Gauge Invariant Formalism for Perturbations in Quantum Cosmology

Guillermo A. Mena Marugán

(IEM-CSIC)

With Laura Castelló Gomar,
& Mercedes Martin-Benito

CSIC
CONSEJO SUPERIOR DE INVESTIGACIONES CIENTÍFICAS

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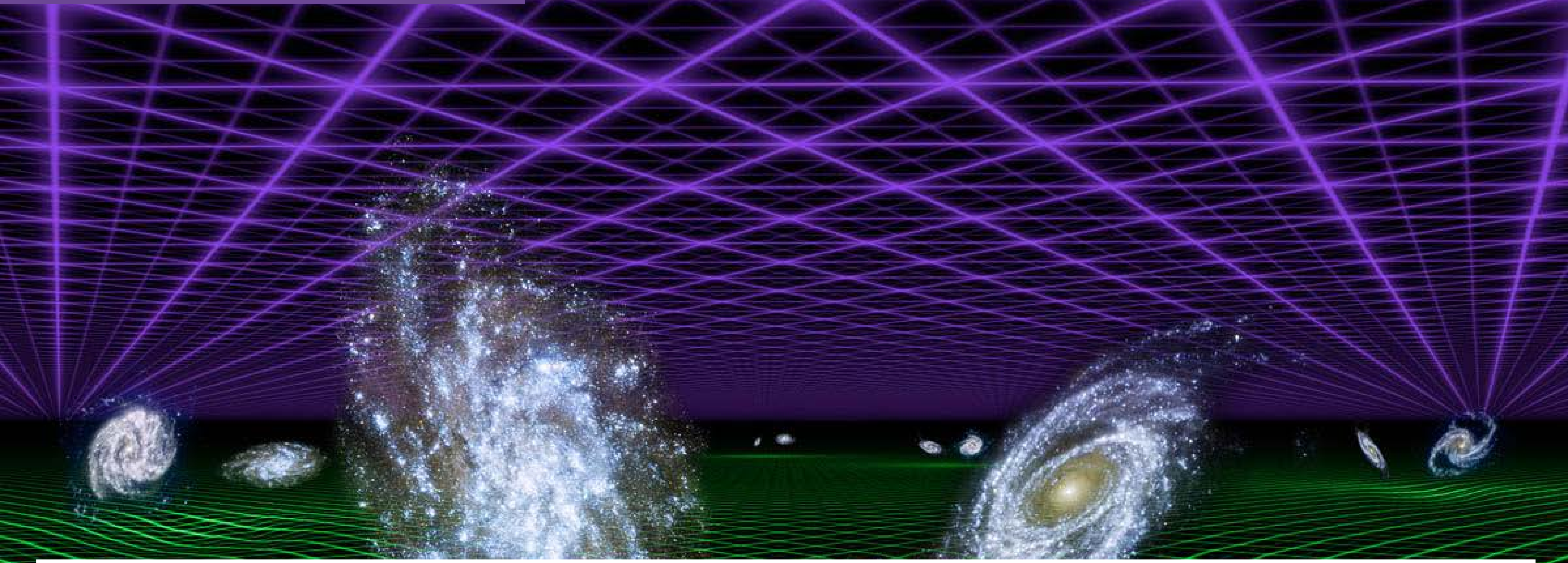
Introduction

