"A primordial, mathematical, logical and computable, demonstration (proof) of the family of conjectures known as Goldbach´s" [1]

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For our seven dear direct predecessors and sons; Carlos, Adrián, Álvaro, Pepita, Antonio, Rosalía and Vicente

*"Whoever understands **Archimedes** and **Apollonius,*** [2]
will admire less the achievements of later men"
Gottfried Wilhelm** (von) **Leibniz
(**1 July 1646, O.S. 21 June – November 14, 1716**)

Summary

In this document, by means of a novel system model and first order topological, algebraic and geometrical free-context formal language (NT-FS&L), first, we describe a new signature for a set of the natural numbers that is rooted in an intensional inductive de-embedding process of both, the tensorial identities of the known as "natural numbers", and the abstract framework of theirs locus-positional based symbolic representations. Additionally, we describe that NT-FS&L is able to: i.- Embed the De Morgan´s Laws and the FOL-Peano´s Arithmetic Axiomatic. ii.- Provide new points of view and perspectives about the succession, precede and addition operations and of their abstract, topological, algebraic, analytic geometrical, computational and cognitive, formal representations. Second, by means of the inductive apparatus of NT-FS&L, we proof that the family of conjectures known as Glodbach's holds entailment and truth when the reasoning starts from the consistent and finitary axiomatic system herein described.

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1. THE FAMILY OF CONJECTURES KNOWN AS "GOLDBACH´S"

On 7th June 1742, Christian Goldbach wrote of a letter to Leonard Euler in Latin-German (*letter XLIII*)[3] in which he proposes the following conjecture about "integers", which today are recognized by the names "positive integers" and "natural numbers":[4]

"Every integer which can be written as the sum of two primes, can also be written as the sum of as many primes as one wishes, until all terms are units"

(*Translated to contemporary English*)

(GC-1)

Additionally, he proposed a second conjecture, which appears in the margin of *letter XLIII*:

"Every integer greater than 2 can be written as the sum of three primes"

(*Translated to contemporary English*)

(GC-II)

The two above conjectures are nowadays considered logically correlated and mathematically connected, but this did not seem to be an issue nowadays.[5] A later corollary of Goldbach´s second conjecture is:

"Every integer greater than 5 can be written as the sum of three primes"

(*Translated to contemporary English*)

(GC-III)

It is documented that Euler replied, in a letter dated 30 June 1742, and reminded C. Goldbach of an earlier conversation they had, in which Goldbach remarked that his original conjecture followed from the statement below:

"Every even integer greater than 2 can be written as the sum of two primes"

(*Translated to contemporary English*)

(GC-IV)

This statement, for centuries named and referred to as Goldbach´s Conjecture, is also known as ***the "strong", "even", or "binary" Goldbach conjecture, in order to distinguish it from other statements, the "weak corollaries"****, such the following:*

"All odd numbers greater than 7 are the sum of three odd primes"

(*Translated to contemporary English*)

(GC-V)

On the other hand, the statement **GC-III** *is today known as the "***weak Goldbach conjecture***", the "odd" Goldbach´s conjecture, or the "ternary" Goldbach conjecture.*

While the "weak Goldbach´s conjecture", a corollary of the strong conjecture, which is nowadays understood as a corollary of the **GC-IV** statement, appears to have been finally proved in 2013,[6] ***to the date, at the beginning of the twenty-first century it is globally accepted that:***

1. ***The "strong Goldbach´s conjecture"* (GC-IV) *remains unsolved.*** [7]

2. ***The "complete family of Conjectures known as Goldbach´s" (set conformed by statements CG I-V) has remained unsolved as a "set of theorems, at least, for more than twenty-three centuries.*** [8, 9]

Hence, let us highlight that:

1. after 23 centuries from the first systematic definition of a plethora of numerical, calculus, algebraic, analytic, topological and geometrical concepts as unity, part, even number, odd number and prime number, even parity, odd parity, even parity number, odd parity number, composed number, measuring, prime number, magnitude, point, part, figure, equality, proportion, symmetry, etc.,

2. after 275 years from the enunciation a collectivity of statements, herein named as "the family of mathematical conjectures known as Goldbch´s (statements CG 1-4)", about both; first, elemental numerical concepts and basic properties (even number, odd number and prime number) and, second, the elemental and primordial operation of addition (also, globally called and nicknamed as "the sum of positive integers numbers" and "the sum of natural numbers"), and, three, their networking correlations, combinations;

the collective human effort known as Science & Technology,[10] ***lacks a finite set of statements that can simultaneously structure*** *mathematical, linguistic,* [11] *algebraic,*[12] *analytical,*[13] *computational,*[14] *topological and geometrical (the two last also analytical or algebraic)* [15] *human cognitive concepts with their conventional and globally accepted symbolic representations* ***at any and every of the possible individual and collective levels of diversity and complexity.***

As a direct consequence, ***a plethora of "long-term unsolved problems"*** [16] *closely related with the logical-mathematical-computational and cognitive human structure of even, odd, prime and composite numerical concepts,* ***remain as the driving force of the collective*** *philosophical illusion and passion to reach progress and human welfare by means of the attempts to understand, by means of the scientific and technological methodologies,* ***our own living nature and both, our cognitive and our epistemological environments; the so-named "virtual" (also conceptual) and the so-named "real" (also physical).***

2. ON SYMBOLIC REPRESENTATIONS, NOMENCLATURE AND FORMULATION OF THE NATURAL NUMBER (ELEMENTS) CONCEPT AND THEIR COLLECTIVITIES (SETS AND SUBSETS), AND THEIR ARITHMETIC

In the beginning of 2017 (Gregorian calendar), after more than a century and a quarter of century since the primordial, colossal, monumental, historic, paradigmatic and fundamental documents "Arithmetices principia, nova method" and "Sur une courbe, qui remplit toute une aire plane", (Giuseppe Peano, 1889, in Latin and French, respectively), [17] *both; i.- every scientific-technological formal language* [18] *(Number Theory, included),*[19] *and ii.- every one of their logical, topological, algebraic, analytic geometrical, symbolic and semiotic, representations and interpretations,*[20] *are "ultimately referred in sense and intension"* [21] *to the human numerical concepts named for centuries as "integers" (more recently, math-nicknamed as "positive integers"; hereinafter, "natural numbers").*

Nowadays, the collectivity conformed by "each and every each natural number" human mental concept as element (also, member) is conventionally recognized as ***"the set of the natural numbers" concept*** *(herein after, the set of natural number concepts will be* ***represented*** *by the globally accepted symbol $\mathcal{N}$).* [22]

Two equivalent logical-mathematical-computational-epistemological axiomatic formulations[23] *referred to both, the natural numbers set $\mathcal{N}$ and their elements, rooted in Peano´s axiomatic description of $\mathcal{N}$ (briefly $\mathcal{N}$* **(P)***) remain valid as the starting point for any systematic reasoning about the concept of natural number:*

2.0.0. **If "0" is considered as a natural number concept; then**, the current global accepted representation of every and each of the numerical natural number rooted in Peano´s concept of "***successor of a natural number***" is conformed by the following set of ***five statements that will sustain and/or hold "entailment " and "truth"*** (herein after, re-named as "five axioms" and/or "**2.0.1, 2.0.2 2.03, 2.0.4, 2.0.5,** respectively"):

2.0.1.- **0** is a number natural.

2.0.2.- If **n** is a natural number, then ***a successor of n*** is also a natural number.

2.0.3.- **0** is not ***a successor of any natural number***.

2.0.4.- If there are two natural numbers **n** and **m** with ***the same successors***, then **n** and **m** are the same natural number.

2.0.5. - The statement named "***principle of mathematical induction*** [24] generation of the natural number concepts and of their ***"locus-based symbolic representations"***: If **0** belongs to the collectivity of the set of natural numbers, and given any natural number, ***the successors of that number also belongs to this collectivity***, then all natural numbers belong to that set $\mathcal{N}$ **(P)**.

Hereinafter, the above formulation of the set of the natural numbers will be represented and referred to by the following formula: [25]

$$\mathcal{N}_{[0 \subset PA]}$$

The "natural number "0" concept" identity is "intentionally de-embedded (also, decoupled, descaled the subset of five axioms ***2.0.1-2.0.5****) as the reference of the* *"identity" of every and each "successor of a natural number concept" in the "P´s axiomatic" of set $\mathcal{N}$, which is conformed by comprehension and extension" by "**the five**" statements enumerated above –and/or named; also, nicknamed- from 2.0.1 to 2.0.5 of the set of the customary natural numbers "$\mathcal{N}$ " set, and symbolically represented by* $\mathcal{N}$ **(P)**.

2.1.0 If <u>"0" is not considered as a natural number concept; then,</u> the current global accepted representation of every and each of the numerical natural number rooted in Peano´s concept of "***successor of a natural number***" is conformed by the following set of ***five statements that will sustain and/or hold "entailment " and "truth"*** (herein after, re-named as "five axioms" and/or "**2.1.1, 2.1.2 2.1.3, 2.1.4, 2.1.5,** respectively"):

> **2.1.1.- 1** is a natural number. **1** belongs to ***N***, the set of natural numbers.
>
> **2.1.2.-** Every natural number **n** has at least ***"a successor of n"***.
>
> **2.1.3.- 1** is not "***successor of any natural number***".
>
> **2.1.4.-** If there are two natural numbers **n** and **m** with ***the same successors***, then **n** and **m** are the same natural number;
>
> **2.1.5.-** The "**principle of mathematical induction"** generation of the natural number concepts and of their "***locus-based symbolic representations***": If the **1** belongs to a set ***N*** of natural numbers, and given one element either **n**, ***the successors of n*** also belongs to the set **N**, then all natural numbers belong to the set **N** and number cero concept is not considered a natural number concept belonging to the set of the natural numbers $\mathcal{N}$ **(P).**

Hereinafter, this second formulation will be represented by the following formula:

$$\mathcal{N}_{[0\not\subset PA]}$$

*The "natural number "1" concept"-identity is "intentionally de-embedded (also, decoupled descaled; subset of axioms 2.1.1-2.1.5) as the reference of the "identity" of every and each "successor of a natural number concept" in the P´s axiomatic of set $\mathcal{N}$, which conformed "by comprehension and extension" by "the five" statements enumerated -and named; also, nicknamed- from **2.1.1** to **2.1.5** of the set of the customary natural numbers "$\mathcal{N}$" set, and symbolically represented by* $\mathcal{N}$ **(P)**.

In both collectivities (sets) conformed by natural numbers, $\mathcal{N}_{[0\not\subset PA]}$ *and* $\mathcal{N}_{[0\subset PA]}$, ***natural number 1** and **its successors** sustain their identities by ensuring an identical (also equal) symbolic representation for every natural number concept that, in turn, in* $\mathcal{N}_{[0\subset PA]}$, ***natural number 0** and **its successors** sustain, ensures and shares with the natural number **1** and **theirs successors** in* $\mathcal{N}_{[0\subset PA]}$**.**

Thus, we are able to state about the natural numbers zero (**0**) and one (**1**) the following:

$$\Big(\big((0 \in \mathcal{N}[0{\subset}PA] \wedge 0 \notin \mathcal{N}[0{\not\subset}PA]\big) \wedge (1 \in \mathcal{N}[0{\subset}PA] \wedge \mathcal{N}[0{\not\subset}PA])$$
$$\vee$$
$$(0 \notin \mathcal{N}[0{\subset}PA] \cap \mathcal{N}[0{\not\subset}PA]) \wedge (1 \in \mathcal{N}[0{\subset}PA] \cap \mathcal{N}[0{\not\subset}PA])\big)$$
$$\rightarrow$$
$$(\mathcal{N}[0{\not\subset}PA] \subset \mathcal{N}[0{\subset}PA])\Big)$$

Then, taking in account De Morgan´s Laws: [26, 27]

$$\mathcal{N}[0{\subset}PA] \cup \mathcal{N}[0{\not\subset}PA] = \overline{\mathcal{N}[0{\subset}PA]} \cap \overline{\mathcal{N}[0{\not\subset}PA]}$$

$$\mathcal{N}[0{\subset}PA] \cap \mathcal{N}[0{\not\subset}PA] = \overline{\mathcal{N}[0{\subset}PA]} \cup \overline{\mathcal{N}[0{\not\subset}PA]}$$

We are able to define both, the union and the intersection of both $\mathcal{N}[0{\subset}PA]$ *and* $\mathcal{N}[0{\not\subset}PA]$:

$$\mathcal{N}[0{\subset}PA] \cup \mathcal{N}[0{\not\subset}PA] := \{\, n \mid 0 \in \mathcal{N}[0{\subset}PA] \,\}_{\forall n \in \mathcal{N}[0{\subset}PA]} \wedge \{0\}_{0 \in \mathcal{N}[0{\subset}PA]}$$
$$\mathcal{N}[0{\subset}PA] \cap \mathcal{N}[0{\not\subset}PA] := \{\, n \mid 0 \notin \mathcal{N}[0{\not\subset}PA] \,\}_{\forall n \in \mathcal{N}[0{\not\subset}PA]} \wedge \{1\}_{0 \notin \mathcal{N}[0{\subset}PA]}$$

Hence,

$$\mathcal{N}[0{\not\subset}PA] \subset \mathcal{N}[0{\subset}PA] \rightarrow \{0\}_{0 \in \mathcal{N}[0{\subset}PA]} \cap \{1\}_{1 \in \mathcal{N}[0{\subset}PA]}$$
$$\&$$
$$\mathcal{N}[0{\subset}PA] = \{0\} \cup \{1\}_{0 \in \mathcal{N}[0{\subset}PA]}$$
$$\&$$
$$\mathcal{N}[0{\not\subset}PA] = \{0\}_{0 \in \mathcal{N}[0{\subset}PA]} \cap \{1\}_{0 \notin \mathcal{N}[0{\subset}PA]}$$

𝒩[0⊂PA]** is univocally identified and well symbolically represented by comprehension as the set of natural numbers defined by extension and conformed by the union of both**, the subset of 𝒩[0⊂PA] which is conformed by the <u>natural number cero concept subset conformed by a unique element (a singleton), and the subset conformed by every natural number with the identity of successor of the natural number when referred to zero (0) natural number concept</u>.

Hence, we are able to establish the next considerations:

*1. **𝒩[0⊂PA]** is defined as the set conformed by the union between the singleton*[28] *of the natural number zero (0) and the subset of the natural numbers identified and symbolically represented as successors of natural number zero **(0)**, which is, in turn, the set **𝒩[0⊄PA].***[29]

2. *$\mathcal{N}$[0⊄PA]* *is univocally identified and symbolically represented and named as the subset of natural numbers* *$\mathcal{N}$[0⊂PA]* *defined by comprehension and conformed by* ***every natural number sustaining the identity of successor of the natural number zero (0) concept.***

3. *$\mathcal{N}$[0⊄PA]* *is defined, in turn, as the set conformed by the union between the singleton of the natural number one (**1**) and the subset of the natural numbers* *$\mathcal{N}$[0⊂PA]* *identified, defined by comprehension and symbolically represented as successors of natural number one (**1**), which is named onwards the set* *$\mathcal{N}$[0⊄PA].*

4. *By means of the principle of mathematical induction of the both above described formulations,* *$\mathcal{N}$[0⊂PA]* and *$\mathcal{N}$[0⊄PA], please, let us generalize above definition:*

$\mathcal{N}$[0⊂PA] *is defined as the set conformed by the union between both; the subset defined by extension and conformed by natural numbers* ***0, 1, …,*** *and* ***"n"****, and the subset of the natural numbers identified, defined by comprehension and well symbolically represented as the* **successors of natural number** *"**n**" (**n**) by formula* *$\mathcal{N}$[0, 1, …, n⊄PA]. Thus,*

$$\textit{subset-}\mathcal{N}[0\subset PA] := \{0\}\forall n \in \mathcal{N}[0\subset PA] \,;$$

$$\textit{subset-}\mathcal{N}[0, 1\subset PA] := \{0, 1\}\forall n \in \mathcal{N}[0\subset PA] \,;$$

……….

$$\textit{subset-}\mathcal{N}[0, 1, \ldots, n \subset PA] := \{0, 1, \ldots, n\}\forall n \in \mathcal{N}[0\subset PA]$$

Hence, allow us to point out that:

$$\textit{subset-}\{\mathcal{N}[0, 1, \ldots, n \subset PA]\}_{\forall n \in \mathcal{N}[0\subset PA]} \subseteq \mathcal{N}[0\subset PA]$$

&

$$((0, 1, 2, \ldots n \in \mathcal{N}[0\subset PA])_{\forall n \in \mathcal{N}[0\subset PA]} \;\wedge\vee\; (1, 2, \ldots n \in \mathcal{N}[0\subset PA] \cap \mathcal{N}[0\not\subset PA]))$$

On the other hand, *$\mathcal{N}$[0, 1, …, n⊄PA]* *is univocally identified and symbolically represented as the subset of natural numbers* *$\mathcal{N}$[0⊂PA]* *which is defined by comprehension and conformed by* ***every natural number with the identity of successor of the natural number "n" (n) concept.***

Please, allow us to point out that for every subset-$\mathcal{N}$**[0, 1, …, n ⊂ PA],** *the next statement are true (theorems):*

$$\#\text{- } \textit{subset-}\mathcal{N}[0 \subset PA] = 1, \forall n\; (n \in \mathcal{N}[0\subset PA])$$

$$\#\text{- } \textit{subset-}\mathcal{N}[0, 1 \subset PA] = 2, \forall n\; (n \in \mathcal{N}[0\subset PA])$$

………..

$$\#\text{- } \textit{subset-}\mathcal{N}[0, 1, ..., n \subset PA] = n + 1, \forall n\ (n \in \mathcal{N}[0 \subset PA])$$

Hence,

$$\#\text{- } \textit{subset-}\mathcal{N}[0, 1, ..., n \subset PA] := \{\ n+1 \mid n+1 \in \mathcal{N}[0 \not\subset PA]\ \}, \forall n\ (n \in \mathcal{N}[0 \subset PA])$$

$$\wedge \vee$$

$$\#\text{-}\textit{subset-}\mathcal{N}[0,1,..., n \subset PA] := \{n+1 \mid n+1 \in \mathcal{N}[0 \not\subset PA] \cap \mathcal{N}[0 \subset PA]\}, \forall n (n \in \mathcal{N}[0 \subset PA])$$

2.1. NT-FS&L General Considerations about First Order Logic Peano´s Arithmetic. On the FOL-succession operation.

In the following, ***first order-logic of Peano´s Arithmetic axiomatic*** (**FOL-PA;** briefly, also **PA)**[30] will be the starting formal background language used (syntax, grammar and semantic) to the formulations of both; i.- well formed statements (w.f.s.) concerning nomenclature and formulation referred to the natural number concept, their collections and their arithmetic, and, ii.- well formed symbolic representations (w.f.-sr) statements of new concepts correlated with the natural number concept.

In the following entailment will be sustained by both; the next list of six statements set (**2.1.1-2.1.6** axioms) and the induction schema (also named principle of mathematical induction; statement formula **2.1.7**): [31, 32]

2.1.1 $\forall x\ (x \in \mathcal{N}[0 \subset FOL\text{-}PA])(S(x) \neq 0)$

2.1.2 $\forall x \forall y (x, y \in \mathcal{N}[0 \subset FOL\text{-}PA])(S(x) = S(y) \rightarrow x = y)$

2.1.3. $\forall x\ (x \in \mathcal{N}[0 \subset FOL\text{-}PA])(\ x + 0 = x)$,

2.1.4. $\forall x \forall y (x, y \in \mathcal{N}[0 \subset FOL\text{-}PA])(\ x + S(y) = S(x + y))$

2.1.5. $\forall x\ (x \in \mathcal{N}[0 \subset FOL\text{-}PA])(\ x \cdot 0 = 0)$,

2.1.6. $\forall x \forall y (x, y \in \mathcal{N}[0 \subset FOL\text{-}PA])(\ x \cdot S(y) = x \cdot y + x)$

2.1.7. $\forall x \forall y\ (x, y_1 \ldots y_n \in \mathcal{N}[0 \subset FOL\text{-}PA])$ [33]

$$(((\varphi(x|0) \wedge (\varphi \rightarrow \varphi(x|S(x), \forall x)) \rightarrow \varphi\ (x), \forall x), \forall y_1 \ldots \forall y_n$$

PA set of axioms incorporates axioms **2.1.3-2.1.4** and **2.1.5-2.1.6** as formal recursive-inductive references of the addition and multiplication operations, respectively, which preserve their usual logical, linguistic, scientific-technological customary meaning.

Please, take into account the following; first, it is allowed to refer the addition (symbolized by "**+**") and multiplication (symbolized by "·") binary operations only if both operation are sustained by the operation "successor of a natural number named **x** " (above symbolized by "**S(x)**"); second, entailment refers only to a first order logic PA w.f. and a w.f.s-r statements referred only to both identities of the concept of natural number (as natural number and as successor of a natural number.

Finally, allow us to point out that:

i.- For every natural number **x** **(**x $\in\mathcal{N}$**[0⊂PA]),** the statement **(S(x) = 0)** is false*.* That is, there is no natural number whose successor is **0.**

ii.- For every natural number **x** **(**x $\in\mathcal{N}$**[0⊂PA]),** the statement **(S(x) = x)** is false*.* That is, there is no natural number whose successor is itself.

iii.- The operation (also function) symbolized by "**S(x)**" (meaning "*successor of natural number* **x**", ∀**x** **(**x $\in\mathcal{N}$**[0⊂FOL-PA])** is an injection as a direct consequence of axiom **2.1.2** and the mathematical induction axiom **(2.1.7)**.

First Highlighted NT-FS&L-Considerations

From this point, we will adopt intended interpretation of ***FOL-PA*** *that holds for* $\mathcal{N}$**[0⊂FOL-PA]***:*

i.- Its individuals are recognized and named as the customary natural numbers (both the linguistic & mathematical perspectives of natural number concept hold, sense and intension, identical "identities" of the natural number concept).

ii.- ***Natural numbers*** *are the members of the set named and symbolized by* $\mathcal{N}$**[0⊂FOL-PA]**, *which* ***is equivalent to the set*** $\mathcal{N}$**[0⊄FOL-PA]** ***from a logical perspective, if and only if, natural number one (1) is successor of natural number zero (0) in the*** *subset-*$\mathcal{N}$**[0, 1 ⊂FOL-PA].** [34, 35]

iii.- The natural number named "zero" (symbolized by ***0****) is, in turn, the "identity conserving element" (also named neutral element) for the binary operation (also named function) addition on* $\mathcal{N}$**[0⊂FOL-PA]***.*

iv.- The next statement is a theorem (a w.f.s and a w.f.s-r statement which sustains truth).

S(x) = x + n, ∀x∀n(x ∈ $\mathcal{N}$[0⊂FOL-PA], n ∈ $\mathcal{N}$[0⊂FOL-PA] ∩ $\mathcal{N}$[0⊄FOL-PA])

Please, allow us to point out both; first, in the present intended interpretation of ***FOL-PA*** *"equality" holds the customary meaning generally accepted of "identity" in a first order logic, and second, its embedded entailment with first order axiom* ***2.1.4*** *of* ***FOL-PA****:*

∀x∀y(x, y ∈ $\mathcal{N}$[0⊂PA])(x + S(y) = S(x + y))

v.- The next statement is true.

1·S(y) = 1·x+1 → S(y) = x+1, ∀x∀y(x, y ∈$\mathcal{N}$[0⊂PA])

Please, take in account both, that: first, the **axiom *2.2.6* FOL-PA**´s referred to the multiplication operation of ***every couple*** of natural numbers that belong to $\mathcal{N}$**[0⊂PA]**, and, second, ***the above statement iv referred to the binary addition of***

a couple of natural numbers, in which one of them belongs to $\mathcal{N}$**[0⊂FOL-PA]** ***and the other one belongs to the*** $\mathcal{N}$**[0⊂FOL-PA]** ∩ $\mathcal{N}$**[0⊄FOL-PA]** ***set***, which is the case for natural number **1**. That said:

$$\forall x \forall y (x, y \in \mathcal{N}[0 \subset \text{FOL-PA}])\big(x \cdot S(y) = x \cdot (y + 1) \ \wedge \ (x \cdot y + x) \ \rightarrow \ x \neq y \wedge S(y)\big)$$

&

$$\forall x \forall y (x, y \in \mathcal{N}[0 \subset \text{FOL-PA}])\big((1 \cdot S(x) = 1 \cdot S(y) \rightarrow 1 \cdot x = x \ \wedge \ 1 \cdot y) \ \rightarrow x = y\big)$$

Additionally,

$$\forall x \forall y \ (x, y \in \mathcal{N}[0 \subset \text{FOL-PA}])\big(\big((x \cdot S(y) = y \cdot S(x)) \rightarrow$$

$$\big(x = 0 \ \rightarrow \ y = 0\big)$$

$$\vee$$

$$((x = 1 \rightarrow \ \big(1 \cdot S(y) = y \cdot S(x) \rightarrow \ y = 1 \wedge \ y \cdot S(x) \wedge \ S(x) \wedge S(y) \wedge x \,))$$

$$\vee$$

$$((x \neq 1) \rightarrow \ (\, x = y \wedge S(x) \wedge S(y)) \ \rightarrow \ y \neq 0 \wedge 1)$$

*Hence, the natural number one (**1**) has to be intended recognized as the identity of natural number "one" (symbolized by constant **1**), which belongs to both* $\mathcal{N}$**[0⊂FOL-PA]** *and* $\mathcal{N}$**[0⊄FOL-PA])** *and, in turn, with the embedded identity of the neutral element (also identity conserving element) for the binary multiplication operation on the* $\mathcal{N}$**[0⊂FOL-PA]** *set.*

<u>FIRST FOUNDAMENTAL NT-FS&L THEOREM</u>

$$\forall n \ (n \in \mathcal{N}[0 \subset \text{FOL-PA}])$$

$$\big(\textit{subset-}\mathcal{N}[0 \subset \text{FOL-PA}] \subseteq \textit{subset-}\mathcal{N}[0,1 \subset \text{FOL-PA}] \subseteq \textit{subset-}\mathcal{N}[0, 1, \ldots, n \subset \text{FOL-PA}] \subseteq$$

$$\mathcal{N}[0 \subset \text{FOL-PA}] \ \rightarrow$$

$$\#\textit{-subset-}\mathcal{N}[0 \subset \text{FOL-PA}] = 1 \ \wedge \ \#\textit{-subset-}\mathcal{N}[0,1 \subset \text{FOL-PA}] = 2 \ \wedge \ \#\textit{-subset-}\mathcal{N}[0, 1, \ldots n \subset \text{FOL-PA}] = n + 1\big)$$

(NT-FS&L F.Th. 1)

Second Highlighted NT-FS&L-Consideration [36]

From this point, we will adopt the next two couples of symbolic *representations,* ***(a)*** *and (****b****), which symbolize their corresponding embedded definitions by comprehension and extension of their identity, of each of the two equivalent logical-intended axiomatic* ***(a.1 and a.2, and b.1 and b.2, respectively)*** *for the above descriptions of the above set of natural numbers* $\mathcal{N}$**[0 ⊂PA]**: [37]

a) For the case that in which ***0*** *is intended-considered a natural number:*

$$\mathcal{N}_{(0\subset PA)} := \{\mathbf{0}, \mathbf{1}, 2, 3, 4, \ldots, x{+}0, x{+}1, x{+}2, \ldots\} \qquad (a.1)$$ [38]

$$\mathcal{N}_{(0\subset FOL\text{-}PA)} := \{\mathbf{0}, S(\mathbf{0}), S(S(\mathbf{0})), S(S(S(\mathbf{0}))), \ldots\} \qquad (a.2)$$

b) For the case that ***0*** *is not intended-considered a natural number:*

$$\mathcal{N}_{(0\not\subset FOL\text{-}PA)} := \{\mathbf{1}, 2, 3, 4, 5, \ldots, \mathbf{x}{+}\mathbf{1}, \mathbf{x}{+}2, \ldots\} \qquad (b.1)$$

$$\mathcal{N}_{(0\not\subset FOL\text{-}PA)} := \{S(\mathbf{0}), S(S(\mathbf{0})), S(S(S(\mathbf{0})), \ldots\} \qquad (b.2)$$

*FOL-PA axiomatic (****2.1.1-2.1.7*** *axioms set) defines, on both cases a) and b), a unary irreducible representation of the natural numbers; the natural number* ***0*** *can be well defined and symbolically represented as* ***0****; the natural number* ***1*** *can be well defined and symbolically represented as S(0); the natural number* ***2*** *as S(S(0)) and by S(1), and so on.*

Third Highlighted NT-FS&L-Consideration

Hereinafter, in the NT-FS&L first order language, every natural number ***x*** *will be intended-considered as the result of "****x-fold (also, x-times) application(s)" of "the succession operation" (abbreviated by symbol "S") to natural number 0*** [39] ***which*** *hereinafter will be noted as* $\boldsymbol{S_x(0)}$. [40]

Then, entailment from ***FOL-PA*** *(axioms* ***2.1.1-2.1.7****) sustaining formula* ***a.3*** *and* ***b.3*** *and by means of indexing representation, will be considered definitions by comprehension and extension of both, natural numbers as elements, and two logical formalized collections of them, the set* $\mathcal{N}$ **(0⊂NT-FOL-PA)** *and the set* $\mathcal{N}$ **(0⊂NT-FOL-PA):**

$$\mathcal{N}_{(0\subset NT\text{-}FOL\text{-}PA)} := \{S_0(\mathbf{0}), S_1(\mathbf{0}), S_2(\mathbf{0}), S_3(\mathbf{0}), \ldots, S_x(\mathbf{0}), \ldots\} \qquad (a.3)$$

$$\mathcal{N}_{(0\not\subset NT\text{-}FOL\text{-}PA)} := \{S_1(\mathbf{0}), S_2(\mathbf{0}), S_3(\mathbf{0}), S_4(\mathbf{0}), \ldots, S_{x+1}(\mathbf{0}), \ldots\} \qquad (b.3)$$

Hereinafter, it will be shown that the NT-FS&L allows referring to the natural number concept zero (**0**), the ordered enumeration and, in turn, the ordered counting of:

- The ordered enumeration and, in turn, the ordered counting of the elements (also members and links) belonging to any subset of natural numbers.
- The amount of elements belonging to any subset of natural numbers.
- The enumeration, counting and ordering of the operations succession, addition and multiplications by the same logic, mathematical and computable formal system and language.

- The enumeration, counting and ordering of axioms and theorems by a unique logic, mathematical and computable formal system and language.

*Hence, $\mathcal{N}$**(0⊂NT-FOL-PA)** will be redefined as the set of natural numbers, which is referred to the natural number concept zero (**0**), by the seven-membered set of axioms named **FOL-PA,** to the first fundamental theorem **NT-FS&L F.Th. 1. Additionaly,** the ordered counting and enumeration of its elements under its identities as natural number concepts, allows access to their complete ordering structure under their identities of successors, and vice verse.*

$$\mathcal{N}_{(0\subset \text{NT-FOL-PA})} := \{\mathbf{0, 1, 2, 3, \ldots, S_x(0), S_{x+1}(0), S_{x+2}(0), x+3, \ldots, n}\}, \quad \textbf{(c.1)}$$

2.2. NT-FS&L-Considerations, definitions and theorems about natural numbers. On Elements and Collectives of successors and predecessors.

First Highlighted NT-Consideration on natural numbers and collectives thereof

*In NT-FS&L FOL-PA, (in the following abbreviated as NT-PA and $\mathcal{N}$**(0⊂NT-FOL-PA)** abbreviated as $\mathcal{N}$**(0⊂NT-PA)**), the next representation meta-operation **A-B** will be operative:* [41, 42]

A. *Meta-exchange index-variable and variable-index representing elements of $\mathcal{N}$**(0⊂NT-PA)**.*

$\mathcal{N}_{(0\subset \text{NT-PA})}=\{0,1,2,3,4, \ldots, \mathbf{x+0}, \mathbf{x+1},\ldots\}$, then $\forall \mathbf{x},\forall \mathbf{y}(x,y\in\mathcal{N}_{(0\subset \text{NT-PA})})$:

i. $\mathcal{N}_{(0\subset \text{NT-PA})} := \{ S_x(\mathbf{0}) \mid (\mathbf{0} = S_0(\mathbf{0}))\land (\mathbf{x} = S_x(\mathbf{0}) \land S_0(\mathbf{x}))\}\forall \mathbf{x}$

ii. $(S_x(\mathbf{y}) = S_y(\mathbf{x})) \rightarrow \mathbf{x = y}$

iii. $S_x(\mathbf{1}) = S_{y+1}(\mathbf{0})) \rightarrow \mathbf{x = y}$

B. *Meta-exchange index-variable and variable-index between $\mathcal{N}$**(0⊂PA)** and $\mathcal{N}$**(0⊄NT-PA)***

$\mathcal{N}_{(0\not\subset \text{NT-PA})} = \{\mathbf{1}, 2, 3, 4, 5,\ldots, \mathbf{x+0}, \mathbf{x+1},\ldots\}$, then:

i. $\mathcal{N}_{(0\not\subset \text{NT-PA})} = \{ S_{x+1}(\mathbf{0}) \mid (\mathbf{1} = S_1(\mathbf{0}))\land(x = S_{x+1}(\mathbf{0}))\}\forall \mathbf{x}$

ii. $\forall x, (x \in \mathcal{N}_{(0\not\subset \text{NT-PA})})\land\forall y(y \in \mathcal{N}_{(0\subset \text{NT-PA})}). ((S_x(\mathbf{1}) = S_{y+1}(\mathbf{0}) \rightarrow (\mathbf{x = y+1}))$

iii. $\forall x, (x \in \mathcal{N}_{(0\not\subset \text{NT-PA})})\land\forall y(y \in \mathcal{N}_{(0\subset \text{NT-PA})}). ((S_{x+1}(\mathbf{0}) = S_y(\mathbf{1})) \rightarrow (\mathbf{x = y}))$

Hence, the next two statements summarize the above NT-considerations:

$$\mathcal{N}(\mathbf{0 \subset NT\text{-}PA}) = \{0, 1, 2, \ldots, x, \ldots\} = \{0, S(0), S(S(0)), \ldots\} \wedge \{S_0(0), S_1(0), S_2(0), \ldots\},$$

if an only if, $\mathcal{N}(\mathbf{0 \subset NT\text{-}PA}) = \{S_{x+0}(0)\} \subseteq \{S_x(\mathbf{0})\}\ \ \forall \boldsymbol{x}(x \in \mathcal{N}(\mathbf{0 \subset NT\text{-}PA}))$

$$\mathcal{N}(\mathbf{0 \not\subset NT\text{-}PA}) = \{1, 2, \ldots, x, \ldots\} = \{1, S(1), S(S(1)), S(S(S(1)), \ldots\} = \{S_1(0), S_2(0), S_3(0), \ldots\}$$

if an only if, $\mathcal{N}(\mathbf{0 \not\subset NT\text{-}PA}) = \{S_x(\mathbf{1})\} \subseteq \{S_{y+1}(\mathbf{0})\},$

$\forall \boldsymbol{x}, \forall \boldsymbol{y}\ (x \in \mathcal{N}(\mathbf{0 \not\subset NT\text{-}PA}) \wedge y \in \mathcal{N}(\mathbf{0 \subset NT\text{-}PA}))$

Let us considered the next NT-conclusion unavoidable: we can use the same well formed format symbols rooted in a common symbolic representation of the operation S (successor of a natural number) identity to represent any natural number identity, which satisfices as element of any of the two ***PA*** *axiomatic initially described, and we can use said format in order to define by extension and comprehension any set or collectivity of them. However, said representation does not provide us a common and in turn intended (sense and intension) unique definition by comprehension of such set or collectivity thereof.*

On the other hand, it is possible to symbolized and enumerate the elements of $\mathcal{N}(\boldsymbol{0 \subset PA})$ and $\mathcal{N}(\mathbf{0 \not\subset PA})$ *by using the alphabet of either of them, as well as reference for enumeration and counting of the elements.*[43]

Second Highlighted NT-Consideration on meta-representations of natural numbers and collectives thereof

$\forall \boldsymbol{x} \forall \boldsymbol{y}\ (\boldsymbol{x} \in \mathcal{N}(\mathbf{0 \subset FOL\text{-}PA}) = \{0, 1, 2, 3, 4 \ldots\} = \{\mathbf{0}, S(\mathbf{0}), S(S(\mathbf{0})), \ldots\}\ \mathbf{\&}\ \ \forall \boldsymbol{y} \in \mathcal{N}(\mathbf{0 \not\subset FOL\text{-}PA}) = \{1, 2, 3, 4 \ldots\} = \{S(\mathbf{0}), S(S(\mathbf{0})), S(S(S(\mathbf{0})), \ldots\} = \{\mathbf{1}, S(\mathbf{1}), S(S(\mathbf{1})), S(S(S(\mathbf{1})), \ldots\})$.

$$\mathbf{((S_0(1) = S_1(0))\ \&\ ((S_y(1) = S_{x+1}(0))} \rightarrow x = y$$

That is to say, both set $\mathcal{N}(\boldsymbol{0 \subset NT\text{-}PA})$ ***and*** $\mathcal{N}(\boldsymbol{0 \not\subset NT\text{-}PA})$ ***require the same amount of natural***[44] ***number symbols (alphabet) for complete representation of their constants, variables and expressions without altering the topological property of non-dense order***[45] ***common from their axiomatic logic and mathematical construction, and definition. Besides, this is extensible to any of its possible subsets.***

Please, take into account that the order theory herein suggested does not correlate with the ubiquitous concept (idea) of "the cardinality" (neither to the "ordinality" concept) used by others authors. Hence, we suggest a deep revision of the central role that this concepts of cardinal and ordinal numbers play in the state of the art of the past and current Number Theory. [46]

Third Highlighted NT-Consideration on meta-representations of natural numbers and collectives thereof

The first Highlighted NT-Consideration on Meta-representations of natural numbers and collectives thereof is also "true" (which has two possible intentional meanings, excluding those that come from the unintentional models: it is a "well formed expression" and, hence, the statement remains as a theorem), for any positional number system (also numbering system) symbolic representation and for any numerical construction and lexical numbering system. [47]

Thus, in NT-FS&L the following statement is invariant (also, independent) of the symbolic number representation system used:

$\forall x \forall y \ (x \in \mathcal{N}_{(0 \subset PA)} = \{0, 1, 01, 11, \ldots\} = \{\mathbf{0}, S(\mathbf{0}), S(S(\mathbf{0})), \ldots\} \ \& \ \forall y \in \mathcal{N}_{(0 \not\subset PA)} = \{1, 01, 11, \ldots\} = \{S(\mathbf{0}), S(S(\mathbf{0})), S(S(S(\mathbf{0})), \ldots\} = \{\mathbf{1}, S(\mathbf{1}), S(S(\mathbf{1})), S(S(S(\mathbf{1})), \ldots\})$.

$$((S_0(1) = S_1(0)) \ \& \ ((S_y(1) = S_{x+1}(0)) \rightarrow x = y$$

In the example, a customary binary symbolic expressions for the natural number concept are used instead of the initial proposition expressed in decimal number system. The "non-dense order relation" holds from ***the operation "successor of a natural number" when expressed in any numbering system" (an invariant property) and it is compatible, with the current accepted "rules and machinery of construction" of the accepted symbols to represented well-formed sentences (expressions) in any number system.***

$$\forall x \forall y \ (x, y \in \mathcal{N}_{(0 \subset PA)} = \{0, 1, 2, 3, 4 \ldots\} = \{\mathbf{0}, S(\mathbf{0}), S(S(\mathbf{0})), \ldots\}, \text{ then}$$

$$S_{x+y}(1) = S_{(x+1)+y}(0) = S_y(x+1) = S_{x+1}(y) = S_{(y+1)+x}(0) = S_{y+1}(x) = S_{x+y+1}(0),$$

if an only if

$$(((S_0(1) = S_1(0)) \ \& \ ((S_x(1) = S_{y+1}(0) \rightarrow ((x = y+1))) \ \forall x \forall y \ (x \in \mathcal{N}_{(0 \subset NT\text{-}PA)})$$

First Highlighted Set of NT-Definitions and NT-Theorems on natural numbers and collectives thereof

From this point, we will considerer the next set of definitions and w.f.s as the *core* of NT-FS&L:

<u>HL-1-NT-Definition</u>: A natural number y $(y \in \mathcal{N}_{(0 \subset NT\text{-}PA)})$ ***is named (also, called, renamed, nicknamed and defined) "<u>direct successor of natural number x</u>", if an only if, the next statement is a wfs and wfs-r statement:***

$$\forall x \forall y \ (x, y \in \mathcal{N}_{(0 \subset NT\text{-}PA)})(y = S_1(x) \wedge (S_{x+1}(0) \rightarrow y = x+1) \qquad \text{(def. 1)}$$ [48]

We will provide a new identity to natural number **y**, which will be well symbolically represented (w.f.s-r) by formulas: **"S_1(x)", "1-*suc*-x"** and **y-sucd-x**, by the next three equivalent equalities:

$$y = S_1(x)\ ;\quad y = 1\text{-}suc\text{-}x;\quad y = sucd\text{-}x;$$

Please, allow us to indicate that **def.1** is sustained on: i.- Previous concept of identity (also, equality in **FOL-PA**); ii.- Previously described theorem **S(x) = x + 1**, which is coming from PA axiomatic; iii.- Equivalence of the next equalities representing both of the next well formed symbolic-representations (w.f.s-r.) identity of natural numbers **x** and **y** $\forall x \forall y\ (x, y \in \mathcal{N}$**(0⊂NT-PA).**

$$0 = S_0(0) \wedge y = S_x(1) \wedge y = S_{x+1}(0) \wedge x = S_{x+1}(0) \wedge y = S_y(0)$$

<u>HL-2-NT Definition</u>: A natural number n (n $\in \mathcal{N}$(0⊂NT-PA)***) is named (also, called renamed, nicknamed and defined) as <u>"y-successor of natural number</u> x<u>"</u>, if and only if the next statement is a wfs and wfs-r statement:***

$$\forall x \forall y \forall n\ (x, y, n) \in \mathcal{N}(0 \subset \text{NT-PA})\ (n = S_x(y) \wedge S_y(x) \wedge (S_{x+y}(0) \rightarrow n = x+y\ ) \quad \textbf{(def. 2)}$$

We provide of a new identity to natural number **n**, which will be well symbolically represented (w.s.r) by formulas: **"S_y(x)"** and **"y-*suc*-x"**, by the next set of equivalent well symbolic (nomenclature and formulation) formed (entailment from FOL-PA´s axiomatic): [49]

n = y-*suc*-x ($n = S_y(x) \wedge S_x(y)$), if and only if:

n = (y+x)-*suc*-0 ($n = S_{x+y}(0) \wedge S_{y+x}(0) \wedge S_0(x+y)$), if and only if:

n = n-*suc*-0 ($n = S_n(0) \wedge S_0(n)$), $\forall n \forall x \forall y\ (x, y, n \in \mathcal{N}$(0⊂PA))

Please, allow us to highlight that in first order logic, "y-successor of natural number x" and "direct-successor of a natural number x", define two different partitions of the set of natural numbers under a unique and well defined NT-FOL-PA´s axiomatic referred to the concept of natural number zero (0). Additionally is very import remember that *zero (0) is member and symbol on* $\mathcal{N}$***(0⊂ NT-PA)) and natural number zero concept only allows to the symbolic of the corresponding first order alphabet of*** $\mathcal{N}$**(0⊄NT-PA).**

*Please, take in account that in NT-FS&L, from the perspective of the symbolic representation of a natural number concept, "**S_1(x)**" and "**1-suc-x**" are obviously indistinguishable for two meaning (semantics) "**1-successor of x**" and "**direct 1-succesor of cero**"; but this does not impose any "logic limitation", it is only a direct consequence of the fact that the symbolic representation of **0** and **1** is a "convention" (please note that only if we see the symbol **0** we can not unsure if the statement referrers about* $\mathcal{N}$*(0⊂NT-PA) or about* $\mathcal{N}$***(0⊄NT-PA)*** *but this does not change the truth sustained by the next statement:*

$[(0 \in \mathcal{N}(0{\subset}NT\text{-}PA) \wedge 0 \notin \mathcal{N}(0{\not\subset}NT\text{-}PA))$ **&** $(1 \in \mathcal{N}(0{\subset}NT\text{-}PA) \wedge \mathcal{N}(0{\not\subset}NT\text{-}PA)) \rightarrow 0 \neq 1]$

& "1-successor of 0 is a identity of natural number one (1)"
& "the direct 1-successor of natural number 0" is one of the identities of natural number one (1)" embedded in PA.

