time as well as κ, the sample from which the process parameter has changed. Panel (iii) examines the effect of the v. Note that for the special case when individual samples are considered (i.e when $n = 1$ so that $v = 2$), the shape is different.

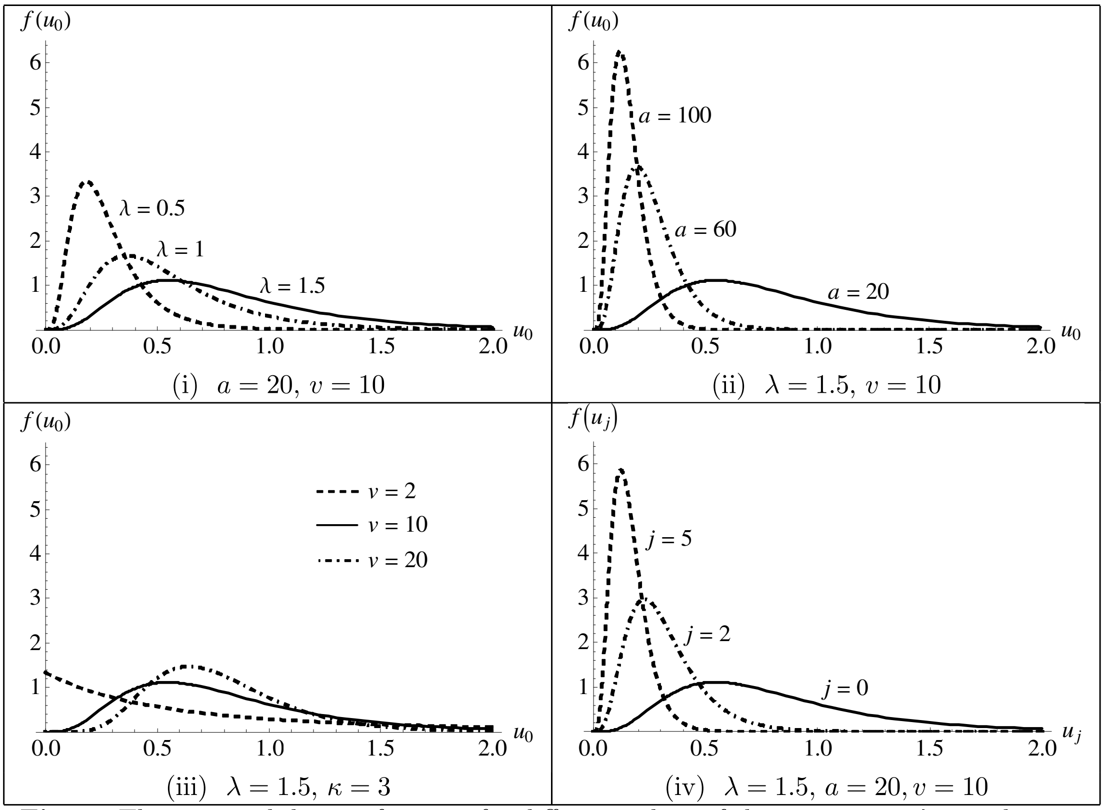

Fig. 1 The marginal density function for different values of the parameters λ, a and v

Figure 2 plots that bivariate density (see (13)) for different values of the parameters λ, a and v. In each panel the reference case ($\lambda = 1.5$, $a = 20$, $v = 10$) will be included for easy comparison. Panels (i) to (iii) consider the two consecutive random variables U_0 and U_1, while panel (iv) illustrates the bivariate density for consecutive random variables further along in the process (for example (U_1, U_2)) and random variables that are not consecutively observed (for example (U_0, U_3)). Panel (i) shows the effect of λ. For $\lambda < 1$ there was a downward shift in the process parameter, while for $\lambda > 1$ an upward shift occurred. The role of a is investigated in panel (ii), where a has to do with when the shift took place. For bigger values of a, the process was longer in control. Note that for $v = 2$ (i.e. individual samples) in panel (iii) the shape is different.