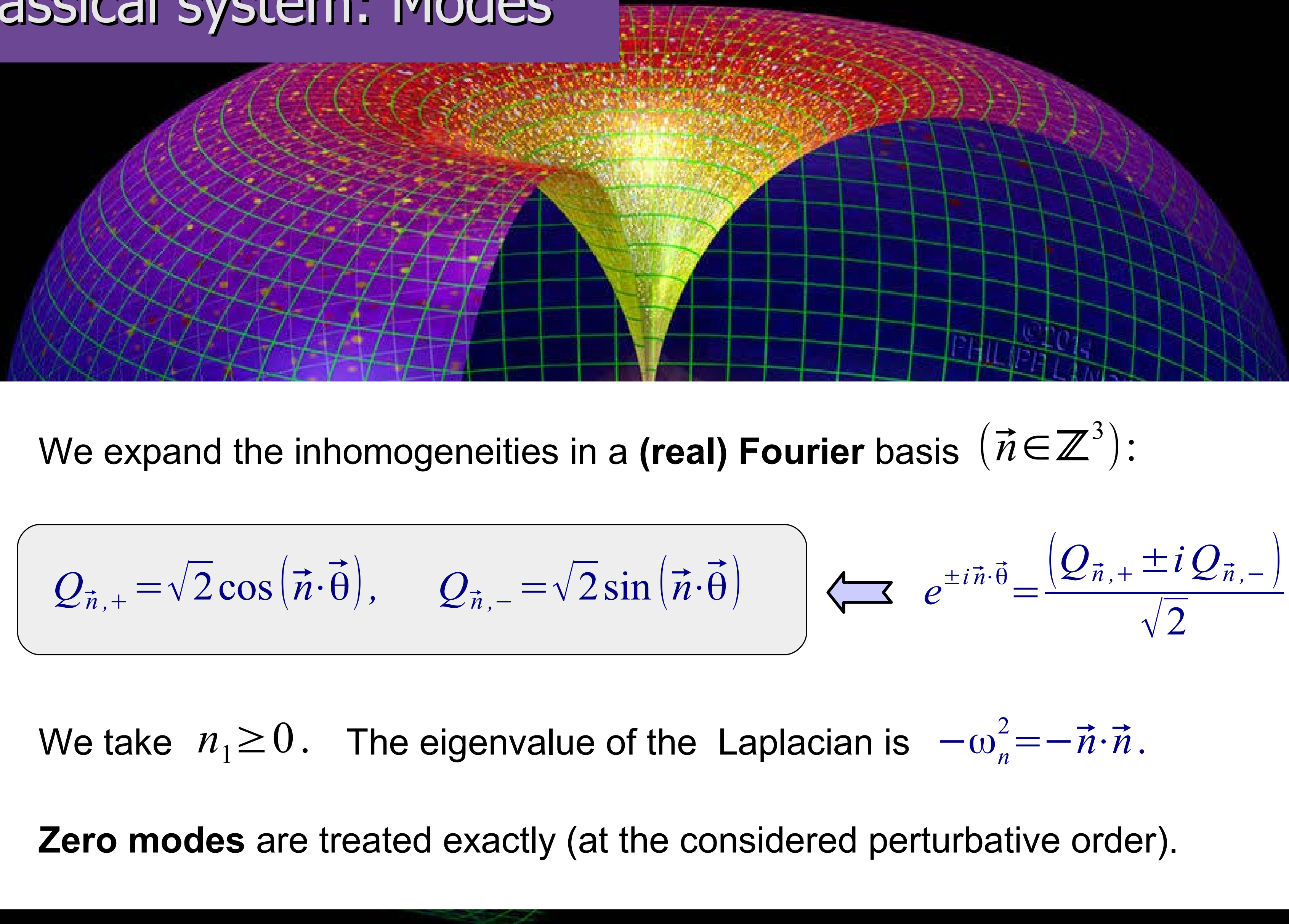
- Our Universe is approximately homogeneous and isotropic: Background with **perturbations**.
- Need of **gauge invariant** descriptions *(Bardeen, Mukhanov-Sasaki).*
- **Canonical formulation** with constraints *(Langlois, Pinto-Neto)*.
- **Quantum** treatment including the background *(Halliwell-Hawking, Shirai-Wada)*.
- Recently studied in ***Loop Quantum Cosmology***.

Classical system



- We consider a **FLRW** universe with **compact flat** topology.
- We include a **scalar field** subject to a potential (e.g. a mass term).
- For simplicity, we analyze only **SCALAR pertubations**.

