Theory of the spin-galvanic effect and the anomalous phase shift φ_0 in superconductors and Josephson junctions with intrinsic spin-orbit coupling

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Due to the spin-orbit coupling (SOC) an electric current flowing in a normal metal or semiconductor can induce a bulk magnetic moment. This effect is known as the Edelstein (EE) or magnetoelectric effect. Similarly, in a bulk superconductor a phase gradient may create a finite spin density. The inverse effect, also known as the spin-galvanic effect, corresponds to the creation of a supercurrent by an equilibrium spin polarization. Here, by exploiting the analogy between a linear-in-momentum SOC and a background SU(2) gauge field, we develop a quasiclassical transport theory to deal with magnetoelectric effects in superconducting structures. For bulk superconductors this approach allows us to easily reproduce and generalize a number of previously known results. For Josephson junctions we establish a direct connection between the inverse EE and the appearance of an anomalous phase shift φ_0 in the current-phase relation. In particular we show that φ_0 is proportional to the equilibrium spin current in the weak link. We also argue that our results are valid generically, beyond the particular case of linear-in-momentum SOC. The magnetoelectric effects discussed in this study may find applications in the emerging field of coherent spintronics with superconductors.

I. INTRODUCTION

Over the past decades, superconductor-ferromagnetic (S-F) structures have been studied extensively [1,2]. The spatial oscillatory behavior of the superconducting condensate induced in the ferromagnet leads to interesting effects as oscillations of the density of state in F/S [3–5] and F/S/F [6] structures, oscillations of the Josephson current in S/F/S Josephson junction [7–10], and oscillations of the critical temperature [1]. Moreover, in the case of multidomain ferromagnets or artificial multilayer structures with inhomogeneous magnetization, the singlet Cooper pairs from a superconductor can be transformed into long-range triplet pairs that may explain the long-range Josephson coupling observed in S/F/S structures [11–25]. Triplet correlations also lead to a dependence of the critical current on the magnetic configuration of diverse S/F structures [26–33]. Such phenomena suggest interesting perspectives of exploiting triplet correlations for the emerging field of coherent superspintronics [34,35]. Also promising applications might be found by using superconducting materials in combination with ferromagnetic insulators that may act as spin filters [36–38]. In particular several thermal effects related to these material combinations have been studied in recent works [39–44].

All the above mentioned phenomena in S/F structures originate from the interaction between the superconducting correlations and the exchange field of the ferromagnet. However it has recently been shown that spin-orbit coupling (SOC) in S/F structures will also lead to, for example, a long-range triplet component [45,46] and peculiarities in the density of states [47–49]. On the other hand, transport properties of nonsuperconducting structures with strong SOC are being intensively studied because of their potential application in a novel direction of spintronics, which exploits the coupling between spin and charge currents [50–53].

In particular, the SOC in semiconductors and normal metals is at the root of a number of interesting phenomena that originate from the coupling between the charge and spin degrees of freedom. The prototype of these phenomena is the spin Hall effect (SHE) [54–64] which consists of the creation of a spin-polarized current by an electric field. Reciprocally, by means of the inverse SHE a spin current can create an electric field [65–67]. These effects allow one to generate and detect spin-polarized currents in nonmagnetic materials [68–72].

There is also another relevant effect in normal systems related to the SOC. It consists of creating a stationary spin density S^a along the a direction in response to an electric field E_k applied in the k direction [73,74]. Within linear response, this effect is described by

$$S^a(\omega) = \sigma_k^a(\omega)E_k(\omega), \qquad (1.1)$$

where the sum over repeated indexes is implied here, and throughout this paper. In particular, in 2D systems with Rashba SOC, the applied electric field and the generated spin density are perpendicular to each other. This magnetoelectric effect, also called the *Edelstein effect* (EE), has been observed in experiments [60,75]. The Edelstein conductivity $\sigma_k^a(\omega)$ in Eq. (1.1) is related to the Kubo correlator $\chi_k^a(\omega) = \langle\langle\hat{S}^a; \hat{j}_k\rangle\rangle_\omega$ of the spin and current operators via $\sigma_k^a(\omega) = \chi_k^a(\omega)/\mathbf{i}\omega$ [76]. Because of the gauge invariance in normal systems the function $\chi_k^a(\omega)$ should vanish in the limit $\omega \to 0$ reflecting the fact that there is no response to a static vector potential. Therefore the $\sigma_k^a(0) = \sigma_k^a$ remains finite and describes the dc EE. It has been pointed out in Ref. [76] that this property, together with the Onsager reciprocity principle, implies that the inverse dc EE, also referred to as the spin-galvanic effect, consists of generating a charge current j_k by a steady spin generation induced by a time-dependent magnetic field via the

paramagnetic effect:

$$j_k = \sigma_k^a [g \mu_B \dot{B}^a], \qquad (1.2)$$

with the Landé g factor, μ_B the Bohr magneton, and $\dot{B}^a$ the time derivative of the magnetic field along the a axis. The inverse EE effect has also been observed in experiments [77,78].

Similar magnetoelectric and spin-galvanic effects should also exist in superconductors [79,80]. However, there the physical situation is different because in the presence of the superconducting condensate the gauge invariance does not forbid the existence of a finite static current-spin response function χ_k^a. In contrast to the normal case, in a superconductor an equilibrium electric (super)current can flow in the absence of an external electric field. The supercurrent $\boldsymbol{j} = n_s \mathbf{v}_s$ (here n_s is the density of superconducting electrons and $\mathbf{v}_s$ the superfluid velocity) is proportional to the gradient of the macroscopic gauge-invariant phase $\nabla \tilde{\varphi} = \nabla \varphi - e\mathbf{A} \sim \mathbf{v}_s$, which is the physical field coupled to the current operator in the Hamiltonian of a superconductor. The existence of such a gauge-invariant field implies that the static response function $\chi_k^a = \langle\langle \hat{S}^a; \hat{j}_k \rangle\rangle_{\omega=0}$ can be nonzero without violating the gauge invariance. In principle, a supercurrent can thus generate an equilibrium spin polarization according to the general linear response relation:

$$S^a = \chi_k^a \partial_k \tilde{\varphi}, \qquad (1.3)$$

where $\partial_k = \partial/\partial x_k$. This effect has been indeed theoretically demonstrated by Edelstein for a 2D superconductor with Rashba SOC, who calculated the proportionality tensor χ_k^a at temperatures T close to the critical superconducting temperature T_c, in both pure ballistic [79] and diffusive [80] limits.

Because in the superconducting state the response function χ_k^a at $\omega \to 0$ is finite, the reciprocity of the EE effect becomes complete. In contrast to the normal case, in superconductors a static Zeeman field $\mathbf{B}$ can induce a supercurrent j_k. Therefore, instead of Eq. (1.2) the following relation holds:

$$j_k = e \chi_k^a h^a, \qquad (1.4)$$

where $h^a = g \mu_B B^a$. An explicit expression of this type has been obtained in a particular case of a 2D ballistic superconductor with intrinsic Rashba SOC [81,82].

It is then clear that the free energy of a superconductor with a SOC must have a term of the Lifshitz type,

$$F_L = h^a \chi_k^a \partial_k \tilde{\varphi}, \qquad (1.5)$$

and Eqs. (1.3) and (1.4) follow directly from the general thermodynamic definitions of the spin and current densities, $\mathbf{S} = \delta F/\delta \mathbf{h}$ and $\boldsymbol{j} = -\delta F/\delta \mathbf{A}$.

In principle, Eqs. (1.3) and (1.4) apply for bulk superconductors, but one can expect similar effects to occur also in an S-X-S Josephson junction, between two massive superconductors (S) and a normal or ferromagnetic bridge X with an intrinsic SOC. In a Josephson junction the supercurrent depends on the phase difference φ between the superconducting electrodes. In the particular cases of a weak proximity effect between the S and the X, or in the high-temperature regime ($T \lesssim T_c$), the current-phase relation is given by $j = j_c \sin \varphi$, where j_c is the critical Josephson current.

When the SOC competes with a Zeeman effect, the natural conjectures following Eqs. (1.3) and (1.4) are as follows: (i) In accordance with Eq. (1.3), the flow of a supercurrent may generate a spin polarization in the X bridge (the Edelstein effect). (ii) In turn, from Eq. (1.4), a Zeeman (spin-splitting) field may induce a supercurrent through the junction, even if the phase difference between the electrodes vanishes (the inverse Edelstein effect).

In other words, the inverse EE is presumably the cause of an anomalous phase φ_0 which modifies the current-phase relation according to $j = j_c \sin(\varphi - \varphi_0)$, with a nontrivial (i.e., nonequal to 0 or π) equilibrium phase φ_0. This defines the so called φ_0 junctions, a subject that has been extensively studied in the past years in different systems, including conventional superconductors with SOC [83–95], with triplet correlations [96–101] or in contact with topological materials [102,103], and also in hybrid systems with nonconventional superconductors [104–112], quantum dots [113–115], and hybrid (0-π) structures [116–119]. φ_0 junctions may produce a self-sustained flux when embedded in a SQUID geometry [120], act as phase batteries in coherent circuits [121,122], present a current asymmetry, and act as supercurrent rectifiers [113].

In the present work we develop a complete theory of the magnetoelectric and spin-galvanic effects in hybrid superconducting structures and confirm the above conjectures. We focus on systems with linear-in-momentum SOC that can be conveniently described in terms of an effective background SU(2) gauge field. This allows us to use the SU(2) covariant quasiclassical equations for the Green's functions (GFs) derived in Refs. [45,46,123]. We establish a connection between the tensor χ_k^a in Eqs. (1.3) and (1.4) and the equilibrium spin current $\mathcal{J}_j^a$ [124,125]. We show that in a generic S-X-S Josephson junction the condition for a nontrivial anomalous phase φ_0 to appear is that $\mathcal{J}_j^a h^a \neq 0$, where h^a can be either an external Zeeman field or the internal exchange field of a ferromagnet. Our SU(2) covariant formulation results in a simple and tractable system of equations to describe hybrid structures with arbitrary linear-in-momentum SOC, temperatures, degree of disorder, and quality of the hybrid interfaces. We also show that qualitatively our results are generically valid beyond the particular case of the linear-in-momentum SOC.

The structure of the paper is the following: In the next section we present a qualitative discussion of the superconducting proximity effect in structures with SOC and its connection with the spin diffusion in normal systems. This qualitative analysis allows us to guess the form of the quasiclassical equations for superconducting structures in the presence of generic spin fields, and in particular to explicitly show the analogy between the charge-spin coupling in normal systems and the singlet-triplet coupling in superconducting ones. In Sec. III we present our model, discuss the associated symmetries, and derive microscopically the quasiclassical equations for generic linear-in-momentum SOC. In Sec. IV we use the derived equations to explore the magnetoelectric effects in bulk superconductors. We generalize the previously known results for the EE and its inverse obtained for 2D Rashba SOC [79–81,126] to generic linear-in-momentum SOC, and relate them to the spin current and the SU(2) gauge fields. In Sec. V we explore the Josephson effect through an S-X-S

diffusive junction and in Sec. VI through a ballistic one. In both cases we show that the anomalous phase φ_0 is proportional to $\partial_i^a h^a$ and determine its dependence on other parameters of the structure, such as temperature and length. We finally present our conclusions and discuss possible experimental setups to verify our predictions in Sec. VII.

II. DIFFUSION OF SUPERCONDUCTING CONDENSATE IN THE PRESENCE OF SPIN-ORBIT COUPLING: HEURISTIC ARGUMENTS

Before presenting the full quantum kinetic theory it is instructive to discuss at the qualitative level the main features of the proximity-induced superconductivity in the presence of an intrinsic SOC. For this sake we present a simple heuristic derivation of the equations describing the coupled motion of the singlet and triplet components induced in a ferromagnet from a bulk s-wave superconductor.

Let us consider an S-X-S junction, where X is a diffusive ferromagnet. We assume that the system is at equilibrium, and that the proximity effect between S and X is weak. In such a case the junction is fully described by the quasiclassical anomalous Green's function $\hat{f}(\mathbf{r})$, which describes the superconducting condensate in X. In general $\hat{f}(\mathbf{r})$ is a 2×2 matrix in the spin space $\hat{f} = f_s \hat{1} + f_t^a \sigma^a$. Here the scalar f_s and the vector with components f_t^a describe the singlet and the triplet components of the condensate, respectively. In this section we show that the functions $f_s(\mathbf{r})$ and $\mathbf{f}_t(\mathbf{r})$ are reminiscent of the charge and spin density in the normal systems.

In the absence of SOC, but in the presence of the exchange field $\mathbf{h}$, the diffusion of the condensate is described by the well known linearized Usadel equations (see, e.g., Ref. [2]),

$$D\nabla^2 f_s - 2|\omega_n|f_s + 2\mathbf{i}\,\mathrm{sgn}(\omega_n)h^a f_t^a = 0, \quad (2.1)$$

$$D\nabla^2 f_t^a - 2|\omega_n|f_t^a + 2\mathbf{i}\,\mathrm{sgn}(\omega_n)h^a f_s = 0, \quad (2.2)$$

where D is the diffusion constant and ω_n is the Matsubara frequency. The terms proportional to $2|\omega_n|$ are responsible for the decay of the superconducting correlations in the normal metal. The last terms on the left-hand sides of Eqs. (2.1) and (2.2) describe the usual singlet-triplet coupling coming from the exchange field. It is worth emphasizing the presence of imaginary unit $\mathbf{i}$ in the exchange field terms, which reflects the breaking of the time-reversal symmetry. Because of this, the singlet-triplet conversion due to the exchange field is always accompanied with a phase shift of $\pi/2$. This point will be of primary importance in the following for understanding the origin of the anomalous phase φ_0.

