Foundations of a Bicoprime Factorisation Theory: A Robust Control Perspective

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A thesis submitted to the University of Manchester for the degree of Doctor of Philosophy in the Faculty of Engineering and Physical Sciences.

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Abstract

This thesis investigates Bicoprime Factorisations (BCFs) and their possible uses in robust control theory. BCFs are a generalisation of coprime factorisations, which have been well known and widely used by the control community over the last few decades. Though they were introduced at roughly the same time as coprime factorisations, they have been largely ignored, with only a very small number of results derived in the literature.

BCFs are first introduced and the fundamental theory behind them is developed. This includes results such as internal stability in terms of BCFs, parametrisation of the BCFs of a plant and state space constructions of BCFs. Subsequently, a BCF uncertainty structure is proposed, that encompasses both left and right coprime factor uncertainty. A robust control synthesis procedure is then developed with respect to this BCF uncertainty structure. The proposed synthesis method is shown to be advantageous in the following two aspects: (1) the standard assumptions associated with $\mathscr{H}_\infty$ control synthesis are directly fulfilled without the need of loop shifting or normalisation of the generalised plant and (2) any or all of the plant's unstable dynamics can be ignored, thus leading to a reduction in the dimensions of the Algebraic Riccati Equations (AREs) that need to be solved to achieve robust stabilisation.

Normalised BCFs are then defined, which are shown to provide many advantages, especially in the context of robust control synthesis. When using a normalised BCF of the plant, lower bounds on the achievable BCF robust stability margin can be easily and directly computed *a priori*, as is the case for normalised coprime factors. Although the need for an iterative procedure is not completely avoided when designing an optimal controller, it is greatly simplified with the iteration variable being scalar. Unlike coprime factorisations where a single ARE needs to be solved to achieve normalisation, two coupled AREs must be satisfied for a BCF to be normalised. Two recursive methods are proposed to solve this problem.

Lastly, an example is presented where the theory developed is used in a practical scenario. A quadrotor Unmanned Aerial Vehicle (UAV) is considered and a normalised BCF controller is designed which in combination with feedback linearisation is used to control both the attitude and position of the vehicle.

Declaration

No portion of the work referred to in this thesis has been submitted in support of an application for another degree or qualification of this or any other university or other institute of learning.

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in memory of
Minos Stylianou
1957–2014

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Publications

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Nomenclature

Sets & Relations

$x \in \mathcal{X}$	x belongs to $\mathcal{X}$.
$\mathcal{X} \cup \mathcal{Y}$	The union of the sets $\mathcal{X}$ and $\mathcal{Y}$.
$\mathcal{X} \cap \mathcal{Y}$	The intersection of the sets $\mathcal{X}$ and $\mathcal{Y}$.
$\mathcal{X} \subseteq \mathcal{Y}$	$\mathcal{X}$ is a subset of $\mathcal{Y}$.
$x < (\leq) y$	x is less than (or equal to) y.
$x > (\geq) y$	x is greater than (or equal to) y.
$x \approx y$	x is approximately equal to y.
$x \neq y$	x is not equal to y.
$x \to y$	x tends to y.
$x \mapsto y$	x maps to y.
$A \Leftrightarrow B$	A is equivalent to B.
$A \Leftarrow B$	A is implied by B.
$A \Rightarrow B$	A implies B.
$\emptyset$	The empty set.
$\mathbb{R}$	Real numbers.
$\mathbb{C}$	Complex numbers.
$\mathbb{R}_+$ ($\bar{\mathbb{R}}_+$)	Positive (non-negative) real numbers.
$\mathbb{R}_-$ ($\bar{\mathbb{R}}_-$)	Negative (non-positive) real numbers.
$\mathbb{C}_-$ ($\bar{\mathbb{C}}_-$)	Open (closed) left-half complex plane.
$\mathbb{C}_+$ ($\bar{\mathbb{C}}_+$)	Open (closed) right-half complex plane.
$\mathbb{R}^{m\times n}$	$m \times n$ real valued matrices.
$\mathbb{C}^{m\times n}$	$m \times n$ complex valued matrices.

Matrices & Operations

0	Zero matrix of appropriate dimensions.
$\mathbf{0}_n$	Zero vector in $\mathbb{R}^n$.
$0_{n\times m}$	$n \times m$ zero matrix.
0_n	$n \times n$ zero matrix.
I	Identity matrix of appropriate dimensions.
I_n	$n \times n$ identity matrix.
A^T	Transpose of $A \in \mathbb{C}^{m\times n}$.
A^*	Complex conjugate transpose of $A \in \mathbb{C}^{m\times n}$.
A^{-1}	Inverse of $A \in \mathbb{C}^{n\times n}$.
A^{-*}	Conjugate transpose inverse of $A \in \mathbb{C}^{n\times n}$. $A^{-*} = (A^{-1})^*$.
$A^\dagger$	Pseudo-inverse of $A \in \mathbb{C}^{m\times n}$.
$\det(A)$	Determinant of $A \in \mathbb{C}^{n\times n}$.
$\mathrm{rank}(A)$	Rank of $A \in \mathbb{C}^{m\times n}$.
$\Lambda(A)$	Spectrum of $A \in \mathbb{C}^{n\times n}$.
$\mathrm{In}(A)$	Inertia of $A \in \mathbb{C}^{n\times n}$.
λ_A^i	i^{th} eigenvalue of $A \in \mathbb{C}^{n\times n}$.
γ_A^i	Geometric multiplicity of λ_A^i.
μ_A^i	Algebraic multiplicity of λ_A^i.
μ_A^{ij}	Dimension of j^{th} Jordan block associated with λ_A^i.
$\rho(A)$	Spectral radius of $A \in \mathbb{C}^{n\times n}$.
$\bar{\lambda}(A)$	Maximum eigenvalue of $A \in \mathbb{C}^{n\times n}$ (used when $\Lambda(A) \subseteq \mathbb{R}$).
$\underline{\lambda}(A)$	Minimum eigenvalue of $A \in \mathbb{C}^{n\times n}$ (used when $\Lambda(A) \subseteq \mathbb{R}$).
$\bar{\sigma}(A)$	Maximum singular value of $A \in \mathbb{C}^{m\times n}$.
$\underline{\sigma}(A)$	Minimum singular value of $A \in \mathbb{C}^{m\times n}$.
$\|x\|$	Euclidean norm of $x \in \mathbb{C}^n$.
$\|A\|$	Induced 2-norm of $A \in \mathbb{C}^{m\times n}$.
$A > (\geq) 0$	Positive (semi)definite matrix.
$A < (\leq) 0$	Negative (semi)definite matrix.
$A^{\frac{1}{2}}$	Square root of $A \geq 0$.
$\mathrm{Im}(A)$	Image (range) of $A \in \mathbb{C}^{m\times n}$.
$\ker(A)$	Kernel (null space) of $A \in \mathbb{C}^{m\times n}$.
$\mathrm{span}\{x_i\}$	Vector space spanned by the set of vectors x_i.

$\mathrm{Ric}(H)$	Riccati operator.
$\mathrm{dom}(\mathrm{Ric})$	Domain of the Riccati operator.
$\mathcal{X}_-(H)$	Conjugate symmetric, stable, invariant spectral subspace of H.

Function Spaces

$\mathbb{R}[s]$	Ring of polynomials.
$\mathscr{R}$	Space of real, rational, proper transfer matrices.
$\mathscr{L}_\infty$	Space of matrix valued functions, essentially bounded on $j\mathbb{R}$.
$\mathscr{H}_\infty$	Subspace of $\mathscr{L}_\infty$ with functions, analytic and bounded in $\mathbb{C}_+$.
$\mathscr{RL}_\infty$	Real, rational, proper subspace of $\mathscr{L}_\infty$.
$\mathscr{RH}_\infty$	Real, rational, proper subspace of $\mathscr{H}_\infty$.
$\mathscr{GH}_\infty$	Space of units in $\mathscr{RH}_\infty$. $\{P \in \mathscr{RH}_\infty : P^{-1} \in \mathscr{RH}_\infty\}$

Linear Systems

P^T	Transpose of $P \in \mathscr{R}$.
$P^\sim$	Conjugate system of $P \in \mathscr{R}$.
P^{-1}	Inverse of $P \in \mathscr{R}$.
$\mathrm{n\,rank}\,(P)$	Normal rank of $P \in \mathscr{R}$.
$\left[\begin{array}{c\|c} A & B \\ \hline C & D \end{array}\right]$	Shorthand notation for $C\,(sI - A)^{-1}\,B + D$.
$\\|P\\|_\infty$	Infinity norm of $P \in \mathscr{RL}_\infty$.
$\\|P\\|_H$	Hankel norm of $P \in \mathscr{RH}_\infty$.
$[P, C]$	Standard positive feedback interconnection of P and C.
$\mathcal{F}_l(H, C)$	Lower linear fractional transformation.
$\mathcal{F}_u(H, \Delta)$	Upper linear fractional transformation.
$H_1 \star H_2$	Redheffer star product.
$\mathscr{C}_r\ (\mathscr{C}_l)$	Set of right (left) coprime pairs in $\mathscr{RH}_\infty$.
$\mathscr{B}$	Set of bicoprime quads in $\mathscr{RH}_\infty$.
$\mathscr{C}_r(P)\ (\mathscr{C}_l(P))$	Set of right (left) coprime factorisations of $P \in \mathscr{R}$ over $\mathscr{RH}_\infty$.
$\mathscr{B}(P)\ (\mathscr{B}_r(P))$	Set of bicoprime factorisations of $P \in \mathscr{R}$ over $\mathscr{RH}_\infty$ (with internal dimension r).
$\widetilde{\mathscr{B}}(P)\ (\widetilde{\mathscr{B}}_r(P))$	Set of BCF symbols associated with $\mathscr{B}(P)$ ($\mathscr{B}_r(P)$).
$\mathscr{C}^\dagger \left[\begin{smallmatrix} M \\ N \end{smallmatrix}\right]$	Set of Bézout factors associated with $\{N, M\} \in \mathscr{C}_r$.
$\mathscr{C}^\dagger \left[\begin{smallmatrix} M & L \end{smallmatrix}\right]$	Set of Bézout factors associated with $\{L, M\} \in \mathscr{C}_l$.

Miscellaneous

j	Imaginary unit, $j^2 = -1$.
$\bar{x}$	Complex conjugate of x.
$\exists x$	There exists an x.
$\nexists x$	There does not exist an x.
:	Such that.
$\forall x$	For all x.
$\square$	End of proof.
$\blacklozenge$	End of remark.
$\lim_{x\to b} f(x)$	Limit of $f(x)$ as x tends to a.
$\max(\mathcal{X})$	Maximal element of the set $\mathcal{X}$.
$\min(\mathcal{X})$	Minimal element of the set $\mathcal{X}$.
$\inf_{x\in\mathcal{X}} f(x)$	Infimum of $f(x)$ over $x \in \mathcal{X}$.
$\sup_{x\in\mathcal{X}} f(x)$	Supremum of $f(x)$ over $x \in \mathcal{X}$.
$\operatorname{ess\,sup}_{x\in\mathcal{X}} f(x)$	Essential supremum of $f(x)$ over $x \in \mathcal{X}$.
$\operatorname{sgn}(x)$	Signum function. Applied element-wise to matrices. For $x \in \mathbb{R}$, $\operatorname{sgn}(x) = \begin{cases} 1 & \text{if } x > 0, \\ 0 & \text{if } x = 0, \\ -1 & \text{if } x < 0. \end{cases}$
$\operatorname{diag}\{A_1, \ldots, A_m\}$	Block diagonal matrix.

Acronyms

ARE Algebraic Riccati Equation

BC Bicoprime

BCF Bicoprime Factorisation

LC Left Coprime

LCF Left Coprime Factorisation

LFT Linear Fractional Transformation

LMI Linear Matrix Inequality

MIMO Multiple Input-Multiple Output

PBH Popov-Belevitch-Hautus

PMD Polynomial Matrx Description

RC Right Coprime

RCF Right Coprime Factorisation

SISO Single Input-Single Output

UAV Unmanned Aerial Vehicle

Chapter 1

Introduction

1.1 Background & Motivation

Many areas of control theory make extensive use of coprime factorisations. This is especially true for fields of robust control theory such as $\mathscr{H}_\infty$ loopshaping (Glover and McFarlane, 1989; McFarlane and Glover, 1992) and distance measures (Vidyasagar, 1984; Georgiou and Smith, 1990; Vinnicombe, 1993; Lanzon and Papageorgiou, 2009). Integer coprimeness is a property studied since ancient times. Two integers are said to be coprime if their greatest common divisor is 1. This notion can be extended to many types of mathematical objects including polynomials and matrices. The case of polynomial coprimeness was studied by the French mathematician Étienne Bézout in the 18^{th} century, who showed that if two polynomials a and b have greatest common divisor d then there exist polynomials x and y such that the linear Diophantine equation $ax + by = d$ is satisfied. Such an equation is now commonly referred to as Bézout's identity, a version of which is used as a coprimeness test for polynomial matrices.

A coprime factorisation is one where a rational object is decomposed into two factors satisfying the coprimeness condition over some set, which in the context of control theory is usually $\mathscr{RH}_\infty$. One of the most important aspects of coprime factorisations is the fact that every object in $\mathscr{R}$ admits a coprime factorisation over $\mathscr{RH}_\infty$, which can be easily constructed from state space data using the methods of Nett et al. (1984). This allows for the development of theories that can be applied to a large class of systems.

One of the most prevalent uses of coprime factorisations in control theory is found in the $\mathscr{H}_\infty$ loop shaping design procedure proposed by McFarlane and Glover (1988) and McFarlane and Glover (1992). In this situation, normalised coprime factors are used to derive a robustly stabilising controller for a plant, which has been shaped using the classical loop shaping procedure. Using coprime factors for robust control synthesis, as outlined by Doyle et al. (1989), leads to significant simplifications and advantages. The need to solve one of the standard Algebraic Riccati Equations (AREs) is removed as it admits the trivial solution, which also guarantees that the spectral radius condition is automatically satisfied

(Zhou et al., 1996). Furthermore, when using normalised coprime factors for synthesis, an infimum for the achievable norm can be explicitly calculated (Glover and McFarlane, 1988, 1989). This removes the need for iterative procedures to obtain the optimal controller. Hence, synthesis of an optimal robust controller that stabilises perturbations on normalised coprime factors of the plant can be achieved via the solution of a single ARE.

The main focus of this thesis is the investigation of Bicoprime Factorisations (BCFs) and their uses in robust control theory. BCFs are a generalisation of the aforementioned coprime factorisations. They were briefly introduced by Vidyasagar (2011)[a] with only a small number of results given. Two motivating points given therein for the study of BCFs are that they naturally arise in closed loop transfer matrices (when starting with coprime factorisations of the plant and controller) and the fact that a state space representation of a plant is itself a BCF over $\mathbb{R}(s)$. Both of these claims will be proven in this thesis. Such factorisations appear in many areas of interest such as J-spectral factorisations (Green et al., 1990) or chain scattering theory (Lanzon et al., 2004) – both of which can be used to solve the standard $\mathscr{H}_\infty$ control problem. Unfortunately, the study of BCFs was largely abandoned when their Left Coprime (LC) and Right Coprime (RC) counterparts started to yield some powerful results, however they were recently reintroduced by Tsiakkas and Lanzon (2015).

Past studies of BCFs commonly assumed a special structure. The relation between such BCFs and classical coprime factorisations was studied by Desoer and Gündeş (1988) (though the assumption was lifted for some cases). A set of simple preliminary results was derived, including internal stability for the feedback interconnection of a plant, given as a BCF, and a controller expressed as a Right Coprime Factorisation (RCF) or Left Coprime Factorisation (LCF). Another interesting result involved a transformation mapping a special BCF set into the classical RCF and LCF, making use of the Bézout factors associated with the BCF. Those results were extended by Gündeş and Desoer (1990) and given a decentralised control context.

It has also been shown that BCFs can be useful in the study of decentralised or distributed control problems. For example, Ünyelioğlu et al. (2000) showed that BCFs can be used to characterise the location of decentralised fixed zeros of a plant, and thus deduce the existence of a decentralised controller. Furthermore, BCFs were used in the design of a decentralised stabilising controller for a plant by Baski et al. (1999).

Another interesting result from BCFs relates to internal stability tests. It will be shown herein that using BCF representations of a plant and controller can, in certain cases, result in reduced dimension tests for internal stability. Standard coprime factor results provide stability tests that require the inversion of a matrix with dimension p or q where p and q are the number of outputs and inputs of the plant respectively. Using BCFs, a test is obtained requiring the inversion of a matrix with dimensions equal to the internal dimension of the BCF, which can always be chosen to be no greater than $\min\{p, q\}$. Such

[a]This is a reprint of the 1985 original.

cases are likely to occur for rank deficient plants, for example in redundant control systems where, for fault tolerance purposes, more actuators and sensors are used than necessary.

Unlike the case of LCFs or RCFs, where robust control synthesis involves the solution of only one ARE, BCFs require the solution of two. However, the structure of the BCF generalised plant allows the designer to selectively ignore some (or all) of the $\mathbb{C}_-$ poles of the plant under consideration, resulting in reduced dimension AREs in $\mathscr{H}_\infty$ robust control synthesis. This can be useful for high order systems with many well-damped poles in $\mathbb{C}_-$, where solving the AREs associated with $\mathscr{H}_\infty$ synthesis can become computationally expensive and at times numerically intractable. Through the use of BCF uncertainty, the computational burden of robust stabilisation can therefore be drastically reduced.

Similarly to classical coprime factorisations, a normalisation property can be imposed on to the Bicoprime (BC) factors of a plant. Methods for obtaining normalised coprime factorisations of a plant where first presented by Meyer and Franklin (1987) for strictly proper systems and later extended by Vidyasagar (1988) to the non-strictly proper case. In the classical case, normalisation is achieved via the solution of an ARE with a sign-definite quadratic term. In a similar manner, a normalised BCF can be obtained via the solution of two coupled AREs, leading to the need for an iterative procedure.

As previously mentioned, normalisation leads to significant advantages in coprime factor synthesis, namely the explicit computation of the lowest achievable robust stability margin. A parallel result is obtained given a normalised BCF of a plant, where lower bounds on the achievable robust stability margin with respect to BCF uncertainty can be easily computed. These bounds however, are not guaranteed to produce the infimum as in the classical case, since an additional condition must also be satisfied. Solving this issue is simple and can be achieved with techniques as trivial as a line search, though root finding methods such as the Newton-Raphson or bisection algorithms would be a more suitable choice.

This thesis begins by providing the foundations to the general study of BCF theory and its applicability to control related problems. The results presented herein cover a range of topics including internal stability in terms of BCFs of the plant and controller, state space parametrisations of BCFs for a given plant, BCF uncertainty characterisation and BCF robust control synthesis. It will be shown in multiple instances how results that have been known to the control community for a long time are actually founded on BCFs. This helps in gaining some intuitive understanding of these results. The aim of this thesis is to establish BCFs as an integral part of control theory. It will become apparent through this body of work that there is a considerable amount of mathematical richness associated with BCFs that so far remains unexplored. Though the more abstract nature of BCFs leads to an increase in complexity (in comparison to LCFs and RCFs), this should not serve as a deterrent to the study of BCFs as the potential advantages to control theory necessitate further exploration of the subject.

1.2 Thesis Organisation

This thesis is organised as follows:

Chapter 2: Preliminaries
In this chapter, some preliminary mathematical tools are presented. First, useful concepts from linear algebra are given including a brief treatment of AREs, followed by the definition of function spaces, which will be used throughout this thesis. Results pertaining to linear systems theory are then presented including operations on state space systems, Lyapunov theory and internal stability. Following this, classical coprime factorisations are introduced, with the associated sets defined and some basic results are presented. Finally, the standard 2-ARE solution to the $\mathscr{H}_\infty$ problem of Doyle et al. (1989) is reiterated for reference.

Chapter 3: Foundations of Bicoprime Factorisations
Here, BCFs of a plant and associated sets are defined. Some basic properties are discussed, such as system stability and zero characterisation using BCFs. The notion of BCF internal dimension is then introduced and a lower bound on the achievable dimension is given. Furthermore, internal stability tests using BCFs of the plant and controller are outlined. Numerous results are presented, each with different levels of constraints, ranging from no assumptions (the plant and controller are allowed to have arbitrary BCFs), to much more specific cases such as a stable plant or controller. It is shown how classical coprime factor internal stability results found in the literature are based on BCFs.

Chapter 4: State Space Formulations of BCFs
In this chapter, methods of constructing BCFs based on state space data of the plant are presented. The first approach is related to observer form controllers, mirroring the full state feedback and estimation interpretations of LCFs and RCFs. Subsequently, a more abstract method, the QR-BCF parametrisation, is given and shown to cover the coprime factor parametrisations found in the literature. Finally, bounds on the minimum internal dimension BCF achievable using the QR-BCF parametrisation are derived and a procedure is proposed that can be used to construct a BCF that achieves this minimum.

Chapter 5: BCF Uncertainty and Robust Stabilisation
The uncertainty structure corresponding to BCFs is presented in this chapter. Similarly to coprime factor uncertainty, stable additive perturbations on the BC factors of the plant are considered. The associated generalised plant and uncertainty matrix are derived. The robust stability margin corresponding to this uncertainty structure is then defined. Again, it is shown how classical results can be obtained via simplifications of the more general BCF results. Robust control synthesis theorems are given based on BCFs of the plant.

Special cases are also considered including symmetric systems and the case where robust stabilisation is achieved via the solution of reduced dimension AREs.

Chapter 6: Normalised BCFs

The notion of normalised BCFs is introduced in this chapter. Iterative methods for obtaining such BCFs are outlined. It is shown how using normalised BCFs allows for an easily computable lower bound on the achievable robust stability margins. Special cases are investigated including unilaterally normalised BCFs (which form a superset for normalised LCFs and RCFs) and a set of symmetric systems for which robust control synthesis reduces down to the solution of a single Lyapunov equation.

Chapter 7: Application: Control of a Quadrotor UAV

A practical example is presented in this chapter. The BCF theory previously developed is used in the design of a robustly stabilising controller for a quadrotor Unmanned Aerial Vehicle (UAV). It is shown by way of simulations how the proposed control strategy robustly stabilises the nonlinear plant.

Chapter 8: Conclusion

In the final chapter the contributions of this thesis are summarised and possible directions of future research are discussed.

Chapter 2

Preliminaries

In this chapter the mathematical results needed to develop the main ideas of this thesis are presented. The results are mostly standard and can be found in most relevant books such as Laub (2005) or Horn and Johnson (2012) for the first section and Zhou et al. (1996) for the rest.

The following topics are covered herein. First some results from linear algebra are presented including a brief study of AREs. Function spaces frequently used in this thesis are then defined, followed by some linear systems theory. Subsequently, classical coprime factorisations are formally defined and some well known internal stability results are given. Finally, the 2-ARE solution to the standard $\mathscr{H}_\infty$ control problem is presented.

2.1 Linear Algebra

Definition 2.1 *A matrix $A \in \mathbb{R}^{n\times n}$ is said to be Hurwitz if all its eigenvalues have a strictly negative real part, or equivalently if $\Lambda(A) \subseteq \mathbb{C}_-$.*

Every square matrix has a Jordan Canonical Form, given by the following theorem.

Theorem 2.1 (Horn and Johnson (2012) Theorem 3.1.15) *Let $A \in \mathbb{C}^{n\times n}$ have m distinct eigenvalues. Then there exists a nonsingular $T \in \mathbb{C}^{n\times n}$ such that*

$$TAT^{-1} = \mathrm{diag}\left(J_1, \ldots, J_m\right),$$
$$J_i = \mathrm{diag}\left(J_{i1}, \ldots, J_{i\gamma_A^i}\right) \in \mathbb{C}^{\mu_A^i \times \mu_A^i} \quad \forall i \in \{1, \ldots, m\},$$
$$J_{ij} = \begin{bmatrix} \lambda_A^i & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 1 \\ 0 & \cdots & \cdots & \lambda_A^i \end{bmatrix} \in \mathbb{C}^{\mu_A^{ij} \times \mu_A^{ij}} \quad \forall j \in \{1, \ldots, \gamma_A^i\},$$

where γ_A^i is the geometric multiplicity of λ_A^i, $\mu_A^i = \sum_{j=1}^{\gamma_A^i} \mu_A^{ij}$ is the algebraic multiplicity of λ_A^i and $n = \sum_{i=1}^{m} \mu_A^i$.

Remark 2.1 *For any $A \in \mathbb{R}^{n\times n}$, a real Jordan canonical for can be obtained where $J_{ij} \in \mathbb{R}^{\mu_A^{ij}\times\mu_A^{ij}}$ and $T \in \mathbb{R}^{n\times n}$. See Laub (2005, Theorem 9.22) for details.* ♦

Linear Fractional Transformations (LFTs) are mathematical tools that allow for seemingly different problems to be formulated in to the same framework. LFTs enable any feedback interconnection to be analysed using the same methods. Figure 2.1 shows upper and lower LFTs in block diagram form.

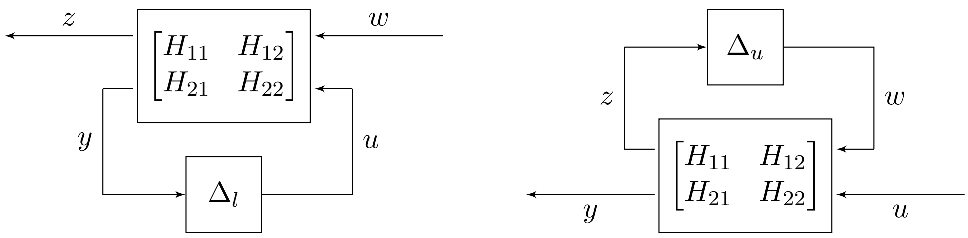


Figure 2.1: Lower (left) and upper (right) linear fractional transformations.

Definition 2.2 (Doyle et al. (1991)) *Let $H = \left[\begin{smallmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{smallmatrix}\right] \in \mathcal{S}^{(p_1+p_2)\times(q_1+q_2)}$, $\Delta_l \in \mathcal{S}^{p_1\times q_1}$ and $\Delta_u \in \mathcal{S}^{p_2\times q_2}$. Then*

- *the lower LFT of H with respect to Δ_l represents the transfer matrix from w to z and is given by*

$$\mathcal{F}_l(H, \Delta_l) = H_{11} + H_{12}\Delta_l(I - H_{22}\Delta_l)^{-1}H_{21};$$

- *the upper LFT of H with respect to Δ_u represents the transfer matrix from u to y and is given by*

$$\mathcal{F}_u(H, \Delta_u) = H_{22} + H_{21}\Delta_u(I - H_{11}\Delta_u)^{-1}H_{12}.$$

2.1.1 Algebraic Riccati Equations

Algebraic Riccati Equations (AREs) of the form

$$A^*X + XA + XRX + Q = 0 \tag{2.1}$$

where Q and R are Hermitian matrices, are commonly encountered in robust and optimal control theory. For every ARE such as (2.1), a Hamiltonian matrix can be defined as

$$H = \begin{bmatrix} A & R \\ -Q & -A^* \end{bmatrix}. \tag{2.2}$$

Let $\mathcal{X}_-(H)$ denote a conjugate symmetric[a], stable, H-invariant spectral subspace of H, that is, $\mathcal{X}_-(H)$ is the eigenspace of H that corresponds to its $\mathbb{C}_-$ eigenvalues. Furthermore,

[a]A vector space $\mathcal{V}$ is said to be conjugate symmetric if $v \in \mathcal{V}$ implies that $\bar{v} \in \mathcal{V}$.

suppose that H has no eigenvalues on $j\mathbb{R}$ and that $\mathcal{X}_-(H)$ is complementary[b] to $\operatorname{Im}\begin{bmatrix} 0 \\ I \end{bmatrix}$. These are called the stability and complementarity properties respectively. Finally, let $\mathcal{X}_-(H)$ be given by $\mathcal{X}_-(H) = \operatorname{Im}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$. Then the stabilising solution to (2.1) is uniquely determined by H and is given by $X = X_2 X_1^{-1}$. The operator mapping H to X is denoted by Ric. Additionally, the set of all Hamiltonian matrices for which the stability and complementarity properties hold is defined as dom(Ric).

Theorem 2.2 (Zhou et al. (1996) Theorem 13.5) *Suppose that* $H \in \operatorname{dom}(\operatorname{Ric})$ *and* $X = \operatorname{Ric}(H)$. *Then*

(a) X *is real symmetric;*

(b) X *is a solution to the ARE* (2.1)*;*

(c) $A + RX$ *is Hurwitz.*

The following theorem considers the case where Q and R are sign-definite.

Theorem 2.3 (Zhou et al. (1996) Theorem 13.7) *Suppose that a Hamiltonian matrix H has the form*

$$H = \begin{bmatrix} A & -BB^* \\ -C^*C & -A^* \end{bmatrix}.$$

Then $H \in \operatorname{dom}(\operatorname{Ric})$ *if and only if* (A, B) *is stabilisable and* (C, A) *has no unobservable modes on* $j\mathbb{R}$. *Furthermore,* $\operatorname{Ric}(H) > 0$ *if and only if* (C, A) *is observable.*

The following lemma will be used in several instances to derive conditions that guarantee that the stabilising solution of an ARE is positive semidefinite.

Lemma 2.4 *Let* $X \geq 0$, $Y \geq 0$ *and* $\alpha \in \mathbb{R}_+$. *Then* $I - \alpha XY$ *is nonsingular and* $(I - \alpha XY)^{-1}X \geq 0$ *if and only if*

$$\alpha\rho(XY) < 1.$$

Proof. First note that $I - \alpha XY$ is invertible if and only if $\alpha^{-1} \notin \Lambda(XY)$. Thus $\alpha\rho(XY) < 1$ implies that $I - \alpha XY$ is invertible. Then, since $I - \alpha XY$ is nonsingular,

$$\begin{aligned}(I - \alpha XY)^{-1}X \geq 0 &\Leftrightarrow X^{\frac{1}{2}}(I - \alpha X^{\frac{1}{2}}YX^{\frac{1}{2}})^{-1}X^{\frac{1}{2}} \geq 0 \\ &\Leftrightarrow I - \alpha X^{\frac{1}{2}}YX^{\frac{1}{2}} > 0 \\ &\Leftrightarrow \alpha\rho(X^{\frac{1}{2}}YX^{\frac{1}{2}}) < 1 \\ &\Leftrightarrow \alpha\rho(XY) < 1.\end{aligned}$$

□

[b]Two n-dimensional vector spaces are said to be complementary if their direct sum gives $\mathbb{R}^n$. In other words, $\mathcal{X} \subseteq \mathbb{R}^n$ and $\mathcal{Y} \subseteq \mathbb{R}^n$ are complementary if $\mathcal{X} \cap \mathcal{Y} = \emptyset$ and $\exists x \in \mathcal{X}, y \in \mathcal{Y} : v = x + y \ \forall v \in \mathbb{R}^n$.

2.2 Function Spaces

Some function spaces often utilised in robust control theory, and in this thesis, are defined in this section. For further information see (Francis, 1987).

Definition 2.3 *$\mathscr{L}_\infty$ is a Banach space of complex matrix valued functions that are essentially bounded on the j$\mathbb{R}$ axis. The $\mathscr{L}_\infty$ norm is defined as*

$$\|P\|_\infty = \operatorname*{ess\,sup}_{\omega \in \mathbb{R}} \bar{\sigma}\left[P(j\omega)\right].$$

Definition 2.4 *$\mathscr{H}_\infty \subseteq \mathscr{L}_\infty$ is the subset of $\mathscr{L}_\infty$ with functions that are analytic and bounded in $\mathbb{C}_+$. The $\mathscr{H}_\infty$ norm is defined as*

$$\|P\|_{\mathscr{H}_\infty} = \sup_{s \in \mathbb{C}_+} \bar{\sigma}\left[P(s)\right].$$

Remark 2.2 *By the maximum modulus theorem $\|P\|_\infty = \|P\|_{\mathscr{H}_\infty}$.* ♦

Definition 2.5 *$\mathscr{R}$ is the space of complex matrix valued functions defined on $\mathbb{C}$ and it consists of all real, rational, proper functions. When used as a prefix, $\mathscr{R}$ denotes subspace of real, rational, proper functions.*

Definition 2.6 *$\mathscr{GH}_\infty$ is the subspace of $\mathscr{RH}_\infty$ with transfer matrices invertible in $\mathscr{RH}_\infty$.*

$$\mathscr{GH}_\infty = \{P \in \mathscr{RH}_\infty : \det P(\infty) \neq 0, P^{-1} \in \mathscr{RH}_\infty\}.$$

2.3 Linear Systems Theory

In this section some fundamental results from linear systems theory are presented. Further details can be found in books such as Zhou et al. (1996) or Ogata (2010).

2.3.1 State Space Systems

Throughout this section it is assumed that $P = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \in \mathscr{R}^{p \times q}$ unless otherwise stated. Furthermore, for the entirety this thesis, the state space matrices A, B, C and D (also referred to as state space data) are assumed to be real valued.

Operations on Systems

Definition 2.7 (Zhou et al. (1996) Definition 3.7) *The transpose of $P(s)$ is given by*

$$P^T(s) = \left[\begin{array}{c|c} A^* & C^* \\ \hline B^* & D^* \end{array}\right].$$

Definition 2.8 (Zhou et al. (1996) Definition 3.8) *The conjugate system of $P(s)$ is given by $P^{\sim}(s) = P^T(-s)$, or in state space*

$$P^{\sim}(s) = \left[\begin{array}{c|c} -A^* & -C^* \\ \hline B^* & D^* \end{array}\right].$$

Lemma 2.5 *Suppose the D has full column (resp. row) rank and let $D^\dagger$ be its left (resp. right) inverse. Then*

$$P^\dagger = \left[\begin{array}{c|c} A - BD^\dagger C & -BD^\dagger \\ \hline D^\dagger C & D^\dagger \end{array}\right]$$

is a left (resp. right) inverse of P satisfying $P^\dagger P = I$ (resp. $PP^\dagger = I$).

Definition 2.9 *Every plant $P \in \mathscr{R}$ has the Gilbert realisation*

$$P = \left[\begin{array}{cc|c} A_+ & & B_+ \\ & A_- & B_- \\ \hline C_+ & C_- & D \end{array}\right]$$

where A_- is Hurwitz and $\Lambda(A_+) \cup \Lambda(A_-) = \Lambda(A)$.

Controllability & Observability

Controllability is a fundamental concept of state space realisations. A pair (A, B) is said to be controllable if there exists an unconstrained control signal u that can transfer any initial state $x(0)$ of $\dot{x} = Ax + Bu$ to any desired location $x(t)$ (Dorf and Bishop, 1998).

Theorem 2.6 (Zhou et al. (1996) Theorem 3.1) *Let $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times q}$, then the following statements are equivalent:*

(i) (A, B) *is controllable;*

(ii) $\begin{bmatrix} A - \lambda I & B \end{bmatrix}$ *has full row rank for all $\lambda \in \Lambda(A)$;*

(iii) For all $\lambda \in \Lambda(A)$ and $x \in \mathbb{C}^n$ such that $x^*A = \lambda x^*$, $x^*B \neq 0$;

(iv) The eigenvalues of $A + BF$ can be freely assigned by a suitable choice of $F \in \mathbb{R}^{q\times n}$.

Corollary 2.7 (Zhou et al. (1996) Theorem 3.2) *Let $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times q}$, then the following statements are equivalent:*

(i) (A, B) *is stabilisable;*

(ii) $\begin{bmatrix} A - \lambda I & B \end{bmatrix}$ *has full row rank for all $\lambda \in \Lambda(A) \cap \bar{\mathbb{C}}_+$;*

(iii) For all $\lambda \in \Lambda(A) \cap \bar{\mathbb{C}}_+$ and $x \in \mathbb{C}^n$ such that $x^*A = \lambda x^*$, $x^*B \neq 0$;

(iv) There exists a matrix $F \in \mathbb{R}^{q\times n}$ such that $A + BF$ is Hurwitz.

A notion dual to controllability is observability, which relates to determining the initial state of a system given only input and output data.

Theorem 2.8 (Zhou et al. (1996) Theorem 3.3) *Let $A \in \mathbb{R}^{n\times n}$ and $C \in \mathbb{R}^{p\times n}$, then (C, A) is observable if and only if (A^*, C^*) is controllable.*

Corollary 2.9 (Zhou et al. (1996) Theorem 3.4) *Let $A \in \mathbb{R}^{n\times n}$ and $C \in \mathbb{R}^{p\times n}$, then (C, A) is detectable if and only if (A^*, C^*) is stabilisable.*

Conditions (ii) and (iii) of Theorem 2.6 and Corollary 2.7 along with their observability duals, are often collectively referred to as the Popov-Belevitch-Hautus (PBH) tests.

Definition 2.10 *A state space realisation P is said to be minimal if (A, B) is controllable and (C, A) is observable.*

Definition 2.11 *The treble (C, A, B) is said to have no modes that are both controllable and observable when all controllable modes in (A, B) are unobservable in (C, A) and all observable modes in (C, A) are uncontrollable in (A, B).*

Normal Rank & System Zeros

The normal rank and zeros of a plant are now characterised based on its state space data.

Definition 2.12 *The normal rank of P is defined as*

$$\operatorname{nrank}(P) = \max_{s\in\mathbb{C}\cup\{\infty\}} \operatorname{rank}(P(s)).$$

Lemma 2.10 (Zhou et al. (1996) Lemma 3.29) *Suppose that $z_0 \in \mathbb{C}$ is not a pole of P. Then z_0 is a transmission zero of P if and only if* $\operatorname{rank}(P(z_0)) < \operatorname{nrank}(P)$.

Definition 2.13 (Zhou et al. (1996) Definition 3.16) *A complex number $z_0 \in \mathbb{C}$ is an invariant zero of P if*

$$\operatorname{rank}\left(\begin{bmatrix} A - z_0 I & B \\ C & D \end{bmatrix}\right) < \operatorname{nrank}\left(\begin{bmatrix} A - sI & B \\ C & D \end{bmatrix}\right).$$

Theorem 2.11 (Zhou et al. (1996) Theorem 3.34) *A complex number $z_0 \in \mathbb{C}$ is a transmission zero of P if and only if it is an invariant zero of a minimal realisation.*

It can be shown that for any given plant P,

$$\operatorname{nrank}\left(\begin{bmatrix} A - sI & B \\ C & D \end{bmatrix}\right) = n + \operatorname{nrank}(P),$$

which leads to the following lemma.

Lemma 2.12 (Zhou et al. (1996) Lemma 3.33) *P has full column (resp. row) normal rank if and only if*

$$\begin{bmatrix} A - sI & B \\ C & D \end{bmatrix}$$

full column (resp. row) normal rank.

2.3.2 Lyapunov Theory

Lyapunov equations of the form

$$A^*X + XA + Q = 0 \tag{2.3}$$

have many applications in control theory. The two following lemmas relate Lyapunov equations to stability and observability (or by duality to controllability).