<u>HL-3-NT Definition</u>: *The set conformed by every natural number* ***y (y*** $\in \mathcal{N}(0{\subset}NT\text{-}PA)$***)****, which is successor of a natural number* ***x (x***$\in\mathcal{N}(0{\subset}NT\text{-}PA)$***)****, is named (also, called, renamed, nicknamed and defined) as "<u>set of successors of natural number x</u>", if and only if the next statement is a wfs and wfs-r statement:* **50**

$$\mathcal{S}uc_{\mathbf{y}}(\mathbf{x}) := \{\, \mathbf{y} \in \mathcal{N}(\mathbf{0}{\subset}\mathrm{NT}\text{-}\mathbf{PA}) \mid \mathbf{x+y} = \mathbf{S_{x+y}(0)} \wedge [\mathbf{S_x(0) + y}] \wedge (\mathbf{y} = \mathbf{S_y(0)}\} \quad \textbf{(def. 3)}$$

<u>HL-4-NT Definition</u>: *The singleton formed by the natural number* ***y*** *as unique member, which is the direct successor of a natural number* ***x (x*** $\in \mathcal{N}(0{\subset}PA)$***)****, is named (also, called, renamed, nicknamed and defined) <u>"set of direct successors of natural number</u>* <u>**x**"</u>*, if and only if the next statement is a wfs and wfs-r statement:* 51

$$\mathcal{S}ucd(\mathbf{x}) := \{\, \mathbf{x+1} \in \mathcal{N}(\mathbf{0{\subset}NT\text{-}PA}) \mid \mathbf{x} = \mathbf{S_0(x)} \wedge \mathbf{S_x(0)}\} \quad \textbf{(def. 4)}$$

Please, allow us to point out both the next two statements NT-FOL-PA about sets of natural numbers:

$$\forall \boldsymbol{x}\, (\boldsymbol{x} \in \mathcal{N}(\mathbf{0{\subset}PA}))\, (\mathcal{S}ucd(\mathbf{x}) \subseteq \overline{\mathcal{S}uc_{\mathbf{x}}(\mathbf{0}) \cap \mathcal{S}uc_{\mathbf{x+1}}(\mathbf{0})})$$

$$\forall \boldsymbol{x}\, (\boldsymbol{x} \in \mathcal{N}(\mathbf{0{\subset}NT\text{-}PA}))\, (\mathcal{S}ucd(\mathbf{x}) \subset \mathcal{S}uc_{\mathbf{x}}(\mathbf{0}))$$

both of them are a direct consequence of initial FOL-PA´s axiomatic definition of the *set of natural numbers referred to natural number zero*:52

$$\mathcal{N}(\mathbf{0{\subset}NT\text{-}PA}) = \mathcal{S}uc_{\mathbf{x}}(\mathbf{0}) \wedge \mathcal{S}uc_{\mathbf{0}}(\mathbf{x}),\ \forall \boldsymbol{x}\, (\boldsymbol{x} \in \mathcal{N}(\mathbf{0}{\subset}\mathrm{NT}\text{-}\mathbf{PA}))$$

<u>HL-1-NT Theorem</u>: *The* $\mathcal{N}(\mathbf{0{\subset}NT\text{-}PA})$ *set*, which has been renamed as $\mathcal{S}uc_{\mathbf{x}}(\mathbf{0})$, is conformed by the union of every singleton $\mathcal{S}ucd(\mathbf{x})$ conformed by one defined and symbolized natural number **x**

$$\bigcup\, [\mathcal{S}ucd(\mathbf{x})]_{\forall \mathbf{x} \in \mathcal{N}(\mathbf{0{\subset}NT\text{-}PA})} \subseteq \mathcal{N}(\mathbf{0{\subset}NT\text{-}PA}) \qquad \textbf{(NT-th. 1)}$$

$\mathcal{S}ucd(\mathbf{x})$ customary represent a partition "member to member" (only singletons are involved) of a set, $\mathcal{N}(\mathbf{0{\subset}NT\text{-}PA})$ which is a now "an infinite enumerable and countable element by element" set as a direct consequence of its complete order.

<u>HL-5-NT Definition</u>: A natural number n (n ∈ $\mathcal{N}$(0⊂NT-PA)) is named (also, called renamed, nicknamed and defined) <u>"y-predecessor of natural number x"</u>, if and only if the next statement is a wfs and wfs-r statement: [53]

$$\forall x\forall y\forall n\ (x, y, n) \in \mathcal{N}(0\subset \text{NT-PA})$$
$$(x := P_0(x) \wedge\ y := P_0(y) \wedge\ n := P_0(n)) \rightarrow$$
$$(x = S_x(0) \wedge y = S_y(0) \wedge n = S_n(0) \rightarrow\ x = n + y \wedge n = y) \quad \text{(def.5)}$$

We will provide of a new identity to natural number **n**, which will be well symbolically represented (w.f.s.r) by both formulas: "$P_y(x)$" and "**y-*pre*-x**", by the next two equivalent equalities:

$$n = P_y(x)\ ; \quad n = y\text{-}pre\text{-}x$$

<u>HL-6-NT Definition</u>: A natural number y is named (also, called renamed, nicknamed and defined) "<u>direct predecessor of natural number x</u>", if an only if the next statement is a wfs and wfs-r statement:

$$\forall x\forall y\ (x, y \in \mathcal{N}(0\subset \text{NT-PA}))\ (x = P_1(x) \wedge (S_y(0) \rightarrow\ x = y+1) \qquad \text{(def. 6)}$$

That is to say, natural number y is the direct predecessor of natural number x, if an only if, x is the direct successor of y when both of them are members of $\mathcal{N}$(0⊂NT-PA).

We provide of a new identity to natural number **x**, which will be well symbolically represented (w.f.s-r) by both formulas: "$P_1(x)$" and "**1-*pre*-x**", by the next statement: [54]

$$\forall x\forall y\ (x, y \in \mathcal{N}(0\subset \text{NT-PA})\ (y = P_1(x) \wedge\ 1\text{-}pre\text{-}x) \rightarrow (x = 0\text{-pre-}x)\ \&\ (y = 0\text{-pre-}y)$$

Please, take in account that the fact that symbolic representations "$P_1(x)$" and "**1-*pre*-x**" are indistinguishable (also, degenerated symbolic representations) for the two meaning (semantic identities) ***1-predecessor of x*** and/or **direct 1-predecesor of x**; this has to be considered as a direct consequence correlated with the fact that the customary symbolic representation of "cero" (**0**) and "one" (**1**) is just a convention. Thus, this "degeneration" has never to be considered "non-context free" from a logic perspective. Thus, please allow us to point out that: first, the presence of the symbol **0** by itself cannot ensure whether we are reasoning about $\mathcal{N}$**(0⊂NT-PA)** and/or about $\mathcal{N}$**(0⊄NT-PA)** and, second, this is only true if a previous axiom (theorem) ensures entailment for the statement: 0 ∈ $\mathcal{N}$(0⊂NT-PA).

Hence, for every natural number concept, the statements "zero is the direct predecessor of one" and "the unique predecessor, direct and non-direct, of 1 is cero" do not transfer any first logic inconsistence; because only translate an "enumerable and countable" ordered symbolic degeneration, which always appeared in "the semantic of symbolic conventions" by means of non-free-

context logical-based statements.

Please, allow us to present as formal coherent *conceptual core in **NT-FOL-FS&L** and* supported by **def. 1-4** the next select set of NT-theorems on $\mathcal{N}$ **(0⊂NT-PA)** *about the "precede operation" (also function):*

1. $\forall x \neq 0\ \mathbf{(}P_0(x) \land S_0(x)) \land S_x(0) \neq 0\mathbf{)}$
2. $\forall x\ \mathbf{(}(P_0(x+0) = \mathbf{x + 0} \land S_x(0) \land S_0(x) \rightarrow (P_0(x) = 0 \land S_0(x) \land S_x(0) \rightarrow x = 0)\mathbf{)}$
3. $\forall x\ \mathbf{(}\ P_0(x+1) = \mathbf{x + 1} \land S_0\ \mathbf{(x+1)} \land \mathbf{S_{x+1}(0))}$
4. $\forall x\ \mathbf{(}\ P_0(x+2) = \mathbf{x + 2} \land (S_0\ \mathbf{(x+1) +1}) \land \mathbf{S_1(x+1)} \land \mathbf{S_x(2)} \land \mathbf{S_{x+2}(0))}$

5. $\forall x\ \mathbf{(}P_0(x) \mathbf{+ 0 =} S_0(x+0) \land S_{x+0}\mathbf{(0))}$
6. $\forall x\ \mathbf{(}P_0(x) \mathbf{+ 1 =} S_0(x+1) \mathbf{+ 0} \land S_0(x\mathbf{+1}) \land S_{x+1}(0)\mathbf{)}$
7. $\forall x\ \mathbf{(}P_0(x) \mathbf{+ 2 =} S_0(x+1) \mathbf{+ 1} \land S_0(x\mathbf{+2}) \land S_{x+2}(0) \land S_0(x+2) \mathbf{+ 0)}$
8. $\forall x\ \mathbf{(}P_0(x) \mathbf{+ 3 =} S_0(x+1) \mathbf{+ 2} \land S_0(x\mathbf{+3}) \land S_{x+3}(0) \land S_0(x+2) \mathbf{+ 1)}$
9. $\forall x\ \mathbf{(}P_0(x) \mathbf{+ 4 =} S_0(x+1) \mathbf{+ 3} \land S_0(x\mathbf{+4})) \land S_{x+4}(0) \land S_0(x+2) \mathbf{+ 2)}$

10. $\forall x \forall y\ \mathbf{(}P_0(x\mathbf{+0}+y) = \mathbf{x+y} \land S_x(y) \land S_y(x) \land S_0(x+\mathbf{0}+y) \land (S_{x+y}(0) + \mathbf{0)}$
11. $\forall x \forall y\ \mathbf{(}P_0(x+\mathbf{1}+y) = \mathbf{(x+0+y)+1} \land S_1(x+0+y) \land S_0(x+\mathbf{1}+y) \land (S_{x+y}(0) + \mathbf{1)}$
12. $\forall x \forall y\ \mathbf{(}P_0(x+\mathbf{2}+y) = \mathbf{(x+1+y)+1} \land S_2(x+0+y) \land S_0(x+\mathbf{2}+y) \land (S_{x+y}(0) + \mathbf{2)}$
13. $\forall x \forall y\ \mathbf{(}P_0(x+\mathbf{3}+y)= \mathbf{(x+2+y)+1} \land S_3(x+0+y) \land S_0(x+\mathbf{3}+y) \land (S_{x+y}(0) + \mathbf{3)}$

14. $\forall x\ (P_0(x+y) = x+y \land 2\cdot x \land S_1(x+y) \land S_0(x+y+1)) \rightarrow x = y)$
15. $\forall x \forall y\ (P_0(2\cdot x+y) = 2\cdot x+y \land (P_0(2\cdot x)+y \land S_1(2\cdot x+y) \land S_0(2\cdot x+y+1)) \rightarrow x = y)$
16. $\forall x \forall y \forall z\ (P_0(x+y+z) = x+y+z \land 3\cdot x \land S_3(x+y+z) \land S_0(x+y+z+3) \rightarrow x = y = z)$
17. $\forall x\ (P_0(2\cdot x+2\cdot y) = 2\cdot x+2\cdot y \land 2\cdot(x+y) \land 2\cdot P_0(x+y) \land S_1(2\cdot x+2\cdot y) \land$
$2S_0(x+y)+1)) \rightarrow x = y)$

Hence, allow us to point out that:

i.- Operation $\mathbf{P_x(y)}$ (**x**-predecessor of natural number **y**) could be understood (also translated) as a uniary injective operation (function) for which the domain is the set of any and every successor element **y** of $\mathcal{N}$**(0⊂NT-PA)** and of course of **Q** (Robinson sense and intension). $\mathbf{P_x(y)}$ is a well defined and symbolically represented formula for any of the natural number if and only $S_y(x)$ it is defined element in NT-FOL-PA´s ($x \in \mathcal{N}$(0⊂NT-PA) $\forall$**x.**

ii.- Operation $\mathbf{P_x(y)}$ identifies each natural number with a unique finite, enumerable and countable set and/or subset of natural numbers $\mathcal{N}$(0⊂NT-PA), and vice versa.

iii.- $\mathbf{P_x(y)}$ has not the significance (sense and intension) of the customary "subtraction operation" nor of the customary currently accepted "subtraction logic".[55]

iv.- $P_x(y)$ as symbolic representation of the operation "precede" referred to

zero ($P_x(y)$∈ 𝒩(0⊂NT-PA), which is the irreducible representation of the meta-exchange index and variable symbolic operation (a tensorial operation)[56] described for the operation "succession" referred to natural number zero, if and only if, x=0

$$((S_y(x) = S_x(y) \wedge P_0(x) \rightarrow x = 0 \wedge y)$$

iv.- ***The natural number symbolically represented by $P_1(x+1)$, which is a well symbolic represented formula when referred to natural number zero as the natural number symbolically represented by $P_0(x)$, is both: the "direct successor of x" $S_1(x)$ ($S_{x+1}(0)$ referred to natural number zero)) and x-predecessor of natural number y ($S_x(y)$) Thus, we are capable of stating:***

$$\forall x \forall y\ (x,y \in \mathcal{N}(0 \subset \text{NT-PA}))\ (x = P_1(x+1) \wedge P_x(y) \wedge S_0(x) \rightarrow 0 = P_0(0) \wedge S_0(0))$$

<u>HL-2-NT Theorem</u>: Every natural number n (n∈𝒩(0⊂NT-PA)) properly described by the addition of two natural numbers (n = x+y), both of them belonging to 𝒩(0⊂NT-PA)), at least accept to be NT-PA represented by the next two NT-FOL-PA identities as: i.- the natural number represented as (x+y)-successor of natural zero, and ii.- the natural number (x+y) predecessor of zero.

$$\forall x \forall y \forall n\ (x, y, n \in \mathcal{N}(0 \subset \text{NT-PA}))$$

$$((n = P_0(n) \wedge S_{x+y}(0) \wedge P_0(x+y) \rightarrow n = x + y) \rightarrow$$

$$(n = x + y \Rightarrow n = P_0(n) \wedge S_{x+y}(0) \wedge P_0(x+y)) \qquad \text{(NT-th. 2)}$$

<u>HL-7-NT Definition</u>: *The set conformed by the natural numbers which are the predecessors of a natural number* ***x (x ∈ 𝒩(0⊂NT-PA))*** *referred to natural number zero, is named (also, called, renamed, nicknamed and defined) <u>"set of predecessors of natural number **x**"</u>, if and only if, the next statement is a wfs and wfs-r statement:*

$$\mathcal{P}re_0(x) := \{x \in \mathcal{N}(0 \subset \text{NT-PA}) \mid x = P_0(x)\}_{\forall x\ (x \in \mathcal{N}(0 \subset \text{NT-PA}))} \qquad \text{(def. 7)}$$

<u>HL-3-NT Theorem</u>: [57]

$$\forall x(x \in \mathcal{N}(0 \subset \text{NT-PA}))(\mathcal{N}(0 \subset \text{NT-PA}) \subseteq \mathcal{P}re_0(0) \cup \mathcal{S}uc_{x+1}(0)) \qquad \text{(NT-th. 3)}$$

Additionally, the next statement is in NT-FOL-PA an equivalent theorem referred to 𝒩**(0⊂PA)**:

$$\forall x(x \in \mathcal{N}(0 \subset \text{NT-PA}))(\mathcal{N}(0 \subset \text{PA}) \subseteq \mathcal{P}re_0(0) \cup \mathcal{S}uc_x(0)) \qquad \text{(NT-th. 3bis)}$$

<u>HL-8-NT Definition</u>: *The singleton conformed by the natural number* ***y****, which is the* *direct predecessor of a natural number* ***x*** ***(x*** $\in$ **𝒩(0⊂NT-PA)*)***, *is named (also, called, renamed, nicknamed and defined) as* *"set of direct predecessors of natural number* ***x****"*, *if and only if the next statement is a wfs and wfs-r statement:*

$$\forall x \forall y\ (x, y \in \mathcal{N}(0 \subset \text{NT-PA}))$$

$$\mathcal{P}red(x) := \{ x \in \mathcal{N}(0 \subset \text{NT-PA}) \mid x = y+1 \ \wedge P_0(x) \wedge \ P_0(y)+1 \} \forall x \forall y \qquad \textbf{(def. 8)}$$

Hence the next statement is a wfs and a wfs-r:

$$\forall x (x \in \mathcal{N}(0 \subset \text{NT-PA}))$$
$$(\mathcal{P}red(x) \subset \mathcal{P}re_0(x) \subset \mathcal{S}uc_x(0) \rightarrow (\mathcal{S}uc_x(0) \subseteq \mathcal{N}(0 \subset \text{NT-PA})) \forall x)$$

HL-4-NT Theorem:[58] *The* 𝒩**(0⊂NT-PA)** *set*, which has been renamed (named called, defined, etc) as $\mathcal{S}uc_x(\mathbf{0})$, can also be conformed by the union of every singleton $\mathcal{P}red_0(\mathbf{x+1})$ which, in turn, is conformed by one defined and symbolized by the natural number **x** identity referred to natural number zero

$$\bigcup [\mathcal{P}red_0(x+1)]_{\forall x \in \mathcal{N}(0 \subset \text{NT-PA})} \subseteq \mathcal{N}(0 \subset \text{NT-PA}) \qquad \textbf{(NT-th. 4)}$$
$$\bigcup [\mathcal{P}re_0(x)]_{\forall x \in \mathcal{N}(0 \subset \text{NT-PA})} \subseteq \mathcal{N}(0 \subset \text{NT-PA}) \qquad \textbf{(NT-th. 4 bis)}$$

Hence, taking in account the nomenclature and notation of the FOL-PA inductions axioms and the *NT-SF&L tensorial formulation;* ***NT-th. 4 and 4 bis*** *can be rewritten as:*

$$\forall x \forall y \forall n\ (x, y_1 \ldots y_n, y_{n+1} \in \mathcal{N}[0 \subset \text{NT-PA}], n \in \mathcal{N}[0 \subset \text{NT-PA}] \cap \mathcal{N}[0 \not\subset \text{NT-PA}])$$

$$\bigcup_{x+0\,=\,y_{x+1}}^{x+1\,=\,y_{n+1}} [\mathcal{P}re_0(x)]_{\forall x \in \mathcal{N}(0 \subset \text{NT-PA})} \subseteq \mathcal{N}(0 \subset \text{NT-PA})$$

(NT-th. 4 tris)

$\mathcal{P}red_0(\mathbf{x+1})$, as well as $\mathcal{P}re_0(\mathbf{x})$, represents "a partition member to member" (only singletons are involved; hence, also, "a partition singleton to singleton") of a set which is "enumerable and countable" by this NT-methodology (Cantor´s, von Neumann´s and Gödel´s numeration and counting methodologies imply different sense and intension concepts for enumeration, numeration and countability), 𝒩 **(0⊂NT-PA)**.

HL-9-NT Definition: *The singletons conformed by the natural number* ***x*** **(x∈ 𝒩(0⊂NT-PA))** *as its unique member, which is the element of the intersection of* $\mathcal{S}uc_x(\mathbf{0})$ *and* $\mathcal{P}re_0(\mathbf{x})$ sets, *will be named (called, defined) "primordial of natural number* ***x*** *primordial set", if and only if the next statement is a wfs and wfs-r statement:*

$$\mathcal{PN}_0(\mathbf{x}) := \{\, \mathbf{x} \in \mathcal{N}(\mathbf{0 \subset NT\text{-}PA}) \mid \mathbf{x} \in \mathit{Suc}_0(\mathbf{x}) \cap \mathit{Pre}_0(\mathbf{x}) \}, \forall \mathbf{x}\, (\mathbf{x} \in \mathcal{N}(\mathbf{0 \subset NT\text{-}PA})) \quad \textbf{(def. 9)}$$

Hence;

$$\forall \mathbf{x}\, (\mathbf{x} \in \mathcal{N}[\mathbf{0 \subset NT\text{-}PA}])$$
$$(\#\text{-}\mathcal{PN}_0(\mathbf{x}) = \mathbf{1} \wedge \#\text{-}\mathcal{PN}_0(\mathbf{x+1}) \rightarrow (\mathbf{1} \wedge (\mathbf{x+1}) \in \mathcal{N}[\mathbf{0 \subset NT\text{-}PA}] \cap \mathcal{N}[\mathbf{0 \not\subset NT\text{-}PA}])$$

<u>HL-5-NT Theorem</u>: $\mathcal{N}(\mathbf{0 \subset NT\text{-}PA})$ *set*, which has been renamed (named, called and, defined) as $\mathit{Suc}_{\mathbf{x}}(\mathbf{0})$ and the set resulting of the union of every direct predecessor of a natural number **x**, is equal (identity), in turns, to the set conformed by the union of $\mathcal{PN}_0(\mathbf{x})$ primordial set of every natural number.

$$\bigcup\, [\mathcal{PN}_0(\mathbf{x})]_{\forall \mathbf{x} \in \mathcal{N}(\mathbf{0 \subset NT\text{-}PA})} \subseteq \mathcal{N}(\mathbf{0 \subset NT\text{-}PA}) \qquad \textbf{(NT-th. 5)}$$

<u>HL-10-NT Definition</u>: *The singleton conformed by the natural number* ***cero*** *as unique member, which is the direct predecessor of natural number* ***1*** ***(***$\forall \mathbf{x} \in \mathcal{N}(0 \subset NT\text{-}PA)$***)****, is named (also, called, renamed, nicknamed and defined)* *<u>"set of predecessors of natural number</u>* **<u>1</u>***"*, *if and only if the next statement is a wfs and wfs-r statement:*

$$\mathcal{P}\mathit{red}(\mathbf{1}) := \{\, \mathbf{0} \mid \mathbf{0} \in \mathcal{N}(\mathbf{0 \subset NT\text{-}PA}) \,\} \forall \mathbf{x}\, (\mathbf{x} \in \mathcal{N}(\mathbf{0 \subset NT\text{-}PA})) \qquad \textbf{(def. 10)}$$

And, hence:

$$\#\text{-}\mathcal{P}\mathit{red}(\mathbf{1}) = \mathbf{1}$$

Please, take in account next NT-theorem:[59]

$$\forall \mathbf{x}\, (\mathbf{x} \in \mathcal{N}(\mathbf{0 \subset NT\text{-}PA}))(\mathcal{P}\mathit{red}(\mathbf{1}) = \mathcal{PN}_0(\mathbf{x}) \rightarrow \mathbf{x} = \mathbf{0} \wedge \mathbf{x} \in \mathit{Suc}_0(\mathbf{x}) \cap \mathit{Pre}_0(\mathbf{x})\})$$

That is say in NT FS&L;

$$\mathcal{P}\mathit{red}(\mathbf{1}) = \mathcal{PN}_0(\mathbf{0})$$

<u>HL-6-NT Theorem</u>:

$$\mathcal{N}(\mathbf{0 \not\subset NT\text{-}PA}) \cup \mathcal{PN}_0(\mathbf{0}) \subseteq \mathcal{N}(\mathbf{0 \subset NT\text{-}PA})) \qquad \textbf{(NT-th. 6)}$$

(Entailment: NT-PA axioms and definitions:

$$\mathcal{N}(\mathbf{0 \not\subset NT\text{-}PA}) := \{S_0(\mathbf{1}), S_1(S_0(\mathbf{1})), S_1(S_1(S_0(\mathbf{1})), \ldots,\} \rightarrow$$
$$\mathcal{N}(\mathbf{0 \subset NT\text{-}PA}) := \{\mathbf{0}, S(\mathbf{0}), S(S(\mathbf{0})), \ldots\} \Rightarrow \mathcal{N}(\mathbf{0 \not\subset NT\text{-}PA}) \cup \mathcal{PN}_0(\mathbf{0}))$$

<u>HL-7-NT Theorem</u>: *#-*$\mathcal{P}\mathit{re}_0(\mathbf{x})$ *is a first order logic universal and uniary natural* *number enumeration and counting operation (remember, also the function in first order typed category language) for every* $\mathcal{Suc}_{\mathbf{x}}(\mathbf{0})$ and for every subset or equivalence class (quotient class) arising from every partition of both $\mathcal{N}$**(0⊄NT-PA)** *and* $\mathcal{N}$**(0⊂NT-PA))**.

$$\forall \mathbf{x}\ (\mathbf{x} \in \mathcal{N}\,(\mathbf{0} \subset \mathbf{NT\text{-}PA}))\ (\mathbf{x} \in \mathcal{PN}_0(\mathbf{x}) \rightarrow$$
$$\#\text{-}\mathcal{Pre}_{\,0}(\mathbf{x}) = (\mathbf{x+1}) \wedge \mathbf{S_1}(\mathbf{x}) \wedge \mathbf{S_x}(\mathbf{1}) \wedge \mathbf{P_0}(\mathbf{x+1}) \wedge \mathbf{P_1}(\mathbf{x+2}) \wedge \mathbf{P_1}(\mathbf{x+1+1}) \wedge \ldots \wedge$$
$$\mathbf{P_x}((\mathbf{x+1})+\mathbf{x}) \wedge \ldots \wedge \mathbf{P_x}((\mathbf{x+x})+\mathbf{1}))$$

(NT-th. 7)

$$\{\ \#\text{-}\mathcal{Pre}_{\,0}(\mathbf{x})\ \} = \{\mathbf{x+1} \in \mathcal{N}(\mathbf{0} \not\subset \mathbf{NT\text{-}PA}) \cup \mathcal{PN}_0(\mathbf{0}))\ |\ \mathbf{x} \in \mathcal{PN}_0(\mathbf{x})\ \}, \forall \mathbf{x}(\mathbf{x} \in \mathcal{N}(\mathbf{0} \not\subset \mathbf{NT\text{-}PA}))$$

(NT-th. 7 bis; *entailment from NT-th. 3, 6 & 7***)**

3. NT-FS&L DEFINITIONS AND THEOREMS ABOUT THE CONCEPTS SUCCESSOR AND PREDECESSOR, COVARIANCE AND CONTRAVARIANCE. NT-FS&L REPRESENTATIONS OF NATURAL NUMBERS AS ELEMENTS, AND OF THEIRS COLLECTIVITIES [60]

Please, allow us to consider ∀x∀y **(x, y ∈** $\mathcal{N}$ **(0⊂NT-PA))**

A. On natural numbers as elements of $\mathcal{N}$(0⊂NT-PA) referred to natural number zero (0) successor identity.

$$\mathbf{S_{x+y}}\,(\mathbf{0}) \wedge \mathbf{S_{y+x}}\,(\mathbf{0}) = (\mathbf{x+y}) \wedge (\mathbf{y+x})$$
&
$$\mathbf{S_{x+y}}\,(\mathbf{0}) = (\mathbf{x+y}) \wedge (\mathbf{y+x}) \rightarrow \mathbf{S_{x+y}}\,(\mathbf{0}) = \mathbf{S_x}\,(\mathbf{y}) \wedge \mathbf{S_y}\,(\mathbf{x})$$
&
$$\mathbf{S_x}\,(\mathbf{0}) = \mathbf{y} \wedge \mathbf{S_y}\,(\mathbf{0}) \rightarrow x = y,$$
&
$$\mathbf{S_{x+y}}\,(\mathbf{0}) = \mathbf{P_0}(\mathbf{x+x}) \rightarrow x = y,$$

B. On $\mathcal{N}$(0⊂NT-PA) natural number set as a collectivity of natural numbers referred to natural number zero (0).

$$\mathcal{Suc}_{\mathbf{x+y}}(\mathbf{0}) \subseteq \mathcal{Suc}_{\mathbf{y+x}}(\mathbf{0}) \wedge \mathcal{Suc}_{\mathbf{y}}(\mathbf{x}) \wedge \mathcal{Suc}_{\mathbf{y}}(\mathbf{x})$$
&
$$\mathcal{PN}_0(\mathbf{x}) \subseteq \mathcal{PN}_0(\mathbf{y}) \rightarrow x = y$$
&
$$\mathcal{PN}_0\,(\mathbf{x+x}) \subseteq \mathcal{PN}_0\,(\mathbf{y+x}) \wedge \mathcal{PN}_0\,(\mathbf{x+y}) \wedge \mathcal{PN}_0\,(\mathbf{y+y}) \rightarrow x = y$$

Thus, let us redefine and enunciate the next NT-methodological theorem:

$\forall x \forall y$ (x, y ∈ $\mathcal{N}$(0⊂NT-PA))

$$\mathcal{S}uc_{\mathbf{y}}(\mathbf{x}) := \{ (x+y)+0, (x+y)+1, (x+y)+2, \ldots \}$$

$$\wedge$$

$$\mathcal{S}uc_{\mathbf{x}}(\mathbf{y}) := \{ (y+x)+0, (y+x)+1, (y+x)+1, \ldots \}$$

$$\wedge$$

$$\mathcal{S}uc_{\mathbf{0}}(\mathbf{x+y}) := \{ (x+y)+0, (x+y)+1, (x+y)+2, \ldots \};$$

$$\rightarrow$$

$$\mathcal{P}re_{\mathbf{0}}(\mathbf{x}) = \{ x, \ldots, 3, 2, 1, 0 \}, \forall x\ (x \in \mathcal{N}(0 \subset \text{NT-PA}))$$

$$\#\text{-}\mathcal{P}re_{\mathbf{0}}(\mathbf{x}) = x+1$$

$$\wedge$$

$$\mathcal{PN}_{0}(\mathbf{x}) = \{ x \mid x \in \mathcal{N}(0 \subset \text{NT-PA}) \} \ \& \ \mathcal{PN}_{0}(\mathbf{y}) := \{ y \mid y \in \mathcal{N}(0 \subset \text{NT-PA}) \}$$

$$\wedge$$

$$\mathcal{PN}_{\mathbf{0}}(\mathbf{x+y}) = \{ x+y \mid x+y \in \mathcal{N}(0 \subset \text{NT-PA}) \}$$

$$\wedge$$

$$\#\text{-}\mathcal{PN}_{0}(x+y) = 1$$

From this point onwards we are able to define in the first order NT-PA language:

HL-10-NT Definition: *Every "presentation"*[61] *(also, representation) of:*

> ***A.*** *The set of successors of a number* ***x referred to zero*** **($\forall$x, x ∈ $\mathcal{N}$(0⊂NT-PA))** $\mathcal{S}uc_{\mathbf{x}}(\mathbf{0})$, *will be intended named (also defined renamed and called,...) as* ***a covariant representation of the successors of a number x referred to zero,***

$$\mathcal{S}uc^{\alpha}{}_{\mathbf{x}}(\mathbf{0})$$

if and only if, we represent such set in the sense and intension that on its representation by extension, each member (natural number) is followed by its direct successor:

$$\mathcal{S}uc^{\alpha}{}_{\mathbf{x}}(\mathbf{0}) := \{ S_x(0), (x)+1, (x+1)+1, (x+1+1)+1, \ldots \}$$

$$\forall x\ (x \in \mathcal{N}(0 \subset \text{NT-PA}) \ \& \ x = S_x(0))$$

(def. 10)

> ***B.*** *The set of successors of a number* ***x referred to zero*** **($\forall$x (x∈$\mathcal{N}$(0⊂NT-PA))** $\mathcal{S}uc_{\mathbf{x}}(\mathbf{0})$, *will be intended named (also defined renamed, called...) as a* ***contravariant representation of the successors of a number x referred to zero,***

$$\mathcal{S}uc^{\beta}{}_{\mathbf{x}}(\mathbf{0})$$

if and only if, we represent such set in the sense and intension that on its representation by extension, each member (natural number) is followed by its direct *predecessor:*

$$\mathcal{Suc}^{\beta}{}_{x}(0) := \{ P_x(x), ..., P_0(2), P_0(1), P_0(0) \}$$

$$\forall x \ (x \in \mathcal{N}(0 \subset NT\text{-}PA) \ \& \ x = P_0(x) \wedge S_0(x))$$

(def. 10bis)

Hence, for example, both previous definitions ***a.2*** *of* $\mathcal{N}(0 \subset PA))$ *and* ***b.2*** *of* $\mathcal{N}(0 \not\subset PA)$*, respectively, have to be considered as covariant presentations of both sets,* $\mathcal{N}^{\alpha}(0 \subset NT\text{-}PA)$ and $\mathcal{N}^{\alpha}(0 \not\subset NT\text{-}PA)$, *respectively. The corresponding contravariant representations and definitions of* $\mathcal{N}^{\beta}(0 \subset NT\text{-}PA))$ and $\mathcal{N}^{\beta}(0 \not\subset NT\text{-}PA)$, *respectively, are:* [62]

i.- $\forall x \ (x \in \mathcal{N}(0 \subset NT\text{-}PA))$

$$\mathcal{N}^{\beta}(0 \subset NT\text{-}PA)) := \{ P_x(x) ...., P_2(x), P_1(x), P_0(x) \} \qquad (a.2\beta)$$

ii.- $\forall x \ (x \in \mathcal{N}^{\beta}(0 \not\subset NT\text{-}PA))$

$$\mathcal{N}^{\beta}(0 \not\subset NT\text{-}PA) := \{ P_x(x+1), ..., P_{1+1}(x+1), P_1(x+1), P_0(x+1), \} \qquad (b.2\beta)$$

HL-8-NT Theorem: [63]

$$\forall x \ (x \in \mathcal{N}(0 \subset NT\text{-}PA))(\mathcal{Suc}^{\alpha}{}_{0}(x) \subseteq \mathcal{Pre}^{\alpha}{}_{0}(x) \ \& \ \mathcal{Suc}^{\beta}{}_{x}(0) \subseteq \mathcal{Pre}^{\beta}{}_{0}(x) \ \rightarrow$$

$$x+1 = \#\text{-}\mathcal{Suc}^{\alpha}{}_{x}(0) \wedge \#\text{-}\mathcal{Suc}^{\beta}{}_{x}(0) \wedge \#\text{-} \mathcal{Pre}^{\beta}{}_{0}(x) \wedge \#\text{-}\mathcal{Pre}^{\beta}{}_{0}(x))$$

[For example, for the case **x =11** $(x \in \mathcal{N}^{\alpha}(0 \subset NT\text{-}PA) \cap \mathcal{N}^{\beta}(0 \subset NT\text{-}PA))$

$$\mathcal{Suc}^{\alpha}{}_{11}(0) := \{ 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11 \}$$
$$\mathcal{Suc}^{\beta}{}_{11}(0) := \{11, 10, 9, 8, 7, 6, 5, 4, 3, 2, 1, 0 \}$$
$$\mathcal{Pre}^{\alpha}{}_{11}(0) := \{ 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11 \}$$
$$\mathcal{Pre}^{\beta}{}_{11}(0) := \{11, 10, 9, 8, 7, 6, 5, 4, 3, 2, 1, 0 \}$$

Then:

$$\mathcal{Suc}_{11}(0) = \mathcal{Suc}^{\alpha}{}_{11}(0) = \mathcal{Pre}^{\alpha}{}_{11}(0) \ \& \ \mathcal{Suc}^{\beta}{}_{11}(0) = \mathcal{Pre}^{\beta}{}_{11}(0) = \mathcal{Pre}_{11}(0)$$

$$\#\text{-}\mathcal{Suc}^{\alpha}{}_{11}(0) \wedge \#\text{-}\mathcal{Suc}^{\beta}{}_{11}(0) \wedge \#\text{-} \mathcal{Pre}^{\beta}{}_{0}(x) \wedge = \#\text{-}\mathcal{Pre}^{\beta}{}_{0}(x) = 12]$$

HL-9-NT Theorem (set of lemmas): *The next statements sustain entailment in first order NT-FS&L from PA axiomatic for the natural number identity concept referred to natural number zero identity:*

$$\forall x \ (x \in \mathcal{N}(0 \subset NT\text{-}PA)) \ \& \ (x \in \mathcal{Suc}^{\alpha}{}_{x}(0) \wedge \mathcal{Suc}^{\beta}{}_{x}(0) \wedge \mathcal{Pre}^{\alpha}{}_{0}(x) \wedge \mathcal{Pre}^{\beta}{}_{0}(x))$$

$$\left(x \in \mathcal{Suc}_{x}(0) \wedge \mathcal{Pre}_{0}(x) \quad \rightarrow \quad \mathcal{Pre}_{0}(x) \cup \mathcal{Suc}_{x+1}(0) \subseteq \mathcal{N}(0 \subset \text{NT-PA})\right)$$

(NT-th. 3)

The next 8-membered set of wfs and wfs-r statements (also, each of them in turn nicknamed as "lemmas" in state of "theorems") [64]*, have considered as first order NT-FS&L definition by extension of the HL-9-NT Theorem:*

1. $\forall x\, \forall y\, (x \in \mathcal{Suc}^{\alpha}{}_{x}(0) \wedge y \in \mathcal{Suc}^{\alpha}{}_{x}(0))(x + x = (S_x(0) + S_y(0)) \wedge S_{x+x}(0) \rightarrow x = y)$
2. $\forall x\, \forall y\, (x \in \mathcal{Suc}^{\alpha}{}_{x}(0) \wedge y \in \mathcal{Suc}^{\beta}{}_{x}(0))(x + y = S_x(0) + S_y(0) \wedge S_{x+y}(0) \rightarrow x = y)$
3. $\forall x\, \forall y\, (x \in \mathcal{Suc}^{\beta}{}_{x}(0) \wedge y \in \mathcal{Suc}^{\alpha}{}_{x}(0))(x + x = S_x(0) + S_y(0) \wedge S_{x+x}(0) \rightarrow x = y)$
4. $\forall x\, \forall y\, (x \in \mathcal{Suc}^{\beta}{}_{x}(0) \wedge y \in \mathcal{Suc}^{\beta}{}_{x}(0))(x + y = S_x(0) + S_y(0) \wedge S_{x+x}(0) \rightarrow x = y)$
5. $\forall x\, \forall y\, (x \in \mathcal{Pre}^{\alpha}{}_{0}(x) \wedge y \in \mathcal{Pre}^{\alpha}{}_{0}(x))(x + x = P_0(x) + P_0(y) \wedge P_0(x+x) \rightarrow x = y)$
6. $\forall x\, \forall y\, (x \in \mathcal{Pre}^{\alpha}{}_{0}(x) \wedge y \in \mathcal{Pre}^{\beta}{}_{0}(x))(x + y = P_0(x) + P_0(y) \wedge P_0(x+y) \rightarrow x = y)$
7. $\forall x\, \forall y\, (x \in \mathcal{Pre}^{\beta}{}_{0}(x) \wedge y \in \mathcal{Pre}^{\alpha}{}_{0}(x))(x + x = P_0(x) + P_0(y) \wedge P_0(x+x) \rightarrow x = y)$
8. $\forall x\, \forall y\, (x \in \mathcal{Pre}^{\beta}{}_{0}(x) \wedge y \in \mathcal{Pre}^{\beta}{}_{0}(x))(x + y = P_0(x) + P_0(x) \wedge P_0(y+x) \rightarrow x = y)$

HL-10-NT Theorem:

$$\forall x \forall y\, (x, y \in \mathcal{Suc}_{x}(0) \wedge \mathcal{Suc}_{0}(x) \wedge \mathcal{Pre}_{0}(x))$$

$$(S_x(0) + S_y(0) = P_0(x) + P_0(y)) \wedge (S_x(0) + P_0(y) = S_y(0) + P_0(y)) \rightarrow x = y)$$

(NT-th. 10)

4. NT-FS&L CONSIDERATIONS AND THEOREMS ABOUT PARITY OF THE NATURAL NUMBERS OF NATURAL NUMBERS AS ELEMENTS, AND OF COLLECTIVITIES THEREOF

Let us considered the next currently accepted definitions in order to adopt Number Theory (Typed Category Theory) nomenclature for the set of even natural numbers (also, the even operation) and the set operation of odd natural numbers (also, the odd operation): [65]

$$\forall x\ (x \in \mathcal{N}_{(0\subset NT\text{-}PA)})$$

$$\mathcal{E}ven(x) := \{\ 2\cdot x\ |\ x \in \mathcal{N}_{(0\subset NT\text{-}PA)}\},\ \forall x \quad \textbf{(def. 0 Parity)}$$

$$\mathcal{O}dd(x) := \{\ 2\cdot x + 1\ |\ x \in \mathcal{N}_{(0\subset NT\text{-}PA)}\},\ \forall x \quad \textbf{(def. 0 Parity)}$$

HLParity-1-NT Even Definition: *The set of successors of a number* ***x referred to zero*** $(\forall x,\ x \in \mathcal{N}_{(0\subset NT\text{-}PA)})$ $\mathcal{S}uc_{x}(\mathbf{0})$, *will be intended named (also defined renamed,called…) as* ***the set even successors referred to zero of a natural number x,***

$$\mathcal{E}ven(x) \vee \mathcal{E}ven(\mathcal{S}uc_{0}(x)) \rightarrow \mathcal{E}ven(x) = \mathcal{E}ven(\mathcal{S}uc_{0}(x))$$

if and only if the next statement is a wfs and wfs-r in firs order NT-FS&L:

$$\mathcal{E}ven(\mathcal{S}uc_{0}(x)) := \{\ x+x \in \mathcal{S}uc_{0}(x)\ |\ x \in \mathcal{N}_{(0\subset NT\text{-}PA)}\},\ \forall x\ (x \in \mathcal{N}_{(0\subset PA)})$$

(def-P. 1)

Please allow us to remind the reader that ***NT-th. 10,*** *only holds true for every couple of natural numbers* ***x*** *and* ***y*** $(\forall x \forall y\ (x \in \mathcal{N}_{(0\subset NT\text{-}PA)})$*, if and only if,* $x = y$. [66]

HLParity-2-NT Odd Definition: [67] *The set of successors of a number* ***x referred to zero*** $(\forall x(x \in \mathcal{N}_{(0\subset NT\text{-}PA)})$ $\mathcal{S}uc_{x}(\mathbf{0})$, *will be (intended) named (also defined renamed, called…) as* ***the set odd of successors referred to zero of a natural number x,***

$$\mathcal{O}dd(x) \vee \mathcal{O}dd(\mathcal{S}uc_{0}(x)) \rightarrow \mathcal{O}dd(x) \subseteq \mathcal{O}dd(\mathcal{S}uc_{0}(x))$$

if and only if, the next definition is a wfs and wfs-r in firs order NT-FS&L statement:

$$\mathcal{O}dd(\mathcal{S}uc_{0}(x)) := \{\ (x+x)+1 \in \mathcal{S}ucd_{0}(x)\ |\ x \in \mathcal{N}_{(0\subset NT\text{-}PA)}\},\ \forall x\ (x \in \mathcal{N}_{(0\subset PA)})$$

(def-P. 2)

That statement is equivalent to:

$$\mathcal{O}dd(\mathcal{S}uc_{0}(x)) := \{\ x \in \mathcal{S}ucd_{0}(x+x)\ |\ x \in \mathcal{N}_{(0\subset NT\text{-}PA)}\},\ \forall x\ (x \in \mathcal{N}_{(0\subset NT\text{-}PA)})$$

(def-P 2bis)

*Hence, the parity of the set of "predecessors" and "successors" of natural number "**x**": will be represented, considered and defined as follows in terms of the "parity" of every* ***element of*** $\mathcal{E}ven(\mathbf{x})$ *and* $\mathcal{O}dd(\mathbf{x})$ *sets:*

i.- **"e-$P_x(0)$"** is in turn an ***even-successor of*** $\mathcal{E}ven(\mathcal{S}uc_{0}(x))$***,*** if and only if, when the natural number represented and identified by **x** is an ***even natural number.***

ii.- "**e-S$_y$(x)**" is an ***even-successor of*** $\mathcal{E}ven(\mathcal{S}uc_0(x))$, when referred to zero is **S$_{x+y}$(0)**, if and only if, ***x+y is an even natural number when both, x and y, are each of them even or odd natural numbers.***

Additionally, each of the elements of $\mathcal{O}dd$**(x)** *will be represented as:*

i.- **o-S$_x$(0),** is an ***odd successor of*** $\mathcal{O}dd(\mathcal{S}uc_0(x))$, when **x** of symbol **o-S$_x$(0),** is representing a ***odd natural number.***

ii.- **o-S$_y$(x),** ***odd-y-successor of x*** , when referred to zero is **S$_{x+y}$(0)**, if and only if: ***x+y is an even natural number when x is an even natural number <u>and</u> y is an odd natural number, <u>or</u> vice versa.***

ii.- o-P$_x$ (0) is an ***odd predecessor of*** $\mathcal{O}dd(\mathcal{S}uc_0(x))$, when **x of symbol o-P$_x$ (0)** ***is an odd natural number.***

<u>HLParity-3-NT Definition</u>:

A primordial $\mathcal{PN}_0(x)$ *set of a natural number* ***x*** **(*x*** $\in \mathcal{N}$ **(0⊂NT-PA))** *is defined (called, renamed, nicknamed) :*

A. "<u>Even primordial of x</u>",

$$e\text{-}\mathcal{PN}_0(x) := \{ x \in \mathcal{E}ven(x) \wedge \mathcal{PN}_0(x) \mid x \in \mathcal{N}_{(0\subset NT\text{-}PA)} \},\ \forall x \ \textbf{(def-P. 3)}$$

&

$$e\text{-}\mathcal{PN}_0(x) := \{ x \in \mathcal{E}ven(x) \cap \mathcal{PN}_0(x) \mid x \in \mathcal{N}_{(0\subset NT\text{-}PA)} \},\ \forall x \ \textbf{(def-P. 3bis)}$$

B. "<u>Odd primordial of x</u>"

$$o\text{-}\mathcal{PN}_0(x) := \{ x \in \mathcal{O}dd(x) \wedge \mathcal{PN}_0(x) \mid x \in \mathcal{N}_{(0\subset NT\text{-}PA)} \} \forall x \ \textbf{(def-P. 3-tris)}$$

&

$$o\text{-}\mathcal{PN}_0(x) := \{ x \in \mathcal{O}dd(x) \cap \mathcal{PN}_0(x) \mid x \in \mathcal{N}_{(0\subset NT\text{-}PA)} \} \forall x \ \textbf{(def-P. 3-tetra)}$$

Therefore, according to the theorem **NT-th. 5,** the next wfs and wfs-r statement is a new definition of the set $\mathcal{N}$**(0⊂NT-PA):**

$$\mathcal{N}_{(0\subset PA)} := [\, [e\text{-}\mathcal{PN}_0(x)] \cup [o\text{-}\mathcal{PN}_0(x)] \,]_{\forall x\, (x \in \mathcal{N}(0\subset NT\text{-}PA)}$$

Hence,

1. In a covariant presentation of $\mathcal{N}$**(0⊂NT-PA)** ***by means of*** $\mathcal{PN}_0(x)$ ***referred to zero (0*** $\in e\text{-}\mathcal{PN}_0(0)$**):**

$\mathcal{N}^{\alpha}$**(0⊂NT-PA)** $:= \{\mathcal{PN}_0(0),\ \mathcal{PN}_0(1),\ \mathcal{PN}_0(2),\ \mathcal{PN}_0(3),\ \mathcal{PN}_0(4),\ \ldots,\ \mathcal{PN}_0(x)\}$

$:= \{\mathcal{PN}_0(0),\ \mathcal{PN}_1(0),\ \mathcal{PN}_2(0),\ \mathcal{PN}_3(0),\ \mathcal{PN}_4(0),\ \ldots, \mathcal{PN}_x(0)\}$

$:= \{\ 0, 1, 2, 3, 4, \ldots, x\ \}$

2. In a contravariant representation of $\mathcal{N}$**(0⊂NT-PA)** ***by means of*** $\mathcal{PN}_0(x)$ ***referred to zero*** $(0 \in e\text{-}\mathcal{PN}_0(0))$**:**

$\mathcal{N}^{\beta}$**(0⊂NT-PA)** $:= \{\ \mathcal{PN}_0(x),\ \ldots,\ \mathcal{PN}_0(4),\ \mathcal{PN}_0(3),\ \mathcal{PN}_0(2), \mathcal{PN}_0(1),\ \mathcal{PN}_0(0)\}$

$:=\{\mathcal{PN}_x(0)\ \ldots,\ \mathcal{PN}_4(0),\ \mathcal{PN}_3(0),\ \mathcal{PN}_2(0),\ \mathcal{PN}_1(0),\ \mathcal{PN}_0(0)\ \}$

$:= \{\ x, \ldots, 4, 3, 2, 1, 0\ \}$

Only,

$$\mathcal{PN}^{\alpha}{}_0(0) = \mathcal{PN}^{\beta}{}_x(0) \wedge \mathcal{PN}^{\alpha}{}_x(0) = \mathcal{PN}^{\beta}{}_0(0) \Rightarrow$$

$$\mathcal{PN}^{\alpha}{}_0(x) = \mathcal{PN}^{\beta}{}_x(0), \forall x\ (x \in \mathcal{N}\ (0\subset \text{NT-PA}))$$

Then we are able to state:

$\forall x,\ (x \in \mathcal{N}\ (0\subset \text{NT-PA}))$

$\mathcal{PN}^{\alpha}{}_0(x) = \mathcal{PN}^{\beta}{}_x(0)$ **has the parity of** $x\ (x \in \mathcal{N}(0\subset \text{NT-PA}))$**, if and only if,**

$\mathcal{PN}^{\alpha}{}_0(x+1) = \mathcal{PN}^{\beta}{}_{x+1}(0)$ **has parity of** ***x+1, which is the 1-succesor of x (its direct successor), and vice versa***

<u>HLParity-1-NT Theorem</u>:

$\forall x\ (\ x \in \mathcal{N}\ (0\subset \text{NT-PA}))$

$$e\text{-}\mathcal{PN}_0\ ((x+x)+0) \subseteq \mathcal{E}ven(\mathcal{S}uc_{x+x}(0)) \wedge \mathcal{E}ven(\mathcal{S}uc_0(x+x))$$

$$\wedge$$

$$o\text{-}\mathcal{PN}_0((x+x)+1) \subseteq \mathcal{O}dd(\mathcal{S}uc_{x+x}(1)) \wedge \mathcal{O}dd\ (\mathcal{S}uc_0(x+x)+1)) \wedge \mathcal{O}dd\ (\mathcal{S}uc_1(x+x)\ +0))$$

Additionally;

1. $\forall x\ (\ x \in \mathcal{N}\ (0\subset \text{NT-PA}))$

$$1 = \#\text{-}\mathcal{PN}_0(x) \wedge \#\text{-}\mathcal{PN}_{x+1}(0) \wedge \#\text{-}\mathcal{PN}_{x+2}(0) \wedge \ldots\ \#\text{-}\mathcal{PN}_{x+x}(0) \wedge \ldots$$

$$2 = \#\text{-}\mathcal{PN}_0(x) + \#\text{-}\mathcal{PN}_0\ (x+1)$$

$$3 = \#\text{-}\mathcal{PN}_0(x) + \#\text{-}\mathcal{PN}_0\ (x+1) + \#\text{-}\mathcal{PN}_0\ (x+2)$$

... ...

