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Localizing Dirac field states

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Non-locality and *quantum* appear together entitling scientific projects and papers quite usually. These are two hardly detachable concepts and modern quantum theories are built up taking into account this aspect explicitly. However, that characteristic turns challenging from a fundamental point of view and can be unhelpful for practical purposes. Here, we construct quantum field theory with deliberately local aspects for a 1+1 dimensional Dirac field in flat spacetime generalizing a recently developed formalism for scalar fields. That purpose is achieved exploiting the non-uniqueness of quantization process. This local construction leads to a natural notion of local quanta and provides a local algebra of operators. As an application, it is shown that the standard vacuum state is a boiling soup of local particles and this fact is connected with a still Gedankenexperiment consisting in slamming down and removing a mirror in an empty cavity.

I. MOTIVATION AND BACKGROUND

Non-locality is a feature as intuitively challenging as genuine in the quantum world. The so-successful ordinary representation of quantum field theory (QFT) is based on the information carried by global operators (creation-annihilation operators) related to field excitations with well defined momentum, and therefore, completely delocalized in space. This picture accounts for an innumerable amount of phenomena with an astonishing accuracy, as particle physics or solid state physics shows. However, this conventional formalism can hardly deal with tasks related to local properties of quantum field states. Simple questions as "where is that particle?" or "how are these particles distributed in space?" are unsuccessfully answered in this background. What is more, it is difficult to reconcile the common understanding of a particle as a minute entity in space with the particle notion arising from the ordinary Fock construction, which rather describes excitations spreading in whole space. Furthermore, no localized states can be constructed in such QFT where occupation number representation of a quantum field can be understood as a "momentum" representation in the sense that the number of particles informs about the momentum carried by the field. The problem at this point is that it does not exist a reciprocal "position" representation, we have no local (in space) creation-annihilation operators. This result can be understood as a consequence of Hegerfeldt's theorem.

Theorem I.1. *(Hegerfeldt's theorem.) Let the operator H be self-adjoint and bounded from below in a Hilbert space $\mathcal{H}$. Let $\mathcal{O}$ be a positive operator, and let Ψ_0 be any vector in $\mathcal{H}$ such that*

$$\Psi_t = e^{-iHt}\Psi_0.$$

Then one of the following two alternatives holds:
i) $\langle\Psi_t|\mathcal{O}|\Psi_t\rangle \neq 0$ for almost all t, or
ii) $\langle\Psi_t|\mathcal{O}|\Psi_t\rangle = 0$ for all t.

The main content of the theorem, as we have just said, is that there are no localized states in standard representation of QFT. Following the original Hegerfeldt's argument[1], let us consider a field state completely localized in some finite region $\mathcal{R}$ fullfilling the starting hypothesis of the theorem. Let us call the state at any time $|\Psi_t\rangle$. Consider some operator $E_{\mathcal{R}}$ which represents the certainty of localizing a particle in $\mathcal{R}$[1]. Construct now the operator $\mathcal{O}_{\mathcal{R}} = \mathbb{1} - E_{\mathcal{R}}$. What the theorem tells us for $\mathcal{O}_{\mathcal{R}}$ is that if $\langle\Psi_0|\mathcal{O}_{\mathcal{R}}|\Psi_0\rangle = 0$ at the initial time, then,

$$\langle\Psi_t|\mathcal{O}_{\mathcal{R}}|\Psi_t\rangle \neq 0 \text{ or } \langle\Psi_t|\mathcal{O}_{\mathcal{R}}|\Psi_t\rangle = 0 \text{ for any time } t.$$

The implication i) entails superluminical propagation from region $\mathcal{R}$ to the whole space. In the second case, $\langle\Psi_t|\mathcal{O}_{\mathcal{R}}|\Psi_t\rangle = 0$ for any time t, we have that the state $|\Psi_t\rangle$ is localized in $\mathcal{R}$ eternally, i.e., $|\Psi_t\rangle$ is a stationary state of H. This discussion has been extended and formalized along last decades[2–4]. Note that the scheme depicted in this paragraph rules in the space of classical solutions with positive frequency and can be summarized saying that every superposition of positive frequency solutions to a field equation cannot be localized. That issue has a straightforward translation into the ordinary quantized field theory where the one particle Hilbert space, from which the quantum Fock space is built, is exactly the space of positive frequency solutions.

Anyhow, to make more precise the discussion shown above, we need to examine more carefully the precise notion of localization. Several localization schemes can be found in the literature but none of them is completely satisfactory. Knight introduced in [5] possibly the most natural and elegant language to treat the problem in the QFT framework. According to Knight, a state $|\Psi_{\mathcal{R}}\rangle$ is *strictly localized* within the region $\mathcal{R} \subset \mathbb{R}^3$ at some time t_0 if the expectation value of any local operator $\mathcal{O}(x)$ outside $\mathcal{R}$ (i.e. $x \notin \mathcal{R}$) is identical to that of the vacuum:

$$\langle\Psi_{\mathcal{R}}|\mathcal{O}(x)|\Psi_{\mathcal{R}}\rangle = \langle 0|\mathcal{O}(x)|0\rangle \text{ if } x \notin \mathcal{R}.$$

[1] With some additional conditions, $E_{\mathcal{R}}$ constitute what is called a localization system[3]

Knight also probed that in the usual Fock representation there are no N-particle states strictly localized with N finite. That is the precise translation of Hegerfeld's theorem to ordinary QFT. At this point we note that localizability is closely related to the election of the one particle Hilbert space, and therefore, to the definition of a particle in our theory. This observation will lead us to take advantage of the particle ambiguity to built a QFT in flat spacetime with local features built into the construction.

In this work, a recently local formalism for scalar fields in Minkowski spacetime[6], inspired by an original idea of local quanta from D. Colosi and C. Rovelli[7], is generalized to the case of a Dirac field. We deal here with the non-trivial role played by the spinorial structure of the Dirac field and its fermionic nature. We think that this work lays the basis to the generalization of this local QFT to arbitrary field representations of the Lorentz group.

The present text is structured as follows: In sec.II the basics of the 1+1-dim. Dirac field are presented, carrying out the quantization of such field confined within a cavity. In sec.III the quantization based on local modes is developed. In sec.IV we determine the transformation between both quantizations and unitary inequivalence is probed. Sec.V is devoted to the application of the formalism developed here to the study of local features of the ordinary vacuum state. The text closes with a discussion of the attained results and the possibility of experimentally checking them. As well, some open questions and potential applications are also spotted.

II. 1-D DIRAC FIELD IN A CAVITY

A. The Dirac field

Consider $\Psi(\mathbf{x},t)$ being a Dirac spinorial field, i.e., an object in the so-called representation $(\frac{1}{2},0)\oplus(0,\frac{1}{2})$ of the Lorentz group (then, Ψ is some combination of left-handed and right-handed Weyl spinors, ψ_L and ψ_R respectively). A Lorentz invariant action for this field in Minkowskian spacetime can be constructed as:

$$S^D=\int \mathrm{d}^3x\ \bar{\Psi}(i\gamma^\mu\partial_\mu-m)\Psi=\int \mathrm{d}^3x\ \bar{\Psi}(i\partial\!\!\!/-m)\Psi, \quad (1)$$

where γ^μ are the γ matrices, which have to obey the Clifford algebra:

$$\{\gamma^\mu,\gamma^\nu\}=2\eta^{\mu\nu}, \quad (2)$$

denoting by braces the anticommutator, $\{\mathcal{O}_1,\mathcal{O}_2\}=\mathcal{O}_1\mathcal{O}_2+\mathcal{O}_2\mathcal{O}_1$. Meanwhile, μ, ν are Lorentz indexes. Furthermore, $\bar{\Psi}=\Psi^\dagger\gamma^0$ is the Dirac adjoint. In this essay, natural units $\hbar=c=1$ and metric signature (+,-,-,-) will be used.