To understand how the Usadel equations (2.1) and (2.2) are modified in the presence of SOC we recall the description of the diffusion of spin $\mathbf{S}(\mathbf{r})$ and charge $n(\mathbf{r})$ densities in normal systems. The general spin diffusion equation in a normal conductor with SOC takes the form

$$\partial_t S^a - D\nabla^2 S^a = \mathcal{T}^a, \quad (2.3)$$

where $\mathcal{T}^a$ is a so called spin torque. In the absence of SOC, $\mathcal{T}^a = 0$ and hence spin is a conserved quantity which satisfies the usual spin diffusion equation. In noncentrosymmetric materials SOC acts as an effective momentum-dependent Zeeman

field that causes precession of spins of *moving* electrons. This precession breaks conservation of the average spin, and shows up formally as a finite torque $\mathcal{T}^a \neq 0$ in Eq. (2.3). In the diffusive regime the motion of the electrons consists of a random motion superimposed on an average drift caused by the density gradients. The spin precession related to these types of motion generates the corresponding contributions to the spin torque. To the lowest order in gradients the general expression for the torque can be written as follows [127–129],

$$\mathcal{T}^a = D\big[-\Gamma^{ab}S^b + 2P_k^{ab}\partial_k S^b + C_k^a \partial_k n\big]. \quad (2.4)$$

Here the first term describes the Dyakonov-Perel (DP) spin relaxation that originates from the spin precession of randomly moving electrons [54]. The positive-definite matrix Γ^{ab} is the DP relaxation tensor with the eigenvalues equal to the inverse squares of the DP spin relaxation lengths. The other two contributions to the torque are related to the average motion of spins. In particular, the second term on the right-hand side of Eq. (2.4) originates from the diffusive motion of spins caused by inhomogeneities of the spin density distribution. The corresponding spin precession is described by antisymmetric (spin rotation) matrices $P_k^{ab} = -P_k^{ba}$ with $\|\hat{P}\| \sim 1/\ell_{\mathrm{so}}$, where ℓ_{so} is the spin precession length.

The last term in Eq. (2.4), which is proportional to the charge density gradient, can be called the spin Hall torque. The charge density gradient generates the charge current which is then transformed to the spin current via the spin Hall effect. Precession of the spins driven by the charge density gradient, via the spin Hall effect, is the origin of the spin Hall torque in Eq. (2.4). The spin Hall torque is parametrized by the tensor C_k^a which is proportional to $\theta_{\mathrm{sH}}/\ell_{\mathrm{so}}$, where θ_{sH} is the spin Hall angle—the conversion coefficient between the charge and the spin currents.

Equation (2.3) with the spin torque of Eq. (2.4) is commonly used in a spintronics context to describe spin dynamics in semiconductors with intrinsic SOC [127–129] (for a discussion between intrinsic and extrinsic SOC, see, e.g., [64]). In the stationary case the diffusion equations for the spin and charge densities reduce to

$$\nabla^2 n + C_k^a \partial_k S^a = 0, \quad (2.5)$$

$$\nabla^2 S^a - \Gamma^{ab}S^b + 2P_k^{ab}\partial_k S^b + C_k^a \partial_k n = 0. \quad (2.6)$$

It is important to emphasize here that spin-charge coupling mediated by the spin Hall torque (C_k^a) is responsible for the EE. This can be seen directly from Eq. (2.6): A uniform charge density gradient produces a uniform spin density given by $S^a = (\hat{\Gamma}^{-1})^{ab} C_k^b \partial_k n$.

We can now construct the Usadel equations in the presence of SOC in analogy to the normal case. Since SOC does not violate the time-reversal symmetry it acts in exactly the same way on the time-reversal conjugated states composing the Cooper pair. Therefore the diffusion of the singlet and the triplet condensates should be modified by SOC in complete analogy with the diffusion of the charge and spin densities in normal systems. The formal connection between the diffusion of the triplet condensate function f_t^a in superconductors and the spin density S^a in normal metals has been discussed recently in Ref. [46], and it has been also

noticed in Ref. [86]. Hence, in order to include the effects of SOC in the Usadel equations all we need to do is to replace the diffusion operators (the Laplacians) in Eqs. (2.1) and (2.2) with the diffusion operators entering Eqs. (2.5) and (2.6), respectively. The result is the following system of equations describing a coupled diffusion of the singlet and triplet condensates in the presence of SOC,

$$\nabla^2 f_s - \kappa_\omega^2 f_s + \text{sgn}(\omega_n)\left[\mathbf{i}\frac{2h^a}{D}f_t^a + C_k^a \partial_k f_t^a\right] = 0, \quad (2.7)$$

$$\nabla^2 f_t^a - \left(\kappa_\omega^2 \delta^{ab} + \Gamma^{ab}\right)f_t^b + 2P_k^{ab}\partial_k f_t^b$$
$$+ \text{sgn}(\omega_n)\left[\mathbf{i}\frac{2h^a}{D}f_s + C_k^a \partial_k f_s\right] = 0. \quad (2.8)$$

In contrast to the normal case, in addition to the DP relaxation, both the f_s and $\mathbf{f}_t$ experience an additional decay proportional to the inverse decay length $\kappa_\omega = \sqrt{2|\omega_n|/D}$, due to the finite lifetime of the superconducting condensate in the normal metal.

The most important feature of Eqs. (2.7) and (2.8) is the presence of two mechanisms for the singlet-triplet coupling which are described by the two terms in the square brackets. The first mechanism is the above discussed Zeeman coupling related to the modification of the internal structure of the Cooper pair by the spin-splitting field **h** [see Eqs. (2.1) and (2.2)]. The second channel of singlet-triplet coupling comes from the spin Hall torque, which converts the gradient of f_s into $\mathbf{f}_t$ and vise versa, in a complete analogy with the EE in normal systems. The corresponding singlet-triplet "conversion amplitudes" have a relative phase shift of $\pi/2$, which is related to the different transformation properties of the Zeeman and spin-orbit fields with respect to the time reversal. We will see in the next sections that the interference of these two singlet-triplet conversion channels is indeed responsible for the magnetoelectric/spin-galvanic effects in superconductors, and, in particular, for the appearance of the intrinsic anomalous phase φ_0 in Josephson junctions.

Although the present heuristic derivation of Eqs. (2.7) and (2.8) may seem imprecise, it uncovers a simple but deep connection between the physics of inhomogeneous superconductors with SOC and the well known spintronics effects, such as the spin Hall effects and direct and inverse magnetoelectric effects (EE). In Sec. III we present a rigorous derivation of the quasiclassical kinetic equations for superconductors with a linear-in-momentum SOC, which in the diffusive limit confirms the correctness of Eqs. (2.7) and (2.8). In the rest of the article we study in detail the physical consequences of the interference of the two singlet-triplet conversion channels and their connection with the theory of φ_0-Josephson junctions.

III. THE MODEL AND BASIC EQUATIONS

In this section we introduce our model and discuss the symmetries associated with superconducting systems in the presence of spin-orbit coupling (SOC). We also present the derivation of the quasiclassical equations in the presence of linear-in-momentum SOC.

A. The Hamiltonian in the presence of generic SOC and symmetry arguments for the appearance of an anomalous phase

Our starting point is a general Hamiltonian describing a metal or a semiconductor with a linear-in-momentum SOC, an exchange field, and superconducting correlations

$$H = \int d\mathbf{r}[\Psi^\dagger H_0 \Psi + V\psi_\uparrow^\dagger \psi_\downarrow^\dagger \psi_\downarrow \psi_\uparrow], \quad (3.1)$$

where $\psi_{\uparrow,\downarrow}(\mathbf{r})$ are the annihilation operators for spin up and down at position $\mathbf{r}$, and $\Psi^\dagger = (\psi_\uparrow^\dagger, \psi_\downarrow^\dagger)$ is the spinor of creation operators. H_0 is the free electron part [149]

$$H_0 = \frac{(\hat{p}_i - \mathcal{A}_i)^2}{2m} - \mu + \mathcal{A}_0 + V_{\text{imp}}, \quad (3.2)$$

where μ the chemical potential and V_{imp} the potential induced by nonmagnetic impurities. The magnetic interactions appear in two places: as an SU(2) scalar potential $\mathcal{A}_0 \equiv \mathcal{A}_0^a \sigma^a/2$, describing for example the intrinsic exchange field in a ferromagnet or a Zeeman field in a normal metal, and as an SU(2) vector potential $\mathcal{A}_i \equiv \mathcal{A}_i^a \sigma^a/2$, describing the SOC. The latter is associated with the momentum operator [150] $\hat{p}_i = -\mathbf{i}\partial_i$ in the form of a minimal coupling $\hat{p}_i - \mathcal{A}_i$. In practice, all the linear-in-momentum SOC can be represented as a gauge potential (see, e.g., [130] or [131] and references therein). In the widely studied case of a free electron gas with Rashba SOC, $\mathcal{A}_x^y = -\mathcal{A}_y^x = -\alpha$, while in the presence of Dresselhaus SOC $\mathcal{A}_x^x = -\mathcal{A}_y^y = \beta$. Finally, $V = V(\mathbf{r}) < 0$ in the second term on the right-hand side of Eq. (3.1) describes the coupling strength which gives rise to superconductivity in some regions of space.

In analogy to electrodynamics one can define the four-potential $\mathcal{A}_\mu$, with space components ($\mu = 1,2,3$ or $\mu = x,y,z$) given by the SOC and the time component ($\mu = 0$) by the Zeeman field. Following the analogy one can define the strength tensor

$$\mathcal{F}_{\mu\nu} = \tfrac{1}{2}\mathcal{F}_{\mu\nu}^a \sigma^a = \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu - \mathbf{i}[\mathcal{A}_\mu, \mathcal{A}_\nu] \quad (3.3)$$

and the electric and magnetic SU(2) fields

$$\mathcal{E}_k^a = \mathcal{F}_{0k}^a \text{ and } \mathcal{B}_i^a = \varepsilon_{ijk}\mathcal{F}_{jk}^a, \quad (3.4)$$

where ε_{ijk} is the Levi-Civitta symbol.

In normal metals and semiconductors, the SHE and EE are consequences of the existence of a finite SU(2) magnetic field. For a pure-gauge vector potential the SOC can be gauged out [46], the SU(2) magnetic field is zero, and hence the SHE and EE do not appear [151]. Following our analogy, in the superconducting case an anomalous phase can only appear if the SU(2) magnetic field is finite. This explains why S-F-S junctions without SOC do not present any magnetoelectric effect, or equivalently, no anomalous phase. As is well known, the ground state of S-F-S junctions corresponds to a phase difference either equal to 0 or to π [1,2,132].

A simple way to describe qualitatively magnetoelectric effects in a superconductor is to provide simple symmetry arguments. Let us consider a ballistic superconductor at T close to its critical temperature T_c and focus on the Ginzburg-Landau free energy. In such an expansion, a SOC is responsible for the presence of a first-order derivative of the order parameter, the so-called Lifshitz invariant which describes

most of the original phenomenology of noncentrosymmetric superconductors [82]. Assuming that the amplitude of the order parameter is constant but its phase position-dependent, the Lifshitz invariant reads $F_L \propto T_i \partial_i \varphi$ where T_i is a vector which has to be odd with respect to the time-reversal operation, and SU(2) invariant. As discussed in Ref. [92], to the lowest order in SOC the Lifshitz invariant for a superconductor can be expressed in terms of the SU(2) fields:

$$F_L \propto \mathrm{Tr}\{\mathcal{F}_{0j}\mathcal{F}_{ji}\}\partial_i\varphi = (\mathcal{E}^a \times \mathcal{B}^a) \cdot \nabla\varphi. \quad (3.5)$$

If we focus on the static case, the electric field is given by $\mathcal{F}_{0j} = -\partial_j \mathcal{A}_0$. Moreover we define the equilibrium spin current [125] in terms of the SU(2) magnetic field as $\mathcal{J}_j = \tilde{\nabla}_i \mathcal{F}_{ij} = \partial \mathcal{F}_{ij}/\partial x_i - \mathbf{i}[\mathcal{A}_i, \mathcal{F}_{ij}]$. If $\mathcal{A}_0$ is spatially homogenous, for example induced by an external magnetic field, Eq. (3.5) reads [92]

$$F_L \propto \mathcal{A}_0^a \mathcal{J}_i^a \partial_i \varphi. \quad (3.6)$$

This Lifshitz invariant agrees with the ones derived from microscopic considerations [133] or quasiclassical expansions [134] for a particular sort of SOC.

Equation (3.6) confirms our guessed Eq. (1.5) and demonstrates that the Edelstein response tensor χ_k^a behaves like the spin current tensor $\mathcal{J}_i^a$. The form of F_L in Eq. (3.6), in terms of the equilibrium spin current, suggests that our results remain valid for any momentum dependence of the SOC. We now proceed to derive the quasiclassical equations and provide a microscopic description of the magnetoelectric effects in superconductors.

B. The quasiclassical equations in the presence of SOC

In order to describe the transport properties of hybrid structures containing superconducting, normal (N), and/or ferromagnetic (F) layers with interfaces and arbitrary temperature and degree of disorder, we have to go beyond the Ginzburg-Landau limit. We present here the quasiclassical equations [135–138] for the Green's functions in the presence of a non-Abelian gauge field [45,46,123] (for a similar discussion in normal metal, see [139]).

We follow here the derivation presented in Ref. [46]. The basic transport equation derived from Hamiltonian (3.1) for the Wigner-transformed covariant Green's functions $\check{G}(\mathbf{p},\mathbf{r})$ in the time-independent limit reads

$$\frac{p_i}{m}\tilde{\nabla}_i \check{G} + \left[\tau_3(\omega_n - \mathbf{i}\mathcal{A}_0) - \mathbf{i}\check{\Delta} + \frac{\langle\check{g}\rangle}{2\tau}, \check{G}\right]$$
$$- \frac{1}{2}\left\{\tau_3\mathcal{F}_{0j} + v_i\mathcal{F}_{ij}, \frac{\partial\check{G}}{\partial p_j}\right\} = 0, \quad (3.7)$$

where $\omega_n = 2T\pi(n + 1/2)$ is the fermionic Matsubara frequency, and $\check{\Delta} = \Delta\begin{pmatrix} 0 & e^{\mathbf{i}\varphi} \\ -e^{-\mathbf{i}\varphi} & 0 \end{pmatrix}$ is the (s-wave) gap parameter of amplitude Δ and phase φ. The scattering at impurities is described within the Born approximation, where τ is the elastic scattering time, $\langle\check{g}\rangle$ is the GF matrix integrated over the quasiparticle energy, and $\langle\cdots\rangle$ describes the average over the Fermi momentum direction.

After integration of (3.7) over the quasiparticle energy and by using the fact that $\check{G}$ is peaked at the Fermi level, one

obtains the generalized Eilenberger equation [46,92]:

$$v_F(n_i\tilde{\nabla}_i)\check{g} + [\tau_3(\omega_n - \mathbf{i}\mathcal{A}_0) - \mathbf{i}\check{\Delta}, \check{g}] - \frac{1}{2m}\left\{n_i\mathcal{F}_{ij}, \frac{\partial\check{g}}{\partial n_j}\right\}$$
$$= -\frac{1}{2\tau}[\langle\check{g}\rangle, \check{g}], \quad (3.8)$$

where n_i, $i = x, y, z$ are the components of the Fermi velocity vector. When deriving (3.8) we have neglected corrections to the exchange term $\mathcal{A}_0$ of the order of $|\mathcal{A}_j|/p_F \ll 1$. In fact, one sees from (3.7) that $\{\tau_3\mathcal{F}_{0j}, \partial\check{G}/\partial p_j\}$ scales like $\{\mathcal{A}_j \partial/\partial p_j, -\mathbf{i}[\tau_3\mathcal{A}_0, \check{G}]\}$ since $\mathcal{F}_{0j} = -\mathbf{i}[\mathcal{A}_0, \mathcal{A}_j]$, and so it renormalizes the term $-\mathbf{i}[\tau_3\mathcal{A}_0, \check{G}]$ already present in (3.7). The correction to $\mathcal{A}_0$ is of the order $\mathcal{A}_j/p_F \ll 1$ and we neglect it from now on.