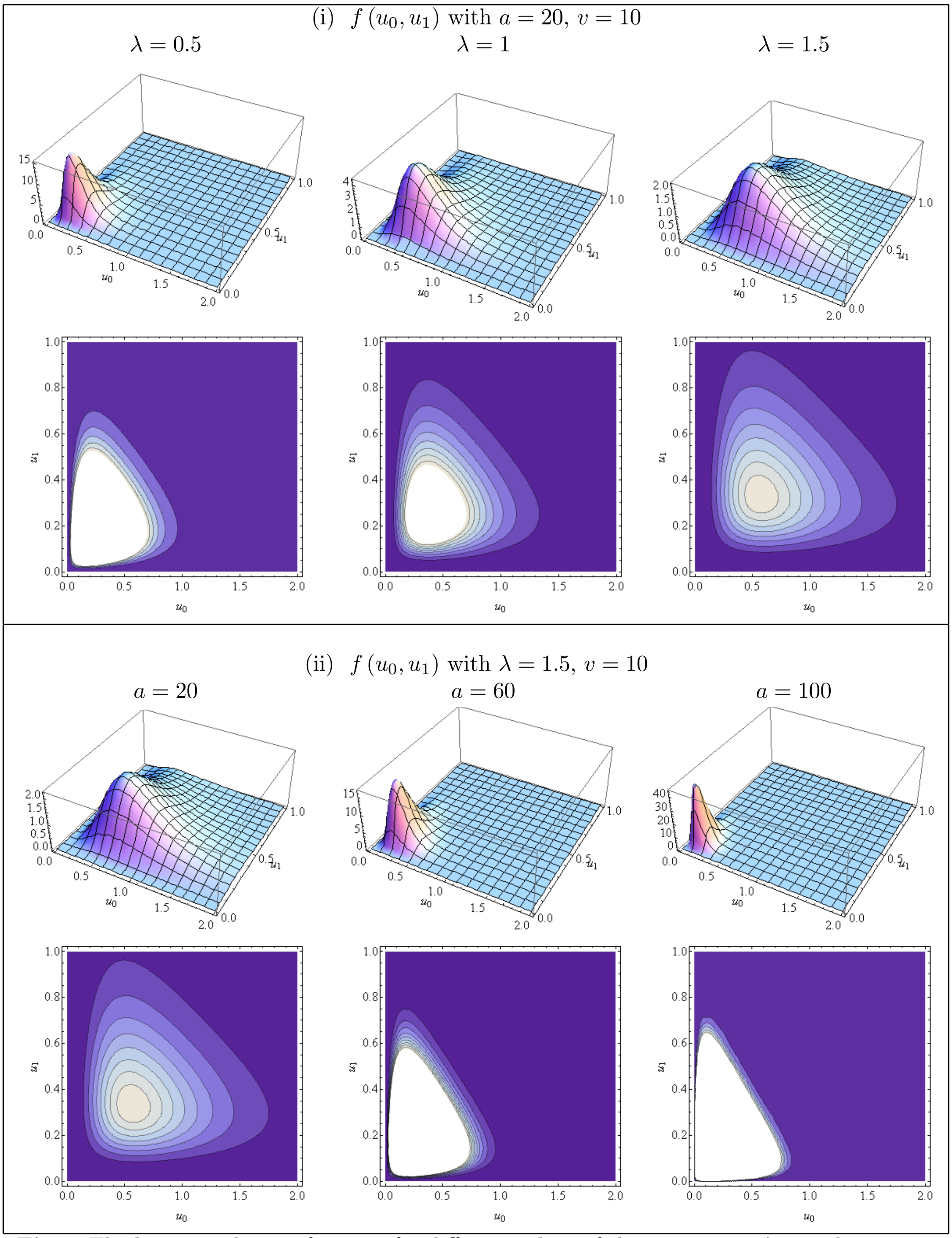

Fig.2 The bivariate density function for different values of the parameters λ, a and v