Lemma 2.13 (Zhou et al. (1996) Lemma 3.18) *Consider the Lyapunov equation* (2.3) *and suppose that A is Hurwitz, then*

(a) $X > 0$ if $Q > 0$ and $X \geq 0$ if $Q \geq 0$.

(b) If $Q \geq 0$, then (Q, A) is observable if and only if $X > 0$.

Lemma 2.14 (Zhou et al. (1996) Lemma 3.19) *Suppose that X is the solution to* (2.3)*, then*

(a) $\Lambda(A) \subseteq \bar{\mathbb{C}}_-$ if $X > 0$ and $Q \geq 0$.

(b) A is Hurwitz if $X > 0$ and $Q > 0$.

(c) A is Hurwitz if $X \geq 0$, $Q \geq 0$ and (Q, A) is detectable.

Definition 2.14 *Let $P \in \mathscr{RH}_\infty$, which implies that A is Hurwitz. Then the controllability and observability Gramians of P are given by the solutions $X \geq 0$ and $Y \geq 0$ to the Lyapunov equations*

$$AX + XA^* + BB^* = 0 \text{ and } YA + A^*Y + C^*C = 0$$

respectively. Furthermore, the Hankel norm of P is given by

$$\|P\|_H = \rho(XY)^{\frac{1}{2}}.$$

2.3.3 Internal Stability

Let $P \in \mathscr{R}$ and $C \in \mathscr{R}$, then the standard positive feedback interconnection of the two is denoted by $[P, C]$ and is shown in Figure 2.2. Internal stability is a core concept of

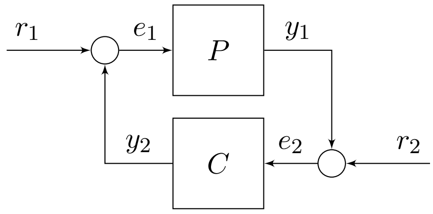


Figure 2.2: Standard feedback interconnection.

control theory and is concerned with the boundedness of all signals from (r_1, r_2) to (e_1, e_2) in Figure 2.2.

Definition 2.15 (Zhou et al. (1996) Definition 5.1) *Consider the standard positive feedback interconnection of a plant $P \in \mathscr{R}$ and controller $C \in \mathscr{R}$ depicted in Figure 2.2. Then $[P, C]$ is said to be well-posed if all closed-loop transfer matrices from (r_1, r_2) to (e_1, e_2) are well-defined and proper.*

Definition 2.16 *Consider the standard positive feedback interconnection of a plant $P \in \mathscr{R}$ and controller $C \in \mathscr{R}$ depicted in Figure 2.2. Then the transfer function matrix from $(-r_2, r_1)$ to (y_1, e_1) is denoted by $H(P, C)$. Furthermore, $H(P, C)$ is given by*

$$H(P, C) = \begin{bmatrix} P \\ I \end{bmatrix} (I - CP)^{-1} \begin{bmatrix} -C & I \end{bmatrix}.$$

The following lemma gives necessary and sufficient conditions for well-posedness and internal stability of a standard positive feedback interconnection.

Lemma 2.15 (Zhou et al. (1996) Lemma 5.3) *Consider the standard positive feedback interconnection of a plant $P \in \mathscr{R}$ and controller $C \in \mathscr{R}$ depicted in Figure 2.2. Then $[P, C]$ is well-posed if and only if $\det(I - C(\infty)P(\infty)) \neq 0$. Furthermore, $[P, C]$ is internally stable if and only if it is well-posed and*

$$\begin{bmatrix} I & -C \\ -P & I \end{bmatrix}^{-1} \in \mathscr{RH}_\infty,$$

or equivalently $H(P, C) \in \mathscr{RH}_\infty$.

The small gain theorem provides a robust stability criterion based only on the size of the systems being considered. This was first proposed by Zames (1966a) and Zames (1966b) and forms a central part of robust control theory.

Theorem 2.16 (Zhou et al. (1996) Theorem 9.1) *Suppose that $H_1 \in \mathscr{RH}_\infty$ and $H_2 \in \mathscr{RH}_\infty$. Then the standard feedback interconnection of H_1 and H_2 as shown in Figure 2.2 is well-posed and internally stable for all H_2 such that*

(a) $\|H_2\|_\infty < 1/\gamma$ if and only if $\|H_1\|_\infty \leq \gamma$;

(b) $\|H_2\|_\infty \leq 1/\gamma$ if and only if $\|H_1\|_\infty < \gamma$.

2.4 Coprime Factorisations

As mentioned earlier, coprime factorisations are an important part of control theory. The following definitions introduce, in a formal way, left and right coprimeness over $\mathscr{RH}_\infty$ as well as RCFs and LCFs of a plant over $\mathscr{RH}_\infty$.

Definition 2.17 (Green and Limebeer (2012) Definition A.2.2) *The ordered pair $\{N, M\}$ is RC in $\mathscr{RH}_\infty$ if $N, M \in \mathscr{RH}_\infty$ and there exist $Y_r, Z_r \in \mathscr{RH}_\infty$ such that $Z_r M + Y_r N = I$. Furthermore, the pair is a RCF of a plant $P \in \mathscr{R}$ over $\mathscr{RH}_\infty$ if M is square,* $\det M(\infty) \neq 0$ *and $P = NM^{-1}$.*

Definition 2.18 (Green and Limebeer (2012) Definition A.2.3) *The ordered pair $\{L, M\}$ is LC in $\mathscr{RH}_\infty$ if $L, M \in \mathscr{RH}_\infty$ and there exist $Y_l, Z_l \in \mathscr{RH}_\infty$ such that $MZ_l + LY_l = I$. Furthermore, the pair is a LCF of a plant $P \in \mathscr{R}$ over $\mathscr{RH}_\infty$ if M is square,* $\det M(\infty) \neq 0$ *and $P = M^{-1}L$.*

The matrices Y_r, Z_r, Y_l and Z_l in the above definitions are known as the Bézout factors (or coefficients) of their respective coprime pairs.

It is convenient to define sets of coprime pairs as well as coprime factorisations of a plant and associated Bézout factor pairs, as in the following definitions.

Definition 2.19 *The set of all RC (resp. LC) pairs in $\mathscr{RH}_\infty$ is defined as $\mathscr{C}_r$ (resp. $\mathscr{C}_l$). Similarly, the set of all RCFs (resp. LCFs) of a plant $P \in \mathscr{R}$ over $\mathscr{RH}_\infty$ is defined as $\mathscr{C}_r(P)$ (resp. $\mathscr{C}_l(P)$).*

Definition 2.20 *Let $\{N, M\} \in \mathscr{C}_r$ and $\{L, \tilde{M}\} \in \mathscr{C}_l$. The associated Bézout factor sets are defined as*

$$\mathscr{C}^\dagger \left[\begin{smallmatrix} M \\ N \end{smallmatrix}\right] = \{\{Y_r, Z_r\} : Y_r, Z_r \in \mathscr{RH}_\infty, Z_r M + Y_r N = I\},$$
$$\mathscr{C}^\dagger \left[\begin{smallmatrix} \tilde{M} & L \end{smallmatrix}\right] = \{\{Y_l, Z_l\} : Y_l, Z_l \in \mathscr{RH}_\infty, \tilde{M} Z_l + L Y_l = I\}.$$

The graph of a plant in $P \in \mathscr{R}$ is defined as the set of all possible bounded input-output pairs corresponding to P. As shown by Vidyasagar (2011, Lemma 7.2.1), the graph of a plant can be generated using its coprime factors. This gives rise to the following definition.

Definition 2.21 (Vinnicombe (2001)) *Let $P \in \mathscr{R}$ and suppose that $\{N, M\} \in \mathscr{C}_r(P)$ and $\{L, \tilde{M}\} \in \mathscr{C}_l(P)$. Then the right and inverse left graph symbols of P are given by*

$$G = \begin{bmatrix} M \\ N \end{bmatrix} \text{ and } \tilde{G} = \begin{bmatrix} \tilde{M} & -L \end{bmatrix}$$

respectively.

Some well known coprime factor stability results are listed in following lemma.

Lemma 2.17 (Zhou et al. (1996) Lemma 5.2) *Let $P \in \mathscr{R}$ and $C \in \mathscr{R}$ and suppose that $\{N, M\} \in \mathscr{C}_r(P)$, $\{L, \tilde{M}\} \in \mathscr{C}_l(P)$, $\{U, V\} \in \mathscr{C}_r(C)$ and $\{W, \tilde{V}\} \in \mathscr{C}_l(C)$. Then the following statements are equivalent:*

(a) $[P, C]$ *is internally stable;*

(b) $\tilde{M}V - LU \in \mathscr{G}\mathscr{H}_\infty$*;*

(c) $\tilde{V}M - WN \in \mathscr{G}\mathscr{H}_\infty$*;*

(d) $\begin{bmatrix} M & U \\ N & V \end{bmatrix} \in \mathscr{G}\mathscr{H}_\infty$*;*

(e) $\begin{bmatrix} \tilde{M} & -L \\ -W & \tilde{V} \end{bmatrix} \in \mathscr{G}\mathscr{H}_\infty$*.*

Before defining normalised coprime factorisations, the following definition of inner and co-inner systems is needed.

Definition 2.22 *A system $P \in \mathscr{R}\mathscr{H}_\infty$ is said to be inner (resp. co-inner) if $P^\sim P = I$ (resp. $PP^\sim = I$). If P is square and inner (or equivalently co-inner), then it is said to be all-pass.*

Definition 2.23 (Vinnicombe (2001) Definition 1.7) *The coprime pairs $\{N, M\} \in \mathscr{C}_r$ and $\{L, \tilde{M}\} \in \mathscr{C}_l$ are said to be normalised if they satisfy*

$$M^\sim M + N^\sim N = I \text{ and } \tilde{M}\tilde{M}^\sim + LL^\sim = I,$$

or equivalently if the associated right and inverse left graph symbols are respectively inner and co-inner.

2.5 $\mathscr{H}_\infty$ Control Synthesis

In this section the 2-ARE based approach of Glover and Doyle (1988) and Doyle et al. (1989) to solving the standard $\mathscr{H}_\infty$ control problem is presented. This will form a basis for developing robust stabilisation results pertaining to BCF uncertainty in the later chapters.

Theorem 2.18 (Zhou et al. (1996) Theorem 17.1) *Consider the generalised plant*

$$G = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right] \in \mathscr{R},$$

where $A \in \mathbb{R}^{n \times n}$, $B_1 \in \mathbb{R}^{n \times q_1}$, $B_2 \in \mathbb{R}^{n \times q_2}$, $C_1 \in \mathbb{R}^{p_1 \times n}$ and $C_2 \in \mathbb{R}^{p_2 \times n}$, and assume that

(A1) (A, B_2) *is stabilisable and* (C_2, A) *is detectable;*

(A2) $D_{12} = \begin{bmatrix} 0 \\ I \end{bmatrix}$, $D_{21} = \begin{bmatrix} 0 & I \end{bmatrix}$ *and* $D_{22} = 0$;

(A3) $\begin{bmatrix} A - j\omega I & B_2 \\ C_1 & D_{12} \end{bmatrix}$ *has full column rank for all* $\omega \in \mathbb{R}$;

(A4) $\begin{bmatrix} A - j\omega I & B_1 \\ C_2 & D_{21} \end{bmatrix}$ *has full row rank for all* $\omega \in \mathbb{R}$.

Let $D_{1\bullet} = \begin{bmatrix} D_{11} & D_{12} \end{bmatrix}$, $D_{\bullet 1} = \begin{bmatrix} D_{11} \\ D_{21} \end{bmatrix}$, $\gamma \in \mathbb{R}_+$ *and define*

$$
\begin{aligned}
R &= D_{1\bullet}^* D_{1\bullet} - \begin{bmatrix} \gamma^2 I_{q_1} & \\ & 0 \end{bmatrix}, \quad \tilde{R} = D_{\bullet 1} D_{\bullet 1}^* - \begin{bmatrix} \gamma^2 I_{p_1} & \\ & 0 \end{bmatrix}, \\
H_\infty &= \begin{bmatrix} A & \\ -C_1^* C_1 & -A^* \end{bmatrix} - \begin{bmatrix} B \\ -C_1^* D_{1\bullet} \end{bmatrix} R^{-1} \begin{bmatrix} D_{1\bullet}^* C_1 & B^* \end{bmatrix}, \\
J_\infty &= \begin{bmatrix} A^* & \\ -B_1 B_1^* & -A \end{bmatrix} - \begin{bmatrix} C^* \\ -B_1 D_{\bullet 1}^* \end{bmatrix} \tilde{R}^{-1} \begin{bmatrix} D_{\bullet 1} B_1^* & C \end{bmatrix}, \\
X_\infty &= \mathrm{Ric}(H_\infty), \ Y_\infty = \mathrm{Ric}(J_\infty), \\
F &= \begin{bmatrix} F_{1\infty} \\ F_{2\infty} \end{bmatrix} = -R^{-1}(D_{1\bullet}^* C_1 + B^* X_\infty), \\
L &= \begin{bmatrix} L_{1\infty} & L_{2\infty} \end{bmatrix} = -(B_1 D_{\bullet 1}^* + Y_\infty C^*) \tilde{R}^{-1}
\end{aligned}
$$

and partition D, F *and* L *as*

$$
\left[\begin{array}{c|ccc}
 & F_{11\infty}^* & F_{12\infty}^* & F_{2\infty}^* \\
\hline
L_{11\infty}^* & D_{1111} & D_{1112} & 0 \\
L_{12\infty}^* & D_{1121} & D_{1122} & I \\
L_{2\infty}^* & 0 & I & 0
\end{array}\right].
$$

Then there exists a controller $C_\infty \in \mathscr{R}^{q \times p}$ *satisfying* $\|\mathcal{F}_l(G, C_\infty)\|_\infty < \gamma$ *if and only if*

(a) $\gamma > \max\left\{ \bar{\sigma}\left(\begin{bmatrix} D_{1111} & D_{1112} \end{bmatrix}\right), \bar{\sigma}\left(\begin{bmatrix} D_{1111}^* & D_{1121}^* \end{bmatrix}\right) \right\}$;

(b) $H_\infty \in \mathrm{dom}(\mathrm{Ric})$ *and* $X_\infty = \mathrm{Ric}(H_\infty) \geq 0$;

(c) $J_\infty \in \mathrm{dom}(\mathrm{Ric})$ *and* $Y_\infty = \mathrm{Ric}(J_\infty) \geq 0$;

(d) $\rho(X_\infty Y_\infty) < \gamma^2$.

If the above conditions hold, the set of all controllers that satisfy $\|\mathcal{F}_l(G, C_\infty)\|_\infty < \gamma$ *is given by* $C_\infty = \mathcal{F}_l(M_\infty, \Phi)$ *where* $\Phi \in \{\Phi \in \mathscr{RH}_\infty, \|\Phi\|_\infty < \gamma\}$,

$$M_\infty = \left[\begin{array}{c|cc} \hat{A} & \hat{B}_1 & \hat{B}_2 \\ \hline \hat{C}_1 & \hat{D}_{11} & \hat{D}_{12} \\ \hat{C}_2 & \hat{D}_{21} & 0 \end{array}\right],$$

$$\hat{D}_{11} = -D_{1121}D_{1111}^* \left(\gamma^2 I - D_{1111}D_{1111}^*\right)^{-1} D_{1112} - D_{1122},$$

$\hat{D}_{12} \in \mathbb{R}^{q_1 \times q_1}$ *and* $\hat{D}_{21} \in \mathbb{R}^{p_1 \times p_1}$ *satisfy*

$$\hat{D}_{12}\hat{D}_{12}^* = I - D_{1121}\left(\gamma^2 I - D_{1111}D_{1111}^*\right)^{-1} D_{1121}^*,$$
$$\hat{D}_{21}\hat{D}_{21}^* = I - D_{1112}\left(\gamma^2 I - D_{1111}D_{1111}^*\right)^{-1} D_{1112}^*,$$

and

$$\hat{B}_2 = Z_\infty(B_2 + L_{12\infty})\hat{D}_{12},$$
$$\hat{C}_2 = -\hat{D}_{21}(C_2 + F_{12\infty}),$$
$$\hat{B}_1 = -Z_\infty L_{2\infty} + \hat{B}_2\hat{D}_{12}^{-1}\hat{D}_{11},$$
$$\hat{C}_1 = F_{2\infty} + \hat{D}_{11}\hat{D}_{21}^{-1}\hat{C}_2,$$
$$\hat{A} = A + BF + \hat{B}_1\hat{D}_{21}^{-1}\hat{C}_2$$

where

$$Z_\infty = \left(I - \gamma^{-2}Y_\infty X_\infty\right)^{-1}.$$

Remark 2.3 *Assumption (A2) of the above theorem can be relaxed. Safonov et al. (1989) suggest various loop-shifting transformations that can be performed to normalise* D_{12} *or* D_{21} *and enforce* $D_{22} = 0$. ♦

Chapter 3

Foundations of Bicoprime Factorisations

3.1 Introduction

In this chapter the concept of bicoprimeness is introduced. Necessary definitions are outlined, followed by the basic properties of the BCFs of a plant; including pole and zero characterisations, parametrisation of a BCF set for a plant and the notion of internal dimension. A series of internal stability tests are then presented based on BCFs of the plant and controller. The material presented in this chapter forms the foundations of BCFs upon which many of the results developed in the subsequent chapters are based.

Polynomial methods received considerable attention by the control community in the 1960's and 1970's, with the Polynomial Matrx Description (PMD) results of Rosenbrock (see Rosenbrock (1970) for a comprehensive study of the field) being a driving force behind this movement. This seminal work then gave rise to state space methods and coprime factor theory, both of which are widely used in many areas of control theory. The BCF theory developed herein can be viewed as a merging of these two fields, dealing with the aspects of PMD theory that were sidelined over the past few decades.

In the early 1980's, Vidyasagar (2011) proposed a functional analysis approach to the study of control theory, introducing coprime factorisations as they are known today. Many of the results developed therein make use of PMDs, often restricting the structure of the PMD (to obtain a factorisation satisfying a coprimeness condition) and the set over which the plant is factorised. Since coprimeness requires the satisfaction of a Bézout identity, it follows that coprime factorisations are only possible over Bézout domains (Vidyasagar, 2011, Lemma 8.1.4), thus necessitating the set restriction.

BCFs over $\mathscr{RH}_\infty$ first appeared in the literature in Vidyasagar (2011) where their existence was acknowledged, though no significant results were given. BCFs, being a generalisation of standard LCFs and RCFs, can act as a link between the two factorisations, explaining their duality (see Vidyasagar (2011, Corollay 4.3.10) for one such case). In the original definition, BCFs of a plant were presented as a quad of objects in $\mathscr{RH}_\infty$.

Definition 3.1 (Vidyasagar (2011) Definition 4.3.1) *The ordered quad* $\{N, M, L, K\}$ *is BC in* $\mathscr{RH}_\infty$ *if* $\{L, M\} \in \mathscr{C}_l$*,* $\{N, M\} \in \mathscr{C}_r$ *and* $K \in \mathscr{RH}_\infty$*. Furthermore, the quad is a BCF of a plant* $P \in \mathscr{R}$ *over* $\mathscr{RH}_\infty$ *if* M *is square,* $\det M(\infty) \neq 0$ *and* $P = NM^{-1}L + K$*.*

From this definition, the claim that LCFs and RCFs of a plant are just special cases of the more abstract BCFs is substantiated. In fact, it is clear that any BCF of a plant $P \in \mathscr{R}$ with $L = I$ (resp. $N = I$) and $K = 0$ defines a RCF (resp. LCF) of P. This now implies that most results derived for BCFs can be easily extended to LCFs and RCFs; a fact demonstrated by many of the results in this section.

As was the case for LC and RC pairs and factorisations, the following definition presents the notation used for the sets of all BC quads and BCFs of a plant.

Definition 3.2 *The set of all BC quads in* $\mathscr{RH}_\infty$ *is defined as* $\mathscr{B}$*. The set of all BCFs of a plant* $P \in \mathscr{R}$ *over* $\mathscr{RH}_\infty$ *is defined as* $\mathscr{B}(P)$*.*

Many coprime factor results use graph symbols of the plant (Vidyasagar, 1984; Vinnicombe, 1993; Lanzon and Papageorgiou, 2009; Dehghani et al., 2009) as defined in Definition 2.21, usually for notational brevity. Similarly, packing a BC quad into a matrix, as in the following definition, is often convenient.

Definition 3.3 *The set* $\widetilde{\mathscr{B}}$ *is defined as*

$$\widetilde{\mathscr{B}} = \left\{ \begin{bmatrix} M & -L \\ N & K \end{bmatrix} : \{N, M, L, K\} \in \mathscr{B} \right\}.$$

The set of all objects in $\widetilde{\mathscr{B}}$ *that define a BCF of a plant* $P \in \mathscr{R}$ *is denoted by* $\widetilde{\mathscr{B}}(P)$*.*

Objects in $\widetilde{\mathscr{B}}(P)$ will henceforth be referred to as the BCF symbols of P. This naming is chosen to parallel that of graph symbols. It is important to note however that such objects are *not* graph symbols of P. That is, they cannot be used to generate the set of all possible bounded input-output pairs of P. It should be noted that any BCF symbol $G \in \widetilde{\mathscr{B}}(P)$ of a plant $P \in \mathscr{R}$ is also a system matrix of P (often referred to as a Rosenbrock matrix) as defined by Rosenbrock (1967).

As mentioned previously, Vidyasagar (2011) suggests that there are two good reasons to study BCFs, the first of which is that state space realisations are in fact BCFs over $\mathbb{R}[s]$, where $\mathbb{R}[s]$ denotes the ring of polynomials. Though this fact is not of any direct importance to the results presented in this thesis, it is proven next as a matter of general interest.

Suppose that a plant $P \in \mathscr{R}$ has a minimal state space realisation $P = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ where $A \in \mathbb{R}^{n \times n}$. Then the quad $\{C, sI - A, B, D\}$ defines a BCF of P over $\mathbb{R}[s]$. This fact follows directly from the PBH tests for controllability and observability. To prove this first note that since all factors belong to $\mathbb{R}[s]$, each pair is coprime in $\mathbb{R}[s]$ if and only if it satisfies the Bézout identity over $\mathbb{R}[s]$, or equivalently if the matrix formed by packing the factors is invertible in $\mathbb{R}[s]$. Therefore,

$$\begin{aligned}\{C, sI - A\} \text{ is RC over } \mathbb{R}[s] &\Leftrightarrow \begin{bmatrix} sI - A \\ C \end{bmatrix} \text{ is invertible in } \mathbb{R}[s] \\ &\Leftrightarrow \operatorname{n rank}\left(\begin{bmatrix} sI - A \\ C \end{bmatrix}\right) = n \\ &\Leftarrow (C, A) \text{ is observable,}\end{aligned}$$

which is true by the minimality assumption. The fact that $\{B, sI - A\}$ is LC over $\mathbb{R}(s)$ can be proven similarly using the controllability of (A, B).

By some minor alterations one could generate P from its BCF symbols using a LFT. Suppose for example that $\{N, M, L, K\} \in \mathscr{B}(P)$ then it is easy to show that

$$\begin{aligned}P &= \mathcal{F}_u\left(\begin{bmatrix} M - I & -L \\ N & K \end{bmatrix}, -I\right) \\ &= \mathcal{F}_u\left(\begin{bmatrix} M & L \\ N & K \end{bmatrix}, \frac{1}{2}M^{-1}\right).\end{aligned}$$

This could be put forth as an argument for redefining BCF symbols. However as will be shown in the next chapter, this arrangement is more convenient in giving state space characterisations of BCFs. Furthermore, this structure mirrors that of standard coprime factor graph symbols given in Definition 2.21.

An alternative method of reconstructing a plant from its BCF symbols as defined in Definition 3.3 is as follows. Let $\begin{bmatrix} M & -L \\ N & K \end{bmatrix} \in \widetilde{\mathscr{B}}(P)$, then P is given by the Schur complement around K.

3.2 Basic Properties of BCFs

Some fundamental features of BCFs are presented in this section; including pole/zero characterisations and the introduction a BCF's internal dimension.

3.2.1 Poles & Zeros

It is a well known result that any plant $P \in \mathscr{R}$ with a RCF $\{N, M\} \in \mathscr{C}_r(P)$ is stable if and only if $M \in \mathscr{G}\mathscr{H}_\infty$[a] (Green and Limebeer, 2012, Lemma A.2.1). The following lemma presents an equivalent result for BCFs.

Lemma 3.1 *Let $P \in \mathscr{R}$ have a BCF $\{N, M, L, K\} \in \mathscr{B}(P)$. Then*

$$P \in \mathscr{R}\mathscr{H}_\infty \Leftrightarrow M \in \mathscr{G}\mathscr{H}_\infty.$$

Proof. This follows from Vidyasagar (2011, Theorem 4.3.12) which states that $p_0 \in \bar{\mathbb{C}}_+$ is a pole of P if and only if it is a transmission zero of M. $\square$

Ünyelioğlu et al. (2000) show that given a BCF of a plant with no additive term, then the plant and its BCF symbol share $\bar{\mathbb{C}}_+$ blocking zeros. The following lemma relates the invariant zeros of a plant to those of its BCF symbols.

Lemma 3.2 *Let $P \in \mathscr{R}$ and $G \in \widetilde{\mathscr{B}}(P)$. Then a complex number $z_0 \in \mathbb{C}$ is an invariant zero of P if and only if it is an invariant zero of G.*

Proof. Let G be given by

$$\begin{aligned} G &= \left[\begin{array}{c:c} M & -L \\ \hdashline N & K \end{array}\right] \\ &= \left[\begin{array}{c|c:c} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ \hdashline C_2 & D_{21} & D_{22} \end{array}\right] \in \widetilde{\mathscr{B}}(P) \end{aligned}$$

and note that $\det D_{11} \neq 0$ since by definition $\det M(\infty) \neq 0$. Then using simple linear algebra it can be shown that

$$\begin{aligned} P &= NM^{-1}L + K \\ &= \left[\begin{array}{c|c} A - B_1 D_{11}^{-1} C_1 & B_2 - B_1 D_{11}^{-1} D_{12} \\ \hline C_2 - D_{21} D_{11}^{-1} C_1 & D_{22} - D_{21} D_{11}^{-1} D_{12} \end{array}\right]. \end{aligned}$$

[a]Sufficiency is obvious. To prove necessity, suppose that $z_0 \in \bar{\mathbb{C}}_+$ is a transmission zero of M but not a pole of P. Then z_0 must also be a transmission zero of N, but that implies that $\begin{bmatrix} M \\ N \end{bmatrix}$ is not invertible in $\mathscr{R}\mathscr{H}_\infty$ and therefore $\{N, M\} \notin \mathscr{C}_r$, which is a contradiction

Now define $\hat{A} = A - B_1 D_{11}^{-1} C_1$, $\hat{B} = B_2 - B_1 D_{11}^{-1} D_{12}$, $\hat{C} = C_2 - D_{21} D_{11}^{-1} C_1$, $\hat{D} = D_{22} - D_{21} D_{11}^{-1} D_{12}$,

$$T_1 = \begin{bmatrix} I & -B_1 D_{11}^{-1} & 0 \\ 0 & -D_{21} D_{11}^{-1} & I \\ 0 & I & 0 \end{bmatrix} \text{ and } T_2 = \begin{bmatrix} I & 0 & 0 \\ -D_{11}^{-1} C_1 & -D_{11}^{-1} D_{12} & D_{11}^{-1} \\ 0 & I & 0 \end{bmatrix}$$

and note that T_1 and T_2 have full rank. Then for all $s \in \mathbb{C}$

$$\begin{aligned} \operatorname{rank}\left(\begin{bmatrix} A - sI & B_1 & B_2 \\ C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{bmatrix}\right) &= \operatorname{rank}\left(T_1 \begin{bmatrix} A - sI & B_1 & B_2 \\ C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{bmatrix} T_2\right) \\ &= \operatorname{rank}\left(\begin{bmatrix} \hat{A} - sI & \hat{B} & \\ \hat{C} & \hat{D} & \\ & & I \end{bmatrix}\right). \end{aligned}$$

The conclusion then follows by the definition of invariant zeros (Definition 2.13). □

Another interesting property of BCF symbols, relating their normal rank to that of their associated plant, is outlined in the following lemma.

Lemma 3.3 *Let $P \in \mathscr{R}$ and $G = \begin{bmatrix} M & -L \\ N & K \end{bmatrix} \in \widetilde{\mathscr{B}}(P)$. Then*

$$\operatorname{n rank}(G) = r + \operatorname{n rank}(P),$$

where $r = \operatorname{n rank}(M)$.

Proof. A Schur decomposition of G gives

$$\begin{aligned} \operatorname{n rank}(G) &= \operatorname{n rank}\left(\begin{bmatrix} I & 0 \\ NM & I^{-1} \end{bmatrix} \begin{bmatrix} M & -L \\ 0 & P \end{bmatrix}\right) \\ &= \operatorname{n rank}\left(\begin{bmatrix} M & -L \\ 0 & P \end{bmatrix}\right) \\ &= \operatorname{n rank}(M) + \operatorname{n rank}(P) \\ &= r + \operatorname{n rank}(P). \end{aligned}$$

□

Both Lemma 3.2 and Lemma 3.3 are expected properties of BCF symbols, since as mentioned earlier they are Rosenbrock matrices of their associated plants (see Rosenbrock (1970, Thoerem 5.2) in combination with Definition 2.13 and Theorem 2.11).

Corollary 3.4 *Let $P \in \mathscr{R}$ and $G \in \widetilde{\mathscr{B}}(P)$. Then P is invertible in $\mathscr{R}$ if and only if G is invertible in $\mathscr{R}$.*

Corollary 3.5 *Let $P \in \mathscr{RH}_\infty$ and $G \in \widetilde{\mathscr{B}}(P)$. Then P is invertible in $\mathscr{RH}_\infty$ if and only if G is invertible in $\mathscr{RH}_\infty$.*

3.2.2 Internal Dimension

It is simple to show that the dimensions of the coprime factors of a plant are fixed and dictated by the number of inputs and outputs of the plant. Suppose that $\{N, M\} \in \mathscr{C}_r(P)$ where $P \in \mathscr{R}^{p\times q}$, then it follows trivially from the definition of RCFs that $N \in \mathscr{RH}_\infty^{p\times q}$ and $M \in \mathscr{RH}_\infty^{q\times q}$. An equivalent fact holds for LCFs of the plant.

Such a restriction does not apply to BCFs (with the exception of the additive term which always has the same dimensions as the plant). Let $P \in \mathscr{R}$ and suppose that $\{N, M, L, K\} \in \mathscr{B}(P)$. Furthermore, let $\{Y_r, Z_r\} \in \mathscr{C}^\dagger \left[\begin{smallmatrix} M \\ N \end{smallmatrix}\right]$, $\{Y_l, Z_l\} \in \mathscr{C}^\dagger \left[\begin{smallmatrix} M & L \end{smallmatrix}\right]$ and define

$$\tilde{N} = \begin{bmatrix} N & 0 \end{bmatrix}, \tilde{M} = \begin{bmatrix} M & \\ & I \end{bmatrix}, \tilde{L} = \begin{bmatrix} L \\ 0 \end{bmatrix}.$$

Then

$$\begin{bmatrix} Y_r & 0 & Z_r \\ 0 & I & 0 \end{bmatrix} \begin{bmatrix} \tilde{M} \\ \tilde{N} \end{bmatrix} = I \text{ and } \begin{bmatrix} \tilde{M} & \tilde{L} \end{bmatrix} \begin{bmatrix} Y_l & 0 \\ 0 & I \\ Z_l & 0 \end{bmatrix} = I.$$

Therefore $\{\tilde{N}, \tilde{M}, \tilde{L}, K\} \in \mathscr{B}(P)$ is also a BCF of P with arbitrarily inflated factor dimensions. This fact gives rise to the following definition.

Definition 3.4 *The internal dimension of a BC quad $\{N, M, L, K\} \in \mathscr{B}$ is defined as the number of rows/columns of M. The set of all BC quads (resp. BCFs of a plant $P \in \mathscr{R}$) of internal dimension r is defined as $\mathscr{B}_r$ (resp. $\mathscr{B}_r(P)$).*

Clearly, the internal dimension of a BCF is given by $\operatorname{rank}(M(\infty)) = \operatorname{n rank}(M)$ since by definition $\det(M(\infty)) \neq 0$.

An interesting case arises when considering BCFs whose additive term is restricted to be zero. This is outlined in the following lemma.

Lemma 3.6 *Let $P \in \mathscr{R}$ and suppose that it has a BCF $\{N, M, L, 0\} \in \mathscr{B}_r(P)$. Then*

$$\operatorname{n rank}(P) \leq r.$$

Before proving Lemma 3.6 the following result is needed.

Lemma 3.7 *Let $A \in \mathscr{R}^{p\times n}$ and $B \in \mathscr{R}^{n\times q}$ with $n \leq \min\{p, q\}$. Then*

$$\operatorname{n rank}(AB) = n \Leftrightarrow \operatorname{n rank}(A) = \operatorname{n rank}(B) = n.$$

Proof.
($\Rightarrow$) Suppose that $\operatorname{nrank}(AB) = n$, then for some $s_0 \in \mathbb{C}$ $\operatorname{rank}(A(s_0)B(s_0)) = n$ and the result follows from Sylvester's rank inequality (Laub, 2005, Theorem 3.19).
($\Leftarrow$) Suppose that $\operatorname{nrank}(AB) < n$ while $\operatorname{nrank}(A) = \operatorname{nrank}(B) = n$, then for all $s \in \mathbb{C}$ $\operatorname{rank}(A(s)B(s)) < n$. This implies that for all $s_0 \in \mathbb{C}$ where $\operatorname{rank}(A(s_0)) = n$, $\operatorname{rank}(B(s_0)) < n$ and *vice versa*. By noting that a system can only have a finite number of transmission zeros a contradiction arises which concludes the proof. □

Proof of Lemma 3.6. Suppose that contrary to the lemma statement $r < \operatorname{nrank}(P)$. Then using Lemma 3.7 $\operatorname{nrank}\left(NM^{-1}L\right) \leq r < \operatorname{nrank}(P)$ which is a contradiction since $P = NM^{-1}L$ and the proof is complete. □

The following theorem utilises Lemma 3.6 to establish a lower bound on the internal dimension of the BCFs of a plant.

Theorem 3.8 *Let $P \in \mathscr{R}$ and suppose that it has a BCF $\{N, M, L, K\} \in \mathscr{B}_r(P)$. Then*

$$\inf_{\tilde{P} \in \mathscr{RH}_\infty} \operatorname{nrank}\left(P - \tilde{P}\right) \leq r.$$

Proof. Since $\{N, M, L, 0\} \in \mathscr{B}_r(P - K)$ it follows from Lemma 3.6 that

$$\operatorname{nrank}(P - K) \leq r.$$

Now suppose that

$$r < \inf_{\tilde{P} \in \mathscr{RH}_\infty} \operatorname{nrank}\left(P - \tilde{P}\right).$$

Then

$$\operatorname{nrank}(P - K) < \inf_{\tilde{P} \in \mathscr{RH}_\infty} \operatorname{nrank}\left(P - \tilde{P}\right)$$

which is a contradiction since $K \in \mathscr{RH}_\infty$ and the proof is complete. □

3.2.3 Parametrisation of BCFs

As shown by Green and Limebeer (2012, Lemma A.2.1) and Vidyasagar (2011, Theorem 4.1.13) LCFs or RCFs of a plant are unique up to pre- or post-multiplication of the factors by an object in $\mathscr{GH}_\infty$. For example, let $P \in \mathscr{R}$ and suppose that $\{N, M\} \in \mathscr{C}_r(P)$, then $\{NQ, MQ\} \in \mathscr{C}_r(P)$ for any $Q \in \mathscr{GH}_\infty$ of compatible dimensions. On the other hand, parametrising BCFs of a plant is not as simple. A set of BCFs of a given plant will be parametrised in this section.

The following lemma gives sufficient conditions for a BC quad to retain its bicoprimeness under predefined stable perturbations of the factors.

Lemma 3.9 *Consider the BC quad* $\{N, M, L, K\} \in \mathscr{B}$ *and let* $Q, R, S, T \in \mathscr{RH}_\infty$ *and* $U, V \in \mathscr{GH}_\infty$*. Then*

$$\begin{bmatrix} V(M - LSN)U & -V(L - MR) \\ (N - QM)U & K + T \end{bmatrix} \in \mathscr{B}^m$$

if $[Q, LS]$ *and* $[SN, R]$ *are internally stable.*

Proof. Since $U, V \in \mathscr{GH}_\infty$, it follows that they can always be absorbed into the Bézout factors[b], hence

$$\begin{aligned}
\{(N - QM)U, V(M - LSN)U\} \in \mathscr{C}_r \\
&\Leftrightarrow \{N - QM, M - LSN\} \in \mathscr{C}_r \\
&\Leftrightarrow \exists \tilde{Y}_r, \tilde{Z}_r \in \mathscr{RH}_\infty : \tilde{Z}_r(M - LSN) + \tilde{Y}_r(N - QM) = I \\
&\Leftrightarrow \exists \tilde{Y}_r, \tilde{Z}_r \in \mathscr{RH}_\infty : \begin{bmatrix} \tilde{Z}_r & \tilde{Y}_r \end{bmatrix} \begin{bmatrix} I & -LS \\ -Q & I \end{bmatrix} \begin{bmatrix} M \\ N \end{bmatrix} = I \\
&\Leftrightarrow \exists \tilde{Y}_r, \tilde{Z}_r \in \mathscr{RH}_\infty : \begin{bmatrix} \tilde{Z}_r & \tilde{Y}_r \end{bmatrix} \begin{bmatrix} I & -LS \\ -Q & I \end{bmatrix} \in \mathscr{C}^\dagger \left[\begin{smallmatrix} M \\ N \end{smallmatrix} \right] \\
&\Leftarrow \begin{bmatrix} I & -LS \\ -Q & I \end{bmatrix} \in \mathscr{GH}_\infty \\
&\Leftrightarrow [Q, LS] \text{ is internally stable.}
\end{aligned}$$

An alternative proof is provided for the LC pair.

$$\begin{aligned}
\{V(L - MR), V(M - LSN)U\} \in \mathscr{C}_l &\Leftrightarrow \{L - MR, M - LSN\} \in \mathscr{C}_l \\
&\Leftrightarrow \mathscr{C}^\dagger \left[\begin{smallmatrix} M - LSN & L - MR \end{smallmatrix} \right] \neq \emptyset \\
&\Leftrightarrow \mathscr{C}^\dagger \left(\left[\begin{smallmatrix} M & L \end{smallmatrix} \right] \left[\begin{smallmatrix} I & -R \\ -SN & I \end{smallmatrix} \right] \right) \neq \emptyset \\
&\Leftarrow \begin{bmatrix} I & -R \\ -SN & I \end{bmatrix} \in \mathscr{GH}_\infty \\
&\Leftrightarrow [R, SN] \text{ is internally stable.}
\end{aligned}$$

Finally, since $K + T \in \mathscr{RH}_\infty$ the conclusion follows. □

Lemma 3.9 in combination with an initial BCF of a plant $P \in \mathscr{R}$ enables the parametrisation of a set of BCFs of P as given in the following lemma.