In the Nambu space $\check{g}$ reads

$$\check{g} = \begin{pmatrix} g & f \\ -\bar{f} & -\bar{g} \end{pmatrix}, \quad (3.9)$$

where the g, f are matrices in the spin space which depend on the spaces coordinates x_i, the momentum direction n_i, and the Matsubara frequency. The time-reversal conjugate $\bar{g}$ and $\bar{f}$ are defined as $\bar{g}(\mathbf{n}) = \sigma^y g^*(-\mathbf{n})\sigma^y$ and $\bar{f} = \sigma^y f^*(-\mathbf{n})\sigma^y$. The latter is the anomalous GF which describes the superconducting correlations.

From the knowledge of $\check{g}$ one can calculate the charge current (density)

$$\mathbf{j} = -\frac{\mathbf{i}\pi e N_0 T}{2}\sum_{\omega_n}\mathrm{Tr}\langle\mathbf{v}_F\tau_3\check{g}\rangle, \quad (3.10)$$

with e the electron charge and N_0 the normal density of states for each spin. The spin polarization is given by

$$\mathbf{S} = \frac{\mathbf{i}\pi N_0 T}{2}\sum_{\omega_n}\mathrm{Tr}\langle\tau_3\boldsymbol{\sigma}\check{g}\rangle. \quad (3.11)$$

C. Linearized quasiclassical equations in diffusive and purely ballistic limits

In the present work we mainly consider two limiting cases: the purely ballistic one in which $\tau \to \infty$ and the diffusive limit in which τ is a small parameter. The transport equation in the ballistic limit is directly obtained from (3.8) by neglecting the right-hand side. The diffusive limit is a bit more puzzling. Because of the anticommutator on the left-hand side of Eq. (3.8), the normalization condition $\check{g}^2 = 1$ does not hold directly and therefore the usual derivation of the Usadel equations cannot be carried out [140]. There is, however, a way out of this puzzle if one assumes that the amplitude of the anomalous GFs, f in (3.9), is small. Then the matrix GF (3.9) can be written as $\check{g} \approx \mathrm{sgn}(\omega_n)\tau_3 + \begin{pmatrix} 0 & f \\ -\bar{f} & 0 \end{pmatrix}$ and the linearized Eilenberger equation becomes an equation for f:

$$(v_F n_i\tilde{\nabla}_i + 2\omega_n)f - \{\mathbf{i}\mathcal{A}_0, f\} + 2\mathbf{i}\Delta\,\mathrm{sgn}(\omega_n)$$
$$- \frac{1}{2m}\left\{n_i\mathcal{F}_{ij}, \frac{\partial f}{\partial n_j}\right\} = -\frac{\mathrm{sgn}(\omega_n)}{\tau}(f - \langle f\rangle). \quad (3.12)$$

This linearization procedure is justified in two cases: either for temperatures close to the critical temperature T_c when

the amplitude of the order parameter Δ is small, or in S-X structures when the proximity effect is weak due to a finite interface resistance for arbitrary temperature.

In the diffusive limit one can expand $f \approx f_0 + n_k f_k + \cdots$, in angular harmonics where $\langle f \rangle = f_0 \gg f_k$. We first average (3.12) over the momentum direction:

$$\frac{\boldsymbol{v}_F}{\dim} \tilde{\nabla}_k f_k + \{\omega_n - \mathbf{i}\mathcal{A}_0, f_0\} = -2\mathbf{i}\Delta \,\mathrm{sgn}\,(\omega_n), \qquad (3.13)$$

where $\dim = 1,2,3$ is the dimension of the system. Next we multiply Eq.(3.12) by n_k and average over the momentum direction to obtain

$$v_F \tilde{\nabla}_k f_0 + \{\omega_n - \mathbf{i}\mathcal{A}_0, f_k\} - \frac{1}{2m}\{\mathcal{F}_{kj}, f_j\} = -\frac{\mathrm{sgn}\,(\omega_n)}{\tau} f_k. \tag{3.14}$$

Equations (3.13) and (3.14) constitute a closed set of coupled differential equations for f_0 and f_k. In particular from Eq. (3.14) we can write f_k in terms of f_0 up to terms of second order in τ:

$$f_k \approx -\tau \,\mathrm{sgn}\,(\omega_n) v_F \tilde{\nabla}_k f_0 - \tau^2 \frac{v_F}{2m}\{\mathcal{F}_{kj}, \tilde{\nabla}_j f_0\}$$
$$+ \tau^2 v_F \{\omega_n - \mathbf{i}\mathcal{A}_0, \tilde{\nabla}_k f_0\} + \cdots. \tag{3.15}$$

Note that the Usadel equation was obtained in several works in the absence of gauge fields, where one skipped the terms of the order τ^2. We keep here these terms since they are crucial for the description of magnetoelectric effects [86,92,141].

The equations can be further simplified by noticing that the anticommutator in the second line of Eq. (3.15) can be written as

$$\{\omega_n - \mathbf{i}\mathcal{A}_0, \tilde{\nabla}_k f_0\} = \tilde{\nabla}_k\{\omega_n - \mathbf{i}\mathcal{A}_0, f_0\} + \mathbf{i}\{\tilde{\nabla}_k \mathcal{A}_0, f_0\}. \tag{3.16}$$

In virtue of (3.13), the first term on the right-hand side of the last equation is in fact of order τ and so this term in (3.15) is of order τ^3 and can be neglected. The second term reads $\tilde{\nabla}_k \mathcal{A}_0 = -\mathbf{i}[\mathcal{A}_k, \mathcal{A}_0] = \mathcal{F}_{k0}$ for a space-independent gauge potential. This electric field renormalizes the paramagnetic effects $\mathcal{A}_0$, and is neglected in the following. Finally, we replace (3.15) into (3.13) to obtain the Usadel equation for f_0:

$$-\,\mathrm{sgn}\,(\omega_n)D\tilde{\nabla}^2 f_0 + \{\omega_n - \mathbf{i}\mathcal{A}_0, f_0\}$$
$$-\frac{\tau D}{2m}\{\tilde{\nabla}_i \mathcal{F}_{ij}, \tilde{\nabla}_j f_0\} = -2\mathbf{i}\Delta \,\mathrm{sgn}(\omega_n) \tag{3.17}$$

with $D = v_F^2 \tau / \dim$ the diffusion constant. This equation is supplemented by the generalized Kupriyanov-Lukichev boundary condition [142]

$$\mathcal{N}_i\left[\tilde{\nabla}_i f_0 + \frac{\tau \,\mathrm{sgn}\,(\omega_n)}{2m}\{\mathcal{F}_{ij}, \tilde{\nabla}_j f_0\}\right]_{x_0} = -\gamma f_{\mathrm{BCS}} \tag{3.18}$$

at an interface located at position x_0 between a bulk superconductor described by the anomalous GF f_{BCS} and the X bridge. The interface is characterized by the transparency γ and normal vector of component $\mathcal{N}_i$. For a fully transparent interface, we impose the continuity of the GFs.

We now need to write the current and spin density in terms of the isotropic anomalous GFs. It is easy to verify, by checking

its conservation, that in the linearized case the electric current, Eq. (3.10), is given by

$$\boldsymbol{j} = \frac{\mathbf{i}\pi e N_0 T}{2} \sum_{\omega_n} \mathrm{Tr}\langle \boldsymbol{v}_F f \bar{f} \rangle \,\mathrm{sgn}\,\omega_n, \qquad (3.19)$$

and correspondingly in the diffusive limit

$$j_i = \mathbf{i}\pi e N_0 DT \sum_{\omega_n} \mathrm{Tr}\Bigg\{ f_0 \tilde{\nabla}_i \bar{f}_0 - \bar{f}_0 \tilde{\nabla}_i f_0$$
$$+ \frac{\tau \,\mathrm{sgn}(\omega_n)}{2m}(f_0\{\mathcal{F}_{ij}, \tilde{\nabla}_j \bar{f}_0\} + \bar{f}_0\{\mathcal{F}_{ij}, \tilde{\nabla}_j f_0\})\Bigg\}. \tag{3.20}$$

The spin polarization (3.11) is more subtle to deal with in the linearized approximation, since the normalization condition does not apply in our case. In accordance with the case without SOC, one may assume that it can be expressed in terms of the isotropic anomalous f as

$$S^a = \mathbf{i}\pi N_0 T \sum_{\omega_n} \mathrm{Tr}\langle \sigma^a f \bar{f} \rangle \,\mathrm{sgn}\,(\omega_n) \tag{3.21}$$

with $\langle \sigma^a f \bar{f} \rangle = \sigma^a f_0 f_0$ in the diffusive limit. In the next section we will show *a posteriori* that these expressions lead to the known results in bulk systems in the presence of Rashba SOC.

For the following discussions it is convenient to write the anomalous GF f as the sum of singlet (scalar) and triplet (vector in spin space) $f = f_s + f_t^a \sigma^a$, and to expand all the spin variables in term of Pauli matrices: $\mathcal{F}_{ij} = \mathcal{F}_{ij}^a \sigma^a / 2$, $\mathcal{A}_\mu = \mathcal{A}_\mu^a \sigma^a / 2$. From Eqs. (3.12) we obtain the equations for the singlet and triplet components in the ballistic case:

$$(v_F n_i \partial_i + 2\omega_n)f_s = -2\mathbf{i}\,\mathrm{sgn}\,(\omega_n)\Delta$$
$$+ \left(\mathbf{i}\mathcal{A}_0^a + \frac{n_i \mathcal{F}_{ij}^a}{2m}\frac{\partial}{\partial n_j}\right)f_t^a, \tag{3.22}$$

$$v_F n_i (\tilde{\nabla}_i f_t)^a + 2\omega_n f_t^a = \left(\mathbf{i}\mathcal{A}_0^a + \frac{n_i \mathcal{F}_{ij}^a}{2m}\frac{\partial}{\partial n_j}\right)f_s. \tag{3.23}$$

Equivalently, from Eq. (3.17) one obtains the equations for the isotropic part of the singlet f_{s0} and triplet f_{t0} components in the diffusive case (for simplicity we skip the subindex 0):

$$\left(\partial_i^2 - \kappa_\omega^2\right)f_s - 2\mathbf{i}\frac{\Delta}{D} + \mathrm{sgn}(\omega_n)\left[\mathbf{i}\frac{\mathcal{A}_0^a}{D} + \frac{\tau}{2m}(\mathcal{J}_i \partial_i)^a\right]f_t^a = 0, \tag{3.24}$$

$$(\tilde{\nabla}_i \tilde{\nabla}_i f_t)^a - \kappa_\omega^2 f_t^a + \mathrm{sgn}(\omega_n)\left(\mathbf{i}\frac{\mathcal{A}_0^a}{D} + \frac{\tau}{2m}\mathcal{J}_i^a \partial_i\right)f_s = 0. \tag{3.25}$$

We write the covariant derivative as $\tilde{\nabla}_i \cdot = \partial_i \cdot -\mathbf{i}[\mathcal{A}_i, \cdot] \equiv \partial_i + \hat{P}_i$, where $\hat{P}_i$ is a tensor dual to $\mathcal{A}_i$ with components $P_i^{ab} = \varepsilon^{abc}\mathcal{A}_i^c$. Thus, $\tilde{\nabla}_i \tilde{\nabla}_i = \partial_i^2 + 2\hat{P}_i \partial_i - \hat{\Gamma}$, where $\hat{P}_i \hat{P}_i = -\hat{\Gamma}$. By noticing that $(\tau/2m)\mathcal{J}_k^a = C_k^a$, it is easy to verify that the diffusive equations (3.24) and (3.25) are identical to those derived in Sec. II from heuristic arguments [Eqs. (2.7) and (2.8)]. One should emphasize though that while Eqs. (3.24) and (3.25) are derived for the particular case of linear-in-momentum SOC, Eqs. (2.7) and (2.8) suggest that the form

of the diffusion equations remain the same for arbitrary momentum dependence.

In particular the form of Eq. (3.25) proves the full analogy between singlet-triplet and charge-spin coupling in diffusive systems [cf. Eqs. (2.5) and (2.6)]. In Ref. [46], the analogy between the diffusion of spin in normal systems and the triplet components was discussed. Here we can extend this result and find that the tensor C_k^a, responsible for the SHE in normal systems, is an additional source for the singlet-triplet conversion and, as we will see in the next sections, is at the root of magnetoelectric effects and the anomalous phase. Equations (3.22)–(3.23) and (3.24)–(3.25) are the central equations of this work, which we now solve for different situations. In Sec. VI B we go beyond this linear approximation.

IV. THE EDELSTEIN EFFECT IN BULK SUPERCONDUCTORS FOR $T \to T_c$

In order to illustrate the usefulness of the SU(2) covariant quasiclassical equations presented above, we study here the magnetoelectric effect and its inverse in bulk superconductors with an intrinsic SOC linear in momentum and derive the response coefficients in (1.3) and (1.4).

We assume that the superconducting order parameter Δ is constant in magnitude but has a spatially dependent phase $\Delta(\mathbf{r}) = |\Delta|e^{i\varphi(\mathbf{r})}$, where $\nabla\varphi$ is assumed to be a constant vector.

Let us first consider a diffusive superconductor. From (3.24) in the lowest order of $\nabla\varphi$ one obtains

$$f_s \approx -\mathbf{i}\frac{|\Delta|}{|\omega_n|}e^{i\varphi}, \qquad (4.1)$$

and hence one can easily obtain the lowest order correction to the triplet component from (3.25):

$$f_t^a = \frac{|\Delta|}{|\omega_n|}\frac{\tau}{2m}\,\mathrm{sgn}\,(\omega_n)\big[(\hat{\Gamma} + \kappa_\omega^2)^{-1}\big]^{ab}\partial_j^b\partial_j\varphi. \qquad (4.2)$$

From Eq. (3.21) it becomes clear that the spin density is determined by the product of the singlet (4.1) and triplet (4.2) components which results in $S^a = \chi_i^a\partial_i\varphi$ with

$$\chi_i^a = 4\pi N_0\frac{\tau}{2m}T\sum_{\omega_n>0}\frac{\Delta^2}{\omega_n^2}\big[(\hat{\Gamma} + \kappa_\omega^2)^{-1}\big]^{ab}\partial_i^b. \qquad (4.3)$$

This is the Edelstein result generalized for arbitrary linear-in-momentum SOC.