$\mathbf{x+1} = \#\text{-}\mathcal{PN}_0(x+0) + \#\text{-}\mathcal{PN}_0(x+1) + \#\text{-}\mathcal{PN}_0(x+2) + \dots + \#\text{-}\mathcal{PN}_0(x+x)$

2. In first order NT-FS&L,

$\forall \mathbf{x}(\mathbf{x} \in \mathcal{N}(\mathbf{0 \subset NT\text{-}PA}))$

$\mathcal{N}^{\alpha}(\mathbf{0 \subset NT\text{-}PA}) := \{\mathbf{0}, \mathbf{1}, 2, 3, 4, \dots, x+\mathbf{0}, x+\mathbf{1}, \dots, \mathbf{e\text{-}(x+x)}\}$ ***(a.1);***

$\forall \mathbf{x}(\mathbf{x} \in \mathcal{N}(\mathbf{0 \not\subset NT\text{-}PA}))$

$\mathcal{N}^{\alpha}(\mathbf{0 \not\subset NT\text{-}PA}) := \{\mathbf{1}, 2, 3, 4, 5, \dots, \mathbf{x+1}, \mathbf{x}+2, \dots, \mathbf{o\text{-}(x+x+1)}\}$ ***(b.1);***

Hence, the next statement is a theorem:

$\forall \mathbf{x}(\mathbf{x} \in \mathcal{N}(\mathbf{0 \subset NT\text{-}PA}))$

$e\text{-}\mathcal{PN}_{\mathbf{0}}((x+x)+\mathbf{0})$ *is an even primordial*

&

$o\text{-}\mathcal{PN}_0((x+x)+\mathbf{1})$ *is an odd primordial*

(NT-P-th. 0)

The distribution of even and odd natural numbers (e-x and o-x, respectively) in NLSF&L PA according to the complete partition named as "parity" is: [68]

$\forall \mathbf{x}(\mathbf{x} \in \mathcal{N}(\mathbf{0 \subset NT\text{-}PA}))$

$\mathcal{N}^{\alpha}(\mathbf{0 \subset NT\text{-}PA}) = \{ 0, 1, 2, 3, 4, 5, 6, 7, 9, 10, 11, \dots, \dots \} \rightarrow$

$\mathcal{N}^{\alpha}(\mathbf{0 \subset NT\text{-}PA}) = \{ S_0(0), S_1(0), S_2(0), S_3(0), S_4(0), \dots, \dots \} \land$
$\{ S_0(0), S_0(1), S_0(2), S_0(3), S_0(4), \dots, \dots \} \land$
$\{ P_0(0), P_0(1), P_0(2), P_0(3), P_0(4), \dots, \dots \} \Rightarrow$

$\mathcal{N}^{\alpha}(\mathbf{0 \subset NT\text{-}PA}) = \{ (0+0), (0+0)+1), (1+1), ((1+1)+1), (2+2), ((2+2)+1),$
$(3+3), ((3+3),+1), (4+4), ((4+4),+1), \dots, \dots \} \rightarrow$

$\mathcal{N}^{\alpha}(\mathbf{0 \subset NT\text{-}PA}) = \{\dots ((x)+(x)), (x+x)+1), \dots,$
$\dots, ((x+x)+(x+x)), ((x+x+x+x)+1), \dots,$
$\dots, ((x+x+x)+(x+x+x), ((x+x+x+x+x+x)+1), \dots,$
$\dots, ((x+x+x+x)+(x+x+x+x)), ((x+x+x+x+x+x+x+x)+1) \dots, \dots \} \Rightarrow$

$\mathcal{N}^{\alpha}(\mathbf{0 \subset NT\text{-}PA}) = \{((1\cdot x)+(1\cdot x)), (2\cdot x)+1), \dots,$
$\dots, ((2\cdot x)+(2\cdot x)), (4\cdot x)+1), \dots,$
$\dots, ((3\cdot x)+(3\cdot x)), (6\cdot x)+1), \dots,$

$$\ldots, ((4\cdot x)+(4\cdot x)), (8\cdot x)+1) \ldots, \ldots \}^{69} \rightarrow$$

$$\mathcal{N}^{\alpha}{}_{(0\subset NT\text{-}PA)} = \{ e\text{-}(0+0), o\text{-}(0+0)+1),$$
$$e\text{-}(1+1),\ o\text{-}((1+1)+1),$$
$$e\text{-}(2+2),\ o\text{-}((2+2)+1),$$
$$e\text{-}(3+3),\ o\text{-}((3+3),+1), \ldots, \ldots \} \Rightarrow$$

$$\mathcal{N}^{\alpha}{}_{(0\subset NT\text{-}PA)} := \{ e\text{-}(0), o\text{-}(1), e\text{-}(2), o\text{-}(3), e\text{-}(4), o\text{-}(5), e\text{-}(6), o\text{-}(7), \ldots, \ldots \}$$

$$\wedge$$

$$\mathcal{N}^{\alpha}{}_{(0\subset NT\text{-}PA)} = \{e\text{-}S_0(0),\ o\text{-}S_0(1), e\text{-}S_0(2), o\text{-}S_0(3), e\text{-}S_0(4), \ldots, \ldots \} \wedge$$
$$\{e\text{-}P_0(0), o\text{-}P_0(1), e\text{-}P_0(2), o\text{-}P_0(3), e\text{-}P_0(4), \ldots, \ldots \} \wedge$$
$$\{e\text{-}S_0(0), o\text{-}S_1(0),\ e\text{-}S_2(0), o\text{-}S_3(0),\ e\text{-}S_4(0), \ldots, \ldots \}$$

$$\wedge$$

$$\mathcal{N}^{\alpha}{}_{(0\subset NT\text{-}PA)} = \{ e\text{-}0, o\text{-}1, e\text{-}2, o\text{-}3, e\text{-}4, o\text{-}5, e\text{-}6, 7, 9, 10, 11, \ldots, \ldots \}$$

$$\wedge$$

$$\mathcal{N}^{\alpha}{}_{(0\subset NT\text{-}PA)} = \{ \text{even, odd, even, odd, even, odd, even, odd}, \ldots, \ldots \}$$

<u>HLParity-1-NT Theorem</u>:

$$\forall x \forall y\ (x, y \in \mathcal{N}_{(0\subset NT\text{-}PA)})$$

$$\mathcal{PN}_0(x) = \mathcal{PN}_y(0) \rightarrow y = x$$

&

$$((x\in \mathcal{E}ven(\mathcal{S}uc_0(x)) \Rightarrow y\in \mathcal{O}dd\,(\mathcal{S}uc_0(x)) \rightarrow (x\in \mathcal{O}dd\,(\mathcal{S}uc_0(x)) \Rightarrow y\in \mathcal{E}ven(\mathcal{S}uc_0(x))$$

$$\vee$$

$$(x\in \mathcal{O}dd(\mathcal{S}uc_0(x)) \Rightarrow y\in \mathcal{E}ven\,(\mathcal{S}uc_0(x)) \rightarrow (x\in \mathcal{E}ven\,(\mathcal{S}uc_0(x)) \Rightarrow y\in \mathcal{O}dd(\mathcal{S}uc_0(x)))$$

&

$$(\sum_{x}^{x+x} [\,\#\text{-}\mathcal{PN}_0\,(x)\,] = y \rightarrow y = x+1, y \in \mathcal{N}_{(0\subset NT\text{-}PA)} \cap \mathcal{N}_{(0\not\subset NT\text{-}PA)})$$

(NT-P-th. 1)

<u>HLParity-2-NT Theorem</u>:

$$\forall x\ \forall y\ \forall\varnothing\ (x, y \in \mathcal{N}_{(0\subset NT\text{-}PA)}, \varnothing \in \Theta)$$

&

$$\forall x \forall y(x, y \in \mathcal{S}uc_x(0) \wedge \mathcal{S}uc_0(x) \wedge \mathcal{P}re_0(x))$$

$$\mathcal{E}ven(\mathcal{S}uc_{0}(0)) = \{\, 0 \mid 0 \in \mathcal{N}\,(0\subset \text{NT-PA}) \,\} \;\&\; \mathcal{O}dd(\mathcal{S}uc_{0}(0)) \subseteq \{\, \varnothing \mid \varnothing \in \Theta \wedge \varnothing \notin \mathcal{N}\,(0\subset \text{NT-PA}) \},$$

Then:

1. $(\forall x \in \mathcal{N}\,(0\subset \text{NT-PA}))$

$$(\, x \in \mathcal{O}dd(x) \rightarrow \#\text{-}\ \mathcal{E}ven(\mathcal{S}uc_{x}(0)) = \#\text{-}\ \mathcal{O}dd(\mathcal{S}uc_{x}(0)) + 0)$$

$$(\, x \in \mathcal{E}ven(x) \rightarrow \#\text{-}\ \mathcal{E}ven(\mathcal{S}uc_{x}(0)) = \#\text{-}\ \mathcal{O}dd(\mathcal{S}uc_{x}(0)) + 1)$$

(NT-P-th. 2)

2. $(\forall x \in \mathcal{N}(0\subset \text{NT-PA}) \Rightarrow (S_{x}(x+x) \wedge S_{x+x}(x) = S_{0}(x+x+x) \wedge S_{x+x+x}(0) \in \mathcal{S}uc_{0}(x))$

$$S_{0}((x+x)+x) \wedge S_{(x+x)+x}(0) \in \mathcal{O}dd(\mathcal{S}uc_{0}(x+x+x)) \rightarrow x \in \mathcal{O}dd(\mathcal{S}uc_{0}(x))$$

$$\vee$$

$$S_{0}(x+x+x) \wedge S_{x+x+x}(0) \in \mathcal{E}ven\,(\mathcal{S}uc_{0}(x+x+x)) \rightarrow x \in \mathcal{E}ven\,(\mathcal{S}uc_{0}(x))$$

(NT-P-th. 2 bis)

3. $\forall x \forall y\ (x, y \in \mathcal{N}(0\subset \text{NT-PA}))$

$$(y = x + 1 \in \mathcal{N}(0\subset \text{NT-PA}) \Rightarrow \text{e-}S_{0}(y+y) \in \mathcal{E}ven\,(\mathcal{S}uc_{0}(y)) \rightarrow$$

$$S_{(x+2)}(x+0) \wedge S_{(x+1)}(x+1) \wedge S_{(x)}(x+2) \in \mathcal{E}ven\,(\mathcal{S}uc_{0}(x))$$

$$\&$$

$$(P_{0}(x+x+2) = P_{0}((x+1)+(x+1)) \wedge S_{0}((x+1)+(x+1)) \wedge S_{(x+1)}(x+1)$$
$$\in \mathcal{E}ven\,(\mathcal{S}uc_{0}(x)))$$

(NT-P-th. 3 tris)

Hence, we are able to state:[70]

$\forall x \forall y\ (x, y \in \mathcal{N}(0\subset \text{NT-PA}))$

$$((S_{x}(0) + S_{0}(0) = ((x + x) \wedge (x \cdot y + x)) \rightarrow x = S_{x}(0) \wedge y = S_{0}(0) \wedge 0)$$

$$\&$$

$$(S_{x}(0) + S_{x}(0) = ((x + x) \wedge (x \cdot y + x)) \rightarrow x = S_{x}(0) \;\&\; y = S_{1}(0) \wedge 1)$$

$$\&$$

$$((S_{x}(0) + S_{x}(0) \wedge S_{x}(0) \cdot S_{y+1}(0) \wedge S_{x}(0) \cdot S_{0}(y+1) \wedge S_{x}(0) \cdot S_{y}(1)) =$$
$$((x + x) \wedge (x \cdot y + x) \wedge (x \cdot (y+1))) \rightarrow$$
$$x = S_{x}(0) \;\&\; y+1 = S_{1}(0) + S_{1}(0) \wedge 1 + 1 \Rightarrow y = S_{2}(0) \wedge 2)$$

$$x = S_{0}(x) \;\&\; y = 1 \wedge S_{1}(0) \vee x = S_{0}(x) \;\&\; y+1 = 2 \wedge S_{1}(1)$$

Thus,

$$x \Rightarrow x\cdot y+x \wedge x\cdot(y+1) \Rightarrow x\cdot S_1(x) \wedge x+x \rightarrow S_x(0)\cdot(S_1(0)+S_1(0))=S_x(0)\cdot S_2(0)=S_x(0)\cdot(2)$$

$$0 \Rightarrow 0\cdot y+0 \wedge 0\cdot(y+1) \Rightarrow 0\cdot S_1(x) \wedge 0+0 \rightarrow S_0(0)\cdot(S_0(1)+S_1(0))= S_0(0)\cdot S_2(0)=(2)\cdot(0)$$
$$1 \Rightarrow 1\cdot y+1 \wedge 1\cdot(y+1) \Rightarrow 1\cdot S_1(x) \wedge 1+1 \rightarrow S_1(0)\cdot(S_0(1)+S_1(0))= S_1(0)\cdot S_2(0)=(2)\cdot(1)$$
$$2 \Rightarrow 2\cdot y+2 \wedge 2\cdot(y+1) \Rightarrow 2\cdot S_1(x) \wedge 2+2 \rightarrow S_2(0)\cdot(S_0(1)+S_1(0))= S_2(0)\cdot S_2(0)=(2)\cdot(2)$$
$$3 \Rightarrow 3\cdot y+3 \wedge 3\cdot(y+1) \Rightarrow 3\cdot S_1(x) \wedge 3+3 \rightarrow S_3(0)\cdot(S_0(1)+S_1(0))= S_3(0)\cdot S_2(0)=(2)\cdot(3)$$
$$4 \Rightarrow 4\cdot y+4 \wedge 4\cdot(y+1) \Rightarrow 4\cdot S_1(x) \wedge 4+4 \rightarrow S_4(0)\cdot(S_0(1)+S_1(0))= S_4(0)\cdot S_2(0)=(2)\cdot(4)$$
$$5 \Rightarrow 5\cdot y+5 \wedge 5\cdot(y+1) \Rightarrow 5\cdot S_1(x) \wedge 5+5 \rightarrow S_5(0)\cdot(S_0(1)+S_1(0))= S_5(0)\cdot S_2(0)=(2)\cdot(5)$$
$$\ldots\ldots\ldots \quad \ldots\ldots\ldots \quad \ldots\ldots\ldots \quad \ldots\ldots\ldots \quad \ldots\ldots\ldots$$
$$n \Rightarrow x\cdot y+x \wedge x\cdot(y+1) \Rightarrow x\cdot S_1(x) \wedge x+x \rightarrow S_x(0)\cdot(S_1(0)+S_1(0))=S_x(0)\cdot S_2(0)=S_x(0)\cdot(2)$$
$$\&$$
$$n \Rightarrow x\cdot y+x \wedge x\cdot(y+1) \Rightarrow x\cdot P_1(x) \wedge x+x \rightarrow P_0(x)\cdot(P_0(1)+P_0(1))=P_0(x)\cdot P_2(0)=P_0(x)\cdot(2)$$

and, then in NT-FS&L, next logical substitution set operations are allowed:

$$0 \Rightarrow 0\cdot y+0 \wedge 0\cdot(y+1) \Rightarrow (0\cdot(x+1) \wedge 0+0) \rightarrow (0\cdot(1+1)) = (0\cdot 2)=(2\cdot 0) \ \&$$
$$1 \Rightarrow 1\cdot y+1 \wedge 1\cdot(y+1) \Rightarrow (1\cdot(x+1) \wedge 1+1) \rightarrow (1\cdot(1+1)) = (1\cdot 2)=(2\cdot 1) \ \&$$
$$2 \Rightarrow 2\cdot y+2 \wedge 2\cdot(y+1) \Rightarrow (2\cdot(x+1) \wedge 2+2) \rightarrow (2\cdot(1+1)) = (2\cdot 2)=(2\cdot 2) \ \&$$
$$3 \Rightarrow 3\cdot y+3 \wedge 3\cdot(y+1) \Rightarrow (3\cdot(x+1) \wedge 3+3) \rightarrow (3\cdot(1+1)) = (3\cdot 2)=(2\cdot 3) \ \&$$
$$4 \Rightarrow 4\cdot y+4 \wedge 4\cdot(y+1) \Rightarrow (4\cdot(x+1) \wedge 4+4) \rightarrow (4\cdot(1+1)) = (4\cdot 2)=(2\cdot 4) \ \&$$
$$5 \Rightarrow 5\cdot y+5 \wedge 5\cdot(y+1) \Rightarrow (5\cdot(x+1) \wedge 5+5) \rightarrow (5\cdot(1+1)) = (5\cdot 2)=(2\cdot 5) \ \&$$
$$\ldots\ldots\ldots \quad \ldots\ldots\ldots \quad \ldots\ldots\ldots \quad \ldots\ldots\ldots \quad \ldots\ldots\ldots \quad \&$$
$$n \Rightarrow x\cdot y+x \wedge x\cdot(y+1) \Rightarrow ((x\cdot S_1(x) \wedge (x+x)) \rightarrow (S_x(0)\cdot(1+1)) = (S_x(0)\cdot 2)=(2\cdot S_x(0))$$
$$\&$$
$$n \Rightarrow x\cdot y+x \wedge x\cdot(y+1) \Rightarrow ((x\cdot P_0(x+1)) \wedge (x+x)) \rightarrow (P_0(x)\cdot(1+1)) = (P_0(x)\cdot 2)=(2\cdot P_0(x))$$

Then, the next w.f. formula in NT-FS&L is representing ***the accumulated addition of n+1 natural numbers referred to zero****, which in turn, are both: the elements successors of* ***natural number zero*** *(**0**) and the predecessor of a* ***natural number n****:*

$$\sum_{x=0}^{x=n}[(x)] = \Big(S_0(0)+S_0(1)+\ldots + S_0(n)\big) \wedge \big(P_0(0)+P_0(1)+\ldots + P_0(n)\big)$$
$$\wedge$$
$$\big(S_0(0)+S_0(1)+\ldots + S_0(n)\big)$$

Please, allow us to point out, that in NT-FS&L:

$$0 = S_0(0) \wedge P_0(0) \ \& \ 1 = S_0(1) \wedge P_0(1) \ \rightarrow \ \sum_{x=0}^{(x)=(n)}[(x)] = \sum_{x=1}^{(x)=(n)}[(x)]$$

SECOND FOUNDAMENTAL NT-FS&L THEOREM

$\forall n \forall x \forall y \ (n, x \in \mathcal{N}(0 \subset \text{NT-PA}), \ y \in \mathcal{N}(0 \subset \text{NT-PA}))$

$$\left((x \cdot y) = \left(\sum_{x=0}^{x=n}[(x)] + \sum_{x=0}^{x=n}[(x)]\right) \rightarrow y = 2 \wedge S_0(1) + S_1(0)\right)$$

$$\wedge$$

$$\left((x \cdot y) = \left(\sum_{x=0}^{x=n}[(x)] + \sum_{x=0}^{x=n}[(x)]\right)\right) \rightarrow y = S_0(0) + S_1(x)\Big)$$

$$\wedge$$

$$\left((x \cdot y) = \left(\sum_{x=1}^{x=n}[(x)] + \sum_{x=1}^{x=n}[(x)]\right) \rightarrow y = 2 \wedge S_0(1) + P_0(1)\right)$$

$$\wedge$$

$$\left((x \cdot y) = \left(\sum_{x=1}^{x=n}[(x)] + \sum_{x=1}^{x=n}[(x)]\right) \rightarrow y = 2 \wedge P_0(1) + P_0(1)\right)$$

$$\wedge$$

$$\left((x \cdot y) = \left(2 \cdot \sum_{x=0}^{x=n}[(x)]\right) \rightarrow y = n+1 \in \mathcal{N}(0 \subset \text{PA}) \cap \mathcal{N}(0 \not\subset \text{NT-PA}) \wedge \mathit{Sucd}(n)) \wedge \mathcal{N}(0 \subset \text{NT-PA})\right)$$

$$\forall y \in \mathcal{N}(0 \subset \text{PA}) \cap \mathcal{N}(0 \not\subset \text{NT-PA}) \wedge \mathit{Sucd}(n)) \wedge \mathcal{N}(0 \subset \text{NT-PA})$$

(NT-FS&L F.Th. 2) [71]

Please, allow us to point out that:

$\mathcal{N}(0 \subset \text{NT-PA}) := \{0,1,2,3,4,\ldots,x+0,x+1, x+2, x+3,\ldots,(x+x)+0, (x+x)+1,\ldots\}_{x \in \mathcal{N}[0 \subset \text{NT-PA}]}$

*(Entailment from the **axiom 7 of PA** expressed in first order logic; "the substitution operation" is intended embedded in every first order logic statement in NT-FS&L)*

Then,

$$\mathit{Subsets}\text{-}\mathcal{N}(0 \subset \text{NT-PA}) := \{0,1,2,3,4,\ldots, x\}_{x \in \mathcal{N}[0 \subset \text{NT-PA}]}$$

$x=(n+0) \rightarrow \{0,1,2,3,4, \ldots,(n+0)\} \wedge \#\text{-}\mathit{Subset}\text{-}\mathcal{N}(0 \subset \text{NT-PA}) = n+1$

$x=(n+1) \rightarrow \{0,1,2,3,4, \ldots,(n+1)\} \wedge \#\text{-}\mathit{Subset}\text{-}\mathcal{N}(0 \subset \text{NT-PA}) = n+2$

$x=(n+2) \rightarrow \{0,1,2,3,4, \ldots,(n+2)\} \wedge \#\text{-}\mathit{Subset}\text{-}\mathcal{N}(0 \subset \text{NT-PA}) = n+3$

…… ………………. ………………. …………

$x=(n+n)+0 \rightarrow \{0,1,2,3,4, \ldots, (n+n)+0\} \wedge \#\text{-}\mathit{Subset}\text{-}\mathcal{N}(0 \subset \text{NT-PA}) = (n+n)+1 = 2 \cdot n+1$

$x=(n+n)+1 \rightarrow \{0,1,2,3,4, \ldots, (n+n)+1\} \wedge \#\text{-}\mathit{Subset}\text{-}\mathcal{N}(0 \subset \text{NT-PA}) = (n+n)+2 = 2 \cdot n+2$

x=(n+n)+2 →{0,1,2,3,4, ..., (n+n)+2} ∧#-*Subset-𝒩*(0⊂NT-PA) = (n+n)+3 = 2·n+3

…… ……………… ……………… ……… ………………… ……………… …………

x=(n+n)+n→{0,1,2,3,4, ..., (n+n)+n}∧#-*Subset-𝒩*(0⊂NT-PA) =(n+n+n)+1= 3·n+1

…… ……………… ……………… ……… ………………… ……………… …………

x=((n+n)+((n+n)+0)+0) → {0,1,2,3,4, ..., ((n+n+n+n))+0)} ∧
#-*Subset-𝒩*(0⊂NT-PA) = ((n+n+n+n)+0)= 4·n+1

x=((n+n)+((n+n)+0)+1) → {0,1,2,3,4, ..., ((n+n+n+n))+1)} ∧
#-*Subset-𝒩*(0⊂NT-PA)= ((n+n+n+n))+1)= 4·n+2

x=((n+n)+((n+n)+0)+2) → {0,1,2,3,4, ..., ((n+n+n+n))+2)} ∧
#-*Subset-𝒩*(0⊂NT-PA)= ((n+n+n+n)+2)= 4·n+3

Additionally, please note the following:

(*Subset-𝒩*(0⊂NT-PA))=({0,1,2,3,4, ..., (x+n)})⊆ (*Subset-𝒩*(0⊂NT-PA)= {0,1,2,3,4, ...,(2·n+0)})

({0,1,2,3,4, ...,(2·n+0)}) = ({0,1,2,3,4, ..., (x)}) ∧ (x)= (n)
({0,1,2,3,4, ...,(2·n+1)}) = ({0,1,2,3,4, ..., (x+1)}) ∧ (x)= ((n)+(n+1))
({0,1,2,3,4, ...,(2·n+2)}) = ({0,1,2,3,4, ..., (x+2)}) ∧ (x) = (((n)+(n+1))+1)
({0,1,2,3,4, ...,(2·n+3)}) = ({0,1,2,3,4, ..., (x+3)}) ∧ (x) = ((((n)+(n+1))+1)+1)

Finally, [72]

1.Case: *Subset-𝒩*[0, 1 ⊂NT-PA] ⊆ 𝒩(0⊂NT-PA)):=
{0, 1, 2, 3, 4, 5, ..., x }∀x=n ∈𝒩[0⊂NTPA];
#-*subset-𝒩*[0, 1 ⊂NT-PA] ∀x=n ∈𝒩[0⊂NTPA] = **n+1**

$$2\cdot\sum_{x=0}^{(x)=(n)} [(x)] = 2\cdot(0+1+2+3+4+5 \ldots + (n)) \wedge x\cdot(2\cdot n+1) \rightarrow x = S_0(2\cdot n+0)$$

2.Case: *subset-𝒩*[0,1⊂NT-PA] ⊆𝒩(0⊂NT-PA)):=
{ 0, 1, 2, 3, 4, 5, ..., x }∀x=n+(n+1) ∈𝒩[0⊂NTPA];
#-*subset-𝒩*[0, 1 ⊂NT-PA] ∀x=n+(n+1) ∈𝒩[0⊂NTPA]= **(n+(n+1)+1)= 2n+2**

$$2\cdot\sum_{x=0}^{(x)=(n+(n+1))} [(x)] = 2\cdot(0+1+2+3+4+5 \ldots + (n+(n+1))) \wedge (x)\cdot(2\cdot n+2) \rightarrow x = S_0(2n+1)$$

3.Case: *subset-𝒩*[0,1⊂NT-PA]⊆𝒩(0⊂NT-PA)):=
{0, 1, 2, 3, 4, 5, ..., x }∀x=(n+(n+1))+1∈𝒩[0⊂NTPA];

$\#\text{-}subset\text{-}\mathcal{N}[0, 1 \subset \text{NT-PA}]\ \forall x=(n+(n+1))+1 \in \mathcal{N}[0 \subset \text{NTPA}] = ((n+(n+1)+1))+1) = 2n+3$

$$2\cdot\sum_{x=0}^{(x)=(n+(n+1))+1} [(x)] = 2\cdot(0+1+2+3+4+5....+((n+(n+1))+(n+2)) \wedge (x)\cdot(2\cdot n+3) \rightarrow x=S_0(2n+2)$$

Hence, by means of the corresponding NT-FS&L notation and formulation of a natural number, which is member of $\mathcal{N}$ ***(0⊂NT-PA)*** *holding the identities of successor and/or predecessor, we are able to state:*

$$((P_0(x))\cdot(S_y(0)) = \sum[(P_0(x))(S_0(y)] + \sum((P_0(x))(S_0(y))) \rightarrow$$

$$P_0(y) = P_0(x+1) \in \mathcal{N}(0 \subset PA)) \cap \mathcal{N}(0 \not\subset PA))$$

&

$$(2\cdot\sum((P_0(x))(S_0(y)) = (P_0(x))\cdot(S_0(y)) \rightarrow S_0(y) = P_0(x+1) \in \mathcal{N}(0 \subset PA) \cap \mathcal{N}(0 \not\subset PA))$$

4.Case: $(x = S_0(x) \wedge y = S_1(x) \wedge (x+1), \forall x \forall y\ (x,y \in \mathcal{N}(0 \subset \text{NT-PA}))$ **&** *Entailment from First Fundamental NT-FS&L Theorem)***)**

$$(x+x) = S_0(x)+S_0(x) = S_0(x+x) \wedge S_x(x) \wedge S_{x+x}(0) \rightarrow 2\cdot x = 2\cdot S_0(x) = S_0(x+x)$$

$$(x+1)+(x+1) = S_1(x)+S_1(x) = S_0(x+x+1+1) \wedge S_0(x+1+x+1) \wedge S_{x+1}(x+1)$$
$$S_{1+1}(x+x) \wedge S_1(x+x+1) \wedge S_{x+x+1}(1) \rightarrow S_1(x+x+1) \wedge S_1(x+x)+1 \wedge S_0(x+x)+2 \wedge$$
$$S_0(2x+2) \wedge 2\cdot S_0(x+1) \wedge 2\cdot S_0(x+1) \wedge 2\cdot S_1(x) \wedge S_2(x+x)$$

$$S_0(x)\cdot S_0(x) = (x\cdot x) = S_0(x\cdot x)\ \&$$
$$x\cdot S_0(x) = x\cdot(x) \wedge x\cdot x + 0 \rightarrow x = S_0(x) \wedge y = S_1(x)$$
$$x\cdot S_0(x) + x\cdot S_0(x) = (x\cdot x + 0) + (x\cdot x + 0) \wedge x\cdot x + x\cdot x) = S_0(x\cdot x + x\cdot x) = 2\cdot S_0(x\cdot x)$$
$$2\cdot x\cdot S_0(x) = 2\cdot S_0(x\cdot x) \rightarrow x\cdot S_0(x) = S_0(x\cdot x)$$
$$x\cdot S_0(x) + x\cdot S_0(x) = S_0(x)\cdot(x+x) \wedge x\cdot(S_0(x)+S_0(x)) \wedge x\cdot S_0(x)(1+1) \wedge 2\cdot x\cdot S_0(x)$$

$$x\cdot(x+1) = x\cdot S_1(x) \wedge x\cdot x + x \rightarrow S_0(x\cdot x) + x = S_x(x\cdot x) \wedge S_0(x\cdot x)+S_0(x)$$

(1.) $x\cdot S_1(x) + x\cdot S_1(x) = (x\cdot x + x) + (x\cdot x + x) \wedge x\cdot x\,(x\cdot x + 1) \rightarrow$
$(x\cdot x)\cdot S_0(x\cdot x+1) \wedge (x\cdot x)\cdot(S_0(x\cdot x)+1) \wedge (x\cdot x)\cdot S_1(x\cdot x)$

(2.) $x\cdot S_1(x) + x\cdot S_1(x) = (x\cdot x + x) + (x\cdot x + x) \wedge (2\cdot x\cdot x + 2\cdot x) \wedge 2\cdot x\cdot(x+1) \rightarrow$
$2\cdot S_0(x)\cdot(x+1) \wedge 2\cdot S_x(0)\cdot(x+1) \wedge 2\cdot S_0(x)\cdot S_1(x)$

(3.) $x\cdot S_1(x) + x\cdot S_1(x) = 2\cdot(x\cdot S_1(x)) \wedge 2\cdot(S_0(x)\cdot S_1(x)) \rightarrow x\cdot S_1(x) = S_0(x)\cdot S_1(x)$

(1 & 2 & 3) $x\cdot S_1(x) + x\cdot S_1(x) = (x\cdot x)\cdot S_1(x\cdot x) \wedge 2\cdot S_0(x)\cdot S_1(x)$

$$(x+1)\cdot(x+1) \wedge S_1(x)\cdot S_1(x) = (x\cdot x + 1\cdot x + 1\cdot x + 1\cdot 1) \wedge (x\cdot x + 1\cdot x + 1) + (x\cdot x + 1\cdot x + 1)$$
$$= x\cdot S_1(x) + x\cdot S_1(x) \wedge x\cdot(S_1(x) + S_1(x)) \wedge x\cdot(S_1(x+x))$$
$$(x+1)\cdot(x+1) \wedge S_1(x)\cdot S_1(x) = (x\cdot x + 1\cdot x + 1\cdot x + 1\cdot 1) \wedge (x\cdot x + x + 1) =$$
$$S_1(x\cdot x + x) \wedge S_0(x\cdot x + x)+1 \rightarrow S_0(x\cdot x)+ S_0(x)+1$$
$$\wedge S_0(x\cdot x)+ S_1(x) \wedge S_1(x\cdot x)+ S_0(x)$$

$$(x+x)\cdot(x+x) \wedge S_1(x)\cdot S_1(x) = (x\cdot x + x + x + x\cdot x) \wedge (x\cdot x + x\cdot x) + (x+x)$$
$$\wedge x(x+x) + (x+x) \wedge (x+x)\cdot(x+1) \wedge S_0(x+x)\cdot S_1(x) \wedge$$
$$2\cdot S_0(x)\cdot S_1(x) \wedge (S_0(x)\cdot S_1(x)+ S_0(x)\cdot S_1(x))$$
$$(x+x)\cdot(x+x) = (x\cdot x + x + x + x\cdot x) = S_0(x\cdot x) + S_0(x) + S_0(x) + S_0(x\cdot x) =$$
$$2\cdot S_0(x\cdot x) + 2\cdot S_0(x) = 2\cdot(S_0(x\cdot x) + S_0(x))$$

Then,

$$((x+x) = (S_0(x) + S_0(x))) \,\&\, ((y+y) = (S_0(y) + S_0(y)) \wedge (S_0(x+1) + S_0(y))$$
$$(y\cdot y) = (x+1)\cdot(x+1) \wedge S_1(x\cdot x) + S_0(x) \rightarrow$$

$$S_1(x\cdot x) = S_0(y) \wedge y = x+1$$

Based on all of the above, we can establish that:

<u>THIRD FOUNDAMENTAL NT-FS&L THEOREM</u>

(NT-PA *Pythagorean Successors & Predecessors*)

The next statement is true, which is a theorem in NT-FS&L (**Entailment from First Fundamental NT-FS&L Theorem**):

$$(\forall x\forall y\ (x,y \in \mathcal{N}(0\subset \text{NT-PA}) \mid x = S_0(x) \wedge y = S_1(x) \wedge (x+1))$$

$$\left(S_1(x+x) = S_0(y\cdot y) \rightarrow (x = S_0(x) \wedge y = S_1(x)\ \forall x\forall y(x,y \in \mathcal{N}(0\subset \text{NT-PA})\right)$$
$$\&$$
$$\left(S_{y+y}(1) = S_{x\cdot x}(0) \rightarrow (x = S_0(x) \wedge y = S_1(x)\ \forall x\forall y(x,y \in \mathcal{N}(0\subset \text{NT-PA})\right)$$
$$\&$$
$$\left(S_0(x+x) = P_1(y\cdot y) \rightarrow (x = P_0(x) \wedge x = P_1(y)\ \forall x\forall y(x,y \in \mathcal{N}(0\subset \text{NT-PA})\right)$$
$$\&$$
$$\left(P_0(x+x) = P_1(y\cdot y) \rightarrow (x = S_0(x) \wedge x = P_1(y)\ \forall x\forall y(x,y \in \mathcal{N}(0\subset \text{NT-PA})\right)$$

$$\rightarrow$$

$$\left(x = P_1(y) \wedge y = P_0(y) \wedge (x+1)\right), \forall x\forall y$$

(NT-FS&L F.Th. 3)

Additionally, please, allow us to point out that:

$\forall x \forall y$ **(x,y ∈ $\mathcal{N}$(0⊂NT-PA) &**

$$x = S_0(x) \wedge y = S_1(x) \wedge (x+1), \forall x \forall y\ (x,y \in \mathcal{N}(0 \subset NT\text{-}PA))$$

&

$$\mathcal{N}^{\alpha}(0 \subset NT\text{-}PA) = \{\text{ even, odd, even, odd, even, odd, even, odd, ..., ... }\}$$

$$\rightarrow$$

$$\big((x \in \mathcal{E}ven(\mathcal{S}uc_0(x)) \Rightarrow y = S_1(x) \in \mathcal{O}dd(\mathcal{S}uc_0(x)) \rightarrow$$
$$(x \in \mathcal{O}dd(\mathcal{S}uc_0(x)) \Rightarrow y = S_1(x) \in \mathcal{E}ven(\mathcal{S}uc_0(x))$$

$$\vee$$

$$(x \in \mathcal{O}dd(\mathcal{S}uc_0(x)) \Rightarrow y = S_1(x) \in \mathcal{E}ven\ (\mathcal{S}uc_0(x)) \rightarrow$$
$$(x \in \mathcal{E}ven\ (\mathcal{S}uc_0(x)) \Rightarrow y = S_1(x) \in \mathcal{O}dd(\mathcal{S}uc_0(x))\big)$$

&

$$((y + y) \wedge (x + x) \wedge S_0(x+x) \wedge P_0(x+x) \in \mathcal{E}ven(\mathcal{S}uc_0(x)))$$

$$x = S_0(x) \wedge y = S_1(x) \wedge (x+1), \forall x \forall y\ (x,y\ \mathcal{N}(0 \subset NT\text{-}PA))$$

$$x = S_0(x)\ \&\ S_0(x) \in \mathcal{E}ven\text{-}\mathcal{S}uc\ (S_0(x)) \Rightarrow y = x + 1\ \&\ S_1(x) \in \mathcal{O}dd\text{-}\mathcal{S}uc\ (S_0(x))$$

$$e\text{-}S_0(x) + e\text{-}S_0(x) = e\text{-}S_0(x+x) \wedge e\text{-}S_0(x+x+0) \Rightarrow S_0(x+x+0) \in \mathcal{E}ven\text{-}\mathcal{S}uc\ (S_0(x))$$
$$e\text{-}S_0(x) + o\text{-}S_1(x) = o\text{-}S_1(x+x) \wedge o\text{-}S_0(x+x+1) \Rightarrow S_0(x+x+1) \in \mathcal{O}dd\text{-}\mathcal{S}uc\ (S_0(x))$$
$$e\text{-}S_0(x) \cdot o\text{-}S_1(x) = e\text{-}S_0(x \cdot (x+1)) \wedge e\text{-}S_0(x \cdot x+x) \Rightarrow S_0(x \cdot x+x) \in \mathcal{E}ven\text{-}\mathcal{S}uc\ (S_0(x))$$

$$x = S_0(x)\ \&\ S_0(x) \in \mathcal{O}dd\text{-}\mathcal{S}uc\ (S_0(x)) \Rightarrow y = x + 1\ \&\ S_1(x) \in \mathcal{E}ven\text{-}\mathcal{S}uc\ (S_0(x))$$

$$o\text{-}S_0(x) + o\text{-}S_0(x) = e\text{-}S_0(x+x) \wedge e\text{-}S_0(x+x+0) \Rightarrow S_0(x+x+0) \in \mathcal{E}ven\text{-}\mathcal{S}uc\ (S_0(x))$$
$$o\text{-}S_0(x) + e\text{-}S_0(x) = o\text{-}S_0(x+x+1) \wedge o\text{-}S_1(x+x) \Rightarrow S_0(x+x+1) \in \mathcal{O}dd\text{-}\mathcal{S}uc\ (S_0(x))$$
$$o\text{-}S_0(x) \cdot e\text{-}S_1(x) = e\text{-}S_0(x \cdot (x+1)) \wedge e\text{-}S_0(x \cdot x+x) \Rightarrow S_0(x \cdot x+x) \in \mathcal{E}ven\text{-}\mathcal{S}uc\ (S_0(x))$$

Finally, please, allow us to point out that in $\mathcal{N}$(0⊂NT-PA)):

$$\forall x \forall y \forall n\ \big(x,y,n \in \mathcal{N}(0 \subset NT\text{-}PA) \mid x = S_0(x) = S_x(0) \wedge P_0(x), y \in \mathcal{S}uc(S_0(x)), n \in$$
$$\mathcal{S}ucc(S_0(y))\ \&\ \mathcal{E}ven(x) := \{(x+x) = (2 \cdot x) \mid x \in \mathcal{N}(0 \subset NT\text{-}PA))\}, \forall x\ \&$$
$$\mathcal{O}dd(x) := \{(x+x+1) = (2 \cdot x + 1) \mid x \in \mathcal{N}(0 \subset NT\text{-}PA))\}, \forall x\big)$$

$$\big(e\text{-}(x) + e\text{-}(y) = e\text{-}(n)\quad \&\quad e\text{-}(x) \cdot e\text{-}(y) = e\text{-}(n) \wedge$$
$$e\text{-}(x) + o\text{-}(y) = o\text{-}(n)\quad \&\quad e\text{-}(x) \cdot o\text{-}(y) = e\text{-}(n) \wedge$$

$$\text{o-}(x) + \text{e-}(y) = \text{o-}(n) \quad \& \quad \text{o-}(x)\cdot\text{e-}(y) = \text{e-}(n) \wedge$$
$$\text{o-}(x) + \text{o-}(y) = \text{e-}(n) \quad \& \quad \text{o-}(x)\cdot\text{o-}(y) = \text{o-}(n)\Big) \longrightarrow$$

$$\Big(\big(\mathcal{N}_{(0\subset NT\text{-}PA)}\big) = \Big(\{\text{e-}0, \text{o-}1, \text{e-}2, \text{o-}3,\ldots, x+(\text{e-}0), x+(\text{o-}1), x+(\text{e-}2), x+(\text{o-}3), \ldots,$$
$$\text{e-}((2\cdot x)+(\text{e-}0)), \text{o-}((2\cdot x)+(\text{o-}1)), \text{e-}((2\cdot x)+(\text{e-}2)), \text{o-}((2\cdot x)+(\text{e-}3)),$$
$$\text{e-}((2\cdot 2\cdot x)+(\text{e-}0)), \text{o-}((2\cdot 2\cdot x)+(\text{o-}1)), \text{e-}((4\cdot x)+(\text{e-}2)), \text{o-}((4\cdot x)+(\text{e-}3)), \ldots,$$
$$\text{e-}((2\cdot 2\cdot 2\cdot x)+(\text{e-}0)),\text{o-}((2\cdot 2\cdot 2\cdot x)+ (\text{o-}1)),\text{e-}((8\cdot x)+(\text{e-}2)),\text{o-}((8\cdot x)+(\text{o-}3)), \ldots,\}\forall x\Big)$$