From the variational principle for the action (1) we obtain the Dirac equation:

$$\left(i\partial\!\!\!/-m\right)\Psi=0. \quad (3)$$

The probability current associated to the field Ψ, which is the Noether current associated to invariance respect $U(1)$ transformations, is:

$$j^\mu=\overline{\Psi}\gamma^\mu\Psi. \quad (4)$$

In a similar way to the K-G field, the first component of this current induces the definition of the inner product in the space of solutions, $\mathcal{S}^D$:

$$(\Psi|\Psi')=\int \mathrm{d}^3x\ \overline{\Psi}\gamma^0\Psi'=\int \mathrm{d}^3x\ \Psi^\dagger\Psi'. \quad (5)$$

This inner product is conserved through the (unitary) evolution generated by the Dirac Hamiltonian:

$$H_D=\left(-i\alpha^i\frac{\mathrm{d}}{\mathrm{d}x^i}+m\beta\right),\quad \beta=\gamma^0,\quad \alpha^i=\beta\gamma^i, \quad (6)$$

and makes it self-adjoint with respect to that inner product[8].

The spectrum of this Hamiltonian is composed by orthogonal eigenspinors which constitute a complete set in $\mathcal{S}^D$ in virtue of the spectral theorem. Then, the set of normalized eigenspinors of H_D is an orthonormal basis of the space of solutions of the Dirac equation, $\mathcal{S}^D$.

Restricting ourselfs to the 1+1 dimensional case, in a similar way as was done in [9], the Clifford algebra can be satisfied by 2×2 matrices, therefore we can choose a 2 dimensional representation for the field Ψ. Then, working in the Dirac representation, we construct the field as

$$\Psi=\frac{1}{\sqrt{2}}\begin{pmatrix}\psi_L+\psi_R\\ \psi_L-\psi_R\end{pmatrix}\equiv\begin{pmatrix}\phi\\ \chi\end{pmatrix}, \quad (7)$$

obtaining the following γ matrices

$$\gamma^0=\begin{pmatrix}1&0\\0&-1\end{pmatrix},\ \gamma^1=\begin{pmatrix}0&1\\-1&0\end{pmatrix}. \quad (8)$$

Setting this representation, independent plane wave solutions to the Dirac equation can be written as:

$$U_P(x,t)=e^{-i(\Omega t-Px)}u(P), \quad (9)$$
$$V_P(x,t)=e^{i(\Omega t-Px)}v(P), \quad (10)$$

where the normalized spinors

$$u(P)=\sqrt{\frac{\Omega+m}{2\Omega}}\begin{pmatrix}1\\ \frac{P}{\Omega+m}\end{pmatrix},\ v(P)=\sqrt{\frac{\Omega+m}{2\Omega}}\begin{pmatrix}\frac{P}{\Omega+m}\\ 1\end{pmatrix},$$

make (9), (10) eigenstates of the Hamiltonian (6).

Here, the wave functions (9), (10) corresponds to solutions with energy Ω, $-\Omega$ respectively, with $\Omega=\sqrt{P^2+m^2}$. This distinction between positive and negative energy solutions at classical level leads, after quantization, to the ordinary interpretation of the field in terms of particles and antiparticles.

The spinors u, v fulfill the relations:

$$u^\dagger(P)u(P) = 1, \qquad v^\dagger(P)v(P) = 1,$$
$$u^\dagger(P)v(-P) = 0. \tag{11}$$

Therefore, any solution to the Dirac equation, $\Psi \in \mathcal{S}^D$, can be spanned in terms of (9) and (10):

$$\Psi = N \int_{-\infty}^{\infty} \mathrm{d}P \;\; (a(P)U_P + b^*(P)V_P)\,, \tag{12}$$

where N is a normalization constant and $a(P)$, $b^*(P)$ are complex functions playing the role of expansion coefficients.

To close this preliminar section, let us point out that the notation introduced in (5) for the inner product encourages to denote vectors in the Hilbert space $\mathcal{S}^D$ as $|\Psi)$. In the same way, elements $(\Psi|$ can be seen as forms in our Hilbert space.

B. Stationary solutions in a cavity

1. Boundary conditions

Now we turn into finding the eigenspinors of the Dirac Hamiltonian for a Dirac field constrained to a cavity, in our case, a segment of the real axis, $\mathcal{C} = (0, R)$.

The naivë choice of Dirichlet conditions at the boundary of $\mathcal{C}$:

$$\Psi(x = 0, t) = 0 = \Psi(x = R, t), \tag{13}$$

leads uniquely to the trivial solution $\Psi(x, t) = 0$ [10, 11]. It is easy to understand this situation. The Dirac equation (in one-dimensional case) is composed by two coupled first order equations for two fields (the two components of Ψ), and imposing the vanishing values of both fields and their stationarity is too restrictive. Thus, this situation lacks of any physical interest and we have to consider more general localization schemes.

Then, the physical criterion to describe a Dirac field in a cavity will consist in imposing the vanishing value of the *probability* current through the boundaries:

$$j^1(x = 0, t) = 0 = j^1(x = R, t), \tag{14}$$

where,

$$j^1(x, t) = \overline{\Psi}(x, t)\gamma^1\Psi(x, t). \tag{15}$$

With the aim of clarifying the starting point as much as possible, we begin with a general analysis of the compatibility of the boundary conditions with the Dirac Hamiltonian (6).

In order to obtain a meaningful description of the system, the self-adjointness of the Hamiltonian is mandatory. Now, it is crucial to point out that the Hamiltonian H_D is defined by (6) but also by its domain. Then, we define the Hamiltonian of the system by (6) acting in the following domain[2] :

$$\mathcal{D}_{H_D} = \left\{ \Psi \in \mathcal{S}^D \quad \Big| \quad \begin{pmatrix} \psi_L(R) \\ \psi_R(0) \end{pmatrix} = \Lambda \begin{pmatrix} \psi_R(R) \\ \psi_L(0) \end{pmatrix} \right\}, \tag{16}$$

where Λ is a 2×2 matrix which encodes the most general boundary conditions (b.c., from now on). In virtue of the Von Neumman's theory of self-adjoint operator extensions[3], this matrix Λ must be unitary, and then, we can parameterize it as follows [11, 13]:

$$\Lambda = \begin{pmatrix} e^{i\mu}e^{i\tau}\cos\theta & e^{i\mu}e^{i\gamma}\sin\theta \\ e^{i\mu}e^{-i\gamma}\sin\theta & -e^{i\mu}e^{i\tau}\cos\theta \end{pmatrix},$$

with $0 \le \theta < \pi$, $0 \le \mu,\ \tau,\ \gamma < 2\pi$.

Now, the b.c. (14) translates into $\theta = 0$. We see that we have a whole family of solutions for the b.c. which are compatible with our picture of a field constrained in a cavity.

Nevertheless, there is one preferred choice, consisting in imposing at the boundary, ∂, the following condition:

$$(1 + i\mathbf{n}\cdot\gamma)\,\Psi|_\partial = 0, \tag{17}$$

where $\mathbf{n}$ is the normal vector to the boundary. (Explicitly, this choice correspond to $\mu = 0$, $\tau = \pi/2$ and $0 \le \gamma \le 2\pi$.)