With the help of Eqs. (3.24) and (3.25) we can also describe the inverse EE, the so-called spin-galvanic effect. We now assume a finite and spatially homogenous $\mathcal{A}_0^a$ and a zero phase gradient. In such a case one can obtain f_t directly from Eq. (3.25), which is now proportional to $\mathcal{A}_0^a$. By substitution of this result into the expression for the current, Eq. (3.20), and by noticing that only the second line contributes to the current we obtain $j_i = e\chi_i^a\mathcal{A}_0^a$, with χ_i^a given by Eq. (4.3) in agreement with Onsager reciprocity.

In short, we are able to derive in a few lines the tensor (4.3), which describes the EE and inverse EE in superconductors. Moreover, the expression (4.3) is valid for arbitrary linear-in-momentum spin-orbit effect and generalizes the result obtained in Ref. [80] for the particular case of a Rashba SOC. If one assumes the same here, i.e., $\mathcal{A}_x^y = -\alpha = -\mathcal{A}_y^x$ and all the other

components equal to zero, one obtains from Eq. (4.3)

$$\chi_i^a = \big(\delta_{ix}^{ay} - \delta_{iy}^{ax}\big)4\pi N_0\frac{D\tau}{2m}T\sum_{\omega_n>0}\frac{\Delta^2}{\omega_n^2}\frac{\alpha^3}{2|\omega_n| + D\alpha^2} \qquad (4.4)$$

that coincides with the expression obtained in Ref. [80].

If we neglect in Eq. (4.2) the Dyakonov-Perel relaxation, then the triplet component is simply proportional to $f_t^a \sim \mathcal{A}_0^a(\mathbf{r})$. By substituting this into the expression for the current Eq. (3.20) one can easily show that

$$j_i = 4e\pi N_0\frac{\tau}{2m}T\sum_{\omega_n>0}\frac{\Delta^2}{\omega_n^2\kappa_\omega^2}\mathcal{F}_{0j}^a\mathcal{F}_{ji}^a. \qquad (4.5)$$

This expression suggests that a spatially inhomogeneous magnetization together with SOC may also induce a finite supercurrent. In this case the spin-galvanic effect scales with the square of the SOC parameter, in contrast to the α^3 dependency found previously for spatially uniform magnetization.

The same effects can be explored in the pure ballistic limit, for which Eqs. (3.22) and (3.23) apply. The singlet component in the lowest order in the SOC is given by

$$f_s \approx -\mathbf{i}\frac{\Delta}{|\omega_n|}\bigg(1 - \mathbf{i}\frac{v_F n_i}{2\omega_n}\partial_i\varphi\bigg), \qquad (4.6)$$

whereas the triplet component can be obtained easily from Eq. (3.23):

$$f_t^a = -\frac{\Delta v_F}{2|\omega_n|\omega_n}[(v_F n_k\hat{P}_k + 2\omega_n)^{-1}]^{ab}\frac{n_i\mathcal{F}_{ij}^b}{2m}\partial_j\varphi. \qquad (4.7)$$

By using Eq. (3.21) we obtain the Edelstein result $S^a = \chi_j^a\partial_j\varphi$ but now for an arbitrary linear-in-momentum SOC

$$\chi_i^a = -2\pi\frac{N_0 v_F}{2m}T\sum_{\omega_n>0}\frac{\Delta^2}{|\omega_n|^3}\big\langle[(v_F n_k\hat{P}_k + 2\omega_n)^{-1}]^{ab}n_j\mathcal{F}_{ji}^b\big\rangle. \qquad (4.8)$$

Identically, we find $j_i = e\chi_i^a\mathcal{A}_0^a$.

In the particular case of a 2D systems with Rashba SOC we recover the Edelstein result for a ballistic superconductor [79]:

$$\chi_i^a = \frac{\pi N_0\Delta^2}{4v_F m}T\sum_{\omega_n>0}\frac{(v_F\alpha)^3}{|\omega_n|^3[(2\omega_n)^2 + (v_F\alpha)^2]}\big(\delta_{ix}^{ay} - \delta_{iy}^{ax}\big). \qquad (4.9)$$

The agreement between our and Edelstein results proves the validity of the expression (3.21) in the linearized approximation.

To conclude this section we note that for Rashba SOC in both cases, diffusive (4.4) and ballistic (4.9), χ_i^a is proportional to α in the strong SOC limit (see also [134]), and to α^3 for weak spin-orbit interaction (see also [92]). So, the quasiclassical formalism is able to recover in an elegant way some well established results obtained after a cumbersome diagrammatic [79,80], and it also allows some easy generalizations of them.

V. MAGNETOELECTRIC EFFECTS IN DIFFUSIVE JOSEPHSON JUNCTIONS

We now turn to the central topic of the present work which is the description of magnetoelectric effects in S-X-S Josephson junctions, and demonstrate their connection to the anomalous phase problem. We first consider the diffusive limit and postpone the discussion of ballistic junctions for the next section.

In particular we consider an S-X-S Josephson junction with an interlayer X of length L. We assume that the magnetic interactions are only finite in X and vanish in the S electrodes. Moreover, we assume that the structure has infinite dimensions in the y-z plane and therefore the GFs only depend on the x coordinate. The superconducting bulk solutions in the leads are written as $f_L = f_{\mathrm{BCS}} e^{-\mathbf{i}\varphi/2}$ and $f_R = f_{\mathrm{BCS}} e^{\mathbf{i}\varphi/2}$, in the left ($x \leqslant -L/2$) and right ($x \geqslant L/2$) electrodes, respectively, with

$$f_{\mathrm{BCS}} = \frac{\Delta}{\sqrt{\omega_n^2 + \Delta^2}}, \tag{5.1}$$

whereas the normal metal fills the region $-L/2 \leqslant x \leqslant L/2$.

We will consider both the highly resistive and the perfectly transparent interfaces between the S and X parts. When the barrier transparency is low, the linearized approximation is justified for all temperatures, whereas for transparent barriers, one is limited to temperatures close to the critical temperature of the junction.

For the particular case of Rashba SOC in the X region and an in-plane exchange field the Josephson current has been calculated in Ref. [92]. It has been shown explicitly that the current-phase relation is given by $I = I_c \sin(\varphi - \varphi_0)$. The anomalous phase φ_0 was calculated as a function of the strength of the spin fields, the temperature, and the junction parameters. Here instead we focus on a generic linear-in-momentum SOC and we derive the expressions for the anomalous Josephson current in the lowest order of the spin fields. This will allow us to understand the link between the inverse EE and the φ_0 junctions.

A. Diffusive junction with low-transparency interfaces

We first consider an S-X-S diffusive Josephson junction with highly resistive S-X interfaces. In this limit the linearization of the quasiclassical equations is justified for all temperatures. Our goal here is to determine the Josephson current through the junction, which in the linearized regime is given by Eq. (3.20). The components of the condensate function f_s entering this expression have to be obtained by solving the system (3.24) and (3.25) in the normal metal which couples the singlet with the triplet component. For the specific S-X-S geometry considered here these equations read

$$\left(\partial_x^2 - \kappa_\omega^2\right) f_s + \mathrm{sgn}\,(\omega_n) \left[\mathbf{i}\frac{\mathcal{A}_0^a}{D} + \frac{\tau}{2m}(\mathcal{J}_i\partial_i)^a\right] f_t^a = 0, \tag{5.2}$$

$$(\tilde{\nabla}_i \tilde{\nabla}_i f_t)^a - \kappa_\omega^2 f_t^a + \mathrm{sgn}\,(\omega_n)\left(\mathbf{i}\frac{\mathcal{A}_0^a}{D} + \frac{\tau}{2m}\mathcal{J}_i^a\partial_i\right) f_s = 0, \tag{5.3}$$
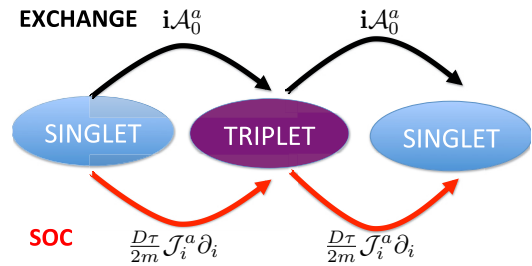

FIG. 1. (Color online) Schematic representation of the singlet-triplet-singlet conversion process at the lowest order with respect to the spin fields. Black arrows represent the action of the exchange field, whereas red arrows encode the effect of the singlet-triplet coupling term due to the SOC. Only mixed red-black paths lead to the appearance of an anomalous phase φ_0 in the singlet component and hence to a supercurrent in an S-X-S junction even without a phase bias between the S electrodes.

and the boundary conditions for the resistive interface [cf. Eq. (3.18)]:

$$\left(\partial_x f_s + \mathrm{sgn}(\omega_n)\frac{\tau}{2m}\mathcal{J}_x^a f_t^a\right)_{x=\pm L/2} = \pm\gamma f_{R,L},$$

$$\partial_x f_t^a|_{x=\pm L/2} = 0. \tag{5.4}$$

The expression Eq. (3.20) can be simplified by calculating the current at the right interface ($x = L/2$) and by using the boundary condition (5.4):

$$j_x = \mathbf{i}e\pi D N_0 T_c \gamma \sum_{\omega_n > 0} \mathrm{Tr}\{f_s \bar{f}_R - \bar{f}_s f_R\}_{x=L/2}. \tag{5.5}$$

It is clear from this equation that the correction to the current due to the spin fields (the anomalous current) is proportional to $\mathrm{Im}[f_R^* \delta f_s(L/2)]$, where δf_s is the first correction to the singlet component due to the gauge potentials. In the absence of a phase difference between the S electrodes f_R is real and the anomalous current is proportional to the imaginary part of the singlet component. According to Eq. (5.2), in the absence of spin fields (exchange and SOC), there is no triplet component and the singlet component is real. Therefore no supercurrent flows at zero phase difference.

In the presence of spin fields there are two sources for singlet-triplet conversion, as seen from the second term on the left-hand side of Eq. (5.3). The first one is the extensively studied mechanism for singlet-triplet conversion in S/F junctions via the intrinsic exchange field $\mathcal{A}_0$ [1,2]. Inclusion of SOC leads to an additional singlet-triplet conversion mechanism described by the last term on the left-hand side of Eq (5.3). As discussed in Sec. II, the singlet-triplet conversion in this case corresponds to the charge-spin conversion in normal systems with SOC. Conversely, once the triplet component is created, both mechanisms will convert it back to singlet, as can be seen in Eq. (5.2).

The singlet-triplet-singlet conversion at the lowest orders in perturbation with respect to the spin fields is schematized in Fig. 1. The black arrows represent the singlet-triplet conversion due to the exchange field which implies a $\pi/2$ phase shift due to the $\mathbf{i}$ factor in front of $\mathcal{A}_0$ in Eqs. (5.2) and (5.3). The red arrows represent the singlet-triplet conversion due to the

SOC, specifically due to the coupling term in Eqs. (5.2) and (5.3) proportional to $\mathcal{J}_i^a \partial_i$. No additional phase is associated with this latter process. If one follows the black path, i.e., the singlet-triplet-singlet conversion only due to the exchange field, the resulting contribution to the singlet component acquires a minus sign (a π shift) and it is proportional to $\mathcal{A}_0^2$. This means that there is no anomalous phase $0 < \varphi_0 < \pi$ induced and hence no Josephson current flows when $\varphi = 0$. Similarly, if one follows the red path the resulting singlet component also remains real with no change of sign. From Fig. 1 it becomes clear that a nontrivial φ_0 only appears from the "cross-term" path that consists of one black and one red arrow. In other words, the mutual action of exchange field and SOC leads to a finite φ_0 and hence to a supercurrent even at zero phase difference. In this case the contribution to this current in the lowest order of the spin fields is proportional to $\mathcal{A}_0^a \mathcal{J}_i^a \partial_i f_s$ between the exchange field and the spin current tensor, as anticipated in the introduction.

In order to quantify this effect and calculate φ_0 in the S-X-S junctions it is convenient to introduce the singlet and triplet propagators associated with Eqs. (5.2)–(5.4):

$$\left(\partial_x^2 - \kappa_\omega^2 \right) K_s(x, x') = -\delta(x - x'),$$
$$\partial_x K_s(x, x') \big|_{x = \pm L/2} = 0, \qquad (5.6)$$

and

$$\left[(\partial_x + \hat{P}_x)^2 + \hat{P}_y^2 + \hat{P}_z^2 - \kappa_\omega^2 \right] \hat{K}_t(x, x') = -\delta(x - x'),$$
$$(\partial_x + \hat{P}_x) \hat{K}_t(x, x') = 0. \qquad (5.7)$$

Thus, Eqs. (5.2)–(5.4) can be rewritten as a set of integral equations:

$$f_s(x) = f_s^{(0)}(x) - \frac{\tau \mathcal{J}_x^a}{2m} \left[K_s(x, L/2) f_t^a - K_s(x, -L/2) \right]$$
$$+ \operatorname{sgn}(\omega_n) \int_{-L/2}^{L/2} dx_1 K_s(x, x_1)$$
$$\times \left[\mathbf{i} \frac{\mathcal{A}_0^b}{D} K_s(x_1, x) + \frac{\tau}{2m} \mathcal{J}_x^b \partial_{x_1} \right] f_t^b(x_1) \qquad (5.8)$$

and

$$f_t^a(x) = \operatorname{sgn}(\omega_n) \int_{-L/2}^{L/2} dx_1$$
$$\times \left[K_t^{ab}(x, x_1) \left(\mathbf{i} \frac{\mathcal{A}_0^b}{D} + \frac{\tau}{2m} \mathcal{J}_1^a \partial_{x_1} \right) f_s(x_1) \right]. \qquad (5.9)$$

Here $f_s^{(0)} = \gamma (K_s(x, \frac{L}{2}) f_R + K_s(x, -\frac{L}{2}) f_L)$ and the second term in Eq. (5.8) takes into account the boundary condition (5.4).

The K_s propagator can be obtained from Eq. (5.6),

$$K_s(x_1, x_2) = \frac{\cosh \kappa_\omega (L - |x_1 - x_2|) + \cosh \kappa_\omega (x_1 + x_2)}{2 \kappa_\omega \sinh \kappa_\omega L}, \qquad (5.10)$$

whereas the equations for the triplet kernel, Eq. (5.7), can be written in the form of an integral equation which is convenient

for the subsequent perturbative analysis:

$$\hat{K}_t(x_1, x_2) = e^{-\hat{P}_x x_1} K_s(x_1, x_2) e^{\hat{P}_x x_2}$$
$$- e^{-\hat{P}_x x_1} \int_{-L/2}^{L/2} dy\, K_s(x_1, y) e^{\hat{P}_x y} \hat{\Gamma}_\perp \hat{K}_t(y, x_2), \qquad (5.11)$$

where $\hat{\Gamma}_\perp = -\hat{P}_y^2 - \hat{P}_z^2$.