As summary of the parity (and non-parity) NTFS&L-properties of the natural numbers and their collectivities:

***A.** Covariant (contravariant) "succession" and "precede" operations and addition operation PA´s:*

$$((P_0(x) \in \mathcal{E}ven\,(\mathcal{P}re_0(x)) \Rightarrow (P_0(x+1) \in \mathcal{O}dd\,(\mathcal{P}re_0(x))$$
$$\wedge$$
$$(S_0(x) \in \mathcal{E}ven\,(\mathcal{S}uc_0(x))) \Rightarrow (S_0(x+1) \in \mathcal{O}dd\,(\mathcal{S}uc_0(x)))$$

***B.** On covariant (contravariant) "succession" and "predecession" operations and multiplication operation PA´s:*

$$\textit{i.-}\ (P_0(x))\cdot(S_0(y)) \in \mathcal{E}ven^{\alpha}(\mathcal{S}uc_{x+y}(0)) \wedge \mathcal{E}ven^{\alpha}(\mathcal{S}uc_{x}(y)) \wedge \mathcal{E}ven^{\alpha}(\mathcal{S}uc_{y}(x))$$
$$\wedge$$
$$\mathcal{E}ven^{\beta}(\mathcal{S}uc_{x+y}(0)) \wedge \mathcal{E}ven^{\beta}(\mathcal{S}uc_{x}(y)) \wedge \mathcal{E}ven^{\beta}(\mathcal{S}uc_{y}(x)) \wedge$$
$$\mathcal{E}ven^{\alpha}(\mathcal{P}re_0(x+y)) \wedge \mathcal{E}ven^{\beta}(\mathcal{P}re(x+y))$$

$$\rightarrow \quad y = x+1 \in \mathcal{N}_{(0\subset NT\text{-}PA))} \cap \mathcal{N}_{(0\not\subset NT\text{-}PA)}\big)$$

$$\textit{ii.-}\ (P_0(x))\cdot(S_0(y))=(P_0(x))\cdot(P_0(y)) \wedge x\cdot y \rightarrow y = x+1 \in \mathcal{N}_{(0\subset NT\text{-}PA))} \cap \mathcal{N}_{(0\not\subset PA)}\big)$$
$$\textit{iii.-}(P_0(x))\cdot(S_0(y))=(P_0(x))\cdot(P_0(y)) \wedge x\cdot x + x \rightarrow y=x+1 \in \mathcal{N}_{(0\subset NT\text{-}PA))} \cap \mathcal{N}_{(0\not\subset NT\text{-}PA)}\big)$$

On the other hand and as direct consequence of both: first, "intended first logic des-embedding" of axiom 6 of PA ($\forall x \forall y(x, y \in \mathcal{N}[0\subset PA])(x \cdot S(y) = x \cdot y + x)$), and second, entailment from First Fundamental NT-FS&L Theorem; we are able to state:

$$\forall x\forall y\forall n\,\big(x, n \in \mathcal{N}[0\subset NT\text{-}PA]\big), y \in \big(\mathcal{S}uc_1(x)\big) \wedge \big(\mathcal{N}[0\subset NT\text{-}PA]\big) \cap \mathcal{N}[0\not\subset NT\text{-}PA]\big)\big)$$
$$\big((x\cdot y+x) \wedge x\cdot(y+1) \longrightarrow$$

$$\mathbf{x\cdot S_1(x) = 2\cdot\sum_{x=0}^{(x)=(n)} [(x)] = 2\cdot\big(0+1+2+3+4+5.....+(n)\big) \wedge x\cdot(2\cdot n+1) \rightarrow x= S_0\,(2\cdot n+0)\big)}$$

$$\Rightarrow$$

$$\big(\mathbf{x\cdot S_1(x)} \in \mathcal{E}ven\text{-}\mathcal{PN}_0(\mathbf{x\cdot x+x}) \cap \mathcal{E}ven\text{-}\mathcal{PN}_0(\mathbf{x\cdot(x+1)})_{\forall x})\big)$$ [73]

5. NT-FS&L CONSIDERATIONS, DEFINITIONS AND THEOREMS ABOUT PRIME AND NATURAL NUMBERS, AND ON THEIR ADDITION AND MULTIPLICATION OPERATIONS (ELEMENTS and COLLECTIVITIES)

The next set of well formed formula (also wff) and wfs-r statements, by the natural number $\mathcal{N}$**(0⊂NT-PA)** set summarizes, *"by comprehension and by extension; sense and intension")* some of the identities of the main collectivities of natural numbers found by "intentional de-embedding methodology" from NT-FOL-PA´s axiomatic (briefly referred as NT-PA):

$\forall\mathbf{x(x\in}\mathcal{N}\mathbf{(0\subset NT\text{-}PA))}$

$$\{ S_0(0), S_1(0), S_2(0), ..., S_x(0), ... \} \subseteq \mathcal{N}^{\alpha}(0\subset NT\text{-}PA)$$

$$\{ S_0(0), S_0(1), S_0(2), ..., S_0(x), ... \} \subseteq \mathcal{N}^{\alpha}(0\subset NT\text{-}PA)$$

$$\{ P_0(0), P_0(1), P_0(2), ..., P_0(x), ... \} \subseteq \mathcal{N}^{\alpha}(0\subset NT\text{-}PA)$$

$$\{ ..., P_0(x), ..., P_0(2), P_0(1), P_0(0)\} \subseteq \mathcal{N}^{\beta}(0\subset NT\text{-}PA)$$

&

$$\{ S_x(0), ..., S_{x+1}(0), S_{x+2}(0), ..., S_{x+x}(0), ...\} \subseteq \mathcal{S}uc_{\,x}(0)$$

$$\{ S_0(x), ..., S_0(x+1), S_0(x+2), ..., S_0(x+x), ...\} \subseteq \mathcal{S}uc_{\,0}(x)$$

$$\{P_0(0), P_0(1), P_0(2), ..., P_0(x), ... P_0(x+x), ...\} \subseteq \mathcal{P}re_x(0)$$

Additionally, for example, when referred to zero by means of the customary decimal numerical system, we are able to state $\forall\mathbf{x(x\in}\mathcal{N}\mathbf{(0\subset NT\text{-}PA)):}$

$$\{ 0, 1, 2, 3, ..., x, ... \} \subseteq \mathcal{N}^{\alpha}(0\subset NT\text{-}PA)$$

$$\{ ..., x, ..., 3, 2, 1, 0 \} \subseteq \mathcal{N}^{\beta}(0\subset NT\text{-}PA)$$

$$\{ 0, 1, 2, 3, ..., x, ... \} \subseteq \mathcal{S}uc^{\alpha}{}_{x}(0)$$

$$\{ ..., x, ..., 3, 2, 1, 0 \} \subseteq \mathcal{S}uc^{\beta}{}_{x}(0)$$

$$\{ 0, 1, 2, 3, ..., x \} \subseteq \mathcal{P}re^{\alpha}{}_{0}(x)$$

$$\{ x, ..., 3, 2, 1, 0 \} \subseteq \mathcal{P}re^{\beta}{}_{0}(x)$$

Therefore, in NT-FOL-PA, the following statements are wff and wfs-r statements: [74]

∀x(x∈𝒩(0⊂NT-PA))

$$\{ S_0(0), S_1(0), S_2(0), \ldots, S_x(0), \ldots \} \subseteq \mathcal{N}^{\alpha}(0 \subset \text{NT-PA})$$

$$\wedge \vee$$

$$\{ S_0(0), S_0(1), S_0(2), \ldots, S_0(x), \ldots \} \subseteq \mathcal{N}^{\alpha}(0 \subset \text{NT-PA})$$

$$\wedge \vee$$

$$\{ P_0(0), P_0(1), P_0(2), \ldots, P_0(x), \ldots \} \subseteq \mathcal{N}^{\alpha}(0 \subset \text{NT-PA})$$

$$\wedge \vee$$

$$\{ 0, 1, 2, 3, \ldots, x, \ldots \} \subseteq \mathcal{N}^{\alpha}(0 \subset \text{NT-PA})$$

Hence, in NT-FOL-PA, the identity by comprehension (by means of intended de-embed logic operation and/or) of a natural number as "element" **(**(∀x)(x∈𝒩(0⊂NT-PA))**)** *can be well represented by:*

$$(0 = 0) \wedge\vee (1 = 1) \wedge\vee (2 = 2) \wedge\vee (3 = 3) \wedge\vee (4 = 4) \wedge\vee \ldots \wedge\vee (x = x) \ldots$$

$$\wedge \vee$$

$$(x+0 = 0+x) \wedge\vee (x+1 = 1+x) \wedge\vee (x+2 = x+2) \wedge\vee \ldots \wedge\vee (x+x = x+x) \ldots$$

$$\wedge \vee$$

$$(x+0 = 0+x) \wedge\vee (x+1 = 1+x) \wedge\vee (x+2 = x+2) \wedge\vee \ldots \wedge\vee (x+x = 1\cdot x+1\cdot x) \ldots$$

$$\wedge \vee$$

$$\wedge\vee (x = 1\cdot x) \ \ldots \wedge\vee (x+x = 2\cdot x) \ldots \wedge\vee (x+x+x = 3\cdot x) \ldots$$

Hence;

$$\left(\forall x\, (x \in \mathcal{N}(0 \subset \text{NT-PA})\right)\left(\left(\mathcal{N}(0 \subset \text{NT-PA}) :=\right.\right.$$

$$\{ (0=0+0) \wedge (1=1+0) \wedge (2=2+0) \wedge (3=3+0) \wedge (x+0=1\cdot x+0) \wedge (x+1=1\cdot x+1) \wedge (x+2=1\cdot x=1\cdot x+2) \wedge ((x+x)=2\cdot x+0) \wedge ((x+x)+1=2\cdot x+1) \wedge ((x+x+x)+0=3\cdot x+0); ((x+x+x)+1=3\cdot x+1) \wedge ((x+x+x+x)+0=4\cdot x+0) \wedge ((x+x+x+x+x)+0=5\cdot x+0) \wedge ((x+x+x+x+x+x)+0= 6\cdot x+0) \wedge \ldots \wedge \ldots)$$

$$\longrightarrow$$

$$\left(\mathcal{S}uc^{\alpha}{}_{0}(x) :=\right.$$

$$\{ 0, 1, 2, 3, \ldots, x+0, x+1, x+2, \ldots, (x+x), (x+x)+1, \ldots,(x+x+x),(x+x+x)+1, \ldots\}$$

$$\wedge \vee$$

$$\{ 0, 1, 2, 3, \ldots, 1\cdot x , 1\cdot x+1, 1\cdot x+2 \ldots, 2\cdot x+0, 2\cdot x+1, \ldots, 3\cdot x+0, 3\cdot x+1, \ldots, \ldots \})$$

$$\&$$

$$\left(\mathcal{S}uc^{\beta}{}_{x}(0) :=\right.$$

$$\{\ldots,(x+x+x)+1, (x+x+x), \ldots, (x+x)+1, (x+x), \ldots, \ldots, x+2, x+1, x+0, \ldots, 3, 2, 1, 0\}$$

$$\{ \ldots, 3\cdot x+1, 3\cdot x+0, \ldots, 2\cdot x+1, 2\cdot x+0, \ldots, 1\cdot x+2,1\cdot x+1, 1\cdot x , \ldots, 3, 2,1, 0 \})$$

&

($\mathcal{P}re^{\alpha}{}_{0}(x)$ **:=**

{ 0, 1, 2, 3, …, x+0, x+1, x+2, …, (x+x), (x+x)+1, …,(x+x+x),(x+x+x)+1, ….}

∧

{ 0, 1, 2, 3, …, 1·x , 1·x+1, 1·x+2 …, 2·x+0, 2·x+1, …, 3·x+0, 3·x+1, …, … })

&

($\mathcal{P}re^{\beta}{}_{0}(x)$ **:=**

{…,(x+x+x)+1, (x+x+x), …, (x+x)+1, (x+x), …, …, x+2, x+1, x+0, …, 3, 2, 1, 0}

∧∨

{ …., 3·x+1, 3·x+0, …, 2·x+1, 2·x+0, …, 1·x+2,1·x+1, 1·x , …, 3, 2,1, 0 }))

In view of all the above, the elements (also, called until now, as natural numbers or integers) that conform non-dense, parity complete ordered $\mathcal{P}re^{\alpha}{}_{0}(x)$, $\mathcal{S}uc^{\alpha}{}_{0}(x)$ *and* $\mathcal{S}uc^{\alpha}{}_{x}(0)$ *collectivities of natural numbers referred to natural number zero are the natural number concepts referred as the elements conforming the set of the natural numbers herein called* $\mathcal{N}$***(0⊂NT-PA)****.*

Said natural numbers and collectivities referred to natural number zero, results from an "intentional de-embedding" (sense and intension of NT-SF&L) operation by means of the next two operational processes of the inductive-deductive logical apparatus of NT-SF&L language:

i.- First order logic translation (briefly, FOL-T) of the set of the higher order logic axioms supporting $\mathcal{N}$***(0⊂PA)*** *, (hereinafter symbolized by* ***SPA(0⊂PA)****).*

ii.- First order logic inductive entailment (briefly, FOL-I) from the set supporting $\mathcal{N}$***(FOL-SPA(0⊂PA))*** *axiomatic definition, as well as of their elements and operations, and subsets.*

<u>HLPrime-1-NT Theorem</u>:

∀x ($x \in \mathcal{P}re^{\alpha}{}_{0}(x) \wedge \mathcal{S}uc^{\alpha}{}_{0}(x) \wedge \mathcal{S}uc^{\alpha}{}_{x}(0) \wedge \mathcal{N}^{\alpha}$**(0⊂NT-PA))**

((1. ({ e-0, o-1, e-2, o-3, …, e-(x), o-(x+1), e-(x+2),…, e-(x+x),

o-(x+x+1)… }⊆ $\mathcal{N}^{\alpha}$**(0⊂PA)))⟶**

({ …, o-(x+x+1), e-(x+x), …, e-(x+2), o-(x+1), e-(x), …,

e-4, o-3, e-2, o-1, e-0 } ⊆ $\mathcal{N}^{\beta}$**(0⊂PA)))**

&

2. ({ e-0, o-1, e-2, o-3, e-4..., e-(x+x), o-(x+x+1), e-(x+x+2),

o(x+x+3), e-(x+x+4),... } ⊆ $Suc^{\alpha}_{x}(0)$) →

{ ..., e-(x+x+4), o(x+x+3), e-(x+x+2), o-(x+x+1), e-(x+x),

..., e-4, o-3, e-2, o-1, e-0}⊆ ($Suc^{\beta}_{x}(0)$)

&

3. ({ e-0, o-1, e-2, o-3, e-4..., e-(x+x), o-(x+x+1), e-(x+x+2),

o(x+x+3), e-(x+x+4),... } ⊆ $Pre^{\alpha}_{0}(x)$) →

($Pre^{\beta}_{0}(x)$ ⊆ { ..., e-(x+x+4), o(x+x+3), e-(x+x+2), o-(x+x+1), e-(x+x),

..., e-4, o-3, e-2, o-1, e-0}))

→

(*𝒩(FOL-SP(0⊂PA) ⊆ 𝒩(SPA(0⊂PA) → 𝒩(0⊂NT-PA) ⊆ 𝒩(FOL-SP(0⊂PA))))*)

(NT-Pr-th. 1) [75]

FOURTH FOUNDAMENTAL NT-FS&L THEOREM

∀x (x ∈ 𝒩(0⊂NT-PA))

(($𝒩^{\alpha}$(0⊂NT-PA))={ e-0, o-1, e-2, o-3, ..., e-(x+x),... } ⇒

$Pre^{\beta}_{0}(x)$ ∧ $Suc^{\alpha}_{x}(0)$ ⊂ $𝒩^{\alpha}$(0⊂NT-PA)) →

((e-P_0(x+x) = e-(x+x) ∧ e- $S_{x+x}(0)$) ∧ (o-P_0(x+x+1) = o-(x+x+1) ∧ o-$S_{x+x+1}(0)$)

⇒

((e-P_0(x) = e- $S_x(0)$ ∧ e-(x) ∧ (o-P_0(x) = o- $S_x(0)$ ∧ o-(x))

&

((#- $Pre^{\alpha}_{0}(x)$ = x+1 ⇒ x ∈ *ℰven*(Suc_0(x)) → (x+1) ∈ *Odd*(Suc_0(x+1))

∨

#- $Pre^{\alpha}_{0}(x)$ = x+1 ⇒ x ∈ *Odd*(Suc_0(x)) → (x+1) ∈ *ℰven*(Suc_0(x+1)))

&

((x+1) ∈ *ℰven*(Suc_0(x+1)) → (x) ∈ *Odd*(Suc_0(x))

∧

$$(x) \in \mathcal{Even}(\mathcal{Suc}_0(x)) \rightarrow (x+1) \in \mathcal{Odd}(\mathcal{Suc}_0(x+1)))$$

(NT-FS&L F.Th. 4)

Please, take in account that:

1. $(\forall x \in \mathcal{N}(0 \subset NT\text{-}PA))$

$$0 \in \mathcal{Suc}_0(x) \cap \mathcal{N}_{0 \subset NT\text{-}PA)} \ \& \ 0 \notin \mathcal{N}_{0 \subset NT\text{-}PA)} \cap \mathcal{N}_{0 \not\subset NT\text{-}PA)} \wedge \mathcal{Suc}_1(x) \wedge \mathcal{Suc}_x(1)$$

$$\wedge$$

$$\mathcal{Suc}_0(1) \wedge \mathcal{Suc}_1(x) \subseteq \mathcal{N}_{0 \subset NT\text{-}PA)} \cap \mathcal{N}_{0 \not\subset NT\text{-}PA)}$$

$$\wedge$$

$$\mathcal{Suc}_x(x) \wedge \mathcal{Suc}_0(x+x) \wedge \mathcal{Suc}_{x+x}(0) \subseteq \mathcal{N}_{0 \subset NT\text{-}PA)} \cap \mathcal{N}_{0 \not\subset NT\text{-}PA)}$$

2. $(\forall x \in \mathcal{N}(0 \subset NT\text{-}PA))$

$$0 \in \mathcal{Even}(\mathcal{Suc}_0(x)) \wedge 1 \in \mathcal{Odd}(\mathcal{Suc}_0(x)) \Rightarrow$$

$$((0=e\text{-} S_0(0) \wedge 1=o\text{-} S_1(0) \wedge o\text{-} S_0(1)) \ \& \ (0= e\text{-} P_0(0) \wedge 1= o\text{-} P_0(1))$$

$$\rightarrow$$

$$(x+x+0) = (x+x) + 0 \in \mathcal{Even}(\mathcal{Suc}_0(x)) \wedge (x+x+1) = (x+x)+1 \in \mathcal{Odd}(\mathcal{Suc}_0(x))$$

$$\&$$

$$\{0, 1,\ldots, (x+0), (x+1),\ldots, \ldots, \ldots, (x+x), (x+x+1), \ldots, \ldots\})$$

3. $(\forall x \in \mathcal{N}(0 \subset NT\text{-}PA))$

$$(\mathit{1.}\ ((S_0(x) \wedge S_x(0) \in \mathcal{Even}(\mathcal{Suc}_0(x)) \vee S_0(x) \wedge S_x(0) \in \mathcal{Odd}(\mathcal{Suc}_0(x)))$$

$$\wedge$$

$$((S_0(x) +1) \in \mathcal{Even}(\mathcal{Suc}_0(x)) \vee (S_0(x) +1) \in \mathcal{Odd}(\mathcal{Suc}_0(x)))$$

$$\&$$

$$\mathit{2.}\ ((S_0(x+x) \wedge S_x(x) \wedge S_{x+x}(0) \in \mathcal{Even}(\mathcal{Suc}_0(x))$$

$$\wedge$$

$$(S_0(x+x+1) \wedge S_{x+1}(x) \wedge S_{x+1}(x) \wedge S_{x+x+1}(0) \wedge S_{x+x}(1) \wedge S_x(x+1) \wedge S_1(x+x) \in \mathcal{Odd}(\mathcal{Suc}_0(x))$$

$$\wedge$$

$$((S_0(x+x)+1 \wedge S_x(x)+1 \wedge S_{x+x}(0)+1 \in \mathcal{Odd}(\mathcal{Suc}_0(x)))$$ [76]

<u>HLPrime-2-NT Theorem</u>:

$\forall x\ (x \in \mathcal{N}(0 \subset NT\text{-}PA))$

A. Succession and covariance referred to the collectivity:

$$\Big[\mathit{Suc}^{\alpha}{}_{x}(0) := \{ S^{\alpha}{}_{0}(0),\ S^{\alpha}{}_{0}(1), S^{\alpha}{}_{0}(2), S^{\alpha}{}_{0}(3), \ldots, S^{\alpha}{}_{0}(x) \}$$

$$\wedge$$

$$\mathit{Suc}^{\alpha}{}_{x}(0) := \{ S^{\alpha}{}_{0}(0),\ S^{\alpha}{}_{1}(0), S^{\alpha}{}_{2}(0), S^{\alpha}{}_{3}(0), \ldots, S^{\alpha}{}_{x}(0) \} \Big]$$

$$\rightarrow$$

$$\mathit{Suc}^{\alpha}{}_{x}(0) := \{e\text{-}0, o\text{-}1, e\text{-}2, o\text{-}3, \ldots, x\} \Rightarrow \mathit{Pre}^{\alpha}{}_{0}(x) := \{e\text{-}0, o\text{-}1, e\text{-}2, o\text{-}3, \ldots, x\}$$

$$\rightarrow$$

$$\mathit{Pre}^{\alpha}{}_{0}(x) := \{ e\text{-}0, o\text{-}1, e\text{-}2, o\text{-}3, \ldots, x \} = \{ P^{\alpha}{}_{0}(0), P^{\alpha}{}_{0}(1), P^{\alpha}{}_{0}(2), P^{\alpha}{}_{0}(3), \ldots, P^{\alpha}{}_{0}(x) \}$$

(*Nomenclature set referred to natural number **0** as index-enumerator**)***

$$\wedge$$

$$\{ P^{\alpha}{}_{x}(x), \ldots, P^{\alpha}{}_{3}(x), P^{\alpha}{}_{2}(x), P^{\alpha}{}_{1}(x), P^{\alpha}{}_{0}(x) \}$$

(*Nomenclature set referred to natural number (**x**) as enumerator, which in turn is referred to natural number zero**)***

B. Succession and contravariance referred to the collectivity:

$$\Big[\mathit{Suc}^{\beta}{}_{x}(0) := \{S^{\beta}{}_{x}(0), \ldots, S^{\beta}{}_{3}(0), S^{\beta}{}_{2}(0), S^{\beta}{}_{1}(0), S^{\alpha}{}_{0}(0)\}$$

$$\wedge$$

$$\mathit{Suc}^{\beta}{}_{x}(0) := \{S^{\alpha}{}_{0}(x), \ldots, S^{\alpha}{}_{0}(3), S^{\alpha}{}_{0}(2), S^{\alpha}{}_{0}(1), S^{\alpha}{}_{0}(0)\} \Big]$$

$$\rightarrow$$

$$\mathit{Suc}^{\beta}{}_{x}(0) := \{x, \ldots, o\text{-}3, e\text{-}2, o\text{-}1, e\text{-}0\} \Rightarrow \mathit{Pre}^{\beta}{}_{0}(x) := \{x, \ldots, o\text{-}3, e\text{-}2, o\text{-}1, e\text{-}0\}$$

$$\rightarrow$$

$$\mathit{Pre}^{\beta}{}_{0}(x) := \{x, \ldots, o\text{-}3, e\text{-}2, o\text{-}1, e\text{-}0,\} = \{P^{\beta}{}_{0}(x) \ldots, P^{\beta}{}_{0}(3), P^{\beta}{}_{0}(2), P^{\beta}{}_{0}(1), P^{\beta}{}_{0}(0)\}$$

(*Nomenclature set referred to natural number **0** as index-enumerator)**)***

$$\wedge$$

$$\{P^{\beta}{}_{0}(x), P^{\beta}{}_{1}(x), P^{\beta}{}_{2}(x), P^{\beta}{}_{3}(x), \ldots, P^{\beta}{}_{x}(x)\}$$

(*Nomenclature set referred to natural number (**x**) as enumerator, which in turn is referred to natural number zero**)***

Hence, $\forall x\ (x \in \mathit{Suc}^{\alpha}{}_{x}(0) \wedge \mathit{Suc}^{\beta}{}_{x}(0) \wedge \mathit{Pre}^{\alpha}{}_{0}(x) \wedge \mathit{Pre}^{\beta}{}_{0}(x) \wedge \mathcal{N}(0 \subset \text{NT-PA}))$

$$\big((S^{\alpha}{}_{x}(0) + S^{\alpha}{}_{x}(0) = 0 \wedge P^{\alpha}{}_{0}(x) + P^{\beta}{}_{x}(x) \rightarrow e\text{-}0 = e\text{-}S^{\alpha}{}_{0}(x) \wedge e\text{-}P^{\alpha}{}_{0}(x) \wedge e\text{-}P^{\beta}{}_{x}(x) \big)$$

$$\rightarrow$$

$$(x=0 \wedge 0+0=0) \Rightarrow 0 \in \mathcal{E}\mathit{ven}(x))$$ [77]

$$\Longrightarrow$$

$$\left[\left[S_x(0) = S^{\alpha}{}_x(0) \wedge S^{\alpha}{}_0(x) \wedge S^{\beta}{}_0(x) \wedge S^{\beta}{}_x(0) \rightarrow P_0(x) = P^{\beta}{}_0(x) \wedge P^{\alpha}{}_0(x)\right]\right.$$

$$\rightarrow$$

1. $S^{\alpha}{}_0(x) + S^{\beta}{}_x(0) \wedge S^{\beta}{}_0(x) + S^{\alpha}{}_x(0) = x$

&

$$\forall x\, \forall y\, (y \in \mathcal{P}re^{\alpha}{}_x(0))(x = P^{\alpha}{}_0(y) + P^{\beta}{}_0(y) = P^{\alpha}{}_0(x) \wedge P^{\alpha}{}_y(y+y) \rightarrow x=y)$$

2. $S^{\alpha}{}_0(x) + S^{\alpha}{}_x(0) \wedge S^{\beta}{}_0(x) + S^{\beta}{}_x(x) = x + x$

&

$$\forall x\, \forall y\, (y \in \mathcal{P}re^{\alpha}{}_x(0))(x = P^{\alpha}{}_0(y) + P^{\alpha}{}_0(y) = P^{\beta}{}_0(y) \wedge P^{\beta}{}_y(y+y) \rightarrow x=y+y)$$

&

$$S^{\alpha}{}_0(x) + S^{\alpha}{}_x(0) \wedge S^{\beta}{}_0(x) + S^{\beta}{}_x(0) \in \mathcal{E}ven(x),\ \forall x\ (x \in \mathcal{N}\,(0 \subset PA))$$

[In multiplicative notation, only alphabetic meaning of symbol "·":
$x + x = 2 \cdot x = 2 \cdot S^{\alpha}{}_x(0) \wedge 2 \cdot S^{\beta}{}_x(0) \wedge 2 \cdot P^{\beta}{}_0(x) \wedge P^{\alpha}{}_0(x) + P^{\beta}{}_0(x) \wedge 2 \cdot P^{\beta}{}_x(2 \cdot x)$]

Hence, $(\forall x \in \mathcal{N}\,(0 \subset NT\text{-}PA))$

$$\mathcal{S}uc^{\alpha}{}_x(0) \oplus \mathcal{S}uc^{\beta}{}_x(0) := \{x \mid x \in \mathcal{PN}_0(x)\},\ \forall x \in \mathcal{E}ven(x) \vee \mathcal{O}dd(x)$$

&

$$\mathcal{S}uc^{\beta}{}_x(0) \oplus \mathcal{S}uc^{\alpha}{}_x(0) := \{x \mid x \in \mathcal{PN}_0(x)\},\ \forall x \in \mathcal{E}ven(x) \vee \mathcal{O}dd(x)$$

$$\rightarrow$$

$$\mathcal{S}uc^{\alpha}{}_x(0) \oplus \mathcal{S}uc^{\beta}{}_x(0) = \mathcal{S}uc^{\beta}{}_x(0) \oplus \mathcal{S}uc^{\alpha}{}_x(0)$$

$$\mathcal{S}uc^{\alpha}{}_x(0) \oplus \mathcal{S}uc^{\alpha}{}_x(0) := \{x+x \in \mathcal{E}ven(x) \mid x \in \mathcal{PN}_0(x)\},\ \forall x \in \mathcal{E}ven(x) \vee \mathcal{O}dd(x)$$

&

$$\mathcal{S}uc^{\beta}{}_x(0) \oplus \mathcal{S}uc^{\beta}{}_x(0) := \{x+x \in \mathcal{E}ven(x) \mid x \in \mathcal{PN}_0(x)\},\ \forall x \in \mathcal{E}ven(x) \vee \mathcal{O}dd(x)$$

$$\rightarrow$$

$$\mathcal{S}uc^{\alpha}{}_x(0) \oplus \mathcal{S}uc^{\alpha}{}_x(0) = \mathcal{S}uc^{\beta}{}_x(0) \oplus \mathcal{S}uc^{\beta}{}_x(0) \subset \mathcal{E}ven(x)$$

(NT-Primes-th. 2)

Additionally, we can define a new operation, which will be herein after named as "the scalar multiplication of a natural number times a set of successors" of natural numbers referred to natural number **0** by the next statement:

$$\mathbf{2} \otimes \{ \mathbf{S}^{\alpha}_{\mathbf{x}}(\mathbf{0}) \} := \mathit{Suc}^{\alpha}_{\mathbf{x}}(\mathbf{0}) \oplus \mathit{Suc}^{\alpha}_{\mathbf{x}}(\mathbf{0})$$

&

$$\mathbf{2} \otimes \{ \mathbf{S}^{\beta}_{\mathbf{x}}(\mathbf{0}) \} := \mathit{Suc}^{\beta}_{\mathbf{x}}(\mathbf{0}) \oplus \mathit{Suc}^{\beta}_{\mathbf{x}}(\mathbf{0})$$

$$\rightarrow$$

$$\mathit{Suc}^{\alpha}_{\mathbf{x}}(\mathbf{0}) \oplus \mathit{Suc}^{\alpha}_{\mathbf{x}}(\mathbf{0}) = \mathbf{2} \otimes \{ \mathbf{S}^{\alpha}_{\mathbf{x}}(\mathbf{0}) \} = \{ \mathbf{2} \cdot \mathbf{S}^{\alpha}_{\mathbf{x}}(\mathbf{0}) \}$$

$$\wedge$$

$$\mathit{Suc}^{\beta}_{\mathbf{x}}(\mathbf{0}) \oplus \mathit{Suc}^{\beta}_{\mathbf{x}}(\mathbf{0}) = \mathbf{2} \otimes \{ \mathbf{S}^{\beta}_{\mathbf{x}}(\mathbf{0}) \} = \{ \mathbf{2} \cdot \mathbf{S}^{\beta}_{\mathbf{x}}(\mathbf{0}) \}$$

$$\forall \mathbf{x} \forall \mathbf{y} \ (\mathbf{n}, \mathbf{x} \in \mathcal{N}(\mathbf{0} \subset \mathbf{PA}))(\mathbf{n} \otimes \{ \mathbf{S}_{\mathbf{x}}(\mathbf{0}) \} = \{ \mathbf{n} \cdot \mathbf{S}^{\alpha}_{\mathbf{x}}(\mathbf{0}) \} \wedge \{ \mathbf{n} \cdot \mathbf{P}^{\alpha}_{\mathbf{x}}(\mathbf{0}) \})$$

&

$$\forall \mathbf{x} \forall \mathbf{y} \ (\mathbf{n}, \mathbf{x} \in \mathcal{N}(\mathbf{0} \subset \mathbf{NT\text{-}PA}))(\mathbf{n} \otimes \{ \mathbf{S}_{\mathbf{x}}(\mathbf{0}) \} = \{ (\mathbf{S}^{\alpha}_{\mathbf{x}}(\mathbf{0})) \cdot \mathbf{n} \} \wedge \{ (\mathbf{P}^{\alpha}_{\mathbf{x}}(\mathbf{0})) \cdot \mathbf{n} \})$$

First Highlighted NT-FS&L Considerations and Observations on first order logic (FOL) and Peano Arithmetic axiomatic of the collectivity named set of the natural numbers and their Arithmetic.

*The set of "statements" (statements) conforming the Peano´s axiomatic of natural numbers and of their Arithmetic, when expressed in first order Logic, herein symbolized by **FOL-SPA(0⊂PA),** referrers to:*

*i.- The two allowed identities, as elements and as successors of elements, of the natural numbers that conforms the collectivity defined by comprehension as "set of the natural numbers, natural number zero included as element" (briefly, symbolized by $\mathcal{N}$**(FOL-SPA⊂PA)**). (Hereinafter, the subset conformed by Axioms **1** and **2**).*

*ii.- The "set of allowed exchanges of above identities" induced by two arithmetic operations of the natural numbers, the addition and the multiplication, respectively. (Hereinafter; by addition the subset conformed by Axioms **3** and **4**, and by multiplication, the subset conformed by Axioms **5** and **6**).*

*iii.- An induction logical & mathematical schema, which is engrained in the FOL logical substitution operation, and allows and ensures logic entailment processes from the elements of $\mathcal{N}$**(FOL-SPA⊂PA) and FOL-SPA(0⊂PA)**, the natural numbers and the statements that hold "truth", respectively. (Hereinafter, the subset conformed by Axiom **7**).* [78] *"intended first logic de-embedding".*

***FOL-SPA(0⊂PA)**;* $\forall \mathbf{x} \forall \mathbf{y} \ (\mathbf{x}, \mathbf{y} \in \mathcal{N}(\mathit{FOL\text{-}SPA}(0 \subset PA)))$

1. ***On the two identities of the natural numbers; elements and successors***

of elements.

1.1. $\forall x\ (S(x)\neq 0)$
1.2. $\forall x \forall y\ (S(x) = S(y) \rightarrow x=y)$

2. ***On the addition operation and the exchange of identities of natural numbers:***

2.1. $\forall x\ (x + 0 = x)$
2.2. $\forall x \forall y\ (x + S(y) = S(x+y))$

3. ***On the multiplication operation (between a successor and a natural number, not between two natural numbers) and the exchange of identities of natural numbers:*** [79]

3.1. $\forall x\ (x\cdot 0=0)$
3.2 $\forall x \forall y\ (x\cdot S(y) = x\cdot y + x)$

4. ***On induction schema by substitution operation:***

4.1. $\forall y_1 \ldots \forall y_n\ ((\varphi(x|0)\ \&\ \forall x(\varphi\rightarrow\varphi(x|S(x)))) \rightarrow \forall x\ \varphi)$, whenever φ is a formula whose free variables (also named occurrences) are among x_0, $y_1, \ldots, y_n$.

NT-FOL-SPA(0⊂PA); **$\forall x \forall y$ $(x, y \in \mathcal{N}_{(0\subset NT\text{-}PA)})$**

1. ***On the two identities of the natural numbers; elements as successors of a natural number:***

1.1. $\forall x\ (S_0(x) = x)$
1.2. $\forall x \forall y\ (S_0(x) = S_0(y) \rightarrow x=y)$

2. ***On the addition operation and the exchange of identities of natural numbers:***

2.1. $\forall x\ (x + 0 = x)$
2.2. $\forall x \forall y\ (x + S_0(y) = S_0(x+y))$

3. ***On the multiplication operation (between a successor and a natural number, not between two natural numbers) and the exchange of identities of natural numbers:***

3.1. $\forall x\ (x\cdot 0=0)$
3.2. $\forall x \forall y\ (x\cdot S_0(y) = x\cdot y + x)$

4. ***On induction schema by substitution operation:***

4.1. $\forall y_1 \ldots \forall y_n\ ((\varphi(x|0)\ \&\ \forall x(\varphi\rightarrow\varphi(x|S_0(x)))) \rightarrow \forall x\ \varphi)$, whenever φ is a formula whose free variables (also named occurrences) are among x_0,

$y_1, \ldots, y_n$.

5. ***On the two identities of the natural numbers; elements as predecessors of natural numbers:***

   1.1. $\forall x\ (P_0(x) = x) \rightarrow x \neq 0)$
   1.2. $\forall x \forall y\ (P_0(x) = P_0(y) \rightarrow x=y)$

6. ***On the addition operation and the exchange of identities of natural numbers:***

   2.1. $\forall x\ (x + 0 = x)$
   2.2. $\forall x \forall y\ (x + P_0(y) = P_0(x+y))$

7. ***On the multiplication operation (between a successor and a natural number, not between two natural numbers!) and the exchange of identities of natural numbers:***

   3.1. $\forall x\ (x \cdot 0 = 0)$
   3.2. $\forall x \forall y\ (x \cdot P_0(y) = x \cdot y + x)$

8. ***On induction schema by substitution operation:***

   4.1. $\forall y_1 \ldots \forall y_n\ ((\varphi(x|0)\ \&\ \forall x(\varphi \rightarrow \varphi(x|P_0(x)))) \rightarrow \forall x\ \varphi)$, whenever φ is a formula whose free variables (also named occurrences) are among x_0, $y_1, \ldots, y_n$.

Thus, we are able to represent ("substitution operation" is allowed as "logic operation" by induction statement 7 of initial PA´s, which represent a successor statement of zero statement) in first order language the next definition about the collectivity named (called) "set of axioms of PA´s" (abbreviated as **SPA(0⊂PA)** definition by extension and comprehension) in which the elements are, in turn, "wfs" and "wfs-r statements: $\forall x\ (x \in \mathcal{N}_{(0\subset PA)} \cap \mathcal{N}_{(0\subset NT\text{-}PA)})$,

1. SPA(0⊂PA) := ({ S(0), S(1)), S(2), S(3), S(4), S(5), S(6), S(7)})

&

({1, 2, 3, 4, 5, 6, 7})

&

(#- SPA´s$_{(0\subset PA)}$ = 7, (7 ∈ $\mathcal{N}_{(0\subset PA)}$))

2. SPA$^{\alpha}$ (0 ⊂FOL-PA) := ({ S(11), S(12)), S(21), S(22), S(31), S(32), S(41)})

&

$$\left(\{ P_0(11), P_0(12)), P_0(21), P_0(22), P_0(31), P_0(32), P_0(41)\}\right)$$

&

$$\left(\{11, 12, 21, 22, 31, 32, 41\}\right)$$

&

$$\left(\#\text{- } SPA(0 \subset NT\text{-}PA) = 7,\ (7 \in \mathcal{N}_{(0 \subset PA)})\right)$$

Please, allow us to state both:

***i.- The next statement is "true" when referred to "statements" as statements referred to the elements of* SPA^{α} (0 ⊂FOL-PA) *and/or* SPA^{α} (0⊂PA) by means of SPA^{α} (0 ⊂NT-PA)**

$$\left(\{ S(0), S(1)), S(2), S(3), S(4), S(5), S(6), S(7)\}\right)$$

$$=$$

$$\left(\{ P_0(11), P_0(12)), P_0(21), P_0(22), P_0(31), P_0(32), P_0(41)\}\right)$$

$$\rightarrow$$

$$(S(0) = P^{\alpha}{}_0(11)) \wedge (S(1) = P^{\alpha}{}_0(12)) \wedge (S(2) = P^{\alpha}{}_0(21)) \wedge (S^{\alpha}(3)) = (P^{\alpha}{}_0(22)) \wedge$$

$$S(4) = P^{\alpha}{}_0(22) \wedge S(5) = P^{\alpha}{}_0(31) \wedge S(6) = P^{\alpha}{}_0(32) \wedge S(7) = P^{\alpha}{}_0(41)$$

***ii.- The next statement is "false" when referred to "natural numbers" as elements of* $\mathcal{N}^{\alpha}$0⊂PA) *and/or* $\mathcal{N}^{\alpha}$(0⊂FOL-PA) *and/or* SPA^{α}(0 ⊂NT-PA) *and/or* SPA(0⊂PA) :**

$$\left(\{ S(0), S(1)), S(2), S(3), S(4), S(5), S(6), S(7)\}\right)$$

$$=$$

$$\left(\{ P_0(11), P_0(12)), P_0(21), P_0(22), P_0(31), P_0(32), P_0(41)\}\right)$$

$$\rightarrow$$

$$\left(((0)= (11)) \wedge ((1) = (12)) \wedge ((2) = (21)) \wedge ((3) = (22)) \wedge\right.$$

$$\left.(4) = (22) \wedge ((5) = (31)) \wedge ((6) = (32)) \wedge ((7) = (41))\right)$$

As a direct consequence, the statement referred to the collectivity* SPA^{α} (0⊂NT-PA) *set

$$\forall x(x \in \mathcal{N}^{\alpha}{}_{(0\subset PA)})\ \textit{and/or}\ \mathcal{N}^{\alpha}{}_{(0\subset FOL\text{-}PA)}\ \textit{and/or}\ SPA^{\alpha}{}_{(0\subset NT\text{-}PA)})$$

$$(\forall x\ (S(x) \neq 0) \rightarrow S(x) = 0),$$

is "false" in NT-FS&L. Please, take in account that for the last statement to "be true", for example, requires that "the conservation of parity" has no entailment.

$$((o\text{-}0) = (11)) \rightarrow ((o\text{-}1)=(e\text{-}12))\ \&\ ((o\text{-}5)=(o\text{-}31))$$

Second Highlighted NT-Considerations on Succession, Precede, Covariance, Contravariance, Partition, Addition, Multiplication of even and odd natural numbers

In NT–PA´s (NT-FS&L), which incorporate the first order logic framework, the former statements of PA´s axiomatic (**1**-**7**), when we are using notation and formulation referred to natural number zero and by using "precede" operation as intended[80] (sense and intension) operation, should be translated as:[81]

1. ***On the element zero as natural number concept and "gauge" reference).***

1.1. $\forall x\ [(0 = S_0(0)\ \wedge\ P_0(0)\ \wedge\ P_x(x))\ \vee\ ((S_1(x) \neq 0\ \rightarrow\ P_1(x) \neq 0)\ \wedge\ (P_1(x) \neq 0\ \rightarrow S_1(x) \neq 0)]]$.

1.2. $\forall x\ [\ [(0 = S_x(0)\ \wedge\ P_0(x) \rightarrow\ x = 0\ ]\ \wedge\ [(S_1(x) \neq 0\ \rightarrow\ P_1(x) \neq 0)\ \wedge\ (P_1(x) \neq 0\ \rightarrow S_1(x) \neq 0)]]$.

1.1. $\forall x \forall y\ [S_0(x) = S_0(y) \rightarrow\ x = y\ ]$

1.2. $\forall x \forall y\ [(S_0(x) = S_y(0)\ \wedge\ (P_0(x) = P_0(y)) \rightarrow\ x = y\ ]$

2. ***On addition operation:***

2.1. $\forall x\ [(x + 0 =\ x)\ \wedge\ (0 + x = x)]$

2.2. $\forall x \forall y[(x + S_y(0) =\ S_{x+y}(0)\ \wedge\ S_0(x+y)\ \wedge\ P_0(x+y))\ \rightarrow\ (x + P_0(y) =\ S_0(x+y)\ \wedge\ S_{x+y}(0)\ )]$

3. ***On multiplication operation:***

3.1. $\forall x\ [(x\cdot 0 =\ 0)\ \wedge\ (0\cdot x = 0)]$

3.2. $\forall x \forall y[((x\cdot S_y(0) =\ (x\cdot S_y(0) + x)\ \wedge\ (x\cdot S_0(y))\ \wedge\ (x\cdot P_0(y))$

4. **On induction schema by substitution operation:**

4.1. $\forall y_1 \ldots \forall y_n\ ((\varphi(x|0)\ \&\ \forall x(\varphi \rightarrow \varphi(x|S(x)))) \rightarrow \forall x\ \varphi)$, whenever φ is a formula whose free variables (also named occurrences) are among $x_0, y_1, \ldots, y_n$.