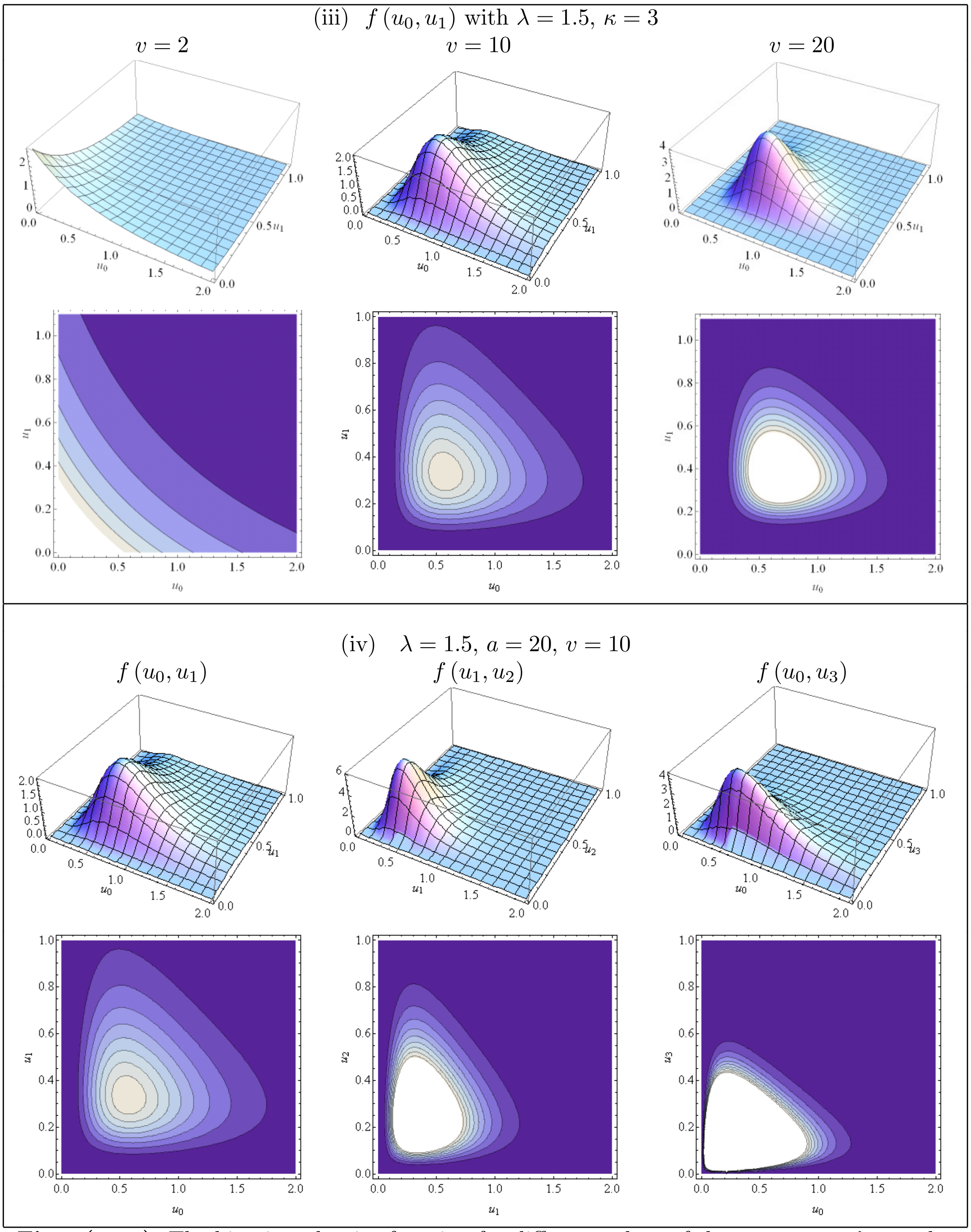

Fig.2 (cont) The bivariate density function for different values of the parameters λ, a and v

Substituting the appropriate values for r and s in (17) it is straightforward to calculate the correlation between U_j and U_{j+m}. Take note that j represents the number of samples after the change in the parameter value and m indicates how far apart the two random variables are. The software package Mathematica was used to compute these correlations. In Figure 3 panel (i) the correlation is plotted as a function of λ for $j = 0, m = 1$, $a = 20$, $v = 10$. The shape will be similar for other values of the parameters j, m, a and v. The sign of the correlation depends on the value of λ, for $\lambda < 1$ (downwards shift in the process parameter) the correlation is positive while for values of $\lambda > 1$ (upwards shift) the correlation is negative. For $\lambda = 1$, the random variables U_j and U_{j+m}, $m > 0$, are uncorrelated. Panels (ii) to (iv) investigate the influence of the other parameters on the correlation for the cases where $\lambda = 0.5$ and $\lambda = 1.5$. Panel (ii) plots the correlation between consecutive observations since $m = 1$. For larger values of j (long time after the change in the parameter took place) the correlation gets very small in absolute terms. Panel (iii) shows that the further apart the two random variables are, the smaller the correlation in absolute terms. Panel (iv) looks at the influence of κ, the sample number when the parameter changed. Only values for $\kappa > 1$ were considered, since it is assumed that the process started in control.