In the lowest order of the gauge potentials one can obtain the correction δf_s to the singlet component by substituting the result (5.10) into Eqs. (5.8) and (5.9). We consider here only the "cross-term" correction δf_s proportional to both the exchange field $\mathcal{A}_0$ and the spin current $\mathcal{J}_i$ and which is responsible for the anomalous phase shift:

$$\delta f_s(L/2) = \mathbf{i} \mathcal{A}_0^a \mathcal{J}_x^b \frac{\tau \gamma}{2mD} f_L \int_{-L/2}^{L/2} dy_1$$
$$\times \int_{-L/2}^{L/2} dy_2 K_t^{ab}(y_2, y_1) \frac{\cosh[\kappa_\omega(y_1 - y_2)]}{\kappa_\omega \sinh \kappa_\omega L}. \qquad (5.12)$$

In principle, one has all the elements to solve Eqs. (5.8) and (5.9), for example recursively by performing a perturbative expansion in the gauge potentials. Here, in order to get analytical compact expressions we restrict our analysis to the short junction limit, i.e., $L \ll \min(\kappa_\omega^{-1}, |\mathcal{A}_k|^{-1})$. In this case $K_s \approx \kappa_\omega^{-2} L^{-1}$ [cf. Eq. (5.10)] and from Eq. (5.11) it is easy to verify that K_t reads

$$\hat{K}_t \approx \frac{\left(\kappa_\omega^2 + \hat{\Gamma}_\perp \right)^{-1}}{L}. \qquad (5.13)$$

We are interested in calculating the anomalous phase φ_0 which can be obtained by noticing that the current (5.5) can be written as

$$j_x = j_c \sin(\varphi - \varphi_0) \approx j_c \sin \varphi - \varphi_0 j_c \cos \varphi \qquad (5.14)$$

for a small φ_0. The anomalous phase φ_0 can be obtained by setting $\varphi = 0$ and dividing by the critical current j_c in the absence of SOC. In the short junction limit this is given by

$$j_c = 4e\pi D N_0 T_c \gamma^2 \sum_{\omega_n > 0} \frac{f_{\mathrm{BCS}}^2}{\kappa_\omega^2 L}. \qquad (5.15)$$

We follow this procedure and from Eq. (5.5) and Eqs. (5.12) and (5.13) we obtain

$$\varphi_0 \approx \frac{\tau}{2mD} L \frac{\sum_{\omega_n > 0} \frac{f_{\mathrm{BCS}}^2}{\kappa_\omega^2} \mathcal{A}_0^a \left[\left(\kappa_\omega^2 + \hat{\Gamma}_\perp \right)^{-1} \right]^{ab} \mathcal{J}_x^b}{\sum_{\omega_n > 0} \frac{f_{\mathrm{BCS}}^2}{\kappa_\omega^2}}. \qquad (5.16)$$

This expression clearly shows the relation between the appearance of the anomalous phase, φ_0, and the inverse Edelstein effect in bulk systems. Both the Josephson current (proportional in the linearized case to φ_0) and the bulk supercurrent are proportional to $\mathcal{A}_0 \mathcal{J}_x$; i.e., both are generated from the mutual action of the exchange field and the SOC.

It is worth noticing that in the present case of low transparent interfaces, the anomalous phase grows linearly with L, the length of the junction (5.16). In the next subsection we show

that in the case of a transparent barrier the anomalous phase behaves like L^3.

In the particular case of a 2D situation, with a SOC coupling of Rashba (described by the parameter α) and Dresselhaus (β) type we obtain from Eq. (5.16)

$$\varphi_0 \approx \frac{\tau L}{2m} \frac{\sum_{\omega_n > 0} \frac{f_{\mathrm{BCS}}^2}{\kappa_\omega^2} \frac{(\beta \mathcal{A}_0^x - \alpha \mathcal{A}_0^y)(\alpha^2 - \beta^2)}{2\omega_n + D(\alpha^2 + \beta^2)}}{\sum_{\omega_n > 0} \frac{f_{\mathrm{BCS}}^2}{\kappa_\omega^2}}. \quad (5.17)$$

Besides controlling the anomalous phase and hence the Josephson current by tuning the external magnetic field, this expression also suggests that the current can be controlled by tuning the Rashba SOC by means of an external gate. In the particular case that $\alpha = \beta$ the anomalous phase is zero and no supercurrent will flow.

B. Diffusive junction with transparent interfaces

We now briefly consider the limit of a full transparent barrier. In that case one assumes continuity of the quasiclassical GFs at the S-X interfaces. The problem is then formally the same as in the previous section, except that the second equations in (5.6) and (5.7) for the propagators K_s and $\hat{K}_t$ are replaced by

$$\hat{K}_{s,t}(x_1, x_2)\big|_{x_1 = \pm L/2} = 0, \quad (5.18)$$

respectively. In this case one should remove the second term in Eq. (5.8) and $f_s^{(0)}(x) = f_L \sinh(L/2 - x)/\sinh(\kappa_\omega L) + f_R \sinh(L/2 + x)/\sinh(\kappa_\omega L)$.

Now the singlet propagator is given by

$$K_s(x_1, x_2) = \frac{\cosh \kappa_\omega(x_1 + x_2) - \cosh \kappa_\omega(L - |x_1 - x_2|)}{2\kappa_\omega \sinh \kappa_\omega L}. \quad (5.19)$$

In the short junction limit K_s is proportional to L and it is temperature independent. From Eq. (5.11) $\hat{K}_t \sim K_s$. Thus, in this case the anomalous phase shift is also temperature independent and proportional to

$$\varphi_0 \propto \frac{\tau L^3}{mD} \mathcal{A}_0^a \mathcal{J}_x^a. \quad (5.20)$$

In contrast to the case of finite barrier resistance, Eq. (5.16), the anomalous phase scale with L^3. This means that in short junctions a finite barrier resistance between the S and the normal metal favors the growth of φ_0. These results generalize those presented recently in Ref. [92] for the particular case of Rashba SOC.

We can then conclude that the anomalous phase, at lowest order in the gauge potentials, is proportional to $\mathcal{A}_0^a \mathcal{J}_x^a$, independently of the type of interface.

VI. MAGNETOELECTRIC EFFECTS IN BALLISTIC JOSEPHSON JUNCTIONS

In this section we consider a pure ballistic S-X-S junction; i.e., we solve (3.8) in the limit $\tau \to \infty$. As before, the junction is along the x axis and the two superconducting electrodes at position $x \leqslant -L/2$ and $x \geqslant L/2$. The spin fields, both exchange and SOC, are only finite in the X region. We assume

that the transverse dimensions of the junction are very large, and therefore the GFs depend on x and only weakly on y, z. We also assume that the interfaces between X and S are perfectly transparent.

In the next subsection we first analyze the Josephson current for temperatures close to the superconducting critical temperature T_c, and make a connection with the diffusive structures studied in the previous section. In the second subsection we derive analytical expressions for the anomalous current at arbitrary temperature for the case of small spin fields.

A. Ballistic junction at $T \to T_c$

In the case of large enough temperatures we analyze the linearized Eilenberger equation. The solutions for the singlet and triplet components in Eqs. (3.22) and (3.23) can be written as propagation in two directions $f_{s,t}(-L/2 \leqslant x \leqslant L/2) = f_{s,t}^>(x)\Theta(\omega_n/n_x) + f_{s,t}^<(x)\Theta(-\omega_n/n_x)$ with

$$f_s^< = \frac{\Delta(L/2)}{|\omega|} e^{-2\omega_n(x - L/2)/v_F n_x} + \int_{L/2}^x \frac{dy}{v_F n_x}$$
$$\times e^{-2\omega_n(x-y)/v_F n_x} \left(i\mathcal{A}_0^a + \frac{n_i \mathcal{F}_{ij}^a}{2m} \frac{\partial}{\partial n_j} \right) f_t^{a<}(y) \quad (6.1)$$

and

$$f_t^{a<} = \int_{L/2}^x \frac{dy}{v_F n_x} e^{-2\omega_n(x-y)/v_F n_x} \left(e^{-\hat{P}_i n_i(x-y)/n_x} \right)^{ab}$$
$$\times \left(i\mathcal{A}_0^b + \frac{n_i \mathcal{F}_{ij}^b}{2m} \frac{\partial}{\partial n_j} \right) f_s^<(y). \quad (6.2)$$

In the opposite propagation direction $f_{s,t}^>$ are found from $f_{s,t}^<$ by substituting $L/2 \to -L/2$.

In analogy with the diffusive case (cf. Fig. 1), expressions (6.1) and (6.2) show explicitly the effect of the SOC on the condensate function. In the absence of SOC the exchange field $\mathcal{A}_0$ is the only source for singlet-triplet conversion. The manifestation of the triplet component in S-F-S junctions has been extensively studied in the past (see [1,2] for reviews). As discussed in Sec. II, the imaginary unit i in front of the $\mathcal{A}_0$ terms leads to a $\pi/2$ phase shift. In the case of a finite SOC the gauge field, $\mathcal{F}_{ij}$, is an additional source of triplet correlations. Notice that in the ballistic case, $\mathcal{F}_{ij}$ not only couples the singlet and triplet components, but also the s-p-wave components of the condensate [143]. Moreover, the term $e^{-\hat{P}_i n_i x/n_x}$ in (6.2) leads to a momentum-dependent rotation of the triplet component in the spin space.

The origin of the anomalous phase φ_0 can be easily understood in the lowest order in the spin fields. Assuming a vanishing phase difference between the superconductors and combining Eqs. (6.1) and (6.2) with the expression for the current (3.19), one obtains for the first nontrivial contribution to the current $\mathcal{F}_{ij}^a \partial_{n_j} (e^{-\hat{P}_i n_i x/n_x})^{ab} \mathcal{A}_0^b \propto \mathcal{F}_{ij}^a P_j^{ab} \mathcal{A}_0^b$. This correction is proportional to $\mathcal{J}_i^b \mathcal{A}_0^b$ and coincides with those obtained in bulk superconductors with SOC (Sec. IV) and a diffusive S-X-S junction (Sec. V).

Quantitatively, a compact analytical solution for the current at zero phase difference can be obtained from Eqs. (6.1) and

(6.2) in the short junction limit; i.e., for $L \ll v_F/2\omega_n$,

$$
j_x(\varphi = 0) = -\frac{e\pi N_0 v_F}{E_F}\frac{L^3}{3}\Delta^2 T_c \mathcal{F}^a_{xi}\mathcal{F}^a_{i0}
$$

$$
\times \sum_{\omega_n > 0}\left\langle \frac{e^{-2\omega_n L/v_F|n_x|}}{\omega_n^2 |n_x|}\left(1 + 2\frac{n_i^2}{n_x^2}\right)\right\rangle, \quad (6.3)
$$

where $\mathcal{F}^a_{xi}\mathcal{F}^a_{i0} = \mathcal{J}^a_x \mathcal{A}^a_0$. Thus the anomalous current is generated by the spin polarization $\mathcal{A}^a_0$ via the spin current $\mathcal{J}^a_i$. This is the spin-galvanic effect, discussed in the previous sections, for a ballistic S-X-S junction.

B. Arbitrary temperatures

The previous result for the current has been obtained at temperatures close to the critical one. We now consider an arbitrary temperature and calculate the current up to the lowest order in the gauge field $\mathcal{F}_{ij}$. In order to calculate the current from Eq. (3.10) to the lowest order in $\mathcal{F}_{ij}$ we need to compute the first two components' matrix $\check{g} = \check{g}^{(0)} + \check{g}^{(1)} + \cdots$.

At zeroth order in $\mathcal{F}_{ij}$ the ballistic equation reduces to

$$
v_F n_x \frac{\partial \check{g}^{(0)}}{\partial x} = \mathbf{i}[\tau_3(\mathbf{i}\omega_n + \mathcal{A}_0) + v_F n_j \mathcal{A}_j + \check{\Delta}, \check{g}^{(0)}], \quad (6.4)
$$

which admits for solution

$$
\check{g}^{(0)} = \check{u}(x)\check{g}_0^{(0)}\check{u}^{-1}(x) + \check{g}_\infty, \quad (6.5)
$$

with $\check{g}_0^{(0)}$ a constant matrix found from the boundary conditions. The propagator $\check{u}(x)$ in Eq. (6.5) is given by

$$
\check{u}(x) = \exp\left[\mathbf{i}\frac{\tau_3(\mathbf{i}\omega_n + \mathcal{A}_0) + v_F n_j \mathcal{A}_j + \check{\Delta}}{v_F n_x}x\right], \quad (6.6)
$$

when we assume that neither $\mathcal{A}_0$ nor $\check{\Delta}$ nor $\mathcal{A}_j$ depend on the position. $\check{u}$ describes how the function $\check{g}_0$ "propagates" from its value at $x = 0$, $\check{g}_0^{(0)}$, to any point x.

The constant $\check{g}_\infty$ in (6.5) satisfies

$$
[\tau_3(\mathbf{i}\omega_n + \mathcal{A}_0) + v_F n_j \mathcal{A}_j + \check{\Delta}, \check{g}_\infty] = 0, \quad (6.7)
$$

and describes the bulk contribution deep inside the superconductor. Notice that according to Eqs. (6.7) and (6.6), $[\check{g}_\infty, \check{u}] = 0$, hence $\check{g}_\infty$ cannot be obtained by the application of (6.6). As we will see below [see (6.13)], the solutions of $\check{u}$ in a superconductor are evanescent waves, so the contribution $\check{u}\check{g}_0^{(0)}\check{u}^{-1}$ vanishes deep inside the superconductor, whereas the contribution $\check{g}_\infty$ remains finite.

The first-order correction with respect to the gauge-field satisfies

$$
v_F n_x \frac{\partial \check{g}^{(1)}}{\partial x} - \left\{\frac{n_i \mathcal{F}_{ij}}{2m}, \frac{\partial \check{g}^{(0)}}{\partial n_j}\right\}
$$

$$
= \mathbf{i}\left[\tau_3(\mathbf{i}\omega_n + \mathcal{A}_0) + v_F n_j \mathcal{A}_j + \check{\Delta}, \check{g}^{(0)}\right] \quad (6.8)
$$

and so

$$
\check{g}^{(1)} = \check{u}(x)\check{g}_0^{(1)}(x)\check{u}^{-1}(x), \quad (6.9)
$$

with a position-dependent $\check{g}_0^{(1)}$ matrix, which reads

$$
\check{g}_0^{(1)} = \check{g}_1 + \int_0^x \frac{dz}{v_F n_x}\left[\check{u}^\dagger(z)\left\{\frac{n_i \mathcal{F}_{ij}}{2m}, \frac{\partial \check{g}^{(0)}}{\partial n_j}\right\}\check{u}(z)\right], \quad (6.10)
$$

where $\check{g}_1$ is a constant matrix.