[b]Let $\{N, M\} \in \mathscr{C}_r$, $U, V \in \mathscr{GH}_\infty$, suppose that $\{Y_r, Z_r\} \in \mathscr{C}^\dagger \left[\begin{smallmatrix} M \\ N \end{smallmatrix} \right]$ and define $\tilde{Y}_r = U^{-1}Y$ and $\tilde{Z}_r = U^{-1}Z_rV^{-1}$. Then $\tilde{Z}_rVMU + \tilde{Y}_rNU = I$, hence $\{NU, VMU\} \in \mathscr{C}_r$.

Lemma 3.10 *Let $P \in \mathscr{R}$ have the BCF $\{N, M, L, K\} \in \mathscr{B}_r(P)$, then*

$$\begin{bmatrix} \tilde{M} & -\tilde{L} \\ \tilde{N} & \tilde{K} \end{bmatrix} = \begin{bmatrix} Q_l M Q_r & -Q_l(L + MR_r) \\ (N + R_l M)Q_r & K - NR_r - R_l L - R_l M R_r \end{bmatrix}$$
$$= \begin{bmatrix} Q_l & 0 \\ R_l & I \end{bmatrix} \begin{bmatrix} M & -L \\ N & K \end{bmatrix} \begin{bmatrix} Q_r & -R_r \\ 0 & I \end{bmatrix} \in \widetilde{\mathscr{B}}_r(P)$$

for all $Q_l, Q_r \in \mathscr{GH}_\infty$ and $R_l, R_r \in \mathscr{RH}_\infty$ with compatible dimensions.

Proof. That $\{\tilde{N}, \tilde{M}, \tilde{L}, \tilde{K}\} \in \mathscr{B}_r$ follows from Lemma 3.9 with $S = 0$, $V = Q_l$, $U = Q_r$, $Q = -R_l$, $R = -R_r$ and $T = -NR_r - R_l L - R_l M R_r$. Then $P = \tilde{N}\tilde{M}^{-1}\tilde{L} + \tilde{K}$ can be shown by direct calculation. □

Observe that the above parametrisation does not allow for a variation of the internal dimension of the BCFs; it is therefore immediate that it does not cover the entire set of BCFs for a given plant. It is interesting however that

$$\begin{bmatrix} Q_l & 0 \\ R_l & I \end{bmatrix}, \begin{bmatrix} Q_r & -R_r \\ 0 & I \end{bmatrix} \in \mathscr{GH}_\infty$$

which parallels the fact that coprime factorisations can by parametrised be pre- or post-multiplication of the graph symbols by objects in $\mathscr{GH}_\infty$. It is also notable that the above proposed parametrisation covers all RCFs and LCFs of a plant by an appropriate selection of Q_l, Q_r and R_l, R_r. This can be seen by considering the BCF $\{N, M, I, 0\} \in \mathscr{B}(P)$ (which gives $\{N, M\} \in \mathscr{C}_r(P)$) and setting $R_l = R_r = 0$ and $Q_l = I$.

Note that the parametrisation of Lemma 3.10 is a strict system equivalence as defined by Rosenbrock (1977) and Fuhrmann (1977). Furthermore, Rosenbrock (1970, Theorem 3.5) suggests that two system matrices are equivalent if and only if they give rise to the same transfer function matrix, which seems to contradict the claim that Lemma 3.10 does not parametrise all BCFs of a plant. This problem can be circumvented by padding one of the BCF symbols with an identity matrix such that the internal dimensions of the two factorisations match (see for example Rosenbrock (1970, Theorem 3.2)). Finally, the works of Coppel (1974) and Smith (1986) (which extend the PMD results of Rosenbrock to factorisations over Bézout domains) suggest that Lemma 3.10 can in fact be used to parametrise all BCFs of a plant.

3.3 Internal Stability

As is the case for RCFs and LCFs in Lemma 2.17, BCFs can be used to establish the internal stability of a feedback interconnection. This proves to be an important use of coprime factorisations, giving rise to many powerful results including the well known Youla parametrisation (Youla et al., 1976) of all stabilising controllers. This parametrisation can be used to solve a number of problems in control theory including the $\mathscr{H}_\infty$ robust stabilisation problem (Zhou et al., 1996, Section 17.6).

When starting with coprime factorisations of the plant and controller, a BCF of the closed loop transfer matrix is naturally obtained. This was the second point given by Vidyasagar (2011) in an attempt to motivate the study of BCFs. This was partly proven by Gündeş (1996), where it was shown that given a plant $P \in \mathscr{R}$ and controller $C \in \mathscr{R}$ with $\{N, M\} \in \mathscr{C}_r(P)$ and $\{W, V\} \in \mathscr{C}_l(C)$, then $\{N, VM - WN, W, 0\} \in \mathscr{B}(P\,(I - CP)^{-1})$. The following lemma provides a more comprehensive proof giving a BCF of $H(P, C)$ in terms of the right and inverse left graph symbols of P and C respectively.

Lemma 3.11 *Let $G \in \mathscr{RH}_\infty$ be a right graph symbol of a plant $P \in \mathscr{R}$ and $\tilde{K} \in \mathscr{RH}_\infty$ an inverse left graph symbol of a controller $C \in \mathscr{R}$. Then $\{G, \tilde{K}G, \tilde{K}, 0\} \in \mathscr{B}(H(P, C))$.*

Proof. The fact that $H(P, C) = G(\tilde{K}G)^{-1}\tilde{K}$ is shown by Vinnicombe (2001, p. 18). Since both G and $\tilde{K}$ are graph symbols associated with a RCF and LCF respectively, they are by definition left and right invertible in $\mathscr{RH}_\infty$ respectively. Therefore $\{G, \tilde{K}G\} \in \mathscr{C}_r$ and $\{\tilde{K}, \tilde{K}G\} \in \mathscr{C}_l$ which concludes the proof. □

Remark 3.1 *It is important to note that Lemma 3.11 is compatible with previous coprime factor results. In fact, the standard coprime factor result given by Lemma 2.17(c) (resp. Lemma 2.17(b)) is just a straight forward combination of Lemma 3.1 and Lemma 3.11 (resp. the dual to Lemma 3.11) (Vidyasagar, 2011, Theorem 5.1.6 and Lemma 5.1.7). This further demonstrates how existing coprime factor results are rooted in BCFs.* ♦

A number of internal stability results making use of BCFs of the plant were derived by Desoer and Gündeş (1988). The results presented therein considered the feedback interconnection of a plant $P \in \mathscr{R}$ and controller $C \in \mathscr{R}$ in terms of a BCF of P and LCF or RCF of C. Initially, the restriction that the additive term of the BCF is zero was imposed but later lifted. The approach taken by Desoer and Gündeş (1988) to prove internal stability was via a transformation from BCF to a LCF or RCF of the plant, which as a mathematical result is interesting in its own right.

Internal stability results based on BCFs of the plant and controller will now be presented. The first considers the case where the additive terms of the BCFs of both the plant and controller are restricted to zero. This is then used to derive a more generalised condition where the restriction is removed.

Theorem 3.12 *Consider the standard positive feedback interconnection of a plant $P \in \mathscr{R}$ and controller $C \in \mathscr{R}$ depicted in Figure 2.2 and suppose that $\{N, M, L, 0\} \in \mathscr{B}(P)$ and $\{U, V, W, 0\} \in \mathscr{B}(C)$. Then*

$$[P, C] \text{ is internally stable} \Leftrightarrow \begin{bmatrix} M & -LU \\ -WN & V \end{bmatrix} \in \mathscr{GH}_\infty. \tag{3.1}$$

Proof. For simplicity, define

$$\tilde{M} = \begin{bmatrix} & M \\ V & \end{bmatrix}, \ \tilde{N} = \begin{bmatrix} U & \\ & N \end{bmatrix} \text{ and } \tilde{L} = \begin{bmatrix} L & \\ & W \end{bmatrix}$$

and note that $\{\tilde{N}, \tilde{M}, \tilde{L}, 0\} \in \mathscr{B}$. Then

$$\begin{aligned} \begin{bmatrix} I & -C \\ -P & I \end{bmatrix}^{-1} &= (I - \tilde{N}\tilde{M}^{-1}\tilde{L})^{-1} \\ &= I + \tilde{N}(\tilde{M} - \tilde{L}\tilde{N})^{-1}\tilde{L}. \end{aligned}$$

Now, $\{\tilde{N}, \tilde{M}, \tilde{L}, 0\} \in \mathscr{B}$ and Lemma 3.9 imply that $\{\tilde{N}, \tilde{M} - \tilde{L}\tilde{N}, \tilde{L}, I\} \in \mathscr{B}$ and hence that

$$\begin{bmatrix} \tilde{M} - \tilde{L}\tilde{N} & -\tilde{L} \\ \tilde{N} & I \end{bmatrix} \in \mathscr{B}^m \left(\begin{bmatrix} I & -C \\ -P & I \end{bmatrix}^{-1} \right).$$

Then from Lemma 3.1 it follows that $[P, C]$ is internally stable if and only if $\tilde{M} - \tilde{L}\tilde{N} \in \mathscr{G}\mathscr{H}_\infty$. The proof is then concluded by a simple column exchange. □

It is simple to see that in the special cases where $L = I$ and $W = I$ (resp. $N = I$ and $U = I$), (3.1) reduces to the standard coprime factor result given by Lemma 2.17(d) (resp. Lemma 2.17(e)). Furthermore, the main stability results of Desoer and Gündeş (1988) follow via trivial simplifications of Theorem 3.12.

The following theorem presents the most abstract stability test attainable using BCFs. Both the plant and controller are given full BCFs and no restrictions are imposed on any of the factors.

Theorem 3.13 *Consider the standard positive feedback interconnection of a plant $P \in \mathscr{R}$ and controller $C \in \mathscr{R}$ depicted in Figure 2.2 and suppose that $\{N, M, L, K\} \in \mathscr{B}(P)$ and $\{U, V, W, X\} \in \mathscr{B}(C)$. Then*

$$[P, C] \text{ is internally stable} \Leftrightarrow \begin{bmatrix} 0 & M & L & 0 \\ V & 0 & 0 & W \\ U & 0 & I & -X \\ 0 & N & -K & I \end{bmatrix} \in \mathscr{G}\mathscr{H}_\infty. \tag{3.2}$$

Proof. First define

$$\begin{aligned} \tilde{N} &= \begin{bmatrix} -N & K \end{bmatrix}, \ \tilde{M} = \begin{bmatrix} M & \\ & I \end{bmatrix}, \ \tilde{L} = \begin{bmatrix} -L \\ I \end{bmatrix}, \\ \tilde{U} &= \begin{bmatrix} -U & X \end{bmatrix}, \ \tilde{V} = \begin{bmatrix} V & \\ & I \end{bmatrix}, \ \tilde{W} = \begin{bmatrix} -W \\ I \end{bmatrix} \end{aligned}$$

and note that $\{\tilde{N}, \tilde{M}, \tilde{L}, 0\} \in \mathscr{B}(P)$ and $\{\tilde{U}, \tilde{V}, \tilde{W}, 0\} \in \mathscr{B}(C)$. Then using Theorem 3.12, $[P, C]$ is internally stable if and only if

$$\begin{bmatrix} \tilde{M} & -\tilde{L}\tilde{U} \\ -\tilde{W}\tilde{N} & \tilde{V} \end{bmatrix} \in \mathscr{GH}_\infty \Leftrightarrow \begin{bmatrix} M & 0 & -LU & LX \\ 0 & I & U & -X \\ -WN & WK & V & 0 \\ N & -K & 0 & I \end{bmatrix} \in \mathscr{GH}_\infty$$

$$\Leftrightarrow \begin{bmatrix} -LU & M & 0 & LX \\ V & -WN & WK & 0 \\ U & 0 & I & -X \\ 0 & N & -K & I \end{bmatrix} \in \mathscr{GH}_\infty$$

$$\Leftrightarrow \begin{bmatrix} 0 & M & L & 0 \\ V & 0 & 0 & W \\ U & 0 & I & -X \\ 0 & N & -K & I \end{bmatrix} \in \mathscr{GH}_\infty.$$

The last equivalence follows by pre-multiplying with

$$\begin{bmatrix} I & \operatorname{diag}(L, W) \\ 0 & I \end{bmatrix} \in \mathscr{GH}_\infty,$$

which concludes the proof. $\square$

A series of results will now be presented each considering a special case. First however, the following lemma is given as it is useful in proving many of the subsequent results.

Lemma 3.14 *Suppose that*

$$S = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \in \mathscr{RH}_\infty$$

where $S_{22} \in \mathscr{GH}_\infty$. *Then*

$$S^{-1} \in \mathscr{GH}_\infty \Leftrightarrow S_{11} - S_{12}S_{22}^{-1}S_{21} \in \mathscr{GH}_\infty.$$

Proof. From Schur complement decomposition

$$S = \begin{bmatrix} I & S_{12}S_{22}^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} S_{11} - S_{12}S_{22}^{-1}S_{21} & 0 \\ 0 & S_{22} \end{bmatrix} \begin{bmatrix} I & 0 \\ S_{22}^{-1}S_{21} & I \end{bmatrix}.$$

The supposition $S_{22}^{-1} \in \mathscr{GH}_\infty$ implies that

$$\begin{bmatrix} I & S_{12}S_{22}^{-1} \\ 0 & I \end{bmatrix}, \begin{bmatrix} I & 0 \\ S_{22}^{-1}S_{21} & I \end{bmatrix} \in \mathscr{GH}_\infty$$

and the result follows. $\square$

Case 1: $P \in \mathscr{RH}_\infty$ or $C \in \mathscr{RH}_\infty$

In the first case, a necessary and sufficient internal stability condition is presented for when the plant or controller is stable.

Lemma 3.15 *Consider the standard positive feedback interconnection of a plant $P \in \mathscr{R}$ and controller $C \in \mathscr{RH}_\infty$ depicted in Figure 2.2 and suppose that $\{N, M, L, 0\} \in \mathscr{B}(P)$. Then*

$$[P, C] \text{ is internally stable} \Leftrightarrow M - LCN \in \mathscr{GH}_\infty. \tag{3.3}$$

Proof. Let $\{U, V, W, 0\} \in \mathscr{B}(C)$. The proof follows by noting that $C \in \mathscr{RH}_\infty \Leftrightarrow V \in \mathscr{GH}_\infty$ by Lemma 3.1 and then applying Lemma 3.14 to (3.1). $\square$

Lemma 3.16 (Dual to Lemma 3.15) *Consider the standard positive feedback interconnection of a plant $P \in \mathscr{RH}_\infty$ and controller $C \in \mathscr{R}$ depicted in Figure 2.2 and suppose that $\{U, V, W, 0\} \in \mathscr{B}(C)$. Then*

$$[P, C] \text{ is internally stable} \Leftrightarrow V - WPU \in \mathscr{GH}_\infty. \tag{3.4}$$

Proof. The proof follows by duality to Lemma 3.15. $\square$

Case 2: $[K, X]$ internally stable

The following lemma considers the special case where the additive terms of the plant and controller BCFs satisfy an internal stability condition.

Lemma 3.17 *Consider the standard positive feedback interconnection of a plant $P \in \mathscr{R}$ and controller $C \in \mathscr{R}$ depicted in Figure 2.2. Furthermore, let $\{N, M, L, K\} \in \mathscr{B}(P)$ and $\{U, V, W, X\} \in \mathscr{B}(C)$ and suppose that $[K, X]$ is internally stable. Then $[P, C]$ is internally stable if and only if*

$$\begin{bmatrix} M - LX\left(I - KX\right)^{-1} N & -L\left(I - XK\right)^{-1} U \\ -W\left(I - KX\right)^{-1} N & V - WK\left(I - XK\right)^{-1} U \end{bmatrix} \in \mathscr{GH}_\infty. \tag{3.5}$$

Proof. Since $[K, X]$ is assumed to be internally stable and $K, X \in \mathscr{RH}_\infty$ we have

$$\begin{bmatrix} I & -X \\ -K & I \end{bmatrix} \in \mathscr{GH}_\infty$$

from Lemma 2.15. The result then follows by applying Lemma 3.14 to (3.2). $\square$

The result of Lemma 3.17 can be restated more succinctly using the Redheffer star product and BCF symbols of the plant and controller. This is given in the following corollary.

Corollary 3.18 *Consider the standard positive feedback interconnection of a plant* $P \in \mathscr{R}$ *and controller* $C \in \mathscr{R}$ *depicted in Figure 2.2. Furthermore, let* $G_P \in \widetilde{\mathscr{B}}(P)$ *and* $G_C \in \widetilde{\mathscr{B}}(C)$ *and suppose that the standard positive feedback interconnection of the additive terms of the two BCFs is internally stable. Then*

$$[P, C] \text{ is internally stable} \Leftrightarrow G_P \star \left(\left[\begin{smallmatrix} & I \\ I & \end{smallmatrix}\right] G_C \left[\begin{smallmatrix} & I \\ I & \end{smallmatrix}\right] \right) \in \mathscr{G}\mathscr{H}_\infty. \tag{3.6}$$

Proof. The proof follows trivially by calculation of (3.6) which gives (3.5). □

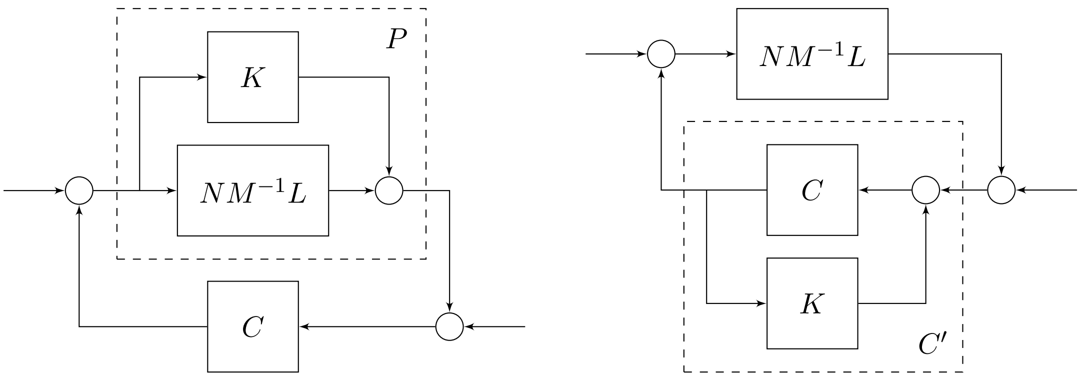
Case 3: $X = 0$ **or** $K = 0$

The following lemma considers the case where only the plant BCF is allowed to have an additive component.

Lemma 3.19 *Consider the standard positive feedback interconnection of a plant* $P \in \mathscr{R}$ *and controller* $C \in \mathscr{R}$ *depicted in Figure 2.2 and suppose that* $\{N, M, L, K\} \in \mathscr{B}(P)$ *and* $\{U, V, W, 0\} \in \mathscr{B}(C)$*. Then*

$$[P, C] \text{ is internally stable} \Leftrightarrow \begin{bmatrix} M & -LU \\ -WN & V - WKU \end{bmatrix} \in \mathscr{G}\mathscr{H}_\infty. \tag{3.7}$$

Proof. The proof follows by noting that $[K, 0]$ is internally stable since $K \in \mathscr{R}\mathscr{H}_\infty$ and then using Lemma 3.17 with $X = 0$. □



(a) $[P, C]$.　(b) $[P - K, C']$

Figure 3.1: Linear loop shifting transformation.

It is worth noting that the above is equivalent to applying Theorem 3.12 after a loop shifting operation absorbing K into the controller. A pictorial representation of this transformation is shown in Figure 3.1. To observe this first consider $C' = C\,(I - KC)^{-1}$ which is a standard linear shift loop transformation as described by Green and Limebeer

(2012, Lemma 3.5.3). Then

$$\begin{aligned} C' &= C\left(I - KC\right)^{-1} \\ &= UV^{-1}W\left(I - KUV^{-1}W\right)^{-1} \\ &= UV^{-1}\left(I - WKUV^{-1}\right)^{-1}W \\ &= U\left(V - WKU\right)^{-1}W. \end{aligned}$$

Then by applying Lemma 3.9 it follows that $\{U, V - WKU, W, 0\} \in \mathscr{B}(C')$. Finally, as mentioned above, the result follows via Theorem 3.12 with the BCFs $\{N, M, L, 0\} \in \mathscr{B}(P - K)$ and $\{U, V - WKU, W, 0\} \in \mathscr{B}(C')$.

Lemma 3.20 (Dual to Lemma 3.19) *Consider the standard positive feedback interconnection of a plant $P \in \mathscr{R}$ and controller $C \in \mathscr{R}$ depicted in Figure 2.2 and suppose that $\{N, M, L, 0\} \in \mathscr{B}(P)$ and $\{U, V, W, X\} \in \mathscr{B}(C)$. Then*

$$[P, C] \text{ is internally stable} \Leftrightarrow \begin{bmatrix} M - LXN & -LU \\ -WN & V \end{bmatrix} \in \mathscr{G}\mathscr{H}_\infty. \tag{3.8}$$

Proof. The proof follows by duality to Lemma 3.19. □

Case 4: $[K, C]$ or $[P, X]$ internally stable

Lemma 3.21 *Consider the standard positive feedback interconnection of a plant $P \in \mathscr{R}$ and controller $C \in \mathscr{R}$ depicted in Figure 2.2. Furthermore, let $\{N, M, L, K\} \in \mathscr{B}(P)$ and suppose that $[K, C]$ is internally stable. Finally, define $C' = C\left(I - KC\right)^{-1} \in \mathscr{R}\mathscr{H}_\infty$. Then*

$$[P, C] \text{ is internally stable} \Leftrightarrow M - LC'N \in \mathscr{G}\mathscr{H}_\infty. \tag{3.9}$$

Proof. First let $\{U, V, W, 0\} \in \mathscr{B}(C)$ and note that since $K \in \mathscr{R}\mathscr{H}_\infty$, $[K, C]$ is internally stable if and only if $V - WKU \in \mathscr{G}\mathscr{H}_\infty$ by Lemma 3.16. Then applying Lemma 3.14 to (3.7), $[P, C]$ is internally stable if and only if

$$\begin{aligned} M - LU\left(V - WKU\right)^{-1}WN \in \mathscr{G}\mathscr{H}_\infty &\Leftrightarrow M - LC\left(I - KC\right)^{-1}N \in \mathscr{G}\mathscr{H}_\infty \\ &\Leftrightarrow M - LC'N \in \mathscr{G}\mathscr{H}_\infty \end{aligned}$$

which concludes the proof. □

Lemma 3.22 (Dual to Lemma 3.21) *Consider the standard positive feedback interconnection of a plant $P \in \mathscr{R}$ and controller $C \in \mathscr{R}$ depicted in Figure 2.2. Furthermore, let $\{U, V, W, X\} \in \mathscr{B}(C)$ and suppose that $[P, X]$ is internally stable. Finally, define*

$P' = P\,(I - XP)^{-1} \in \mathscr{RH}_\infty$. Then

$$[P, C] \text{ is internally stable} \Leftrightarrow V - WP'U \in \mathscr{GH}_\infty. \tag{3.10}$$

Proof. The proof follows by duality to Lemma 3.21. □

3.4 Numerical Examples

Example 1

A simple numerical example is presented below to demonstrate how the results of this chapter can be applied in a practical situation.

Consider the plant

$$P = \begin{bmatrix} \frac{3(s+1)}{(s+2)(s-1)} & \frac{4}{s-1} \\ \frac{2s+1}{(s+2)(s-1)} & \frac{2}{s-1} \end{bmatrix} \in \mathscr{R},$$

a BCF of which is given by

$$G = \left[\begin{array}{c|c} M & -L \\ \hline N & K \end{array}\right] = \left[\begin{array}{c|cc} \frac{s-1}{s+1} & -\frac{2}{s+1} & -\frac{4}{s+1} \\ \hline \frac{3}{s+2} & \frac{3}{s+2} & \frac{4}{s+2} \\ \frac{2s+1}{(s+2)(s+1)} & \frac{2s+1}{(s+1)(s+2)} & \frac{2s}{(s+2)(s+1)} \end{array}\right] \in \widetilde{\mathscr{B}}_1(P).$$

Clearly, the above BCF of P has internal dimension 1. Hence this BCF of the plant cannot be obtained by applying Lemma 3.10 to a LCF or RCF of P, which would give an internal dimension of 2.

Furthermore, note that in accordance to Lemma 3.3 both the plant and its BCF symbol have full normal rank. However, neither is invertible in $\mathscr{R}$ since $\det(P(\infty)) = \det(G(\infty)) = 0$, a fact also implied by Corollary 3.4.

Now let a candidate controller for P be given by

$$C = -\begin{bmatrix} \frac{(s+1)^2}{s(2s+1)} & \frac{s+1}{s(2s+1)} \\ \frac{1}{4s} & \frac{s+1}{4s} \end{bmatrix} \in \mathscr{R}$$

and consider the standard positive feedback interconnection of P and C.

The internal stability of $[P, C]$ can be established using most of the results presented in the preceding section. However, an interesting case arises when using Lemma 3.21. First note that the supposition that $[K, C]$ is internally stable is satisfied since

$$C' = C\,(I - KC)^{-1} = -\begin{bmatrix} 0.5 & \\ & \frac{s+1}{2(2s+1)} \end{bmatrix} \in \mathscr{RH}_\infty.$$

Then (3.9) can be evaluated as

$$M - LC'N = \frac{s^2+s+3}{(s+1)(s+2)} \in \mathscr{GH}_\infty.$$

It can be therefore concluded that C is an internally stabilising controller for P. The internal stability of $[P, C]$ has been established by a scalar test, despite the fact that both P and C are Multiple Input-Multiple Output (MIMO) objects.

Example 2

One could argue that the claim in example 1 (that internal stability of a MIMO object was established via a Single Input-Single Output (SISO) test) is not completely accurate since the stability of C' had also to be tested. However, it is assumed that this is known *a priori*. There are cases where using BCFs gives a truly SISO test for internal stability. Consider for example the standard feedback interconnection of a plant given as a BCF with internal dimension 1 and zero additive term[c] and a controller in $\mathscr{RH}_\infty$, then using Lemma 3.15 the internal stability of the feedback interconnection can be established by a SISO test.

Suppose that the transfer matrix of a plant is given by

$$P = \begin{bmatrix} \frac{2(s+1)}{(s-1)(s+2)} & \frac{s+1}{s+2} \\ \frac{2(2s+1)}{(s-1)(s+2)} & \frac{2s+1}{s+2} \end{bmatrix} \in \mathscr{R}$$

and note that

$$\left[\begin{array}{c|c} M & -L \\ \hline N & K \end{array}\right] = \left[\begin{array}{c|cc} \frac{s-1}{s+1} & -\frac{2}{s+1} & \frac{1-s}{s+1} \\ \hline \frac{s+1}{s+2} & 0 & 0 \\ \frac{2s+1}{s+2} & 0 & 0 \end{array}\right] \in \widetilde{\mathscr{B}}_1(P).$$

Now consider the standard positive feedback interconnection of P and a controller $C \in \mathscr{RH}_\infty$ given by

$$C = \begin{bmatrix} \frac{3(2s+1)}{s+1} & -\frac{2s+1}{s+1} \\ \frac{3(2s+1)}{s+1} & -\frac{2s+1}{s+1} \end{bmatrix}.$$

Since $C \in \mathscr{RH}_\infty$, Lemma 3.15 can be used to test the internal stability of $[P, C]$. Using the above BCF of P, (3.3) can be evaluated as

$$M - LCN = -\frac{s+2}{s+1} \in \mathscr{GH}_\infty,$$

which shows that $[P, C]$ is internally stable.

[c]Using Lemma 3.6 this implies that the plant has normal rank 1. Such cases could be encountered when multiple sensors or actuators are attached to a SISO system for redundancy.

3.5 Summary & Conclusion

The foundations of BCF theory were developed in this chapter. Bicoprimeness over $\mathscr{RH}_\infty$ was first introduced followed by the definition of BCFs of a plant. The basic properties of such factorisations were presented; including a characterisation of plant poles and zeros based on a given BCF and the parametrisation of a fixed internal dimension BCF set. The concept of internal dimension was also introduced and lower bounds for it were derived. A series of internal stability tests of increasing specificity were then presented based on BCFs of the plant, controller or both.

It was shown in numerous instances how BCFs generalise existing results in the literature utilising LCFs and RCFs of the plant. Finally, it was shown via numerical examples that in some cases using a BCF of the plant can lead to advantages such as a reduced dimension stability test.

Chapter 4

State Space Formulations of BCFs

4.1 Introduction

An important and useful aspect of coprime factorisations is that every plant in $\mathscr{R}$ admits both a LCF and RCF over $\mathscr{RH}_\infty$. In fact, such factorisations can be easily obtained from state space data of the plant, subject to a stabilisability and detectability assumption. This allows results derived based on coprime factorisations to be systematically applied to any plant in $\mathscr{R}$. In this chapter state space methods for constructing BCFs of a plant will be presented.

Formulae for generating LCFs and RCFs of a plant were first given by Nett et al. (1984) and were quickly established in the literature as the standard method of obtaining such factorisations. The LCF construction presented therein is now restated for reference.

Lemma 4.1 (Nett et al. (1984)) *Let $P \in \mathscr{R}^{p\times q}$ have a stabilisable and detectable state space realisation $P = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ where $A \in \mathbb{R}^{n\times n}$. Furthermore, suppose that $H \in \mathbb{R}^{n\times p}$ is such that $A + HC$ is Hurwitz and define*

$$\begin{bmatrix} M & -L \end{bmatrix} = \left[\begin{array}{c|cc} A + HC & H & -(B + HD) \\ \hline C & I & -D \end{array}\right]. \tag{4.1}$$

Then, $M \in \mathscr{RH}_\infty^{p\times p}$, $L \in \mathscr{RH}_\infty^{p\times q}$ and $\{L, M\} \in \mathscr{C}_l(P)$.

One of many methods of acquiring a BCF of a plant is by trivially extending one of its LCFs or RCFs. Consider again the LCF of P given by (4.1) and define

$$\begin{aligned} \begin{bmatrix} M & -L \\ N & K \end{bmatrix} &= \left[\begin{array}{c|cc} A + HC & H & -(B + HD) \\ \hline C & I & -D \\ 0 & I & 0 \end{array}\right] \\ &= \begin{bmatrix} M & -L \\ I & 0 \end{bmatrix}. \end{aligned} \tag{4.2}$$

Since $\{L, M\} \in \mathscr{C}_l(P)$, it follows that $\{I, M, L, 0\} \in \mathscr{B}_p(P)$.

An alternative state space realisation of a BCF symbol of P is given by

$$\begin{bmatrix} M & -L \\ N & K \end{bmatrix} = \left[\begin{array}{c|cc} A+BF & BF & -B \\ \hline I & I & 0 \\ 0 & C & D \end{array}\right] \in \widetilde{\mathscr{B}}(P), \tag{4.3}$$

where $A + BF$ is Hurwitz. Note that unlike (4.2), the above has no trivial factors.

To see that $\{L, M\} \in \mathscr{C}_l$, first note that

$$(sI - (A + BF))^{-1}BF + I = (sI - (A + BF))^{-1}(sI - A).$$

Then,

$$\begin{bmatrix} M & -L \end{bmatrix} = -(sI - (A + BF))^{-1} \begin{bmatrix} A - sI & B \end{bmatrix}.$$

Since $A + BF$ is Hurwitz it follows that $sI - (A + BF)$ has full rank for all $s \in \bar{\mathbb{C}}_+$. Then, using Sylvester's rank inequality gives

$$\operatorname{rank}\left(\begin{bmatrix} M & -L \end{bmatrix}\right) = \operatorname{rank}\left(\begin{bmatrix} A - sI & B \end{bmatrix}\right) \quad \forall s \in \bar{\mathbb{C}}_+,$$

which has full row rank since (A, B) is stabilisable. Furthermore,

$$\lim_{s\to\infty} \begin{bmatrix} M & -L \end{bmatrix} = \begin{bmatrix} I & 0 \end{bmatrix},$$

which also has full row rank. Therefore, $\begin{bmatrix} M & -L \end{bmatrix}$ has full row rank for all $s \in \bar{\mathbb{C}}_+ \cup \{\infty\}$ which implies that it has an inverse in $\mathscr{RH}_\infty$, and hence leads to the conclusion that $\{L, M\} \in \mathscr{C}_l$. The fact that $\{N, M\} \in \mathscr{C}_r$ can be proven by construction of the associated Bézout pair. Let H be a stabilising state estimation gain matrix (i.e. H is such that $A + HC$ is Hurwitz), then

$$\left[\begin{array}{c|cc} A+HC & BF-HC & H \\ \hline -I & I & 0 \end{array}\right] \begin{bmatrix} M \\ N \end{bmatrix} = I.$$

Finally, $P = NM^{-1}L + K$ can be trivially shown via simple algebra.

Duals to both (4.2) and (4.3) can be easily derived based on an LCF of the plant and full state estimation respectively.

Although the BCF given by (4.3) is simple to construct and is non-trivial, it still suffers from the drawback that two of its factors are constant (N and K). Additionally, as with BCFs obtained from a RCF, the internal dimension is fixed. In fact, the internal dimension of the BCF given by (4.2) is dictated by the number of outputs of the plant, whereas in the case of (4.3) it is increased to match the number of states of the plant.

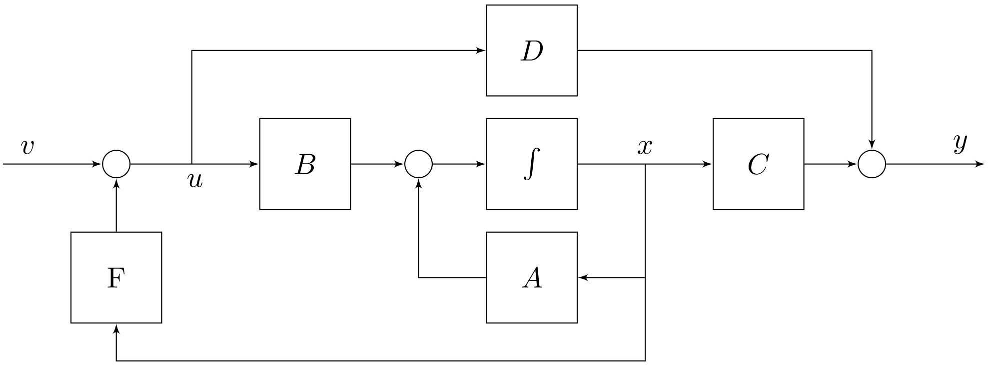


Figure 4.1: Full state feedback controller.

The state space BCF parametrisations given in the subsequent sections of this chapter will in general not suffer from these issues (if the parameters are reasonably chosen), leading to more "balanced" BCFs of the plant.

4.2 An Observer Form BCF

In addition to their simple nature, the coprime factor constructions of Nett et al. (1984) have an appealing control theoretic interpretation, which is outlined by Zhou et al. (1996, Remark 5.3). Figure 4.1 depicts the block diagram of a plant $P \in \mathscr{R}$ with a stabilisable state space realisation $P = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ under full state feedback. Let $\{N, M\} \in \mathscr{C}_r(P)$ be given by the RCF dual to (4.2), then from Figure 4.1 it becomes apparent that M corresponds to the transfer matrix from v to u, while N corresponds to that from v to y, giving RCFs of this structure a full state feedback interpretation. The LCF construction of (4.2) can be given a dual full state estimation interpretation.

In this section a BCF parametrisation based on state space data is presented. This construction is shown to have a similar, yet generalised, interpretation; being representative of an observer form controller.

Theorem 4.2 *Let $P \in \mathscr{R}^{p\times q}$ have a stabilisable and detectable state space realisation $P = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ where $A \in \mathbb{R}^{n\times n}$. Furthermore, suppose that $F \in \mathbb{R}^{q\times n}$ and $H \in \mathbb{R}^{n\times p}$ are such that $\hat{A} = A + BF + HC + HDF$ is Hurwitz. Then,*

$$\left[\begin{array}{c:c} M & -L \\ \hdashline N & K \end{array}\right] = \left[\begin{array}{c|cc:c} \hat{A} & B+HD & H & B+HD \\ \hline F & I & 0 & 0 \\ C+DF & D & I & D \\ \hdashline C+DF & D & 0 & D \end{array}\right] \in \widetilde{\mathscr{B}}_{p+q}(P). \tag{4.4}$$

Proof. Let $\tilde{H} \in \mathbb{R}^{n\times p}$ and $\tilde{F} \in \mathbb{R}^{q\times n}$ be such that $A+B\tilde{F}$ and $A+\tilde{H}C$ are Hurwitz. Then it can be shown that the Bézout factors associated with (4.4) are given by

$$\left[\begin{array}{c|ccc} A+\tilde{H}C & -(B+\tilde{H}C) & -H & \tilde{H} \\ \hline F & I & 0 & 0 \\ 0 & 0 & I & -I \end{array}\right] \begin{bmatrix} M \\ N \end{bmatrix} = I$$

and

$$\begin{bmatrix} M & -L \end{bmatrix} \left[\begin{array}{c|cc} A+B\tilde{F} & 0 & H \\ \hline -F & I & 0 \\ -(C+D\tilde{F}) & 0 & I \\ \tilde{F} & -I & 0 \end{array}\right] = I.$$

Hence, $\{N, M\} \in \mathscr{C}_r$ and $\{L, M\} \in \mathscr{C}_l$. The fact that $P = NM^{-1}L + K$ is easy to show and thus omitted. □

Remark 4.1 *The BCF presented in Theorem 4.2 does not directly reduce to the LCF construction given in* (4.2) *or its RCF dual by any selection of* F *and* H*. Although by selecting* $F = 0$ *(resp.* $H = 0$*) the LC (resp. RC) factors of the plant given by* (4.2) *appear as part of the resulting BCF.* ♦

As previously mentioned, an attractive feature of the parametrisation given in Theorem 4.2 is its practical interpretation. Consider the block diagram of an observer form controller as shown in Figure 4.2. Then it is simple to show that

$$\left[\begin{array}{c:c} M & -L \\ \hdashline N & K \end{array}\right] = \left[\begin{array}{c:c} T_{\left(\begin{smallmatrix} v \\ y \end{smallmatrix}\right)\mapsto\left(\begin{smallmatrix} u \\ e \end{smallmatrix}\right)} & T_{v\mapsto\left(\begin{smallmatrix} u-v \\ e \end{smallmatrix}\right)} \\ \hdashline T_{\left(\begin{smallmatrix} v \\ y \end{smallmatrix}\right)\mapsto\hat{y}} & T_{v\mapsto\hat{y}} \end{array}\right],$$

where $T_{\alpha\mapsto\beta}$ denotes the transfer function matrix mapping signal α to signal β.