Classical system: Modes



- We expand the inhomogeneities in a **(real) Fourier** basis $\left(\vec{n}\in\mathbb{Z}^3\right)$:

$$Q_{\vec{n},+}=\sqrt{2}\cos\left(\vec{n}\cdot\vec{\theta}\right), \qquad Q_{\vec{n},-}=\sqrt{2}\sin\left(\vec{n}\cdot\vec{\theta}\right) \quad \Longleftarrow \quad e^{\pm i\vec{n}\cdot\vec{\theta}}=\frac{\left(Q_{\vec{n},+}\pm i\,Q_{\vec{n},-}\right)}{\sqrt{2}}.$$

- We take $n_1\geq 0$. The eigenvalue of the Laplacian is $-\omega_n^2=-\vec{n}\cdot\vec{n}$.
- **Zero modes** are treated exactly (at the considered perturbative order).

Classical system: Inhomogeneities

- Scalar perturbations: **metric and field.**

$$h_{ij}=\sigma^2 e^{2\alpha}\left[{}^0h_{ij}+2\sum\left\{\boldsymbol{a}_{\vec{\boldsymbol{n}},\pm}(t)Q_{\vec{n},\pm}\,{}^0h_{ij}+\boldsymbol{b}_{\vec{\boldsymbol{n}},\pm}(t)\left(\frac{3}{\omega_n^2}(Q_{\vec{n},\pm})_{;ij}+Q_{\vec{n},\pm}\,{}^0h_{ij}\right)\right\}\right],$$

$$N=\sigma\left[N_0(t)+e^{3\alpha}\sum \boldsymbol{g}_{\vec{\boldsymbol{n}},\pm}(t)Q_{\vec{n},\pm}\right],\qquad N_i=\sigma^2 e^{2\alpha}\sum\frac{\boldsymbol{k}_{\vec{\boldsymbol{n}},\pm}(t)}{\omega_n^2}(Q_{\vec{n},\pm})_{;i},$$

$$\Phi=\frac{1}{\sigma(2\pi)^{3/2}}\left[\varphi(t)+\sum \boldsymbol{f}_{\vec{\boldsymbol{n}},\pm}(t)Q_{\vec{n},\pm}\right].\qquad \sigma^2=\frac{G}{6\pi^2},\qquad \tilde{m}=m\sigma.$$

- Truncating at **quadratic perturbative order** in the action:

$$H=N_0\left[H_0+\sum H_2^{\vec{n},\pm}\right]+\sum g_{\vec{n},\pm}H_1^{\vec{n},\pm}+\sum k_{\vec{n},\pm}\tilde{H}_{\uparrow 1}^{\vec{n},\pm}.$$

$$H_0=\frac{e^{-3\alpha}}{2}\left(-\pi_\alpha^2+\pi_\varphi^2+e^{6\alpha}\tilde{m}^2\varphi^2\right).$$

Gauge invariant perturbations

- Consider the sector of zero modes as describing a fixed **background.**

- Look for a transformation of the **perturbations --canonical only** with respect to their symplectic structure-- adapted to gauge invariance:

a) Find new variables that **abelianize** the perturbative constraints.

$$\breve{\boldsymbol{H}}_{\mathbf{1}}^{\vec{n},\pm}=H_1^{\vec{n},\pm}-3e^{3\alpha}H_0 a_{\vec{n},\pm}.$$

b) Include the gauge-invariant **Mukhanov-Sasaki** variable.

$$\boldsymbol{v}_{\vec{n},\pm}=e^{\alpha}\left[f_{\vec{n},\pm}+\frac{\pi_\varphi}{\pi_\alpha}\left(a_{\vec{n},\pm}+b_{\vec{n},\pm}\right)\right].$$

c) Complete the transformation with suitable **momenta**.

Gauge invariant perturbations

- **Mukhanov-Sasaki** momentum (*removing ambiguities*):

$$\bar{\pi}_{v_{\vec{n},\pm}} = e^{-\alpha}\left[\pi_{f_{\vec{n},\pm}} + \frac{1}{\pi_\varphi}\left(e^{6\alpha}\tilde{m}^2\varphi f_{\vec{n},\pm} + 3\pi_\varphi^2 b_{\vec{n},\pm}\right)\right] - e^{-2\alpha}\left(\frac{1}{\pi_\varphi}e^{6\alpha}\tilde{m}^2\varphi + \pi_\alpha + 3\frac{\pi_\varphi^2}{\pi_\alpha}\right)v_{\vec{n},\pm}.$$

- The Mukhanov-Sasaki momentum is independent of $\left(\pi_{a_{\vec{n},\pm}}, \pi_{b_{\vec{n},\pm}}\right)$.