The current can be written in powers of $\mathcal{F}_{ij}$, $j_x = j_x^{(0)} + j_x^{(1)} + \cdots$ with [see (3.10)]

$$
j_x^{(0)} = \frac{\mathbf{i}e\pi N_0 v_F}{2}\sum_{\omega_n}\mathrm{Tr}\langle n_x \check{g}_0^{(0)}\tau_3\rangle \quad (6.11)
$$

and the first-order correction

$$
j_x^{(1)} = \frac{\mathbf{i}e\pi N_0 v_F}{2}\sum_{\omega_n}\mathrm{Tr}\langle n_x \check{g}_1 \tau_3\rangle. \quad (6.12)
$$

Notice that the second line in (6.10) vanishes after the angular average. We then need to obtain $\check{g}_0^{(0)}$ and $\check{g}_1$ to determine the current through the S-X-S Josephson junction.

We separate the solution of the problem in the three regions: the two superconductors ($x \geqslant L/2$ and $x \leqslant -L/2$) and the normal region ($-L/2 \leqslant x \leqslant L/2$). One can check that in the superconductors:

$$
\check{g}\left(x \leqslant -\frac{L}{2}\right) = e^{-\mathbf{i}\tau_3 \frac{\varphi}{4}}[S_L g_L \tau_+ S_L^\dagger + \check{g}_\infty]e^{\mathbf{i}\tau_3 \frac{\varphi}{4}},
$$

$$
\check{g}\left(x \geqslant \frac{L}{2}\right) = e^{\mathbf{i}\tau_3 \frac{\varphi}{4}}[S_R g_R \tau_- S_R^\dagger + \check{g}_\infty]e^{-\mathbf{i}\tau_3 \frac{\varphi}{4}}, \quad (6.13)
$$

with $\tau_\pm = (\tau_1 \pm \mathbf{i}\tau_2)/2$ and

$$
\check{g}_\infty = \frac{\tau_3 \omega_n + \tau_2 \Delta}{\sqrt{\omega_n^2 + \Delta^2}} = \frac{\tau_3 \sinh\eta + \tau_2}{\cosh\eta}, \quad (6.14)
$$

$$
S_{L,R} = \frac{e^{\eta/2} + \mathbf{i}\tau_1 e^{-\eta/2}}{\sqrt{2\cosh\eta}}e^{\tau_3 \cosh\eta(x\pm L/2)/\xi_0}, \quad (6.15)
$$

where $\sinh\eta = \hbar\omega_n/\Delta$, and $\xi_0 = \hbar v_F/\Delta$ is the superconducting coherence length. The matrices $g_{L,R} \approx g_{L,R}^{(0)} + g_{L,R}^{(1)} + \cdots$ have been expanded in powers of $\mathcal{F}_{ij}$; $g_{L,R}^{(0,1,\cdots)}$ are constant matrices found from boundary conditions order by order. $\check{g}_\infty$ is present at the zeroth order only.

In the normal region, the solution reads

$$
\check{g}\left(-\frac{L}{2} \leqslant x \leqslant \frac{L}{2}\right)
$$

$$
= \check{u}_0(x)\check{g}_0\check{u}_0^\dagger(x) + \check{u}_0(x)\check{g}_0^{(1)}(x)\check{u}_0^\dagger(x) + \cdots, \quad (6.16)
$$

where

$$
\check{g}_0^{(1)}(x) = \check{g}_1 + \int_0^x \check{\mathfrak{G}}(z)dz,
$$

$$
\check{\mathfrak{G}}(z) = \frac{1}{v_F n_x}\check{u}_0^\dagger(z)\left\{\frac{n_i \mathcal{F}_{ij}}{2m}, \frac{\partial \check{u}_0(z)\check{g}_0\check{u}_0^\dagger(z)}{\partial n_j}\right\}\check{u}_0(z), \quad (6.17)
$$

where $\check{u}_0 = \check{u}(\Delta = 0)$ [see (6.6)] is a unitary matrix that can be written as

$$\check{u}_0(x) = e^{-\omega_n \tau_3 x / v_F n_x} \begin{pmatrix} u & 0 \\ 0 & \bar{u} \end{pmatrix}.$$

The spin matrices u and $\bar{u}$ are defined as

$$u(x,\mathbf{n}) = \exp\left[\mathbf{i}\frac{\mathcal{A}_0 + v_F n_j \mathcal{A}_j}{v_F n_x} x\right],$$

$$\bar{u}(x,\mathbf{n}) = \sigma^y u^*(x, -\mathbf{n})\sigma^y = \exp\left[\mathbf{i}\frac{-\mathcal{A}_0 + v_F n_j \mathcal{A}_j}{v_F n_x} x\right]. \quad (6.18)$$

The matrices $\check{g}_0$ and $\check{g}_1$ in Eq. (6.16) are obtained from the boundary conditions, assuming continuity of the GFs at the left and right boundaries. At zeroth order we obtain

$$\check{g}_0 = \begin{pmatrix} g_0 & f_0 \\ -\bar{f}_0 & -\bar{g}_0 \end{pmatrix}, \quad (6.19)$$

$$g_0 = \frac{U\left(-\frac{L}{2}\right)\bar{U}\left(\frac{L}{2}\right) - U\left(\frac{L}{2}\right)\bar{U}\left(-\frac{L}{2}\right) + 2\sinh 2\chi}{2\cosh 2\chi + \operatorname{Tr}\{U(L)\}}, \quad (6.20)$$

$$f_0 = -2\mathbf{i}\frac{e^{\chi} U\left(\frac{L}{2}\right) + e^{-\chi} U\left(-\frac{L}{2}\right)}{2\cosh 2\chi + \operatorname{Tr}\{U(L)\}}, \quad (6.21)$$

$$\chi = \frac{\omega_n L}{v_F n_x} + \operatorname{arcsinh}\frac{\omega_n}{\Delta} + \mathbf{i}\frac{\varphi}{2}, \quad (6.22)$$

$$U(x) = u(x)\bar{u}(-x), \quad (6.23)$$

whereas $\bar{U}(x) = \bar{u}(x)u(-x)$ is its time-reversal conjugate. We here give only the contribution corresponding to the positive projection of the Fermi velocity; the negative projection can be found straightforwardly.

The matrices entering the first-order correction, Eq. (6.17), have the following form in Nambu space:

$$\check{g}_1 = \begin{pmatrix} g_1 & f_1 \\ -\bar{f}_1 & -\bar{g}_1 \end{pmatrix} \quad \text{and} \quad \check{\mathfrak{G}} = \begin{pmatrix} \mathfrak{G} & \mathfrak{F} \\ -\bar{\mathfrak{F}} & -\bar{\mathfrak{G}} \end{pmatrix}, \quad (6.24)$$

with

$$g_1 = -\frac{1}{2}\left[\int_0^{-L/2} + \int_0^{L/2}\right]\mathfrak{G}dz$$
$$+ \frac{1}{2}\int_{-L/2}^{L/2}\mathfrak{G}\cdot g_0 dz - \frac{1}{2}\int_{-L/2}^{L/2}\mathfrak{F}\cdot\bar{f}_0 dz. \quad (6.25)$$

After multiplication by n_x and taking the angular average the first line of this equation vanishes. The second line of (6.25) can be simplified using the normalization condition $g_0^2 - f_0\bar{f}_0 = 1$ available for the zeroth-order correction. We obtain

$$j_x^{(1)} = \mathbf{i}\frac{e\pi N_0 T v_F}{2}\sum_{\omega_n > 0}\int_{-L/2}^{L/2}dz\sum_{\alpha=\pm}$$
$$\times \operatorname{Tr}\left\langle\frac{n_i \mathfrak{F}_{ij}}{2m}\left(\mathfrak{f}_\alpha\frac{\partial\bar{\mathfrak{f}}_\alpha}{\partial n_j} - \bar{\mathfrak{f}}_\alpha\frac{\partial\mathfrak{f}_\alpha}{\partial n_j}\right)\operatorname{sgn}(n_x)\right\rangle, \quad (6.26)$$

with

$$\mathfrak{f}_\pm(z) = -2\mathbf{i}\frac{e^{\pm\chi_\pm} U\left(z + \frac{L}{2}\right) + e^{\mp\chi_\pm} U\left(z - \frac{L}{2}\right)}{2\cosh 2\chi_\pm + \operatorname{Tr}\{U(L)\}},$$
$$\chi_\pm = \frac{\omega_n L}{v_F|n_x|} + \operatorname{arcsinh}\frac{\omega_n}{\Delta} \pm \mathbf{i}\frac{\varphi}{2}, \quad (6.27)$$

and $\bar{\mathfrak{f}}_\pm(z,\mathbf{n}) = \sigma^y \mathfrak{f}_\pm^*(z, -\mathbf{n})\sigma^y$ its time-reversal conjugate.

If $\mathcal{A}_0$ commutes with $\mathcal{A}_i$, then $u(x)\bar{u}(-x) = \exp[\mathbf{i}\mathcal{A}_0 x/v_F n_x]$ becomes independent of the SOC [see the definitions (6.18)], and the contribution (6.26) vanishes. Therefore we expect (6.26) to be proportional to $\mathfrak{F}_{ij}[\mathcal{A}_0,\mathcal{A}_j] \propto \mathfrak{F}_{ij}\mathfrak{F}_{0j}$ at the smallest order in the gauge fields.

By expanding the expression (6.26) in the gauge potentials, up to the term proportional to the electric-like field one obtains

$$\mathfrak{f}_\pm(z) \approx \frac{-\mathbf{i}e^{\pm\chi_\pm}}{2\cosh^2\chi_\pm}\left\{1 + \frac{2\mathbf{i}A_0}{v_x}\left(z + \frac{L}{2}\right) + \frac{1}{2}\left[\frac{2\mathbf{i}A_0}{v_x}\left(z + \frac{L}{2}\right)\right]^2 - \frac{(z+L/2)^2}{v_x^2}v_k[A_k,A_0]\right\}$$
$$+ \frac{-\mathbf{i}e^{\mp\chi_\pm}}{2\cosh^2\chi_\pm}\left\{1 + \frac{2\mathbf{i}A_0}{v_x}\left(z - \frac{L}{2}\right) + \frac{1}{2}\left[\frac{2\mathbf{i}A_0}{v_x}\left(z - \frac{L}{2}\right)\right]^2 - \frac{(z-L/2)^2}{v_x^2}v_k[A_k,A_0]\right\} + \cdots, \quad (6.28)$$

where $\operatorname{Tr}\{U(L)\} \approx 2$ in the small gauge-field limit. Besides the terms proportional to $\mathcal{A}_0$ only, responsible for the oscillations of the S/F proximity effect, the SOC $\mathcal{A}_j$ only appears in the electric-field construction (the last term on each line), due to symmetry with respect to the time reversal. After angular averaging only the last contributions of the two lines are nonzero. This leads to

$$\frac{j_x^{(0)}}{j_0} = 4\langle|n_x|M\rangle, \quad (6.29)$$

$$\frac{j_x^{(1)}}{j_0} = -2\frac{L^3}{3E_F}\times\operatorname{Tr}\{\mathfrak{F}_{xj}\mathfrak{F}_{j0}\}\left\langle\frac{1}{|n_x|}\left(1 + 2\frac{n_j^2}{n_x^2}\right)\frac{\partial M}{\partial\varphi}\right\rangle \quad (6.30)$$

with $j_0 = \pi e v_F N_0 T$ and

$$M = \sum_{\omega_n > 0}\Im\left\{\tanh\left(\frac{\omega_n L}{v_F|n_x|} + \operatorname{arcsinh}\frac{\omega_n}{\Delta} + \mathbf{i}\frac{\varphi}{2}\right)\right\} \quad (6.31)$$

(note that the sum over j applies inside the angular averaging as well). As in all previous examples the anomalous current is proportional to $\operatorname{Tr}\{\mathfrak{F}_{xk}\mathfrak{F}_{k0}\} = \mathfrak{J}_x^a \mathcal{A}_0^a$, where the later form suggests our expressions are valid beyond the linear-in-momentum-SOC approximation, given any spin current $\mathfrak{J}_i^a$ and paramagnetic interaction $\mathcal{A}_0^a$.

Close to the critical temperature $M \approx \Delta^2 \sin\varphi \sum_{\omega_n \geqslant 0} e^{-2\omega L/v_F|n_x|}/2\omega^2$ and we recover (6.3).