In our case, that means:

$$\Psi|_{x=0} = i\gamma^1\Psi|_{x=0}, \qquad \Psi|_{x=R} = -i\gamma^1\Psi|_{x=R}. \tag{18}$$

The conditions (17) are the b.c. derived from the M.I.T. bag model, which originally was developed as a hadronic model in the 1970's[14]. Those b.c. can be seen as the result of coupling the free field in the cavity with an infinite massive field outside. In that sense, they are analogue to the Dirichlet b.c. in non-relativistic quantum mechanics. On the other hand, the CPT symmetry[4] of the Dirac field is respected by (17) (which is not true in general [11]), which is a very desirable fact from a fundamental point of view. Furthermore, this model is widely used to describe a Dirac field in a finite region in the Casimir community [15, 16] and in relativistic quantum information[17].

2. Modes in the cavity

Now, let us construct the orthonormal stationary modes in our cavity, i.e., let us solve the eigenproblem

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[2] Here, the use of Weyl representation is motivated only by simplicity, compactness and clarity. In any case, once the b.c. are known in Weyl representation, the translation into Dirac representation is immediate.

[3] For a pedagogical introduction to this subject see e.g. [13]. In [12] it is found a more profound treatment.

[4] Considering parity transformations with the center of the cavity as symmetry axes.

for the Dirac hamiltonian (6) in the domain:

$$\mathcal{D}_{H_D} = \left\{\Psi \in \mathcal{S}^D \quad | \quad (1 + i\mathbf{n}\cdot\gamma)\,\Psi|_\partial = 0\right\} \equiv \mathcal{S}^D_{\mathcal{C}}. \quad (19)$$

In this case, the boundaries ∂ are the edges of our cavity $\mathcal{C}$, i.e., $x = 0,\ R$.

Modes with definite positive frequency Ω_P are constructed as the linear combination:

$$\Psi^{(+)}_P(x,t) = A_+ U_P(x,t) + B_+ U_{-P}(x,t), \quad (20)$$

while negative energy modes are:

$$\Psi^{(-)}_P(x,t) = A_- V_P(x,t) + B_- V_{-P}(x,t), \quad (21)$$

being $A_\pm,\ B_\pm$ some complex coefficients, determined by the b.c..

Explicitly, imposing the b.c. (17) on these modes, we obtain:

$$\Psi^{(+)}_P|_{x=0} = i\gamma^1 \Psi^{(+)}_P|_{x=0} \ \Rightarrow\ B_+ = -A_+ \frac{1 - \frac{iP}{\Omega+m}}{1 + \frac{iP}{\Omega+m}},$$

$$\Psi^{(-)}_P|_{x=0} = i\gamma^1 \Psi^{(-)}_P|_{x=0} \ \Rightarrow\ B_- = -A_- \frac{1 + \frac{iP}{\Omega+m}}{1 - \frac{iP}{\Omega+m}},$$

and

$$\Psi^{(\pm)}_P|_{x=R} = -i\gamma^1 \Psi^{(\pm)}_P|_{x=R} \ \Rightarrow\ \tan(PR) = -\frac{P}{m}.$$

Therefore, modes in the cavity have the following structure:

$$\Psi^{(+)}_N(x,t) = (1 + \frac{iP_N}{\Omega+m})U_{P_N}(x,t) - (1 - \frac{iP_N}{\Omega+m})U_{-P_N}(x,t),$$

$$\Psi^{(-)}_N(x,t) = (1 - \frac{iP_N}{\Omega+m})V_{P_N}(x,t) - (1 + \frac{iP_N}{\Omega+m})V_{-P_N}(x,t),$$

where the discrete spectrum $\{P_N\}$ is determined by the solutions to the trascendental equation:

$$\tan(P_N R) = -\frac{P_N}{m}. \quad (22)$$

After normalizing these modes with respect to the inner product (5) we finally can write:

$$\Psi^{(+)}_N(x,t) = \sqrt{\frac{\Omega_N^2}{2R(\Omega_N^2 + m/R)}}\, e^{-i\Omega_N t}\left(e^{i(P_N x + \Delta_N)}\, u(P_N) - e^{-i(P_N x + \Delta_N)}\, u(-P_N)\right), \quad (23)$$

$$\Psi^{(-)}_N(x,t) = \sqrt{\frac{\Omega_N^2}{2R(\Omega_N^2 + m/R)}}\, e^{i\Omega_N t}\left(e^{-i(P_N x + \Delta_N)}\, v(P_N) - e^{i(P_N x + \Delta_N)}\, v(-P_N)\right), \quad (24)$$

where we have defined:

$$\Delta_N = \arctan\left(\frac{P_N}{\Omega_N + m}\right). \quad (25)$$

Now, the orthonormality relations:

$$(\Psi^{(+)}_N|\Psi^{(+)}_{N'}) = \delta_{NN'}, \quad (\Psi^{(-)}_N|\Psi^{(-)}_{N'}) = \delta_{NN'}$$
$$(\Psi^{(+)}_N|\Psi^{(-)}_{N'}) = 0, \quad (26)$$

can be directly checked.

With the decomposition of the identity

$$\mathbb{1} = \sum_N |\Psi^{(+)}_N)(\Psi^{(+)}_N| + |\Psi^{(-)}_N)(\Psi^{(-)}_N|, \quad (27)$$

they built up an orthonormal basis[5] of $\mathcal{S}^D_{\mathcal{C}}$. Thus, any solution to the Dirac equation in the cavity can be spanned as:

$$\Psi(x,t) = \sum_{N=1}^{\infty} A_N \Psi^{(+)}_N(x,t) + B^*_N \Psi^{(-)}_N(x,t). \quad (28)$$

The plus sign in last two expressions is quite remarkable because, as we will see, it encodes the Fermi-Dirac statistics of the system. Its presence is directly due to the positivity of the inner product in $\mathcal{S}^D$, (5).

[5] As we have just noted before, the orthogonality is guaranteed by the self-adjointness of H_D and the completeness follows from the spectral theorem.

C. Quantization

Now, we can proceed to the quantization of the theory. This issue is carried out building an antisymmetric Fock space from the one particle Hilbert space, $\mathfrak{H}$. Then, first of all, we have to isolate in $\mathcal{S}^D_{\mathcal{C}}$ this one particle Hilbert space. With this aim, we introduce the complex structure

$$\mathfrak{J} = i\sum_N |\Psi^{(+)}_N)(\Psi^{(+)}_N| - |\Psi^{(-)}_N)(\Psi^{(-)}_N|, \quad (29)$$

which is a linear map in $\mathcal{S}_{\mathcal{C}}^D$ satisfying $\mathfrak{J}^2 = -\mathbb{1}$ and, therefore, with eigenvalues i and $-i$. This fact allows us to decompose $\mathcal{S}_{\mathcal{C}}^D$ in the eigenspaces of $\mathfrak{J}$, i.e., to understand the solutions space as $\mathcal{S}_{\mathcal{C}}^D = \mathfrak{H} \oplus \mathfrak{H}^*$. Those eigenspaces are identified as the one particle and antiparticle Hilbert spaces, formally, $\mathfrak{H} := Span[\{\Psi_N^{(+)}\}]$ and $\mathfrak{H}^* := Span[\{\Psi_N^{(-)}\}]$. Let us note that this separation is the natural decomposition in positive and negative frequency modes.

Once the solutions space is splitted as we have just seen, the quantum states space is the so called antisymmetric Fock space[18]:

$$\mathfrak{F}^G = \left(\bigoplus_{N=0}^{\infty} \bigotimes_{A}^{N} \mathfrak{H}\right) \otimes_A \left(\bigoplus_{N=0}^{\infty} \bigotimes_{A}^{N} \mathfrak{H}^*\right). \tag{30}$$

The subindex A refers to the fact that we are considering the antisymmetrized tensor product.