Therefore, only in terms of the precede operation referred to zero (axioms 3-1 and 3-2 NT-PA) can be "expanded" (also, translated) in NTS&L :

$\forall x \forall y\ (x, y \in \mathcal{N}_{(0 \subset NT\text{-}PA)})$

3.1 $\forall x((x \otimes \mathcal{S}uc^{\alpha}{}_{0}(x) \subseteq (\mathcal{S}uc^{\alpha}{}_{0}(x) \otimes x) \wedge 1 \otimes \mathcal{S}uc^{\alpha}{}_{x}(0) \wedge x \otimes \mathcal{P}re^{\alpha}{}_{0}(x)) \rightarrow x=1))$

3.2 $\forall x((1 \otimes \mathcal{S}uc^{\alpha}{}_{0}(x) \subseteq (\mathcal{S}uc^{\alpha}{}_{0}(x) \otimes 1) \wedge \mathcal{S}uc^{\alpha}{}_{1\cdot x}(0) \wedge \mathcal{P}re^{\alpha}{}_{0}(1\cdot x)))$

3.3 $\forall x((1 \otimes \mathcal{S}uc^{\alpha}{}_{0}(x) \subset (\mathcal{S}uc^{\alpha}{}_{0}(x) \otimes 2) \wedge \mathcal{S}uc^{\alpha}{}_{2\cdot x}(0) \wedge \mathcal{P}re^{\alpha}{}_{0}(2\cdot x)))$

3.4 $\forall x\ (x\cdot 0 = 0 \wedge 0\cdot x \wedge P_x(x)\ P_0(0) \wedge S_0(0) \wedge P_{x+x}(x+x) \wedge P_{x+x+x}(x+x+x))$

3.5 $\forall x((x\cdot S_0(x) = x\cdot P_0(x) \wedge x\cdot P_x(x+x) \wedge x\cdot(P_x(x)+x)) \wedge (x\cdot P_x(x) + x\cdot x)$

$\wedge\ x\cdot 0 + x\cdot x \wedge x\cdot x)) \wedge$

$(x\cdot S_0(x) = P_0(x)\cdot P_0(x) \wedge x\cdot x)$

3.6 $\forall x \forall y((x\cdot S_0(y) = (x\cdot y+x) \wedge x\cdot P_0(y) \wedge x\cdot P_y(y+y) \wedge$

$x\cdot(P_y(y)+y) \wedge (x\cdot P_y(y) + x\cdot y) \rightarrow (x\cdot S_0(y) = x\cdot 0 + x\cdot y \wedge x\cdot y)) \wedge$

$(x\cdot S_0(y) = P_0(x)\cdot P_0(y) \wedge P_0(y)\cdot P_0(x) \wedge x\cdot y))$

3.7 $\forall x \forall y((x\cdot S_0(x) = x\cdot S_0(y) \rightarrow P_0(x)\cdot P_0(x) = P_0(x)\cdot P_0(y) \wedge$

$(x\cdot P_0(x) = x\cdot P_0(y) \wedge (P_0(x)\cdot P_0(x) = x\cdot x)) \wedge (P_0(x) = P_0(y) \wedge (x = y)$

3.8 $\forall x((x\cdot S_1(x) = x\cdot x+x \wedge x\cdot(x+1) \wedge (P_0(x)\cdot P_0(x)+ P_0(x)) = (P_0(x)\cdot(P_0(x) + 1)) = (P_0(x)\cdot(P_0(x+1) \wedge (P_0(x)\cdot(P_1(x+1) \rightarrow$

$x\cdot S_1(x) = P_0(x)\cdot P_1(x+1) \wedge (P_0(x)\cdot P_0(x) + 1\cdot P_0(x))$

$\wedge$

$x\cdot S_1(x) = (1\cdot x + x\cdot x) \wedge (x\cdot x + 1\cdot x)$

3.9 $\forall x \forall y\ ((x\cdot S_1(y) = x\cdot(y+1) \wedge (P_0(x)\cdot(P_0(y+1)) = (P_0(x)\cdot(P_0(y)+1) \wedge (P_0(x)\cdot(P_0(y)+ 1\cdot P_0(x) \rightarrow$

$x\cdot S_1(y) = (P_0(x)\cdot(P_0(y+1)) \wedge P_0(x)\cdot P_0(y) + 1\cdot P_0(x)$

$\wedge$

$x\cdot S_1(y) = (1\cdot x + x\cdot y) \wedge (x\cdot y + 1\cdot x)$

3.10 $\forall x \forall y\ ((x\cdot S_1(x) = x\cdot S_1(y) \rightarrow$

$x\cdot S_1(x) = ((P_0(x)\cdot P_1(x+1) \wedge (P_0(x)\cdot P_0(x) + 1\cdot P_0(x)) \wedge$

$x\cdot S_1(y) = ((P_0(x)\cdot(P_0(y+1)) \wedge P_0(x)\cdot P_0(y) + 1\cdot P_0(x)) \wedge$

$$\mathbf{P_0(x) = P_0(y) \wedge (P_1(x+1) = (P_0(y+1)) \wedge}$$

$$\mathbf{x \cdot S_1(x) = x \cdot y \rightarrow y = x+1 \wedge (S_0(x+1)) \wedge (P_0(x+1)) \wedge (P_1(x+2)) \wedge}$$
$$\mathbf{(P_{1+x}(x+x+2)) \wedge (P_{1+x+x}(x+x+x+2)) \wedge (P_{1+x+x+x}(x+x+x+x+2))}$$
$$\mathbf{\wedge (P_{1+(x+x)+(x+x)+x}((x+x+x)+(x+x+x)+2))}$$

Please, before defining prime natural numbers, let us point out the next NT-PA considerations and observations about "parity and imparity, covariance and contravariance, in relation with the operations addition, multiplication of both; natural numbers and their "successor and predecessor" identities, and with theirs collectivities.

1. $\forall \mathbf{x}(\mathbf{x} \in \mathcal{N}_{(\mathbf{0 \subset NT\text{-}PA})})$

$$\mathit{Subset}\text{-}\mathcal{N}^{\alpha}{}_{(\mathbf{0 \subset NT\text{-}PA})}\,\mathbf{(x)} \subset \mathit{Subset}\text{-}\mathcal{N}^{\alpha}{}_{(\mathbf{0 \subset NT\text{-}PA})}\,\mathbf{(x+1)}$$
$$\mathit{Subset}\text{-}\,\mathcal{P}re^{\alpha}{}_{\mathbf{0}}\,\mathbf{(x)} \subset \mathit{Subset}\text{-}\,\mathcal{P}re^{\alpha}{}_{\mathbf{0}}\,\mathbf{(x+1)}$$
$$\mathit{Subset}\text{-}\,\mathcal{P}re^{\alpha}{}_{\mathbf{0}}\,\mathbf{(x)} \subseteq \mathit{Subset}\text{-}\,\mathcal{P}re^{\alpha}{}_{\mathbf{1}}\,\mathbf{(x+1)}$$
$$\mathit{Subset}\text{-}\mathcal{N}_{(\mathbf{0 \subset NT\text{-}PA})}\,\mathbf{(x+x)} \subset \mathit{Subset}\text{-}\mathcal{N}_{(\mathbf{0 \subset NT\text{-}PA})}\,\mathbf{(x+x+1)}$$
$$\mathit{Subset}\text{-}\,\mathcal{P}re^{\alpha}{}_{\mathbf{0}}\,\mathbf{(x+x)} \subset \mathit{Subset}\text{-}\,\mathcal{P}re^{\alpha}{}_{\mathbf{0}}\,\mathbf{(x+x+1)}$$
$$\mathit{Subset}\text{-}\,\mathcal{P}re^{\alpha}{}_{\mathbf{0}}\,\mathbf{(x+x)} \subseteq \mathit{Subset}\text{-}\,\mathcal{P}re^{\alpha}{}_{\mathbf{1}}\,\mathbf{(x+x+1)}$$
$$\mathcal{E}\mathit{ven}\text{-}\mathit{Subset}\text{-}\mathcal{N}_{(\mathbf{0 \subset NT\text{-}PA})}\,\mathbf{(x+x+1)} \subseteq \mathcal{E}\mathit{ven}\text{-}\mathit{Subset}\text{-}\mathcal{N}_{(\mathbf{0 \subset NT\text{-}PA})}\,\mathbf{(x+x)}$$
$$\mathcal{E}\mathit{ven}\text{-}\mathit{Subset}\text{-}\,\mathcal{P}re^{\alpha}{}_{\mathbf{0}}\,\mathbf{(x+x+1)} \subseteq \mathcal{E}\mathit{ven}\text{-}\mathit{Subset}\text{-}\,\mathcal{P}re^{\alpha}{}_{\mathbf{0}}\,\mathbf{(x+x)}$$
$$\#\text{-}\mathcal{E}\mathit{ven}\text{-}\mathit{Subset}\text{-}\,\mathcal{P}re^{\alpha}{}_{\mathbf{0}}\,\mathbf{(x+x+1)} = \#\text{-}\mathcal{E}\mathit{ven}\text{-}\mathit{Subset}\text{-}\,\mathcal{P}re^{\alpha}{}_{\mathbf{0}}\,\mathbf{(x+x)}$$

2. $\forall \mathbf{x}(\mathbf{x} \in \mathcal{N}_{(\mathbf{0 \subset NT\text{-}PA})})$

$$\mathcal{E}\mathit{ven}\text{-}\mathit{Subset}\text{-}\,\mathcal{P}re^{\alpha}{}_{\mathbf{0}}\,\mathbf{(x+x)} \subseteq \{\mathbf{e\text{-}(0+0), e\text{-}(1+1), e\text{-}(2+2), \ldots, e\text{-}(x+x)}\}$$
$$\wedge$$
$$\{\mathbf{0, 2, 4, 6, 8, 10\ldots, e\text{-}(x+x)}\} \subseteq \{\mathbf{0, 2, 4, 6, 8, 10\ldots, e\text{-}(x+x)}\}$$

$$\mathcal{E}\mathit{ven}\text{-}\mathit{Subset}\text{-}\mathcal{P}re^{\alpha}{}_{\mathbf{0}}\mathbf{(x+x)} \subseteq \mathcal{P}re^{\alpha}{}_{\mathbf{0}}\mathbf{(2 \cdot x)} \wedge \mathbf{2} \otimes \mathcal{P}re^{\alpha}{}_{\mathbf{0}}\mathbf{(x)}$$
$$\#\text{-}\mathcal{E}\mathit{ven}\text{-}\mathit{Subset}\text{-}\mathcal{P}re^{\alpha}{}_{\mathbf{0}}\mathbf{(x+x)} = \#\text{-}\,\mathcal{P}re^{\alpha}{}_{\mathbf{0}}\mathbf{(2 \cdot x)} \wedge \#\text{-}\mathbf{2} \otimes \mathcal{P}re^{\alpha}{}_{\mathbf{0}}\mathbf{(x)}$$

$$\mathcal{E}\mathit{ven}\text{-}\mathit{Subset}\text{-}\mathcal{P}re^{\alpha}{}_{\mathbf{0}}\mathbf{(x+x+1)} \subseteq \mathcal{P}re^{\alpha}{}_{\mathbf{0}}\mathbf{(2 \cdot x)} \wedge \mathbf{2} \otimes \mathcal{P}re^{\alpha}{}_{\mathbf{0}}\mathbf{(x)}$$

$$\#\text{-}\mathcal{E}\mathit{ven}\text{-}\mathit{Subset}\text{-}\mathcal{P}re^{\alpha}{}_{\mathbf{0}}\mathbf{(x+x+1)} = \#\text{-}\mathcal{P}re^{\alpha}{}_{\mathbf{0}}\mathbf{(2 \cdot x)} \wedge \#\text{-}\mathbf{2} \otimes \mathcal{P}re^{\alpha}{}_{\mathbf{0}}\mathbf{(x)}$$
$$\#\text{-}\mathit{Subset}\text{-}\mathcal{P}re^{\alpha}{}_{\mathbf{0}}\mathbf{(x+x) + 1} = \mathit{Subset}\text{-}\mathcal{P}re^{\alpha}{}_{\mathbf{0}}\mathbf{(x+x+1)}$$

3. $\forall \mathbf{x}(\mathbf{x} \in \mathcal{N}_{(\mathbf{0 \subset NT\text{-}PA})})$

$$\textit{Odd-Subset-}\ \mathit{Pre}^{\alpha}_{0}\ (x+x) \subset \{\ e\text{-}(0+0)+1,\ e\text{-}(1+1)+1,\ e\text{-}(2+2)+1,\ \ldots,\ e\text{-}(x+x)+1 \}$$

$$\wedge$$

$$\{\ 1, 3, 5, 7, 9, 11\ldots,\ (e\text{-}(x+x))+1\ \} \subseteq \textit{Odd-Subset-}\ \mathit{Pre}^{\alpha}_{0}\ ((x+x)+1))$$

$$\wedge$$

$$\{\ 0+1, 2+1, 4+1, 6+1, 8+1, 10+1\ldots,\ e\text{-}(x+x)+1\ \}$$

$$\wedge$$

$$\{\ 1, 3, 5, 7, 9, 11\ldots,\ e\text{-}(x+x+1)\ \}$$

$$P_{x}(x) \wedge\ P_{x+1}(x+1) \wedge\ P_{x+x}(x+x) \wedge P_{x+x+1}(x+x+1) \wedge \notin\ \textit{Odd-}\mathit{Pre}^{\alpha}_{0}\ ((x+1))$$

$$\textit{Even-Subset-}\ \mathit{Pre}^{\alpha}_{0}\ (x+x) \cap \textit{Even-Subset-}\ \mathit{Pre}^{\alpha}_{0}\ (x+x+1) \subseteq \mathit{Pre}^{\alpha}_{0}(x+x)$$

$$\textit{Even-Subset-}\ \mathit{Pre}^{\alpha}_{0}\ (x+x) \cup \textit{Odd-Subset-}\ \mathit{Pre}^{\alpha}_{0}\ (x+x+1) \subseteq \mathit{Pre}^{\alpha}_{0}(x+x+1)$$

$$\mathit{Pre}^{\alpha}_{0}(x+x) \subseteq \mathit{Pre}^{\alpha}_{0}(2\cdot x) \wedge\ 2 \otimes \mathit{Pre}^{\alpha}_{0}(x)$$

$$\mathit{Pre}^{\alpha}_{0}(x+x) \subseteq 1 \otimes \mathit{Pre}^{\alpha}_{0}(x+x) \cap 2 \otimes \mathit{Pre}^{\alpha}_{0}(x)$$

$$1 \otimes \mathit{Pre}^{\alpha}_{0}(x+x) \subset 1 \otimes \mathit{Pre}^{\alpha}_{0}(x+x+1)$$

$$P_{0}((x+x+1) \notin 1 \otimes \mathit{Pre}^{\alpha}_{0}(x+x) \cap 2 \otimes \mathit{Pre}^{\alpha}_{0}(x)$$

$$P_{0}((x+x+1) \notin 1 \otimes \mathit{Pre}^{\alpha}_{0}(x+x) \cap\ y \otimes \mathit{Pre}^{\alpha}_{0}(x)\ (\forall y \neq 1, y \in \mathcal{N}(0 \subset \text{NT-PA}))$$

$$P_{0}((x+x+1) \in 1 \otimes \mathit{Pre}^{\alpha}_{0}(x+x+1) \subseteq\ 1 \otimes (1 \oplus 2 \otimes \mathit{Pre}^{\alpha}_{0}(x\ )))$$

Hence: **[82]**

- Every natural even number x represented by its identity of predecessor and/or successor of a natural number referred to the even-natural number x+x **($P_0(x+x)$), which is referred in turn to natural number 0,** ***is an element of the collectivity*** $1 \otimes \mathit{Pre}^{\alpha}_{0}(x+x) \subseteq 1 \otimes \mathit{Pre}^{\alpha}_{0}(x+x) \cap 2 \otimes \mathit{Pre}^{\alpha}_{0}(1\cdot x)$

- Every natural odd number x+1 represented by its identity of predecessor and/or successor of a natural number referred to the even-natural number x+x **($P_1(x+x)$), which is referred in turn to natural number 0,** ***is an element of the collectivity*** $1 \otimes \mathit{Pre}^{\alpha}_{0}(x+x) \subseteq (1 \otimes (1 \oplus 2 \otimes \mathit{Pre}^{\alpha}_{0}(x\ ) \cap 2 \otimes \mathit{Pre}^{\alpha}_{0}(x))$

- Every natural odd number x+1 represented by its identity of predecessor and/or successor of a natural number referred to the odd-natural number x+x+1 **($(P_0(x+x+1) \wedge P_1(x+1+x+1))$, which is referred in turn to natural number 0,** ***is not an element of either a collectivity*** $1 \otimes \mathit{Pre}^{\alpha}_{0}(x+x) \cap\ y \otimes \mathit{Pre}^{\alpha}_{0}(1\cdot x)$ **($\forall y \neq 1$, $y \in \mathcal{N}(0 \subset \text{NT-PA})$) or, a collectivity** $1 \otimes \mathit{Pre}^{\alpha}_{0}(x+x) \cap\ 1 \otimes \mathit{Pre}^{\alpha}_{0}(x\cdot y)$ **($\forall y \neq 1$, $y \in \mathcal{N}(0 \subset \text{NT-PA})$)**

4. Please, allow us to remind the reader that **NT-FS&L F.Th 2**, is a true statement, [83] if and only if,

$$\forall x \forall y\ (x \in \mathcal{Pre}^{\alpha}{}_{1}\ (x+1)) \wedge \mathcal{N}^{\alpha}\ (0 \subset \text{NT-PA}),\ y \in \mathcal{N}(0 \subset \text{NT-PA}) \wedge$$

$$y \in \mathcal{N}(0 \subset \text{PA}) \cap \mathcal{N}(0 \not\subset \text{NTPA}) \wedge\ \mathcal{Pre}^{\alpha}{}_{0}\ (x)) \wedge\ \mathcal{Pre}^{\alpha}{}_{x+x}\ (x+x+1)) \wedge \mathcal{PN}\ ((P_0(y))$$

$$x \cdot (1 \otimes P_{x+x}(x+x+1) = (2 \otimes \sum_{x=P_{x+x}(x+x)}^{x=P_0(x)} [(x)]) \rightarrow\ y = P_0(x) + 1\ \wedge\ P_{x+x}(x+x+1),$$

$$\forall y \in \mathcal{Sucd}\,(P_0(x)) \wedge\ (\mathcal{N}(0 \subset \text{PA}) \cap \mathcal{N}(0 \not\subset \text{NT-PA}))$$

&

$$P_0(x) \cdot (1 \otimes P_{x+x}(2 \cdot x+1) = (2 \otimes \sum_{x=P_{x+x}(x+x)}^{x=P_0(x)} [(x)]) \rightarrow\ y = P_0(x) + 1\ \wedge\ P_{x+x}(x+x+1),$$

$$\forall y \in (\mathcal{Pre}^{\alpha}{}_{0}\ (x) \cap\ \mathcal{Pre}^{\alpha}{}_{0}\ (P_0(x+1)) \wedge \mathcal{Pre}^{\alpha}{}_{0}\ (P_1(x+2)) \wedge \mathcal{Pre}^{\alpha}{}_{0}\ \ (P_{x+x}(x+x+1))$$

$$\wedge$$

$$\forall x \in\ \mathcal{Pre}^{\alpha}{}_{0}\ \ (P_0(x)) \wedge \mathcal{Pre}^{\alpha}{}_{0}\ \ (P_1(x+1)) \wedge \mathcal{Pre}^{\alpha}{}_{0}\ (P_x(x+x)),$$

$$x, y\ (\mathcal{N}(0 \subset \text{PA}) \cap \mathcal{N}(0 \not\subset \text{NT-PA}))$$

&

Additionally, we have proof that in NT-FS&L, is in turn true that:

$$\forall x \in \mathcal{N}^{\alpha}\ (0 \subset \text{NT-PA})\ (e\text{-}(x \in \mathcal{Even}\text{-}(P_0(x)) \rightarrow$$

1. $o\text{-}y \in \mathcal{Even}\text{-}(((P_0(x+1) \cap \mathcal{Even}\text{-}(P_0(2 \cdot x)) \cap\ \mathcal{Even}\text{-}(P_0(2 \cdot x+1))$ **&**

2. $o\text{-}y = P_0(x+1) \in \mathcal{Odd}\text{-}(P_1(x+2)) \wedge\ o\text{-}y \notin\ \mathcal{Pre}^{\alpha}{}_{1}(x+1) \wedge\ \mathcal{Pre}^{\alpha}{}_{1}(x+x+1)$

   $\wedge\ \mathcal{Odd}\text{-}(P_0(2 \cdot x)) \cap\ \mathcal{Odd}\text{-}(P_0(2 \cdot x+1))$

HLPrime-1-NT Definition: [84] *A natural number y* $(y \in (\mathcal{Pre}^{\alpha}{}_{0}\ (x) \cap\ \mathcal{Pre}^{\alpha}{}_{0}\ (P_0(x+1))\ \wedge \mathcal{Pre}^{\alpha}{}_{0}\ \ (P_1(x+2))\ \wedge \mathcal{Pre}^{\alpha}{}_{0}\ \ (P_{x+x}(x+x+1)\ \wedge \mathcal{Sucd}\ (P_0(x))\ \wedge\ \mathcal{PN}\ ((P_0(y))\ \wedge\ (\mathcal{N}(0 \subset \text{PA}) \cap \mathcal{N}(0 \not\subset \text{NT-PA}))$ ***is named (also, called renamed, nicknamed and defined) both: "prime predecessor of even natural number x" and will be represented as p-*** $P^{\alpha}{}_{0}(x)$ $(x \in \mathcal{N}(0 \subset \text{PA}) \wedge \mathcal{PN}^{\alpha}{}_{0}\ (x)\ \wedge\ x = p\text{-}\ P^{\alpha}{}_{0}(x))$, ***or "prime natural number y", and then represented by "p" and formulated as:***

$$y = p\ \wedge\ \ (1 \otimes P_{x+x}(x+x+1))$$

if and only if, then:

$$x \cdot y = x \cdot (1 \otimes P_{x+x}(x+x+1)$$

$$\rightarrow$$

$$\big(y = (1 \otimes P_{x+x}(x+x+1) \wedge p\text{-}P^{\alpha}{}_0(p) \wedge p\text{-} P^{\alpha}{}_0(y) \wedge p\text{-} P^{\alpha}{}_0(y)\big) \rightarrow x = 1$$

$$\wedge$$

$$\big(y = 1 \wedge p \wedge (1 \otimes P_{x+x}(x+x+1) \rightarrow x = 0, x \notin (\mathcal{N}(0 \subset PA) \cap \mathcal{N}(0 \not\subset NT\text{-}PA))\big)$$

when NT-FS&L F.Th 2 is a true statement (wfs and wfs-rs) and every natural number is referred to its predecessor (and/or successor) identity is , in turn, referred to the natural number cero concept (0) identity: [85]

<u>HLPr-2-NT Definition</u>: *The set of natural numbers* $\mathbf{p}$ $(p \in \mathcal{P}red^{\alpha}{}_0(x) \wedge p \in \mathcal{N}(0 \subset NT\text{-}PA))$, *which is conformed by* ***"prime predecessors of every natural number x"*** $(x \in \mathcal{N}(0 \subset NT\text{-}PA) \wedge \mathcal{PN}_0(x), x = p\text{-} P^{\alpha}{}_0(x) \wedge p\text{-} S^{\alpha}{}_x(0) \wedge p\text{-} S^{\alpha}{}_0(x))$ ***is named (also, called renamed, nicknamed and defined) the "set of primes of natural number x", if an only if the next statement is a wfs and wfs-r statement and every natural number as their predecessor (and/or successor) identity is, in turn, referred to natural the number cero concept (0) identity:***

$$\mathcal{P}rimes(p\text{-}P^{\alpha}{}_0(x)) := \{ x = p \wedge (1 \otimes P_{x+x}(x+x+1)) \in (\mathcal{N}(0 \subset PA) \cap \mathcal{N}(0 \not\subset NT\text{-}PA)) \mid p = x \wedge p\text{-}P^{\alpha}{}_0(x) \wedge (\mathcal{P}re^{\alpha}{}_0(x) \cap \mathcal{P}re^{\alpha}{}_0(P_0(x+1)) \wedge \mathcal{P}re^{\alpha}{}_0(P_1(x+2)) \wedge \mathcal{P}re^{\alpha}{}_0(P_{x+x}(x+x+1) \wedge \mathcal{S}ucd(P_0(x)) \wedge \mathcal{PN}((P_0(y)) \}, \forall x(x \in \mathcal{N}(0 \subset NT\text{-}PA))$$

(def-Pr. 2)

NT-FS&L re-PRESENTATION (Decimal Number System) OF SET OF THE PRIMES NATURAL NUMBER SET in the Decimal System:

$$\mathcal{P}rimes(p\text{-}P^{\alpha}{}_0(x)) := \{1, 2, 3, 5, 7, 11, 13, 17, \ldots\}$$

On the other hand, in NTFS&L is able to establish a complete (two equivalence classes or quotient sets; also, referred as " a two color partition", two collectivities whose union is an "all" and the intersection is the non-numerable and non-countable previously referred and defined set $\Theta := \{ \varnothing \mid \varnothing \notin \mathcal{N}[0 \subset PA] \wedge \mathcal{N}[0 \not\subset PA]\}$, *partition of the natural number* $\mathcal{N}(0 \subset NT\text{-}PA)$ *set collectivity.*
Hence,

$$\mathcal{C}lass\ \mathit{I} := \{ x \in \mathcal{N}(0 \subset NT\text{-}PA)) \mid x \in \mathcal{P}rimes(p\text{-}P^{\alpha}{}_0(x)) \wedge \mathcal{P}(0 \subset NT\text{-}PA) \} \ \&$$

$$\mathcal{P}(0 \subset NT\text{-}PA) := \mathcal{C}lass\ \mathit{I}$$

$$\&$$

$$\mathcal{C}lass\ \mathit{II} : \{ x \in \mathcal{N}(0 \subset NT\text{-}PA)) \mid x \notin \mathcal{P}rimes(p\text{-}P^{\alpha}{}_0(x)) \wedge \mathcal{P}(0 \subset NT\text{-}PA) \}$$

$$\mathcal{NP}_{(0\subset NT\text{-}PA)} := \mathcal{C}lass\ II$$

(def-Pr. 2 bis)

Hence, in NT-FS&L the new identity of $\mathcal{N}_{(0\subset NT\text{-}PA)}$ ***is well symbolically represented and referred by the following statement:***

$$\mathcal{N}_{(0\subset NT\text{-}PA)} := \{ x \in \mathcal{N}_{(0\subset NT\text{-}PA)} \mid x \in \mathcal{C}lass\ I \cup \mathcal{C}lass\ II \}, \forall x\ (x \in \mathcal{N}_{(0\subset NT\text{-}PA)})$$

Finally, as a direct consequence, we are able to state: [86]

1. $\forall x\, \forall p\ (x \in \mathcal{N}_{(0\subset NT\text{-}PA)}, p \in \mathcal{P}_{(0\subset NT\text{-}PA)})\ (\mathcal{PN}_0(x) = \mathcal{PN}_0(p) \rightarrow x = p)$
2. $\forall p\ (p \in \mathcal{N}_{(0\subset NT\text{-}PA)} \cap \mathcal{P}_{(0\subset NT\text{-}PA)})\ (\mathcal{PN}_0(x) = \mathcal{PN}_0(p) \rightarrow x = p)$
3. $$\bigcup\ [\mathcal{PN}_0(p)]_{\forall p \in \mathcal{P}_{(0\subset NT\text{-}PA)}} \subseteq \mathcal{P}_{(0\subset NT\text{-}PA)} \rightarrow$$
   $$((\mathcal{P}_{(0\subset NT\text{-}PA)} \subset \mathcal{N}_{(0\subset NT\text{-}PA)} \cap \mathcal{N}_{(0\not\subset NT\text{-}PA)}) \rightarrow$$
   $$\mathcal{N}_{(0\subset NT\text{-}PA)}) \subseteq \mathcal{N}_{(0\not\subset NT\text{-}PA)} \cup \mathcal{PN}_0(0)))$$
   NT-th. 6 bis)
4. $(\mathcal{PN}_0(0) \not\subset \mathcal{P}_{(0\subset NT\text{-}PA)} \rightarrow \mathcal{P}_{(0\subset NT\text{-}PA)} \subset \mathcal{N}_{(0\subset NT\text{-}PA)}) \cap \mathcal{N}_{(0\not\subset NT\text{-}PA)})$
5. $\forall p\ (p \in \mathcal{N}_{(0\subset NT\text{-}PA)} \cap \mathcal{P}_{(0\subset NT\text{-}PA)})\ (p \in \mathcal{E}ven\ \mathcal{P}_{(0\subset NT\text{-}PA)} \rightarrow p = 2)$
   $$\wedge$$
   $\forall p\ (p \in \mathcal{N}_{(0\subset NT\text{-}PA)} \cap \mathcal{P}_{(0\subset NT\text{-}PA)})\ (p \in \mathcal{O}dd\ \mathcal{P}_{(0\subset NT\text{-}PA)} \rightarrow p \neq 2)$
6. $\forall p\ (p \in \mathcal{N}_{(0\subset NT\text{-}PA)} \cap \mathcal{P}_{(0\subset NT\text{-}PA)})\ (\#\text{-}\ \mathcal{E}ven\ (\mathcal{P}_{(0\subset NT\text{-}PA)}) = 1 \rightarrow$
   $$(\#\text{-}\ \mathcal{P}_{(0\subset NT\text{-}PA)} = \#\text{-}(\mathcal{O}dd\text{-}\mathcal{P}_{(0\subset NT\text{-}PA)}) + 1)$$
   $$\wedge$$
   $$(\#\text{-}\ \mathcal{P}_{(0\subset PA)}(x) = S^{\alpha}{}_1\ (\#\text{-}\ \mathcal{O}dd\text{-}\mathcal{P}rimes(p\text{-}P^{\alpha}{}_0(x)))$$
   $$\wedge$$
   $$(\#\text{-}\ \mathcal{P}_{(0\subset PA)}(x) = P^{\alpha}{}_1\ (\#\text{-}\ \mathcal{O}dd\text{-}\mathcal{P}rimes(p\text{-}P^{\alpha}{}_0(x))) + 2)$$
7. $\forall p\ (p \in \mathcal{N}_{(0\subset NT\text{-}PA)} \cap \mathcal{P}_{(0\subset NT\text{-}PA)})$
   $(\mathcal{E}ven\text{-}\mathcal{P}_{(0\subset PA)} \cap \mathcal{O}dd\ \mathcal{P}_{(0\subset PA)} = e\text{-}\mathcal{PN}_0(2) \subset \mathcal{N}_{(0\subset NT\text{-}PA)} \rightarrow$
   $$e\text{-}\mathcal{PN}_0(2) \cup \mathcal{O}dd\text{-}\mathcal{P}_{(0\subset PA)} \subseteq \mathcal{P}_{(0\subset NT\text{-}PA)}$$
   &
   $$(e\text{-}\mathcal{PN}_0(0) \cup e\text{-}\mathcal{PN}_0(2)) \cup \mathcal{O}dd\ \mathcal{P}_{(0\subset PA)} \subseteq \mathcal{N}_{(0\subset NT\text{-}PA)}$$

<u>HLPr-3-NT Definition</u>: The natural number 2 $\left(2 \in \mathcal{N}(0 \subset \text{NT-PA}) \wedge \mathcal{E}ven(\mathbf{x})\right)$, ***which is in NT-FS&L the unique element of the "primordial of natural number two"*** $e\text{-}\mathcal{PN}^{\alpha}{}_{0}(2)$ ***(briefly:*** $\mathcal{PN}(2)$***) is named (also, called renamed, nicknamed and defined) "the even-prime number" and hence, if an only if, the next statement is a wfs and wfs-r statement:*** $(\forall \mathbf{x}, \mathbf{x} \in \mathcal{N}(0 \subset \text{NT-PA}))$

$$e\text{-}\mathcal{PN}(2) \subset \left(\left(\mathcal{N}(0\subset\text{NT-PA}) \cap \mathcal{N}(0\not\subset\text{NT-PA})\right) \cap \mathcal{P}(0\subset\text{NT-PA})\right) \rightarrow$$
$$\left(2 = p\text{-}P_0(2) \wedge p\text{-}S_0(1+1) \wedge p\text{-}S_{1+0}(1) \wedge p\text{-}S_{1+1}(0) \wedge p\text{-}S_2(0) \wedge p\text{-}S_1(1)\right) \in$$
$$\mathcal{E}ven\text{-}\left(\mathcal{N}(0\subset\text{NT-PA}) \cap \mathcal{N}(0\not\subset\text{NT-PA})\right) \cap \mathcal{P}(0\subset\text{NT-PA})) \wedge e\text{-}\mathcal{PN}^{\alpha}{}_{0}(2)$$
$$\wedge$$
$$\left((2\cdot x = (1+1)\cdot x) \rightarrow \left(1\cdot x = (0\cdot 1 + 1)\cdot x\right) \wedge (x\cdot 1 = x\cdot(1\cdot 0 + 1)) \wedge (x\cdot 1 = x\cdot 1\cdot(0+1)\right)$$
$$\wedge$$
$$\left(1 + 1 = 2 \rightarrow \left(2 + 2 = (2\cdot 2 + 0\cdot 2)\right) \wedge 2\cdot(2+0) \wedge ((2\cdot 1+1)+1) \wedge (3+1) \wedge (4)\right)$$ [87]

(def-Pr. 3)

$$e\text{-}\{\mathcal{PN}(2)\} \subseteq \{\mathcal{E}ven\text{-}\left(\mathcal{N}(0\subset\text{NT-PA}) \cap \mathcal{N}(0\not\subset\text{NT-PA})\right) \cap \mathcal{P}(0\subset\text{NT-PA})\}$$

(def-Pr. 3 bis)

$$e\text{-}\{\mathcal{PN}(2)\} := \{\mathcal{P}(0\subset\text{NT-PA}) \cap \mathcal{O}dd\text{-}\mathcal{P}rimes(x)\}$$

(def-Pr. 3 tris)

The natural number p $\left(\mathbf{p} \in \mathcal{N}(0\subset\text{NT-PA}) \cap \mathcal{P}(0\subset\text{NT-PA}) \cap \mathcal{O}dd(x)\right)$, ***is named (also, called renamed, nicknamed and defined) "odd-prime number" and hence, element of the "set of odd -prime natural numbers"*** $\mathcal{O}dd\text{-}\mathcal{P}rimes(x)$ ***if an only if, the next statement is a wfs and wfs-r statement:***

$$\forall \mathbf{p}\left(p \in \mathcal{P}(0\subset\text{NT-PA})\right)\left(\mathbf{p} \in \mathcal{O}dd(x)\right) \rightarrow p \neq 2$$

Hence, we can adopt the following definition for the prime natural prime set collectivity:

$$\mathcal{P}(0\subset\text{NT-PA}) := \{\left(even\text{-}\mathcal{PN}(2)\right) \cup \left(\mathcal{O}dd\text{-}\mathcal{P}rimes(x) \subset \mathcal{N}(0\subset\text{NT-PA})\right)\}$$

(def-Pr. 3 tris)

<u>HLPr-4-NT Definition</u>: The natural number x $\left(\mathbf{x} \in \mathcal{N}(0\subset\text{PA}) \wedge \mathcal{O}dd(x)\right)$, ***is named (also, called renamed, nicknamed and defined) "odd non-prime number" and hence, element of the "set of odd non-prime natural numbers"*** $\mathcal{O}dd\text{-}\mathcal{NP}(x)$

(briefly: o-$\mathcal{NP}(x)$), if an only if, the next statement is a wfs and wfs-r statement:

$$\mathcal{O}dd\text{-}\mathcal{NP}(x) := \{\, x \in \mathcal{O}dd(x) \mid x \notin \mathcal{P}(0\subset NT\text{-}PA) \cap o\text{-}\mathcal{P}rimes(p\text{-}P^{\alpha}{}_{0}(x))\},$$
$$\forall \mathbf{x} \in \mathcal{N}(0\subset NT\text{-}PA) \cap \mathcal{O}dd(\mathbf{x})\,)$$
$$\forall \mathbf{x}\, \forall \mathbf{p}\ (\mathbf{x} \in \mathcal{N}(0\subset NT\text{-}PA),\, \boldsymbol{p} \in \mathcal{P}(0\subset NT\text{-}PA) \wedge \mathcal{S}uc^{\alpha}{}_{0}(\mathbf{x}) \wedge \mathcal{P}re^{\alpha}{}_{0}(\mathbf{x})\,)$$

(def-Pr. 4)

<u>HLPr-5-NT Definition</u>: The natural number x $(\mathbf{x} \in \mathcal{N}(\mathbf{0\subset NT\text{-}PA}) \cap \mathcal{O}dd(x))$***, is named (also, called renamed, nicknamed and defined) "even non-prime number" and hence, element of the "set of even non-prime natural numbers"*** $\mathcal{E}ven\text{-}\mathcal{NP}(x)$ ***(briefly:*** $\mathcal{E}\text{-}\mathcal{NP}(x)$***) if an only if, the next statement is a wfs and wfs-r statement:***

$$\mathcal{E}ven\text{-}\mathcal{NP}(x) := \{\, x \in \mathcal{E}ven(x) \mid x \notin \mathcal{P}(0\subset NT\text{-}PA) \cap \mathcal{P}rimes(p\text{-}P^{\alpha}{}_{0}(x))\,\},$$
$$\forall \mathbf{x} \in \mathcal{N}(0\subset PA) \cap \mathcal{E}ven(x)\,)$$
$$\forall \mathbf{x}\, \forall \mathbf{p}\ (\mathbf{x} \in \mathcal{N}(0\subset PA),\, \boldsymbol{p} \in \mathcal{P}(0\subset PA) \wedge \mathcal{S}uc^{\alpha}{}_{0}(\mathbf{x}) \wedge \mathcal{P}re^{\alpha}{}_{0}(\mathbf{x})\,)$$

(def-Pr. 5)

<u>HLPrime-3-NT Theorem</u>: $(\forall \mathbf{x} \in \mathcal{N}\,(\mathbf{0\subset NT\text{-}PA}))$ [88]

The natural number 2 (2 $\in \mathcal{N}^{\alpha}(0\subset NT\text{-}PA)$ ***is the unique natural number, which in turn is element of the set*** $\mathcal{P}(\mathbf{0\subset NT\text{-}PA})$ ***and*** $e\text{-}\mathcal{PN}_{0}(\mathbf{x})$.

$$\forall x\ (\mathbf{x} \in \mathcal{N}\,(\mathbf{0\subset NT\text{-}PA}))(\mathbf{x} \in e\text{-}\mathcal{PN}_{0}(\mathbf{x}) \cap \mathcal{P}(0\subset NT\text{-}PA) \rightarrow$$
$$(\mathbf{x = 2 \Rightarrow (0{+}1{+}1) \wedge (1{+}1) \wedge 1\cdot(0{+}1) \wedge 1\cdot(1{+}1) \wedge (1\cdot(0{+}1) \wedge ((1{+}0)\cdot 1))}$$
$$\rightarrow$$
$$\mathbf{x = S_2(0) \wedge S_1(1) \wedge S_0(2) \wedge S_0(1{+}1{+}0) \wedge S_1(0{+}1) \wedge S_1(1{+}0)}$$

(NT-Pr-th. 3)

Hence, $\forall$**x, (x** $\in \mathcal{N}$ **(0⊂NT-PA)),**

The "one-time addition of natural number one (1)" is natural number two (2), which is the "one-successor of the number one (1)", which is both: the "one-successor of natural number zero (0)" and the "one predecessor of number two (2)".

Please allow us to point the following statement: $(\forall \mathbf{x} \in \mathcal{N}(\mathbf{0\subset NT\text{-}PA}))$ [89]

$$(\mathcal{E}ven^{\alpha}{}_{0}\,(\mathbf{x + x + 2}) \subseteq \mathcal{E}ven^{\alpha}{}_{0}((\mathbf{x{+}1}) + (\mathbf{x{+}1})) \rightarrow$$

$$\left(\mathcal{E}ven^{\alpha}{}_{0}\,(x + x) \subseteq \mathcal{E}ven^{\alpha}(x + x + 2) \rightarrow x = 1\right)\right)$$

&

$$\left((x+1) \in \mathcal{E}ven^{\alpha}{}_{0}\,(x + 1) \vee \mathcal{O}dd^{\alpha}{}_{0}(x + 1)\right) \,\&\, \left((x+x) \in \mathcal{E}ven^{\alpha}(S_{x}(0))\right)$$

(NT-Pr-th. 3 bis)

HLPrime-4-NT Theorem: $(\forall x \in \mathcal{N}_{(0 \subset \text{NT-PA})})$

Lemma. I

$$o\text{-}\mathcal{PN}_{0}\,(0+(1)) = 1) \subseteq \mathcal{O}dd^{\alpha}{}_{0}(x) \cap \mathcal{P}^{\alpha}{}_{0\,(0 \subset \text{NT-PA})}(x)$$

$$o\text{-}\mathcal{PN}_{0}\,(1+(0)) = 1) \subseteq \mathcal{O}dd^{\alpha}{}_{0}(x) \cap \mathcal{P}^{\alpha}{}_{0\,(0 \subset \text{NT-PA})}(x)$$

$$e\text{-}\mathcal{PN}_{0}\,(0) \cup o\text{-}\mathcal{PN}_{0}\,(x) \subseteq \mathcal{O}dd^{\alpha}{}_{0}\,(x+1)$$

$$\mathcal{PN}_{0}\,(1) \subseteq \mathcal{O}dd^{\alpha}{}_{0}\,(x) \cap o\text{-}\mathcal{NP}^{\alpha}{}_{0\,(0 \subset \text{NT-PA})}(x) \rightarrow$$

$$\mathcal{PN}_{0}\,(1) \subseteq \mathcal{O}dd^{\alpha}{}_{0}\,(x) \cap \mathcal{P}^{\alpha}{}_{0\,(0 \subset \text{NT-PA})}(x)$$

Lema. II

$$e\text{-}\mathcal{PN}_{0}\,(0+1)+1 = 2) \subseteq \mathcal{E}ven^{\alpha}{}_{0}\,(x) \cap \mathcal{P}^{\alpha}{}_{0\,(0 \subset \text{NT-PA})}(x) \,\&$$

$$e\text{-}\mathcal{PN}_{0}\,(1+(1+0) = 2) \subseteq \mathcal{E}ven^{\alpha}{}_{0}\,(x) \cap \mathcal{P}^{\alpha}{}_{0\,(0 \subset \text{NT-PA})}(x)$$

$$e\text{-}\mathcal{PN}_{0}\,(0) \cup e\text{-}\mathcal{PN}_{0}\,(2) \subseteq \mathcal{E}ven^{\alpha}{}_{0}\,(x+2) \,\&$$

$$e\text{-}\mathcal{PN}_{0}\,(2) \subseteq \mathcal{E}ven^{\alpha}{}_{0}\,(x) \cap \mathcal{P}^{\alpha}{}_{0\,(0 \subset \text{PA})}(x) \rightarrow$$

$$e\text{-}\mathcal{PN}_{0}\,(2) \subseteq \mathcal{E}ven^{\alpha}{}_{0}\,(x) \cap e\text{-}\mathcal{NP}^{\alpha}{}_{0\,(0 \subset \text{PA})}(x)$$

Lemma. III

$$\mathcal{E}ven^{\alpha}(x + x + 2) \subseteq \mathcal{E}ven^{\alpha}{}_{0}\left((x+1) + (x+1)\right) \rightarrow$$

$$\left(\mathcal{E}ven^{\alpha}{}_{0}\left((x + x) + 2\right) \subseteq \mathcal{E}ven^{\alpha}{}_{0}\,(y + y) \rightarrow y = x+1 \wedge S_{1}(x) \wedge S_{0}(x+1)\right)$$

(NT-Pr-th. 4) [90]

HLPr-6-NT Set of Definitions about NT-tensorial natural number nomenclature, formulation and notation: [91]

Hereinafter:

"z[+x]$_t$" (*also*" [x+]$_t$ z"), [+x]$_t$ (*also* "[x+]$_t$") *and* " [+x]" (*also* "[x+]$_t$") ∈
{*Symbols and formulas of the alphabet, lexical, syntax, grammar and semantic of NT-FS&L PA´s , which are referring to the addition, succession and precede identities operations in turn to conserving and breaking parity, imparity, covariance, contravariance and primality properties of their symbolic locus-based representation identities* **}**

&

z[·x]$_t$" (*also*" [x·]$_t$ z"), [·x]$_t$ (*also* "[x·]$_t$") *and* " [·x]" (*also* "[x·]$_t$") ∈
{*Symbols and formulas of the alphabet, lexical, syntax, grammar and semantic of NT-FS&L, which are referring to the multiplication, succession and precede, identities-operations in turn to conserving and breaking parity, imparity, covariance, contravariance and primality properties of their symbolic locus-based representation identities* **}**

Then;

$\forall x \left(x \in \mathcal{N}^{\alpha}{}_{0(1\otimes,\, 0\oplus,\, 0\subset \text{NT- PA})} \right)$

$\left(\{\mathcal{N}(0\subset\text{PA}) \cap \mathcal{N}(0\not\subset\text{PA})\}\forall x \in (\mathcal{N}(0\subset\text{PA}) \cap \mathcal{N}(0\not\subset\text{NT-PA})) \subseteq \mathcal{PN}^{\alpha}{}_{0(1\otimes,\, 0\oplus,\, 0\subset\text{NT-FOL-PA})}(0)\right.$

$$\rightarrow$$

$\left.\left(x \in \{\mathcal{N}(0\subset\text{PA})\ \mathcal{N}(1\otimes, 0\oplus, 0\subset\text{NT- PA})\right) \rightarrow 0 \in \{\mathcal{PN}^{\alpha}{}_{0(1\otimes,\, 0\oplus,\, 0\subset\text{NT-FOL-PA})}(0)\}\right))$

Def-NT-6.1. $\forall x\forall y\forall z\forall t \left(x, y, z, t \in \mathcal{N}(1\otimes, 0\oplus, 0\subset\text{NT- FOL-PA})\right)$ [92]

Every natural number x $(x \in \mathcal{N}(0\subset\text{NT- PA}))$ ***resulting from (also, obtained by) the addition of natural number z*** $(z \in \mathcal{N}(0\subset\text{NT- PA}))$ ***and the natural number resulting from adding t-times*** $(t \in \mathcal{N}(0\subset\text{NT- PA}))$ ***the natural number y*** $(y \in \mathcal{N}(0\subset\text{NT- PA}))$, ***if an only if, the next statement is a wfs and wfs-r statement:***

$$\left(\left(x := z[+x]_t \;\&\; y := 0[+x]_t \right)\right) \Rightarrow$$
$$\left(x = x[+1]_0 \;\wedge\; x = 0[+x]_1 \;\wedge\; x = x[+0]_t \;\wedge\; x = S_x(0)\right) \;\&$$
$$\left(y = y[+1]_0 \;\wedge\; y = 0[+x]_1 \;\wedge\; y = y[+0]_t \;\wedge\; y = S_y(0)\right) \;\&$$
$$\left(z = z[+z]_0 \;\wedge\; z = 0[+z]_1 \;\wedge\; z = z[+0]_t \;\wedge\; z = S_z(0)\right) \;\&$$
$$\left(t = t[+1]_0 \;\wedge\; t = 0[+x]_1 \;\wedge\; t = x[+0]_t \;\wedge\; t = S_t(0)\right))$$

(def-Pr. 6.1)

Def. NT-6.2. $\forall x \forall y \forall z \forall t$ **(x, y, z, *t* ∈ $\mathcal{N}$(1⊗, 0⊕, 0⊂NT- PA)) (z[·x]$_t$, [·x]$_t$, [·x] ∈ {*Tensorial Symbols of the NTS&L Alphabet for natural number concept-identity*}**

Hereinafter, every natural number x (x ∈$\mathcal{N}$ (0⊂NT- PA)) resulting from (obtained by) the multiplication of natural number z (z∈$\mathcal{N}$(0⊂NT-PA)) and the natural number resulting from the multiplication t-times (t∈$\mathcal{N}$(0⊂NT-PA)) the natural number y (y∈ $\mathcal{N}$(0⊂NT- PA)), if an only if, the next statement is a wfs and wfs-r statement:

((x := z[·x]$_t$ & y := 1[·x]$_t$)) ⇒

(x = x[·1]$_1$ ∧ x = 1[·x]$_1$ ∧ x = x[·1]$_t$ ∧ x = S_x(0)) &

(y = y[·1]$_1$ ∧ y = 1[·y]$_1$ ∧ y = y[·1]$_t$ ∧ y = S_y(0)) &

(z = z[·1]$_1$ ∧ z = 1[·z]$_1$ ∧ z = z[·1]$_t$ ∧ z = S_z(0)) &

(*t* = *t*[+1]$_1$ ∧ t = 1[1x]$_1$ ∧ *t* = x[·1]$_t$ ∧ *t* = S_t(0)))

(def-Pr. 6.2)

Hence, hereinafter the notation, nomenclature and formulation in the first order language NFS&L of every natural number and every addition symbolic-representation by the next w.f.s and w.f.s-r statement about of symbols and/or formulas in both $\mathcal{N}$(0⊂NT-FOL-PA) and $\mathcal{N}$(1⊗, 0⊕, 0⊂NT- FOL-PA):

$\forall x \forall y \forall z \forall t$ **(x, y, z, *t* ∈ $\mathcal{N}$(1⊗, 0⊕, 0⊂NT- PA) ∧ $\mathcal{N}$(0⊂NT-FOL-PA))**

((z[+x]$_y$ = z[x+]$_y$)→([+x]$_y$ = [x+]$_y$)

→

$\mathcal{N}$(z[·x]$_t$, z[+x]$_t$, 1⊗, 0⊕, 0⊂NT-PA) ⊆ $\mathcal{N}$(0⊂NT-FOL-PA))

<u>HLPrime-7-NT Theorem</u>: **(∀x ∈ $\mathcal{N}$(0⊂NT-FOL-PA))**

7.1.