Commonly, the concept of a φ_0 junction is defined for junctions with a sinusoidal current-phase relation. This is valid
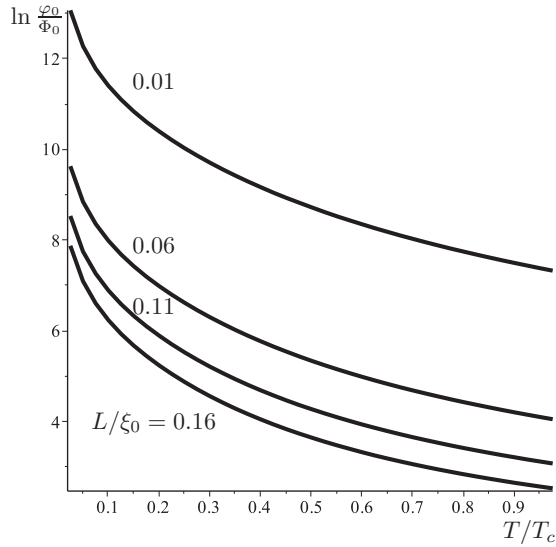
FIG. 2. The temperature dependence of the averaged anomalous phase shift (6.33) on a log scale $\ln(\varphi_0(T)/\Phi_0)$, where $\Phi_0 = -\hbar L^3 \, \mathrm{Tr} \{\mathcal{F}_{xy}\mathcal{F}_{y0}\}/6E_F$. We have approximated $\Delta(T) \approx 1{,}764 T_c \tanh(1{,}74\sqrt{T_c/T-1})$, which is the usual interpolation for the temperature dependence of the superconducting gap. The curves are given for different ratios of $L/\xi_0 = \{0.01, 0.06, 0.11, 0.16\}$, with $\xi_0 = \hbar v_F/\Delta_0$ and $\Delta_0 = 1.764 T_c$ the gap at zero temperature. Note that φ_0 does not vanish when $T \to T_c$.

at temperatures close to the critical temperature or in the case of a weak proximity effect between the S electrodes and the X bridge. However, in several cases the current-phase relation is more complex and higher harmonics are involved [132]. This is the case of the ballistic junction studied here with a current-phase relation given by the sum of Eqs. (6.29) and (6.30). In such cases the φ_0 is defined as the phase difference across the junction that minimize the energy, or equivalently, as the phase difference imposed on the junction in order to get a zero current state, i.e., $j(\varphi_0) = 0$. In our perturbative analysis φ_0 is small and hence

$$\varphi_0 = -\frac{j(\varphi=0)}{\partial_\varphi j\big|_{\varphi=0}}. \tag{6.32}$$

It is clear that $j(\varphi=0) = j_x^{(1)}$, whereas $\partial_\varphi j\big|_{\varphi=0} = \partial_\varphi j_x^{(0)}\big|_{\varphi=0}$, and from Eqs. (6.29) and (6.30) we obtain

$$\varphi_0 = -\frac{\hbar L^3}{6E_F} \frac{\mathcal{F}_{xj}^a \mathcal{F}_{j0}^a \frac{\partial}{\partial\varphi}\big\langle \frac{M}{|n_x|}\big(1+2\frac{n_j^2}{n_x^2}\big)\big\rangle\big|_{\varphi=0}}{\frac{\partial}{\partial\varphi}\langle|n_x|M\rangle\big|_{\varphi=0}}. \tag{6.33}$$

In Fig. 2 we show the temperature dependence of φ_0 for the ballistic junction for a 2D system when only $\mathcal{F}_{xy}^a \mathcal{F}_{y0}^a$ is nonzero. We assume a circular Fermi surface, $n_x = \cos\theta$ and $n_y = \sin\theta$. We plot the anomalous phase for different junction lengths.

VII. DISCUSSION AND CONCLUSIONS

In order to verify our findings and prove the existence of the anomalous φ_0 phase one can design a superconducting ring interrupted by a semiconducting link with a strong SOC, similar to the one used recently in Ref. [144] for the characterization
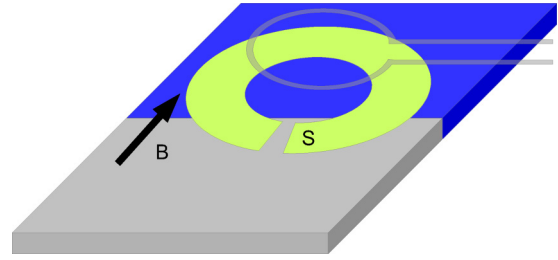

FIG. 3. (Color online) One possible experimental setup to verify the generation of a spontaneous current in S/X/S Josephson junction. It consists of a truncated superconducting loop (in green) deposited on top of a material exhibiting SOC (in gray). In order to isolate electrically part of the S loop from the conducting substrate we assume an insulating layer between them (in blue). By applying an in-plane magnetic field a circulating supercurrent might be generated which in turn induces a magnetic flux that can be measured via an extra pickup coil (shaded gray). For more details see discussion in the main text.

of the current-phase relation of a Nb/3D-HgTe/Nb junction or in [145] for the observation of a spontaneous supercurrent induced by a ferromagnetic π junction. A schematic view of the proposed setup is shown in Fig. 3: It consists of a superconducting ring (green) grown on top of a semiconductor or a metallic substrate with strong SOC (gray). In order to isolate electrically the S ring from the semiconductor one can for example add an insulating barrier (blue) under the ring.

If a magnetic field is applied in the plane of the ring, it will act as a Zeeman field and hence, according to our previous results, it will create a spontaneous circulating supercurrent; see [7,146] for more details. This supercurrent will generate a magnetic flux that in principle can be measured by a second loop [144] or a micro-Hall sensor [145].

In the case when the bridge is made of a 2D semiconductor with a generic SOC described by a combination of Rashba and Dresselhaus terms, $\mathcal{A}_x = -\alpha\sigma^y + \beta\sigma^x$ and $\mathcal{A}_y = \alpha\sigma^x - \beta\sigma^y$, the generated supercurrent should be proportional to

$$j_s \propto (\alpha^2 - \beta^2)(h_x\beta + h_y\alpha).$$

Thus the current depends on the direction of the applied magnetic field. In particular, for a field perpendicular to the 2D gas the effect should vanish. In addition by applying a gate voltage one could modify the ratio between Dresselhaus and Rashba interactions and hence control the supercurrent flow. We thus expect that the dependency of the spontaneous supercurrent with respect to the orientation of the magnetic field and/or the gate voltage realizes a clear demonstration of the spin-galvanic effect in Josephson systems.

Instead of using a semiconducting bridge one could grow the superconducting loop on top of a metallic substrate. Metals with strong SOC, such as Pt and Ta, are good candidates to observe the φ_0-junction behavior, but also an ultrathin layer of Pb might be used [147]. In such a case probably one cannot control the φ_0 shift using a gate, but a spontaneous circulating current might still be controlled by switching the in-plane external field on and off.

Eventually, the existence of a magnetoelectric phase shift φ_0 can be probed by measuring the Shapiro steps in S-X-S Josephson junctions as suggested in Ref. [148].

In conclusion, we have demonstrated that the inverse Edelstein effect, also called spin-galvanic effect, and the appearance of an anomalous phase shift φ_0 in Josephson junctions are the two sides of the same coin. We presented a full SU(2) covariant quasiclassical formalism that allows us to study these magnetoelectric phenomena in bulk and hybrid superconducting structures with arbitrary linear-in-momentum SOC (Sec. III).

With the help of our quasiclassical transport formalism we derived the Edelstein effect close to the critical temperature of a bulk superconductor, recovering the Edelstein result in a very compact way (Sec. IV) and generalizing it for the case of an arbitrary linear-in-momentum SOC. We have shown that the Edelstein effect and its inverse are reciprocal in the sense of the Onsager relations, both in ballistic and diffusive superconducting systems: A static supercurrent can induce a finite magnetization due to the presence of a spin-orbit coupling, and reciprocally a finite magnetization produces a finite supercurrent in a bulk system. We have demonstrated that the linear-response tensor is directly proportional to the equilibrium spin current tensor $\mathcal{J}_i^a$.

We have also generalized this result to inhomogeneous systems. In particular we have studied the current-phase relation of a Josephson junction consisting of two superconductors coupled via a normal metal with both SOC and spin-splitting field. We have demonstrated that a supercurrent can flow even if the phase difference between the S electrodes is zero. This current is associated with an anomalous phase shift φ_0. This result holds for both ballistic (Sec. VI) and diffusive systems (Sec. V), for arbitrary linear-in-momentum spin-orbit coupling, and for arbitrary barrier resistance between the superconductor and the normal metal. For all these situations we have demonstrated that SU(2) gauge fields are the only objects of relevance in the phenomenology of the φ_0 shift, and in particular we have shown that $\varphi_0 \propto \mathcal{A}_{0}^a \mathcal{J}_i^a = \mathcal{F}_{0j}^a \mathcal{F}_{ji}^a$; i.e., the anomalous phase shift is proportional to the SU(2) electric and magnetic fields, or equivalently to the spin current tensor. We thus directly linked the anomalous phase shift in superconducting systems to the inverse Edelstein effect (also known as the spin-galvanic effect) extensively studied in normal systems.

[1] A. I. Buzdin, Rev. Mod. Phys. **77**, 935 (2005).

[2] F. S. Bergeret, A. F. Volkov, and K. B. Efetov, Rev. Mod. Phys. **77**, 1321 (2005).

[3] P. San Giorgio, S. Reymond, M. R. Beasley, J. H. Kwon, and K. Char, Phys. Rev. Lett. **100**, 237002 (2008).

[4] K. M. Boden, W. P. Pratt, and N. O. Birge, Phys. Rev. B **84**, 020510 (2011).

[5] T. Kontos, M. Aprili, J. Lesueur, and X. Grison, Phys. Rev. Lett. **86**, 304 (2001).

[6] A. Cottet, Phys. Rev. B **76**, 224505 (2007).

[7] L. Bulaevskii, V. V. Kuzii, and A. A. Sobyanin, JETP Lett. **25**, 290 (1977).

[8] A. I. Buzdin, L. Bulaevskii, and S. V. Panyukov, Sov. Phys. JETP **35**, 178 (1982).

[9] V. V. Ryazanov, V. A. Oboznov, A. Yu. Rusanov, A. V. Veretennikov, A. A. Golubov, and J. Aarts, Phys. Rev. Lett. **86**, 2427 (2001).

[10] T. Kontos, M. Aprili, J. Lesueur, F. Genêt, B. Stephanidis, and R. Boursier, Phys. Rev. Lett. **89**, 137007 (2002).

[11] F. S. Bergeret, A. F. Volkov, and K. B. Efetov, Phys. Rev. B **64**, 134506 (2001).

[12] F. S. Bergeret, A. F. Volkov, and K. B. Efetov, Phys. Rev. Lett. **86**, 3140 (2001).

[13] J. W. A. Robinson, J. D. S. Witt, and M. G. Blamire, Science **329**, 59 (2010).

[14] T. S. Khaire, M. A. Khasawneh, W. P. Pratt, and N. O. Birge, Phys. Rev. Lett. **104**, 137002 (2009).

[15] M. S. Anwar, F. Czeschka, M. Hesselberth, M. Porcu, and J. Aarts, Phys. Rev. B **82**, 100501(R) (2010).

[16] M. S. Anwar, M. Veldhorst, A. Brinkman, and J. Aarts, Appl. Phys. Lett. **100**, 052602 (2012).

[17] I. T. M. Usman, K. A. Yates, J. D. Moore, K. Morrison, V. K. Pecharsky, K. A. Gschneidner, T. Verhagen, J. Aarts, V. I. Zverev, J. W. A. Robinson, J. D. S. Witt, M. G. Blamire, and L. F. Cohen, Phys. Rev. B **83**, 144518 (2011).

[18] C. Klose, T. S. Khaire, Y. Wang, W. P. Pratt, N. O. Birge, B. J. McMorran, T. P. Ginley, J. A. Borchers, B. J. Kirby, B. B. Maranville, and J. Unguris, Phys. Rev. Lett. **108**, 127002 (2012).

[19] E. C. Gingrich, P. Quarterman, Y. Wang, R. Loloee, W. P. Pratt, and N. O. Birge, Phys. Rev. B **86**, 224506 (2012).

[20] J. D. S. Witt, J. W. A. Robinson, and M. G. Blamire, Phys. Rev. B **85**, 184526 (2012).

[21] J. W. A. Robinson, F. Chiodi, M. Egilmez, G. B. Halász, and M. G. Blamire, Sci. Rep. **2**, 699 (2012).

[22] Y. Kalcheim, O. Millo, M. Egilmez, J. W. A. Robinson, and M. G. Blamire, Phys. Rev. B **85**, 104504 (2012).

[23] Y. Kalcheim, I. Felner, O. Millo, T. Kirzhner, G. Koren, A. Di Bernardo, M. Egilmez, M. G. Blamire, and J. W. A. Robinson, Phys. Rev. B **89**, 180506 (2014).

[24] M. Alidoust, K. Halterman, and O. T. Valls, Phys. Rev. B **92**, 014508 (2015).

[25] N. Banerjee, J. Robinson, and M. G. Blamire, Nat. Commun. **5**, 4771 (2014).

[26] P. V. Leksin, N. N. Garif'yanov, I. A. Garifullin, Y. V. Fominov, J. Schumann, Y. Krupskaya, V. Kataev, O. G. Schmidt, and B. Büchner, Phys. Rev. Lett. **109**, 057005 (2012).

[27] N. Banerjee, C. B. Smiet, R. G. J. Smits, A. Ozaeta, F. S. Bergeret, M. G. Blamire, and J. W. A. Robinson, Nat. Commun. **5**, 3048 (2014).

[28] X. L. Wang, A. Di Bernardo, N. Banerjee, A. Wells, F. S. Bergeret, M. G. Blamire, and J. W. A. Robinson, Phys. Rev. B **89**, 140508 (2014).

[29] F. Chiodi, J. D. S. Witt, R. G. J. Smits, L. Qu, G. B. Halász, C.-T. Wu, O. T. Valls, K. Halterman, J. W. A. Robinson, and M. G. Blamire, Europhys. Lett. **101**, 37002 (2013).

[30] Y. Gu, J. W. A. Robinson, M. Bianchetti, N. A. Stelmashenko, D. Astill, F. M. Grosche, J. L. MacManus-Driscoll, and M. G. Blamire, APL Mater. **2**, 046103 (2014).

[31] A. A. Jara, C. Safranski, I. N. Krivorotov, C.-T. Wu, A. N. Malmi-Kakkada, O. T. Valls, and K. Halterman, Phys. Rev. B **89**, 184502 (2014).

[32] A. Yu. Rusanov, M. Hesselberth, J. Aarts, and A. I. Buzdin, Phys. Rev. Lett. **93**, 057002 (2004).

[33] A. Yu. Rusanov, S. Habraken, and J. Aarts, Phys. Rev. B **73**, 060505(R) (2006).

[34] M. Eschrig, Physics Today **64**, 43 (2011).

[35] J. Linder and J. W. A. Robinson, Nat. Phys. **11**, 307 (2015).

[36] F. S. Bergeret, A. Verso, and A. F. Volkov, Phys. Rev. B **86**, 060506(R) (2012).

[37] A. Pal, Z. H. Barber, J. W. A. Robinson, and M. G. Blamire, Nat. Commun. **5**, 3340 (2014).

[38] D. Massarotti, A. Pal, G. Rotoli, L. Longobardi, M. G. Blamire, and F. Tafuri, Nat. Commun. **6**, 7376 (2015).

[39] F. S. Bergeret and F. Giazotto, Phys. Rev. B **88**, 014515 (2013).

[40] F. S. Bergeret and F. Giazotto, Phys. Rev. B **89**, 054505 (2014).

[41] A. Ozaeta, P. Virtanen, F. S. Bergeret, and T. T. Heikkilä, Phys. Rev. Lett. **112**, 057001 (2014).

[42] F. Giazotto, J. W. A. Robinson, J. S. Moodera, and F. S. Bergeret, Appl. Phys. Lett. **105**, 062602 (2014).

[43] F. Giazotto, T. Heikkilä, and F. S. Bergeret, Phys. Rev. Lett. **114**, 067001 (2015).

[44] U. Nasti, L. Parlato, M. Ejrnaes, R. Cristiano, T. Taino, H. Myoren, R. Sobolewski, and G. Pepe, Phys. Rev. B **92**, 014501 (2015).