In that space, the field Ψ is promoted into an operator. Then, following the decomposition (28), we arrive to:

$$\widehat{\Psi}(x,t) = \sum_N \widehat{A}_N \Psi_N^{(+)}(x,t) + \widehat{B}_N^\dagger \Psi_N^{(-)}(x,t). \tag{31}$$

Henceforth, the hat over operators will be omitted when there will not be confusion.

At this point, the canonical quantization recipe orders to impose the canonical anticommutators:

$$\{\Psi(x,t), \Psi^\dagger(x',t)\} = \delta(x-x'). \tag{32}$$

These anticommutation relations are satisfied only if:

$$\{A_N, A_M^\dagger\} = \delta_{NM}, \quad \{B_N, B_M^\dagger\} = \delta_{NM}. \tag{33}$$

From that algebra follows the customary creation-annihilation interpretation of the operators A_N, $A_N^\dagger$, B_N, $B_N^\dagger$.

Now, the vacuum state $|0^G\rangle$ is defined as the unique element in $\mathfrak{F}^G$ annihilated by every annihilation operators:

$$A_N|0^G\rangle = 0,\ B_N|0^G\rangle = 0 \quad \forall N. \tag{34}$$

The field states in the Fock space are constructed as:

$$|n_1, n_2, \ldots, \bar{n}_1, \bar{n}_2, \ldots\rangle = \prod_M (A_M^\dagger)^{n_M} (B_M^\dagger)^{\bar{n}_M} |0^G\rangle, \tag{35}$$

where n_M, $\bar{n}_M$ only can take the values 1 or 0 because of the anticommutation relations (33). That means we have at most one particle and one antiparticle for each frequency (recall that in one dimension there are no different spin orientations). That is the well known Fermi-Dirac statistics.

In that sense, the state $|0,0,1,0,\ldots\rangle$, for example, corresponds to a single particle in the cavity with energy Ω_3, while $|0,\ldots,\bar{0},\bar{1},\bar{0},\ldots\rangle$ corresponds to an antiparticle with energy Ω_2. In general, the quantum Hamiltonian of the system can be expressed as:

$$H = \sum_N \Omega_N \left(A_N^\dagger A_N + B_N^\dagger B_N\right), \tag{36}$$

where the normal ordering prescription was considered in order to renormalize the infinite vacuum energy.

Let us point out that the one particle Hilbert space was built by modes with definite momenta and therefore, completely delocalized. Then, the resulting concept of a particle is related to a field excitation completely *delocalized* in the cavity. In the next section, a local notion of a particle will be developed.

III. LOCAL DIRAC FIELD

The theory exposed up to this line gives us a global notion of particles with the associated creation-annihilation operators, and then, does not allow the existence of localizad states in its formal body. Actually, it provides very few tools to deal with any notion of local operations. For example, what does this formalism tell us about what is happening in a portion of the cavity?

Imagine splitting the cavity in two pieces, $[0,r]$ and $[r,R]$. Then, in completely analogy with (23), (24), in each subcavity we will leave with an orthonormal basis of stationary modes: $\{\psi_n^{(+)}, \psi_n^{(-)}\}$ in the left side and $\{\tilde{\psi}_n^{(+)}, \tilde{\psi}_n^{(-)}\}$ in the right side.

Turning again into the study of the entire cavity, we can draw inspiration from the stationary modes $\{\psi_n^{(+)}, \psi_n^{(-)}\}$ to define the (*non-stationary*) modes which are the solution to the Cauchy problem defined by the following initial conditions:

$$\psi_n^{(+)}(x,t=0) = \sqrt{\frac{\omega_n^2}{2r(\omega_n^2 + \frac{m}{r})}} \left(e^{i(p_n x + \delta_n)}\ u(p_n) - e^{-i(p_n x + \Delta_n)}\ u(-p_n)\right) \Theta(r-x), \tag{37}$$

$$\psi_n^{(-)}(x,t=0) = \sqrt{\frac{\omega_n^2}{2r(\omega_n^2 + \frac{m}{r})}} \left(e^{-i(p_n x + \delta_n)}\ v(p_n) - e^{i(p_n x + \Delta_n)}\ v(-p_n)\right) \Theta(r-x), \tag{38}$$

where now p_n are the solutions to

$$\tan(p_n r) = -\frac{p_n}{m}, \tag{39}$$

and

$$\delta_n \equiv \arctan\frac{p_n}{\omega_n + m}, \qquad \omega_n \equiv \sqrt{p_n^2 + m^2}. \tag{40}$$

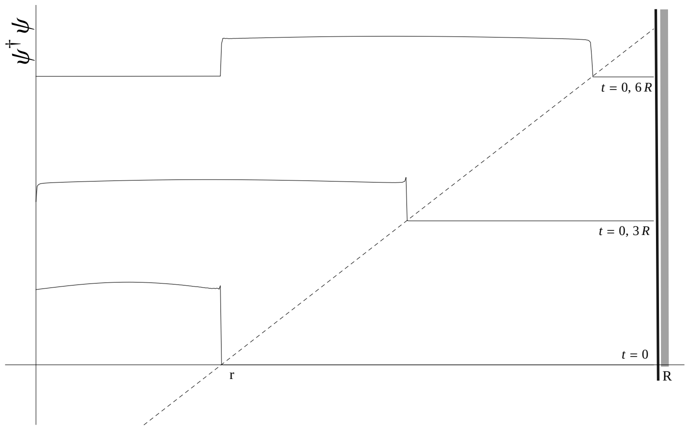

FIG. 1. Causal evolution of local modes. Classical density $\psi_1^{(\pm)\dagger}\psi_1^{(\pm)}$ in the case with $m = 0.5R$, $r = 0.3R$. The three plots correspond to three different times, $t = 0$, $t = 0.3R$, $t = 0.6R$, while the dotted line is the light cone. The mixing of both positive and negative frequencies allowed us to build up a localized mode spreading causally, avoiding the non- causal infinite tails that Hegerfeldt's theorem would imply.

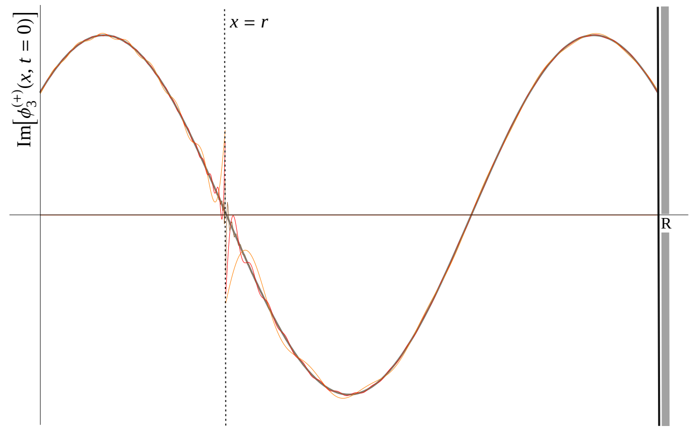

FIG. 2. Expansion of the mode $\Psi_3^{(+)}$ in the local basis. Sum of the first fifteen (in orange), fifty (in red) and two hundred (in brown, hardly distinguishable from the actual curve, in gray) terms of the expansion for the first component.