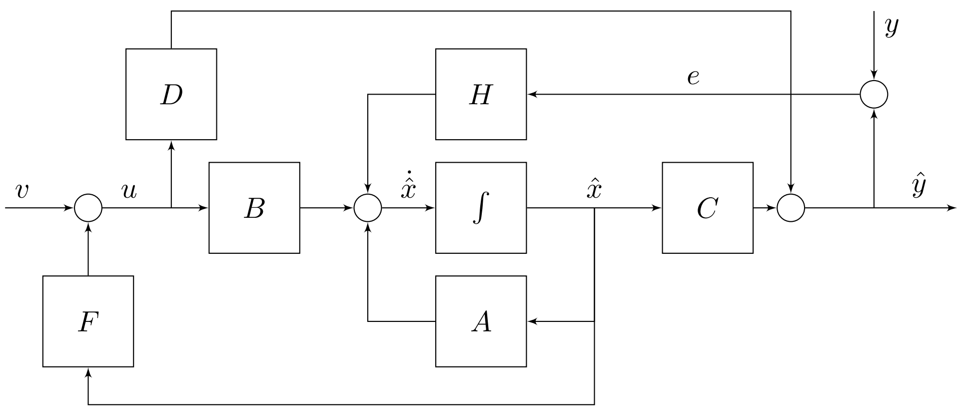


Figure 4.2: Observer form controller.

Remark 4.2 *A deviation of the BCF presented in Theorem 4.2 from its observer form controller interpretation is that F and H must be chosen such that $\hat{A}$ is Hurwitz instead of $A+BF$ and $A+HC$. This is equivalent to the resulting controller being stable instead of the closed loop transfer matrix. Clearly, if the plant is strongly stabilisable*[a]*, F and H can be chosen such that all three are Hurwitz.* ♦

A slightly simplified version of this parametrisation is given by

$$\left[\begin{array}{c|cc:c} A+BF+HC & B & H & B \\ \hline F & I & 0 & 0 \\ C & 0 & I & 0 \\ \hdashline C & 0 & 0 & D \end{array}\right] \in \widetilde{\mathscr{B}}_{p+q}(P).$$

However, the relation to Figure 4.2 is no longer valid. In this case the equivalent block diagram is similar to Figure 4.2 but with D connecting v instead of u to $\hat{y}$. Clearly, if the plant is strictly proper the interpretation still holds.

Even though the BCF parametrisation presented by Theorem 4.2 is appealing due to its practical interpretation, it is rather restrictive as it still results in a fixed internal dimension. Additionally, the fact that $A+BF$ and $A+HC$ are not required to be Hurwitz can be counter-intuitive to a control engineer.

4.3 The QR-BCF Parametrisation

All the state space BCFs presented so far in this chapter suffer from the same restriction, which is the fact that their internal dimension is fixed. A final state space parametrisation for the BCFs of a plant which is even more versatile and useful is given in the following theorem.

Theorem 4.3 *Let $P \in \mathscr{R}^{p\times q}$ have a stabilisable and detectable state space realisation $P = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$. Furthermore, suppose that $Q \in \mathbb{R}^{n\times r}$ and $R \in \mathbb{R}^{r\times n}$ are such that $A+QR$ is Hurwitz. Finally, let $D_N \in \mathbb{R}^{p\times r}$ and $D_L \in \mathbb{R}^{r\times q}$ be arbitrary matrices. Then*

$$\begin{bmatrix} M & -L \\ N & K \end{bmatrix} = \left[\begin{array}{c|cc} A+QR & Q & B+QD_L \\ \hline R & I & D_L \\ C+D_NR & D_N & D+D_ND_L \end{array}\right] \in \widetilde{\mathscr{B}}_r(P). \tag{4.5}$$

Proof. First, it is easy to show that $P = NM^{-1}L + K$. The fact that the factors are indeed coprime can be proven by construction of the associated Bézout pairs as follows. Let $F \in \mathbb{R}^{q\times n}$ and $H \in \mathbb{R}^{n\times p}$ be such that $A+BF$ and $A+HC$ are Hurwitz. Then, after

[a] A plant $P \in \mathscr{R}$ is said to be strongly stabilisable if there exists a controller $C \in \mathscr{RH}_\infty$ such that $[P, C]$ is internally stable.

some linear algebra it can be shown that

$$\left[\begin{array}{c|cc} A+HC & -(Q+HD_N) & H \\ \hline R & I & 0 \end{array}\right]\begin{bmatrix} M \\ N \end{bmatrix} = I$$

and

$$\begin{bmatrix} M & -L \end{bmatrix}\left[\begin{array}{c|c} A+BF & Q \\ \hline -(R+D_LF) & I \\ F & 0 \end{array}\right] = I.$$

Hence, $\{N,M\} \in \mathscr{C}_r$ and $\{L,M\} \in \mathscr{C}_l$ which completes the proof. $\square$

The BCF construction given by (4.5) will henceforth be referred to as the QR-BCF parametrisation. The naming is chosen to reflect that it is parametrised by the matrices Q and R.

Remark 4.3 *The QR-BCF parametrisation given in Theorem 4.3 reduces to the standard LCF and RCF constructions given by Nett et al. (1984) by an appropriate selection of Q, R, D_N and D_L. For example, let $P \in \mathscr{R}$ and $\{N,M,L,K\} \in \mathscr{B}(P)$ given by* (4.5) *with $Q=-H$, $R=-C$, $D_N=I$ and $D_L=-D$. Then $N=I$, $K=0$ and $\{L,M\} \in \mathscr{C}_l(P)$ where M and L coincide with those given in* (4.2). ♦

The QR-BCF parametrisation can be extended to include (4.4), however this introduces an unnecessary level of complexity. One such extension is given by the following corollary to Theorem 4.3.

Corollary 4.4 *Let $P \in \mathscr{R}^{p\times q}$ have a stabilisable and detectable state space realisation $P = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$. Furthermore, suppose that $Q \in \mathbb{R}^{n\times r}$, $S \in \mathbb{R}^{r\times r}$ and $R \in \mathbb{R}^{r\times n}$ are such that $A+QSR$ is Hurwitz, where* $\det S \neq 0$. *Finally, let $D_N \in \mathbb{R}^{p\times r}$ and $D_L \in \mathbb{R}^{r\times q}$ be arbitrarily chosen matrices. Then*

$$\begin{bmatrix} M & -L \\ N & K \end{bmatrix} = \left[\begin{array}{c|cc} A+QSR & QS & B+QSD_L \\ \hline SR & S & SD_L \\ C+D_NSR & D_NS & D+D_NSD_L \end{array}\right] \in \widetilde{\mathscr{B}}_r(P). \tag{4.6}$$

Proof. Let $\hat{R}=SR$, $\hat{D}_L = SD_L$ and suppose that $\{N,M,L,K\} \in \mathscr{B}(P)$ is a QR-BCF of P induced by Q and $\hat{R}$ with D_N and $\hat{D}_L$. The conclusion then follows from Lemma 3.10 with $R_l=0$, $R_r=0$, $Q_l=I$ and $Q_r=S$. $\square$

Note that with the selection $Q = \begin{bmatrix} B & H \end{bmatrix}$, $R = \begin{bmatrix} F \\ C \end{bmatrix}$, $S = \begin{bmatrix} I & 0 \\ D & I \end{bmatrix}$, $D_N = \begin{bmatrix} D & 0 \end{bmatrix}$ and $D_L = \begin{bmatrix} 0 \\ D \end{bmatrix}$, (4.6) is transformed to (4.4).

Remark 4.4 *For any $A \in \mathbb{R}^{n\times n}$ there always exist matrices $Q \in \mathbb{R}^{n\times r}$ and $R \in \mathbb{R}^{r\times n}$ such that $A+QR$ is Hurwitz. This would seem to imply that the stabilisability and detectability assumptions in Theorem 4.3 are not necessary and included simply for ease of proof. This however is not true. Consider again the RC pair given by* (4.5)*, then*

$$\begin{aligned}
\{N, M\} \in \mathscr{C}_r \Leftrightarrow & \begin{bmatrix} M \\ N \end{bmatrix} \text{ has no } \bar{\mathbb{C}}_+ \text{ transmission zeros} \\
\Leftrightarrow & \operatorname{rank}\left(\begin{bmatrix} A+QR-sI & Q \\ R & I \\ C+D_N R & D_N \end{bmatrix}\right) = n+r \quad \forall s \in \bar{\mathbb{C}}_+ \\
\Leftrightarrow & \operatorname{rank}\left(\begin{bmatrix} A-sI & Q \\ 0 & I \\ C & D_N \end{bmatrix}\right) = n+r \quad \forall s \in \bar{\mathbb{C}}_+ \\
\Leftrightarrow & \operatorname{rank}\left(\begin{bmatrix} A-sI \\ C \end{bmatrix}\right) = n \quad \forall s \in \bar{\mathbb{C}}_+ \\
\Leftrightarrow & (C, A) \text{ is detectable.}
\end{aligned}$$

Hence, if (C, A) is not detectable then $\{N, M\} \notin \mathscr{C}_r$ even if $A+QR$ is Hurwitz. It can similarly be shown that (A, B) must be stabilisable for $\{L, M\} \in \mathscr{C}_l$. ♦

By setting $D_N = 0$ and $D_L = 0$, the BCF obtained by the QR-BCF parametrisation is greatly simplified. This selection will be imposed for the rest of this thesis unless otherwise indicated. Any results presented can be generalised to the case where these terms are non-zero after some basic, albeit lengthy and tedious, linear algebra.

4.3.1 Minimal Internal Dimension

Given a stabilisable and detectable state space realisation of $P \in \mathscr{R}^{p\times q}$, using the QR-BCF parametrisation it is always possible to obtain a BCF with internal dimension $r \leq \min\{p, q\}$[b] with one example achieving $r = p$ given by (4.2). It was shown in the previous chapter that reducing the internal dimension of a BCF can be advantageous in deducing the internal stability of a feedback interconnection.

The question that now arises is "what is the minimum internal dimension achievable via a QR-BCF of a plant?". This question of internal dimensional minimality is tantamount to finding the smallest dimension Q and R such that $A+QR$ is Hurwitz; or equivalently, finding the smallest dimension Q such that (A, Q) is stabilisable[c]. This question is answered by a corollary to the following lemma.

[b]Since if $Q = B$ then $r = q$ while if $R = C$ then $r = p$.

[c]Or by duality the smallest dimension R such that (R, A) is detectable.

Lemma 4.5 *Let $A \in \mathbb{R}^{n\times n}$, then there exists a matrix $B \in \mathbb{R}^{n\times q}$ such that (A, B) is controllable if and only if*

$$\max_{\lambda_A^i} \gamma_A^i \leq q.$$

Before proving Lemma 4.5, a few preliminary results are required.

Lemma 4.6 *Let $A \in \mathbb{R}^{n\times n}$ and suppose that* $\operatorname{rank}(A) = r < n$ *or equivalently that* $\dim\ker(A) = n - r$. *Then there exists a matrix $B \in \mathbb{R}^{n\times q}$ such that* $\operatorname{rank}\left(\begin{bmatrix} A & B \end{bmatrix}\right) = n$ *if and only if $q \geq \dim\ker A$.*

Proof. Without loss of generality, via a Jordan form decomposition, suppose that A has the form $A = \begin{bmatrix} J & \\ & 0 \end{bmatrix}$ with $J \in \mathbb{R}^{r\times r}$ having full rank. Then

$$\begin{aligned}
\exists B \in \mathbb{R}^{n\times q} : \operatorname{rank}\left(\begin{bmatrix} A & B \end{bmatrix}\right) = n& \\
&\Leftrightarrow \exists B_1 \in \mathbb{R}^{r\times q}, B_2 \in \mathbb{R}^{(n-r)\times q} : \operatorname{rank}\left(\begin{bmatrix} J & 0 & B_1 \\ 0 & 0 & B_2 \end{bmatrix}\right) = n \\
&\Leftrightarrow \exists B_2 \in \mathbb{R}^{(n-r)\times q} : \operatorname{rank}(B_2) = n - r \\
&\Leftrightarrow q \geq n - r,
\end{aligned}$$

which concludes the proof. □

As a consequence of the above lemma, if $q = \dim\ker A$ then B must have full column rank.

Lemma 4.7 *Let $A \in \mathbb{R}^{n\times n}$ have the block diagonal form* $A = \operatorname{diag}(A_1, \ldots, A_m)$ *with $B \in \mathbb{R}^{n\times q}$ partitioned compatibly and suppose that $\Lambda(A_i) \cap \Lambda(A_j) = \emptyset$ for all $i \neq j$. Then (A, B) is controllable if and only if (A_k, B_k) is controllable for all $k \in \{1, \ldots, m\}$.*

Proof. The proof follows trivially using PBH tests. □

Lemma 4.8 *Suppose that a matrix $A \in \mathbb{R}^{n\times n}$ has only one distinct eigenvalue λ, with algebraic multiplicity n and geometric multiplicity γ. Then there exists a matrix $B \in \mathbb{R}^{n\times q}$ such that (A, B) is controllable if and only if $\gamma \leq q$.*

Proof. By definition, $\gamma = \dim\ker(A - \lambda I)$ and $r = \operatorname{rank}(A - \lambda I) = n - \gamma$. Then as a consequence of Lemma 4.6

$$\exists B \in \mathbb{R}^{n\times q} : \operatorname{rank}\left(\begin{bmatrix} A - \lambda I & B \end{bmatrix}\right) = n \Leftrightarrow q \geq n - r = \gamma$$

which concludes the proof. □

We are now adequately equipped to prove the result of Lemma 4.5.

Proof of Lemma 4.5. Without loss of generality, suppose that A has the Jordan canonical form $A = \operatorname{diag}(J_1, \ldots, J_m)$ and that B is compatibly partitioned. Then

$$\begin{aligned}\exists B \in \mathbb{R}^{n\times q} : (A,B) \text{ is controllable} &\Leftrightarrow \exists B_i \in \mathbb{R}^{\mu_A^i \times q} : (J_i, B_i) \text{ is controllable} \\ &\qquad \forall i \in \{1,\ldots,m\} \text{ by Lemma 4.7} \\ &\Leftrightarrow \gamma_A^i \leq q \quad \forall \lambda_A^i \text{ by Lemma 4.8}\end{aligned}$$

and the result follows by induction. □

The following corollary reduces the result of Lemma 4.5 to stabilisability by only considering the eigenvalues of A that belong to $\bar{\mathbb{C}}_+$.

Corollary 4.9 *Let $A \in \mathbb{R}^{n\times n}$, then there exists a matrix $B \in \mathbb{R}^{n\times q}$ such that (A,B) is stabilisable if and only if*

$$\max_{\lambda_A^i \in \bar{\mathbb{C}}_+} \gamma_A^i \leq q.$$

Proof. Without loss of generality, suppose that A has the form $A = \begin{bmatrix} A_+ & \\ & A_- \end{bmatrix}$ where $\Lambda(A_+) \subseteq \bar{\mathbb{C}}_+$ and A_- is Hurwitz and let B be compatibly partitioned as $B = \begin{bmatrix} B_+ \\ B_- \end{bmatrix}$. Then it is obvious that

$$(A,B) \text{ is stabilisable} \Leftrightarrow (A_+, B_+) \text{ is controllable.}$$

The conclusion then follows from Lemma 4.5 upon noting that $\Lambda(A_+) = \Lambda(A) \cap \bar{\mathbb{C}}_+$. □

Returning to the QR-BCF state space characterisation of Theorem 4.3, by a direct application of Corollary 4.9, it becomes apparent that the minimum internal dimension achievable by a QR-BCF of the plant is given by $r = \max_{\lambda_A^i \in \bar{\mathbb{C}}_+} \gamma_A^i$. One such QR-BCF will now be constructed based on a Gilbert realisation of the plant.

Let a plant $P \in \mathscr{R}$ have a stabilisable and detectable state space realisation in Gilbert form given by

$$P = \left[\begin{array}{cc|c} A_+ & & B_+ \\ & A_- & B_- \\ \hline C_+ & C_- & D \end{array}\right]$$

where $A_+ \in \mathbb{R}^{n_+ \times n_+}$ and A_- is Hurwitz. From Corollary 4.9 it now follows that a BCF can be obtained with internal dimension given by $r = \max_{\lambda_{A_+}^i} \gamma_{A_+}^i$. Let $Q_+ \in \mathbb{R}^{n_+ \times r}$ and $R_+ \in \mathbb{R}^{r \times n_+}$ be such that $A_+ + Q_+R_+$ is Hurwitz. Then $Q = \begin{bmatrix} Q_+ \\ 0 \end{bmatrix}$ and $R = \begin{bmatrix} R_+ & 0 \end{bmatrix}$

induce the QR-BCF

$$\left[\begin{array}{cc} M & -L \\ N & K \end{array}\right] = \left[\begin{array}{cc|cc} A_+ + Q_+R_+ & 0 & Q_+ & B_+ + Q_+D_L \\ 0 & A_- & 0 & B_- \\ \hline R_+ & 0 & I & D_L \\ C_+ + D_NR_+ & C_- & D_N & D + D_ND_L \end{array}\right] \in \widetilde{\mathscr{B}}_r(P). \tag{4.7}$$

Clearly, it is not necessary for the parts of Q and R corresponding to A_- to be selected as zero in (4.7) to achieve the minimal internal dimension. Nonetheless, this choice was made for clarity as it simplifies the resulting BCF.

Since both Q and R are free parameters (subject to the restriction that $A + QR$ is Hurwitz) there are countless ways of selecting them. For example, substituting $Q_+ = A_+ + \epsilon I$ where $\epsilon \in \mathbb{R}_+$ and $R_+ = -I$ in (4.7) yields a valid BCF. Another simple choice is given by $R_+ = I$ and $Q_+ = -kI$ with $k \in \mathbb{R}_+$ such that $\rho(A_+) < k$. It will be shown in Chapter 6 that by selecting Q and R in more strategic ways significant advantages can be obtained, especially in the context of robust control synthesis.

A great advantage of the QR-BCF parametrisation is the fact that the parameters can always be chosen such that the right factor N is tall and the left factor L is wide, since r can always be chosen to satisfy $r \leq \min\{p, q\}$. Additionally, they can always be made to have full column and row normal rank respectively. A simple way of achieving this is by selecting D_N and D_L to have full column and row rank respectively. This then results in a BCF where the factors are invertible in $\mathscr{R}$.

The following lemma gives a result seemingly similar to that of Lemma 4.5. However, its objective is to provide a method of constructing a matrix $B \in \mathbb{R}^{n\times q}$ such that the pair (A, B) is controllable for some $A \in \mathbb{R}^{n\times n}$. This can then be used in the selection of Q when constructing a QR-BCF.

Lemma 4.10 *Let $A \in \mathbb{R}^{n\times n}$ have m distinct eigenvalues and suppose that, without loss of generality, it has a real Jordan canonical form $A = \operatorname{diag}(J_1, \ldots, J_m)$. Furthermore, let $B \in \mathbb{R}^{n\times q}$ be compatibly partitioned and given by*

$$B = \begin{bmatrix} B_1 \\ \vdots \\ B_m \end{bmatrix}, \; B_i = \begin{bmatrix} B_{i1} \\ \vdots \\ B_{i\gamma_A^i} \end{bmatrix} \in \mathbb{R}^{\mu_A^i \times q}, \; B_{ij} = \begin{bmatrix} \star \\ \beta_{ij}^* \end{bmatrix} \in \mathbb{R}^{\mu_A^{ij} \times q},$$

where $\beta_{ij} \in \mathbb{R}^q$ and $\star$ denotes a 'don't care' element.

Then (A, B) is controllable if and only if

$$\operatorname{rank}\left(\begin{bmatrix} \beta_{i1} & \cdots & \beta_{i\gamma_A^i} \end{bmatrix}\right) \geq \gamma_A^i \quad \forall i \in \{1, \ldots, m\}.$$

Proof. First define[d]

$$e_{ij} = \begin{bmatrix} \mathbf{0}_{\sum_{h=1}^{j}\{\mu_A^{ih}\}-1} \\ 1 \\ \mathbf{0}_{\sum_{h=(j+1)}^{\gamma_A^i}\{\mu_A^{ih}\}} \end{bmatrix} \quad \forall j \in \{1, \dots, \gamma_A^i\}$$

and note that $e_{ij}^* B_i = \beta_{ij}^*$. Then it is easy to see that the left eigenspace of J_i is given by $\mathcal{E}_l(J_i) = \operatorname{span}\{e_{ij}\}$.

Since $\Lambda(J_p) \cap \Lambda(J_q) = \emptyset$ for all $p \neq q$, it follows from Lemma 4.7 that

$$\begin{aligned}
(A, B) \text{ is controllable} &\Leftrightarrow (J_i, B_i) \text{ is controllable} \quad \forall i \in \{1, \dots, m\} \\
&\Leftrightarrow \nexists y \in \mathcal{E}_l(J_i) : y^* B_i = 0 \quad \forall i \in \{1, \dots, m\} \\
&\Leftrightarrow \nexists \alpha \in \mathbb{R}^{\gamma_A^i} : \left(\begin{bmatrix} e_{i1} & \cdots & e_{i\gamma_A^i} \end{bmatrix} \alpha\right)^* B_i = 0 \quad \forall i \in \{1, \dots, m\} \\
&\Leftrightarrow \nexists \alpha \in \mathbb{R}^{\gamma_A^i} : \alpha^* \begin{bmatrix} \beta_{i1} & \cdots & \beta_{i\gamma_A^i} \end{bmatrix}^* = 0 \quad \forall i \in \{1, \dots, m\} \\
&\Leftrightarrow \operatorname{rank}\left(\begin{bmatrix} \beta_{i1} & \cdots & \beta_{i\gamma_A^i} \end{bmatrix}\right) \geq \gamma_A^i \quad \forall i \in \{1, \dots, m\},
\end{aligned}$$

which concludes the proof. □

4.4 Numerical Example

A numerical example will now be presented where Theorems 4.2 and 4.3 are used to construct BCFs of a plant. Via this example it will become apparent that a reduction in internal dimension can make a problem much more tractable, reinforcing the conclusion of the previous chapter. On the other hand, a potential flaw of using a QR-BCF will emerge when Q is chosen such that where (A, Q) is only stabilisable (not completely controllable).

Consider the plant

$$P = \begin{bmatrix} \frac{2s-1}{(s+1)(s-2)} & \frac{s-1}{s-2} \\ \frac{s+2}{s+1} & 1 \\ 1 & \frac{1}{s+1} \end{bmatrix} \in \mathscr{R}^{3\times 2},$$

which has the stabilisable and detectable state space realisation

$$P = \left[\begin{array}{ccc|cc} 2 & 0 & 0 & 1 & 1 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 \\ \hline 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right].$$

[d]For notational simplicity, $\mu_A^{ij} = 0$ for all $j \notin \{1, \dots, \gamma_A^i\}$.

By inspection of the above state space realisation it can be seen that the plant has poles at $s = 2$ and $s = -1$ both of which are semi-simple[e] with multiplicities 1 and 2 respectively.

A BCF of P will first be constructed using Theorem 4.2. Let

$$F = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \text{ and } H = \begin{bmatrix} -6.5 & 4 & -1 \\ -3 & 1 & 1 \\ 3 & -2 & -1 \end{bmatrix},$$

then (4.4) yields the BCF

$$\left[\begin{array}{ccc|ccccc:cc} -0.5 & 0 & 0 & 4 & -1.5 & -6.5 & 4 & -1 & 4 & -1.5 \\ 0 & -2 & 0 & 3 & -2 & -3 & 1 & 1 & 3 & -2 \\ 0 & 0 & -1 & -3 & 2 & 3 & -2 & -1 & -3 & 2 \\ \hline 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 2 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 3 & 3 & 1 & 1 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 2 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ \hdashline 1 & 2 & 2 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 3 & 3 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 2 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \in \widetilde{\mathscr{B}}_5(P). \tag{4.8}$$

The QR-BCF will now be used to obtain a BCF of P. From Lemma 4.5 it follows that any matrix Q must have at least 2 columns for (A, Q) to be controllable. However, since the geometric multiplicity of the $\bar{\mathbb{C}}_+$ pole is 1, it follows from Corollary 4.9 that A can be stabilised via a single column Q. One such Q is given by $Q = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^*$ which in combination with $R = \begin{bmatrix} -4 & 0 & 0 \end{bmatrix}$ induces the QR-BCF

$$\left[\begin{array}{ccc|c:cc} -2 & 0 & 0 & 1 & 1 & 1 \\ -4 & -1 & 0 & 1 & 1 & 0 \\ -4 & 0 & -1 & 1 & 0 & 1 \\ \hline -4 & 0 & 0 & 1 & 0 & 0 \\ \hdashline 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{array}\right] \in \widetilde{\mathscr{B}}_1(P). \tag{4.9}$$

Comparing the two BCFs of P constructed, it can be seen that using the QR-BCF to reduce the internal dimension leads to a much more manageable BCF of the plant.

Although the reduced internal dimensions can be beneficial in certain situations, the fact that (A, Q) is only stabilisable implies that the modes of the factorisation cannot be

[e] An eigenvalue is said to be semi-simple if its geometric and algebraic multiplicities are equal.

assigned freely. It will be shown in the next chapter that this might have some adverse effects in the context of control synthesis.

4.5 Summary & Conclusion

State space methods for generating BCFs of a plant were presented in this chapter. First, it was shown how the formulae of Nett et al. (1984), commonly used to obtain RCFs and LCFs, can be extended to generate BCFs of the plant. Similar to the classical case, where the factorisations can be related to full state feedback and estimation, the presented state space parametrisation was demonstrated to have an observer form controller interpretation.

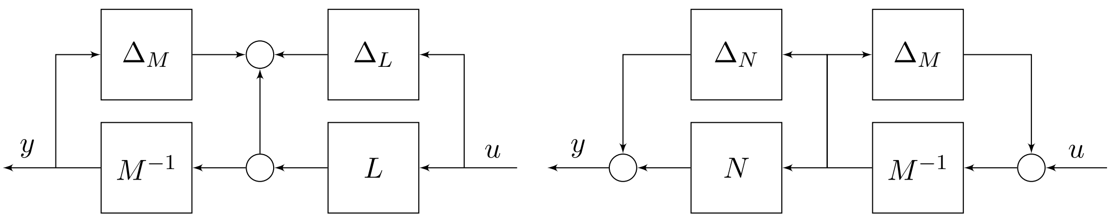
The QR-BCF parametrisation was then introduced, also based on the state space data of the plant. This parametrisation is more abstract than those previously presented, allowing greater flexibility and even variable internal dimension. A lower bound on the minimum internal dimension achievable using this method was derived. It was also shown how the formulae of Nett et al. (1984) can be reconstructed using the QR-BCF parametrisation by an appropriate selection of the parameters.

Chapter 5

BCF Uncertainty and Robust Stabilisation

5.1 Introduction

Stable perturbations on the coprime factor of a plant as a method to model uncertainty was first proposed by Vidyasagar and Kimura (1986). It was argued therein and corroborated by many other authors that coprime factor uncertainty is superior to other structures such as additive or multiplicative (Green and Limebeer, 2012; Vinnicombe, 2001). Suppose that a plant $P \in \mathscr{R}$ has a LCF $\{L, M\} \in \mathscr{C}_l(P)$, then a perturbed plant P_Δ can be defined as $P_\Delta = (M + \Delta_M)^{-1}(L + \Delta_L)$ where $\Delta_M, \Delta_L \in \mathscr{RH}_\infty$. RCF uncertainty can be similarly defined. Block diagram representations of both cases are shown in Figure 5.1. Unlike additive or multiplicative uncertainty, the perturbations mapping P to P_Δ are not unique in the coprime factor case. That is, there are many Δ_M, Δ_L satisfying $P_\Delta = (M + \Delta_M)^{-1}(L + \Delta_L)$ for a given pair $\{L, M\} \in \mathscr{C}_l(P)$ (Vinnicombe, 2001).



(a) LCF uncertainty

(b) RCF uncertainty

Figure 5.1: Block diagram representations of coprime factor uncertainty.

It is important that $\{L + \Delta_L, M + \Delta_M\} \in \mathscr{C}_l(P_\Delta)$ since if this is not true there is a $\bar{\mathbb{C}}_+$ pole/zero cancellation and the resulting perturbed plant is not robustly stabilisable[a] (Glover and McFarlane, 1989, Remark 4.4).

[a]The notion of robust stabilisability is distinct from that of state space stabilisability and refers to the existance of a controller stabilising both the nominal and perturbed plants.

In general, coprime factor uncertainty is well suited to capture low frequency parameter errors, neglected high frequency dynamics and uncertain $\bar{\mathbb{C}}_+$ poles and zeros (Zhou et al., 1996, Table 9.1).

In this chapter uncertainty in terms of a plant's BC factors is defined, followed by associated robust stability conditions. Robust stabilisation results are then presented with respect to the defined uncertainty structure. Finally a practical example is provided to demonstrate the application of the methods developed.

5.2 Uncertainty Characterisation & Robust Stability

In this section, stable perturbations on the BC factors of a plant are examined. Similarly to RCFs and LCFs, a BCF of a plant can be used to define an uncertainty structure and by extent, a robust stability condition.

5.2.1 BCF Uncertainty

Let a plant $P \in \mathscr{R}$ have the BCF $\{N, M, L, K\} \in \mathscr{B}(P)$. Then a perturbed plant P_Δ can be defined by stable additive perturbations on the BC factors of the plant (same as LCF or RCF uncertainty) which yields

$$P_\Delta = (N + \Delta_N)(M + \Delta_M)^{-1}(L + \Delta_L) + (K + \Delta_K). \tag{5.1}$$

As is the case for LCF and RCF uncertainty, we will impose that the bicoprimeness of the factors is preserved under these perturbations; or equivalently that $\{N + \Delta_N, M + \Delta_M, L + \Delta_L, K + \Delta_K\} \in \mathscr{B}(P_\Delta)$. Figure 5.2 shows a block diagram representation of the proposed BCF uncertainty structure. By comparing Figures 5.1 and 5.2, it is easy to see that the uncertainty structure induced by a BCF of the plant contains elements from both its LCF and RCF counterparts. It can therefore be argued that (5.1) is similarly suited to capturing the same types of modelling errors as those listed in the previous section for

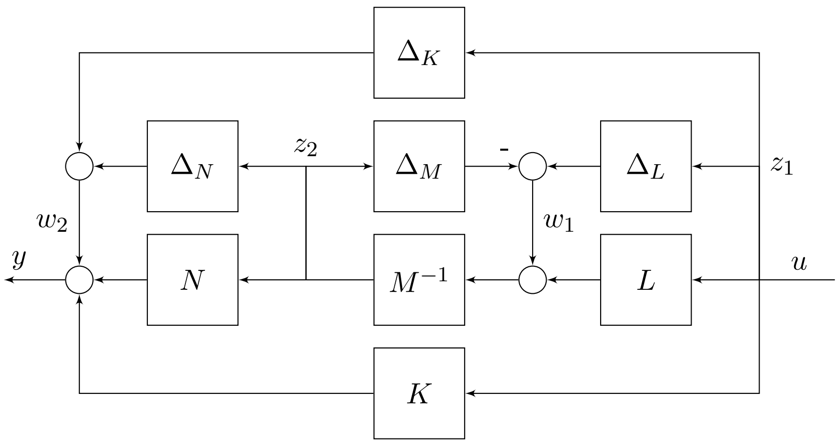


Figure 5.2: Perturbed plant block diagram with BC factor uncertainty.

coprime factor uncertainty. Additionally, additive plant errors are also represented by the Δ_K term. It is interesting to point out that this structure closely resembles the standard four-block problem[b], commonly studied by the robust control community, as is evident from Figure 5.2.

As as a motivation for the study of the BCF uncertainty structure the following example is provided. Suppose a plant $P \in \mathscr{R}$ has a LCF $\{L, M\} \in \mathscr{C}_l(P)$ being perturbed to $P_\Delta^{LCF} = (M+\Delta_M)^{-1}(L+\Delta_L)$. By perturbing the induced BCF $\{I, M, L, 0\} \in \mathscr{B}(P)$, the resulting perturbed plant would be given by $P_\Delta^{BCF} = (I+\Delta_N)(M+\Delta_M)^{-1}(L+\Delta_L) + \Delta_K$ which allows for capturing output multiplicative and additive modelling errors (Zhou et al., 1996, Table 9.1) in addition to the coprime factor errors normally represented by LCF uncertainty. Thus it becomes apparent that LCF and RCF uncertainty is a special structured case of BCF uncertainty. Furthermore, using the distance definition of Lanzon and Papageorgiou (2009)[c], it follows from the above discussion that two plants are always "closer" in terms of BCF distance in comparison to LCF or RCF distance (which in the normalised case corresponds to the ν-gap metric).

A central part in the study of any uncertainty structure is the construction of a generalised plant. In the case of BCF uncertainty this can be achieved as follows. For notational brevity, first define $z = \begin{pmatrix} z_2 \\ z_1 \end{pmatrix}$ and $w = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$. Let $P \in \mathscr{R}^{p\times q}$ and suppose that $\{N, M, L, K\} \in \mathscr{B}_r(P)$. Then from Figure 5.2 a generalised plant $\Pi : \begin{pmatrix} w \\ u \end{pmatrix} \mapsto \begin{pmatrix} z \\ y \end{pmatrix}$ and uncertainty matrix $\Delta : z \mapsto w$ can be obtained as

$$\Pi = \left[\begin{array}{cc:c} M^{-1} & 0 & M^{-1}L \\ 0 & 0 & I \\ \hdashline NM^{-1} & I & P \end{array}\right] \in \mathscr{R}^{(p+q+r)\times(p+q+r)} \tag{5.2}$$

and

$$\Delta = \begin{bmatrix} -\Delta_M & \Delta_L \\ \Delta_N & \Delta_K \end{bmatrix} \in \mathscr{RH}_\infty^{(p+r)\times(q+r)}. \tag{5.3}$$

Finally, it is simple to confirm via routine calculations that the perturbed plant as defined by (5.1) is given by $P_\Delta = \mathcal{F}_u(\Pi, \Delta)$.

5.2.2 Robust Stability in Terms of BC Factors

For a given plant $P \in \mathscr{R}$ and stabilising controller $C \in \mathscr{R}$, the robust stability margin with respect to an uncertainty structure represented by an LFT interconnection is obtained by computing $\|\mathcal{F}_l(\Sigma, C)\|_\infty^{-1}$ when $[P, C]$ is internally stable, where Σ denotes the associated generalised plant (see Lanzon and Papageorgiou (2009) for details). When considering

[b]See Lanzon et al. (2012) for a block diagram representation of the four-block interconnection.
[c]Defined as the smallest size uncertainty matrix mapping the nominal to the perturbed plant.

BCF uncertainty, applying the above procedure with $\Sigma = \Pi$ where Π is defined as (5.2), yields the following theorem.

Theorem 5.1 *Consider the standard positive feedback interconnection of a plant $P \in \mathscr{R}$ and stabilising controller $C \in \mathscr{R}$ depicted in Figure 2.2 and suppose that $\{N, M, L, K\} \in \mathscr{B}(P)$. Furthermore, define $\Delta \in \mathscr{RH}_\infty$ as in (5.3), $P_\Delta = \mathcal{F}_u(\Pi, \Delta)$ with Π as in (5.2). Finally, suppose that $\{N + \Delta_N, M + \Delta_M, L + \Delta_L, K + \Delta_K\} \in \mathscr{B}(P_\Delta)$.*

Then $[P_\Delta, C]$ is internally stable for all $\|\Delta\|_\infty < \gamma$ (resp. $\leq \gamma$) if and only if

$$\left\| \begin{bmatrix} M^{-1}L \\ I \end{bmatrix} C\,(I - PC)^{-1} \begin{bmatrix} NM^{-1} & I \end{bmatrix} + \begin{bmatrix} M^{-1} & \\ & 0 \end{bmatrix} \right\|_\infty \leq \frac{1}{\gamma} \quad \left(\text{resp.} < \frac{1}{\gamma}\right). \tag{5.4}$$

Proof. It follows by direct calculation that

$$\mathcal{F}_l(\Pi, C) = \begin{bmatrix} M^{-1}L \\ I \end{bmatrix} C\,(I - PC)^{-1} \begin{bmatrix} NM^{-1} & I \end{bmatrix} + \begin{bmatrix} M^{-1} & \\ & 0 \end{bmatrix}.$$

The conclusion then follows by an application of the small gain theorem. □

When a BCF of the controller is available the following theorem can be used to obtain a robust stability margin.

Theorem 5.2 *Consider the standard positive feedback interconnection of a plant $P \in \mathscr{R}$ and stabilising controller $C \in \mathscr{R}$ depicted in Figure 2.2 and suppose that $\{N, M, L, K\} \in \mathscr{B}(P)$ and $\{U, V, W, 0\} \in \mathscr{B}(C)$. Furthermore, define $\Delta \in \mathscr{RH}_\infty$ as in (5.3), $P_\Delta = \mathcal{F}_u(\Pi, \Delta)$ with Π as in (5.2). Finally, suppose that $\{N + \Delta_N, M + \Delta_M, L + \Delta_L, K + \Delta_K\} \in \mathscr{B}(P_\Delta)$.*

Then $[P_\Delta, C]$ is internally stable for all $\|\Delta\|_\infty < \gamma$ (resp. $\leq \gamma$) if and only if

$$\left\| \begin{bmatrix} I & \\ & U \end{bmatrix} \begin{bmatrix} M & -LU \\ -WN & V - WKU \end{bmatrix}^{-1} \begin{bmatrix} I & \\ & W \end{bmatrix} \right\|_\infty \leq \frac{1}{\gamma} \quad \left(\text{resp.} < \frac{1}{\gamma}\right).$$

Proof. Since $[P, C]$ is internally stable it follows from Lemma 3.19 that

$$\begin{bmatrix} M & -LU \\ -WN & V - WKU \end{bmatrix} \in \mathscr{GH}_\infty.$$

Then, using Lemma 3.19 again, $[P_\Delta, C]$ is internally stable if and only if

$$\begin{bmatrix} M+\Delta_M & -(L+\Delta_L)U \\ -W(N+\Delta_N) & V-W(K+\Delta_K)U \end{bmatrix} \in \mathscr{GH}_\infty$$
$$\Leftrightarrow \left(\begin{bmatrix} M & -LU \\ -WN & V-WKU \end{bmatrix} - \begin{bmatrix} I & \\ & W \end{bmatrix} \begin{bmatrix} -\Delta_M & \Delta_L \\ \Delta_N & \Delta_K \end{bmatrix} \begin{bmatrix} I & \\ & U \end{bmatrix} \right)^{-1} \in \mathscr{RH}_\infty$$
$$\Leftrightarrow \left(I - \begin{bmatrix} M & -LU \\ -WN & V-WKU \end{bmatrix}^{-1} \begin{bmatrix} I & \\ & W \end{bmatrix} \Delta \begin{bmatrix} I & \\ & U \end{bmatrix} \right)^{-1} \in \mathscr{RH}_\infty$$
$$\Leftrightarrow \left(I - \begin{bmatrix} I & \\ & U \end{bmatrix} \begin{bmatrix} M & -LU \\ -WN & V-WKU \end{bmatrix}^{-1} \begin{bmatrix} I & \\ & W \end{bmatrix} \Delta \right)^{-1} \in \mathscr{RH}_\infty.$$

The conclusion then follows by an application of the small gain theorem. □

Note that the controller in the above theorem is restricted to having no additive term. This can be justified in two ways. First, since it is up to the designer to construct the controller, it can always be chosen such that this supposition is satisfied. Secondly, if the controller is given with a non-zero additive term, the factors can be inflated to accommodate this term. An example of this procedure will be presented in the proof of Lemma 5.5.