  The perturbative Hamiltonian constraint is independent of $\pi_{b_{\vec{n},\pm}}$.
  The perturbative momentum constraint depends through $\pi_{a_{\vec{n},\pm}} - \pi_{b_{\vec{n},\pm}}$.

- It is straightforward to complete the transformation:

$$\tilde{C}^{\vec{n},\pm}_{\uparrow 1} = 3 b_{\vec{n},\pm}, \qquad \breve{C}^{\vec{n},\pm}_{1} = -\frac{1}{\pi_\alpha}\left(a_{\vec{n},\pm} + b_{\vec{n},\pm}\right).$$

Gauge invariant perturbations

- The redefinition of the perturbative Hamiltonian constraint amounts to a **redefinition of the lapse** at our order of truncation in the action**:**

$$H=\breve{N}_0\left[H_0+\sum_{\vec{n},\pm} H_2^{\vec{n},\pm}\right]+\sum_{\vec{n},\pm} g_{\vec{n},\pm}\breve{H}_1^{\vec{n},\pm}+\sum_{\vec{n},\pm} k_{\vec{n},\pm}\widetilde{H}_{\uparrow 1}^{\vec{n},\pm},$$

$$\breve{\mathbf{N}}_{\mathbf{0}}=N_0+3e^{3\alpha}\sum_{\vec{n},\pm} g_{\vec{n},\pm}a_{\vec{n},\pm}.$$

Full system

- We now include the **zero modes** as variables of the system, and complete the **canonical** transformation.

- We re-write the Legendre term of the action, keeping its canonical form at **quadratic perturbative order**:

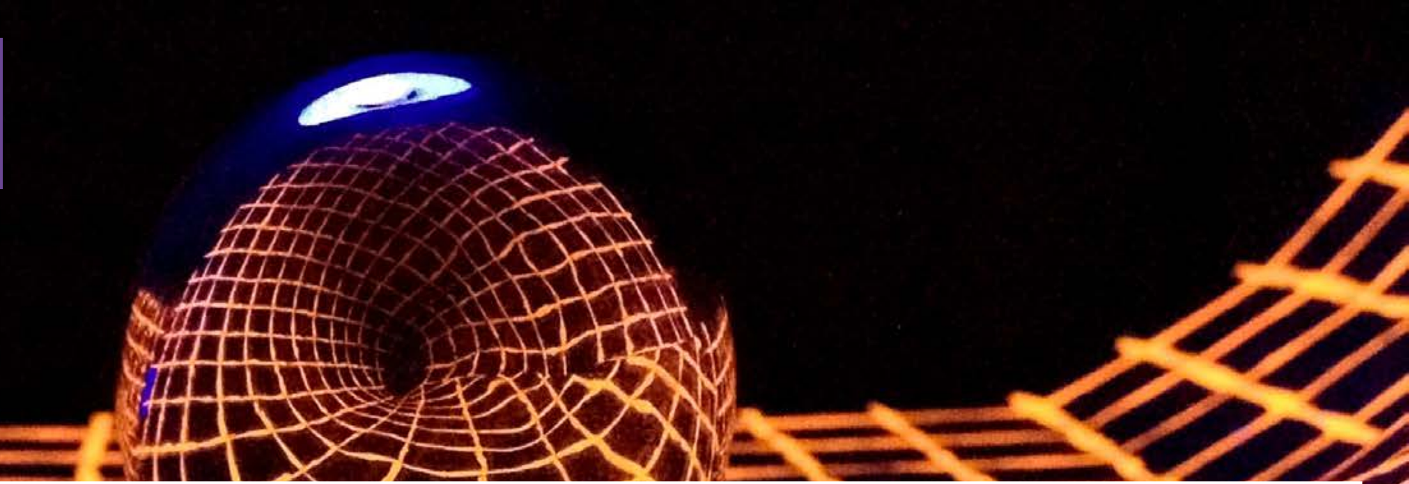
$$\int dt\left[\sum_a \dot{w}_q^a w_p^a + \sum_{l,\vec{n},\pm} \dot{X}_{q_l}^{\vec{n},\pm} X_{p_l}^{\vec{n},\pm}\right] \equiv \int dt\left[\sum_a \dot{\tilde{w}}_q^a \tilde{w}_p^a + \sum_{l,\vec{n},\pm} \dot{V}_{q_l}^{\vec{n},\pm} V_{p_l}^{\vec{n},\pm}\right].$$