The modes defined above are completely localized in $[0,\ r]$ at the initial time and then, they spread in the cavity. Those modes at arbitrary time can be computed by means of the expansion in terms of the well known stationary modes $\{\Psi_N^{(+)}, \Psi_N^{(-)}\}$:

$$\psi_n^{(\pm)}(x,t) = \sum_N (\Psi_N^{(+)}|\psi_n^{(\pm)})\Psi_N^{(+)}(x,t) + (\Psi_N^{(-)}|\psi_n^{(\pm)})\Psi_N^{(-)}(x,t), \tag{41}$$

$$\tilde{\psi}_n^{(\pm)}(x,t) = \sum_N (\Psi_N^{(+)}|\tilde{\psi}_n^{(\pm)})\Psi_N^{(+)}(x,t) + (\Psi_N^{(-)}|\tilde{\psi}_n^{(\pm)})\Psi_N^{(-)}(x,t). \tag{42}$$

The explicit expressions for the coefficients $(\Psi_N^{(\pm)}|\psi_n^{(\pm)})$, $(\tilde{\Psi}_N^{(\pm)}|\psi_n^{(\pm)})$ appear in section IV, expressions (58–61). This solves the Cauchy problem posed by (37), (38). Numerical evaluation of the evolution of these modes is shown in figure 1. We see how the constructed modes spread causally inside the light cone. Actually, they maintain their local nature through their evolution. Let us point out that any other modes built from exclusively positive frequency solutions would spread instantaneously according to Hegerfeldt's theorem.

Similar modes can be defined for the right side of the cavity, $\{\tilde{\psi}_n^{(+)}, \tilde{\psi}_n^{(-)}\}$.

Now, we point out that at the initial time, $\{\psi_n^{(+)}, \psi_n^{(-)}\}$ span the space of spinors subjected to the b.c. (17) at the boundaries $\partial = \{x = 0,\ x = r\}$. Let us call this space $\mathcal{S}^D_{left}$. The same occurs for $\{\tilde{\psi}_n^{(+)}, \tilde{\psi}_n^{(-)}\}$ at $x = r$, $x = R$. We shall say that they span the space $\mathcal{S}^D_{right}$. Hence, observe that for any spinor $\Psi(x, t = 0)$ defined in the entire interval $[0, R]$, $\Psi \in \mathcal{S}^D_{\mathcal{C}}$, there exists a succession of spinors in $\mathcal{S}^D_{left}$ which converges pointwise to Ψ at every point $x \in [0, r)$. The same can be argued in $(r, R]$. Thus, we can construct a linear combination of $\{\psi_n^{(+)}, \psi_n^{(-)}, \tilde{\psi}_n^{(+)}, \tilde{\psi}_n^{(-)}\}$ which converges almost everywhere in $[0, R]$ to any $\Psi \in \mathcal{S}^D_{\mathcal{C}}$ at the initial time. That means we can construct any initial condition in $\mathcal{S}^D_{\mathcal{C}}$, and then, any solution to the Dirac equation in the cavity $\mathcal{C}$. What we are saying is that $\{\psi_n^{(+)}, \psi_n^{(-)}, \tilde{\psi}_n^{(+)}, \tilde{\psi}_n^{(-)}\}$ is another orthonormal basis of $\mathcal{S}^D_{\mathcal{C}}$ with pointwise convergence almost everywhere.

So, we can also span any solution $\Psi \in \mathcal{S}^D_{\mathcal{C}}$ as:

$$\Psi(x,t) = \sum_n (\psi_n^{(+)}|\Psi)\psi_n^{(+)}(x,t) + (\psi_n^{(-)}|\Psi)\psi_n^{(-)}(x,t) + (\tilde{\psi}_n^{(+)}|\Psi)\tilde{\psi}_n^{(+)}(x,t) + (\tilde{\psi}_n^{(-)}|\Psi)\tilde{\psi}_n^{(-)}(x,t), \tag{43}$$

having now the following completeness relation:

$$\mathbb{1} = \sum_n |\psi_n^{(+)})(\psi_n^{(+)}| + |\psi_n^{(-)})(\psi_n^{(-)}| + |\tilde{\psi}_n^{(+)})(\tilde{\psi}_n^{(+)}| + |\tilde{\psi}_n^{(-)})(\tilde{\psi}_n^{(-)}|. \tag{44}$$

A. Local quantization

From the representation of the solution space constructed above, follows another possible quantization of the system. The splitting of $\mathcal{S}^D_{\mathcal{C}}$ into the one particle and antiparticle Hilbert spaces, $\mathfrak{H}^L$, $\mathfrak{H}^{*L}$, is carried out in that case by means of the complex structure:

$$\mathfrak{J} = i\sum_n |\psi_n^{(+)})(\psi_n^{(+)}| - |\psi_n^{(-)})(\psi_n^{(-)}| + |\tilde{\psi}_n^{(+)})(\tilde{\psi}_n^{(+)}| - |\tilde{\psi}_n^{(-)})(\tilde{\psi}_n^{(-)}|, \tag{45}$$

yielding to another Fock space:

$$\mathfrak{F}^L = \left(\bigoplus_{n=0}^{\infty}\bigotimes_{A}^{n}\mathfrak{H}^L\right)\otimes_A\left(\bigoplus_{n=0}^{\infty}\bigotimes_{A}^{n}\mathfrak{H}^{L*}\right). \quad (46)$$

Now, $\mathfrak{H}^L = \mathfrak{H}\otimes\tilde{\mathfrak{H}}$ and $\mathfrak{H}^{*L} = \mathfrak{H}^*\otimes\tilde{\mathfrak{H}}^*$, denoting

$$\mathfrak{H} = Span[\{\psi_n^{(+)}\}], \quad \mathfrak{H}^* = Span[\{\psi_n^{(-)}\}]$$
$$\tilde{\mathfrak{H}} = Span[\{\tilde{\psi}_n^{(+)}\}], \quad \tilde{\mathfrak{H}}^* = Span[\{\tilde{\psi}_n^{(-)}\}].$$

In this *local* Fock space, the field *operator* Ψ is expresed as:

$$\Psi(x,t) = \sum_n a_n\psi_n^{(+)}(x,t) + b_n^\dagger\psi_n^{(-)}(x,t)$$
$$\tilde{a}_n\tilde{\psi}_n^{(+)}(x,t) + \tilde{b}_n^\dagger\tilde{\psi}_n^{(-)}(x,t) \quad (47)$$

providing another set of creation-annihilation operators $\{a_n^\dagger, a_n, b_n^\dagger, b_n, \tilde{a}_n^\dagger, \tilde{a}_n, \tilde{b}_n^\dagger, \tilde{b}_n\}$ satisfying the anticommutation relations:

$$\{a_n, a_m^\dagger\} = \delta_{nm},\ \{b_n, b_m^\dagger\} = \delta_{nm},$$
$$\{\tilde{a}_n, \tilde{a}_m^\dagger\} = \delta_{nm},\ \{\tilde{b}_n, \tilde{b}_m^\dagger\} = \delta_{nm}. \quad (48)$$

Other anticommutators vanish (the commutation between operator related to different sides of the cavity also anticommutes by construction).

In the Fock space constructed now, the vacuum state is the state satisfying:

$$a_n|0^L\rangle = 0,\ b_n|0^L\rangle,\ \tilde{a}_n|0^L\rangle = 0,\ \tilde{b}_n|0^L\rangle\ \forall n, \quad (49)$$

and can be seen as the product of vacuum states in each side of the cavity, $|0^L\rangle = |0^L_{left}\rangle\otimes|0^L_{right}\rangle$.