As expected, the results of Theorems 5.1 and 5.2 reduce to the standard LCF result (and its RCF dual) given by Zhou et al. (1996, Theorem 9.6) when the appropriate factor and uncertainty assumptions are made. For example, let $K = 0$, $N = I$ and $W = I$ which gives $\{L, M\} \in \mathscr{C}_l(P)$ and $\{U, V\} \in \mathscr{C}_r(C)$, and $\Delta_N = 0$ and $\Delta_K = 0$ which corresponds to LCF uncertainty. Then under these restrictions

$$\begin{aligned}
\begin{bmatrix} I & \\ & U \end{bmatrix} & \begin{bmatrix} M & -LU \\ -WN & V-WKU \end{bmatrix}^{-1} \begin{bmatrix} I & \\ & W \end{bmatrix} \\
&= \begin{bmatrix} I & \\ & U \end{bmatrix} \begin{bmatrix} M & -LU \\ -I & V \end{bmatrix}^{-1} \\
&= \begin{bmatrix} I & \\ & U \end{bmatrix} \begin{bmatrix} V\,(MV-LU)^{-1} & V\,(MV-LU)^{-1}M - I \\ (MV-LU)^{-1} & (MV-LU)^{-1}M \end{bmatrix} \\
&= \begin{bmatrix} I \\ C \end{bmatrix} (I-PC)^{-1} \begin{bmatrix} M^{-1} & I \end{bmatrix} + \begin{bmatrix} & -I \\ 0 & \end{bmatrix}.
\end{aligned}$$

The second column is irrelevant due to the structure imposed on the uncertainty matrix and hence the required LCF result is obtained. The RCF dual can be similarly obtained.

5.3 Robust Control Synthesis

Control synthesis is a key aspect of any control theory and one to which coprime factorisations play a pivotal role. As an example, one could consider the Youla parametrisation which uses the coprime factors of a plant to parametrise all stabilising controllers (Youla et al., 1976).

As shown in the previous section, BCF uncertainty has an appealing structure that encompasses both LCF and RCF uncertainty, while closely resembling the standard four-block structure commonly studied by the robust control community. It is therefore reasonable to develop new and alternative control synthesis procedures with respect to this structure; which is the aim of this section.

In the first subsection, a new controller parametrisation method, based on BCFs, is presented for systems without any transmission zeros in $\bar{\mathbb{C}}_+ \cup \{\infty\}$. A robust stability result is given for controllers obtained in this manner, while some special cases of structured uncertainty are also examined.

Then, in the following subsections, the well known $\mathscr{H}_\infty$ synthesis results of Doyle et al. (1989) are adapted to BCF uncertainty. The structure resulting from the use of a BCF is shown to be very appealing as all of the standard assumptions[d] are directly and trivially satisfied when using the QR-BCF parametrisation.

5.3.1 Controller Parametrisation for Systems Without RHP Zeros

A controller parametrisation for systems without any transmission zeros in $\bar{\mathbb{C}}_+ \cup \{\infty\}$ will now be presented along with some associated robust stability conditions. Note that this class contains all minimum phase systems.

Theorem 5.3 *Let $P \in \mathscr{R}^{p\times q}$ have no transmission zeros in $\bar{\mathbb{C}}_+ \cup \{\infty\}$ and suppose that $\{N, M, L, 0\} \in \mathscr{B}_r(P)$ where $r = \operatorname{nrank}(P)$. Furthermore, define the set*

$$\mathcal{C}(P) = \left\{ L^\dagger (Q^{-1} + M) N^\dagger : Q \in \mathscr{RH}_\infty, \det Q(\infty) \neq 0 \right\} \subseteq \mathscr{R}^{q\times p},$$

where $L^\dagger \in \mathscr{RH}_\infty^{q\times r}$ and $N^\dagger \in \mathscr{RH}_\infty^{r\times q}$ satisfy $LL^\dagger = I$ and $N^\dagger N = I$.

Then $[P, C]$ is internally stable for all $C \in \mathcal{C}(P)$.

Before proving the above, the following lemma is needed.

Lemma 5.4 *Let $P \in \mathscr{R}^{p\times q}$ have no transmission zeros in $\bar{\mathbb{C}}_+ \cup \{\infty\}$ and suppose that $\{N, M, L, 0\} \in \mathscr{B}_r(P)$ where $r = \operatorname{nrank}(P)$. Then N and L are invertible in $\mathscr{RH}_\infty$*

Proof. First note that the assumption that P has no transmission zeros in $\bar{\mathbb{C}}_+ \cup \{\infty\}$ implies that the plant achieves its normal rank at infinity; or equivalently $\operatorname{rank}(P(\infty)) = r$. Suppose on the contrary that $\operatorname{rank}(P(\infty)) < r$, then by definition P has a transmission

[d]See Glover and Doyle (1988) for a list of these assumptions.

zero at infinity which contradicts the lemma assumption. This further implies that L and N have full row and column rank respectively at infinity, and therefore have proper rational inverses. Furthermore, since the P has no $\bar{\mathbb{C}}_+$ transmission zeros neither do L and N[e]; or equivalently L and N are minimum phase. Finally, from Zhou et al. (1996, Lemma 3.38), it follows that both L and N are invertible in $\mathscr{RH}_\infty$. □

Theorem 5.3 can now be proven as follows.

Proof of Theorem 5.3. First, note that $\{L^\dagger, Q, N^\dagger, L^\dagger M N^\dagger\} \in \mathscr{B}_r(C)$ is a valid BCF for any $C \in \mathcal{C}(P)$. Then using Lemma 3.20

$$
\begin{aligned}
[P, C] \text{ is internally stable} &\Leftrightarrow \begin{bmatrix} M - LL^\dagger M N^\dagger N & -LL^\dagger \\ -N^\dagger N & Q \end{bmatrix} \in \mathscr{GH}_\infty \\
&\Leftrightarrow \begin{bmatrix} 0 & I \\ I & -Q \end{bmatrix} \in \mathscr{GH}_\infty
\end{aligned}
$$

which completes the proof. □

The following lemma provides a robust stability condition for any feedback interconnection where the plant and controller satisfy the suppositions of Theorem 5.3.

Lemma 5.5 *Consider the standard positive feedback interconnection of a plant $P \in \mathscr{R}$ that has no transmission zeros in $\bar{\mathbb{C}}_+ \cup \{\infty\}$ and a stabilising controller $C \in \mathcal{C}(P)$ depicted in Figure 2.2 and suppose that $\{N, M, L, 0\} \in \mathscr{B}_r(P)$ where $r = \operatorname{nrank}(P)$. Furthermore, define $\Delta \in \mathscr{RH}_\infty$ as in (5.3) and $P_\Delta = \mathcal{F}_u(\Pi, \Delta)$ with Π as in (5.2). Finally, suppose that $\{N + \Delta_N, M + \Delta_M, L + \Delta_L, K + \Delta_K\} \in \mathscr{B}(P_\Delta)$.*

Then $[P_\Delta, C]$ is internally stable for all $\|\Delta\|_\infty < \gamma$ (resp. $\le \gamma$) if and only if

$$
\left\| \begin{bmatrix} I & \\ & L^\dagger \end{bmatrix} \left(\begin{bmatrix} 0 & I \\ I & M \end{bmatrix} + \begin{bmatrix} I \\ M \end{bmatrix} Q \begin{bmatrix} I & M \end{bmatrix} \right) \begin{bmatrix} I & \\ & N^\dagger \end{bmatrix} \right\|_\infty \le \frac{1}{\gamma} \quad \left(\textit{resp.} < \frac{1}{\gamma} \right). \tag{5.5}
$$

Proof. Again we begin by noting that $\{L^\dagger, Q, N^\dagger, L^\dagger M N^\dagger\} \in \mathscr{B}_r(C)$. Since the additive term of this BCF is non-zero, the robust stability result of Theorem 5.2 cannot be applied directly. This problem is circumvented by noting that

$$
\left[\begin{array}{c:c} V & -W \\ \hdashline U & 0 \end{array}\right] = \left[\begin{array}{cc:c} Q & 0 & -N^\dagger \\ 0 & I & -N^\dagger \\ \hdashline L^\dagger & L^\dagger M & 0 \end{array}\right] \in \widetilde{\mathscr{B}}_{2r}(C)
$$

[e]Let $p \ge q$. Suppose that $\exists z_0 \in \bar{\mathbb{C}}_+, y_0 \in \mathbb{R}^r : N(z_0)y_0 = 0$ and $\operatorname{rank}(L(z_0)) = r$. Since $\{N, M\} \in \mathscr{C}_r$, z_0 is not a transmission zero of M and hence $\operatorname{rank}\left(M(z_0)^{-1}L(z_0)\right) = r$. Now let $x_0 = L(z_0)^\dagger M(z_0) y_0$. Then $P(z_0)x_0 = N(z_0)M(z_0)^{-1}L(z_0)(L(z_0)^\dagger M(z_0))y_0 = N(z_0)y_0 = 0$. Therefore z_0 is also a transmission zero of P. Obviously if $\exists \tilde{z}_0 \in \bar{\mathbb{C}}_+, \tilde{y}_0 \in \mathbb{R}^q : L(\tilde{z}_0)\tilde{y}_0 = 0$ then $P(\tilde{z}_0)\tilde{y}_0 = 0$. Therefore, if $z_0 \in \bar{\mathbb{C}}_+$ is a transmission of N or L, then it is also a transmission zero of P. The claim then follows from the contrapositive.

is also a valid BCF[f] for any $C \in \mathcal{C}(P)$. Then

$$\begin{aligned}
&\begin{bmatrix} I & \\ & U \end{bmatrix} \begin{bmatrix} M & -LU \\ -WN & V - WKU \end{bmatrix}^{-1} \begin{bmatrix} I & \\ & W \end{bmatrix} \\
&\quad = \begin{bmatrix} I & \\ & L^\dagger \begin{bmatrix} I & M \end{bmatrix} \end{bmatrix} \begin{bmatrix} M & -I & -M \\ -I & Q & 0 \\ -I & 0 & I \end{bmatrix}^{-1} \begin{bmatrix} I & \\ & \begin{bmatrix} I \\ I \end{bmatrix} N^\dagger \end{bmatrix} \\
&\quad = -\begin{bmatrix} I & \\ & L^\dagger \begin{bmatrix} I & M \end{bmatrix} \end{bmatrix} \begin{bmatrix} Q & I & QM \\ I & 0 & M \\ Q & I & -(I - QM) \end{bmatrix} \begin{bmatrix} I & \\ & \begin{bmatrix} I \\ I \end{bmatrix} N^\dagger \end{bmatrix} \\
&\quad = -\begin{bmatrix} I & \\ & L^\dagger \end{bmatrix} \begin{bmatrix} Q & I + QM \\ I + MQ & M + MQM \end{bmatrix} \begin{bmatrix} I & \\ & N^\dagger \end{bmatrix} \\
&\quad = -\begin{bmatrix} I & \\ & L^\dagger \end{bmatrix} \left(\begin{bmatrix} 0 & I \\ I & M \end{bmatrix} + \begin{bmatrix} I \\ M \end{bmatrix} Q \begin{bmatrix} I & M \end{bmatrix} \right) \begin{bmatrix} I & \\ & N^\dagger \end{bmatrix}
\end{aligned}$$

and the conclusion follows from Theorem 5.2. □

Remark 5.1 *When posed as an optimisation problem, the above can be solved for an optimal Q using the methods presented by Green et al. (1990, Theorem 2.6) or Glover et al. (1991). Note that this is in fact a special case of the bilateral model matching problem since the objects multiplying the argument are both in $\mathscr{RH}_\infty$ as are their inverses, which slightly simplifies the problem. On the other hand, a complication does arise since Q must be invertible.* ♦

When the uncertainty matrix Δ is structured to mirror LCF or RCF uncertainty, that is when $\Delta_K = 0$, and $\Delta_N = 0$ or $\Delta_L = 0$, the result of Lemma 5.5 can be considerably simplified. The former case is given in the following lemma.

Lemma 5.6 *Consider the standard positive feedback interconnection of a plant $P \in \mathscr{R}$ that has no transmission zeros in $\bar{\mathbb{C}}_+ \cup \{\infty\}$ and a controller $C \in \mathcal{C}(P)$ depicted in Figure 2.2 and suppose that $\{N, M, L, 0\} \in \mathscr{B}(P)$. Furthermore, define a structured uncertainty matrix*

$$\Delta = \begin{bmatrix} -\Delta_M & \Delta_L \\ 0 & 0 \end{bmatrix} \in \mathscr{RH}_\infty$$

and a perturbed plant $P_\Delta = \mathcal{F}_u(\Pi, \Delta)$ where Π is given by (5.2). Finally, suppose that $\{N, M + \Delta_M, L + \Delta_L, 0\} \in \mathscr{B}(P_\Delta)$.

---

[f]Since $\begin{bmatrix} 0 & -M & L \\ 0 & I & 0 \end{bmatrix} \begin{bmatrix} Q & 0 \\ 0 & I \\ L^\dagger & L^\dagger M \end{bmatrix} = I$ and $\begin{bmatrix} Q & 0 & -N^\dagger \\ 0 & I & N^\dagger \end{bmatrix} \begin{bmatrix} 0 & 0 \\ I & I \\ -N & 0 \end{bmatrix} = I$.

Then $[P_\Delta, C]$ is internally stable for all $\|\Delta\|_\infty < \gamma$ (resp. $\leq \gamma$) if and only if

$$\left\| \begin{bmatrix} I & 0 \\ L^\dagger M & L^\dagger \end{bmatrix} \begin{bmatrix} Q \\ I \end{bmatrix} \right\|_\infty \leq \frac{1}{\gamma} \quad \left(\textit{resp.} < \frac{1}{\gamma} \right). \tag{5.6}$$

Proof. Using Lemma 3.17, $[P_\Delta, C]$ is internally stable if and only if

$$\begin{aligned}
&\begin{bmatrix} M + \Delta_M - (L + \Delta_L)L^\dagger M N^\dagger N & -(L+\Delta_L)L^\dagger \\ -N^\dagger N & Q \end{bmatrix} \in \mathscr{GH}_\infty \\
&\qquad \Leftrightarrow \begin{bmatrix} \Delta_M - \Delta_L L^\dagger M & -(I + \Delta_L L^\dagger) \\ -I & Q \end{bmatrix} \in \mathscr{GH}_\infty \\
&\qquad \Leftrightarrow \left(\begin{bmatrix} \Delta_M - \Delta_L L^\dagger M & -\Delta_L L^\dagger \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & I \\ I & -Q \end{bmatrix} \right)^{-1} \in \mathscr{RH}_\infty \\
&\qquad \Leftrightarrow \left(I + \begin{bmatrix} -\Delta_M & \Delta_L \\ 0 & 0 \end{bmatrix} \begin{bmatrix} I & 0 \\ L^\dagger M & L^\dagger \end{bmatrix} \begin{bmatrix} Q & I \\ I & 0 \end{bmatrix} \right)^{-1} \in \mathscr{RH}_\infty \\
&\qquad \Leftrightarrow \left(I + \begin{bmatrix} -\Delta_M & \Delta_L \end{bmatrix} \begin{bmatrix} I & 0 \\ L^\dagger M & L^\dagger \end{bmatrix} \begin{bmatrix} Q \\ I \end{bmatrix} \right)^{-1} \in \mathscr{RH}_\infty.
\end{aligned}$$

The conclusion then follows from the small gain theorem and by noting that

$$\|\Delta\|_\infty = \left\| \begin{bmatrix} -\Delta_M & \Delta_L \end{bmatrix} \right\|_\infty. \qquad \square$$

An alternative proof would be to use (5.5) and ignore the right column, since it corresponds to the second row of the uncertainty matrix which in this case is zero and thus does not affect the norm.

A dual result can be derived for the case where Δ is structured to mirror RCF uncertainty. This is given in the following lemma.

Lemma 5.7 *Consider the standard positive feedback interconnection of a plant $P \in \mathscr{R}$ that has no transmission zeros in $\bar{\mathbb{C}}_+ \cup \{\infty\}$ and a controller $C \in \mathcal{C}(P)$ depicted in Figure 2.2 and suppose that $\{N, M, L, 0\} \in \mathscr{B}(P)$. Furthermore, define an uncertainty matrix*

$$\Delta = \begin{bmatrix} -\Delta_M & 0 \\ \Delta_N & 0 \end{bmatrix} \in \mathscr{RH}_\infty$$

and a perturbed plant $P_\Delta = \mathcal{F}_u(\Pi, \Delta)$ where Π is given by (5.2). Finally, suppose that $\{N + \Delta_N, M + \Delta_M, L, 0\} \in \mathscr{B}(P_\Delta)$.

Then $[P_\Delta, C]$ is internally stable for all $\|\Delta\|_\infty < \gamma$ (resp. $\leq \gamma$) if and only if

$$\left\| \begin{bmatrix} Q & I \end{bmatrix} \begin{bmatrix} I & MN^\dagger \\ 0 & N^\dagger \end{bmatrix} \right\|_\infty \leq \frac{1}{\gamma} \quad \left(\textit{resp.} < \frac{1}{\gamma} \right). \tag{5.7}$$

Proof. The proof follows by duality to Lemma 5.6. □

Remark 5.2 *Via trivial manipulations* (5.6) *and* (5.7) *can be restated as*

$$\left\|\begin{bmatrix} I \\ L^{\dagger}M \end{bmatrix} Q + \begin{bmatrix} 0 \\ L^{\dagger} \end{bmatrix}\right\|_{\infty} \quad \textit{and} \quad \left\|Q\begin{bmatrix} I & MN^{\dagger} \end{bmatrix} + \begin{bmatrix} 0 & N^{\dagger} \end{bmatrix}\right\|_{\infty}$$

respectively. Hence finding an optimal (with respect to (5.6) *or* (5.7)*) Q involves solving a unilateral model matching problem, again however the required invertibility of Q poses a complication in using the standard methods of solving such problems.* ♦

Remark 5.3 *It is important to note that the controller set $\mathcal{C}(P)$ defined in Theorem 5.3 does not parametrise all stabilising controllers. Therefore, a controller achieving the infimum of* (5.5) *is not guaranteed to be optimal with respect to* (5.4)*, as the truly optimal controller might not belong to $\mathcal{C}(P)$.* ♦

5.3.2 BCF $\mathscr{H}_\infty$ Control Synthesis

Here, the $\mathscr{H}_\infty$ controller synthesis procedure outlined in Doyle et al. (1989) will be adapted to the context of BCF theory via the QR-BCF parametrisation.

As a first step, a generalised plant needs to be obtained in state space form. Let $P \in \mathscr{R}^{p\times q}$ have a stabilisable and detectable state space realisation $P = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ where $A \in \mathbb{R}^{n\times n}$ and $D = 0$[g]. Let a QR-BCF of P induced by $Q \in \mathbb{R}^{n\times r}$ and $R \in \mathbb{R}^{r\times n}$ be given by (4.5). Now combining this QR-BCF of P with the generalised plant Π in (5.2), a BCF generalised plant can be expressed in state space form as

$$\begin{aligned} \Pi &= \left[\begin{array}{c:c} \Pi_{11} & \Pi_{12} \\ \hdashline \Pi_{21} & \Pi_{22} \end{array}\right] \\ &= \left[\begin{array}{c|cc:c} A & Q & 0 & B \\ \hline -R & I & 0 & -D_L \\ 0 & 0 & 0 & I \\ \hdashline C & D_N & I & 0 \end{array}\right]. \end{aligned} \tag{5.8}$$

Before adapting Theorem 2.18 to develop a BCF $\mathscr{H}_\infty$ control synthesis theorem, it is necessary to show that the generalised plant Π satisfies the standard assumptions given by Glover and Doyle (1988), which can be restated as:

1. (A, B) is stabilisable and (C, A) is detectable;
2. $\Pi_{12}(\infty) = \begin{bmatrix} 0 \\ I \end{bmatrix}$ and $\Pi_{21}(\infty) = \begin{bmatrix} 0 & I \end{bmatrix}$;
3. Π_{12} and Π_{21} have full column and row rank on $j\mathbb{R}$ respectively.

[g]See Remark 2.3 for a justification as to why there is no loss of generality in assuming $D = 0$.

The assumption that (A, B) and (C, A) are stabilisable and detectable respectively is directly satisfied by the suppositions of the QR-BCF parametrisation which requires these properties for the state space realisation of the plant.

From the generalised plant given in (5.8), it is clear that the selection $D_N = 0$ and $D_L = 0$ simplifies the exposition of the results, since in so doing there is no need to normalise the D_{12} and D_{21} terms of the generalised plant; hence the second assumption is satisfied. This selection is subsequently assumed in the development of a BCF robust synthesis theorem. A procedure that can be followed if such a simplification is not desirable is well documented in the literature[h].

The fact that Π_{12} has full column rank on $j\mathbb{R}$ follows by noting that

$$\begin{aligned}
\operatorname{rank}\left(\Pi_{12}(j\omega)\right) = q \quad \forall \omega \in \mathbb{R} &\Leftrightarrow \operatorname{rank}\left(\begin{bmatrix} A - j\omega I & B \\ -R & -D_L \\ 0 & I \end{bmatrix}\right) = n + q \quad \forall \omega \in \mathbb{R} \\
&\Leftrightarrow \operatorname{rank}\left(\begin{bmatrix} A - j\omega I \\ R \end{bmatrix}\right) = n \quad \forall \omega \in \mathbb{R} \\
&\Leftrightarrow (R, A) \text{ has no undetectable modes on } j\mathbb{R} \\
&\Leftarrow (R, A) \text{ is detectable,}
\end{aligned}$$

which is true since $A + QR$ being Hurwitz implies that (R, A) is detectable. The fact that $\Pi_{21}(j\omega)$ has full row rank for all $\omega \in \mathbb{R}$ can be proven similarly using the stabilisability of (A, Q). Hence, the final assumption is also satisfied.

A BCF $\mathscr{H}_\infty$ control synthesis theorem can now be stated as follows.

Theorem 5.8 *Let $P \in \mathscr{R}^{p\times q}$ have a stabilisable and detectable state space realisation $P = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ where $A \in \mathbb{R}^{n\times n}$ and $D = 0$. Furthermore, let a QR-BCF of P induced by $Q \in \mathbb{R}^{n\times r}$ and $R \in \mathbb{R}^{r\times n}$ be given by (4.5) with $D_L = 0$ and $D_N = 0$. Finally, let $\gamma \in \mathbb{R}_+$ and define the Hamiltonian matrices*

$$H_\infty = \begin{bmatrix} A - \frac{1}{\gamma^2-1}QR & \frac{1}{\gamma^2-1}QQ^* - BB^* \\ -\frac{\gamma^2}{\gamma^2-1}R^*R & -\left(A - \frac{1}{\gamma^2-1}QR\right)^* \end{bmatrix} \tag{5.9}$$

and

$$J_\infty = \begin{bmatrix} \left(A - \frac{1}{\gamma^2-1}QR\right)^* & \frac{1}{\gamma^2-1}R^*R - C^*C \\ -\frac{\gamma^2}{\gamma^2-1}QQ^* & -\left(A - \frac{1}{\gamma^2-1}QR\right) \end{bmatrix}, \tag{5.10}$$

and the generalised plant Π as in (5.8).

[h]See Remark 2.3 for further details.

Then there exists a controller $C_\infty \in \mathscr{R}^{q\times p}$ *satisfying* $\|\mathcal{F}_l(\Pi, C_\infty)\|_\infty < \gamma$ *if and only if*

$$(a)\ \gamma > 1; \tag{5.11}$$

$$(b)\ H_\infty \in \mathrm{dom(Ric)}\ \ \textit{with}\ X_\infty = \mathrm{Ric}(H_\infty) \geq 0; \tag{5.12}$$

$$(c)\ J_\infty \in \mathrm{dom(Ric)}\ \ \textit{with}\ Y_\infty = \mathrm{Ric}(J_\infty) \geq 0; \tag{5.13}$$

$$(d)\ \rho(X_\infty Y_\infty) < \gamma^2. \tag{5.14}$$

If the above conditions hold, the set of all controllers that satisfy $\|\mathcal{F}_l(\Pi, C_\infty)\|_\infty < \gamma$ *is given by* $C_\infty = \mathcal{F}_l(\Pi_\infty, \Phi)$ *where* $\Phi \in \{\Phi \in \mathscr{RH}_\infty, \|\Phi\|_\infty < \gamma\}$,

$$\Pi_\infty = \left[\begin{array}{c|cc} A_\infty & -Z_\infty Y_\infty C^* & Z_\infty B \\ \hline B^* X_\infty & 0 & I \\ -C & I & 0 \end{array}\right],$$

$$A_\infty = A - \tfrac{1}{\gamma^2-1} QR + \left(\tfrac{1}{\gamma^2-1} QQ^* - BB^*\right) X_\infty - Z_\infty Y_\infty C^* C$$

and $Z_\infty = \left(I - \gamma^{-2} Y_\infty X_\infty\right)^{-1}$.

Proof. The proof follows by direct application of Theorem 2.18 with the generalised plant Π as defined in (5.8). □

One of the most appealing advantages of using coprime factors for robust control synthesis is the fact that the stabilising solution to one of the AREs is always zero (Glover and McFarlane, 1989). This guarantees that the spectral radius condition in Doyle et al. (1989) (condition (d) in Theorem 5.8) is satisfied and thus $Z_\infty = I$. Hence the procedure is greatly simplified both algebraically and computationally. This unfortunately is not the case for BCF robust control synthesis as is evident from Theorem 5.8.

An alternative approach to obtaining a robustly stabilising BCF controller would be via the Linear Matrix Inequality (LMI) formulations developed by Gahinet and Apkarian (1994) and Gahinet (1996). Applying the theory developed therein to the BCF case is possible as the assumptions outlined above are relaxed in the LMI case.

5.3.3 BCF $\mathscr{H}_\infty$ Synthesis With Reduced Dimension AREs

Due to the structure of the QR-BCF parametrisation and by extent that which it imparts on the generalised plant Π given in (5.8) and the Hamiltonian matrices H_∞ and J_∞ given in (5.9) and (5.10) respectively, it is possible to synthesise a robustly stabilising controller by solving reduced dimension AREs. This property does not hold when coprime factors of the plant are used.

Suppose that a plant $P \in \mathscr{R}$ has n_+ poles in $\bar{\mathbb{C}}_+$ and let it be given a QR-BCF of the form outlined in (4.7) with $D_N = 0$ and $D_L = 0$. Then the design of a robustly stabilising controller would only require the solution of two n_+ dimensional AREs. To prove this fact the following lemma is first required.

Lemma 5.9 *Consider the continuous time ARE*

$$A^*X + XA + XSX + C^*C = 0 \tag{5.15}$$

where $A = \begin{bmatrix} A_+ & \\ & A_- \end{bmatrix}$ *with* A_- *Hurwitz,* $S = \begin{bmatrix} S_{11} & S_{12} \\ S_{12}^* & S_{22} \end{bmatrix}$ *and* $C = \begin{bmatrix} C_1 & 0 \end{bmatrix}$. *Then the stabilising solution to* (5.15) *is given by* $X = \begin{bmatrix} X' & \\ & 0 \end{bmatrix}$ *where* X' *is the stabilising solution to*

$$A_+^*X' + X'A_+ + X'S_{11}X' + C_1^*C_1 = 0. \tag{5.16}$$

Proof. By definition X is a stabilising solution if and only if $A + SX$ is Hurwitz. Let X' satisfy (5.16) which implies that $A_+ + S_{11}X'$ is Hurwitz. Then

$$\begin{aligned} A + SX &= \begin{bmatrix} A_+ & \\ & A_- \end{bmatrix} + \begin{bmatrix} S_{11} & S_{12} \\ S_{12}^* & S_{22} \end{bmatrix} \begin{bmatrix} X' & \\ & 0 \end{bmatrix} \\ &= \begin{bmatrix} A_+ + S_{11}X' & 0 \\ S_{12}^*X' & A_- \end{bmatrix}. \end{aligned}$$

Hence, $A + SX$ is Hurwitz since both A_- and $A_+ + S_{11}X'$ are Hurwitz. Proving that X also satisfies (5.15) is trivial and thus omitted. The result then follows from the uniqueness of the stabilising solution. □

Remark 5.4 *It is easy to show that the result of Lemma 5.9 holds for both control and filtering AREs via duality.* ♦

Given a BCF of the form defined in (4.7), a new generalised plant can be constructed in state space form as

$$\Pi' = \left[\begin{array}{cc|cc:c} A_+ & 0 & Q_+ & 0 & B_+ \\ 0 & A_- & 0 & 0 & B_- \\ \hline -R_+ & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & I \\ \hdashline C_+ & C_- & 0 & I & D \end{array}\right], \tag{5.17}$$

where $A_+ \in \mathbb{R}^{n_+ \times n_+}$ and A_- is Hurwitz.

Now using Π' in Theorem 5.8 results in Hamiltonians where the associated AREs have the structure assumed in Lemma 5.9 and its dual. Therefore, it is only necessary to solve AREs corresponding to A_+, whose dimension is given by n_+. This fact is formally stated in the following corollary to Theorem 5.8.

Corollary 5.10 *Let $P \in \mathscr{R}^{p\times q}$ have a stabilisable and detectable state space realisation in Gilbert form given by*

$$P = \left[\begin{array}{cc|c} A_+ & & B_+ \\ & A_- & B_- \\ \hline C_+ & C_- & D \end{array}\right],$$

where $A \in \mathbb{R}^{n\times n}$, $A_+ \in \mathbb{R}^{n_+\times n_+}$, A_- is Hurwitz and D=0. Furthermore, let a QR-BCF of P induced by $Q = \begin{bmatrix} Q_+ \\ 0_{(n-n_+)\times r}\end{bmatrix} \in \mathbb{R}^{n\times r}$ and $R = \begin{bmatrix} R_+ & 0_{r\times(n-n_+)}\end{bmatrix} \in \mathbb{R}^{r\times n}$ be given by (4.7) *with $D_N = 0$ and $D_L = 0$. Finally, let $\gamma \in \mathbb{R}_+$ and define the Hamiltonian matrices*

$$H'_\infty = \begin{bmatrix} A_+ - \frac{1}{\gamma^2-1}Q_+R_+ & \frac{1}{\gamma^2-1}Q_+Q_+^* - B_+B_+^* \\ -\frac{\gamma^2}{\gamma^2-1}R_+^*R_+ & -\left(A_+ - \frac{1}{\gamma^2-1}Q_+R_+\right)^* \end{bmatrix}$$

and

$$J'_\infty = \begin{bmatrix} \left(A_+ - \frac{1}{\gamma^2-1}Q_+R_+\right)^* & \frac{1}{\gamma^2-1}R_+^*R_+ - C_+^*C_+ \\ -\frac{\gamma^2}{\gamma^2-1}Q_+Q_+^* & -\left(A_+ - \frac{1}{\gamma^2-1}Q_+R_+\right) \end{bmatrix},$$

and the generalised plant Π' as in (5.17).

Then there exists a controller $C'_\infty \in \mathscr{R}^{q\times p}$ satisfying $\|\mathcal{F}_l(\Pi', C'_\infty)\|_\infty < \gamma$ if and only if

(*a*) $\gamma > 1;$ (5.18)

(*b*) $H'_\infty \in \mathrm{dom(Ric)}$ *with* $X'_\infty = \mathrm{Ric}(H'_\infty) \geq 0;$ (5.19)

(*c*) $J'_\infty \in \mathrm{dom(Ric)}$ *with* $Y'_\infty = \mathrm{Ric}(J'_\infty) \geq 0;$ (5.20)

(*d*) $\rho(X'_\infty Y'_\infty) < \gamma^2.$ (5.21)

If the above conditions hold, the set of all controllers that satisfy $\|\mathcal{F}_l(\Pi', C'_\infty)\|_\infty < \gamma$ is given by $C'_\infty = \mathcal{F}_l(\Pi'_\infty, \Phi)$ where $\Phi \in \{\Phi \in \mathscr{RH}_\infty, \|\Phi\|_\infty < \gamma\}$,

$$\Pi'_\infty = \left[\begin{array}{cc|cc} A'_\infty & -Z'_\infty Y'_\infty C_+^* C_- & -Z'_\infty Y'_\infty C_+^* & Z'_\infty B_+ \\ -B_-B_+^*X'_\infty & A_- & 0 & B_- \\ \hline B_+^*X'_\infty & 0 & 0 & I \\ -C_+ & -C_- & I & 0 \end{array}\right],$$

$$A'_\infty = A_+ - \tfrac{1}{\gamma^2-1}Q_+R_+ + \left(\tfrac{1}{\gamma^2-1}Q_+Q_+^* - B_+B_+^*\right)X'_\infty - Z'_\infty Y'_\infty C_+^*C_+$$

and $Z'_\infty = \left(I - \gamma^{-2}Y'_\infty X'_\infty\right)^{-1}.$

Proof. Note that

$$A - \tfrac{1}{\gamma^2-1}QR = \begin{bmatrix} A_+ - \tfrac{1}{\gamma^2-1}Q_+R_+ & \\ & A_- \end{bmatrix},$$

$$R^*R = \begin{bmatrix} R_+^*R_+ & \\ & 0 \end{bmatrix} \text{ and } QQ^* = \begin{bmatrix} Q_+Q_+^* & \\ & 0 \end{bmatrix}.$$

Therefore, the AREs associated with the Hamiltonians H_∞ in (5.9) and J_∞ in (5.10) have the structure necessary to apply Lemma 5.9. The proof then follows by applying Theorem 5.8 to Π'. □

It should be noted that it is up to the designer to choose which eigenvalues of the system to include in A_+. Therefore giving the freedom to "ignore" any stable modes which are not expected to have a significant bearing on the system dynamics.

Although the procedure presented in Corollary 5.10 gives a significant computational advantage, the resulting controller still has the same order as the plant. One disadvantage of "ignoring" any stable dynamics, is their effect on closed loop performance. In fact, the "ignored" modes will also appear in the closed loop transfer matrix, which can be shown by direct calculation. Suppose that a plant $P \in \mathscr{R}$ has a BCF given by (5.17) and that a controller $C'_\infty \in \mathscr{R}$ is synthesised as instructed by Corollary 5.10[i].

It can be shown via routine calculations and a simple similarity transform that

$$P\left(I - C'_\infty P\right)^{-1} = \left[\begin{array}{cccc|c} A_+ & 0 & B_+B_+^*X'_\infty & 0 & B_+ \\ 0 & A_- & B_-B_+^*X'_\infty & 0 & B_- \\ -Z'_\infty Y'_\infty C_+^* C_+ & 0 & A'_\infty & -Z'_\infty Y'_\infty C_+^* C_- & 0 \\ 0 & 0 & 0 & A_- & B_- \\ \hline C_+ & C_- & 0 & 0 & 0 \end{array}\right].$$

It is then apparent that, since the "A"-matrix has a block triangular structure, the eigenvalues of A_- are also poles of $P\left(I - C'_\infty P\right)^{-1}$.

As is often the case in control engineering, a trade-off now arises. In this instance, it is between the dimension of the AREs to be solved and the closed loop performance.

One method of deciding which modes to "ignore" is to simply find a state space realisation of the plant with the form given by Definition 2.9, where all the plant eigenvalues in the shaded $\mathbb{C}_-$ sector shown in Figure 5.3 are placed in A_-, while the rest are placed in A_+. This would ensure that plant eigenvalues in A_+ are moved to appropriate locations via control action while those in A_- that are carried through to the input-output transfer matrix are sufficiently fast and well damped to be relatively irrelevant in the closed loop dynamics.

[i] It is assumed that the central controller is used.

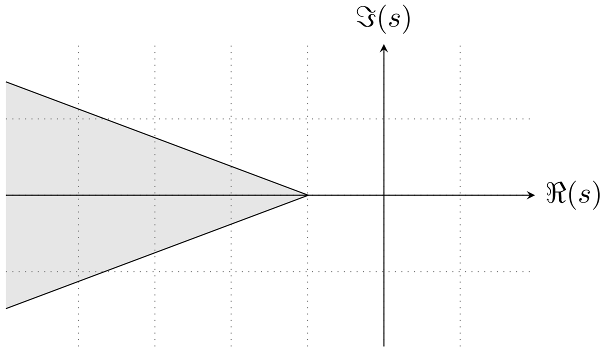


Figure 5.3: Well-damped sector in $\mathbb{C}_-$.

5.3.4 BCF $\mathscr{H}_\infty$ Synthesis For Symmetric Systems

A square system $P \in \mathscr{R}^{p\times p}$ is said to be symmetric if it satisfies $P = P^T$. These form an interesting class containing all SISO systems. In this subsection we consider BCFs of such systems and provide an associated robust control synthesis result.

Control synthesis for symmetric systems was studied in the past with many results available in the literature. Tan and Grigoriadis (2001) present an LMI approach to $\mathscr{H}_\infty$ synthesis while Mahony and Helmke (1995) deal with a more general pole placement problem; it must be noted however that both consider a special class of symmetric systems allowing stabilisation via static output feedback.

For any state space realisation of a symmetric system, a mode is detectable if and only if it is stabilisable. This can be easily seen as follows. Let $P \in \mathscr{R}$ be symmetric and have the state space realisation $P = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$. Since $P = P^T$, it follows that

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \left[\begin{array}{c|c} A^* & C^* \\ \hline B^* & D^* \end{array}\right].$$

Then λ_A^i is detectable if and only if

$$\begin{aligned}\begin{bmatrix} A - \lambda_A^i I \\ C \end{bmatrix} \text{ has full column rank} &\Leftrightarrow \begin{bmatrix} (A - \lambda_A^i I)^* \\ B^* \end{bmatrix} \text{ has full column rank} \\ &\Leftrightarrow \begin{bmatrix} A - \lambda_A^i I & B \end{bmatrix} \text{ has full row rank} \\ &\Leftrightarrow \lambda_A^i \text{ is stabilisable.}\end{aligned}$$

The contrapositive statement now suggests that any unobservable modes will also be uncontrollable (and vice versa) and therefore will not have any effect on the input-output dynamics even with non-zero initial conditions. Due to this fact only minimal state space realisations are considered in this subsection.