- <u>Zero modes</u>: **Old** $\left\{w_q^a, w_p^a\right\} \rightarrow$ **New** $\left\{\tilde{w}_q^a, \tilde{w}_p^a\right\}.$ $\left(\left\{w_q^a\right\} = \left\{\alpha, \varphi\right\}.\right)$

- <u>Inhomogeneities</u>: **Old** $\left\{X_{q_l}^{\vec{n},\pm}, X_{p_l}^{\vec{n},\pm}\right\} \rightarrow$ **New:**

$$\left\{V_{q_l}^{\vec{n},\pm}, V_{p_l}^{\vec{n},\pm}\right\} = \left\{\left(v_{\vec{n},\pm}, \breve{C}_1^{\vec{n},\pm}, \widetilde{C}_{\uparrow 1}^{\vec{n},\pm}\right), \left(\bar{\pi}_{v_{\vec{n},\pm}}, \breve{H}_1^{\vec{n},\pm}, \widetilde{H}_{\uparrow 1}^{\vec{n},\pm}\right)\right\}.$$

Full system



- Using that the change of perturbative variables is linear, it is not difficult to find the **new zero modes**, which include modifications **quadratic in the perturbations**.

- Expressions:

$$w_q^a = \tilde{w}_q^a - \frac{1}{2} \sum_{l,\vec{n},\pm} \left[X_{q_l}^{\vec{n},\pm} \frac{\partial X_{p_l}^{\vec{n},\pm}}{\partial \tilde{w}_p^a} - \frac{\partial X_{q_l}^{\vec{n},\pm}}{\partial \tilde{w}_p^a} X_{p_l}^{\vec{n},\pm} \right],$$

$$w_p^a = \tilde{w}_p^a + \frac{1}{2} \sum_{l,\vec{n},\pm} \left[X_{q_l}^{\vec{n},\pm} \frac{\partial X_{p_l}^{\vec{n},\pm}}{\partial \tilde{w}_q^a} - \frac{\partial X_{q_l}^{\vec{n},\pm}}{\partial \tilde{w}_q^a} X_{p_l}^{\vec{n},\pm} \right].$$

$\left\{ X_{q_l}^{\vec{n},\pm}, X_{p_l}^{\vec{n},\pm} \right\} \rightarrow$ Old perturbative variables in terms of the new ones.

New Hamiltonian

- Since the change of the zero modes is **quadratic in the perturbations**, the new scalar constraint at our **truncation order** is

$$H_0(w^a)+\sum_{\vec{n},\pm} H_2^{\vec{n},\pm}\left(w^a, X_l^{\vec{n},\pm}\right)\Rightarrow$$

$$H_0(\tilde{w}^a)+\sum_b \left(w^b-\tilde{w}^b\right)\frac{\partial H_0}{\partial \tilde{w}^b}(\tilde{w}^a)+\sum_{\vec{n},\pm} H_2^{\vec{n},\pm}\left[\tilde{w}^a, X_l^{\vec{n},\pm}(\tilde{w}^a, V_l^{\vec{n},\pm})\right],$$

$$w^a-\tilde{w}^a=\sum_{\vec{n},\pm} \Delta\tilde{w}^a_{\vec{n},\pm}\,.$$

- The perturbative contribution to the new scalar constraint is:

$$\bar{H}_2^{\vec{n},\pm}=H_2^{\vec{n},\pm}+\sum_a \Delta\tilde{w}^a_{\vec{n},\pm}\frac{\partial H_0}{\partial \tilde{w}^a}.$$