The states in $\mathfrak{F}^L$ are obtained from the vacuum in a similar way to (35). Notice that now the particle notion is associated to one of the both sides of the cavity, it is an actual *local* quanta. Indeed, exploring the local one particle state:

$$|1_m^L\rangle := a_m^\dagger|0^L\rangle, \quad (50)$$

we now can see that this is actually a *strictly localized* state. First of all, at the initial time we see that a local operator $\tilde{\mathcal{O}}$ acting on the region $[r,R]$ is build up by a series expansion of the operators $\{\tilde{a}_n,\ \tilde{a}_n^\dagger, \tilde{b}_n,\ \tilde{b}_n^\dagger\}$, that is, $\tilde{\mathcal{O}} = \tilde{\mathcal{O}}(\tilde{a}_n, \tilde{a}_n^\dagger, \tilde{b}_n,\ \tilde{b}_n^\dagger)$. Then, the anticommutation of a_m, $a_m^\dagger$ with the set $\{\tilde{a}_n,\ \tilde{a}_n^\dagger, \tilde{b}_n,\ \tilde{b}_n^\dagger\}$ guarantees the strict localization of $|1_m^L\rangle$ as follows. First, let see that for any product of the later set of operators, $\mathcal{P}$, we have $a_m\mathcal{P} = (-1)^{\#_\mathcal{P}}\mathcal{P}a_m$, where $\#_\mathcal{P}$ is the power of the product $\mathcal{P}$. When calculating the average on the vacuum $|0^L\rangle$, only terms with even powers of $\{\tilde{a}_n,\ \tilde{a}_n^\dagger, \tilde{b}_n,\ \tilde{b}_n^\dagger\}$ give rise non-vanishing contributions, and for that terms, $(-1)^{\#_\mathcal{P}} = 1$. Then, we learn that:

$$\langle 1_m^L|\mathcal{O}|1_m^L\rangle = \langle 0^L|\mathcal{O}a_m a_m^\dagger|0^L\rangle = \langle 0^L|\mathcal{O}|0^L\rangle,$$

and $|1_m^L\rangle$ is in fact a strict localized state in the sense introduced in section I.

The causal spreading of the local modes depicted in figure 1 also imply that this state remains strictly localized in the light-cone $[0, r+t]$.

IV. RELATING LOCAL AND GLOBAL DESCRIPTIONS

Once local formalism was successfully developed, it is specially interesting to explore the precise relation of our local representation of QFT with the standard representation constructed in sectionII.

We begin spanning a general solution $\Psi\in\mathcal{S}_\mathcal{C}^D$ in both orthonormal basis we have constructed in $\mathcal{S}_\mathcal{C}^D$:

$$\Psi(x,t) = \sum_N(\Psi_N^{(+)}|\Psi)\Psi_N^{(+)}(x,t) + (\Psi_N^{(-)}|\Psi)\Psi_N^{(-)}(x,t)$$
$$= \sum_n(\psi_n^{(+)}|\Psi)\psi_n^{(+)}(x,t) + (\psi_n^{(-)}|\Psi)\psi_n^{(-)}(x,t)$$
$$+ (\tilde{\psi}_n^{(+)}|\Psi)\tilde{\psi}_n^{(+)}(x,t) + (\tilde{\psi}_n^{(-)}|\Psi)\tilde{\psi}_n^{(-)}(x,t). \quad (51)$$

Both basis are related by a Bogoliubov transformation[6]:

$$(\psi_n^{(+)}|\Psi) \equiv a_n = \sum_N(\psi_n^{(+)}|\Psi_N^{(+)})A_N + (\psi_n^{(+)}|\Psi_N^{(-)})B_N^*, \quad (52)$$

$$(\psi_n^{(-)}|\Psi) \equiv b_n^* = \sum_N(\psi_n^{(-)}|\Psi_N^{(-)})B_N^* + (\psi_n^{(-)}|\Psi_N^{(+)})A_N, \quad (53)$$

$$(\tilde{\psi}_n^{(+)}|\Psi) \equiv \tilde{a}_n = \sum_N(\tilde{\psi}_n^{(+)}|\Psi_N^{(+)})A_N + (\tilde{\psi}_n^{(+)}|\Psi_N^{(-)})B_N^*, \quad (54)$$

$$(\tilde{\psi}_n^{(-)}|\Psi) \equiv \tilde{b}_n^* = \sum_N(\tilde{\psi}_n^{(-)}|\Psi_N^{(-)})B_N^* + (\tilde{\psi}_n^{(-)}|\Psi_N^{(+)})A_N, \quad (55)$$

being $A_N \equiv (\Psi_N^{(+)}|\Psi)$, $B_N^* \equiv (\Psi_N^{(-)}|\Psi)$ as in (28). The above relations are translated directly into the quantum domain with the subtlety that now a, $b^\dagger$ acts in a different Fock space than A, $B^\dagger$. Being more specific, in the quantum theory, the Bogoliubov transformation determined by (52–55) is a map between two different Fock representations:

$$\mathfrak{B}:\ \mathfrak{F}^G \longrightarrow \mathfrak{F}^L.$$

Then, a state in $\mathfrak{F}^G$, $|\Psi\rangle$, is mapped in $\mathfrak{F}^L$ as $\mathfrak{B}|\Psi\rangle$. Meanwhile, linear operators in $\mathfrak{F}^G$, $\mathcal{O}$, are mapped in linear operators in $\mathfrak{F}^L$. In matrix language, $\mathcal{O} \longrightarrow \mathfrak{B}\mathcal{O}\mathfrak{B}^{-1}$.

The transformation is completely characterized by the Bogoliubov coefficients

$$\Upsilon_{\epsilon_1 n,\epsilon_2 N} = (\psi_n^{(\epsilon_1)}|\Psi_N^{(\epsilon_2)}), \quad (56)$$
$$\tilde{\Upsilon}_{\epsilon_1 n,\epsilon_2 N} = (\tilde{\psi}_n^{(\epsilon_1)}|\Psi_N^{(\epsilon_2)}), \quad (57)$$

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[6] More formally, a Bogoliubov transformation is a transformation which preserves the classical symplectic structure, which is translated in the preservation of the canonical anticommutation (commutation for bosonic fields) relations.

with $\epsilon_i = +$ or $-$.
There, $\Upsilon_{-n,+N}$, $\Upsilon_{+n,-N}$ are β-type coefficients which relates annihilators with creators, while $\Upsilon_{+n,+N}$, $\Upsilon_{-n,-N}$ are said to be α-type.