The following lemma gives necessary and sufficient conditions for a system to be symmetric based on its state space data.

Lemma 5.11 (Ionescu et al. (2011)) *Let $P \in \mathscr{R}$ have a minimal state space realisation $P = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$. Then P is symmetric (i.e. $P = P^T$) if and only if $D = D^*$ and there exists a nonsingular Hermitian matrix S such that $SA = A^*S$ and $SB = C^*$.*

Remark 5.5 *For any minimal state space realisation of a symmetric system, the matrix $S = S^*$ satisfying the conditions of Lemma 5.11 is given by $S = \mathcal{O}^*\mathcal{C}^\dagger$ where $\mathcal{C}$ and $\mathcal{O}$ are the controllability and observability matrices of the realisation*[j] *(Sorensen and Antoulas, 2002, Lemma 2.3).* ♦

By strategically choosing Q and R, a QR-BCF can be obtained for a symmetric system that provides various advantages. One such QR-BCF is given in the following lemma, which is utilised in Theorem 5.13 to simplify the robust control synthesis procedure of Theorem 5.8.

Lemma 5.12 *Let $P \in \mathscr{R}$ be symmetric and have a minimal state space realisation $P = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ where $A \in \mathbb{R}^{n\times n}$ and suppose that $S = S^*$ satisfies the conditions of Lemma 5.11. Then there exist matrices $Q \in \mathbb{R}^{n\times r}$ and $R \in \mathbb{R}^{r\times n}$ such that*

(a) $A + QR$ is Hurwitz; (5.22)

(b) $SQR = (QR)^*S$; (5.23)

(c) $\exists U : U^*U = I, UQ^*S = R$. (5.24)

Furthermore, let $\{N, M, L, K\} \in \mathscr{B}(P)$ be the QR-BCF induced by such Q and R, then

$$\begin{bmatrix} U & \\ & I \end{bmatrix} \begin{bmatrix} M & -L \\ N & K \end{bmatrix} \begin{bmatrix} U & \\ & I \end{bmatrix} = \begin{bmatrix} M & -L \\ N & K \end{bmatrix}^T.$$

Proof. Proof of (5.22), (5.23) and (5.24) will be achieved constructively. First note that combining (5.23) and (5.24) implies that $SQUQ^*S$ must be Hermitian. A sufficient (though not necessary) condition for this to hold, is that U is also Hermitian. Hence the problem can be restated as follows. Find a pair of matrices Q and R such that $A+QR$ is Hurwitz and there exists a unitary Hermitian matrix U such that $UQ^*S = R$.

Let $Z = Z^*$ be such that $A + ZS$ is Hurwitz and suppose that Z has the eigendecomposition $\hat{Q}\Lambda\hat{Q}^*$ where $\hat{Q}$ is unitary and Λ is real and diagonal (Laub, 2005, Theorem 10.2). Define $U = \operatorname{sgn}(\Lambda)$ and note that $U = U^*$, $U^2 = I$ and $U\Lambda = \Lambda U \geq 0$. Finally define $Q = \hat{Q}(U\Lambda)^{\frac{1}{2}}$ and $R = UQ^*S = U(U\Lambda)^{\frac{1}{2}}\hat{Q}^*S$. Then the constructed Q and R satisfy all suppositions of Lemma 5.12. It is obvious that (5.24) holds, then by noting that $QR = ZS$, both (5.22) and (5.23) follow.

[j] See Ogata (2010) for definitions.

Finally, define $\tilde{U} = \begin{bmatrix} U & \\ & I \end{bmatrix}$, then

$$\begin{aligned}
\tilde{U} \left[\begin{array}{c|cc} A+QR & Q & B \\ \hline R & I & \\ C & & D \end{array}\right] \tilde{U}^* &= \tilde{U} \left[\begin{array}{c|cc} S(A+QR)S^{-1} & SQ & SB \\ \hline RS^{-1} & I & \\ CS^{-1} & & D \end{array}\right] \tilde{U}^* \\
&= \left[\begin{array}{c|cc} (A+QR)^* & R^* & C^* \\ \hline Q^* & I & \\ B^* & & D^* \end{array}\right] \\
&= \left[\begin{array}{c|cc} A+QR & Q & B \\ \hline R & I & \\ C & & D \end{array}\right]^T
\end{aligned}$$

and the proof is complete. □

The above proof assumes the existence of a Hermitian matrix Z such that $A + ZS$ is Hurwitz. Since S is Hermitian and nonsingular, a trivial choice for Z is given by $Z = -kS^{-1}$ where $k > \rho(A)$. With this, the resulting BCF will have internal dimension equal to the number of states of the plant. If a reduced internal dimension is desired, Z can be chosen to be singular and then any columns of $\hat{Q}$ corresponding to eigenvalues of Z at the origin can be ignored.

It should be noted that the above selection is not unique. An alternative method of finding a suitable Z would be to find a matrix $T = T^*$ such that $\mathscr{Z} = \begin{bmatrix} A^* & -S \\ -T & -A \end{bmatrix} \in$ dom(Ric) and then set $Z = \mathrm{Ric}\,(\mathscr{Z})$. In fact, by selecting $T = k(2A + kI)S^{-1}$ yields $Z = -kS^{-1}$ giving the trivial choice mentioned above.

The following theorem provides a robust control synthesis result for symmetric systems using a QR-BCF satisfying the structure defined by Lemma 5.12.

Theorem 5.13 *Suppose that $P = P^T \in \mathscr{R}^{p\times p}$ has a minimal state space realisation $P = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ where $A \in \mathbb{R}^{n\times n}$, $D = 0$ and $S = S^*$ satisfies the conditions of Lemma 5.11. Let $Q \in \mathbb{R}^{n\times r}$ and $R \in \mathbb{R}^{r\times n}$ satisfy the conditions (a)–(c) of Lemma 5.12. Furthermore, let a QR-BCF of P induced by Q and R be given by* (4.5) *with $D_L = 0$ and $D_N = 0$. Finally, let $\gamma \in \mathbb{R}_+$ and define H_∞ as in* (5.9) *and Π as in* (5.2).

Then there exists a controller $C_\infty \in \mathscr{R}^{p\times p}$ satisfying $\|\mathcal{F}_l(\Pi, C_\infty)\|_\infty < \gamma$ if and only if

(a) $\gamma > 1$; (5.25)

(b) $H_\infty \in$ dom(Ric) *and* $X_\infty = \mathrm{Ric}(H_\infty) \geq 0$; (5.26)

(c) $\rho(S^{-1}X_\infty) < \gamma$. (5.27)

Proof. Define J_∞ as in (5.10) and $\tilde{S} = \begin{bmatrix} S & \\ & S^{-1} \end{bmatrix}$. Then

$$\begin{aligned}
\tilde{S} H_\infty \tilde{S}^{-1} &= \begin{bmatrix} SAS^{-1} - \frac{1}{\gamma^2-1} SQRS^{-1} & \frac{1}{\gamma^2-1} SQQ^* S - SBB^* S \\ -\frac{\gamma^2}{\gamma^2-1} S^{-1} R^* R S^{-1} & -\left(SAS^{-1} - \frac{1}{\gamma^2-1} SQRS^{-1}\right)^* \end{bmatrix} \\
&= \begin{bmatrix} A^* - \frac{1}{\gamma^2-1}(QR)^* & \frac{1}{\gamma^2-1} R^* R - C^* C \\ -\frac{\gamma^2}{\gamma^2-1} QQ^* & -\left(A^* - \frac{1}{\gamma^2-1}(QR)^*\right)^* \end{bmatrix} \\
&= J_\infty.
\end{aligned}$$

Hence a conjugate symmetric, stable, invariant spectral subspace of J_∞ is given by

$$\begin{aligned}
\mathcal{X}_-(J_\infty) &= \tilde{S} \mathcal{X}_-(H_\infty) \\
&= \operatorname{Im} \left\{ \tilde{S} \begin{bmatrix} I \\ X_\infty \end{bmatrix} \right\} \\
&= \operatorname{Im} \begin{bmatrix} S \\ S^{-1} X_\infty \end{bmatrix},
\end{aligned}$$

which implies that $Y_\infty = \operatorname{Ric}(J_\infty) = S^{-1} X_\infty S^{-1}$, hence (5.12) and (5.13) are equivalent under the suppositions of the theorem. Furthermore,

$$\begin{aligned}
\rho(Y_\infty X_\infty) &= \rho(S^{-1} X_\infty S^{-1} X_\infty) \\
&= \rho\left\{(S^{-1} X_\infty)^2\right\} \\
&= \rho(S^{-1} X_\infty)^2.
\end{aligned}$$

Thus (5.14) is satisfied if and only if $\rho(S^{-1} X_\infty) < \gamma$ which concludes the proof. □

Remark 5.6 *Suppose that a plant $P \in \mathscr{R}$ can be decomposed as $P = P_s + K$, where P_s is symmetric and $K \in \mathscr{RH}_\infty$. Then Corollary 5.10 can be used to apply Theorem 5.13 even if P itself is not symmetric.* ♦

Remark 5.7 *If $S > 0$, (5.27) can be simplified to $X_\infty < \gamma S$. Systems that satisfy this condition include state-space-symmetric systems such as those considered by Liu et al. (1998) and Tan and Grigoriadis (2001) where $S = I$.* ♦

5.4 Numerical Example

To exemplify the theory developed in this chapter, the classical control experiment of an inverted pendulum pivoting on a moving cart will now be considered. Figure 5.4 depicts one version of this system as given by Ogata (2010, Figure 3-6).

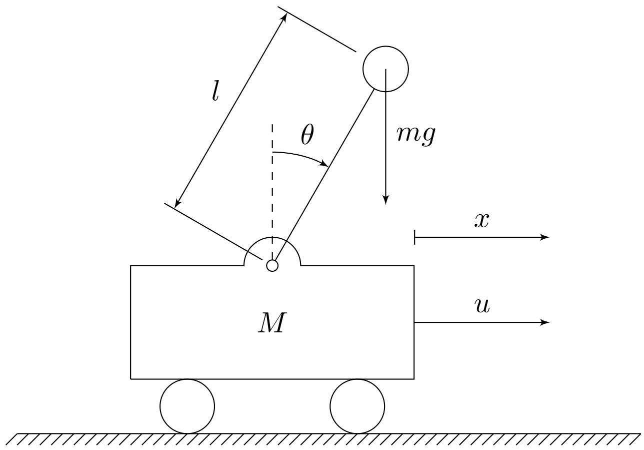


Figure 5.4: Inverted pendulum on a moving cart schematic.

The system consists of a ball attached to the end of a rod which in turn pivots on a cart that is allowed to move horizontally. The pendulum rod is assumed to be massless and have length l, the mass of the ball is denoted by m, while the mass of the cart is given by M. Finally, the surface upon which the cart is moving is assumed to be frictionless. The input to the system is the force u acting on the cart while the outputs are the angle θ and the position of the cart x. After some simplifications via small angle approximations, a linearised model for the plant is given by the differential equations (Ogata, 2010, Example 3-6)

$$Ml\ddot{\theta} = (M+m)g\theta - u \text{ and} \tag{5.28}$$

$$M\ddot{x} = u - mg\theta \tag{5.29}$$

which can be expressed in state space form as

$$P = \left[\begin{array}{cccc|c} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \frac{(M+m)g}{Ml} & 0 & 0 & 0 & -\frac{1}{Ml} \\ -\frac{mg}{M} & 0 & 0 & 0 & \frac{1}{M} \\ \hline 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{array}\right] \in \mathscr{R}^{2\times 1},$$

where $g = 9.81\,\mathrm{ms}^{-2}$ is the acceleration due to gravity.

Parameter	Variable	Value	Units
Mass of cart	M	0.5	kg
Mass of ball	m	0.2	kg
Length of rod	l	0.2	m

Table 5.1: Parameter values for inverted pendulum on a cart system.

Numerical values for the physical parameters of the system are given in Table 5.1. Using these values, the plant can now be expressed numerically as

$$P = \left[\begin{array}{cccc|c} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 68.67 & 0 & 0 & 0 & -10 \\ -3.92 & 0 & 0 & 0 & 2 \\ \hline 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{array}\right].$$

Then via a similarity transform a state space realisation of P can be obtained in Gilbert form as

$$\begin{aligned} P &= \left[\begin{array}{c:c|c} A_+ & & B_+ \\ \hdashline & A_- & B_- \\ \hline C_+ & C_- & D \end{array}\right] \\ &= \left[\begin{array}{ccc:c|c} 0 & 1 & & & 0 \\ & 0 & & & 1.43 \\ & & 8.29 & & -0.57 \\ \hdashline & & & -8.29 & -0.57 \\ \hline 0 & 0 & 1.06 & -1.06 & 0 \\ 1 & 0 & -0.06 & 0.06 & 0 \end{array}\right]. \end{aligned}$$

Now define

$$Q = \left[\begin{array}{c} Q_+ \\ \hdashline 0 \end{array}\right] = \left[\begin{array}{c} 1 \\ -2 \\ 5 \\ \hdashline 0 \end{array}\right] \text{ and } R = \left[\begin{array}{c:c} R_+ & 0 \end{array}\right] = -\left[\begin{array}{ccc:c} 2.1 & 3.3 & 4.7 & 0 \end{array}\right].$$

Then the QR-BCF induced by Q and R is in the form of (4.7) with $D_L = 0$ and $D_N = 0$, hence the associated generalized plant is given by (5.17). Thus Corollary 5.10 can be used to construct a robustly stabilising controller.

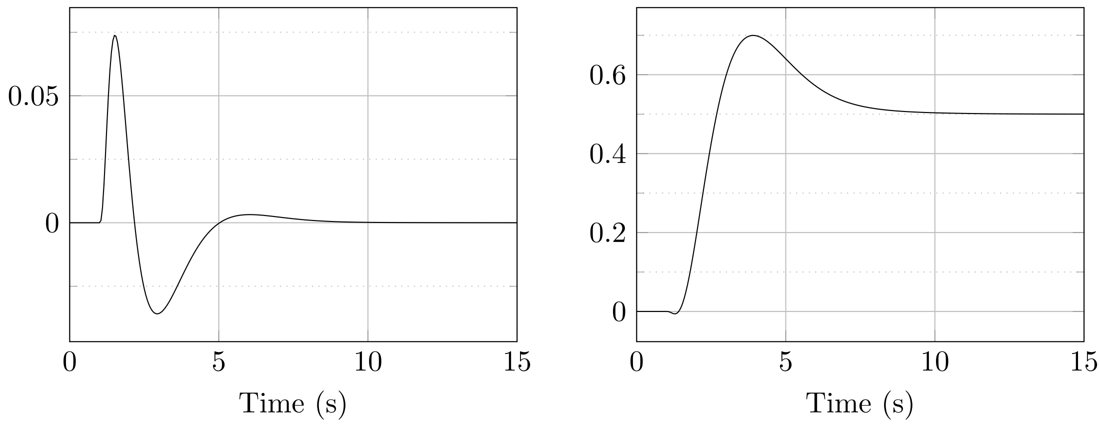
First, the stabilising solutions to the AREs associated with the Hamiltonians H'_∞ and J'_∞ were obtained as

$$X'_\infty = \begin{bmatrix} 1.669 & 2.932 & 11.51 \\ 2.932 & 5.324 & 21.36 \\ 11.51 & 21.36 & 142.226 \end{bmatrix} \text{ and } Y'_\infty = \begin{bmatrix} 2.09 & 1.75 & 0.55 \\ 1.75 & 5.08 & -0.87 \\ 0.55 & -0.87 & 16.38 \end{bmatrix}$$

respectively. The central controller was then constructed as

$$
C'_\infty = \left[\begin{array}{cccc|cc}
-825.407 & & & & -214.391 & 0.112 \\
 & -33.175 & & & -57.141 & 0.51 \\
 & & -1.569 & & 1.646 & -0.477 \\
 & & & 1.224 & 6.849 & 0.479 \\
\hline
-193.429 & 21.05 & -2.671 & 4.274 & 0 & 0
\end{array}\right] \in \mathscr{R}^{1\times 2},
$$

which achieves $\gamma = \|\mathcal{F}_l(\Pi', C'_\infty)\|_\infty = 48.78$. Though this value for γ may seem high in comparison to what is suggested in the literature (usually $\gamma \approx 3$ is considered to achieve a good balance between robustness and performance in the context of $\mathscr{H}_\infty$ loopshaping, see Vinnicombe (2001, Remark 2.11) for details), it must be noted that this was obtained by directly applying the methods developed in this chapter to the plant without any design input (i.e. weighting functions) as would be the case in a practical setting.



(a) Pendulum Arm Angle θ (rads)

(b) Cart Position x (m)

Figure 5.5: Inverted pendulum system with BCF controller simulation results.

Figure 5.5 shows the simulation results of the inverted pendulum system with the designed BCF controller. The reference signal was given as $\theta = 0\,\text{rad}$ for the rod angle while a step input of magnitude $x = 0.5\,\text{m}$ at $t = 1\,\text{s}$ was given to the position channel. It can be seen that the plant is stabilised by the designed controller and that θ remains sufficiently small so as no to invalidate the small angle assumption.

Remark 5.8 *By restricting the output of the system to being just the position of the cart, a SISO system can be obtained allowing the application of Theorem 5.13.* ⧫

5.5 Summary & Conclusion

In this chapter, uncertainty in terms of the BC factors of a plant was presented followed by the associated robust stability tests. The uncertainty structure proposed generalises LCF and RCF uncertainty where the coprime factors of the plant are additively perturbed by stable objects. By the very definition of this type of uncertainty representation, it becomes apparent that it encompasses both its LCF and RCF counterparts and thus inherits many of their features. Hence, BCF uncertainty can accommodate low frequency parameter errors, neglected high frequency dynamics and uncertain $\bar{\mathbb{C}}_+$ poles and zeros, making it an appealing candidate for representing modelling errors.

Robust stabilisation results were then presented for plants with BCF uncertainty. It was shown that via a QR-BCF of the plant, the standard assumptions related to $\mathscr{H}_\infty$ control can be trivially satisfied. Furthermore, the robust stabilisation methods presented afford the designer the freedom to ignore some or all of the stable modes of the plant. As a consequence of this, the dimensions of the AREs that need to be solved can be reduced. Finally, via an appropriate selection of Q and R it was shown that robust control synthesis for symmetric systems can be greatly simplified, eliminating the need to solve one of the two AREs usually associated with the standard $\mathscr{H}_\infty$ robust stabilisation problem.

Chapter 6

Normalised BCFs

6.1 Introduction

Normalised coprime factorisations play an important role in control theory. They are an indispensable part of many powerful results such as $\mathscr{H}_\infty$ loopshaping (McFarlane and Glover, 1992), various distance measures including the gap (Georgiou and Smith, 1990), graph (Vidyasagar, 1984) and ν-gap (Vinnicombe, 2001) metrics, as well as controller validation methods (Dehghani et al., 2009). Another appealing feature of normalised coprime factorisations is the relationship between the robust stability margin they induce and the standard four-block problem (Zhou et al., 1996, Lemma 18.4).

As stated in Definition 2.23, a RCF (resp. LCF) of a plant is said to be normalised if the associated graph symbol is inner (resp. co-inner). Equivalently, let the pair $\{N, M\}$ be a right coprime factorisation of $P = NM^{-1} \in \mathscr{R}$ over $\mathscr{RH}_\infty$, then the factorisation is normalised if $M^\sim M + N^\sim N = I$. The notion of a normalised BCF of a plant is defined in a similar manner with the left and right coprime pairs of the factorisation being independently normalised.

Definition 6.1 *Let $P \in \mathscr{R}$ and suppose that $\{N, M, L, K\} \in \mathscr{B}(P)$. The BCF is said to be normalised if the factors satisfy*

$$M^\sim M + N^\sim N = I \text{ and} \tag{6.1}$$

$$MM^\sim + LL^\sim = I. \tag{6.2}$$

With the above definition, a BCF symbol associated with a normalised BCF is neither inner (if P is tall) nor co-inner (if P is wide) in general. It might seem reasonable to redefine normalised BCFs such that this property holds, similar to LCFs and RCFs. However, a BCF satisfying this cannot be constructed for all plants in $\mathscr{R}$. Consider the $P \in \mathscr{R}^{p\times q}$ where $\operatorname{rank}(P(\infty)) < \min\{p, q\}$. Then, it follows directly from Corollary 3.4 that its BCF symbols are not invertible and hence cannot be inner or co-inner.

In this chapter, state space conditions for normalisation are presented based on the QR-BCF parametrisation, followed by recursive methods of obtaining a normalised BCF of a plant. The use of normalised BCFs as defined above is then examined in the context of robust control synthesis and shown to yield parallel results to those obtained using normalised coprime factorisations (McFarlane and Glover, 1990). Some special cases are also investigated including unilaterally normalised BCFs and symmetric systems. Finally, numerical examples are provided to illustrate the applicability of the theory developed herein.

6.2 State Space Properties of Normalised BCFs

The following theorem gives conditions that are both necessary and sufficient for a QR-BCF of a plant to be normalised.

Theorem 6.1 *Let $P \in \mathscr{R}$ have a stabilisable and detectable state space realisation $P = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ where $A \in \mathbb{R}^{n\times n}$. Let $\{N, M, L, K\} \in \mathscr{B}(P)$ be a QR-BCF of P induced by $Q \in \mathbb{R}^{n\times r}$ and $R \in \mathbb{R}^{r\times n}$ given by (4.5) with $D_L = 0$ and $D_N = 0$. Furthermore, let $G \in \widetilde{\mathscr{B}}(P)$ denote the associated BCF symbol.*

Then the QR-BCF is normalised if and only if each of the trebles $(R, A+QR, Q+XR^*)$ and $(R+Q^*Y, A+QR, Q)$ has no modes that are both controllable and observable, where $X \geq 0$ and $Y \geq 0$ are the controllability and observability Gramians of G respectively.

Proof. By direct calculation and some linear algebra it can be shown that

$$\begin{bmatrix} M \\ N \end{bmatrix}^{\sim} \begin{bmatrix} M \\ N \end{bmatrix} = I + \left[\begin{array}{c|c} A+QR & Q \\ \hline R+Q^*Y & 0 \end{array}\right] + \left[\begin{array}{c|c} A+QR & Q \\ \hline R+Q^*Y & 0 \end{array}\right]^{\sim}. \tag{6.3}$$

Given that for any strictly proper $S \in \mathscr{RH}_\infty$, $S + S^\sim = 0$ if and only if $S = 0$, it follows that (6.3) reduces to the identity matrix if and only if $\left[\begin{array}{c|c} A+QR & Q \\ \hline R+Q^*Y & 0 \end{array}\right] = 0$ or equivalently, using Zhou et al. (1996, Theorem 3.10), the treble $(R + Q^*Y, A + QR, Q)$ has no modes that are both controllable and observable. It can similarly be proven that $\{L, M\}$ is normalised if and only if $(R, A + QR, Q + XR^*)$ has no modes that are both controllable and observable. $\square$

When the treble (R, A, Q) is minimal, Theorem 6.1 can be simplified as follows.

Corollary 6.2 *Let $P \in \mathscr{R}$ have a stabilisable and detectable state space realisation $P = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ where $A \in \mathbb{R}^{n\times n}$. Let $\{N, M, L, K\} \in \mathscr{B}(P)$ be a QR-BCF of P induced by $Q \in \mathbb{R}^{n\times r}$ and $R \in \mathbb{R}^{r\times n}$ given by (4.5) with $D_L = 0$ and $D_N = 0$ and suppose that the treble (R, A, Q) is minimal. Furthermore, let $G \in \widetilde{\mathscr{B}}(P)$ denote the associated BCF symbol.*

Then the QR-BCF is normalised if and only if $Q + XR^* = 0$ and $R + Q^*Y = 0$, where $X \geq 0$ and $Y \geq 0$ are the controllability and observability Gramians of G respectively.

Proof. First note that

$$\begin{bmatrix} M \\ N \end{bmatrix} = \left[\begin{array}{c|c} A + QR & Q \\ \hline R & I \\ C & 0 \end{array}\right]$$

is minimal and that its observability Gramian is given by Y.

Then by definition, the pair $\{N, M\} \in \mathscr{C}_r$ is normalised if and only if

$$\begin{aligned} \begin{bmatrix} M \\ N \end{bmatrix}^{\sim} \begin{bmatrix} M \\ N \end{bmatrix} = I \Leftrightarrow \begin{bmatrix} M \\ N \end{bmatrix} &\text{ is inner} \\ \Leftrightarrow R + Q^*Y = 0 & \end{aligned}$$

where the last equivalence follows from Zhou et al. (1996, Corollary 13.30). The fact that $\{L, M\} \in \mathscr{C}_l$ is normalised if and only if $Q + XR^* = 0$ can be similarly proven. □

Remark 6.1 *It is easy to see that $Q + XR^* = 0$ and $R + Q^*Y = 0$ are sufficient for the induced QR-BCF to be normalised regardless of the minimality of (R, A, Q).* ♦

The following theorem gives a sufficient condition for a QR-BCF to be normalised based on two AREs with sign-definite quadratic terms. The resulting condition, although not necessary, is much easier to evaluate than that of Theorem 6.1, making it more useful in constructing normalised BCFs. It is interesting to note the similarity of this condition to those employed by Meyer and Franklin (1987) to construct normalised coprime factorisations.

Theorem 6.3 *Let $P \in \mathscr{R}$ have a stabilisable and detectable state space realisation $P = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ where $A \in \mathbb{R}^{n \times n}$. Let $X \geq 0$ and $Y \geq 0$ be the stabilising solutions to*

$$XA^* + AX - XR^*RX + BB^* = 0, \tag{6.4}$$

$$YA + A^*Y - YQQ^*Y + C^*C = 0, \tag{6.5}$$

and suppose that $Q \in \mathbb{R}^{n \times r}$ and $R \in \mathbb{R}^{r \times n}$ satisfy $Q + XR^* = 0$ and $R + Q^*Y = 0$. Then the QR-BCF induced by Q and R given by (4.5) with $D_L = 0$ and $D_N = 0$ is normalised.

Proof. First note that $A + QR$ is Hurwitz since X and Y are stabilising solutions, hence Q and R induce a valid BCF.

Then by substituting $R = -Q^*Y$ and $Q = -XR^*$ into (6.4) and (6.5) respectively, it can be seen that X and Y are the controllability and observability Gramians of the associated BCF symbol respectively. The conclusion then follows from Theorem 6.1. □

Remark 6.2 *Selecting* $D_L = 0$ *and* $D_N = 0$ *greatly simplifies the above result. If this selection is not imposed then the AREs* (6.4) *and* (6.5) *would have a structure similar to those used by Vidyasagar (1988).* ♦

The following two lemmas relate the normalising Q and R to the Gramians of the associated BCF symbol. The results prove useful in defining initial conditions and terminating criteria for the iterative algorithms developed in the next section, that aim to generate such a Q and R for a given system. Both conditions are necessary but not sufficient for a QR-BCF to be normalised.

Lemma 6.4 *Let* $P \in \mathscr{R}$ *have a stabilisable and detectable state space realisation* $P = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ *and a normalised* QR*-BCF constructed to satisfy the suppositions of Theorem 6.3. Then*

$$\begin{bmatrix} Q \\ R^* \end{bmatrix} \in \operatorname{Ker} \begin{bmatrix} I & X \\ Y & I \end{bmatrix}. \tag{6.6}$$

Furthermore,

$$Q \in \operatorname{Ker}(I - XY) \text{ and } R^* \in \operatorname{Ker}(I - YX).$$

Proof. The proof follows trivially from $Q + XR^* = 0$ and $R + Q^*Y = 0$. □

Lemma 6.5 *Let* $P \in \mathscr{R}^{p\times q}$ *have a stabilisable and detectable state space realisation* $P = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ *and a normalised* QR*-BCF constructed to satisfy the suppositions of Theorem 6.3. Then there exists a unitary matrix* $U \in \mathbb{R}^{q\times q}$ *such that*

$$CQ = (RBU)^*. \tag{6.7}$$

Proof. Using $Q = -XR^*$, (6.4) can be rearranged into $QQ^* = XA^* + AX + BB^*$. Substituting into (6.5) yields

$$\begin{aligned} Y(XA^* + AX + BB^*)Y &= A^*Y + YA + C^*C \\ YBB^*Y &= (I - YX)A^*Y + YA(I - XY) + C^*C. \end{aligned}$$

Then pre- and post-multiplying by Q^* and Q respectively, gives

$$Q^*YBB^*YQ = Q^*C^*CQ$$

upon using Lemma 6.4, or equivalently

$$RB(RB)^* = (CQ)^*CQ.$$

The conclusion then follows from Horn and Johnson (2012, Theorem 7.3.11). $\square$

6.3 Obtaining a Normalised BCF

Obtaining a normalised LCF or RCF of a plant is simple and can be achieved via the solution of a single ARE with a sign-definite quadratic term as demonstrated by Meyer and Franklin (1987) and Vidyasagar (1988). On the contrary, the equivalent BCF result given by Theorem 6.3 requires the solution of two coupled AREs, again with sign-definite quadratic terms. This is a considerably harder problem, a direct solution to which does not exist in general.

In this section, two iterative algorithms are presented that generate a Q and R satisfying the conditions set forth in Theorem 6.3 and can therefore be used to construct a normalised BCF of a plant.

Algorithm 6.1 *Let $P \in \mathscr{R}$ have a stabilisable and detectable state space realisation $P = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ where $A \in \mathbb{R}^{n\times n}$ and specify a tolerance $\mu \in \mathbb{R}_+$.*

1. *Select $R_0 \in \mathbb{R}^{r\times n}$ such that (R_0, A) is detectable and set $i = 0$.*
2. *Solve the ARE*
$$X_iA^* + AX_i - X_iR_i^*R_iX_i + BB^* = 0 \tag{6.8}$$
for the stabilising solution $X_i \geq 0$ and set $Q_{i+1} = -X_iR_i^*$.
3. *Solve the ARE*
$$Y_iA + A^*Y_i - Y_iQ_{i+1}Q_{i+1}^*Y_i + C^*C = 0 \tag{6.9}$$
for the stabilising solution $Y_i \geq 0$ and set $R_{i+1} = -Q_{i+1}^*Y_i$.
4. *If $i \geq 1$ and $\max\{\|R_i(I - X_iY_i)\|, \|(I - X_iY_{i-1})Q_i\|\} < \mu$, then set $Q = Q_i$, $R = R_i$ and stop. Otherwise increment i and go to (2).*

The following algorithm is inspired by the Kleinman approach to solving AREs (Kleinman, 1968), where the solution is obtained via the recursive solution of Lyapunov equations. Unlike the one presented above, this algorithm requires initial selections for both Q and R.

Algorithm 6.2 *Let $P \in \mathscr{R}$ have a stabilisable and detectable state space realisation $P = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ where $A \in \mathbb{R}^{n\times n}$ and specify a tolerance $\mu \in \mathbb{R}_+$.*

1. *Select $Q_0 \in \mathbb{R}^{n\times r}$ and $R_0 \in \mathbb{R}^{r\times n}$ such that $A + Q_0R_0$ is Hurwitz[a] and set $i = 0$.*

[a]This implies that (A, Q_0) is stabilisable and (R_0, A) is detectable.

2. Solve the Lyapunov equation

$$X_i(A + Q_iR_i)^* + (A + Q_iR_i)X_i + Q_iQ_i^* + BB^* = 0 \tag{6.10}$$

for the solution $X_i \geq 0$ *and set* $Q_{i+1} = -X_iR_i^*$.

3. Solve the Lyapunov equation

$$Y_i(A + Q_{i+1}R_i) + (A + Q_{i+1}R_i)^*Y_i + R_i^*R_i + C^*C = 0 \tag{6.11}$$

for the solution $Y_i \geq 0$ *and set* $R_{i+1} = -Q_{i+1}^*Y_i$.

4. If $i \geq 1$ *and* $\max\{\|R_i(I - X_iY_i)\|, \|(I - X_iY_{i-1})Q_i\|\} < \mu$, *then set* $Q = Q_i$, $R = R_i$ *and stop. Otherwise increment* i *and go to (2).*

The suitability of the stopping conditions of Algorithms 6.1 and 6.2 could be attributed to Lemma 6.4. However, a more intuitive explanation can be derived as follows. Consider Algorithm 6.1, then

$$\begin{aligned} R_{i+1} - R_i &= -Q_{i+1}^*Y_i - R_i \\ &= R_iX_iY_i - R_i \\ &= -R_i(I - X_iY_i). \end{aligned} \tag{6.12}$$

It can similarly be shown that

$$Q_{i+1} - Q_i = -(I - X_iY_{i-1})Q_i. \tag{6.13}$$

Hence, taking the norm of (6.12) or (6.13) gives a measure of the change imparted on Q_i and R_i at the i^{th} iteration. Therefore as (6.12) and (6.13) tend to 0, Q_i and R_i converge to constant values. It can be shown that the same arguments hold for Algorithm 6.2 for all $i \geq 1$. A seemingly reasonable change to Algorithm 6.2 would be to replace (6.11) with

$$Y_i(A + Q_iR_i) + (A + Q_iR_i)^*Y_i + R_i^*R_i + C^*C = 0$$

and then update $R_{i+1} = -Q_i^*Y_i$. However by doing this, (6.12) and (6.13) no longer hold. This means that Q_{i+1} and R_{i+1} are not related to their previous values in a simple way. Though extensive numerical testing indicates that the resulting algorithm also converges to the normalising Q and R, it generally takes longer to converge than the proposed two algorithms and is not as numerically reliable.

An important point to make about the above two algorithms is that a solution always exists at every iteration. For Algorithm 6.1 this is immediately evident by noting that (A, Q_i) is stabilisable for all $i \geq 1$ and (R_{i+1}, A) is respectively detectable for all $i \geq 0$, and then using Theorem 2.3. The same holds for Algorithm 6.2, as suggested by the following lemma.

Lemma 6.6 *Let $P \in \mathscr{R}$ have a stabilisable and detectable state space realisation $P = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$. Now consider Algorithm 6.2 applied to this state space realisation of P. Then there exists a $X_i \geq 0$ and $Y_i \geq 0$ satisfying* (6.10) *and* (6.11) *respectively for all $i \geq 0$.*

Before providing a proof for the above, the following result is needed.

Lemma 6.7 *Let $P \in \mathscr{R}^{p\times q}$ have a stabilisable state space realisation $P = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ where $A \in \mathbb{R}^{n\times n}$ and suppose that $F \in \mathbb{R}^{q\times n}$ is such that $A+BF$ is Hurwitz. Furthermore, let $Y \geq 0$ be the solution to the Lyapunov equation*

$$(A + BF)^*Y + Y(A + BF) + F^*F + C^*C = 0. \tag{6.14}$$

Then $A - BB^*Y$ is also Hurwitz.

Proof. By rearranging (6.14) it can be shown that Y is also a solution to

$$(A - BB^*Y)^*Y + Y(A - BB^*Y) + S^*S = 0$$

where

$$S = \frac{1}{\sqrt{2}} \begin{bmatrix} 2B^*Y + F \\ \sqrt{2}C \\ F \end{bmatrix}.$$

The conclusion then follows from Lemma 2.14 since $(S, A - BB^*Y)$ is detectable[b], $Y \geq 0$ and $S^*S \geq 0$. □

Proof of Lemma 6.6. First note that $R_i^*R_i + C^*C \geq 0$ and $Q_iQ_i^* + BB^* \geq 0$ for all $i \geq 0$. Then using Lemma 6.7 note that $(A + Q_iR_i)$ and $(A + Q_{i+1}R_i)$ are Hurwitz for all $i \geq 0$. This guarantees via Lemma 2.13 that there exist $X_i \geq 0$ and $Y_i \geq 0$ satisfying (6.10) and (6.11) for all $i \geq 0$, which concludes the proof. □

Due to the computational issues related to solving AREs (Kleinman, 1968; Lanzon et al., 2008), Algorithm 6.2 tends to be numerically more reliable. Hence it is a better choice for obtaining a normalised BCF of a plant.

Algorithms used to iteratively solve single AREs typically exhibit monotonicity in the iteration variables (Banks and Ito, 1991; Kleinman, 1968; Lanzon et al., 2008), a fact

[b]The claim follows from the fact that $A + BF$ is Hurwitz, which implies that (F, A) is detectable, and

$$\begin{bmatrix} I & \frac{\sqrt{2}}{2}B & 0 & -\frac{\sqrt{2}}{2}B \\ 0 & \sqrt{2}I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & \sqrt{2}I \end{bmatrix} \begin{bmatrix} A - BB^*Y - \lambda I \\ \frac{1}{\sqrt{2}}(2B^*Y + F) \\ C \\ \frac{1}{\sqrt{2}}F \end{bmatrix} = \begin{bmatrix} A - \lambda I \\ 2B^*Y + F \\ C \\ F \end{bmatrix}.$$

commonly exploited in proving convergence. Unfortunately, this is not the case for either of the proposed algorithms, thus a simple proof of convergence is not available.

The selection of initial conditions for Algorithms 6.1 and 6.2 is important and can affect both the rate of convergence as well as the solution to which the algorithms converge. Suppose that for a plant $P \in \mathscr{R}$, Q and R induce a normalised QR-BCF. If R_0 is chosen to be exactly R, then Algorithm 6.1 would converge after just one iteration. On the other hand, Algorithm 6.2 would need a few iterations depending on how close Q_0 is to the actual solution Q. It is also possible that R_i is changed before re-converging to the solution. If also Q_0 is chosen as Q, then Algorithm 6.2 would also converge after just one iteration. Lemma 6.5 can be used to assist in the selection of Q_0 and R_0 in Algorithm 6.2.

Remark 6.3 *Though the normalisation condition set forth by Theorem 6.3 appears to be simple, it must be noted that it is in fact hard to solve as the coupling between the two AREs leads to a nonlinear problem. This can be observed by substituting for, and eliminating Q and R from* (6.4) *and* (6.5) *yielding*

$$XA_R^* + A_R X - XC^*CX + BB^* = 0,$$
$$A_Q^* Y + YA_Q - YBB^*Y + C^*C = 0$$

where $A_R = (I - XY)A$ and $A_Q = A(I - XY)$, which must be solved for X and Y. No explicit mathematical method exists in the literature to solve such coupled AREs. ♦

An important point to note about the results presented in this section is that none prove the existence of a normalised BCF for a given plant. A possible approach might be via a convergence proof for Algorithms 6.1 or 6.2. Experimental evidence suggests that both algorithms converge to a normalising solution with no counterexamples found. As stated above however, such a proof is not currently available.

6.4 Robust Control Synthesis

Introducing the normalisation property to a coprime factorisation of a plant offers many advantages, with a prominent one being the ability to directly calculate a lower bound on the achievable robust stability margin as shown by Glover and McFarlane (1989) and McFarlane and Glover (1990). This then allows for the synthesis of a robustly stabilising controller without the need of an iterative procedure.