$\mathcal{NP}^{\alpha}_0$ (y[·x]$_t$; z[+x]$_t$, 1⊗, 0⊕, 0⊂NT- PA) ⊆

{ x ∈ $\mathcal{N}$(0⊂NT-PA)| x = S_x(0) ∧ y[·x]$_t$ & x ∉ $\mathcal{P}rimes$(*p*-P^{α}_0(x))},

$\forall x \forall y \forall t \forall p$ (x, y, t ∈ $\mathcal{N}$(0⊂NT-PA) , *p* ∈ $\mathcal{P}$(0⊂NT-PA) ∩ $\mathcal{S}uc^{\alpha}_0$(x))

(NT-Pr-th. 7.1)

7.2.

$\mathcal{NP}^{\alpha}_0$ (y[·x]$_t$; z[+x]$_t$; 1⊗, 0⊕, 0⊂NT- PA) ⊆

$$\{ x \in \mathcal{N}(0\subset\text{NT-PA}) \mid x = S_x(0) \wedge y[\cdot x]_t \;\&\; x \notin \mathcal{Primes}(p\text{-}P^{\alpha}{}_0(x))\},$$

$$\forall x \forall y \forall t \,\forall p \;(x, y, t \in \mathcal{N}(0\subset\text{NT-PA}),\, p \in \mathcal{P}(0\subset\text{NT-PA}) \cap \mathcal{Suc}^{\alpha}{}_0(x))$$

(NT-Pr-th. 7.2)[93]

7.3.

$$\mathcal{P}^{\alpha}(\, y[\cdot x]_t\,;\, 1\otimes, 0\oplus, 0\subset\text{NT-PA}) =$$

$$\{ x \in \mathcal{N}(0\subset\text{NT-PA}) \mid x = S_x(0) \;\wedge\; y[\cdot x]_t \;\&\; x \in \mathcal{Class}\ \text{I})\},$$

$$\forall x \forall y \forall t \;(x, y, t \in \mathcal{N}(0\subset\text{NT-PA}),\, x \in \mathcal{P}(0\subset\text{NT-PA}) \cap \mathcal{Suc}^{\alpha}{}_0(x))$$

FIFTH FUNDAMENTAL NT-FS&L THEOREM

$\forall x \,\forall y \,(x \in \mathcal{N}(0\subset\text{NT-PA}) \wedge y \in \mathcal{P}(0\subset\text{NT-PA}))$

Part.1

$$\mathcal{Even}^{\alpha}{}_0(y+0) \subseteq \mathcal{Even}^{\alpha}{}_0(x+x+0) \subseteq \mathcal{Even}^{\alpha}{}_0((x+0)+(0+x)) \rightarrow y = x+x$$

$$\mathcal{Odd}^{\alpha}{}_0(y+0) \subseteq \mathcal{Odd}^{\alpha}{}_0(x+x+0) \subseteq \mathcal{Odd}^{\alpha}{}_0((x+0)+(0+x)) \rightarrow y = x+x$$

$$\mathcal{Even}^{\alpha}{}_0(y+1) \subseteq \mathcal{Even}^{\alpha}{}_0(x+x+1) \subseteq \mathcal{Even}^{\alpha}{}_0((x+0)+(x+1)) \rightarrow y = x+x$$

$$\mathcal{Even}^{\alpha}{}_0(y+1) \subseteq \mathcal{Even}^{\alpha}{}_0(x+x+1) \subseteq \mathcal{Even}^{\alpha}{}_0((0+x)+(x+1)) =$$

$$\mathcal{Even}^{\alpha}{}_0((x+x)+1)) \subseteq \mathcal{Even}^{\alpha}{}_0(1+(x+x)) \rightarrow y = x+x$$

$$\mathcal{Even}^{\alpha}{}_0(y+2) \subseteq \mathcal{Even}^{\alpha}{}_0(x+x+2) \subseteq \mathcal{Even}^{\alpha}{}_0((x+1)+(x+1)) \rightarrow y = x+x$$

$$\mathcal{Even}^{\alpha}{}_0(y+2) \subseteq \mathcal{Even}^{\alpha}{}_0(x+x+2) \subseteq \mathcal{Even}^{\alpha}{}_0((0+x)+(x+2)) =$$

$$\mathcal{Even}^{\alpha}{}_0((x+x)+2)) \subseteq \mathcal{Even}^{\alpha}{}_0(2+(x+x)) \rightarrow y = x+x$$

&

$$\mathcal{Odd}^{\alpha}{}_0(y+1) \subseteq \mathcal{Odd}^{\alpha}{}_0(x+x+1) \subseteq \mathcal{Odd}^{\alpha}{}_0((x+0)+(x+1)) \rightarrow y = x+x$$

$$\mathcal{Odd}^{\alpha}{}_0(y+1) \subseteq \mathcal{Odd}^{\alpha}{}_0(x+x+1) \subseteq \mathcal{Odd}^{\alpha}{}_0((0+x)+(x+1)) =$$

$$\mathcal{Odd}^{\alpha}{}_0((x+x)+1)) \subseteq \mathcal{Odd}^{\alpha}{}_0(1+(x+x)) \rightarrow y = x+x$$

$$\mathcal{Odd}^{\alpha}{}_0(y+2) \subseteq \mathcal{Odd}^{\alpha}{}_0(x+x+2) \subseteq \mathcal{Odd}^{\alpha}{}_0((x+1)+(x+1)) \rightarrow y = x+x$$

$$\mathcal{Odd}^{\alpha}{}_0(y+2) \subseteq \mathcal{Odd}^{\alpha}{}_0(x+x+2) \subseteq \mathcal{Odd}^{\alpha}{}_0((0+x)+(x+2)) =$$

$$\mathcal{Odd}^{\alpha}{}_0((x+x)+2)) \subseteq \mathcal{Odd}^{\alpha}{}_0(2+(x+x)) \rightarrow y = x+x$$

&

$$\mathcal{Odd}^{\alpha}{}_0\ (x) \subseteq\ [\mathcal{PN}^{\alpha}{}_0\,(p)]_{\forall p\in \mathcal{P}(0\subset NT\text{-}PA)} \cup o\text{-}\mathcal{NP}^{\alpha}{}_0\,(x)\ [_{\forall p\in \mathcal{P}(0\subset NT\text{-}PA)}]$$

$$\mathcal{N}^{\alpha}{}_0(\ y[\cdot x]_t\,;z[+x]_t\,,1\otimes,0\oplus,0\subset NT\text{-}PA):=$$
$$[\mathcal{PN}_0\,(p)]_{\forall p\in \mathcal{P}(0\subset NT\text{-}PA)} \cup o\text{-}\mathcal{NP}(x) \cup [\ \mathcal{E}ven(x)\ )]_{\forall p\in \mathcal{P}(0\subset NT\text{-}PA)}$$

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$$\forall x\ \forall p\ (x\in \mathcal{N}(0\subset NT\text{-}PA) \wedge\ p\in\ \mathcal{N}(0\subset NT\text{-}PA) \cap \mathcal{P}(0\subset NT\text{-}PA))$$

$$\mathcal{P}(0\subset NT\text{-}PA):=\{\ p\in \mathcal{N}(0\subset NT\text{-}PA)|\ p\in \mathcal{P}rimes(p\text{-}P^{\alpha}{}_0(x))\}\ \forall x\ (x\in \mathcal{N}(0\subset NT\text{-}PA))$$

$$x = p+1\ \rightarrow (p\text{-}P^{\alpha}{}_0(x) = (p\text{-}P^{\alpha}{}_0(x+1) \rightarrow\ S^{\alpha}{}_0(p+1) = S^{\alpha}{}_1(p) \wedge$$
$$S^{\alpha}{}_0(p+1) = P^{\alpha}{}_0(x)$$

$$\{\mathcal{NP}^{\alpha}{}_0\{(x+x)^p\}_{\forall (x+x)=p\ (x,\,p\in\ \mathcal{N}(0\subset NT\text{-}PA))} \cap \mathcal{NP}^{\alpha}{}_0\{\ p^N\}_{\forall (x+x)=p\ (x\in\ \mathcal{N}(0\subset NT\text{-}PA))}\}:=$$
$$\{1,\,p\}_{\forall (x+x)=p\ (p\in\ \mathcal{N}(0\subset NT\text{-}PA)\ \cap\ \mathcal{N}(0\not\subset NT\text{-}PA))}\}$$
$$\rightarrow$$
$$\mathcal{P}^{\alpha}{}_0(0\subset NT\text{-}PA) \subseteq (\mathcal{E}ven^{\alpha}{}_0(p+p))_{\ \forall p\in \mathcal{P}(0\subset NT\text{-}A)} \subseteq \mathcal{E}ven^{\alpha}{}_0(x))_{\ \forall x=p+1\ \ p\in \mathcal{P}(0\subset NT\text{-}PA)}$$
$$\&$$
$$(\mathcal{E}ven^{\alpha}{}_0(x))_{\ \forall x=p+1,\ \ p\in \mathcal{P}(0\subset NT\text{-}PA)} \subseteq \mathcal{NP}^{\ \alpha}{}_0\{\ x^N\}_{\forall (x+1)=p\ (x\in\ \mathcal{N}(0\subset NT\text{-}PA))}$$
$$\rightarrow$$
$$(\mathcal{E}ven^{\alpha}{}_0(x+x))_{\ \forall x=p+1(x\in\ \mathcal{N}(0\subset NT\text{-}PA))} \subseteq \mathcal{NP}^{\ \alpha}{}_0\{\ x^N\}_{\forall (x+x)=2x\ (x\in\ \mathcal{N}(0\subset NT\text{-}PA))}$$

(NT-FS&L F.Th. 5)

SIXTH FUNDAMENTAL NT-FS&L THEOREM

$\forall x\forall y\forall z\ \forall t\forall n\forall p_1\forall p_2\ (x,\, t\in \mathcal{P}^{\alpha}{}_0\ (y[\cdot x]_t\,;z[+x]_t\,;\,1\otimes,0\oplus,0\subset NT\text{-}PA) \cap\ \mathcal{N}^{\alpha}{}_0\ (y[\cdot x]_t\,;z[+x]_t\ 1\otimes,0\oplus,0\subset NT\text{-}PA),\, p_n\in \mathcal{P}^{\alpha}{}_0\ (1\otimes,0\oplus,0\subset NT\text{-}PA),\ n\in \mathcal{N}^{\alpha}(\ y[\cdot x]_t\,;z[+x]_t\,;\,1\otimes,0\oplus,0\subset NT\text{-}PA))$

1. $\forall x\forall t\ (x,\, t\in \mathcal{E}ven\text{-}\mathcal{S}uc_{x+x}(1)\ \wedge\ (\mathcal{O}dd\text{-}(\mathcal{P}re_{x+x}(1)) \subseteq \mathcal{P}^{\alpha}{}_0\ (\ y[\cdot x]_t\,;z[+x]_t\,;\,1\otimes,0\oplus,0\subset NT\text{-}PA) \cap\ \mathcal{N}^{\alpha}{}_0\ (\ y[\cdot x]_t\,;z[+x]_t\,;\,1\otimes,0\oplus,0\subset NT\text{-}PA))$

$$(x + t =\ S_0(x+t)\ \wedge\ (o\text{-}S_0(x) + S_0(t))\ \wedge\ (o\text{-}S_0(x) + o\text{-}S_0(x))\ \wedge\ (e\text{-}2\cdot S_0(x))$$

$$\rightarrow$$

1.1. $t = x \rightarrow (o\text{-}S_0(x) = o\text{-}S_0(t)) \wedge (o\text{-}S_x(0) = o\text{-}S_t(0)) \wedge o\text{-}P_t(t+t) \wedge o\text{-}P_x(x+x)$

$$\Rightarrow ((x = p_1) \rightarrow (y = p_1 \wedge p\text{-}P_0(x) \wedge p\text{-}P_x(x+x)))$$

$$\wedge$$

$$x + t \in \mathcal{E}ven^{\alpha}(x+x) \wedge x \in \mathcal{O}dd\text{-}\mathcal{P}re^{\alpha}{}_1(x+1)$$

$$\wedge$$

$$x + t = 2\cdot p_1 \wedge (x + x)$$

$$\vee$$

1.2. $t \neq x \rightarrow (o\text{-}S_0(x) \neq o\text{-}S_0(t) \wedge (o\text{-}S_0(x) \neq o\text{-}S_0(t)) \wedge o\text{-}P_t(t+t) \wedge o\text{-}P_x(x+x)$

$$\wedge$$

$$((o\text{-}S_0(x)+o\text{-}S_t(x))= e\text{-}(S_0(t)+ S_x(t)) \rightarrow (o\text{-}S_0(x) = P_t(x+x)) \wedge (o\text{-}S_t(x) = o\text{-}P_0(t))$$

$$\Rightarrow ((x = p_1) \wedge (y = p_2) \rightarrow ((x = p\text{-}P_0(x)) \wedge (t = p\text{-}P_t(x) \wedge p\text{-}P_t(x+x)))$$

$$\wedge$$

$$x + t \in \mathcal{E}ven^{\alpha}{}_0(x+x) \wedge x \in \mathcal{O}dd\text{-}\mathcal{P}re^{\alpha}{}_1(x+1))$$

$$\wedge$$

$$(x + t)= (p_1 + p_2)$$

2. $\forall x \forall t(x, t \in \mathcal{E}ven^{\alpha}{}_0\text{-}\mathcal{S}uc^{\alpha}{}_{(x+x+x)}(1) \wedge (\mathcal{O}dd^{\alpha}{}_0\text{-}(\mathcal{P}re^{\alpha}{}_{x+x+x}(1)) \subseteq \mathcal{P}^{\alpha}{}_0 (y[\cdot x]_t ;$ $z[+x]_t ;$ $1\otimes, 0\oplus, 0\subset NT\text{-} PA) \cap \mathcal{N}^{\alpha}(y[\cdot x]_t ; z[+x]_t ; 1\otimes, 0\oplus, 0\subset NT\text{-} PA))$

2.0. $(\mathcal{O}dd^{\alpha}{}_0\text{-}(\mathcal{P}re^{\alpha}{}_{x+x+x}(1) \subseteq \mathcal{E}ven^{\alpha}{}_0\text{-}\mathcal{S}uc^{\alpha}{}_{x+x+x}(1) \wedge (\mathcal{O}dd^{\alpha}{}_0\text{-}\mathcal{P}re_{x+x}(1) \subseteq$ $\mathcal{E}ven^{\alpha}{}_0\text{-}\mathcal{S}uc^{\alpha}{}_{x+x}(1))$

$$(x + x + x) = 3\cdot x \wedge (x + x + t) \wedge (x + x)+ t \wedge (x)+ (x + t) \wedge (t) + (x + x)$$

$$\rightarrow$$

$$(t = x) \vee ((x+t) = e\text{-}(x+x) \wedge e\text{-}2\cdot x)$$

$$\Rightarrow$$

$$\exists(S_0(x+x+t) \wedge (S_0(x+x) + S_0(t)) \wedge (o\text{-}S_0(x+x) + o\text{-}S_0(x)) \wedge$$

$$(o\text{-}3\cdot S_0(x) \wedge (o\text{-}(2+1)\cdot S_0(x) \wedge (e\text{-}(2)\cdot S_0(x) + o\text{-}S_0(x)) \wedge (o\text{-}(1+1+1)\cdot S_0(x)))$$

$$S_1((x + x + t) \wedge P_1((x + x + t)$$

2.1. $o\text{-}(x + x + t)$

2.1.1. **$t = x \rightarrow (o\text{-}S_0(x) = o\text{-}S_0(t)) \wedge (o\text{-}S_x(0) = o\text{-}S_t(0)) \wedge o\text{-}P_t(t+t) \wedge o\text{-}P_x(x+x)$**

$$\Rightarrow ((x = p_1) \rightarrow (y = p_1 \wedge p\text{-}P_0(x) \wedge p\text{-}P_{x+x}(x) \wedge p\text{-}P_x(x+x)))$$

$$\wedge$$

$$x + t \in \mathcal{E}ven^{\alpha}{}_0 (x+x) \wedge x \in \mathcal{O}dd^{\alpha}{}_0\text{-}\mathcal{P}re^{\alpha}{}_1(x+1))$$

$$\wedge$$

$$x + t = p_1 + p_1$$

$$\vee$$

2.1.2. **$t \neq x \rightarrow (o\text{-}S_0(x) \neq o\text{-}S_0(t) \wedge (o\text{-}S_0(x) \neq o\text{-}S_0(t)) \wedge o\text{-}P_t(t+t) \wedge o\text{-}P_x(x+x)$**

$$\wedge$$

$$((o\text{-}S_0(x)+o\text{-}S_t(x))= (o\text{-}S_0(t)+ o\text{-}S_x(t)) \rightarrow (o\text{-}S_0(x) = P_t(x+x)) \wedge (o\text{-}S_t(x) = o\text{-}P_0(t))$$

$$\Rightarrow ((x = p_1) \wedge (y = p_2) \rightarrow ((x = p\text{-}P_0(x)) \wedge (t = p\text{-}P_t(x) \wedge p\text{-}P_{x+x}(x)))$$

$$\wedge$$

$$x + t \in \mathcal{E}ven^{\alpha}{}_0(x+x) \wedge x \in \mathcal{O}dd^{\alpha}{}_0\text{-}\mathcal{P}re^{\alpha}{}_1(x+1))$$

$$\wedge$$

$$(x + t) = (p_1 + p_2)$$

(NT-FS&L F.Th. 6)

SEVENTH FUNDAMENTAL NT-FS&L THEOREM

$\forall x_0 \forall x_1 \forall x_2 \ldots \forall x_n, \ \forall p_0 \forall p_1 \forall p_2 \ldots \forall p_n \forall n (x_n \ p_n \ x, \ t \in \mathcal{P}^{\alpha}{}_0(y[\cdot x]_t \ ; \ z[+x]_t \ ; \ 1\otimes, \ 0\oplus, \ 0 \subset NT\text{-}PA) \cap \mathcal{N}^{\alpha}{}_0(y[\cdot x]_t ; z[+x]_t ; 1\otimes, 0\oplus, 0 \subset NT\text{-}PA), p_n \in \mathcal{P}^{\alpha}{}_0 (y[\cdot x]_t ; z[+x]_t ; 1\otimes, 0\oplus, 0 \subset NT\text{-}PA), n \in \mathcal{N}^{\alpha}{}_0 (y[\cdot x]_t ; z[+x]_t ; 1\otimes, 0\oplus, 0 \subset NT\text{-}PA) \cap \mathcal{N}^{\alpha}{}_0(0 \not\subset NT\text{-}PA))$

1.

$$\mathcal{S}ubset\text{-}\{\mathcal{P}^{\alpha}{}_0 [p_0, p_1, p_2 \ldots, p_n]\} \forall n, n \in \mathcal{N}^{\alpha}{}_0 (y[\cdot x]_t ; z[+x]_t \ \ 1\otimes, 0\oplus, 0 \subset PA) \subseteq$$

$$\mathcal{S}ubset\text{-}\{\mathcal{P}^{\alpha}{}_0 [S_0(p_0), S_0(p_1), S_0(p_2), \ldots, S_0(p_n)]\} \forall n, n \in \mathcal{N}^{\alpha}{}_0(y[\cdot x]_t; z[+x]_t ; 1\otimes, 0\oplus, 0 \subset PA)$$

$$\#\text{-}\mathcal{S}ubset\text{-}\{\mathcal{P}^{\alpha}{}_0 [p_0, \ldots, p_n]\} \forall n, n \in \mathcal{N}^{\alpha}{}_0 (y[\cdot x]_t; z[+x]_t ; 1\otimes, 0\oplus, 0 \subset PA) = S_1(n)$$

$$\rightarrow$$

$$\mathcal{S}ubset\text{-}\{\mathcal{P}^{\alpha}{}_0[S_0(p_0), S_0(p_1), S_0(p_2), \ldots, S_0(p_n)]\} \forall n, n \in \mathcal{N}^{\alpha}(y[\cdot x]_t; z[+x]_t ,1\otimes, 0\oplus, 0 \subset PA) \subseteq$$

$Subset\text{-}\{\mathcal{N}^{\alpha}{}_0[P_1(0+1), P_1(1+1), P_1(2+1), \ldots, P_1(x_n+1)]\}\forall n, n \in \mathcal{N}^{\alpha}{}_0\ (y[\cdot x]t; z[+x]t; 1\otimes, 0\oplus, 0\subset PA)$

$\#\text{-}Subset\text{-}\{\mathcal{N}^{\alpha}{}_0\ [P_1(0+1), \ldots, P_1(x_n+1)]\}\forall n, n \in \mathcal{N}^{\alpha}{}_0\ [y[\cdot x]t; z[+x]t; 1\otimes, 0\oplus, 0\subset PA] = S_1(x_n)$

2.

$subset\text{-}\{\mathcal{P}^{\alpha}{}_0[p_0+p_0, p_1+p_1, p_2+p_2, \ldots, p_n+p_n]\}\forall n, n \in \mathcal{N}^{\alpha}{}_0[y[\cdot x]t; z[+x]t; 1\otimes, 0\oplus, 0\subset PA] \subseteq$

$subset\text{-}\{\mathcal{P}^{\alpha}[S_0(p_0+p_0), S_0(p_1+p_1), \ldots, S_0(p_n+p_n)]\}\forall n, n \in \mathcal{N}^{\alpha}[y[\cdot x]t; z[+x]t; 1\otimes, 0\oplus, 0\subset PA]$

$\#\text{-}Subset\text{-}\{\mathcal{P}^{\alpha}[S_0(p_0+p_0), \ldots, S_0(p_n+p_n)]\}\forall n, n \in \mathcal{N}^{\alpha}\ [y[\cdot x]t; z[+x]t; 1\otimes, 0\oplus, 0\subset PA] = S_1(n)$

$\wedge$

$2\otimes subset\text{-}\{Suc^{\alpha}\ S_0(p_n)\}\forall n, n \in \mathcal{N}^{\alpha}[y[\cdot x]t; z[+x]t; 1\otimes, 0\oplus, 0\subset PA] \subseteq$

$2\otimes subset\text{-}\{Suc^{\alpha}\ P_1(x_n+1)]\}\forall n, n \in \mathcal{N}^{\alpha}[y[\cdot x]t; z[+x]t; 1\otimes, 0\oplus, 0\subset PA]$

$\rightarrow$

$subset\text{-}\{\mathcal{P}^{\alpha}[p_0, p_1, \ldots, p_n]\}\forall n, n \in \mathcal{N}^{\alpha}\ [y[\cdot x]t; z[+x]t; 1\otimes, 0\oplus, 0\subset PA] \subseteq$

$subset\text{-}\{Suc^{\alpha}\ P_1(x_n+1)]\}\forall n, n \in \mathcal{N}^{\alpha}[y[\cdot x]t; z[+x]t; 1\otimes, 0\oplus, 0\subset PA]$

$\wedge$

$p_n = P_1(x_n+1)\forall n, n \in \mathcal{N}^{\alpha}[y[\cdot x]t; z[+x]t; 1\otimes, 0\oplus, 0\subset PA]$

$\rightarrow$

$(p_n + p_n) \in \mathcal{E}ven^{\alpha}{}_0(x_n+x_n) \wedge (p \wedge x_n \in \mathcal{O}dd^{\alpha}{}_0\text{-}\mathcal{P}re^{\alpha}{}_1(p-(x+1)))$

$\wedge$

$(S_0(p+p) = S_0(x+x) \wedge S_0(x) + S_p(x+x) \wedge S_0(x) + S_x(p+p))\forall x, \forall p\ (x, p \in \mathcal{N}^{\alpha}[y[\cdot x]t; z[+x]t; 1\otimes, 0\oplus, 0\subset PA] \cap \mathcal{P}^{\alpha}[y[\cdot x]t; z[+x]t; 1\otimes, 0\oplus, 0\subset PA] \rightarrow x_n = p_n$

3.

$subset\text{-}\{\mathcal{N}^{\alpha}{}_0\ [0, 1, 2, 3, \ldots, x_n]\}\forall n, n \in \mathcal{N}^{\alpha}{}_0\ [y[\cdot x]t; z[+x]t; 1\otimes, 0\oplus, 0\subset PA] \subseteq$

$subset\text{-}\{\mathcal{N}^{\alpha}{}_0\ [S_0(0), S_0(1), S_0(2), \ldots, S_0(x_n + x_n)]\}\forall n, n \in \mathcal{N}^{\alpha}{}_0\ [y[\cdot x]t; z[+x]t; 1\otimes, 0\oplus, 0\subset PA] \subseteq$

$subset\text{-}\{\mathcal{N}^{\alpha}{}_0\ [P_1(0+1), P_1(1+1), \ldots, P_1((x_n+1) + (x_n+1))]\}\forall n, n \in \mathcal{N}^{\alpha}{}_0\ [y[\cdot x]t; z[+x]t; 1\otimes, 0\oplus, 0\subset PA] \subseteq$

$subset\text{-}\{\mathcal{N}^{\alpha}{}_0\ [0, 1, 2, 3, \ldots, p_n]\}\forall n, n \in \mathcal{N}^{\alpha}{}_0\ [y[\cdot x]t; z[+x]t; 1\otimes, 0\oplus, 0\subset PA]$

$\rightarrow$

$$subset\text{-}\{Suc^{\alpha}_0\ S_0(x_n)\}\forall n, n \in \mathcal{N}^{\alpha}_0 [\ y[\cdot x]t;\ z[+x]t;\ 1\otimes, 0\oplus, 0\subset PA] \subseteq$$

$$subset\text{-}\{Suc^{\alpha}_0\ P_1(x_n+1)]\}\forall n, n \in \mathcal{N}^{\alpha}_0 [\ y[\cdot x]t;\ z[+x]t;\ 1\otimes, 0\oplus, 0\subset PA]$$

$$subset\text{-}\{\mathcal{P}^{\alpha}_0\ [S_0(p_0), S_0(p_1), S_0(p_2), \ldots, S_0(p_n)]\}\forall n, n \in \mathcal{N}^{\alpha}_0 [\ y[\cdot x]t;\ z[+x]t;\ 1\otimes, 0\oplus, 0\subset PA] \subseteq$$

$$subset\text{-}\{\mathcal{N}^{\alpha}_0\ [P_1(0+1), P_1(1+1), P_1(2+1), \ldots, P_1(x_n+1)]\}\forall n, n \in \mathcal{N}^{\alpha}_0 [\ y[\cdot x]t;\ z[+x]t;\ 1\otimes, 0\oplus, 0\subset PA]$$

$$\wedge$$

$$x_n = P_1(x_n+1)\forall n, n \in \mathcal{N}^{\alpha}_0 [\ y[\cdot x]t;\ z[+x]t;\ 1\otimes, 0\oplus, 0\subset PA]$$

$$\longrightarrow$$

$$(x_n + x_n) \in \mathcal{E}ven^{\alpha}_0(x_n+x_n) \wedge (x_n \wedge p_n \in \mathcal{O}dd^{\alpha}_0\text{-}\mathcal{P}re^{\alpha}_1\ (p\text{-}(x+1)))$$

$$\wedge$$

$$(S_0(p+p) = S_0(x+x) \wedge S_0(x) + S_p(x+x) \wedge S_0(x) + S_x(p+p))$$

$$\forall x, \forall p\ (x, p \in \mathcal{P}^{\alpha}_0 [\ y[\cdot x]t;\ z[+x]t;\ 1\otimes, 0\oplus, 0\subset PA] \cap \mathcal{N}^{\alpha}_0 [\ y[\cdot x]t;\ z[+x]t;\ 1\otimes, 0\oplus, 0\subset PA]$$

$$\longrightarrow\ x_n = p_n$$

(NT-FS&L F.Th. 7)

PRIME NUMBERS FUNDAMENTAL NT-FS&L INDUCTIVE THEOREM

$\forall x_0\forall x_1\forall x_2\ldots\forall x_n, \forall p_0\forall p_1\forall p_2\ldots\forall p_n(x_n\ p_n\ x,\ t \in \mathcal{P}^{\alpha}_0\ (\ y[\cdot x]t;\ z[+x]t;\ 1\otimes, 0\oplus, 0\subset NT\text{-}PA) \cap \mathcal{N}^{\alpha}(\ y[\cdot x]t;\ z[+x]t;\ 1\otimes, 0\oplus, 0\subset NT\text{-}PA), p_n \in \mathcal{P}^{\alpha}_0\ (\ y[\cdot x]t;\ z[+x]t;\ 1\otimes, 0\oplus, 0\subset NT\text{-}PA), \forall n, n \in \mathcal{N}^{\alpha}_0\ (0\subset NT\text{-}PA))$

1. $\forall p_n\ (p_n \in \mathcal{P}^{\alpha}_0\ (\ y[\cdot x]t;\ z[+x]t;\ 1\otimes, 0\oplus, 0\subset NT\text{-}PA), \forall n, n \in \mathcal{N}^{\alpha}_0\ (\ y[\cdot x]t;\ z[+x]t;\ 0\subset NT\text{-}PA))$

$$p_n = p\text{-}S^{\alpha}_0(x_n) \in \mathcal{E}ven^{\alpha}_0\ (S_0(x_n+1)) \wedge \mathcal{E}ven^{\alpha}_0\ (S_0(x_{n+1})) \longrightarrow$$

$$Subset\text{-}\{\mathcal{P}^{\alpha}_0\ [p\text{-}S^{\alpha}_0(p_0), p\text{-}S^{\alpha}_0(p_1), \ldots, p\text{-}S^{\alpha}_0(p_n)]\}\forall n, n \in \mathcal{N}^{\alpha}_0 [\ y[\cdot x]t;\ z[+x]t;\ 1\otimes, 0\oplus, 0\subset NT\text{-}PA]$$

$$\subseteq$$

$$Subset\text{-}\{\mathcal{N}^{\alpha}_0\ [p\text{-}S^{\alpha}_0(x_0), p\text{-}S^{\alpha}_0(x_1), \ldots, p\text{-}S^{\alpha}_0(x_n)]\}\forall n, n \in \mathcal{N}^{\alpha}_0 [\ y[\cdot x]t;\ z[+x]t;\ 1\otimes, 0\oplus, 0\subset NT\text{-}PA]$$

$$\subseteq$$

$$Subset\text{-}\{\mathcal{N}^{\alpha}{}_0\ [p\text{-}P^{\alpha}{}_1(x_0+1),\ p\text{-}P^{\alpha}{}_0(x_1+1),\ \ldots,\ p\text{-}P^{\alpha}{}_0(x_n+1)]\}\forall n,\ n\in\mathcal{N}^{\alpha}{}_0\ [\ y[\cdot x]t;\ z[+x]t\ ;\ 1\otimes,\ 0\oplus,\ 0\subset NT\text{-}PA]$$

$$\wedge$$

$$Subset\text{-}\{\mathcal{N}^{\alpha}{}_0\ [Suc^{\alpha}{}_0\ (p\text{-}{}^{\alpha}P_n(x_n+1)]\}\forall n,\ n\in\mathcal{N}^{\alpha}{}_0\ [\ y[\cdot x]t;\ z[+x]t\ ;\ 1\otimes,\ 0\oplus,\ 0\subset NT\text{-}PA]$$

$$\subseteq$$

$$Subset\text{-}\{\mathcal{N}^{\beta}{}_0\ [\mathcal{P}re^{\beta}{}_0(p\text{-}P_n(x_n+1)]\}\forall n,\ n\in\mathcal{N}^{\alpha}{}_0\ [\ y[\cdot x]t;\ z[+x]t\ ;\ 1\otimes,\ 0\oplus,\ 0\subset NT\text{-}PA]$$

$$\longrightarrow$$

$$Subset\text{-}\{\mathcal{P}^{\beta}{}_0\ [p_n,\ldots,\ p_2,p_1,p_0]\}\forall n,\ n\in\mathcal{N}^{\alpha}{}_0\ [y[\cdot x]t;\ z[+x]t\ ;\ 1\otimes,\ 0\oplus,\ 0\subset NT\text{-}PA]$$

2. $\forall p_n\ (p_n\in\mathcal{P}^{\alpha}{}_0\ (\ y[\cdot x]t;\ z[+x]t\ ;\ 1\otimes,\ 0\oplus,\ 0\subset NT\text{-}\ PA),\ \forall n,\ n\in\mathcal{N}^{\alpha}{}_0\ (\ y[\cdot x]t;\ z[+x]t\ ;\ 0\subset NT\text{-}\ PA))$

$$p_n + p_n = S^{\alpha}{}_1(p_n + p_n) \in \mathcal{E}ven^{\alpha}{}_0\ S_0(2\cdot p_n+ 1) \quad \longrightarrow$$

$$Subset\text{-}\{\mathcal{P}^{\alpha}{}_0\ [p\text{-}S^{\alpha}{}_0(p_0),\ p\text{-}S^{\alpha}{}_0(p_1),\ \ldots,\ p\text{-}S_0(2\cdot p_n+1)]\}\forall n,\ n\in\mathcal{N}^{\alpha}{}_0\ [\ y[\cdot x]t;\ z[+x]t\ ;\ 1\otimes,\ 0\oplus,\ 0\subset NT\text{-}PA]$$

$$\subseteq$$

$$Subset\text{-}\{\mathcal{N}^{\alpha}{}_0\ [p\text{-}S^{\alpha}{}_0(x_0),\ \ldots,\ p\text{-}S^{\alpha}{}_0(x_{2\cdot x+1})]\}\forall n,\ n\in\mathcal{N}^{\alpha}{}_0\ [\ y[\cdot x]t;\ z[+x]t\ ;\ 1\otimes,\ 0\oplus,\ 0\subset NT\text{-}PA]$$

$$Subset\text{-}\{\mathcal{N}^{\alpha}{}_0\ [p\text{-}P^{\alpha}{}_1(x_0+1),\ \ldots,\ p\text{-}P^{\alpha}{}_0(x_{2\cdot n+1})]\}\forall n,\ n\in\mathcal{N}^{\alpha}{}_0\ [\ y[\cdot x]t;\ z[+x]t\ ;\ 1\otimes,\ 0\oplus,\ 0\subset NT\text{-}PA]$$

$$\wedge$$

$$Subset\text{-}\{\mathcal{N}^{\alpha}{}_0\ [Suc^{\alpha}{}_0\ (p\text{-}P^{\alpha}(x_{2\cdot n+1})]\}\forall n,\ n\in\mathcal{N}^{\alpha}{}_0\ [\ y[\cdot x]t;\ z[+x]t\ ;\ 1\otimes,\ 0\oplus,\ 0\subset NT\text{-}PA]$$

$$\subseteq$$

$$Subset\text{-}\{\mathcal{N}^{\beta}{}_0\ [\mathcal{P}re^{\beta}{}_1(p\text{-}P^{\beta}{}_n(x_{2\cdot n+1})]\}\forall n,\ n\in\mathcal{N}^{\alpha}{}_0\ [\ y[\cdot x]t;\ z[+x]t\ ;\ 1\otimes,\ 0\oplus,\ 0\subset NT\text{-}PA]$$

$$\longrightarrow$$

$$Subset\text{-}\{\mathcal{P}^{\alpha}{}_0\ [p_0,\ p_1,\ p_2,\ \ldots,\ p_n\ ,\ p_{n+1}\ \ldots,\ p_{2\cdot n}]\}\forall n,\ n\in\mathcal{N}^{\alpha}{}_0\ [y[\cdot x]t;\ z[+x]t\ ;\ 1\otimes,\ 0\oplus,\ 0\subset NT\text{-}PA]$$

$$\wedge$$

$$Subset\text{-}\{\mathcal{P}^{\beta}{}_0\ [p_{2\cdot n},\ldots,\ p_{n+1},\ p_n,\ \ldots,\ p_2,\ p_1,\ p_0]\}\forall n,\ n\in\mathcal{N}^{\alpha}{}_0\ [y[\cdot x]t;\ z[+x]t\ ;\ 1\otimes,\ 0\oplus,\ 0\subset NT\text{-}PA]$$

In view of all the above, the next well symbolically represented and formulated statements hold true in the herein described NT-FS&L language: $\forall x\forall y\forall t\forall p$ *General* *nomenclature, formulation and notation (Alphabet valid for any numerical system referring the natural number concept)*

∀x, ∀x_n (x = S_0(x) ∧ n = S_0(x+1)∀n, n ∈$\mathcal{N}^{\alpha}$[1⊗, 0⊕, NT-PA⊂NT-FS&L]), ∀y ,∀z, ∀t

(x, x_n, y, z, *t*, *n = S_0(x+1)* ∀n, n ∈$\mathcal{N}^{\alpha}$[1⊗, 0⊕, 0⊂PA] ∈

{$\mathcal{N}^{\alpha}$ (x_n, z[+x]*t*, z[x+]*t*, z[·x]*t*, z[x·]*t*, z[x⊕]*t* z[x⊕]*t*, z[⊗x]*t*,z[x⊗]*t*, {0 ⊂ NT-PA)})[95,96]

1.1. Suc^{α}_0(x) **= { S_0(x), ..., (S_0(x)+1), (S_0(x)+2), ..., (S_0(x)+ x), ...} &**

1.2. $Suc^{\alpha}_{0,x}$(0) **= { S_x(0), ..., S_{x+1}(0), S_{x+2}(0), ..., S_{x+x}(0), ... } &**

1.2. $Suc^{\alpha}_{0,0}$(x) **= { S_0(x), S_0(x+1), S_0(x+2), ..., e-S_0(x+x), ...} &**

1.3. $Pre^{\alpha}_{0,x}$(0) **= {e-P_0(0), o-P_0(1), e-P_0(2), ..., P_0(x), ... e-P_0(x+x), ...} &**

1.4. $\mathcal{N}^{\alpha\alpha}_0$ **(NT-PA⊂NT-FS&L) = { e-P_x(x), o-P_0(1), e-P_0(2), ..., P_0(x), ... } &**

1.5. $\mathcal{N}^{\beta}_0$ **(NT-PA⊂NT-FS&L) = { ..., P_0(x), ..., P_0(2), P_0(1), P_0(0)}**

1.6. Suc^{α}_0(x) **= { e-S_0(x), o-S_0(x+1), e-S_0(x+2), ..., e-S_0(x+x), ...} }**

∩

{$\mathcal{P}^{\alpha}_{0,\{0\subset \text{NT-FOL-PA})\}}$ ⊆ {NT- FS&L}} [97]

⟶

2.1. ∀x_n, **(x_n∈ {*Subset –Even*$^{\alpha}_0$(x_n) ∧ *Subset –Even-*$\mathcal{N}^{\alpha}_0$(x_n+1) ∧**

***Subset –Even*$^{\alpha}_0$(x_n+ x_n) ⊆ $\mathcal{N}^{\alpha}_0${0⊂ PA)} ⊆ {NT- FS&L}},**

∀p_n ∈ {*Subset –Odd*$^{\alpha}_0$(x_n +1) ⊆ $\mathcal{P}^{\alpha}${0⊂ PA)} ⊆ {NT- FS&L}}∀x,

∀n ∈ {*Subset –Suc*$^{\alpha}_0$(x_n +1) ⊆ $\mathcal{N}^{\alpha}${0⊂ PA)} ⊆ {NT- FS&L}}∀x

o-p_n = p_n-P_1(x_n+1) ∧ x_n = e- S_0(x_n) ∧ e- S_0(x_n+0) ∧ e- S_n(x_n+x_n)

∧ e- S_{x_n}(x_n+n) ∧ e- (S_{x_n} (x_n)+ S_0(n)) ∧ e-(S_{x_n} ((x_n) + S_1(n+1))

∧ e- (S_{x_n} (x_n)) + P_1(n+1)

⟹

o-p_n = p_n-S_1(x_n +1) = p_n-(S_1(x_n+1)) = p-(S_0(x_n+2)) = p_n-(S_0(x_n)+2)) =

p_n-(S_0(x_n))+ 2) = p_n-(S_1(x_n)+1) = p_n-(S_0(x_n+1)+1) p_n-(S_0(x_n+2)) ⟶

o-p_n ∈ {*Subset –Pre*$^{\alpha}_0$ p_n-(P_0(x_n+2))}

∧

x_n ∈ {*Subset –Pre*$^{\alpha}_0$ p_n-(S_2(x_n))} ∧ {*Subset –Even*$^{\alpha}_0$(x_n +x_n) ⊆ $\mathcal{N}^{\alpha}_0${0⊂ NT-PA)} ⊆ {NT- FS&L}},

⟶

$$(o\text{-}p_n + o\text{-}p_n) = e\text{-}(x_n) \wedge 2\cdot(o\text{-}p_n) \wedge e\text{-}(x_n + x_n) = e\text{-}(2\cdot x_n)) \Rightarrow$$

$$(e\text{-}x_n \neq o\text{-}p_n) \vee (e\text{-}(2\cdot p_n)) = e\text{-}(2\cdot x_n)) = e\text{-}(x_n+x_n)) \rightarrow$$

2.1.1. $(e\text{-}x_n \neq o\text{-}p_n) \Rightarrow ((o\text{-}p_n = p_n\text{-}P_1(x_n+1)) \wedge (e\text{-}x_n = p_n\text{-}P_1(p_n+1) = p_n\text{-}S_0(x_n)) \rightarrow e\text{-}x_n = p_n\text{-}S_0(x_n) \wedge p_n\text{-}S_n(x_n + x_n) \wedge p_n\text{-}P_0(x_n) \wedge p_n\text{-}P_n(x_n + x_n) \wedge (o\text{-}p_n + o\text{-}p_n) \wedge 2\cdot(o\text{-}p_n)$

$$\Rightarrow e\text{-}x_n = e\text{-}(o\text{-}p_n + o\text{-}p_n) \wedge e\text{-}(P_1(p_n+1) + P_1(p_n+1)) \wedge 2\cdot S_0(x_n) \wedge S_0(x_n) + S_0(x_n) \wedge e\text{-}(S_1(p_n+1) + S_1(p_n+1)) \wedge e\text{-}(S_2(p_n) + S_2(p_n))$$

2.1.2. $e\text{-}(2\cdot p_n)) = e\text{-}(2\cdot x_n)) = e\text{-}(x_n+x_n) \Rightarrow (o\text{-}p_n + o\text{-}p_n) = (p\text{-}P_1(x_n+1) + p\text{-}P_1(x_n+1)) \wedge e\text{-}(p\text{-}P_1(x_n+1 +(x_n+1)) \wedge e\text{-}(p\text{-}P_1(2\cdot x_n+2) \wedge e\text{-}2(p\text{-}P_1(x_n+1)) = 2\cdot(p\text{-}P_1(x_n+1)) \wedge 2\cdot S_0(x_n) \wedge e\text{-}(S_0(x_n) + S_0(x_n)) \wedge e\text{-}(S_2(p_n) + S_2(p_n))$

3.1. $\forall x_n, (x_n \in \{\mathit{Subset\text{-}Odd}^{\alpha}{}_0(x+1) \wedge \mathit{Subset\text{-}Odd}^{\alpha}{}_0(x_n + x_n + 1) \subseteq \mathcal{N}^{\alpha}{}_0\{0 \subset PA)\} \subseteq \{NT\text{-} FS\&L\}\}$

$\forall p_n \in \{\mathit{Subset\text{-}Odd}^{\alpha}{}_0(x_n + 1) \wedge \mathit{Subset\text{-}Odd}^{\alpha}{}_0(2\cdot x_n + 1) \subseteq \mathcal{P}^{\alpha}{}_0\{0 \subset PA)\} \subseteq \{NT\text{-} FS\&L\}\} \forall x,$

$\forall n \in \{\mathit{Subset\text{-}Suc}^{\alpha}{}_0(x_n + 1) \subseteq \mathcal{N}^{\alpha}\{0 \subset PA)\} \subseteq \{NT\text{-} FS\&L\}\} \forall x$

$$o\text{-}p_n = p_n\text{-}P_1(x_n+1) \wedge x_n = e\text{-} S_0(x_n) \wedge e\text{-} S_0(x_n+0) \wedge e\text{-} S_n(x_n+x_n) \wedge e\text{-} S_{x_n}(x_n+n) \wedge e\text{-} (S_{x_n}(x_n) + S_0(n)) \wedge e\text{-}(S_{x_n}((x_n) + S_1(n+1)) \wedge e\text{-} (S_{x_n}(x_n)) + P_1(n+1)$$

$$\Rightarrow$$

$$\mathit{Subset\text{-}Odd}^{\alpha}{}_0(x_n + x_n + 1) \subseteq \mathit{Subset\text{-}Odd}^{\alpha}{}_0((x_n + x_n) + 1) \subseteq \mathit{Subset\text{-}Odd}^{\alpha}{}_0(x_n + x_n) \cup \mathit{Subset\text{-}Odd}^{\alpha}{}_0(1) \wedge \mathit{Subset\text{-}Odd}^{\alpha}{}_0(x_n) \cup \mathit{Subset\text{-}Odd}^{\alpha}{}_0(x_n+1) \rightarrow$$

$$x_n \in \{\mathit{Subset\text{-}Odd}^{\alpha}{}_0(x_n+1) \cup (\mathit{Subset\text{-}Odd}^{\alpha}{}_0(e\text{-}(x_n+x_n)) \subseteq \mathit{Subset\text{-}Even}^{\alpha}{}_0(x_n + x_n))\}$$

$$\rightarrow$$

$$\exists ((o\text{-}p_n + o\text{-}p_n + o\text{-}p_n) = 3\cdot o\text{-}p_n = o\text{-}p_n + e\text{-}(o\text{-}p_n + o\text{-}p_n) = (x_n + x_n) + o\text{-}p_n = (o\text{-}p_n + x_n) + o\text{-}p_n = e\text{-}(o\text{-}x_n + o\text{-}x_n) + o\text{-}p_n = (o\text{-}x_n + o\text{-}x_n) + o\text{-}x_n = o\text{-}x_n + o\text{-}x_n + o\text{-}x_n = 2\cdot(p_n\text{-}x_n) + 1\cdot p_n\text{-}x_n$$

$$\rightarrow$$

$$\exists\, o\text{-}x_n = 2\cdot(p_n - x_n) \wedge o\text{-}x_n = (1\cdot (p_n - x_n)) \Rightarrow$$

$$(p_n - x_n = e\text{-}(o\text{-}p_n + o\text{-}p_n) \wedge e\text{-}(2\cdot (p_n - x_n) \wedge o\text{-}(p_n - p_n)$$

&

Further numerable and countable NT FS&L Theorems by means of " induction"
(**Entailment ensured from the above analyzed cases 2.1.1 & 2.1.2 from NT-PA.)**

$$\rightarrow$$

Further numerable and countable NT FS&L-Theorems by means of "induction"
(Entailment ensured from the above analyzed cases from NT-FOL-PA).