After some tedious algebraic gymnastics one is led to the following expressions for those coefficients:

$$\Upsilon_{+n,+N} = \quad \mathrm{C}_{nN}\left[\frac{p_n\Omega_N \sin P_N r \cos p_n r - P_N\omega_n \cos P_N r \sin p_n r}{\Omega_N - \omega_n} + m \sin P_N r \sin p_n r\right] = [\Upsilon_{-n,-N}]^*, \tag{58}$$

$$\Upsilon_{+n,-N} = -i\mathrm{C}_{nN}\left[\frac{p_n\Omega_N \sin P_N r \cos p_n r + P_N\omega_n \cos P_N r \sin p_n r}{\Omega_N + \omega_n} + m \sin P_N r \sin p_n r\right] = [\Upsilon_{-n,+N}]^*, \tag{59}$$

and similarly, for the Bogoliubov coefficients related with the right partition of the cavity:

$$\tilde{\Upsilon}_{+n,+N} = \quad \mathrm{C}_{nN}\left[\frac{p_n\Omega_N\left(\sin P_N R \cos p_n(R-r) - \sin P_N r\right) - P_N\omega_n \cos P_N R \sin p_n(R-r)}{\Omega_N - \omega_n} + m \sin P_N R \sin p_n(R-r)\right]$$
$$= [\tilde{\Upsilon}_{-n,-N}]^*, \tag{60}$$

$$\tilde{\Upsilon}_{+n,-N} = -i\mathrm{C}_{nN}\left[\frac{p_n\Omega_N\left(\sin P_N R \cos p_n(R-r) - \sin P_N r\right) + P_N\omega_n \cos P_N R \sin p_n(R-r)}{\Omega_N + \omega_n} + m \sin P_N R \sin p_n(R-r)\right]$$
$$= [\tilde{\Upsilon}_{-n,+N}]^*, \tag{61}$$

where it was defined

$$\mathrm{C}_{nN} = \sqrt{\frac{1}{rR(\omega_n^2 + m/r)(\Omega_N^2 + m/R)}}.$$

The orthonormality of local and global modes implies that these coefficients have to satisfy the conditions:

$$\sum_N \Upsilon_{+n+N}\Upsilon^*_{+m+N} + \Upsilon_{+n-N}\Upsilon^*_{+m-N} = \delta_{nm}, \tag{62}$$

$$\sum_N \Upsilon_{+n+N}\Upsilon^*_{-m+N} + \Upsilon_{+n-N}\Upsilon^*_{-m-N} = 0, \tag{63}$$

$$\sum_n \Upsilon_{+n+N}\Upsilon^*_{+n+M} + \Upsilon_{-n+N}\Upsilon^*_{-n+M} = \delta_{nm}, \tag{64}$$

$$\sum_n \Upsilon_{+n+N}\Upsilon^*_{+n-M} + \Upsilon_{-n+N}\Upsilon^*_{-n-M} = 0. \tag{65}$$

In the scalar case, the analog to these expressions carries a minus sign, again, due to the non-positivity of the K-G inner product.

A. Unitary inequivalence

The first task one could wonder about at this point, is if two different quantizations describe the same physical system. If the map $\mathfrak{B} : \mathfrak{F}^G \to \mathfrak{F}^L$ which relates both representations is an unitary map, every observable will take the same value in both quantizations and then they will predict the same physical consequences[19]. In that case it is said that both quantizations are *unitary equivalent*.

A necessary condition for the unitarity of the Bogoliubov transformation between both Fock spaces is given by[17]:

$$\sum_{n,N} |\Upsilon_{+n,-N}|^2 + |\tilde{\Upsilon}_{+n,-N}|^2 < \infty. \tag{66}$$

Inspecting (59) and (61) we can conclude that row and column series are both convergent, i.e,

$$\sum_n |\Upsilon_{+n,-N}|^2 + |\tilde{\Upsilon}_{+n,-N}|^2 < \infty, \tag{67a}$$

$$\sum_N |\Upsilon_{+n,-N}|^2 + |\tilde{\Upsilon}_{+n,-N}|^2 < \infty. \tag{67b}$$

This fact can be seen analyzing the behaviour of $|\Upsilon_{+n,-N}|^2 = |\Upsilon_{-n,+N}|^2$ in the limits $n \to \infty$ and $N \to \infty$.

$$|\Upsilon_{+n,-N}|^2 \sim n^{-2} \text{ and } \sim N^{-2}$$

in each case. Then, the convergence of series (67a), (67b) follows through the Maclaurin–Cauchy convergence test[20]. The situation now is different from what happens for a scalar field, where the respective sum over n was always divergent[6]. This is an interesting feature which will be explored later. Nevertheless, it does not mean that (66) is convergent. The convergence of the double serie (66) requires more attention.

By the Abel's (n, N)-th Term Test[21], a necessary condition for the convergence of a double serie $\sum_{n,N} a_{n,N}$ is given by:

$$\lim_{n,N\to\infty} nN\, a_{n,N} = 0. \tag{68}$$

Let us check this limit for $|\Upsilon_{-n,+N}|^2$. First of all, we will focus on the behaviour of $|\Upsilon_{-n,+N}|^2$ for large values of n and N. Observe that in such limit:

- $p_n, P_N >> m$, then, $\Omega_N \to P_N,\ \omega_n \to p_n$,
- $\cos p_n R = -(\sin p_n r)m/p_n \ \to 0$,
- and, even though, $n >> 1 \Rightarrow p_n \simeq \frac{(2n-1)\pi}{2r}$,
- $N >> 1 \Rightarrow P_N \simeq \frac{(2N-1)\pi}{2R}$.

Therefore, (59) behaves in that limit as:

$$\Upsilon_{+n,-N} \sim -i\sqrt{\frac{1}{rR}}\cos P_N r \sin p_n r \left[\frac{1}{P_N + p_n} + \frac{1}{P_N p_n}\right]$$
$$\simeq i\sqrt{\frac{1}{rR}}\cos\left(\tfrac{(2N-1)\pi r}{2R}\right)(-1)^n \times \left[\frac{rR}{(rN+Rn)\pi} + \frac{rR}{Nn\pi}\right].$$

Taking into account this limiting behaviour, the leading term of $nN|\Upsilon_{-n,+N}|^2$ reads:

$$nN|\Upsilon_{+n,-N}|^2 \sim nN\,\frac{rR}{\pi^2}\cos^2\tfrac{(2N-1)\pi r}{2R}\left[\frac{1}{rN+Rn} + \frac{1}{nN}\right]^2 \sim \frac{rR}{\pi^2}\left[\frac{nN}{r^2N^2 + R^2n^2 + 2rRnN} + \frac{1}{nN} + \frac{2nN}{rN+Rn}\right].$$

Inmediately we see that:

$$\lim_{n,N\to\infty} nN|\Upsilon_{+n,-N}|^2 \neq 0. \qquad (69)$$

Actually this limit does not exist. It is easy to see that the value of (69) depends on how the limit is taken. Particularly, the inequality (69) is clearly satisfied when the limit is taken by the path $n = N$. Otherwise, e.g.

$$\lim_{n,N=n^2\to\infty} nN|\Upsilon_{+n,-N}|^2 = 0. \qquad (70)$$

Therefore we have to conclude that both quantizations constructed here are unitary inequivalent. We can connect with the results due to Knight[5] which forbid the existence of strictly localized states in the ordinary Fock representation. According to this result, the localized states built here must lie outside the global Fock space. As we have just seen, this fact reconciles now with the unitary inequivalence.

V. LOCAL PROPERTIES OF THE GLOBAL VACUUM

Once we have established the main formal results developed in this work, let us digging into the physical significance of our construction.

Due to the convergent behaviour of

$$\sum_N |\Upsilon_{+n,-N}|^2 + |\tilde{\Upsilon}_{+n,-N}|^2 = \langle 0^G|a_n^\dagger a_n + \tilde{a}_n^\dagger \tilde{a}_n|0^G\rangle < \infty,$$
$$= \sum_N |\Upsilon_{-n,+N}|^2 + |\tilde{\Upsilon}_{-n,+N}|^2 = \langle 0^G|b_n^\dagger b_n + \tilde{b}_n^\dagger \tilde{b}_n|0^G\rangle < \infty,$$

the *local* number operators $n_n^{(a)} = a_n^\dagger a_n$, $n_n^{(b)} = b_n^\dagger b_n$, $\tilde{n}_n^{(a)} = \tilde{a}_n^\dagger \tilde{a}_n$, $\tilde{n}_n^{(b)} = \tilde{b}_n^\dagger \tilde{b}_n$ are well defined operators also in $\mathfrak{F}^G$. Then, we can explore the local particle content of the global vacuum $|0^G\rangle$ in terms of those operators. The average number of local particles and antiparticles associated to the left side of the cavity is:

$$\langle n_n^{(a)}\rangle_{|0^G\rangle} = \sum_N |\Upsilon_{+n,-N}|^2 = \sum_N |\Upsilon_{-n,+N}|^2 = \langle n_n^{(b)}\rangle_{|0^G\rangle}.$$

The coincidence between particle and antiparticle spectra was awaited because of the CPT invariance of the vacuum state. This intricate expression can be evaluated numerically. Some examples are shown in figure 3. The most significant difference with the scalar case has to do with the high frequency tails, which decrease more rapidly in this case. This fact makes a qualitative difference between both fields as we will see in section V B.