[45] F. S. Bergeret and I. V. Tokatly, Phys. Rev. Lett. **110**, 117003 (2013).

[46] F. S. Bergeret and I. V. Tokatly, Phys. Rev. B **89**, 134517 (2014).

[47] S. H. Jacobsen, J. A. Ouassou, and J. Linder, Phys. Rev. B **92**, 024510 (2015).

[48] J. Arjoranta and T. T. Heikkila, arXiv:1507.02320.

[49] M. Alidoust and K. Halterman, New J. Phys. **17**, 033001 (2015).

[50] S. Maekawa, S. O. Valenzuela, E. Saitoh, and T. Kimura, *Spin Current* (Oxford University Press, Oxford, 2012).

[51] T. Kuschel and G. Reiss, Nat. Nanotechnol. **10**, 22 (2014).

[52] A. Manchon, Nat. Phys. **10**, 340 (2014).

[53] C. Ciccarelli, K. M. D. Hals, A. Irvine, V. Novak, Y. Tserkovnyak, H. Kurebayashi, A. Brataas, and A. Ferguson, Nat. Nanotechnol. **10**, 50 (2014).

[54] M. I. D'yakonov and V. I. Perel, JETP Lett. **13**, 467 (1971).

[55] M. Dyakonov and V. Perel, Phys. Lett. A **35**, 459 (1971).

[56] J.-N. Chazalviel, Phys. Rev. B **11**, 3918 (1975).

[57] J. E. Hirsch, Phys. Rev. Lett. **83**, 1834 (1999).

[58] M. Dyakonov and A. Khaetskii, in *Spin Physics in Semiconductors*, Springer Series in Solid-State Sciences, Vol. 157,

edited by M. Dyakonov (Springer, Berlin, 2006), Chap. 8, pp. 211–243.

[59] E. G. Mishchenko, A. V. Shytov, and B. I. Halperin, Phys. Rev. Lett. **93**, 226602 (2004).

[60] Y. K. Kato, R. C. Myers, A. C. Gossard, and D. D. Awschalom, Phys. Rev. Lett. **93**, 176601 (2004).

[61] Y. K. Kato, R. C. Myers, A. C. Gossard, and D. D. Awschalom, Science **306**, 1910 (2004).

[62] J. Wunderlich, B. Kaestner, J. Sinova, and T. Jungwirth, Phys. Rev. Lett. **94**, 047204 (2005).

[63] R. Raimondi, C. Gorini, P. Schwab, and M. Dzierzawa, Phys. Rev. B **74**, 035340 (2006).

[64] R. Raimondi, P. Schwab, C. Gorini, and G. Vignale, Ann. Phys. **524**, 153 (2012).

[65] S. O. Valenzuela and M. Tinkham, Nature (London) **442**, 176 (2006).

[66] M. Morota, Y. Niimi, K. Ohnishi, D. H. Wei, T. Tanaka, H. Kontani, T. Kimura, and Y. Otani, Phys. Rev. B **83**, 174405 (2011).

[67] M. Isasa, E. Villamor, L. E. Hueso, M. Gradhand, and F. Casanova, Phys. Rev. B **91**, 024402 (2014).

[68] E. Saitoh, M. Ueda, H. Miyajima, and G. Tatara, Appl. Phys. Lett. **88**, 182509 (2006).

[69] K. Ando, S. Takahashi, K. Harii, K. Sasage, J. Ieda, S. Maekawa, and E. Saitoh, Phys. Rev. Lett. **101**, 036601 (2008).

[70] T. Kimura, Y. Otani, T. Sato, S. Takahashi, and S. Maekawa, Phys. Rev. Lett. **98**, 156601 (2007).

[71] K. Uchida, J. Xiao, H. Adachi, J. Ohe, S. Takahashi, J. Ieda, T. Ota, Y. Kajiwara, H. Umezawa, H. Kawai, G. E. W. Bauer, S. Maekawa, and E. Saitoh, Nat. Mater. **9**, 894 (2010).

[72] Y. Kajiwara, K. Harii, S. Takahashi, J. Ohe, K. Uchida, M. Mizuguchi, H. Umezawa, H. Kawai, K. Ando, K. Takanashi, S. Maekawa, and E. Saitoh, Nature (London) **464**, 262 (2010).

[73] A. G. Aronov and Y. Lyanda-Geller, JETP Lett. **50**, 431 (1989).

[74] V. Edelstein, Solid State Commun. **73**, 233 (1990).

[75] A. Y. Silov, P. A. Blajnov, J. H. Wolter, R. Hey, K. H. Ploog, and N. S. Averkiev, Appl. Phys. Lett. **85**, 5929 (2004).

[76] K. Shen, G. Vignale, and R. Raimondi, Phys. Rev. Lett. **112**, 096601 (2014).

[77] S. Ganichev, E. Ivchenko, V. Bel'kov, S. Tarasenko, M. Sollinger, D. Weiss, W. Wegscheider, and W. Prettl, Nature (London) **417**, 153 (2002).

[78] J. C. Rojas Sánchez, L. Vila, G. Desfonds, S. Gambarelli, J. P. Attané, J. M. De Teresa, C. Magén, and A. Fert, Nat. Commun. **4**, 2944 (2013).

[79] V. M. Edelstein, Phys. Rev. Lett. **75**, 2004 (1995).

[80] V. M. Edelstein, Phys. Rev. B **72**, 172501 (2005).

[81] S. K. Yip, Phys. Rev. B **65**, 144508 (2002).

[82] E. Bauer and M. Sigrist, *Non-Centrosymmetric Superconductors*, Lecture Notes in Physics, Vol. 847 (Springer, Berlin, 2012).

[83] I. V. Krive, A. M. Kadigrobov, R. I. Shekhter, and M. Jonson, Phys. Rev. B **71**, 214516 (2005).

[84] A. I. Buzdin, Phys. Rev. Lett. **101**, 107005 (2008).

[85] J.-F. Liu and K. S. Chan, Phys. Rev. B **82**, 125305 (2010).

[86] A. G. Mal'shukov, S. Sadjina, and A. Brataas, Phys. Rev. B **81**, 060502 (2010).

[87] A. A. Reynoso, G. Usaj, C. A. Balseiro, D. Feinberg, and M. Avignon, Phys. Rev. B **86**, 214519 (2012).

[88] T. Yokoyama, M. Eto, and Y. V. Nazarov, J. Phys. Soc. Jpn. **82**, 054703 (2013).

[89] T. Yokoyama, M. Eto, and Y. V. Nazarov, Phys. Rev. B **89**, 195407 (2014).

[90] T. Yokoyama and Y. V. Nazarov, Europhys. Lett. **108**, 47009 (2014).

[91] G. Campagnano, P. Lucignano, D. Giuliano, and A. Tagliacozzo, J. Phys. Condens. Matter **27**, 205301 (2015).

[92] F. S. Bergeret and I. V. Tokatly, Europhys. Lett. **110**, 57005 (2015).

[93] F. Konschelle, arXiv:1408.4533.

[94] I. Kulagina and J. Linder, Phys. Rev. B **90**, 054504 (2014).

[95] S. V. Mironov, A. S. Mel'nikov, and A. I. Buzdin, Phys. Rev. Lett. **114**, 227001 (2015).

[96] V. Braude and Y. V. Nazarov, Phys. Rev. Lett. **98**, 077003 (2007).

[97] Y. Tanaka, A. A. Golubov, S. Kashiwaya, and M. Ueda, Phys. Rev. Lett. **99**, 037005 (2007).

[98] M. Eschrig, T. Löfwander, T. Champel, J. C. Cuevas, J. Kopu, and G. Schön, J. Low Temp. Phys. **147**, 457 (2007).

[99] R. Grein, M. Eschrig, G. Metalidis, and G. Schön, Phys. Rev. Lett. **102**, 227005 (2009).

[100] J.-F. Liu and K. S. Chan, Phys. Rev. B **82**, 184533 (2010).

[101] I. Margaris, V. Paltoglou, and N. Flytzanis, J. Phys.: Condens. Matter **22**, 445701 (2010).

[102] Y. Tanaka, T. Yokoyama, and N. Nagaosa, Phys. Rev. Lett. **103**, 107002 (2009).

[103] F. Dolcini, M. Houzet, and J. S. Meyer, Phys. Rev. B **92**, 035428 (2015).

[104] V. Geshkenbeim and A. I. Larkin, JETP Lett. **43**, 395 (1986).

[105] S. Yip, Phys. Rev. B **52**, 3087 (1995).

[106] Y. Tanaka and S. Kashiwaya, Phys. Rev. B **56**, 892 (1997).

[107] M. Sigrist, Prog. Theor. Phys. **99**, 899 (1998).

[108] S. Kashiwaya and Y. Tanaka, Rep. Prog. Phys. **63**, 1641 (2000).

[109] Y. Asano, Y. Tanaka, M. Sigrist, and S. Kashiwaya, Phys. Rev. B **67**, 184505 (2003).

[110] Y. Asano, Y. Tanaka, M. Sigrist, and S. Kashiwaya, Phys. Rev. B **71**, 214501 (2005).

[111] P. M. R. Brydon, B. Kastening, D. K. Morr, and D. Manske, Phys. Rev. B **77**, 104504 (2008).

[112] Y. Rahnavard, D. Manske, and G. Annunziata, Phys. Rev. B **89**, 214501 (2014).

[113] A. A. Reynoso, G. Usaj, C. Balseiro, D. Feinberg, and M. Avignon, Phys. Rev. Lett. **101**, 107001 (2008).

[114] A. Zazunov, R. Egger, T. Jonckheere, and T. Martin, Phys. Rev. Lett. **103**, 147004 (2009).

[115] A. Brunetti, A. Zazunov, A. Kundu, and R. Egger, Phys. Rev. B **88**, 144515 (2013).

[116] A. Buzdin and A. E. Koshelev, Phys. Rev. B **67**, 220504(R) (2003).

[117] E. Goldobin, D. Koelle, R. Kleiner, and A. I. Buzdin, Phys. Rev. B **76**, 224523 (2007).

[118] E. Goldobin, D. Koelle, R. Kleiner, and R. G. Mints, Phys. Rev. Lett. **107**, 227001 (2011).

[119] H. Sickinger, A. Lipman, M. Weides, R. G. Mints, H. Kohlstedt, D. Koelle, R. Kleiner, and E. Goldobin, Phys. Rev. Lett. **109**, 107002 (2012).

[120] R. Mints, Phys. Rev. B **57**, R3221 (1998).

[121] A. I. Buzdin, J. Phys. IV **131**, 213 (2005).

[122] A. I. Buzdin, Phys. Rev. B **72**, 100501 (2005).

[123] F. Konschelle, Eur. Phys. J. B **87**, 119 (2014).

[124] E. Rashba, Phys. Rev. B **68**, 241315 (2003).

[125] I. V. Tokatly, Phys. Rev. Lett. **101**, 106601 (2008).

[126] D. F. Agterberg and R. P. Kaur, Phys. Rev. B **75**, 064511 (2007).

[127] A. G. Mal'shukov, L. Y. Wang, C. S. Chu, and K. A. Chao, Phys. Rev. Lett. **95**, 146601 (2005).

[128] T. D. Stanescu and V. Galitski, Phys. Rev. B **75**, 125307 (2007).

[129] M. Duckheim, D. L. Maslov, and D. Loss, Phys. Rev. B **80**, 235327 (2009).

[130] J. Fröhlich and U. Studer, Rev. Mod. Phys. **65**, 733 (1993).

[131] B. Berche and E. Medina, Eur. J. Phys. **34**, 161 (2013).

[132] A. Golubov, M. Kupriyanov, and E. Il'ichev, Rev. Mod. Phys. **76**, 411 (2004).

[133] V. P. Mineev and K. V. Samokhin, Phys. Rev. B **78**, 144503 (2008).

[134] M. Houzet and J. S. Meyer, Phys. Rev. B **92**, 014509 (2015).

[135] G. Eilenberger, Z. Phys. **214**, 195 (1968).

[136] A. I. Larkin and Y. N. Ovchinnikov, Sov. Phys. JETP **28**, 1200 (1969).

[137] K. D. Usadel, Phys. Rev. Lett. **25**, 507 (1970).

[138] N. B. Kopnin, *Theory of Nonequilibrium Superconductivity* (Oxford University Press, Oxford, 2001).

[139] C. Gorini, P. Schwab, R. Raimondi, and A. L. Shelankov, Phys. Rev. B **82**, 195316 (2010).

[140] D. N. Langenberg and A. I. Larkin, *Nonequilibrium Superconductivity* (North-Holland, Amsterdam, 1986).

[141] A. G. Mal'shukov and C. S. Chu, Phys. Rev. B **78**, 104503 (2008).

[142] M. Y. Kupriyanov and V. F. Lukichev, Sov. Phys. JETP **67**, 1163 (1988).

[143] L. P. Gor'kov and E. Rashba, Phys. Rev. Lett. **87**, 037004 (2001).

[144] I. Sochnikov, L. Maier, C. A. Watson, J. R. Kirtley, C. Gould, G. Tkachov, E. M. Hankiewicz, C. Brüne, H. Buhmann, L. W. Molenkamp, and K. A. Moler, Phys. Rev. Lett. **114**, 066801 (2015).

[145] A. Bauer, J. Bentner, M. Aprili, M. L. Della Rocca, M. Reinwald, W. Wegscheider, and C. Strunk, Phys. Rev. Lett. **92**, 217001 (2003).

[146] A. Buzdin, J. Phys.: Conf. Ser. **150**, 052029 (2009).

[147] L. Serrier-Garcia, J. C. Cuevas, T. Cren, C. Brun, V. Cherkez, F. Debontridder, D. Fokin, F. S. Bergeret, and D. Roditchev, Phys. Rev. Lett. **110**, 157003 (2013).

[148] F. Konschelle and A. I. Buzdin, Phys. Rev. Lett. **102**, 017001 (2009).

[149] Lower indices $(i, j, k, ...)$ describe space variables, while upper indices $(a, b, ...)$ will describe spin variables.

[150] We use units in which Planck's constant $\hbar = 1$ and Boltzmann's constant $k_B = 1$.

[151] Note that in a one-dimensional system described by the Hamiltonian Eq. (3.2), the gauge potential is always a pure gauge and therefore the 1D problem should not exhibit any magnetoelectric effect. Nevertheless, the above argument does not apply for the topological 1D channel, where a φ_0 effect has been discussed [102,103], since the dispersion relation is not quadratic in that case [95]. Also, the problem of the spin-active interface is nontrivial when the spin-orbit effect is treated as a gauge potential, and might eventually lead to a φ_0 effect as well [94,99–101].