6. NT-DEMONSTRATION (PROOF) OF THE FAMILY OF CONJECTURES KNOWN AS "GOLDBACH´s"

Please, for all of the above, let us state in NT-SF& that:

$$\forall x, \forall x_n\ (x = S_0(x) \wedge n = S_0(x+1) \forall n,n \in \mathcal{N}^{\alpha}[\,y[\cdot x]t;\ z[+x]t;\ 1\otimes, 0\oplus, \text{NT-PA}\subset\text{NT-FS\&L}]),$$
$$\forall y, \forall z, \forall t$$

$$(x,\ x_n, y, z, t,\ n = S_0(x+1)\ \forall n, n \in \mathcal{N}^{\alpha}{}_0\,[\,y[\cdot x]t;\ z[+x]t;\ 1\otimes, 0\oplus, 0\subset \text{PA}] \in$$
$$\{\mathcal{N}^{\alpha}{}_0(x_n, z[+x]t, z[x+]t, z[\cdot x]t, z[x\cdot]t, z[x\oplus]t\ z[x\oplus]t, z[\otimes x]t, z[x\otimes]t, \{0 \subset \text{NT-PA})\})$$

$$\mathcal{PN}^{\alpha}{}_0(x) \wedge \mathcal{PN}^{\alpha}(x_n) \subseteq \{\mathcal{N}^{\alpha}{}_{0\{0\subset \text{PA})\}} \subseteq \{\text{NT- FS\&L}\}\} \cap \{\mathcal{P}^{\alpha}{}_{0\{0\subset \text{PA})\}} \subseteq \{\text{NT- FS\&L}\}\}$$

$$\Rightarrow$$

$$\mathcal{PN}^{\alpha}{}_0(1) \wedge \mathcal{PN}^{\alpha}{}_0(2) \wedge \mathcal{PN}^{\alpha}{}_0(3) \wedge \mathcal{PN}^{\alpha}{}_0(5) \wedge \mathcal{PN}^{\alpha}{}_0(7)$$

$$\subset$$

$$\{\mathcal{N}^{\alpha}{}_{0,\{0\subset \text{PA})\}} \subseteq \{\text{NT- FS\&L}\}\} \cap \{\mathcal{P}_{0,\{0\subset \text{PA})\}} \subseteq \{\text{NT- FS\&L}\}\}$$

Quod erat demonstrandun [98]

All of the above, we are able to offer a translation, from the Latin-German, 18th century into contemporary English by means of the NT-PA FS&L, of the statements conforming ***the family of conjectures known as Goldbach´s,*** *and formally express and evoke at the same time a logical and mathematical truth, if and only if, the following four NT-FS&L theorems referred to the natural number and to the prime natural number concepts are at once well formed and well symbolically representation statements about the natural number concepts involved:*

0 = (0 + 0); **1** = (0 + **1)**; **2** = ((0 + 2) = (**1 + 1**)); **3** = ((0 + 3) = (2 + 1) = (**1 + 1 +1**))

1.- -(GC-I)- Goldbach-Euler Statement:

"Every integer which can be written as the sum of two primes, can also be written as the sum of as many primes as one wishes, until all terms are units".
(*Translated into contemporary English, from Latin-German, 18th century a.C.)*

*"Every natural number successor of the natural number cero-successor of natural number two (briefly represented in the decimal system by symbol **2**) can be expressed as the sum of two prime natural numbers, and as the sum of as many primes natural numbers as one wishes, until all terms are the prime natural number one (briefly represented in positional-decimal system by the symbol **1**)".*
(*NT-FS&L theorem in contemporary English)*

2.- (GC-IV)-Goldbach-Euler Statement:

"Every even integer greater than 2 can be written as the sum of two primes".
(*Translated into contemporary English, from Latin-German, 18th century a. C.)*

*"Every even "**natural number non-cero-successor**" of the natural number "**two-successor of natural number zero" of prime natural number one**" ("two-successor of natural number zero" hereinafter briefly represented in turn in decimal system by both: i.-the decimal symbol **2 representing the natural number two concept,** and ii.- by the well symbolic representation formula and true statements referred to natural number zero **(0+1)+(0+1)= (1·(1+1) and/or (1·2+0) and/or (1+1**), can be covariant-wise represented and identify as the addition of two prime natural numbers referred to natural number zero"*
(*NT-FS&L theorem in contemporary English)*

3.- (GC-II), (GC-III) and-(GC-V) Goldbach-Euler Statements, respectively:

"Every integer greater than 2 can be written as the sum of three primes" and
"Every integer greater than 5 can be written as the sum of three primes" and
"All odd numbers greater than 7 are the sum of three odd primes"
(*Translated into contemporary English, from Latin-German, 18th century a. C.)*

*"Every **odd** "**natural non-zero-successor**" of the natural number "one-successor of natural number zero" (("one-successor of natural number zero" hereinafter briefly represented in turn in decimal system by both: i.- the decimal symbol **1 representing the natural number one concept,** and, ii.- the well symbolic representation formula and true statements **(0+1)= (1·(0+1) and/or (1·1+0)**), can be covariant-wise represented as the sum of tree prime natural numbers referred to natural number zero and being "three" the "**three-successor of natural number zero**" (briefly represented in turn in decimal system by both: i.- the decimal symbol **3 representing the natural number three concept,** and, ii.- the well symbolic representation formula and true statements **3 = (0+1)+(0+1)+(0+1) and/or (3·(1+0) and/or (1·(1+1)+ 1) and/or (1·(2)+ 1) and/or (1+1+1)**) "*
(*NT-FS&L theorem in contemporary English)*

Hence, we have described that the new system model and formal language, which provide a new signature for the set of the natural numbers, the family of conjectures known as Glodbach's holds entailment and truth and therefore has been proved (proof).

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[1] First contribution to the ***NT Intellectual Creations & Invention Reports*** set titled ***NT-Principia of Human Reasoning and Knowledge*** (P. **N**oheda and N. **T**abarés, Copyright-**NT, 2016**).

NT-I-Intellectual Creation; **Part I.** *On the Noheda-Tabarés Formal System and Language (NT-FS&L)* and **Part II.** *"The first set of Noheda-Tabarés (NT) Theorems. A primordial, mathematical, logical and computable, demonstration (proof) of the family of conjectures known as Goldbach´s",* P. **N**oheda and

N. **T**abarés ; **Part III.** *"The first set of Noheda-Tabarés (NT-FS&L) Algorithms,* P. **N**oheda and N. **T**abarés; VATC/CSIC (Spain), December 15, **2016**.

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of truth"); In Peano´s, in an equation, the equals sign will be "placed" between ***two expressions that are referring to any of their "conventional accepted identities" and to "their conventional accepted symbolic-identities representations"*** (Please, see Jeff Miller. "*Earliest Uses of Symbols of Set Theory and Logic*". (Internet access *http://jeff560.tripod.com/set.html).*

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[25] *Please, let us introduce the following eight general NT-considerations about nomenclature, formulation and notation for the set of natural numbers* $\mathcal{N}$:

1. [**P**], (**P**) and **P**; ***Peano axiomatic*** of the natural number concept and any collectivity of them.

2. [**PA**], (**PA**) and **PA**; ***Peano Arithmetic axiomatic*** of natural number concept and any collectivity of them.

3. In formula, $\mathcal{N}$ **[0⊂PA]**, subindex "**[0⊂PA]**" means that: "natural number zero concept" is included in **axiomatic** briefly named **PA ";** the statement " $0 \in \mathcal{N}$**[P]**" means "natural number zero belongs natural numbers set $\mathcal{N}$ described by set of axioms briefly named **P**".

4.

$$\mathbf{0} \in \mathcal{N}[\mathbf{P}] \Rightarrow \mathcal{N}[\mathbf{0 \subset PA}] := \mathcal{N}[\mathbf{P} \subset \mathbf{2.0.0\text{-}2.1.5}]$$

&

$\mathcal{N}$**[0⊂PA] :=** **{*successors of 0*}0** ⊂ $\mathcal{N}$[0⊂PA] **(***briefly,* **{*suc*(0)}0** ⊂ $\mathcal{N}$[0⊂PA] ***and/or*** **{0}0** ∈ $\mathcal{N}$[0⊂PA]**).**

5.

$$\mathbf{0} \notin \mathcal{N}[\mathbf{P}] \Rightarrow \mathcal{N}[\mathbf{0 \not\subset PA}] := \mathcal{N}[\mathbf{P} \subset \mathbf{2.1.0\text{-}2.0.5}]$$

&

$\mathcal{N}$**[0⊄PA] :=** **{*successors of 1*}0** ⊄ $\mathcal{N}$[0⊂PA] **(***briefly,* **{*suc*(1)}0** ⊄ $\mathcal{N}$[0⊂PA]**)** ***and/or*** **{1}0** ∈ $\mathcal{N}$[0⊂PA]**)**

6.

$$\{0\}_0 \in \mathcal{N}[0\subset PA] := \{suc(0)\}_0 \subset \mathcal{N}[0\subset PA] \cap \{suc(1)\}_0 \not\subset \mathcal{N}[0\subset PA]$$

$$\wedge \vee$$

$$\{0\}_0 \in \mathcal{N}[0\subset PA] := \{\{suc(0)\}_0 \subset \mathcal{N}[0\subset PA] \cap \{suc(1)\}_1 \subset \mathcal{N}[0\subset PA]$$

7.

$$\{1\}_0 \in \mathcal{N}[0\subset PA] := \{suc(0)\}_0 \subset \mathcal{N}[0\subset PA] \cap \{suc(1)\}_0 \not\subset \mathcal{N}[0\subset PA]$$

$$\wedge \vee$$

$$\{1\}_0 \in \mathcal{N}[0\subset PA] := \{\{suc(0)\}_0 \subset \mathcal{N}[0\subset PA] \cap \{suc(1)\}_1 \subset \mathcal{N}[0\subset PA]$$

8.

$$\mathcal{N}[P] := \{\{suc(0)\}_0 \subset \mathcal{N}[0\subset PA] \cap \{suc(1)\}_1 \subset \mathcal{N}[0\not\subset PA]\}$$

$$\wedge \vee$$

$$\mathcal{N}[P] := \{\{suc(1)\}_1 \subset \mathcal{N}[0\subset PA] \cap \{suc(1)\}_0 \not\subset \mathcal{N}[0\not\subset PA]\}$$

$$\Rightarrow \quad \mathcal{N}[\mathbf{P}] \subseteq \mathcal{N}[\mathbf{0\subset PA}] \cap \mathcal{N}[\mathbf{0\not\subset PA}]$$

$$\wedge \vee$$

$$\mathcal{N}[\mathbf{P}] \subseteq \mathcal{N}[\mathbf{PA}]$$

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[28] Hereinafter the natural number concept representing "the amount of elements (also: 1. members and, "entities" (Chemistry, Biology and Statistic), 2. "measurement and/or value of a magnitude" ("amount of unities" quantifying a magnitude in Physics) of a singleton will be the natural number one (**1**); the natural number concept representing the amount of a doubleton is the natural number two (**2**) and so on. The natural number identity of the amount of elements of a collectivity of natural numbers (set and subsets) herein will be symbolically represented by the symbol "#" ("hash" in English; "almohadilla", in Spanish). Please, let us remember to the reader the concept of "mol" in the International System of Measuring as well as the universal constant named Avogadro´s number)

Please, let us observe that the singleton $\{0\}_{0 \in \mathcal{N}[0\subset PA]}$ is conformed by one element, the natural number zero (**0**). Hereinafter, the "empty set" **(Θ) defined by the next statement and formula:**

$$\Theta := \{ \varnothing \mid \varnothing \notin \mathcal{N}[0\subset PA] \wedge \mathcal{N}[0\not\subset PA] \}, \forall n \in \mathcal{N}[0\subset PA] \vee \mathcal{N}[0\not\subset PA] \,;\ \{\varnothing\}_{\varnothing\in\Theta}$$

it is not "a countable set" as a direct consequence (sense and intension) that their elements are not a natural number concept.

$$\varnothing \notin \mathcal{N}[0\subset PA] \wedge \mathcal{N}[0\not\subset PA] \ \ \& \ \ \varnothing \notin \mathcal{N}[0\subset PA] \vee \mathcal{N}[0\not\subset PA]$$

$$\forall n\, (n \in \mathcal{N}[0\subset PA] \vee \mathcal{N}[0\not\subset PA])(\varnothing \neq 0 \wedge n \wedge \text{"successor of } n\text{"})$$

Quotation´s Internet access: http://mathworld.wolfram.com/Zero.html). Kaplan, Robert. (2000). *The Nothing That Is: A Natural History of Zero*. Oxford: Oxford University Press.): "***Zero** is the integer denoted **0** that, when used as a counting number, means that no objects are present ... It is the only integer that is neither (graded, categorized) as negative nor positive. A root of a function f is also customary graded and known as "a zero of f ".*

Please, let us remember two conceptually radical different perspectives (which are globally accepted per years) about the singleton concept and their "core-role" in the state of the art of the current accepted typed framework structures (the Set and Category Theories), from that described in this document (NT-FS&L): 1. *(Please, take into account that "both, Zermelo-Frankel (briefly* ***ZF****) and Zermelo-Fraenkel axiomatic complemented with the "axiom of choice" (briefly* ***ZFC****) set theories, are focused and addressed -sense and intension- on a real number set (also, $\mathcal{R}$) axiomatic).* ***Set Theory****. " Within the framework of Zermelo–Fraenkel set theory, the "axiom of regularity" guarantees that no set is an element of itself. This implies that a singleton is necessarily distinct from the element it contains, thus 1 and {1} are not the same thing, and the empty set is distinct from the set containing only the empty set. A set such as {{1, 2, 3}} is a singleton as it contains a single element (which itself is a set, however, not a singleton). A set is a singleton if and only if its cardinality is 1. Additionally, in the standard set-theoretic construction of the natural numbers set (also, $\mathcal{N}$), the number 1 is defined as the singleton {0}. In axiomatic set theory, the existence of singletons is a consequence of the "axiom of pairing": for any set A, the axiom applied to A and A asserts the existence of {A, A}, which is the same as the singleton {A} (since it contains A, and no other set, as an element). If A is any set and S is any singleton, then there exists precisely one function from A to S, the function sending every element of A to the single element of S. Thus, every singleton is a terminal object in the category of sets. A singleton has the property that every function from it to any arbitrary set is injective. The only non-singleton set with this property is the empty set".* ***2. Category Theory.*** *"Typed structures built on singletons very often serve as "terminal objects" or "zero objects" of various categories: 1. The singleton sets are precisely the terminal objects in the category "set of sets". No other sets are terminal. 2. Usually, any singleton admits a "unique topological space structure". These singleton topological spaces are "terminal objects" in the category of topological spaces and continuous functions. No other spaces are terminal in that category. 3. Any singleton admits a unique group structure (the unique element serving as identity element). These singleton groups are zero objects in the category of groups and group homomorphisms. No other groups are terminal in that category.* (Internet access to the quotation: https://en.wikipedia.org/wiki/Singleton_(mathematics).

[29] $$((\mathbf{0} \in \mathcal{N}[0\subset \mathrm{PA}] \wedge \mathbf{0} \notin \mathcal{N}[0\not\subset \mathrm{PA}]) \wedge (\mathbf{1} \in \mathcal{N}[0\subset \mathrm{PA}] \wedge \mathbf{1} \in \mathcal{N}[0\not\subset \mathrm{PA}])) \Rightarrow (\#\text{-}\, \mathcal{N}[0\subset \mathrm{PA}] = \#\text{-}\, \mathcal{N}[0\not\subset \mathrm{PA}] + \mathbf{1}, \{\mathbf{0}\}_{\mathbf{0} \in \mathcal{N}[0\subset \mathrm{PA}]} \cap \{\mathbf{1}\}_{\mathbf{1} \in \mathcal{N}[0\subset \mathrm{PA}]}\ \mathbf{1} \in \mathcal{N}[0\subset \mathrm{PA}] \wedge \mathcal{N}[0\not\subset \mathrm{PA}])$$

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33 *Please, let us point out that in NT-FS&L, the following statement, which is accepted as the* ***induction schema (7)*** *of* ***PA´s*** *expressed in current first order logic, holds the methodology to count, (also, to enumerate) any infinite collectivity conformed by natural numbers as a evident consequence that it is a well formed statement (w.f.s) if and only if de Morgan Laws are considered embedded. Hence, in*

$$\forall\bar{y}\ \big(\varphi(0,\bar{y}) \wedge \forall x(y(x,\bar{y}) \Rightarrow \varphi(S(x),\bar{y})) \Rightarrow \forall x\varphi(x,\bar{y})\big)$$

$\bar{y}$ *symbolizes an abbreviation for free variables (occurrences)* $y_1, \ldots, y_k$ *, that extents the* ***first-order induction axiom*** *for every* $\varphi(x, y_1, \ldots, y_k)$ *and each of them in a embed "tensorial-like formal symbolical-representation of an identity of the natural numbers".*

34

$$subset\text{-}\mathcal{N}[0, 1 \subset \text{FOL-PA}] := \{0, 1\}\forall n \in \mathcal{N}[0 \subset \text{PA}] \rightarrow \#\text{-}\ subset\text{-}\mathcal{N}[0, 1 \subset \text{FOL-PA}] = 2;$$

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[35] **Ring concept. 1.** Fraenkel, A. (1914). "Über die Teiler der Null und die Zerlegung von Ringen". J. reine angew. Math. 145: 139–176. *(English translation by Laurence Sigler, 2002).* **2.** *Hilbert, David (1897). "Die Theorie der algebraischen Zahlkörper". Jahresbericht der Deutschen Mathematiker Vereinigung. (http://www.fen.bilkent.edu.tr/~franz/publ/hil.pdf)* **3**. Noether, Emmy (1921). "Idealtheorie in Ringbereichen". Math. Annalen. 83: 24–66. doi:10.1007/bf01464225. (please, see: *Fleischmann, Peter (2000), "The Noether bound in invariant theory of finite groups",* Advances in Mathematics, *156 (1): 23–32, doi:10.1006/aima.2000.1952, MR 1800251)* **4.** *Allenby, R. B.* Rings, Fields, and Groups: An Introduction to Abstract Algebra, 2nd ed. *Oxford, England: Oxford University Press, 1991.* **5.** *Birkhoff, G. and Mac Lane, S.* A Survey of Modern Algebra, 5th ed. *New York: Macmillian, 1996.* **6.** *Bronshtein, I. N.; Semendyayev, K. A.; Musiol, G.; and Muehlig, H.* Handbook of Mathematics, 4th ed. *New York: Springer-Verlag, 2004.* **7.** *Fletcher, C. R. "Rings of Small Order."* Math. Gaz. *64, 9-22, 1980. 15. Harris, J. W. and Stocker, H.* Handbook of Mathematics and Computational Science. *New York: Springer-Verlag, 1998.* **8.** *Kleiner, I. "The Genesis of the Abstract Ring Concept."* Amer. Math. Monthly *103, 417-424, 1996.* **9.** *Knuth, D. E.* The Art of Computer Programming, Vol. 2: Seminumerical Algorithms, 3rd ed. *Reading, MA: Addison-Wesley, 1998.* **10.** *Weisstein, Eric W. "Ring." From* MathWorld--*A Wolfram Web Resource. http://mathworld.wolfram.com/Ring.html.* **11.** *Artin, Michael (1991). Algebra. Prentice-Hall.* **12.** *Atiyah, Michael; Macdonald, Ian G. (1969). Introduction to commutative algebra. Addison–Wesley.* **13.** *Bourbaki, N. (1998). Algebra I, Chapters 1-3. Springer.* **14.** *Cohn, Paul Moritz (2003), Basic algebra: groups, rings, and fields, Springer, ISBN 978-1-85233-587-8.* 5. **15.** *Eisenbud, David (1995). Commutative algebra with a view toward algebraic geometry. Springer.* **16.** *Gardner, J.W.; Wiegandt, R. (2003). Radical Theory of Rings. Chapman & Hall/CRC Pure and Applied Mathematics. ISBN 0824750330.* **17.** *Jacobson, Nathan (2009). Basic algebra. (2nd ed.). Dover. ISBN 978-0-486-47189-1.* **18.** *Kaplansky, Irving (1974), Commutative rings (Revised ed.), University of Chicago Press, ISBN 0-226-42454-5, MR 0345945.* **19.** *Lam, Tsit Yuen (2001). A first course in noncommutative rings. Graduate Texts in Mathematics. 131 (2nd ed.). Springer. ISBN 0-387-95183-0.* **20.** *Lang, Serge (2002), Algebra, Graduate Texts in Mathematics, 211 (Revised third ed.), New York: Springer-Verlag, ISBN 978-0-387-95385-4, Zbl 0984.00001, MR1878556.* **21.** *Matsumura, Hideyuki (1989). Commutative Ring Theory. Cambridge Studies in Advanced Mathematics (2nd ed.). Cambridge University Press. ISBN 978-0-521-36764-6.* **22.** *Balcerzyk, Stanisław; Józefiak, Tadeusz (1989), Commutative Noetherian and Krull rings, Mathematics and its Applications, Chichester: Ellis Horwood Ltd., ISBN 978-0-13-155615-7.* **23**. *Balcerzyk, Stanisław; Józefiak, Tadeusz (1989), Dimension, multiplicity and homological methods, Mathematics and its Applications, Chichester: Ellis Horwood Ltd., ISBN 978-0-13-155623-2.* **24.** *Ballieu, R. (1947). "Anneaux finis; systèmes hypercomplexes de rang trois sur un corps commutatif". Ann. Soc. Sci. Bruxelles. I (61): 222–227.* **25**. *Berrick, A. J.; Keating, M. E. (2000). An Introduction to Rings and Modules with K-Theory in View. Cambridge University Press. 5. Cohn, Paul Moritz (1995), Skew Fields: Theory of General Division Rings, Encyclopedia of Mathematics and its Applications, 57, Cambridge University Press, ISBN 9780521432177.* **26**. *Eisenbud, David (1995), Commutative algebra. With a view toward algebraic geometry., Graduate Texts in Mathematics, 150, Springer, ISBN 978-0-387-94268-1, MR 1322960. 7. Gilmer, R.; Mott, J. (1973). "Associative Rings of Order". Proc. Japan Acad. 49: 795–799. doi:10.3792/pja/1195519146.* **27.** *Harris, J. W.; Stocker, H. (1998). Handbook of Mathematics and Computational Science. Springer.* **29.** *Serre, Jean-Pierre (1979), Local fields, Graduate Texts in Mathematics, 67, Springer. 10. Weibel, Charles. "The K-book: An introduction to algebraic K-theory".* **30.** *Zariski, Oscar; Samuel, Pierre (1975). Commutative algebra. Graduate Texts in Mathematics. 28-29. Springer. ISBN 0-387-90089-6. (Please, let us remember* Artin–Wedderburn theorem).

[36] *Please, let us remember the next w.f.s statement: the infinity concept, which nowadays is the customary represented by symbol "∞", is not element of* $\mathcal{N}$(0⊂FOL-PA) *and/or* $\mathcal{N}$(0⊄FOL-PA). Thus, in NT-FS&L first order language:

$$\left(\infty \notin \mathcal{N}(0\not\subset \text{FOL-PA}) \land \mathcal{N}(0\subset \text{FOL-PA})\right) \,\&\, \left(\infty \notin \mathit{Alphabet}\,(\mathcal{N}(\text{PA})) \land \mathcal{N}(0\not\subset \text{FOL-PA}) \land \mathcal{N}(0\subset \text{FOL-PA})\right)$$

$\left(0 \in \mathcal{N}(0 \subset \text{FOL-PA}) \wedge 0 \notin \mathcal{N}(0 \not\subset \text{FOL-PA})\right)$ & $\left(0 \in \mathcal{Alphabet}\ (\mathcal{N}(\text{PA})) \wedge \mathcal{N}(0 \not\subset \text{FOL-PA}) \wedge \mathcal{N}(0 \subset \text{FOL-PA})\right)$

Infinite concept. **1.** Conway, J. H. and Guy, R. K. *The Book of Numbers.* New York: Springer-Verlag, p. 19, 1996. **2.** Courant, R. and Robbins, H. "The Mathematical Analysis of Infinity" in *What Is Mathematics?: An Elementary Approach to Ideas and Methods, 2nd ed.* Oxford, England: Oxford University Press, pp. 77-88, 1996. **3.** Hardy, G. H. *Orders of Infinity: The 'infinitarcalcul' of Paul Du Bois-Reymond, 2nd ed.* Cambridge, England: Cambridge University Press, 1924. (https://www.amazon.com/Orders-Infinity-Infinitarcalcul-Bois-Reymond-Mathematics/dp/1107493668) **4.** Lavine, S. *Understanding the Infinite.* Cambridge, MA: Harvard University Press, 1994. **5.** Maor, E. *To Infinity and Beyond: A Cultural History of the Infinite.* Boston, MA: Birkhäuser, 1987. **6.** Moore, A. W. *The Infinite.* New York: Routledge, 1991. **7.** Morris, R. *Achilles in the Quantum Universe: The Definitive History of Infinity.* New York: Henry Holt, 1997. **8.** Péter, R. *Playing with Infinity.* New York: Dover, 1976. **9.** Rucker, R. *Infinity and the Mind: The Science and Philosophy of the Infinite.* Princeton, NJ: Princeton University Press, 1995. **9.** Smail, L. L. *Elements of the Theory of Infinite Processes.* New York: McGraw-Hill, 1923. **10.** Thomson, J. "Infinity in Mathematics and Logic". *The Encyclopedia of Philosophy, Vol. 4.* New York: Crowell Collier, pp. 183-190, 1967. **11.** Vilenskin, N. Ya. *in Search of Infinity.* Boston, MA: Birkhäuser, 1995. **12.** Weisstein, E. W. "Books about Infinity." http://www.ericweisstein.com/encyclopedias/books/Infinity.html. **13.** Wilson, A. M. *The Infinite in the Finite.* New York: Oxford University Press, 1996. **14.** Zippin, L. *Uses of Infinity.* New York: Random House, 1962. Weisstein, Eric W. "Infinity." From *MathWorld*--A Wolfram Web Resource. http://mathworld.wolfram.com/Infinity.html**)**.(Additional quoation about "**infinity**" from Wofram/Alpha: "Infinity or **∞** is a symbol that represents a positive infinite quantity. Details: **∞** can be entered as \[Infinity] or EscinfEsc.; In StandardForm, Infinity is printed as **∞;** Infinity is converted to DirectedInfinity[1]; Certain arithmetic operations work with Infinity; NumberQ[Infinity] yields False." Internet Access: http://reference.wolfram.com/language/ref/Infinity.html)

[37] *The "axiom of mathematical induction" (**axiom 7** of **PA´s**, also nicknamed "induction schema" and onwards the "pillar" of logic induction-deductive apparatus of NT-FS&L) provides a unique methodology (mainly, by substitution) for reasoning about the set of all natural numbers and their Arithmetic. Please, let us observe that:*

$\mathcal{N}$ **(0⊂FOL-PA)** ⊆ **{**0, S(0), S(S(0)), …**}** (Entailment from **NT-FS&L F.Th. 1**)

$\mathcal{N}$ **(0⊂FOL-PA) :=** {0, S(0), S(S(0)), …} → $\mathcal{N}$ **(0⊂FOL-PA)** ⊆ {0, S(0), S(S(0)), …} ⊆ {**0**, **1**, 2, 3, 4, …, **x+0, x+1**, **x+2,** …}

[38] Please, let us remember statement **2.1.3** of FOL-PA;

$$\forall x\ (x \in \mathcal{N}[0 \subset \text{FOL-PA}])(\ x + 0 = x)$$

[39] *In NT-FS&L, the succession operation referred to natural number zero (**0**) concept, which belongs to* $\mathcal{N}$**[0⊂FOL-PA],** *hereinafter will be noted and formulated as* $S_x(0)$*".* **Fixed Point Brouwer´s Theorem. Luitzen Egbertus Jan Brouwer.** (Harvey Mudd College Mathematics Department. "Mudd Math Fun Facts: Brouwer Fixed Point Theorem." Internet access: http://www.math.hmc.edu/funfacts/ffiles/20002.7.shtml.) **1.** Kannai, Y. "An Elementary Proof of the No Retraction Theorem." *Amer. Math. Monthly* **88**, 264-268, 1981. **2.** Milnor, J. W. *Topology from the Differentiable Viewpoint.* Princeton, NJ: Princeton University Press, 1965. **3.** Samelson, H. "On the Brouwer Fixed Point Theorem." *Portugal. Math.* **22**, 189-191, 1963. (Brown, R. F. (Ed.) (1988). *Fixed Point Theory and Its Applications*. American Mathematical Society. ISBN 0-8218-5080-6; **5.** Dugundji, James; Granas, Andrzej (2003). *Fixed Point Theory*. Springer-Verlag. ISBN 0-387-00173-5.; **6.** Giles, John R. (1987). *Introduction to the Analysis of Metric Spaces*. Cambridge University Press.

ISBN 978-0-521-35928-3.; **7.** Eberhard Zeidler, *Applied Functional Analysis: main principles and their applications*, Springer, 1995; **8.** Solomon Lefschetz (1937). "On the fixed point formula". *Ann. of Math.* **38** (4): 819–822. doi:10.2307/1968838.; **9.** Fenchel, Werner; Nielsen, Jakob (2003). Schmidt, Asmus L., ed. *Discontinuous groups of isometries in the hyperbolic plane*. De Gruyter Studies in mathematics. **29**. Berlin: Walter de Gruyter & Co.; **10.** Alfred Tarski (1955). "A lattice-theoretical fixpoint theorem and its applications". *Pacific Journal of Mathematics*. **5:2**: 285–309.; **11.** Peyton Jones, Simon L. (1987). *The Implementation of Functional Programming*. Prentice Hall International; **12.** Cutland, N.J., *Computability: An introduction to recursive function theory*, Cambridge University Press, 1980. ISBN 0-521-29465-7; **13.** Loeckx, J.; Sieber, K. *The foundations of program verification*, 2nd edition, John Wiley & Sons, ISBN 0-471-91282-4, (http://eu.wiley.com/WileyCDA/WileyTitle/productCd-0471912824.html) **14.** Zagier, D. (1990), "A one-sentence proof that every prime $p \equiv 1 \pmod 4$ is a sum of two squares", *American Mathematical Monthly*, **97** (2): 144, doi:10.2307/2323918, MR 1041893. **15.** Barnsley, Michael. (1988). *Fractals Everywhere*. Academic Press, Inc. ISBN 0-12-079062-9.

[40] *Please, take in account that in NT-FS&L FOL-PA any statement about notation, formulation and nomenclature of succession operation (also, function) of a natural number, zero for example,* ***S_x(0)****, is accepted as a w.f.s-r statement, if and only if :*

numerical index *(also "**x**-fold" the operation & "**x**-times" the operation)* $\in$

$$\{ x \mid x \in \mathcal{N}(0 \subset \text{FOL-PA}) \wedge \mathit{Alphabet}\,(\mathfrak{N}[0 \subset \text{FOL-PA}] \cap \mathcal{N}(0 \not\subset \text{FOL-PA})) \}_{\forall x\,(x \in \mathfrak{N}[0 \subset \text{FOL-PA}])}$$

&

numerical index concept *(for example, "**x**-fold" the operation & "**x**-times" the operation)* $\subseteq$

$$\{ x \mid x \in \mathcal{N}(0 \subset \text{FOL-PA}) \wedge \mathit{Alphabet}\,(\mathfrak{N}[0 \subset \text{FOL-PA}] \cap \mathcal{N}(0 \not\subset \text{FOL-PA})) \}_{\forall x\,(x \in \mathfrak{N}[0 \subset \text{FOL-PA}])}$$

Indicator Function (also "**characteristic function"). 1.** *Lange, Rutger-Jan (2012), "Potential theory, path integrals and the Laplacian of the indicator", Journal of High Energy Physics, Springer, 2012 (11): 29–30, arXiv:1302.0864, Bibcode:2012JHEP...11..032L, doi:10.1007/JHEP11(2012)032.* **2.** *Folland, G.B. (1999). Real Analysis: Modern Techniques and Their Applications (Second ed.). John Wiley & Sons, Inc.* **3.** ***Kleene, Stephen*** *(1971) [1952]. Introduction to Metamathematics (Sixth Reprint with corrections). Netherlands: Wolters-Noordhoff Publishing and North Holland Publishing Company.* ISBN-13: 978-0923891572 andISBN-10: 0923891579. **3.** *Cormen, Thomas H.; Leiserson, Charles E.; Rivest, Ronald L.; Stein, Clifford (2001). Introduction to Algorithms (Second ed.). MIT Press and McGraw-Hill. ISBN 0-262-03293-7.* (http://is.ptithcm.edu.vn/~tdhuy/Programming/Introduction.to.Algorithms.pdf.). **4.** *Davis, Martin, ed. (1965). The Undecidable. New York: Raven Press Books, Ltd.* (https://www.abebooks.com/book-search/title/the-undecidable/author/davis/). **5.** *Boolos, George; Burgess, John P.; Jeffrey, Richard C. (2002). Computability and Logic. Cambridge UK: Cambridge University Press. ISBN 0-521-00758-5.* **6.** *Zadeh, Lotfi A. (June 1965). "Fuzzy sets" (PDF). Information and Control. 8 (3): 338–353. doi:10.1016/S0019-9958(65)90241-X.* **7.** *Goguen, Joseph (1967). "L-fuzzy sets". Journal of Mathematical Analysis and Applications. 18 (1): 145–174. doi:10.1016/0022-247X(67)90189-8.*

[41] *Please let us observe that: whenever* **φ** *is a well-formulated formula, the customary name of "the variables of the* **φ** *operation" referred into the induction schema axiom PA´s in first order language (FOL) are customary nicknamed as "occurrences" and/or "free variables".*

[42] **Albert Einstein.** ***Einstein summation convention. Tensorial Analysis, Differential Geometry and Calculus Concepts*.** **1.** *Einstein, Albert (1916).* Einstein, A. "Die Grundlage der allgemeinen Relativitätstheorie." *Ann. der Physik* **49**, 769-822, 1916). *"The Foundation of the General Theory of Relativity". Annalen der Physik. Bibcode:1916AnP...354..769E. doi:10.1002/andp.19163540702.* **2.** *Ricci, Gregorio; Levi-Civita, Tullio (March 1900). "Méthodes de calcul différentiel absolu et leurs applications" (PDF). Mathematische Annalen. Springer.* ***54*** *(1–2): 125–201. doi:10.1007/BF01454201.* **3.** *Kuptsov, L. P. (2001), "Einstein rule", in Hazewinkel, Michiel, Encyclopedia of Mathematics, Springer, ISBN 978-1-55608-010-4.* **4.** *Bishop, Richard L.; Samuel I. Goldberg (1980) [1968]. Tensor*

Analysis on Manifolds. Dover. ISBN 978-0-486-64039-6. **5.** *Jeevanjee, Nadir (2011). An Introduction to Tensors and Group Theory for Physicists. Birkhauser. ISBN 978-0-8176-4714-8.* **6.** Bishop, R. and Goldberg, S. *Tensor Analysis on Manifolds.* New York: Dover, 1980. **7.** Bott, R. and Tu, L. W. *Differential Forms in Algebraic Topology.* New York: Springer-Verlag, 1995. **8.** Lawden, D. F. *An Introduction to Tensor Calculus, Relativity, and Cosmology, 3rd ed.* Chichester, England: Wiley, 1982. **9.** Wolfram Mathworld; Stover, Christopher and Weisstein, Eric W. "Einstein Summation." From *MathWorld*--A Wolfram Web Resource. *"Einstein Summation". Wolfram Mathworld.* http://mathworld.wolfram.com/EinsteinSummation.html. **10.** Cartan, É. *The Theory of Spinors.* New York: Dover, 1981. **11.** Sokolnikoff, I. S. *Tensor Analysis: Theory and Applications to Geometry and Mechanics of Continua, 2nd ed.* New York: Wiley, 1964. **12.** Abraham, R.; Marsden, J. E.; and Ratiu, T. S. *Manifolds, Tensor Analysis, and Applications, 2nd ed.* New York: Springer-Verlag, 1991. **13.** Schutz, Bernard, *Geometrical methods of mathematical physics*, Cambridge University Press, 1980. **14.** Borisenko, A. I. and Tarpov, I. E. *Vector and Tensor Analysis with Applications.* New York: Dover, 1980. **15.** Parker, L. and Christensen, S. M. *MathTensor: A System for Doing Tensor Analysis by Computer.* Reading, MA: Addison-Wesley, 1994. **16.** *Lawden, D. F. (2003). Introduction to Tensor Calculus, Relativity and Cosmology (3/e ed.). Dover. ISBN 978-0-486-42540-5.* **17.** Cubitt, T. "Einstein Summation Convention and δ-Functions." http://www.drqubit.org/teaching/summation_delta.pdf. **18.** Lovelock, D. and Rund, H. *Tensors, Differential Forms, and Variational Principles.* New York: Dover, 1989. **19.** Nicolaescu, L. I. *Lectures on the Geometry of Manifolds.* Singapore: World Scientific, 1996. **20.** *Dimitrienko, Yuriy (2002). Tensor Analysis and Nonlinear Tensor Functions. Kluwer Academic Publishers (Springer). ISBN 1-4020-1015-X.* **20.** Munkres, James, *Analysis on Manifolds*, Westview Press, 1991. **21.** *Kay, David C (1988-04-01). Schaum's Outline of Tensor Calculus. McGraw-Hill. ISBN 978-0-07-033484-7.* **22.** Schutz, Bernard, *Geometrical methods of mathematical physics*, Cambridge University Press, 1980. **23.** Bishop, R. and Goldberg, S. *Tensor Analysis on Manifolds.* New York: Dover, 1980. **24.** Norton, John D., "The Hole Argument", *The Stanford Encyclopedia of Philosophy* (Fall 2015 Edition), Edward N. Zalta (ed.), URL = https://plato.stanford.edu/archives/fall2015/entries/spacetime-holearg/

[43] Additionally, in current first-order logic, **PA** axiom "$\forall x\ (x \in N)(0 \neq S(x))$" is a symbolic abbreviation for "$\forall x\ (x \in N)(\sim(S(x)=0))$", the negation of the formula $S(x)=0$. Let us observe that a unique (1) symbol "≠" replaces two (2) symbols, "=" and "∼", which means that symbolic length shortens in one unit when an affirmative (positive) enunciate has to be represented from the corresponding negative enunciate. Please, let us point out the following referential examples (symbolically and semiotically) examples in contemporary English, which are widely used in contemporary Mathematical Logic: 1. Countable and un-countable; 2. Completeness and in-completeness; 3. Decidable and un-decidable.

(Please, let us propose to the reader the "on line translation" (sense and intension referred to "Wittgestein´s"concepts" –It is globally accepted two different "Wittgestein´s philosophic periods) in English of the following positive sentence: "***Sobre la dificultad de contar***" (Javier Marías, http://www.javiermarias.es/discurso.pdf) in Spanish. Ours results follows:

INPUT in both (a. Google translator (https://translate.google.com/?hl=es#es/en/google) and b. El País Traductor (http://servicios.elpais.com/traductor/?sl=es&tl=en)); Sobre la dificultad de contar; OUTPUT of Google translator: "***About of the difficulty of counting***". Second: "***On the difficulty of counting***".

Additionaly, by means of the "World Herramientas operating (embed) in a Mac, the results (Estadística, using a Spanish desk), are symbolically represented by natural numbers (concepts) which are expressed by digits according to the locus-positional decimal numerical system (Cambria(Cuerpo; 10; normal)) are the following:

1. Sobre la dificultad de contar

(Páginas **1**; Palabras **5**; Caracteres(sin espacios) **25**; Caracteres(con espacios) **30**; Párrafos **1**; Líneas **1**)

2. About the difficulty of counting

(Páginas **1**; Palabras **5**; Caracteres(sin espacios) **28**; Caracteres(con espacios) **32**; Párrafos **1**; Líneas **1**)

3. On the difficulty of counting

(Páginas **1**; Palabras **5**; Caracteres(sin espacios) **25**; Caracteres(con espacios) **29**; Párrafos **1**; Líneas **1**)

4. "for the present complete (?!) document" , after (?!) to insert these latter results, at this moment (!?),… are: (Páginas **108**; Palabras **37.464**; Caracteres(sin espacios) **195.123**; Caracteres(con espacios) **114.364**; Párrafos **1.888**; Líneas **4.512**)

[44] This statement has to be considered a direct consequence of "the uniary nature" of the formal representation above described. The customary process involved is call "counting", which can be understand as the customary meaning of "measuring the amount of any magnitude" (counting operation as before described) herein represented by symbol "#".