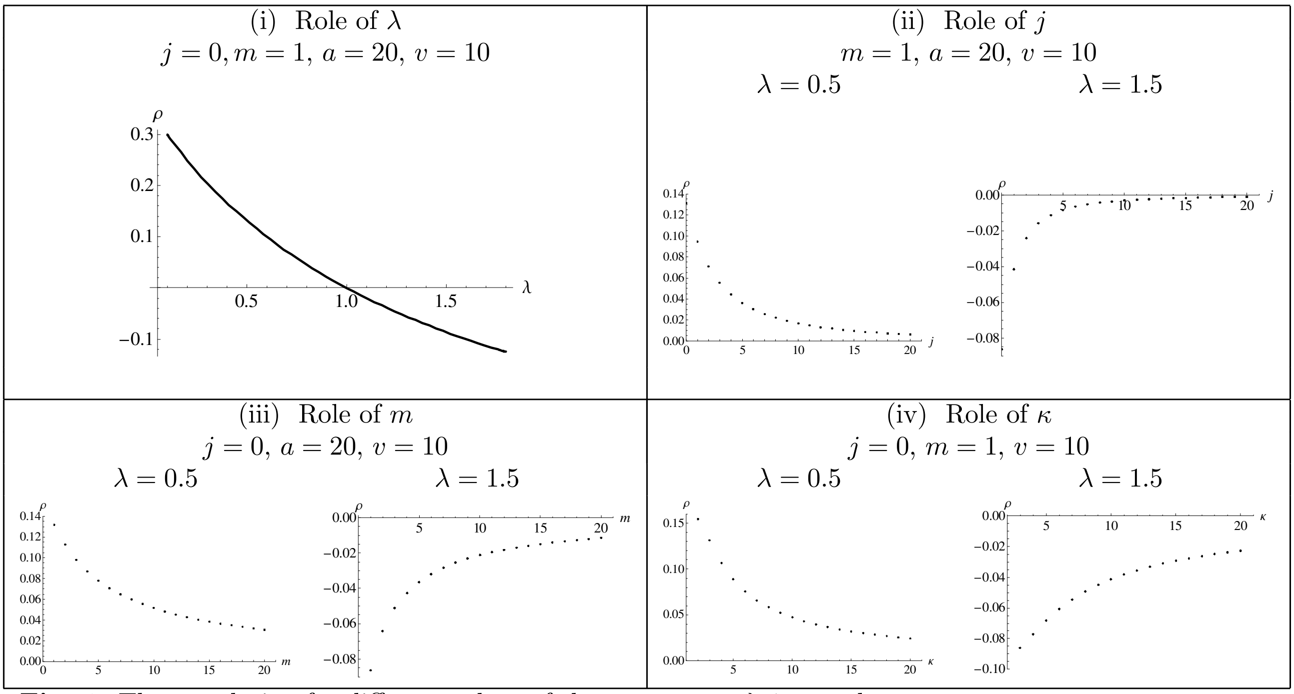

Fig. 3 The correlation for different values of the parameters λ,j, m and κ

5. Conclusion

A new generalized multivariate beta distribution with density in a closed form is proposed, motivated by a Statistical Process Control problem where a distribution is needed for the run-length of a Q-chart that monitors the process average when measurements are from an exponential distribution with unknown parameter. The joint moments of this generalized multivariate beta distribution are derived to shed light on the nature of this distribution, specifically the correlation structure. The computational aspect of the run-length (i.e. evaluating multiple integral expressions as given towards the end of the introduction) will be addressed in a follow-up paper.

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