In this section, normalised BCFs will be used to simplify the result of Theorem 5.8. Theorem 6.8 shows that when using a normalised BCF of the plant, the Hamiltonians associated with BCF $\mathscr{H}_\infty$ synthesis given by (5.9) and (5.10) are guaranteed to belong to dom(Ric) and have a positive semidefinite stabilising solution if γ is chosen to satisfy some simple inequalities. This gives a lower bound on the achievable robust stability margin that, similar to the normalised LCF and RCF case, can be calculated *a priori* using the Hankel norm of the graph symbols associated with the LC and RC pairs of the factorisation. Unlike the classical normalised coprime factor case however, this lower bound is not guaranteed to be the infimum as the spectral radius condition (5.14) still needs to be satisfied.

Theorem 6.8 *Let $P \in \mathscr{R}^{p\times q}$ have a stabilisable and detectable state space realisation $P = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ where $A \in \mathbb{R}^{n\times n}$ and $D = 0$. Suppose that $\{N, M, L, K\} \in \mathscr{B}(P)$ is a normalised QR-BCF of P constructed to satisfy the suppositions of Theorem 6.3. Finally, let $\gamma \in \mathbb{R}_+$ and define Π as in (5.8).*

Then there exists a controller $C_\infty \in \mathscr{R}^{q\times p}$ satisfying $\|\mathcal{F}_l(\Pi, C_\infty)\|_\infty < \gamma$ if and only if

$$\text{(a)} \quad \left(1 - \left\|\begin{bmatrix} M & -L \end{bmatrix}\right\|_H^2\right)^{-\frac{1}{2}} < \gamma \tag{6.15}$$

$$\text{(b)} \quad \left(1 - \left\|\begin{bmatrix} M \\ N \end{bmatrix}\right\|_H^2\right)^{-\frac{1}{2}} < \gamma \tag{6.16}$$

$$\text{(c)} \quad \rho\left\{\Phi(\gamma)\right\} < \gamma^2 \tag{6.17}$$

where $\Phi(\gamma)$ is defined by

$$\Phi(\gamma) = \left(\tfrac{\gamma^2}{\gamma^2-1}\right)^2 \left(I - \tfrac{\gamma^2}{\gamma^2-1}\hat{Y}X\right)^{-1} \hat{Y}\hat{X} \left(I - \tfrac{\gamma^2}{\gamma^2-1}Y\hat{X}\right)^{-1}$$

and $\hat{X} \geq 0$ and $\hat{Y} \geq 0$ are the controllability and observability Gramians of M respectively. Furthermore, the solutions X_∞ and Y_∞ of the AREs associated with the Hamiltonian matrices H_∞ in (5.9) and J_∞ in (5.10) are given by

$$X_\infty = \mathrm{Ric}(H_\infty) = \tfrac{\gamma^2}{\gamma^2-1}\hat{Y}\left(I - \tfrac{\gamma^2}{\gamma^2-1}X\hat{Y}\right)^{-1} \quad \text{and} \tag{6.18}$$

$$Y_\infty = \mathrm{Ric}(J_\infty) = \tfrac{\gamma^2}{\gamma^2-1}\hat{X}\left(I - \tfrac{\gamma^2}{\gamma^2-1}Y\hat{X}\right)^{-1} \tag{6.19}$$

respectively.

Proof. First note that

$$H_\infty = \begin{bmatrix} I & -\frac{\gamma^2}{\gamma^2-1}X \\ 0 & \frac{\gamma^2}{\gamma^2-1}I \end{bmatrix} \hat{H}_\infty \begin{bmatrix} I & -\frac{\gamma^2}{\gamma^2-1}X \\ 0 & \frac{\gamma^2}{\gamma^2-1}I \end{bmatrix}^{-1}$$

where

$$\hat{H}_\infty = \begin{bmatrix} A + QR & 0 \\ -R^*R & -(A+QR)^* \end{bmatrix}.$$

A conjugate symmetric, stable, invariant spectral subspace of $\hat{H}_\infty$ is given by

$$\mathcal{X}_-(\hat{H}_\infty) = \mathrm{Im} \begin{bmatrix} I \\ \hat{Y} \end{bmatrix}$$

and thus

$$\begin{aligned} \mathcal{X}_-(H_\infty) &= \begin{bmatrix} I & -\frac{\gamma^2}{\gamma^2-1} X \\ 0 & \frac{\gamma^2}{\gamma^2-1} I \end{bmatrix} \mathcal{X}_-(\hat{H}_\infty) \\ &= \mathrm{Im} \begin{bmatrix} I - \frac{\gamma^2}{\gamma^2-1} X\hat{Y} \\ \frac{\gamma^2}{\gamma^2-1}\hat{Y} \end{bmatrix}. \end{aligned} \tag{6.20}$$

Since $A + QR$ has no $j\mathbb{R}$ eigenvalues, it follows that $H_\infty \in \mathrm{dom}(\mathrm{Ric})$ if and only if $I - \frac{\gamma^2}{\gamma^2-1} X\hat{Y}$ is nonsingular. Furthermore, when $I - \frac{\gamma^2}{\gamma^2-1} X\hat{Y}$ is invertible, $X_\infty = \mathrm{Ric}(H_\infty)$ exists and is given by

$$X_\infty = \tfrac{\gamma^2}{\gamma^2-1}\hat{Y}\left(I - \tfrac{\gamma^2}{\gamma^2-1} X\hat{Y}\right)^{-1}.$$

Then using Lemma 2.4 when $\gamma > 1$,

$$\begin{aligned} \det\left(I - \tfrac{\gamma^2}{\gamma^2-1} X\hat{Y}\right) \neq 0 \text{ and } X_\infty \geq 0 &\Leftrightarrow \tfrac{\gamma^2}{\gamma^2-1}\rho(X\hat{Y}) < 1 \\ &\Leftrightarrow \gamma > \left(1 - \rho(X\hat{Y})\right)^{-\frac{1}{2}} \\ &\Leftrightarrow \gamma > \left(1 - \left\| \begin{bmatrix} M & -L \end{bmatrix} \right\|_H^2 \right)^{-\frac{1}{2}}. \end{aligned}$$

The last equivalence follows by noting that X is the controllability Gramian of $\begin{bmatrix} M & -L \end{bmatrix}$ and therefore its Hankel norm is given by $\rho(X\hat{Y})^{\frac{1}{2}}$ since the observability Gramian $\hat{Y}$ of M is also the observability Gramian of $\begin{bmatrix} M & -L \end{bmatrix}$.

It can similarly be shown that γ must also satisfy (6.16) and that the solution Y_∞ to the ARE associated with J_∞, if it exists (i.e. if $J_\infty \in \mathrm{dom}(\mathrm{Ric})$), is given by

$$Y_\infty = \mathrm{Ric}(J_\infty) = \tfrac{\gamma^2}{\gamma^2-1}\hat{X}\left(I - \tfrac{\gamma^2}{\gamma^2-1} Y\hat{X}\right)^{-1} \geq 0.$$

Using the expressions derived above for X_∞ and Y_∞, the spectral radius condition (d) in Theorem 5.8 is transformed to (6.17). Finally, since $\{L, M\}$ is normalised, it follows that $\left\| \begin{bmatrix} M & -L \end{bmatrix} \right\|_H < 1$ (Glover and McFarlane, 1989, Lemma 4.2) therefore $\left(1 - \left\| \begin{bmatrix} M & -L \end{bmatrix} \right\|_H^2 \right)^{-\frac{1}{2}} > 1$ and $\gamma > 1$, which concludes the proof. □

It is simple to show that there always exists a γ satisfying the spectral radius condition (6.17) of the above theorem. Define $\hat{\rho} = \lim_{\gamma\to\infty} \rho\{\Phi(\gamma)\}$ and note that

$$\rho_\infty = \rho\left\{\left(I - \hat{Y}X\right)^{-1}\hat{Y}\hat{X}\left(I - Y\hat{X}\right)^{-1}\right\}. \tag{6.21}$$

Then, since $\rho\{\Phi(\gamma)\} \to \rho_\infty < \infty$ as $\gamma \to \infty$ it follows that there exists a γ such that $\rho\{\Phi(\gamma)\} < \gamma^2$.

Finding the smallest γ that satisfies the conditions of Theorem 6.8 is a simple task. A trivial approach is via a line search, with the initial value of γ set to $\max\{\sqrt{\rho_\infty}, \gamma_0\}$ where

$$\gamma_0 = \max\left\{\left(1 - \left\|\begin{bmatrix} M & -L\end{bmatrix}\right\|_H^2\right)^{-\frac{1}{2}}, \left(1 - \left\|\begin{bmatrix} M \\ N\end{bmatrix}\right\|_H^2\right)^{-\frac{1}{2}}\right\} \tag{6.22}$$

and then slowly incremented until the required condition is satisfied. For clarity, Figure 6.1 is provided which depicts a sketch of $\rho\{\Phi(\gamma)\}$ and γ^2 against γ, with some important points marked. This should provide some insight to the line search approach proposed above. The point ρ_0 corresponds to $\rho\{\Phi(\gamma_0)\}$ while $\gamma^\star$ is the smallest value of γ for which the aforementioned conditions are satisfied.

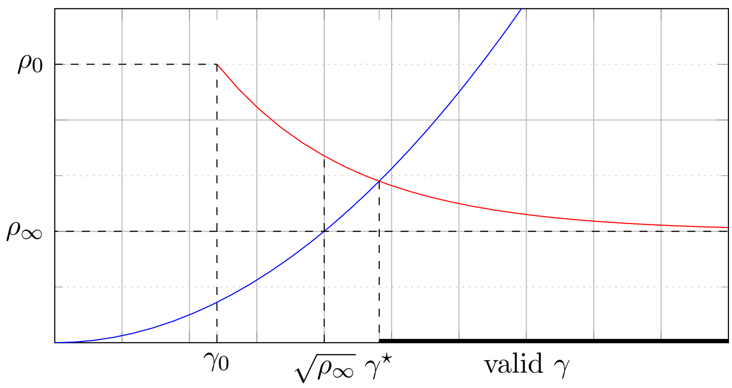


Figure 6.1: Sketch of $\rho\{\Phi(\gamma)\}$ (——) and γ^2 (——) against γ.

Alternatively and more efficiently, root finding methods such as the bisection algorithm or Newton-Raphson iterations can be used to arrive at suitable value for γ.

6.5 Interesting Special Cases

In this section some special cases are considered. As an example of such a special case consider the stable plant $P \in \mathscr{RH}_\infty$. Then it is easy to see that $\{0, U, 0, P\} \in \mathscr{B}_r(P)$ is normalised for any unitary matrix $U \in \mathbb{R}^{r\times r}$. It is important to note however that this is not unique, which is confirmed via the following example. Consider the simple first order system $P = \left[\begin{array}{c|c} -1 & 4 \\ \hline 2 & 0 \end{array}\right] \in \mathscr{RH}_\infty$. Then, in addition to the trivial choice $Q = R = 0$, it is easy to show that $Q = 2\sqrt{3}$ and $R = -\sqrt{3}$ also induce a normalised QR-BCF.

6.5.1 Unilaterally Normalised BCF

Suppose that $P \in \mathscr{R}$ has a BCF given by $\{N, M, L, K\} \in \mathscr{B}(P)$. The BCF is said to be unilaterally normalised if $\{N, M\} \in \mathscr{C}_r$ or $\{L, M\} \in \mathscr{C}_l$ is normalised, but not both. Obviously, standard normalised LCFs or RCFs of a system fall into this category. It should be noted that such BCFs are incompatible with Definition 6.1 and thus are not normalised. Obtaining such a BCF of a plant amounts to selecting a Q and then using the methods of Meyer and Franklin (1987) to generate a R that normalises the RC pair of the factorisations and vice versa.

The following theorem makes use of the partial normalisation property to simplify the result of Theorem 5.8 by providing a direct solution to one of the AREs.

Theorem 6.9 *Let $P \in \mathscr{R}^{p\times q}$ have a stabilisable and detectable state space realisation $P = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ where $A \in \mathbb{R}^{n\times n}$ and $D = 0$. Let $Q \in \mathbb{R}^{n\times r}$ be such that (A, Q) is stabilisable and $R = -Q^*Y$ where $Y \geq 0$ is the stabilising solution to the ARE* (6.5). *Let $\{N, M, L, K\} \in \mathscr{B}(P)$ be a QR-BCF induced by Q and R given by* (4.5) *with $D_L = 0$ and $D_N = 0$. Furthermore, let $\gamma \in \mathbb{R}_+$, define Π as in* (5.8) *and H_∞ as in* (5.9).

Then there exists a controller $C_\infty \in \mathscr{R}^{q\times p}$ such that $\|\mathcal{F}_l(\Pi, C_\infty)\|_\infty < \gamma$ if and only if

(a) $\left(1 - \left\| \begin{bmatrix} M \\ N \end{bmatrix} \right\|_H^2 \right)^{-\frac{1}{2}} < \gamma$;

(b) $H_\infty \in \mathrm{dom}(\mathrm{Ric})$ *and* $X_\infty = \mathrm{Ric}(H_\infty) \geq 0$;

(c) $\rho\left\{ I + X_\infty \hat{X} \left(I - \frac{\gamma^2}{\gamma^2-1} Y \hat{X} \right)^{-1} \right\} < \gamma^2$;

where $\hat{X} \geq 0$ is the controllability Gramian of M.

Proof. Define J_∞ as in (5.10) and note that similarly to Theorem 6.8,

$$\mathcal{X}_-(J_\infty) = \mathrm{Im} \begin{bmatrix} I - \frac{\gamma^2}{\gamma^2-1} Y \hat{X} \\ \frac{\gamma^2}{\gamma^2-1} \hat{X} \end{bmatrix}.$$

Since $A + QR$ has no $j\mathbb{R}$ eigenvalues, it follows that $J_\infty \in \mathrm{dom}(\mathrm{Ric})$ if and only if $I - \frac{\gamma^2}{\gamma^2-1} Y \hat{X}$ is nonsingular. Hence, when $I - \frac{\gamma^2}{\gamma^2-1} Y \hat{X}$ is nonsingular, $Y_\infty = \mathrm{Ric}(J_\infty)$ exists and is given by

$$Y_\infty = \frac{\gamma^2}{\gamma^2-1} \hat{X} \left(I - \frac{\gamma^2}{\gamma^2-1} Y \hat{X} \right)^{-1}.$$

Then $\gamma > 1$, $J_\infty \in \mathrm{dom(Ric)}$ and $Y_\infty \geq 0$ if and only if condition (a) is satisfied.

The rest of the proof then follows by noting that $\left(1 - \left\| \begin{bmatrix} M \\ N \end{bmatrix} \right\|_H^2 \right)^{-\frac{1}{2}} > 1$ and substituting the above expression for Y_∞ in Theorem 5.8. □

A dual to Theorem 6.9 can be trivially derived for the case where LC pair of the factorisation is normalised instead of RC pair, simply by considering P^T. Nevertheless the result is given in the following theorem.

Theorem 6.10 *Let $P \in \mathscr{R}^{p\times q}$ have a stabilisable and detectable state space realisation $P = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ where $A \in \mathbb{R}^{n\times n}$ and $D = 0$. Let $R \in \mathbb{R}^{r\times n}$ be such that (R, A) is detectable and $Q = -XR^*$ where $X \geq 0$ is the stabilising solution to the ARE (6.4). Let $\{N, M, L, K\} \in \mathscr{B}(P)$ be a QR-BCF induced by Q and R given by (4.5) with $D_L = 0$ and $D_N = 0$. Furthermore, let $\gamma \in \mathbb{R}_+$, define Π as in (5.8) and J_∞ as in (5.10).*

Then there exists a controller $C_\infty \in \mathscr{R}^{q\times p}$ such that $\|\mathcal{F}_l(\Pi, C_\infty)\|_\infty < \gamma$ if and only if

(a) $\left(1 - \left\| \begin{bmatrix} M & -L \end{bmatrix} \right\|_H^2 \right)^{-\frac{1}{2}} < \gamma$;

(b) $J_\infty \in \mathrm{dom(Ric)}$ *and* $Y_\infty = \mathrm{Ric}(J_\infty) \geq 0$;

(c) $\rho\left\{ I + Y_\infty \hat{Y} \left(I - \frac{\gamma^2}{\gamma^2 - 1} X\hat{Y} \right)^{-1} \right\} < \gamma^2$;

where $\hat{Y} \geq 0$ is the observability Gramian of M.

Proof. The proof follows by duality to Theorem 6.9. □

6.5.2 Symmetric Systems

It was shown by Theorem 5.13 that BCF robust control synthesis for symmetric systems can be simplified by a strategic selection of Q and R.

We will now restrict out attention to symmetric systems satisfying some additional conditions so as to derive a stronger result. These restrictions ensure that a normalised QR-BCF of the plant satisfying Lemma 5.12 can be trivially constructed. This is then used to deduce the existence of a robustly stabilising controller via the solution of a single Lyapunov equation.

Theorem 6.11 *Suppose that $P = P^T \in \mathscr{R}^{p\times p}$ has a minimal state space realisation $P = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ where $A \in \mathbb{R}^{n\times n}$, $D = 0$, $S > 0$ satisfies the conditions of Lemma 5.11 and*

$$2SA + C^*C \geq 0. \tag{6.23}$$

Let $R \in \mathbb{R}^{r\times n}$ be such that $R^*R = 2SA + C^*C$, $Q = -S^{-1}R^*$ and suppose that $\tilde{X} > 0$ is the solution to the Lyapunov equation

$$\tilde{X}\tilde{A} + \tilde{A}^*\tilde{X} + C^*C = 0 \tag{6.24}$$

where $\tilde{A} = -(A + BC)$. Finally, let $\gamma \in \mathbb{R}_+$ and define Π as in (5.8).

Then the QR-BCF induced by Q and R given by (4.5) with $D_L = 0$ and $D_N = 0$ is normalised. Furthermore, there exists a controller $C_\infty \in \mathscr{R}^{p\times p}$ satisfying $\|\mathcal{F}_l(\Pi, C_\infty)\|_\infty < \gamma$ if and only if $S < \gamma\tilde{X}$.

Proof. First note that (6.23) guarantees the existence of a matrix R such that $R^*R = 2SA + C^*C$. Furthermore, from Lemma 2.14, $\tilde{A} = A + QR = -(A + BC)$ is Hurwitz since $\left(\begin{bmatrix} R \\ C \end{bmatrix}, \tilde{A}\right)$ is observable and $S > 0$ is a solution to the Lyapunov equation

$$S\tilde{A} + \tilde{A}^*S + R^*R + C^*C = 0.$$

Therefore a valid QR-BCF is induced by R and $Q = -S^{-1}R^*$, which also satisfy (5.22), (5.23) and (5.24). It is trivial to show that the BCF is normalised and thus that part of the proof is omitted.

Since $\tilde{X}$ is the solution to (6.24), $\tilde{A}$ is Hurwitz and (C, A) is observable, it follows from Lemma 2.13 that $\tilde{X} > 0$. Now consider the Lyapunov equation

$$(S - \tilde{X})\tilde{A} + \tilde{A}^*(S - \tilde{X}) + R^*R = 0.$$

Since $\tilde{A}$ is Hurwitz and $R^*R \geq 0$ it follows, again from Lemma 2.13, that $S - \tilde{X} \geq 0$ and hence $\tilde{X} \leq S$.

Before proving the main result of the theorem, the assumption that $\gamma > 1$ will be imposed temporarily, which from Theorem 5.13 is a necessary condition for the existence of a robustly stabilising controller. It will be shown that this is guaranteed by the other conditions set forth by this theorem.

Define H_∞ as in (5.9) and $\tilde{H}_\infty$ as

$$\begin{aligned}\tilde{H}_\infty &= \begin{bmatrix} I & -\gamma^2 S^{-1} \\ -\gamma^2 S & \gamma^2 I \end{bmatrix}^{-1} H_\infty \begin{bmatrix} I & -\gamma^2 S^{-1} \\ -\gamma^2 S & \gamma^2 I \end{bmatrix} \\ &= \begin{bmatrix} \tilde{A} & 0 \\ -C^*C & -\tilde{A}^* \end{bmatrix}\end{aligned}$$

and note that a conjugate symmetric, stable, invariant spectral subspace of $\tilde{H}_\infty$ is given by

$$\mathcal{X}_-(\tilde{H}_\infty) = \operatorname{Im}\begin{bmatrix} I \\ \tilde{X} \end{bmatrix}$$

where $\tilde{X} > 0$ is the solution to (6.24). Then

$$\begin{aligned}
\mathcal{X}_-(H_\infty) &= \begin{bmatrix} I & -\gamma^2 S^{-1} \\ -\gamma^2 S & \gamma^2 I \end{bmatrix} \mathcal{X}_-(\tilde{H}_\infty) \\
&= \operatorname{Im}\left\{ \begin{bmatrix} I & -\gamma^2 S^{-1} \\ -\gamma^2 S & \gamma^2 I \end{bmatrix} \begin{bmatrix} I \\ \tilde{X} \end{bmatrix} \right\} \\
&= \operatorname{Im}\begin{bmatrix} I - \gamma^2 S^{-1}\tilde{X} \\ -\gamma^2 (S - \tilde{X}) \end{bmatrix}.
\end{aligned}$$

It now follows that $H_\infty \in \operatorname{dom}(\operatorname{Ric})$ if and only if it has no $j\mathbb{R}$ eigenvalues[c] and $I - \gamma^2 S^{-1}\tilde{X}$ is invertible. Furthermore, when $I - \gamma^2 S^{-1}\tilde{X}$ is nonsingular, $X_\infty = \operatorname{Ric}(H_\infty)$ exists and is given by

$$\begin{aligned}
X_\infty &= -\gamma^2 (S - \tilde{X})(I - \gamma^2 S^{-1}\tilde{X})^{-1} \\
&= \tfrac{\gamma^2}{\gamma^2 - 1}(S - \tilde{X})\left(I - \tfrac{\gamma^2}{\gamma^2-1} S^{-1}(S - \tilde{X})\right)^{-1}.
\end{aligned}$$

From Lemma 2.4 and under the assumption that $\gamma > 1$, we have

$$\begin{aligned}
\det\left(I - \tfrac{\gamma^2}{\gamma^2-1} S^{-1}(S - \tilde{X})\right) \neq 0 \text{ and } X_\infty \geq 0 &\Leftrightarrow \tfrac{\gamma^2}{\gamma^2-1}\rho\left\{S^{-1}(S - \tilde{X})\right\} < 1 \\
&\Leftrightarrow \tfrac{\gamma^2}{\gamma^2-1}(S - \tilde{X}) < S \\
&\Leftrightarrow S < \gamma^2 \tilde{X}.
\end{aligned} \tag{6.25}$$

Suppose that (6.25) is satisfied, then

$$\begin{aligned}
X_\infty < \gamma S &\Leftrightarrow \rho(S^{-1}X_\infty) < \gamma \\
&\Leftrightarrow \rho\left\{\gamma^2 S^{-1}(S - \tilde{X})\left(\gamma^2 S^{-1}\tilde{X} - I\right)^{-1}\right\} < \gamma \\
&\Leftrightarrow \rho\left\{\gamma S^{-1}(S - \tilde{X})\left(\gamma^2 \tilde{X} - S\right)^{-1} S\right\} < 1 \\
&\Leftrightarrow \rho\left\{\gamma (S - \tilde{X})\left(\gamma^2 \tilde{X} - S\right)^{-1}\right\} < 1 \\
&\Leftrightarrow \gamma(S - \tilde{X}) < \gamma^2 \tilde{X} - S \\
&\Leftrightarrow S < \gamma \tilde{X}.
\end{aligned} \tag{6.26}$$

[c]Which is always true since $\tilde{A}$ is Hurwitz.

Combining the fact that $\tilde{X} \leq S$ with (6.26) implies that $S < \gamma S$ and therefore $\gamma > 1$ is implicitly guaranteed. It then becomes apparent that (6.26) also implies (6.25). Thus conditions (5.25), (5.26) and (5.27) are all satisfied if and only if (6.26) holds, which concludes the proof. $\square$

It is simple yet important to show that (6.23) is invariant under a similarity transform. Let a symmetric plant $P \in \mathscr{R}$ have the minimal state space realisations

$$P = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \left[\begin{array}{c|c} \tilde{A} & \tilde{B} \\ \hline \tilde{C} & D \end{array}\right]$$

and suppose that T is nonsingular and satisfies $TAT^{-1} = \tilde{A}$, $\tilde{C} = CT^{-1}$. Furthermore, let $S > 0$ and define $\tilde{S} = T^{-*}ST^{-1} > 0$. Then it is trivial to confirm that S satisfies the conditions of Lemma 5.11 for the first realisation if and only if $\tilde{S}$ does so for the second realisation. Finally,

$$\begin{aligned} 2SA + C^*C \geq 0 &\Leftrightarrow T^{-*}(2ST^{-1}TA + C^*C)T^{-1} \geq 0 \\ &\Leftrightarrow 2\tilde{S}\tilde{A} + \tilde{C}^*\tilde{C} \geq 0 \end{aligned}$$

and the claim follows. This now implies that it is sufficient to check a single state space realisation of a plant for (6.23) to conclude whether or not Theorem 6.11 can be applied.

Remark 6.4 *Any symmetric system that is analytic in $\mathbb{C}_-$ (i.e. all of its poles lie in $\bar{\mathbb{C}}_+$) with $S > 0$ satisfies* (6.23)*. To see this first define $\tilde{A} = S^{\frac{1}{2}}AS^{\frac{1}{2}}$; then using Sylvester's law of inertia (Laub, 2005, Theorem 10.31)* $\mathrm{In}(\tilde{A}) = \mathrm{In}(A)$*. Furthermore, $\Lambda(\tilde{A}) = \Lambda(SA)$ since the two matrices are similar. Then $\Lambda(SA) \subseteq \bar{\mathbb{C}}_+ \Leftrightarrow \Lambda(\tilde{A}) \subseteq \bar{\mathbb{C}}_+ \Leftrightarrow \Lambda(A) \subseteq \bar{\mathbb{C}}_+$. Finally, since SA is Hermitian and $\Lambda(SA) \subseteq \bar{\mathbb{C}}_+$ it follows that $SA \geq 0$ which then implies that $2SA + C^*C \geq 0$.* ♦

6.6 Numerical Examples

Two numerical examples will be presented in this section to demonstrate the theory developed in this chapter. The first deals with a practical case of a mechanical system while the second is purely mathematical in nature.

Example 1

For this example we consider a mass-spring-damper system extensively studied as a benchmark in many areas of control theory (see for example Vinnicombe (2001) or Lanzon and Petersen (2008)).

A schematic representation of the system under consideration is shown in Figure 6.2. Two masses are attached to the walls and coupled together via springs and dampers. It is assumed that the masses are constrained to slide only in the horizontal direction while the surface upon which they do so is frictionless. The inputs to the system are the forces u_1 and u_2 applied to m_1 and m_2 respectively while the outputs are their respective displacements x_1 and x_2.

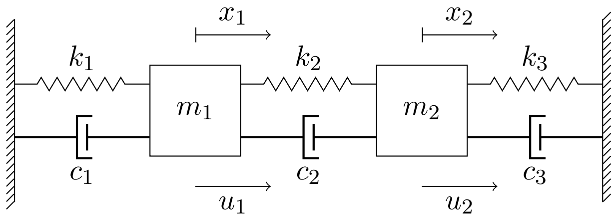


Figure 6.2: Mass-spring-damper system schematic.

Using standard Newtonian mechanics, a model of the above system can be expressed in state space as

$$P = \left[\begin{array}{cccc|cc} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -\frac{k_1+k_2}{m_1} & \frac{k_2}{m_1} & -\frac{c_1+c_2}{m_1} & \frac{c_2}{m_1} & \frac{1}{m_1} & 0 \\ \frac{k_2}{m_2} & -\frac{k_2+k_3}{m_2} & \frac{c_2}{m_2} & -\frac{c_2+c_3}{m_2} & 0 & \frac{1}{m_2} \\ \hline 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{array}\right] \in \mathscr{R}^{2\times 2}.$$

Parameter	Variable	Value	Units
Spring Constants	k_1	0.5	N/m
	k_2	1.75	N/m
	k_3	1.5	N/m
Damping Coefficients	c_1	0.1	Ns/m
	c_2	0.3	Ns/m
	c_3	0.5	Ns/m
Masses	m_1	1.25	kg
	m_2	1	kg

Table 6.1: Numerical parameter values for the mass-spring-damper system.

Numerical values for the parameters of the system are given in Table 6.1, with which the transfer matrix can be expressed as

$$P = \left[\begin{array}{cccc|cc} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -1.8 & 1.4 & -0.32 & 0.24 & 0.8 & 0 \\ 1.75 & -3.25 & 0.3 & -0.8 & 0 & 1 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{array}\right].$$

Algorithm	# of iterations	$\|R_i(I - X_iY_i)\|$	$\|(I - X_iY_{i-1})Q_i\|$
6.1	16	7.248×10^{-4}	6.291×10^{-4}
6.2	7	2.175×10^{-4}	3.646×10^{-4}

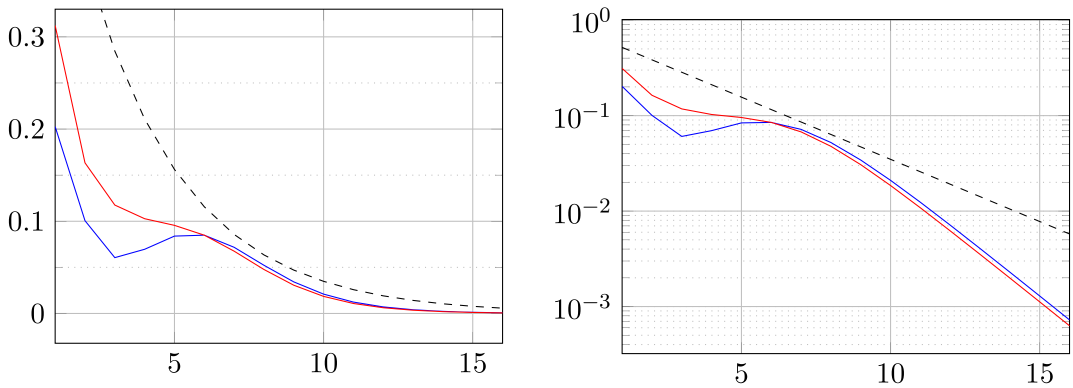
Table 6.2: Execution data for Algorithms 6.1 and 6.2.

Algorithms 6.1 and 6.2 were executed for the above state space realisation of P with the tolerance set to $\mu = 10^{-3}$ and using as initial conditions

$$Q_0 = \begin{bmatrix} 0 & 0 & 1 & 1 \end{bmatrix}^* \text{ and } R_0 = -\begin{bmatrix} 0.15 & 0.1 & 0 & 0 \end{bmatrix}.$$

It can be easily shown that Q_0 and R_0 satisfy the necessary condition given by Lemma 6.5.

Table 6.2 provides some important data obtained from the execution of the two algorithms including the number of iterations it took for them to converge to their final values. Additionally, the trajectories of $\|R_i(I - X_iY_i)\|$ and $\|(I - X_iY_{i-1})Q_i\|$ are shown in Figure 6.3 and Figure 6.4 for Algorithm 6.1 and Algorithm 6.2 respectively. The results are plotted on both linear and logarithmic scales to aid analysis.



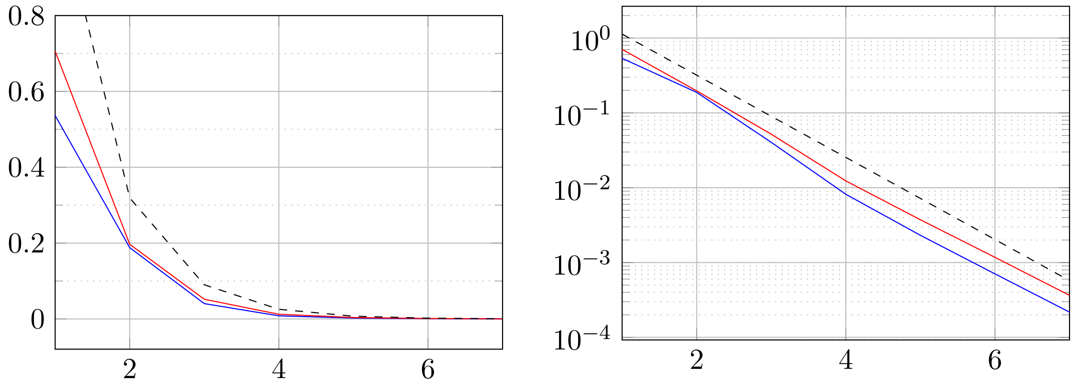
(a) Algorithm 6.1 - Linear scale

(b) Algorithm 6.1 - Logarithmic scale

Figure 6.3: Evolution of $\|R_i(I - X_iY_i)\|$ (——) and $\|(I - X_iY_{i-1})Q_i\|$ (——) using Algorithm 6.1 including exponential upper bounds (- - -).

The first thing to note from the above results is that in both cases the plots are exponentially bounded from above[d]. Additionally, Algorithm 6.2 exhibits monotonicity which is clearly not true for Algorithm 6.1. Finally, both from Table 6.2 and by visual comparison of Figures 6.3 and 6.4 it can be seen that Algorithm 6.2 is more efficient and more than twice as fast to converge. From these observations, the argument that Algorithm 6.2 is a better choice for obtaining a normalised BCF of a plant is further validated.

[d]There exists $\alpha, \beta \in \mathbb{R}_+$ such that $\max\{\|R_i(I - X_iY_i)\|, \|(I - X_iY_{i-1})Q_i\|\} \leq \alpha e^{-\beta i}$ for all $i \geq 1$.



(a) Algorithm 6.2 - Linear scale

(b) Algorithm 6.2 - Logarithmic scale

Figure 6.4: Evolution of $\|R_i(I - X_iY_i)\|$ (——) and $\|(I - X_iY_{i-1})Q_i\|$ (——) using Algorithm 6.2 including exponential upper bounds (- - -).

The resulting matrices, denoted by Q_{ric} and R_{ric} for those obtained via Algorithm 6.1 and Q_{klein} and R_{klein} for Algorithm 6.2, were given by

$$Q_{ric} = \begin{bmatrix} 0.717 & 0.035 & -0.617 & 0.373 \end{bmatrix}^*, \ R_{ric} = \begin{bmatrix} -0.607 & 0.116 & 0.458 & -0.115 \end{bmatrix} \text{ and}$$
$$Q_{klein} = \begin{bmatrix} 0.39 & 0.273 & 0.398 & 0.281 \end{bmatrix}^*, \ R_{klein} = -\begin{bmatrix} 0.572 & 0.383 & 0.487 & 0.273 \end{bmatrix}.$$

It is easy to confirm that both pairs induce a normalised BCF of P.

A robustly stabilising controller was then synthesised using the pair generated by Algorithm 6.2. The Gramians of the BCF symbol associated with the normalised QR-BCF $\{N, M, L, K\} \in \mathscr{B}(P)$ induced by Q_{klein} and R_{klein} were given by

$$X = \begin{bmatrix} 0.432 & 0.232 & 0.076 & 0.062 \\ 0.232 & 0.284 & 0.044 & 0.037 \\ 0.076 & 0.044 & 0.618 & 0.134 \\ 0.062 & 0.037 & 0.134 & 0.61 \end{bmatrix} \text{ and } Y = \begin{bmatrix} 1.048 & 0.171 & 0.248 & 0.063 \\ 0.171 & 0.742 & 0.147 & 0.195 \\ 0.248 & 0.147 & 0.675 & 0.29 \\ 0.063 & 0.195 & 0.29 & 0.284 \end{bmatrix}.$$

Furthermore, the Gramians of M were given by

$$\hat{X} = \begin{bmatrix} 0.237 & 0.167 & 0.003 & 0.007 \\ 0.167 & 0.118 & -0.004 & 0.001 \\ 0.003 & -0.004 & 0.16 & 0.11 \\ 0.007 & 0.001 & 0.11 & 0.076 \end{bmatrix} \text{ and } \hat{Y} = \begin{bmatrix} 0.265 & 0.147 & 0.061 & 0.026 \\ 0.147 & 0.09 & 0.087 & 0.045 \\ 0.061 & 0.087 & 0.37 & 0.209 \\ 0.026 & 0.045 & 0.209 & 0.118 \end{bmatrix}.$$

Using (6.21) and (6.22), ρ_∞ and γ_0 were evaluated as $\rho_\infty = 0.437$ and $\gamma_0 = 1.33$. Newton-Raphson iterations[e] were then used to calculate the smallest value of γ satisfying the spectral radius condition of Theorem 6.8. With the initial condition set to γ_0, the

[e] The step size and tolerance were both set to 10^{-3}.

solution (the smallest valid value of γ) was obtained as $\gamma^\star = 1.685$. Convergence was reached within 17 iterations[f]. Figure 6.5 shows the trajectory of γ over the iterations of the algorithm.

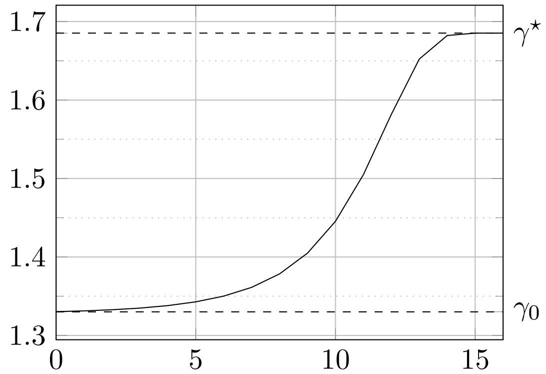


Figure 6.5: Newton-Raphson iterations to find the smallest valid value of γ.

To avoid computational issues[g], the value of γ used for synthesising the controller was chosen to be 1% higher than $\gamma^\star$ giving $\gamma = 1.702$. X_∞ and Y_∞ were constructed via (6.18) and (6.19) using the values for X, Y, $\hat{X}$ and $\hat{Y}$ given above. Finally, using Theorem 5.8, the central controller was obtained as

$$C_{bcf} = \left[\begin{array}{cccc|cc} -12.79 & & & & 3.695 & 2.532 \\ & -0.448 & 2.011 & & -0.016 & -0.011 \\ & -2.011 & -0.448 & & 0.004 & 0.003 \\ & & & -2.261 & -0.657 & -0.457 \\ \hline -5.132 & 0.197 & 0.575 & -5.848 & 0 & 0 \\ -3.517 & 0.127 & 0.399 & -4.069 & 0 & 0 \end{array}\right] \in \mathscr{R}^{2\times 2},$$

achieving a robust stability margin of $\gamma^{-1} = 0.587$.