[45] "Non-dense order property" of the natural number set holds for the concept that between two natural number concepts there is not another natural number concept. Please, let us point out that in NT-FS&L:

$$\forall \mathbf{x}(\mathbf{x} \in \mathcal{N}(\mathbf{0} \subset \mathbf{PA})(\mathbf{S_{x+1}}(\mathbf{0}) = \mathbf{S_x}(\mathbf{1}))$$

Theory of Order. 1. For a representative glossary of terms customary used in Order Theory, please see https://en.wikipedia.org/wiki/Glossary_of_order_theory). **2.** *Birkhoff, Garrett (1940). Lattice Theory.* ***25*** *(3rd Revised ed.). American Mathematical Society. ISBN 978-0-8218-1025-5.* **3.** *Burris, S. N.; Sankappanavar, H. P. (1981). A Course in Universal Algebra. Springer. ISBN 978-0-387-90578-5.* **4.** *Davey, B. A.; Priestley, H. A. (2002). Introduction to Lattices and Order (2nd ed.). Cambridge University Press. ISBN 0-521-78451-4.* **5.** *Gierz, G.; Hofmann, K. H.; Keimel, K.; Mislove, M.; Scott, D. S. (2003). Continuous Lattices and Domains. Encyclopedia of Mathematics and its Applications.* ***93****. Cambridge University Press. ISBN 978-0-521-80338-0.* **5.** *Weisstein, Eric W. "Dense." From MathWorld--A Wolfram Web Resource. http://mathworld.wolfram.com/Dense.html and therein references*: **5.1.** Bailey, D. H. and Crandall, R. E. "On the Random Character of Fundamental Constant Expansions." *Exper. Math.* **10**, 175-190, 2001. **5.2.** Bailey, D. H. and Crandall, R. E. "Random Generators and Normal Numbers." *Exper. Math.* **11**, 527-546, 2002). **6. Nowhere Dense Set: 6.1.** Ferreirós, J. "Lipschitz and Hankel on Nowhere Dense Sets and Integration." in *Labyrinth of Thought: A History of Set Theory and Its Role in Modern Mathematics.* Basel, Switzerland: Birkhäuser. 1999. **6.2.** Milovich, Dave and Weisstein, Eric W. "Nowhere Dense." From *MathWorld*--A Wolfram Web Resource. http://mathworld.wolfram.com/NowhereDense.html. **Ramsey theory** (**Frank Plumpton Ramsey**). **1.** *Ramsey, F. P. (1930), "On a Problem of Formal Logic", Proceedings London Mathematical Society, s2-30 (1): 264–286, doi:10.1112/plms/s2-30.1.264.*; **2.** *Ramsey, F. P. (1930), "On a Problem of Formal Logic", Proceedings London Mathematical Society, s2-30 (1): 264–286, doi:10.1112/plms/s2-30.1.264.*; **3.** *Erdös, P. & Szekeres, G. (1935), "A combinatorial problem in geometry", Compositio Mathematica,* ***2****: 463–470.*; **4.** *Boolos, G.; Burgess, J. P.; Jeffrey, R. (2007), Computability and Logic (5th ed.), Cambridge: Cambridge University Press, ISBN 978-0-521-87752-7*. **5.** *Landman, B. M. & Robertson, A. (2004), Ramsey Theory on the Integers, Student Mathematical Library,* ***24****, Providence, RI: AMS, ISBN 0-8218-3199-2*.

[46] **Georg Ferdinand Ludwig Philipp Cantor** *(1955) [1915], Philip Jourdain, ed., Contributions to the Founding of the Theory of Transfinite Numbers, New York: Dover, ISBN 978-0-486-60045-1*. (Internet access: https://archive.org/details/contributionstot003626mbp) ***2.*** *Ferreirós, José (2007), Labyrinth of Thought: A History of Set Theory and Its Role in Mathematical Thought, Basel, Switzerland: Birkhäuser*. ISBN 3-7643-8349-6. **3.** *Joseph Warren Dauben (1990). Georg Cantor: His Mathematics and Philosophy of the Infinite. Princenton University Press, 2012 ISBN 9780691024479.* **4.** *Ferreirós, José (1995), "'What fermented in me for years': Cantor's discovery of transfinite numbers", Historia Mathematica,* ***22****: 33–42, doi:10.1006/hmat.1995.1003*. **5.** *Ferreirós, José (2007), Labyrinth of Thought: A History of Set Theory and Its Role in Mathematical Thought (2nd revised ed.), Birkhäuser, ISBN 3-7643-8349-6*. **6.** Conway, J. H. and Guy, R. K. "Cantor's Ordinal Numbers." In *The Book of Numbers*. New York: Springer-Verlag, pp. 266–267 and 274, 1996. ***Sierpiński, Wacław*** *(1958) Cardinal and ordinal numbers.*, Polska Akademia Nauk Monografie Matematyczne **34**, Warsaw: Państwowe Wydawnictwo Naukowe, MR 0095787. ***Von Neumann cardinal assignment*** *(http://self.gutenberg.org/articles/eng/John_von_Neumann). "On the introduction of transfinite numbers", in Jean van Heijenoort, From Frege to Gödel: A Source Book in Mathematical Logic, 1879-1931 (3rd ed.), Harvard University Press, (January 2002) ISBN 0-674-32449-8. Philosophia Mathematica (2002) 10 (3): 349-350.* ***DOI:*** *https://doi.org/10.1093/philmat/10.3.349. Published: 01 June 2002).* **Paul Du Bois-Reymond*.* *(1875)*** *"Über asymptotische Werte, infinitäre Approximationen und infinitäre Auflösungen von*

*Gleichungen", Mathematische Annalen, **8**: 363–414, doi:10.1007/bf01443187. (Internet access http://www.wikivisually.com/wiki/Paul_David_Gustav_du_Bois-Reymond)*

[47] **Monoid concept. 1.** *Howie, John M. (1995), Fundamentals of Semigroup Theory, London Mathematical Society Monographs. New Series.* ***12****, Oxford: Clarendon Press, ISBN 0-19-851194-9, Zbl 0835.20077*. **2.** *Hazewinkel, Michiel, ed. (2001), "Monoid", Encyclopedia of Mathematics, Springer, ISBN 978-1-55608-010-4 ;* **3.** Monoid at PlanetMath.org. (http://planetmath.org/Monoid*).* **4.** *Jacobson, Nathan (2009), Basic algebra,* ***1*** *(2nd ed.), Dover, ISBN 978-0-486-47189-1.* **5.** *Kilo, Mati; Knauer, Ulrich; Mikhalev, Alexander V. (2000), Monoids, acts and categories. Gruyter Expositions in Mathematics,* ***29****, Berlin: Walter de Gruyter, ISBN 3-11-015248-7, Zbl 0945.20036.* **6.** *Lothaire, M. (1997), Combinatorics on words, Encyclopedia of Mathematics and Its Applications. Perrin, D.; Reutenauer, C.; Berstel, J.; Pin, J. E.; Pirillo, G.; Foata, D.; Sakarovitch, J.; Simon, I.; Schützenberger, M. P.; Choffrut, C.; Cori, R.; Lyndon, Roger; Rota, Gian-Carlo. Foreword by Roger Lyndon (2nd ed.), Cambridge University Press, doi:10.1017/CBO9780511566097, ISBN 0-521-59924-5, MR 1475463, Zbl 0874.20040.*

[48] Moreover, the statement **def.1** certifies the existence of countable, enumerable and complete ordered amount of "successors" of any element belonging to both $\mathcal{N}$**(0⊂NT-PA)** and $\mathcal{N}$**(0⊄NT-PA),** which could be hereinafter re-presented by $\mathcal{N}$**(0⊂FOL-PA)** ∩ $\mathcal{N}$**(0⊄FOL-PA)**.

[49] Please, take in account that:

$$\forall x \forall y \forall z\ (x, y, z \in \mathcal{N}(0 \subset \mathrm{PA}))\ ((z = S_{x+y}(1) \wedge S_y(x)+1 \wedge S_x(y)+1)\ \&\ (x = S_x(0))\ \&\ (y = S_y(0)) \rightarrow z = (x+1)+y \wedge (y+1)+x \wedge (x+y)+1 \wedge (1+(x+y))$$

[50] *Please let us remember to the reader that natural number zero is not a member of* $\mathcal{N}$*(0⊄NT-PA). Hence, it only has to be considered a symbol of its alphabet as well as the name of the set of all of their successors. Hence, both* $\mathcal{Suc}$**(x)** *and* $\mathcal{Suc}_{\mathbf{x}}(\mathbf{0})$, $\forall x(x \in \mathcal{N}$**(0⊂NT-PA))** *sets could be referred as* ***two*** *"infinite sets" in Cantor countability nomenclature (sense and intension), for which in NT-FS&L can be properly enumerated, numerated and can count every and each of their elements.*

[51] Please, let us remember that in NT-FS&L "*one-membered set" is named "singleton".* Thus,

$$\mathcal{Suc}_{\mathbf{x}}(\mathbf{0}) := \{\, \mathbf{0} \in \mathcal{N}(0 \subset \text{NT-PA}) \mid \mathbf{x} = \mathbf{0} \,\} \forall x\ (x \in \mathcal{N}(0 \subset \text{NT-PA}))$$

[52] On the other hand, the next statements $\forall \mathbf{x}(\mathbf{x} \in \mathcal{N}(0 \subset \text{NT-PA}))$

$$\mathcal{N}(0 \subset \text{NT-PA}) \subseteq \mathcal{N}(0 \not\subset \text{NT-PA}) \cup \mathcal{Suc}_{\mathbf{x}}(\mathbf{0})$$

&

$$\mathcal{Suc}_{\mathbf{x}}(\mathbf{0}) \subseteq \mathcal{N}(0 \subset \text{NT-PA}) \cap \mathcal{N}(0 \not\subset \text{NT-PA})$$

are both "false". They are not w.f.s and w.f.s-r statements in any first order language, in which "equality" means "no-identity" and vice versa.

[53] Please, take in account that:

$$\forall x \forall y(x, y \in \mathcal{N}(0 \subset \text{NT-PA}))\ ((x = S_0(x) \wedge y = S_0(y) \wedge n = S_0(n) \rightarrow (x = n+y)\ \&\ (n \in \mathcal{N}(0 \subset \text{NT-PA}))$$

[54] Please, let us considered and observe:

$$((x = y+1 \rightarrow x = S_{y+1}(0)) \wedge ((x = y+1 \rightarrow x = P_0(x))$$

, if and only if,

$$(x = S(y) +1) \wedge (S_{y+1}(0) = P_y(0) + 1) \wedge (y = P_0(y) = P_0(x+1))$$

[55] *Crolard, Tristan. Substractive Logic. (2001)* Theoretical Computer Science. Vol 254, Issues 1-2, March 2001, pp-151-185. (http://dx.doi.org/10.1016/S0304-3975(99)00124-3)

[56] *Please, see previous title* ***Albert Einstein. Einstein summation convention. Tensorial Analysis, Differential Geometry and Calculus Concepts***.

[57] $\mathcal{P}re_{\ 0}(\mathbf{x})$, $\mathcal{S}uc_{\ \mathbf{x}}(\mathbf{0})$ and $\mathcal{N}$ **(0⊂NT-PA)** sets *had to be grade as "enumerable, complete ordered, countable sets" (different sense and intension that for others authors as Cantor, Von Neumann, Gödel and Ramsey)*

[58] In NT-FS&L, there are two equivalent notations and/or formulations for "the accumulated union operation" of set (or subsets):

Definition (Notation 1)

$$\bigcup[\mathcal{P}red_0(\mathbf{x+1})]_{\forall x \in \mathcal{N}(0\subset NT\text{-}PA)} := \{\mathcal{P}red_0(\mathbf{1}) \cup \mathcal{P}red_0(\mathbf{2}) \cup \ldots \cup \mathcal{P}red_0(\mathbf{x}) \cup \mathcal{P}red_0(\mathbf{x+1})\}$$

Definition (Notation 2)

$$\bigcup[\mathcal{P}re_1(\mathbf{x+1})]_{\forall x \in \mathcal{N}(0\subset NT\text{-}PA)} := \{\mathcal{P}re_1(\mathbf{0}) \cup \mathcal{P}re_0(\mathbf{1}) \cup \ldots \cup \mathcal{P}re_1(\mathbf{x}) \cup \mathcal{P}re_1(\mathbf{x+1})\}$$

[59] Please, let us indicate:

$$\forall x(\, x \in \mathcal{N}(0\subset NT\text{-}PA))(\,\{P_0(x)\} \subseteq \{x\} \rightarrow \{x\} \not\subset \{\emptyset\} \;\&\; \{\emptyset\} \subseteq \Theta\,)$$

[60] *Please, let us remember to the reader the previous note title* ***Albert Einstein. Einstein summation convention. Tensorial Analysis, Differential Geometry and Calculus Concepts***.

[61] *Hereinafter, we will use the customary expression "presentation of" (for example, presentation of an abstract algebraic framework such as "Presentation of a Group" or "Presentation of a Monoid") with the same sense and intension that it is used, for example, in the current "Representation Theory".*

[62] ***Please, let us remember both, that every natural number has a unique direct successor identity, and a unique direct predecessor identity. The primordial of every natural number identity are 1-membered collectivities (subsets, also singletons) of the natural number set*** $\mathcal{N}$**(0⊂NT-PA)).**

[63] Please, let us remember that in NT-FS&L (NT-FOL-PA) are operating the *NT-FOL-de Morgan Laws for the union and intersection of any collectivity of natural numbers* **x** (**x** ∈ $\mathcal{N}$**(0⊂NT-PA)**)

[64] *In NTFS&L, the word "lemma" is a synonymous of the word "theorem" referred to the meaning of "statement" that sustain entailment from the initial axiomatic set of wfs and wfs-r statements. They sustain "truth".*

[65] *Please let us consider that in customary notation the multiplication operation symbol (·) is an element of the alphabet of NT-PA first order language.* Thus, **(x+x)** can be "referred-represented" by the symbolic notations "**2·n**" and/or "**2n**"; on the other hand, **(x+x+1)** can be "referred-

represented" by symbolic notations by "**2·n+1**"and/or "**2n+1**".

66 Please, let us remember:

$$\forall x \forall y\ (x, y \in \mathcal{Suc}^{\alpha}{}_{x}(0) \wedge \mathcal{Suc}^{\beta}{}_{x}(0) \wedge \mathcal{Pre}^{\alpha}{}_{0}(x) \wedge \mathcal{Pre}^{\beta}{}_{0}(x))$$

$$(x + y = (S_x(0) + S_y(0)) = P_0(x) + P_0(y)) \wedge (S_x(0) + P_0(y) = S_y(0) + P_0(y)) \rightarrow x = y); \quad \textbf{NT-th. 10}$$

67 $\mathcal{Odd}(\mathcal{Suc}_0(0)) := \{\Theta \mid \Theta = \text{set}\ \{\varnothing\}\}$

68 In NT-FS&L, "parity" and/or "imparity" induce a nicknamed *"complete alternant-ordered two colors NT-partition"* in $\mathcal{N}$**(0⊂NT-PA),** *which is in turn a non-dense and complete ordered set. Hence, please, let us remember the abstract algebraic structure of the group of the* *set-*$\mathcal{N}$**[0,1⊂NT-PA]:={0,1}∀n ∈$\mathcal{N}$[0⊂NT-PA]** ***in NT-FS&L embed (sense and intension) both: first, a homomorphe*** *"complete alternant-ordered two colors partition"; second****, a homomorphe "****complete ordered two colors-NT Logic and Algebra of Boole"* ***extended field.*** (Please, take under your consideration the current customary "Minor field" concept)

69 Please, let us remember:

i.- **((0, 1, 2, … n ∈ $\mathcal{N}$[0⊂NT-PA]) ∀n ∈$\mathcal{N}$[0⊂NT-PA] ∧(1, 2, … n ∈ $\mathcal{N}$[0⊂NT-PA]) ∩ $\mathcal{N}$[0⊄NT-PA])**

ii.- **∀x (x ∈$\mathcal{N}$[0⊂PA])(x · 0 = 0) & ∀x∀y(x, y ∈$\mathcal{N}$[0⊂PA])(x · S(y) = x · y + x) (axioms 5 and 6 of NT-PA first order axiomatic).**

ii.- natural number **0** is the identity element for addition operation in $\mathcal{N}$**[0⊂NT-PA])** and, in turn, natural number **1** is the identity element for multiplication operation. Natural number **0** and every $\mathbf{S_x(0)}$ as well that $\mathbf{P_0(x)}$ are not elements of $\mathcal{N}$**[0⊄NT-PA].**

Please, take in account that this fact involves an "infinite" collectivity of elements, which are all of the successors of a natural number but a "finitary collectivity of elements", theirs corresponding predecessors. In NT-FS&L, the exchange operation successor of natural number identity for the predecessor of natural number (also, safeguards, confirms, certifies, warrants, make sure, make certain) both, the order (parity, logic and mathematical) structure and the transformation of a "starting infinite no numerable collectivity (the successors of a natural number which are infinity) into a finitary numerable number (the predecessor of a natural number). Hence, in NT-FS&L the processes of counting, enumeration, numeration and ordering are simultaneously all possible.

70 **∀x (x ∈$\mathcal{N}$[0⊂PA])(x · 0 = 0) & ∀x∀y(x, y ∈$\mathcal{N}$[0⊂PA])(x · S(y) = x · y + x) (axioms 5 and 6 of NT-PA first order).** *Please, let us observe that* ***the customary distributive property of the multiplication on addition, only is operating ∀x (for each and every each natural number) if and only if*** $\mathbf{y = S_1(y)\ x = P_1(y)}$

71 *Please, take in account that:*

$$(2\cdot(1+2+3+4+....+(n)+(n+1)+(n+2)+(n+3) = (n+3)\cdot(n+4)=(n+3)\cdot S_1(n+3)) \rightarrow$$
$$(2\cdot(1+2+3+4+....+(n)+(n+1)+(n+2) = (n+2)\cdot(n+3)=(n+2)\cdot S_1(n+2)) \rightarrow$$
$$(2\cdot(1+2+3+4+....+(n)+(n+1) = (n+1)\cdot(n+2)=(n+1)\cdot S_1(n+1)) \rightarrow$$
$$(2\cdot(1+2+3+4+....+(n) = n\cdot(n+1) = n\cdot S_1(n)) \quad \&$$
$$(2\cdot(1+2+3+4+....+(n)+(n+1)+(n+2)+(n+3) = (n+3)\cdot(n+4)=(n+3)\cdot P_0(n+4)) \rightarrow$$
$$(2\cdot(1+2+3+4+....+(n)+(n+1)+(n+2) = (n+2)\cdot(n+3)=(n+2)\cdot P_0(n+3)) \rightarrow$$
$$(2\cdot(1+2+3+4+....+(n)+(n+1) = (n+1)\cdot(n+2)=(n+1)\cdot P_0(n+2)) \rightarrow$$

$$(2\cdot(1+2+3+4+....+(n) \quad = \quad (n)\cdot(n+1) = n\cdot P_0(n+1))\ \&$$
$$(2\cdot(1+2+3+4+....+((3n+0)+(n)) \quad = \quad (2\cdot(1+2+3+4+....+(4n+0)) \rightarrow$$
$$(2\cdot(1+2+3+4+....+((2n+0)+(n)) \quad = \quad (2\cdot(1+2+3+4+....+(3n+0)) \rightarrow$$
$$(2\cdot(1+2+3+4+....+((n)+(n)) \quad = \quad (2\cdot(1+2+3+4+....+(2n+0)) \rightarrow$$
$$(2\cdot(1+2+3+4+....+(n)) \quad = \quad (2\cdot(1+2+3+4+....+(1n+0))$$

[72] **1. *NT reading suggestions.* 1.** *Internet access: http://www.xamuel.com/the-axioms-of-peano-arithmetic-modern-version.* **2.** *Penrose, R., Shimony, A., Cartwright, N. and Hawking, S. The Large, the Small and the Human Mind. Cambrige University Press, 1997. ISBN 0 521 56330 5.* **3. *NT-reading suggestion.*** *Quotation of the Spanish Traslation by Javier García Sanz ("Title: Lo grande, lo pequeño y la mente humana. Apéndice I. Cambridge University Press. ISBN 84 8323 047): "… Lo que es bastante más extraordinario es que el teorema de Goodstein es realmente un teorema de Gödel para el procedimiento que aprendemos en la escuela denominado **inducción matemática.*** Recordemos que la inducción matemática proporciona una manera de demostrar que ciertos enunciados matemáticos S(n) son válidos para cualquier n=1,2,3….. El procedimiento consiste en demostrar , primero, que el enunciado es válido para n=1 y denostrar luego que es válido para n, entonces debe ser válido para n+1. Un ejemplo familiar es el enunciado:*

$$1+2+3+4+5.....+n = (1/2)\ n(n+1)$$

Para demostrar esto por inducción matemática, establecemos que es cierto para n=1 (obvio) y confirmamos luego si la formula funciona para n, entonces también funciona para n+1, lo que es ciertamente verdadero porque tenemos:

$$1+2+3+4+....+n+(n+1) = (1/2)\ n\ (n+1) + (n+1) = (1/2)\ (n+1)\ ((n+1)+1))$$

Lo que Kirby y Paris, demostraron de hecho, era que si P representa el procedimiento de inducción matemática (junto con las operaciones aritméticas y lógicas ordinarias) , entonces podemos volver a expresar G(P) en la forma del teorema de Goodstein. Este nos dice que si creemos que el procedimiento de inducción matemática es digno de confianza(lo que difícilmente es una hipótesis dudosa), entonces debemos creer también la verdad del teorema de Goodstein, pese al hecho de que no es demostrable por inducción matemática!!! (Esto fue demostrado por: L.A.S. Kirby y J.B. Paris en: "Accesible independence results for Peano". Bulletin of the London Mathematical Society, 14, 1982, págs 285-293)… "*

[73] *In "ad hoc" substitution notation:*

$$(\mathbf{x\cdot S_1(x)}) \subseteq (\mathcal{E}ven\text{-}\mathcal{PN}_0(\mathbf{x\cdot x+x}) \cap \mathcal{E}ven\text{-}\mathcal{PN}_0(\mathbf{x\cdot(x+1)})_{\forall x})$$

[74] In NT-FS&L, the logic gate "and/or" is symbolically re-presented by degenerative *locus*-positional symbols "∧∨" and/or "∨∧". **1.** Hagar, Amit and Cuffaro, Michael, "Quantum Computing", *The Stanford Encyclopedia of Philosophy* (Summer 2015 Edition), Edward N. Zalta (ed.), URL = <https://plato.stanford.edu/archives/sum2015/entries/qt-quantcomp/>. **2.** Rescorla, Michael, "The Computational Theory of Mind", *The Stanford Encyclopedia of Philosophy* (Winter 2016 Edition), Edward N. Zalta (ed.),
URL = <https://plato.stanford.edu/archives/win2016/entries/computational-mind/>.

[75] In NTFS&L; the following statement

$$\mathbf{(\mathcal{N}(\textit{FOL-SP}(0\subset PA) \subseteq \mathcal{N}(\textit{SPA}(0\subset PA) \rightarrow \mathcal{N}(0\subset \textit{NT-PA}) \subseteq \mathcal{N}(\textit{FOL-SP}(0\subset PA))}$$

, which is a member (hence, also could be considered its "singleton" identity) of **NT-Pr-th. 1 Set,** *should be read (send and intension):*

"***If the axiomatic 𝒩(FOL-SP(0⊂PA) set has been intended de-embed logically from 𝒩(SPA(0⊂PA), then 𝒩(0⊂NT-PA) has been intended de-embed logically from 𝒩(FOL-***

SP(0⊂PA) and, in turn, from $\mathcal{N}$(SPA(0⊂PA)".

[76] *Additionally, let us observe that parity referred as the addition* ***(x+x)*** *operation in NT-PA´s, is able to reinforce (and/or to "regenerate order" and/or to reestablish) complete order of every disordered subset of natural numbers by means of an inductive-type process into the non-dense but complete ordered set of the natural numbers ($\mathcal{N}$(0⊂NT-PA)). This is achieved (also evinced) operationally by means of the following inductive process; renaming successor of zero as zero and successor of even (0+0) by even (1+1) and in general the natural number x by the even (addition of itself) of its odd-direct successor.*
Hence, operating with the elements of $\mathcal{N}$(0⊂NT-PA):

If:

$$((0) = (0+0)) \rightarrow ((0) = (0+0) +0)) \wedge ((1) = (0+0)) \rightarrow ((2) = (1+1) +1))) \;\&$$
$$((0) = (0+1)) \rightarrow ((0) = (1+1) +0))) \wedge ((1) = (1+1)) \rightarrow ((3) = (1+1) +1))) \;\&$$
$$((0) = (0+2)) \rightarrow ((0) = (2+2) +0))) \wedge ((1) = (2+2)) \rightarrow ((4) = (2+2) +1))) \;\&$$
$$((0) = (0+3)) \rightarrow ((0) = (3+3) +0))) \wedge ((1) = (3+3)) \rightarrow ((7) = (3+3) +1))) \;\&$$
$$\ldots\ldots \quad \ldots\ldots\ldots \ldots\ldots\ldots \quad \ldots\ldots\ldots\ldots \ldots\ldots\ldots \;\&$$

then:

$$(((0)=(S_0(0+x)) \rightarrow ((0)= (S_0(x+x) +0))) \wedge ((1)=(S_0(x+x)) \rightarrow ((2\cdot x)= S_0(2\cdot x+1))))$$

[77] *In NT-FS&L it is enough to fix (also index) "cero as reference", which in turn is the unique natural number, to enable us to index both;*

i.- a complete order and distribution of the complete $\mathcal{N}$(0⊂PA)); and

ii.- the identity of every and any natural number. Hence, both of their identities, "elements" and "part and/or parts of a collectivity", of every "successor" and/or "predecessor" are transformed in "invariants elements" in a ***Projective Space****, $\mathcal{N}$(0⊂NT-PA) generated in the NT-SF&L first order logic language which is free from any topological and geometrical (mathematical) context.*

[78] *Please, let us note that we use $\mathcal{N}$ (0⊂PA)) in state of $\mathcal{N}$ (0∈PA)) because we are indicated that "natural number zero concept" and "the statements about this concept" is a element according to PA´s axiomatic (In NT-FS&L, holds the identity of a set of statements, which in turns is a type of collectivity)*

[79] The former axiom 6 (PA´s) on multiplication operation, which is referred to the multiplication operation:

$$\forall x \forall y\ (x, y \in \mathcal{N}(0 \subset PA))\ (x \cdot S(y) = x \cdot y + x)$$

(axiome 6; PA´s expressed in current first order logic)

$$\forall x \forall y\ (x, y \in \mathcal{N}(0 \subset PA))(x \cdot S(y) = x \cdot (y + 1) \vee \forall x \forall y\ (x, y \in \mathcal{N}(0 \subset PA))(y \cdot S(x) = y \cdot (x + 1)$$
$$\rightarrow$$
$$(y \cdot (x + 1)) = (x \cdot (y + 1)) \Rightarrow (y + 1 = x+1) \rightarrow (x) = (y)$$

(axiome 6; PA´s expressed by first order logic "ad hoc" substitution operation)

[80] *Please, let us observe and remember that "precessor operation" have to be considered in NT-FS&L as a "unintended embedded" (sense and intention) identity in the original "PA´s axiomatic" of G. Peano.*

[81] *Covariance and contravariance concepts have to be considered as above described (sense and intension, succession and precession); thus we should introduce: and, or, and and/or logic operations from notation. The induction schema is expressed by "replacement notation".*

[82] *Please, let us remember that:*

$$\forall x\ (x \in \mathcal{N}(0 \subset \text{NT-PA}))\ ((\mathcal{S}ucd\,(x) \subseteq \overline{\mathcal{S}uc_{x}(0)\ \cap\ \mathcal{S}uc_{x+1}(0)}) \rightarrow$$

$$((\mathcal{S}uc_{x+1}(0)\ \wedge \mathcal{S}uc_{1}(0) = 1) \wedge (\mathcal{S}uc_{x}(0) \wedge \mathcal{P}re_{1}(1)\ \wedge \mathcal{P}re_{x}(x) = 0) \rightarrow (x = 0))$$

[83] Please, take in account that **NT-FS&L F.Th 1-2** and **NT-FS&L F.Th 3 are embed in the previous item natural number (NT-PA)- Pythagorean Successors & Predecessors identities.**

[84] ***NT-FS&L PRESENTATION OF SET OF THE PRIMES NATURAL NUMBER SET:***

$$\mathcal{P}rimes(p\text{-}P^{\alpha}{}_{0}(x)) := \{1, 2, 3, 5, 7, 11, 13, 17, \ldots\}$$

Please, let us highlight that "Christian Goldbach listed natural number ***1*** *as the first prime in his previously referred correspondence with Leonhard Euler; however, Euler himself did not consider 1 to be a prime number. Even more, by the early 20th century, mathematicians began to arrive at the consensus that positive integer number 1 is not a prime number". Nowadays, it is globally accepted that number 1 is not a prime number rooted in the* ***argument from authority*** *(Latin:* ***argumentum ad verecundiam****), which has been rejected for example by* ***Henri León Legendre*** *in the development during the last century of his innovative, efficient and effective Lebesgue Integral Theory). Hence, for a very long period of time a plethora of scholar and academic and professional books (and e-books), computer programs, applications and Computational Knowledge Engines globally reinforce low reasoning procedures rooted on conventions instead of arguments (Sagan, Carl. The Demon-Haunted World: Science as a Candle in the Dark. Ballantine Books. (2011) ISBN 9780307801043).* **For example:**

1. The On-Line Encyclopedia of Integers Sequences. Founded in 1964 by N.J.A. Sloane https://oeis.org/A000040. Additionally, please see the sequences related with the named Mersenne primes (sequence A000043 (p) and A000668 (M_p) in OEIS).

2.1. Wolfram MathWorld: Weisstein, Eric W. "Prime Number." From *MathWorld*--A Wolfram Web Resource. http://mathworld.wolfram.com/PrimeNumber.html and references therein). https://www.wolframalpha.com/examples/PrimeNumbers.html. **2.2.** Wolfram Mathematica 10. (10.4.0.0): **2.2.1.** Input: PrimeQ[1] ¿primo? Result: False; **2.2.2.** Input: Select [table[K, {K, 10}], PrimeQ] Result: {2,3,5,7}; **2.2.3.** Input: a = {1, 2, 3, 4, 5, 6, 7}; Select[a, PrimeQ] Result: {2, 3, 5, 7}. **2.3.** Wolfram ALPHA. Computational Knowledge Engine. "*Question: IS 1 PRIME? INPUT: Is 1 a prime number? Result: Is not a prime number. Nearest primes: 2, 3.*" (https://www.wolframalpha.com/input/?i=Is+1+prime%3F).

3. Number Impire (http://es.numberempire.com/). Number Properties. "*Entry: 1. Is prime? Answer: NO.*"

4. Great Internet Mersenne Prime Search (GIMPS). (https://www.mersenne.org/). The list of the Mersenne Primes Numbers starts with number 3.

5. Conway, John; Guy, Richard, The Book of the Numbers. e-Book, ISBN 978-1-4612-4072-3; https://www.amazon.es/Book-Numbers-John-Horton-Conway/dp/038797993X).

6. Milton Abramowitz and Irene A. Stegun; Handbook of mathematical Functions. National Bureau of Standars. Applied Mathematics Series, 55. Issued June 1964, seventh Printig, May 1968, whith corrections. Pp.- 870-876).

7. Wikipedia (entry Prime Number: https://en.wikipedia.org/wiki/Prime_number).

[85] *Please, note that for more than twenty-five centuries this is the first time that in a first order system and language, the NT-FS&L, which has been built by means of (sense and intension) a primordial intended inductive FOL de-embedding methodology and from a topological-, algebraic- and geometrical-free context referred only to "the natural number concept" (linguistic, logic and mathematical) has been proven, the next set of* ***NT-FS&L F.Th 1-3*** *statement:*

*If and only if, **0 belongs to** $\mathcal{N}$ **(0⊂NT-PA)** is the even "conserving" non-covariant/non-precessor parity/umparity neutral addition element referred to a natural number non-prime concept", then: **1** as element belonging to $\mathcal{N}$ **(0⊂NT-PA)** is both: first, the odd "breaking" covariant umparity universal addition element representative of a natural number concept"; second, the odd "conserving" non-covariant/(successor of natural number **0**) /contravariant predecessor of natural number **2**) parity/umparity neutral multiplicative element referred to natural number prime concept".*

*Hence, in NT-FS&L, if **0** is a natural number, then **0** is even then **1** is odd and then **1** has to be prime, which can be represented by the following theorems: (Einstein-like tensorial addition convention)*

$$0 = 0 + 0;\ 0 + 1 = 1;\ 0 + 1 = 1 + 0;\ (0+1) + (1+0) = 1 + 1;\ 1 + 1 = 2;\ (0+1) + (1+0) = (0+0) + (1+1);\ 1+1+1 = 3;\ (1+1)+1=3:\ 1+(1+1) = (1+1+1)\ \ldots..$$

[86] Please, let us remember the following NT-"entailment-schema":

[0-Entailment:

$$\mathcal{N}(0\subset \text{NT-PA})) \subseteq \mathcal{N}(0\not\subset \text{NT-PA}) \cup \mathcal{PN}_0(0) \qquad \textbf{(NT-th. 6)}$$

1-Entailment:

$$[e\text{-}\mathcal{PN}_0(x)] \cup [o\text{-}\mathcal{PN}_0(x)] \subseteq \mathcal{N}(0\subset\text{NT-PA})$$

$$\mathcal{PN}_0(x) = \mathcal{PN}_{x+1}(0),$$

$$\mathcal{PN}_0(x) = \mathcal{PN}_{x+1}(0) \text{ has parity of } x\ (x \in \mathcal{N}(0\subset\text{PA}))$$

$$\mathcal{PN}_0(x+1) = \mathcal{PN}_{x+2}(0) \text{ has parity of } x+1,$$

which is the 1-succesor of x (its direct successor) **(∀x ∈ $\mathcal{N}$ (0⊂NT-PA))**

$$\#\text{-}\mathcal{P}re_0(x) := x+1 \in \mathcal{N}(0\subset\text{NT-PA})) \rightarrow x \in \mathcal{PN}_0(x),\ \forall x\ (x \in \mathcal{N}(0\subset\text{NT-PA}))$$

(NT-th. 7)

$$\{\#\text{-}\mathcal{P}re_0(x)\} = \{x+1 \in \mathcal{N}(0\not\subset\text{NT-PA}) \cup \mathcal{PN}_0(0)) \mid x \in \mathcal{PN}_0(x)\},$$

$$\forall x\ (x \in \mathcal{N}(0\not\subset\text{NT-PA})) \qquad \textbf{(NT-th. 7 bis)}$$

2-Entailment:

$$\mathcal{N}(0\subset\text{PA}) = \mathcal{P}re_0(x) \cup \mathcal{S}uc_{x+1}(0) \qquad \textbf{(NT-th. 3)}$$

$$\mathcal{N}(0\subset\text{PA})) = \mathcal{N}(0\not\subset\text{NT-PA}) \cup \mathcal{PN}_0(0) \qquad \textbf{(NT-th. 6) \&(NT-th. 7)}$$

3-Entailment :

<u>HLParity-1-NT Theorem</u>: **(∀x ∈ $\mathcal{N}$ (0⊂NT-PA))**

$$\mathcal{PN}_0(x) \cap \mathcal{PN}_{x+1}(0) \subseteq o\text{-}\mathcal{PN}_0(x+x) \quad \text{(always is an even primordial)}$$

$$\mathcal{PN}_0(x) \cap \mathcal{PN}_{x+2}(0) \subseteq o\text{-}\mathcal{PN}_0((x+x)+1) \quad \text{(always is an odd primordial)}]$$

[87] ***Please, it is very important indicate that natural number 2 is the unique natural number that :***

1. $2 = P_0(2) \wedge P_1(1+1) \wedge P_2(2) \wedge P_1(2\cdot1+1) = P_1(2\cdot1+1)$
2. $0 = P_2(2) \wedge P_1(1) \wedge P_0(0) \wedge P_1(1\cdot1+1)$
3. $P_x(x) + 2 = (0+0) + (1+1) = ((0+1) + (0+1)) \wedge ((1+0) + (1+0)) \wedge ((0+0) \wedge (1+1))$
4. $2^{(0+0)+(1+1)} = 2^{(0+1)+(0+1)} \wedge 2^{((1+0)+(1+0))} \longrightarrow 2^{(0+0)} \cdot 2^{(1+1)} \wedge 2^{(1+1)} \cdot 2^{(0+0)}$

[88] ***Archimedean Property and Archimedean Axiom.*** **1.** Kazdan, J.L. "*The Archimedan Property*". September 2014, https://www.math.upenn.edu/~kazdan/508F14/Notes/archimedean.pdf. **2.** http://planetmath.org/Archimedeanproperty. **3.** G. Fisher (1994) in P. Ehrlich(ed.), Real Numbers, Generalizations of the Reals, and Theories of continua, 107-145, Kluwer Academic. **4.** Shell, Niel, Topological Fields and Near Valuations, Dekker, New York, 1990. <u>ISBN 0-8247-8412-X</u>
Boyer, C. B. and Merzbach, U. C. *A History of Mathematics, 2nd ed.* New York: Wiley, pp. 89 and 129, 1991. **4.** *Knopp, Konrad (1951). Theory and Application of Infinite Series (English 2nd ed.). London*

and Glasgow: Blackie & Son, Ltd. p. 7. ISBN 0-486-66165-2. **5.** Neal Koblitz, "p-adic Numbers, p-adic Analysis, and Zeta-Functions", Springer-Verlag,1977. **6.** Weisstein, Eric W. "Archimedes' Axiom." From *MathWorld*--A Wolfram Web Resource. http://mathworld.wolfram.com/ArchimedesAxiom.html

[89] *Entailment from* **(NT-th. 3)** $(\forall x\ (x \in \mathcal{N}(0\subset \text{NT-PA}))(\mathcal{Pre}_0(x) \cup \mathcal{Suc}_{x+1}(0) \subseteq \mathcal{N}(0\subset \text{NT-PA}))$

[90] *Further entailment is ensured by taking in account the last result and the following two definitions:*

$\mathcal{E}ven(\mathcal{Suc}_0(x)) := \{\ x+x \mid x+x\ \in \mathcal{Suc}_0(x)\ \},\ \forall x\ (x \in \mathcal{N}(0\subset \text{NT-PA}))$ **(def-P. 1)**

$\mathcal{O}dd\ (\mathcal{Suc}_0(x)) := \{\ (x+x)+1 \mid (x+x)+1 \in \mathcal{Sucd}_0(x+x)\ ) \mid x\ \in \mathcal{N}(0\subset \text{NT-PA}))\ \},\ \forall x$ **(def-P. 2)**

Additionaly, please let us remember: $\forall x\ (x \in \mathcal{N}(0\subset \text{NT-PA}))$

$$P^{\alpha}{}_0(0)= S^{\alpha}{}_0(0)=0\ \ \&$$
$$0 + S^{\alpha}{}_0(0)= S^{\alpha}{}_0(0)+0\ \ \wedge\ \ 0 + P^{\alpha}{}_0(0)= P^{\alpha}{}_0(0)+0\ \ \&$$
$$0 + S^{\alpha}{}_0(2)= S^{\alpha}{}_2(0)+0\ \ \wedge\ \ 0 + P^{\alpha}{}_2(0)= P^{\alpha}{}_0(1)+ P^{\alpha}{}_0(1)\ \&$$
$$S^{\alpha}{}_1(x) + S^{\alpha}{}_1(x) = P^{\alpha}{}_{(1+1)+(1+1)}(x+x)\ \&$$
$$S^{\alpha}{}_1(x) + S^{\alpha}{}_1(x) = P^{\alpha}{}_{(2)+(2)}(x+x)\ \&$$
$$S^{\alpha}{}_1(x) + S^{\alpha}{}_1(x) = P^{\alpha}{}_{(2+2)}(x+x)\ \&$$
$$S^{\alpha}{}_1(x) + S^{\alpha}{}_1(x) = P^{\alpha}{}_{(4)}(x+x)\ \rightarrow$$
$$S^{\alpha}{}_{x+1}(0) + S^{\alpha}{}_0(x+1) = P^{\alpha}{}_0(x+1) + P^{\alpha}{}_0(x+1),\ \forall x\ (x \in \mathcal{N}(0\subset \text{NT-PA}))$$

[91] *In NT-FS&L, tensorial nomenclature, formulation and notation holds for a topological analytic, algebraic and geometrical and classic-numerical free-context.*

[92] *In NT-FS&L, these names of the natural numbers* **x, y, z, t** $(\forall x\forall y\forall z\forall t\ (x, y, z, t\ \in \mathcal{N}^{\alpha}{}_0(1\otimes, 0\oplus, 0\subset \text{NT-PA}))$ *are intentionally (sense and intension) the nicknames referred-identities to the globally accepted and called as "dimensions" in the customary "tetra-dimensional space-time" concept. Please, take in account that NT-FS&L could be presented as a topological-, algebraic-, analytic -and geometrical free-context FOL language. However, take into account that, also according the authors, NT-FS&L never could be considered as a logical-mathematical-computational-cognitive free-context language.*

[93] ***Please let us observe the following two* NT-theorems :**

1. $\mathcal{P}(0\subset\text{NT-PA}) \cup \mathcal{NP}(0\subset\text{NT-PA}) \subseteq \mathcal{N}(0\subset\text{NT-PA})$
2. $((\mathcal{Primes}(x)) \cup \mathcal{Primes}(p\text{-}P^{\alpha}{}_0(x)) \subseteq \mathcal{P}(0\subset\text{NT-PA}) \rightarrow$

$$p \in \mathcal{Primes}(p\text{-}P^{\alpha}{}_0(x)) \rightarrow p= x)\ \forall x\forall p\ \ (x, p \in \mathcal{N}(0\subset\text{NT-PA})\ )$$

[94] *NT-FS&L nomenclature, formulation and notation of NT-PA primordial sets (singletons):*

$\mathcal{PN}\{1, p\} := \mathcal{PN}\ (\{\ 1\ \} \cup \{\ p\ \})\ \ \wedge\ \ \mathcal{PN}\{1, 2\ \} := \mathcal{PN}(\{\ 1\ \} \cup \{\ 2\ \}) \rightarrow \{0, 2\} := \mathcal{PN}(\{0\} \cup \{(1\oplus 1)\})$

[95] *Translation from NT-FS&L into:*

1.- The decimal numerical positional-based system language:

$$\mathcal{N}^{\alpha}(0 \subset \text{NT-PA})) = \{ \text{e-0}, \text{o-1}, \text{e-2}, \text{o-3}, \ldots, \text{-x}, \ldots \} \ \& $$

$$\mathcal{N}^{\beta}(0 \subset \text{NT-PA})) = \{ \ldots, x, \ldots, \text{o-3}, \text{e-2}, \text{o-1}, \text{e-0} \} ;$$

$$Suc^{\alpha}{}_{x}(0) = \{ \text{e-0}, 1, 2, 3, \ldots, x, \ldots \} \ \& \ Suc^{\beta}{}_{x}(0) = \{ \ldots, x, \ldots, 3, 2, 1, 0 \}$$

$$Pre^{\alpha}{}_{0}(x) = \{ 0, 1, 2, 3, \ldots, x \} \ \& \ Pre^{\beta}{}_{0}(x) = \{ x, \ldots, 3, 2, 1, 0 \}$$

2.- The decimal numerical positional-based system language

Naturals Numbers Set= **{ 0, 1, 2, 3, 4, 5, ... }** ; *Prime Numbers Set* = **{1, 2, 3, 4, 5, ... }**

96 Briefly symbolized by next formula:

$$\{\mathcal{N}^{\alpha}\{0 \subset \text{PA})\} \subseteq \{\text{NT- FS\&L}\}\}$$

97 Expanded formal NT-FS&L formula:

$$\{\mathcal{P}^{\alpha}(\, p\text{-}p_n, z[+x]_t, z[x+]_t, z[\cdot x]_t, z[x\cdot]_t, z[x\oplus]_t\, z[x\oplus]_t, z[\otimes x]_t, z[x\otimes]_t, \{0 \subset \text{NT-PA})\})\}$$

$$p\text{-}p_n, y, z, t, n = (\, S_0(x+1) \ \wedge \ P_0(x+1)) \ \forall n, n \in \mathcal{N}^{\alpha}[1\otimes, 0\oplus, 0 \subset \text{PA}]\,)$$

98 *Please, let us remember that "**Quod erat demonstrandun**"* (Translating from the Latin into contemporary English yields "what was to be demonstrated") *for centuries has been abbreviated by acronyms "Q.e.d" and Q.E.D"; nowadays it could be misunderstood as the abbreviation of "**Q**uantum **E**lectro**D**inamics" which has been proposed and globally "popularized" by the theoretical physicist Richard Feymann.* Feynman, Richard **1.** *QED: The Strange Theory of Light and Matter* (1985), Princeton University Press. ISBN 978-0-691-08388-9. **2.** (1985) W. W. Norton & Company. ISBN 978-0-393-31604-9. *Surely you're joking, Mr. Feynman!*