A. Trapping local quantas slamming down a mirror. Particle creation.

As we have noted before, the local modes we have defined coincide with stationary modes when a mirror is placed at $x = r$ at the initial instant. Then, if this action is actually implemented, the local modes become actual stationary modes in the new cavities created and local particles in the vacuum $|0^G\rangle$ are revealed as real particles in the subcavities. This can be understood as a manifestation of the dynamical Casimir effect [22]. As we have seen, the total number of particles created is infinite, but this divergence is naturally regularized in practice by an ultraviolet cutoff related to the penetrability of the mirror, in analogy with the regularization of the Casimir energy.

B. Particle creation by removing the mirror

In section IV A we have shown that:

$$\sum_n |\Upsilon_{+n,-N}|^2 + |\tilde{\Upsilon}_{+n,-N}|^2,$$

is also convergent. This fact allows to study the operators $N_N^{(A)} = A_N^\dagger A_N$, $N_N^{(B)} = B_N^\dagger B_N$ acting on $\mathfrak{F}^L$. Arguing similarly to the previous section, we can interpret the average values of these operators:

$$\langle N_n^{(A)}\rangle_{|0_L\rangle} = \sum_n |\Upsilon_{-n,+N}|^2 + |\tilde{\Upsilon}_{-n,+N}|^2 = \langle N_N^{(B)}\rangle_{|0_L\rangle},$$

as the average number of particles (and antiparticles) created when a mirror placed at $x = r$ is suddenly removed at $t = 0$. In the scalar case, this number was infinite for every frequency Ω_N in abrupt contrast with the fermionic system discussed here. One can gain more

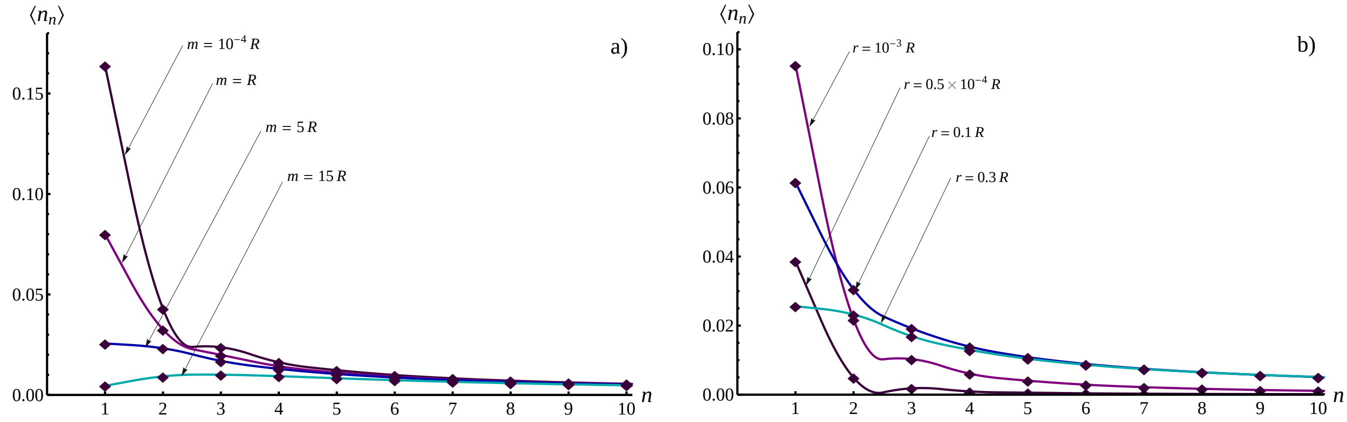

FIG. 3. Local spectra in global vacuum $|0^G\rangle$. Mean values $\langle 0^G|n_n^{(a,b)}|0^G\rangle$ are represented for different mass of the field, with fixed $r = 0.3R$(in the plot a)) and for different localization sizes, r, and fixed $m = 5R$ (in b)).

understanding on the interpretation of that phenomenon attending to the different statistics displayed by both fields. In the case treated by us, it is not possible to create more than one particle with the same energy Ω_N and then, the number of such created particle is strictly limited. This fact, is mathematically expressed in expression (64). This expression bounds the value of $\sum_N |\Upsilon_{-n,+N}|^2 < 1$, constraining necessarily the asymptotical behaviour in the limit of great N, corresponding to the high frequency limit.

Returning to the scalar case, the disconcerting result of infinite creation of any frequency particles could be interpreted meaningfully considering an experimental ultraviolet cutoff as before. In any case, it is noteworthy this completely different ultraviolet behaviour for scalar and fermionic fields. However, these questions escape from the limited scope of this study by the moment and deserve further attention.

VI. CONCLUSIONS

In the work presented, a formalism for the quantization of a Dirac field was developed as a generalization of a recent local formalism for scalar fields. The more complex structure of the field (described by spinors instead of scalar functions) has complicated the, at a first sight, trivial problem of fixing a b.c. in order to describe the field confined within a finite region of space. It remains to elucidate the influence of different b.c. in order to adapt this formalism to particular experimental settings. In any case, the b.c. chosen here are the most physically meaningful and the most used (if not the only one) to model fermionic cavities in the existing literature. Also, the study of the case treated here allows to identify how the fermionic statistics affects to the localization of the field. Interestingly, the resemblance of this fermionic nature of the field appears in a subtle way by means of the positive definite character of the inner product in the classical solutions space, affecting dramatically the behaviour of the local vacuum spectra in the high frequency limit.

The local quantization carried out gives rise to a successful notion of a strictly localized fermion and provides a local algebra of annihilation-creation operators. These tools could be extended to the usual global Fock space, $\mathfrak{F}^G$, allowing the characterization of the local particle and antiparticle content of the vacuum $|0^G\rangle$. The thought experiment consisting in slamming down a mirror in our empty cavity provides a clear physical interpretation of the local spectra obtained for the global vacuum. Experimental checks for such phenomenon using Circuit Quantum Electrodynamics have been just proposed [23]. Actually, these technics were used for the first observation of Dynamical Casimir effect [24].

Finally, some posible applications and prospective development of this work are sketched. The increasing interest of the quantum information community in the deeper understanding provided by QFT could be matched with the tools we provide. In this direction, proposals of using the vacuum for technological purposes are becoming quite popular and the formalism worked out here could be useful for the study and characterization of crucial properties of this state. as the entanglement it displays[25]. Also, the developing of a local notion for a particle could be useful in the framework of curved spacetime where an unambiguous particle conception does not exist[7, 18].

On the other hand, for the immediate future of the study presented, it would very desirable to find the explicit expression for the transformation $\mathfrak{B}$ which would allow to describe the global vacuum in terms of local states in $\mathfrak{F}^L$. Equally, the confrontation of the results obtained from our representation of QFT with existing models of particle detectors would bring light to the physical meaning of our construction.

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