New Hamiltonian

- Carrying out the calculation explicitly, one obtains:

$$\bar{H}_2^{\vec{n},\pm} = \breve{H}_2^{\vec{n},\pm} + F_2^{\vec{n},\pm} H_0 + \breve{F}_1^{\vec{n},\pm} \breve{H}_1^{\vec{n},\pm} + \left(F_{\uparrow 1}^{\vec{n},\pm} - 3\frac{e^{-3\tilde{\alpha}}}{\pi_{\tilde{\alpha}}} \breve{H}_1^{\vec{n},\pm} + \frac{9}{2} e^{-3\tilde{\alpha}} \widetilde{H}_{\uparrow 1}^{\vec{n},\pm} \right) \widetilde{H}_{\uparrow 1}^{\vec{n},\pm},$$

$$\breve{H}_2^{\vec{n},\pm} = \frac{e^{-\tilde{\alpha}}}{2} \left\{ \left[\omega_n^2 + e^{-4\tilde{\alpha}} \pi_{\tilde{\alpha}}^2 + \tilde{m}^2 e^{2\tilde{\alpha}} \left(1 + 15\tilde{\varphi}^2 - 12\tilde{\varphi}\frac{\pi_{\tilde{\varphi}}}{\pi_{\tilde{\alpha}}} - 18 e^{6\tilde{\alpha}} \tilde{m}^2 \frac{\tilde{\varphi}^4}{\pi_{\tilde{\alpha}}^2} \right) \right] (v_{\vec{n},\pm})^2 + (\bar{\pi}_{v_{\vec{n},\pm}})^2 \right\}.$$

- The F 's are well determined functions.
- The term $\breve{H}_2^{\vec{n},\pm}$ is the Mukhanov-Sasaki Hamiltonian.
- It has **no linear contributions** of the Mukhanov-Sasaki momentum.
- It is **linear in the momentum** $\pi_{\tilde{\varphi}}$.

New Hamiltonian

- We re-write the **total Hamiltonian** of the system at our **truncation order**, redefining the Lagrange multipliers:

$$\bar{H}_2^{\vec{n},\pm} = \breve{H}_2^{\vec{n},\pm} + F_2^{\vec{n},\pm} H_0 + \breve{F}_1^{\vec{n},\pm} \breve{H}_1^{\vec{n},\pm} + \left(F_{\uparrow 1}^{\vec{n},\pm} - 3 \frac{e^{-3\tilde{\alpha}}}{\pi_{\tilde{\alpha}}} \breve{H}_1^{\vec{n},\pm} + \frac{9}{2} e^{-3\tilde{\alpha}} \widetilde{H}_{\uparrow 1}^{\vec{n},\pm} \right) \widetilde{H}_{\uparrow 1}^{\vec{n},\pm} \quad \Rightarrow$$

$$H = \bar{N}_0 \left[H_0 + \sum_{\vec{n},\pm} \breve{H}_2^{\vec{n},\pm} \right] + \sum_{\vec{n},\pm} \breve{G}_{\vec{n},\pm} \breve{H}_1^{\vec{n},\pm} + \sum_{\vec{n},\pm} \widetilde{K}_{\vec{n},\pm} \widetilde{H}_{\uparrow 1}^{\vec{n},\pm} .$$

Hybrid quantization

Approximation: Quantum geometry effects are especially relevant in the background

- Adopt a **quantum cosmology** scheme for the zero modes and a **Fock quantization** for the perturbations. The scalar constraint **couples** them.

- We assume:

a) The zero modes **commute** with the perturbations under quantization.

b) Functions of $\tilde{\varphi}$ act by multiplication.

Uniqueness of the Fock description

The **Fock representation** in QFT is fixed (up to unitary equivalence) by:

1) The *background* **isometries**; **2)** The demand of a **UNITARY** evolution.