A second controller was synthesised based on a normalised LCF of the plant using the work of Glover and McFarlane (1989) via Zhou et al. (1996, Corollary 18.2); achieving a robust stability margin of $b_{lcf} = 0.631$. This controller was given by

$$C_{lcf} = \left[\begin{array}{cccc|cc} -24.72 & & & & 3.35 & 2.34 \\ & -1.77 & & & 5.35 & 3.8 \\ & & -0.59 & 2.04 & 0.46 & -0.31 \\ & & -2.04 & -0.59 & 0.74 & -0.53 \\ \hline -5.9 & 0.29 & 0.02 & -0.07 & 0 & 0 \\ -3.88 & 0.19 & -0.02 & 0 & 0 & 0 \end{array}\right] \in \mathscr{R}^{2\times 2}.$$

[f]Given the same initial value and step size, it would take 355 iterations for a line search algorithm to arrive at $\gamma^\star$.

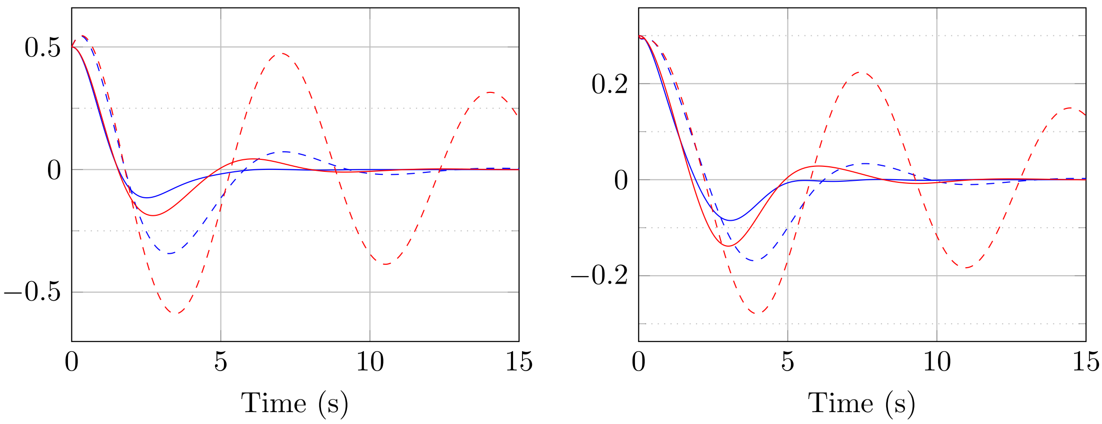
[g]Given $X \geq 0$ and $\alpha > \rho(X)$, as $\alpha \to \rho(X)$, $\underline{\lambda}(I - \alpha X) \to 0$ which can lead to numerical problems when attempting to obtain the inverse of $I - \alpha X$.

Figure 6.6 shows the simulated response of the system under the constructed controllers. The initial conditions for the simulation were set to $x_1 = 0.5\,\mathrm{m}$ and $x_2 = 0.3\,\mathrm{m}$. The controller was then left to stabilise the feedback interconnection and return the positions of both masses to the origin.

In addition to the nominal plant, a perturbed plant was also constructed as $P_\Delta = \mathcal{F}_u(\Pi, \Delta)$ with Π as defined in (5.2) and $\Delta \in \mathscr{RH}_\infty$ given by

$$\Delta = \frac{0.1}{s^2+8s+17}\begin{bmatrix} 5(s^2+8s+17) & 0 & -(s^2+9s+22) \\ s+4 & 1 & -1 \\ -(s^2+10s+30) & -(2s+7) & 2s+8 \end{bmatrix}$$

which satisfies $\|\Delta\|_\infty = 0.549$. Note that with this uncertainty matrix, an unstable complex conjugate pair of poles is introduced at $0.191 \pm 0.755j$. Furthermore, the gap between P and P_Δ as defined by El-Sakkary (1985), is given by $\delta(P, P_\Delta) = 0.626$. The simulation results obtained using the perturbed plant are also shown in Figure 6.6.



(a) Position of mass m_1, x_1 (m). (b) Position of mass m_2, x_2 (m)

Figure 6.6: Simulation results of mass-spring-damper systems with C_{bcf} (——) and C_{lcf} (——). The continuous and dashed lines correspond to the nominal plant perturbed plants respectively.

As expected, the designed controllers successfully stabilise both the nominal and perturbed plants. This should be no surprise since $\gamma\|\Delta\|_\infty = 0.934 < 1$ and $\delta(P, P_\Delta) < b_{lcf}$ (see Georgiou and Smith (1990, Theorem 4) for details). Although the performance of the two controllers is rather similar for the nominal plant, this is not true in the case of the perturbed plant. It can be seen from the simulation results that even though Δ is not enough to destabilise the feedback interconnection when using C_{bcf}, performance is considerably affected with increased oscillations and approximately double the settling time[h]. Performance deterioration for the normalised LCF controller is much worse, as is

[h]Since the response is decaying to zero, the classical measure of settling time cannot be applied. Hence settling time is defined here as $t_s \in \mathbb{R}_+ : \max\{|x_1(t)|, |x_2(t)|\} \leq 0.01\,\mathrm{m}\,\forall t \geq t_s$.

evident from the simulation results. A very oscillatory and slow response was obtained with the settling time being more than 8 times longer.

Example 2

A simple example will now be presented to demonstrate an application of Theorem 6.11. Consider the state-space-symmetric system

$$P = \left[\begin{array}{cc|c} -1 & & 2 \\ & 2 & -2 \\ \hline 2 & -2 & 0 \end{array}\right] \in \mathscr{R}^{1\times 1},$$

where $S = I$ satisfies the conditions on Lemma 5.11. Note that (6.23) is satisfied with a valid R given by $R = \sqrt{2}\begin{bmatrix} 1 & -2 \end{bmatrix}$. Furthermore, the solution to (6.24) is given by $\tilde{X} = \begin{bmatrix} \frac{2}{3} & \\ & \frac{1}{3} \end{bmatrix} > 0$. It can therefore be concluded from Theorem 6.11 that there exists a robustly stabilising controller achieving a robust stability margin γ if and only if $I < \gamma\tilde{X}$ or equivalently $\gamma > 3$. Then, following the procedure of Theorem 5.8 yields the controller

$$C_\infty = \left[\begin{array}{cc|c} -1.081 & & -0.08 \\ & 2.335 & 0.17 \\ \hline 0.08 & -0.17 & 0 \end{array}\right] \in \mathscr{R}^{1\times 1},$$

achieving $\gamma = 3.0455$.

6.7 Summary & Conclusion

In this chapter the concept of normalisation was extended to BCFs of the plant. A number of state space conditions and properties were presented based on normalised QR-BCFs. Unlike LCFs or RCFs where a single ARE (with sign-definite quadratic term) needs to be solved to achieve normalisation, two such AREs must be solved to obtain a normalised QR-BCF of the plant. It was shown that the resulting problem is nonlinear, an explicit solution to which does not exist in the literature. Thus, two iterative procedures to construct a normalised BCF of a plant were proposed, one based on AREs while the second following the Kleinman approach based on Lyapunov equations which are computationally easier to handle.

Robust control synthesis results using normalised BCFs were presented. It was shown that similarly to normalised LCFs or RCFs a lower bound on the achievable robust stability margin can be calculated *a priori*. However, this is not guaranteed to be achievable as an additional spectral radius condition must also be satisfied. These results were then restricted to a special set of symmetric systems where robust stabilisation can be achieved via the solution of a single Lyapunov equation.

Chapter 7

Application: Control of a Quadrotor UAV

7.1 Introduction

Unmanned Aerial Vehicles (UAVs) have many real-world uses in both civilian and military situations. Some example applications include reconnaissance, surveillance, search and rescue or as a simple pass-time for hobbyists. As a result, there is considerable interest in developing new and novel vehicles (Crowther et al., 2011). Control of UAVs has attracted much attention from the control community in recent years. This is in part due to the multitude of uses mentioned above, but also because of their interesting dynamics.

In this chapter, a quadrotor[a] UAV is considered such as the one described by Pounds et al. (2010). See Lanzon et al. (2014) for a schematic representation of such a vehicle, as well as some of the formalisms adopted herein. After mathematically modelling the system, a controller is designed for the attitude of the system using feedback linearisation and the normalised BCF procedure described in Theorem 6.8. This is followed by a simple state feedback controller for the position of the vehicle in Euclidean 3D space. Though many strategies have been successfully employed in the past to tackle this problem (Mokhtari et al., 2005; Raffo et al., 2010; Tzes et al., 2012), the goal here is to demonstrate how BCF theory, as developed in this thesis, can be applied to a practical control situation.

7.2 System Modelling

An important part of controlling a UAV is understanding how it is actuated, in other words, how it generates the necessary forces and torques. This is crucial in developing a successful control strategy. Let the force generated by the i^{th} rotor be given by $f_i \in \mathbb{R}$, then the reaction torque generated by that rotor is assumed to be given by $\tau_i = k_{\tau,i} f_i$ for some $k_{\tau,i} \in \mathbb{R}_+$. A further simplifying assumption commonly made in this context, is

[a]Having four rotors arranged in a planar fashion.

that $k_{\tau,1} = k_\tau$ for all i, implying that all rotors are identical. Now, define $\tau_b \in \mathbb{R}^3$ as the torque vector in the body axes and let $u_f \in \mathbb{R}$ denote the total force generated by the four rotors, that is $u_f = \sum_{i=1}^{4} f_i$. Then for a quadrotor UAV, the torques and total force can be related to the forces generated by the rotors as

$$\begin{bmatrix} \tau_b \\ u_f \end{bmatrix} = \begin{bmatrix} 0 & -l & 0 & l \\ -l & 0 & l & 0 \\ k_\tau & -k_\tau & k_\tau & -k_\tau \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{bmatrix}. \tag{7.1}$$

Modelling the attitude of a rigid body in 3D space is a problem studied extensively in the past. Many methods are available to achieve this; including Euler angles, directional cosine matrices, angle-axis notation and quaternions. Each of these methods has its own advantages and disadvantages. For example, using quaternion notation (Kuipers, 2002) gives a singularity and ambiguity free model, but one that is more mathematically complex. On the other hand, using Euler angles leads to a much simpler and easier to visualise model but prone to problems such as gimbal lock (the loss of a degree of freedom) (Grassia, 1998) and singularities (relies on the trigonometric tangent function).

The Euler angles approach will be used to model the attitude of a UAV for its simplicity. Let the attitude of the UAV be given by the angles roll ϕ, pitch θ and yaw ψ and define $\eta = \begin{bmatrix} \phi & \theta & \psi \end{bmatrix}^* \in \mathbb{R}^3$. Then η is related to the angular rate vector $\omega_b \in \mathbb{R}^3$ (the rate of change of each angle) by

$$\dot{\eta} = \Phi \omega_b, \tag{7.2}$$

where

$$\Phi = \begin{bmatrix} 1 & \sin\phi\tan\theta & \cos\phi\tan\theta \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi\sec\theta & \cos\phi\sec\theta \end{bmatrix}.$$

Note that the singularities mentioned before are obvious here as Φ depends on $\tan\theta$ and $\sec\theta$ which tend to ∞ as θ tends to $\pi/2\,\mathrm{rad}$.

Finally, using standard Newtonian mechanics the dynamics of ω_b can be expressed as

$$J\dot{\omega}_b = -\omega_b \times J\omega_b - k_r\omega_b + \tau_b, \tag{7.3}$$

where $0 < J \in \mathbb{R}^{3\times3}$ is the inertia matrix and $k_r \in \mathbb{R}_+$ is the rotational drag coefficient. The UAV in question is assumed to be symmetric about the x and y axes, with J being diagonal. Furthermore, it is also assumed that the rotational drag is equal in all directions.

Parameter	Variable	Value	Units
Total mass of UAV	m	0.5	kg
Inertia about x-axis	J_{xx}	5.9×10^{-3}	$\mathrm{kg\,m^2}$
Inertia about y-axis	J_{yy}	5.9×10^{-3}	$\mathrm{kg\,m^2}$
Inertia about z-axis	J_{zz}	1.16×10^{-3}	$\mathrm{kg\,m^2}$
Rotor arm length	l	0.255	m
Rotational drag coefficient	k_r	0.01	$\mathrm{N\,m\,s\,rad^{-1}}$
Translational drag coefficient	k_t	0.05	$\mathrm{N\,s\,m^{-1}}$
Torque-to-thrust ratio	k_τ	0.24	m

Table 7.1: Numerical parameter values for quadrotor UAV system.

Modelling the position of a UAV is considerably simpler. Suppose that the position of the UAV in 3D space is given by the vector $\lambda \in \mathbb{R}^3$. Then

$$m\ddot{\lambda} = -k_t\dot{\lambda} + R_{B\mapsto E}u_f\vec{n}_z - mg\vec{n}_z, \tag{7.4}$$

where $m \in \mathbb{R}_+$ is the mass of the UAV, $k_t \in \mathbb{R}_+$ is the translational drag coefficient, $R_{B\mapsto E} \in \mathbb{R}^{3\times 3}$ is the rotation matrix mapping the body to the earth axes (Salazar-Cruz et al., 2009), g is the acceleration due to gravity and finally, $\vec{n}_z = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^*$. As was the case for the rotational drag, it is assumed that the translational drag is equal in all directions. Numerical values for the parameters of the system are given in Table 7.1.

7.3 Control Synthesis

In this section a two stage controller is designed for controlling the UAV system given by (7.1), (7.2), (7.3) and (7.4). In the inner loop, the attitude of the UAV will be stabilised via feedback linearisation and a normalised BCF controller synthesised using Theorem 6.8. In the outer loop, the position will be forced to track a reference signal using a simpler state feedback approach while ignoring the attitude dynamics. For this approach to work, the dynamics of closed loop attitude subsystem must be considerably faster than those of the position subsystem. A high level block diagram of the proposed control structure is shown in Figure 7.1.

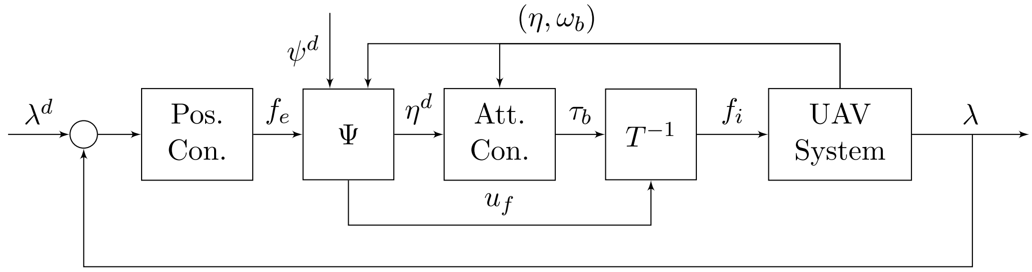


Figure 7.1: UAV control strategy. Ψ denotes the mapping from the desired forces in the earth axes and yaw angle to the attitude and total force required to achieve them.

7.3.1 Attitude Control

As mentioned above, the attitude of the UAV will be controlled via feedback linearisation and a normalised BCF controller. By differentiating (7.2) we have

$$\begin{aligned}\ddot{\eta} &= \dot{\Phi}\omega_b + \Phi\dot{\omega}_b \\ &= \dot{\Phi}\omega_b + \Phi J^{-1}(-\omega_b \times J\omega_b - k_r\omega_b + \tau_b).\end{aligned} \tag{7.5}$$

Before proceeding, it must noted that from the above it follows that the relative degree of of the attitude system is 6, which implies that there are no zero dynamics (Khalil, 2000), hence there are no complicating factors in applying feedback linearisation. It is now easy to see that by setting

$$\tau_b = \omega_b \times J\omega_b + k_r\omega_b + J\Phi^{-1}(u_\eta - \dot{\Phi}\omega_b)$$

where $u_\eta \in \mathbb{R}^3$ is an artificial input, the attitude dynamics are linearised to $\ddot{\eta} = u_\eta$. By strategically designing u_η various objectives can be achieved, for example robust feedback linearisation as described by Franco et al. (2006). A different approach will be taken here and this opportunity will be used to "shape" the linearised plant. By setting $u_\eta = \hat{A}_2\dot{\eta} + \hat{A}_1\eta + k\hat{B}\hat{\tau}_b$ where $\hat{A}_2, \hat{A}_1, \hat{B} \in \mathbb{R}^{3\times3}$ and $k \in \mathbb{R}_+$, the linearised plant is transformed to[b]

$$P = \left[\begin{array}{cc|c} 0 & I_3 & 0 \\ \hat{A}_1 & \hat{A}_2 & \hat{B} \\ \hline kI_3 & 0 & 0 \end{array}\right] \in \mathscr{R}^{3\times3}.$$

Now by an appropriate selection of these parameters, the open loop transfer function of the linearised plant can be given some desirable features that will help to improve the closed loop performance. For example, high gain at low frequencies to reduce steady state error and higher bandwidth for a faster response.

The above matrices were selected as $\hat{A}_1 = 0$

$$\hat{A}_2 = \begin{bmatrix} -2I_2 & \\ & -1 \end{bmatrix} \text{ and } \hat{B} = \begin{bmatrix} 40I_2 & \\ & 16 \end{bmatrix},$$

with $k = 8$. The choice to set $\hat{A}_1 = 0$ was made so that an integrator is retained in each channel to guarantee reference tracking. The singular value plot of the resulting linearised plant is shown in Figure 7.2a.

[b]Note that with the selection $\hat{A}_1 = 0$, $\hat{A}_2 = -k_rJ^{-1}$, $\hat{B} = J^{-1}$ and $k = 1$ the aforementioned robust feedback linearisation control law is obtained.

Using Lemma 4.10, Q_0 and R_0 were constructed as

$$Q_0 = \begin{bmatrix} 0 & 0.5 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } R_0 = \begin{bmatrix} 0.53 & -7.98 & -0.09 & 0.12 & -0.8 & -0.03 \\ -7.93 & -0.53 & 0.9 & -1.33 & -0.07 & 0.27 \\ -0.9 & 0.03 & -7.95 & -0.12 & 0.01 & -0.91 \end{bmatrix}$$

to be used as initial conditions[c] for Algorithm 6.2. The algorithm was then executed, converging to the normalising pair

$$Q = \begin{bmatrix} -0.35 & 1.68 & 0.45 \\ 1.73 & 0.36 & -0.01 \\ 0.06 & -0.28 & 1.09 \\ -7.14 & 34.52 & 9.33 \\ 35.69 & 7.44 & -0.21 \\ 0.86 & -3.71 & 14.38 \end{bmatrix} \text{ and } R = \begin{bmatrix} 1.57 & -7.83 & -0.46 & 0.07 & -0.35 & -0.03 \\ -7.57 & -1.63 & 1.99 & -0.34 & -0.07 & 0.14 \\ -2.05 & 0.05 & -7.73 & -0.09 & 0 & -0.54 \end{bmatrix}$$

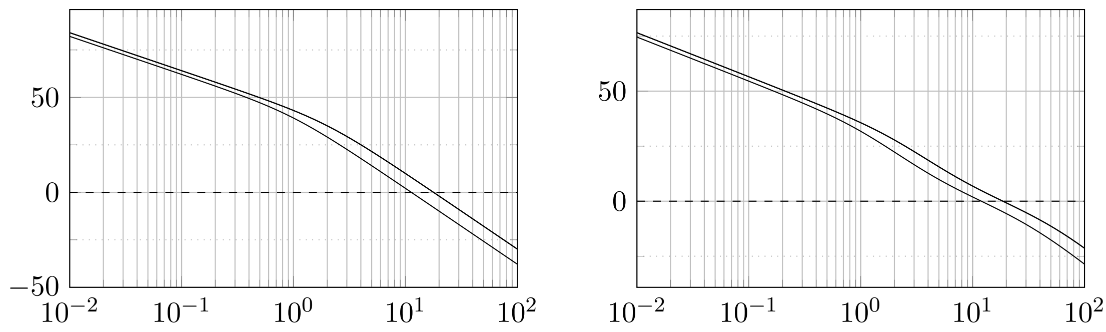
within 12 iterations with $\|R_{12}(I - X_{12}Y_{12})\| = 2.47 \times 10^{-5}$ and $\|(I - X_{12}Y_{11})Q_{12}\| = 4.31 \times 10^{-4}$. The Gramians X, Y, $\hat{X}$ and $\hat{Y}$ are omitted for brevity but can be easily obtained by solving the associated Lyapunov equations.

From (6.21) and (6.22), it was calculated that $\sqrt{\rho_\infty} = 2.001$ and $\gamma_0 = 1.797$. Then, following the same procedure as in the previous chapter, the smallest value satisfying the conditions of Theorem 6.8 was obtained as $\gamma^\star = 3.054$. With γ set to 3.084 (as before 1% higher than $\gamma^\star$), a robustly stabilising controller was synthesised as

$$C_\eta = \left[\begin{array}{cccccc|ccc} -612.5 & & & & & & & 0 & -80.9 \\ & -415.7 & & & & & -55.2 & 25.0 & 0 \\ & & -415.5 & & & & 25.0 & 55.2 & 0 \\ & & & -33.0 & & & 0 & 0 & 1.8 \\ & & & & -53.7 & & -4.5 & 2.0 & 0 \\ & & & & & -53.7 & -2.0 & -4.5 & 0 \\ \hline 0 & 21.9 & -9.9 & 0 & -30.5 & -13.8 & 0 & 0 & 0 \\ 0 & -9.9 & -21.9 & 0 & 13.8 & -30.5 & 0 & 0 & 0 \\ 24.5 & 0 & 0 & 53.1 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

The loop shape obtained using the linearised plant and C_η is shown in Figure 7.2b. By comparing this to Figure 7.2a it can be seen that as expected, the controller imparts some minor changes around the crossover frequency to improve robustness while the rest of the loop shape remains largely unchanged.

[c]Note that these do not satisfy condition of Lemma 6.5.



(a) Singular values (dB) of P.

(b) Singular values (dB) of PC_η.

Figure 7.2: Singular value plots of linearised plant and forward loop transfer function.

7.3.2 Position Control

Due to the simpler nature of the translational dynamics, a more heuristic approach will be taken to control the position of the UAV. Consider again (7.4) and define $f_e = R_{B \mapsto E} \vec{n}_z u_f$ which will temporarily be considered the input to the system. Furthermore, let the desired position be given by $\lambda^d \in \mathbb{R}^3$ and define an error term as $\tilde{\lambda} = \lambda^d - \lambda$. Then, by setting

$$f_e = mg\vec{n}_z + (k_t I + m\Lambda_2)\dot{\lambda} + m\Lambda_1\lambda + m\left(\ddot{\lambda}^d - \Lambda_2\dot{\lambda}^d - \Lambda_1\lambda^d\right) \tag{7.6}$$

where Λ_1, $\Lambda_2 \in \mathbb{R}^{3\times 3}$, the translational error dynamics are linearised to

$$\begin{bmatrix} \dot{\tilde{\lambda}} \\ \ddot{\tilde{\lambda}} \end{bmatrix} = \begin{bmatrix} 0 & I \\ \Lambda_1 & \Lambda_2 \end{bmatrix} \begin{bmatrix} \tilde{\lambda} \\ \dot{\tilde{\lambda}} \end{bmatrix}.$$

By choosing Λ_1 and Λ_2 such that the above is stable, the position error dynamics will asymptotically converge to the origin and thus closed loop tracking will be achieved. In this case, these parameters were chosen as

$$\Lambda_1 = \begin{bmatrix} -6I_2 & \\ & -12 \end{bmatrix} \text{ and } \Lambda_2 = \begin{bmatrix} -5I_2 & \\ & -7 \end{bmatrix},$$

with the reason being the fact that the closed loop position dynamics must be considerably slower that the attitude dynamics such that the latter has little impact and can thus be ignored. With the above selections, the bandwidth[d] of the system is around $1.53\,\mathrm{rad\,s^{-1}}$ for the x and y channels while for z this is increased to about $2.19\,\mathrm{rad\,s^{-1}}$, which is substantially lower (approximately by an order of magnitude) than that achieved by the attitude control system as seen in Figure 7.2b.

The final step in deriving a control law is to find the attitude and total force that map to the value of f_e as generated by (7.6). This is represented by the block Ψ in Figure 7.1.

[d]Frequency at which the gain is at $-3\,\mathrm{dB}$ of its DC value.

First, set the total force to $u_f = \|f_e\|$ and define $u_e = \begin{bmatrix} u_x & u_y & u_z \end{bmatrix}^* = \|f_e\|^{-1} f_e$. Then, let $\psi^d \in \mathbb{R}$ represent an exogenous desired value for the yaw angle and define

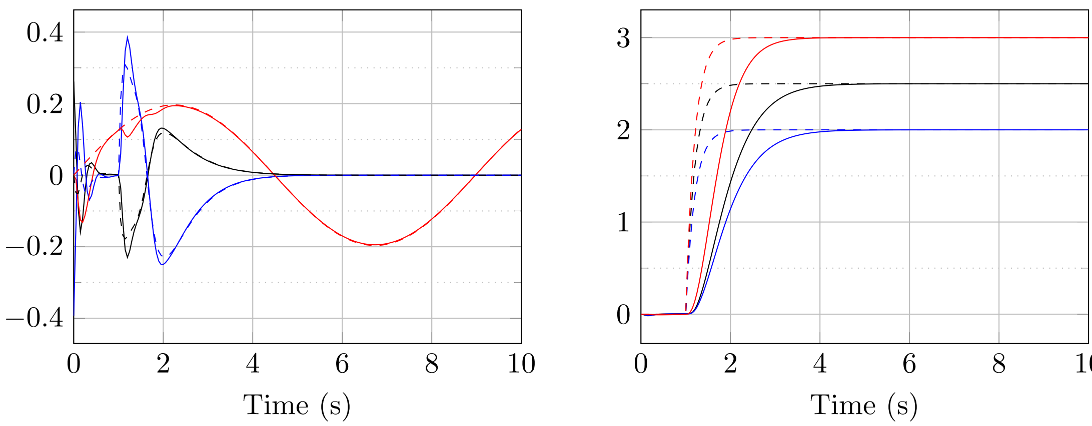
$$\phi^d = \sin^{-1}(u_x \sin\psi^d - u_y \cos\psi^d) \text{ and}$$
$$\theta^d = \sin^{-1}\left(\frac{u_x \cos\psi^d + u_y \sin\psi^d}{\sqrt{1 - (u_x \sin\psi^d - u_y \cos\psi^d)^2}} \right).$$

Finally, combining $\eta = \begin{bmatrix} \phi^d & \theta^d & \psi^d \end{bmatrix}^*$ and $u_f = \|f_e\|$ as defined above, the desired f_e is obtained. This can be validated by back substitution. A similar approach is used by many authors, however it is common to use small angle approximations for simplification, resulting in substantially different equations for ϕ^d and θ^d.

7.4 Simulation Results

The feedback interconnection of the UAV system and designed controller was simulated, with the results shown in Figure 7.3. For the simulation, the position reference was given as step signals to all channels at $t = 1\,\text{s}$, with the magnitude being 2.5 m for x, 2 m for y and 3 m for z, thus $\lambda^d = \begin{bmatrix} 2.5 & 2 & 3 \end{bmatrix}^*$ m. The yaw reference was given as a sinusoidal signal of amplitude $\pi/16$ rad and frequency $0.7\,\text{rad s}^{-1}$, that is $\psi^d = (\pi/16)\sin(0.7t)$ rad. Finally, the initial conditions were set to $\eta = \begin{bmatrix} \pi/12 & -\pi/8 & 0 \end{bmatrix}^*$ rads for the attitude and $\omega_b = \begin{bmatrix} -0.1 & 0.3 & 0 \end{bmatrix}^*$ rad s^{-1} for the angular rates, while the position and velocity were set to zero.

To avoid excessive control action, a prefilter was used to smooth out the position reference signal. This was achieved by setting $t_d \dot{\lambda}^d = \hat{\lambda}^d - \lambda^d$ where $\hat{\lambda}^d$ is the new reference input and $t_d \in \mathbb{R}_+$ is the filtering time constant which was set to 0.2.



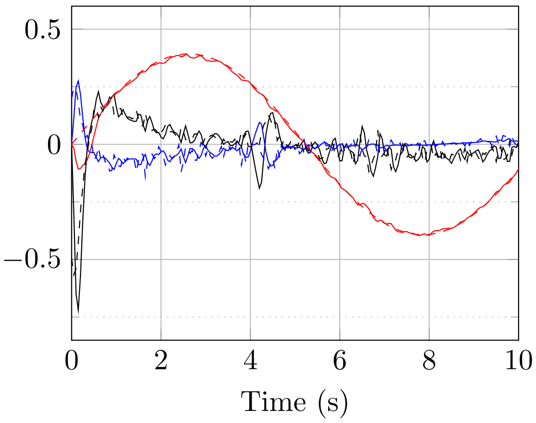
(a) Attitude η (rad) ϕ (——), θ (——), ψ (——).

(b) Position λ (m) x (——), y (——), z (——).

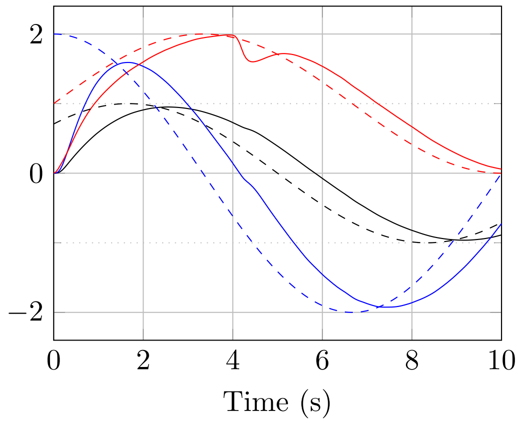
Figure 7.3: Closed loop simulation results of quadrotor UAV. The dashed lines represent the reference signal/desired value.

It can be seen that the UAV is successfully stabilised and that reference tracking is achieved. The attitude subsystem is considerably faster than the position control outer loop and evidently does not negatively impact its performance.

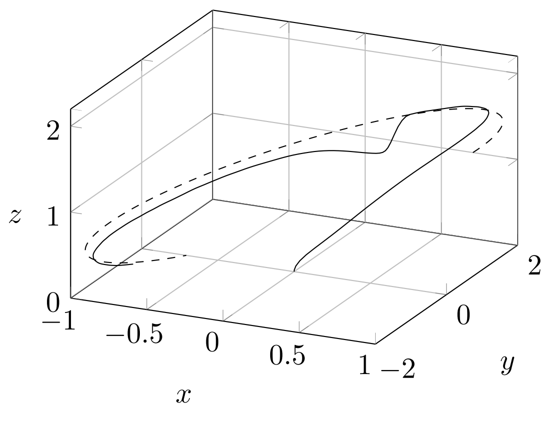
A second simulation was performed with a more complex elliptical desired trajectory. The yaw reference was again a sinusoid given by $\psi^d = (\pi/8)\sin(0.6t)$. In this case, some white noise was added in the feedback path to simulate sensor inaccuracies. Additionally, the input to the system was set to zero at $t = 4\,\mathrm{s}$ for a duration of $0.2\,\mathrm{s}$ to simulate temporary, complete actuator failure. The initial conditions were assumed to be zero for all states. A reference prefilter was not used for this simulation. Figure 7.4 shows the results of this simulation.



(a) Attitude η (rad)
ϕ (——), θ (——), ψ (——).



(b) Position λ (m)
x (——), y (——), z (——).

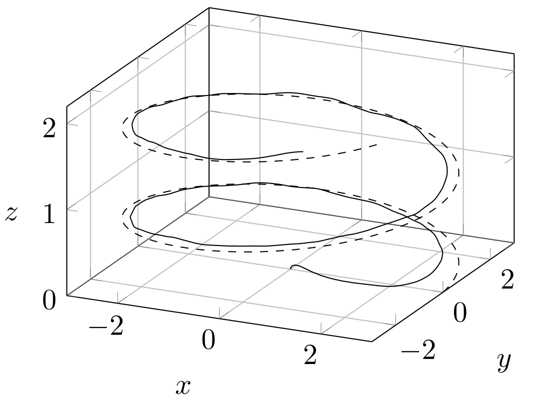


(c) Position 3D (m).

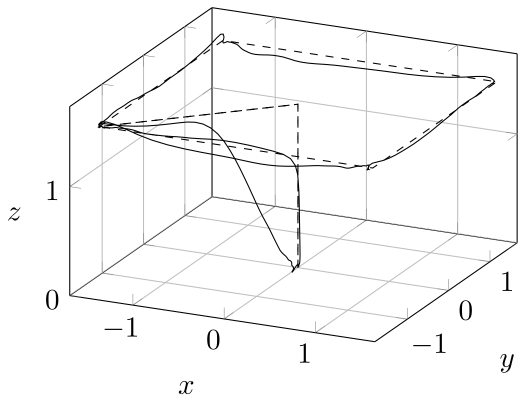
Figure 7.4: Closed loop simulation results of quadrotor UAV with elliptical reference trajectory. The dashed lines represent the reference signal/desired value.

From the results of the simulation, it becomes apparent that the designed control strategy is able to cope with both the actuator failure and sensor noise. When the actuators are disabled, the altitude starts dropping almost immediately, however the horizontal trajectory is mostly maintained due to the momentum of the vehicle. For this reason the attitude is not significantly affected, since the horizontal direction in which the UAV is travelling does not change. The controller attempts to correct the sudden loss of altitude by increasing the total force generated by the rotors but not changing the attitude of the vehicle. Shortly after the actuators are enabled, the system resumes to tracking the reference signal.

Additional simulation results are shown in Figure 7.5; these are provided purely as a matter of interest with only the 3D position of the vehicle plotted. In the first simulation, the desired trajectory was given by an ascending spiral, where the UAV is commanded to follow a circular trajectory in the xy-plane while ascending at a constant rate. For the second simulation, the UAV takes off vertically and follows a horizontal square trajectory before returning to its origin. No reference prefiltering was used for these simulations.



(a) Ascending spiral trajectory (m).



(b) Square trajectory (m).

Figure 7.5: Additional closed loop simulation results of quadrotor UAV. The dashed lines represent the desired trajectories.

7.5 Summary & Conclusion

In this chapter a control strategy for a quadrotor UAV was proposed based on feedback linearisation and normalised BCF theory. It was shown to be an effective approach towards controlling such systems. By way of this example the argument that BCFs deserve additional attention from the control community is further validated.

Though the results obtained exhibit sufficiently good performance with reference tracking and disturbance rejection properties, not much effort was put into selecting the various control parameters. For example trivial choices were made for Λ_1 and Λ_2. It is possible to considerably improve the closed loop properties of the interconnection by more carefully selecting the values of these parameters.

Chapter 8

Conclusion

In this chapter the main contributions of this thesis are summarised and possible directions of future research are explored. The aim of this thesis is to lay the foundations of a BCF theory upon which further results can be based and to examine their possible uses in robust control theory. This is achieved, firstly, by the introduction of systematic methods for obtaining BCFs of a plant, and secondly, by adapting well established robust control synthesis and analysis tool to make use of such factorisations. Several numerical examples are used to demonstrate how BCFs can be used to tackle control problems and in certain situations provide various advantages; demonstrating that BCF theory is viable in a control theoretic environment.

8.1 Contributions

The main contributions of this thesis are summarised below. Square brackets are used to denote novel results.

- The concept of BCFs, largely ignored by the control community, is presented in a comprehensive manner and the notion of internal dimension [Definition 3.4] of a BCF is introduced with lower bounds for it derived [Lemma 3.6 and Theorem 3.8]. Numerous internal stability conditions for a standard positive feedback interconnection are derived [Theorems 3.12 and 3.13, Lemmas 3.15 to 3.17 and 3.19 to 3.22, and Corollary 3.18]; extending the limited number of results already found in the literature. It is shown that through the use of BCFs of the plant, reduced dimension internal stability conditions can be obtained; with an extreme case presented where the internal stability of a MIMO feedback interconnection is established via a scalar test.

- Methods are formulated that can be used to generate a BCF of a plant based on a stabilisable and detectable state space realisation [Theorems 4.2 and 4.3, and Corollary 4.4], which are shown to capture the well known formulae of Nett et al. (1984) [Remark 4.3]. With the introduction of the QR-BCF parametrisation, it is shown that a non-trivial

BCF over $\mathscr{RH}_\infty$ can be constructed for every plant in $\mathscr{R}$; thus any subsequent state space results can be applied without loss of generality.

- BCF uncertainty is introduced [Equation 5.1], proving to have an appealing structure that combines LC, RC and additive uncertainties. Subsequently, robust analysis [Theorems 5.1 and 5.2] and stabilisation results [Theorem 5.8 and Corollary 5.10] are developed for this type of uncertainty. Robust control for special classes of systems is examined based on BCFs including minimum phase [Theorem 5.3 and Lemma 5.5] and symmetric [Theorem 5.13] systems. It is shown that the standard assumptions associated with the 2-ARE solution to the $\mathscr{H}_\infty$ robust stabilisation problem can be easily satisfied using a QR-BCF of the plant.

- A definition of normalised BCFs is proposed [Definition 6.1]. Tests based on state space data that can be used to establish whether a BCF is normalised are presented [Theorem 6.1, Corollary 6.2 and Theorem 6.3] and iterative methods of constructing such factorisations are developed based on two coupled AREs with sign-definite quadratic terms [Algorithms 6.1 and 6.2]. Such factorisations are shown to produce advantages similar to those that arise from classical normalised coprime factorisations, in that a lower bound for the achievable robust stability margin can be calculated *a priori* [Theorem 6.8]. Unilaterally normalised BCFs are also introduced with an associated robust stabilisation result presented [Theorem 6.9]. The special case of symmetric systems is further examined in the context of normalised BCFs [Theorem 6.11].

8.2 Direction of Future Research

Although a substantial number of results pertaining to BCFs were presented in this thesis, the theory developed herein is far from complete. Possible areas of future research are briefly discussed below.

- Distance measures form an area of control theory where coprime factorisations find extensive use. A method for developing a distance measure based on any uncertainty structure was presented by Lanzon and Papageorgiou (2009). Given a nominal and perturbed plant, the set of BCF uncertainty matrices that map the former to the latter can be easily parametrised. However, finding an admissible uncertainty matrix whose norm is the infimum in that set is not trivial. Furthermore, the properties of BCF uncertainty claimed herein are based solely on those derived from coprime factor uncertainty. Further investigation of both issues is needed.

- The choice of coprime factorisation of the plant affects the closed loop properties of a feedback interconnection. For example, if the factorisation has lightly damped poles then the achievable robust stability margin is reduced, as shown by Engelken and Lanzon (2012). It is reasonable to assume (though this must be examined as well) that a similar fact is true for QR-BCFs of the plant. Hence, the potential benefits of using systematic methods such as the one described by Sivashankar et al. (1994) to construct a QR-BCF also need to be explored.

- Further investigation is needed into the problem posed by Theorem 6.3 to establish if a solution always exists. One approach to achieve this would be via a convergence proof for Algorithms 6.1 and 6.2.

- It was alluded to in many instances that loop shaping weights could be used with the robust control synthesis result of Theorem 5.8 to improve the closed loop characteristics of the system. It would be interesting and useful to know what performance/robustness guarantees can be obtained from such a procedure and how they relate to those that emerge from using standard coprime factors derived by McFarlane and Glover (1992).

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