- The introduced **scaling** of the field by the scale factor is essential for unitarity.
- The proposal selects a **UNIQUE canonical pair** for the Mukhanov-Sasaki field, precisely the one we chose to fix the ambiguity in the momentum.
- We can use the massless representation (due to compactness), with its creation and annihilation operators, and the corresponding basis of occupancy number states $|N\rangle$.

Representation of the constraints

- We admit that the operators that represent the linear constraints (or an integrated version of them) act as derivatives (or as translations).

- Then, physical states are independent of $\left(\breve{C}_1^{\vec{n},\pm}, \tilde{C}_{\uparrow 1}^{\vec{n},\pm}\right).$

- We pass to a space of states that depend on the **zero modes** and the **Mukhanov-Sasaki modes**, with **no gauge fixing**.

- In this covariant construction, physical states still must satisfy the scalar constraint given by the FLRW and the Mukhanov-Sasaki contributions.

$$H_S = e^{3\tilde{\alpha}}\left(H_0 + \sum_{\vec{n},\pm} \breve{H}_2^{\vec{n},\pm}\right) = 0.$$

Born-Oppenheimer ansatz

- Consider states whose dependence on the FLRW geometry and the inhomogeneities (N) **split:**

$$\Psi=\Gamma(\tilde{\alpha},\tilde{\varphi})\psi(N,\tilde{\varphi}), \qquad \Gamma(\tilde{\alpha},\tilde{\varphi})=\hat{U}(\tilde{\alpha},\tilde{\varphi})\chi(\tilde{\alpha}).$$

- The FLRW state is **normalized.**
- $\hat{U}$ is a unitary evolution operator close to the **unperturbed** one.

Effective Mukhanov-Sasaki equations

- Using the **Born-Oppenheimer** form of the constraint (diagonal in the FLRW geometry) and assuming a direct **effective** counterpart:

$$d^2_{\eta_\Gamma} v_{\vec{n},\pm} = -v_{\vec{n},\pm}\left[4\pi^2\omega_n^2 + \langle\hat{\theta}\rangle_\Gamma\right],$$

$$[\hat{\pi}_{\tilde{\varphi}}, \hat{U}]\hat{U}^{-1} = \hat{h}_0,$$

$$\langle\hat{\theta}\rangle_\Gamma = 2\pi^2 \frac{\langle 2\hat{\vartheta}_e^q + \hat{\vartheta}_o \hat{h}_0 + \hat{h}_0 \hat{\vartheta}_o + [\hat{\pi}_{\tilde{\varphi}} - \hat{h}_0, \hat{\vartheta}_o]\rangle_\Gamma}{\langle e^{2\hat{\tilde{\alpha}}}\rangle_\Gamma},$$

$$H_0^{(2)} = \pi_{\tilde{\alpha}}^2 - e^{6\tilde{\alpha}}\tilde{m}^2\tilde{\varphi}^2,$$

$$\vartheta_o = -12 e^{4\tilde{\alpha}}\tilde{m}^2 \frac{\tilde{\varphi}}{\pi_{\tilde{\alpha}}},$$

$$\vartheta_e^q = e^{-2\tilde{\alpha}} H_0^{(2)}\left(19 - 18\frac{H_0^{(2)}}{\pi_{\tilde{\alpha}}^2}\right) + \tilde{m}^2 e^{4\tilde{\alpha}}\left(1 - 2\tilde{\varphi}^2\right),$$

where η_Γ is a **state-dependent conformal time.**

- The effective equations are of harmonic **oscillator** type, without dissipation, and **hyperbolic in the ultraviolet.**

Conclusions

- We have considered an FLRW universe with a massive scalar field perturbed at **quadratic** order in the action.
- We have found a canonical transformation for the **full system** that respects *covariance* at the perturbative level of truncation.
- We have discussed the **hybrid quantization** of the system. This can be applied to a variety of quantum FLRW cosmology approaches.
- A **Born-Oppenheimer** ansatz leads to **Mukhanov-Sasaki equations with** quantum corrections.