OPTIMAL STOPPING PROBLEMS WITH APPLICATIONS TO MATHEMATICAL FINANCE

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The main contribution of the present thesis is a solution to finite horizon optimal stopping problems associated with pricing several exotic options, namely the American lookback option with fixed strike, the British lookback option with fixed strike, American swing put option and shout put option. We assume the geometric Brownian motion model and under the Markovian setting we reduce the optimal stopping problems to free-boundary problems. The latter we solve by probabilistic arguments with help of local time-space calculus on curves ([52]) and we characterise optimal exercise boundaries as the unique solution to certain integral equations. Then using these optimal stopping boundaries the option price can be obtained.

The significance of Chapters 2 and 3 is a development of a method of scaling strike which helps to reduce three-dimensional optimal stopping problems, for lookback options with fixed strike, including a maximum process to two-dimensional one with varying parameter. In Chapter 3 we show a remarkable example where, for some values of the set parameters, the optimal exercise surface is *discontinuous* which means that the three-dimensional problem could not be tackled straightforwardly using local time-space calculus on surfaces ([55]). This emphasises another advantage offered by the reduction method.

In Chapter 4 we study the multiple optimal stopping problems with a put payoff associated to American swing option using local time-space calculus. To our knowledge this is the first work where a) a sequence of integral equations has been obtained for consecutive optimal exercise boundaries and b) the early exercise premium representation has been derived for swing option price. Chapter 5 deals with the shout put option which allows the holder to lock the profit at some time τ and then at time T take the maximum between two payoffs at τ and T. The novelty of the work is that it provides a rigorous analysis of the free-boundary problem by probabilistic arguments and derives an integral equation for the optimal shouting boundary along with the shouting premium representation for the option price in some cases. This approach can also be applied to other shout and reset options.

In Chapter 6 we discuss a problem of the smooth-fit property for the American put option in an exponential Lévy model. In [2] the necessary and sufficient condition was obtained for the perpetual case. Recently Lamberton and Mikou [40] covered almost all cases for an exponential Lévy model with dividends on finite horizon and we study remaining cases. Firstly, we take the logarithm of the stock price as a Lévy process of finite variation with zero drift and finitely many jumps, and prove that one has the smooth-fit property without regularity unlike in the infinite horizon case. Secondly, we provide some analysis and calculations for another case uncovered in [40] where the drift is positive but for all maturities and removing the additional condition they used.

The result of Chapter 1 is contained in the publication [33] and results of Chapters 2-5 are exposed in preprints [34], [17] and [35] that are submitted for publication.

Declaration

No portion of the work referred to in this thesis has been submitted in support of an application for another degree or qualification of this or any other university or other institute of learning.

Copyright Statement

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Chapter 1

Introduction

Optimal stopping theory is one of the most developed and exciting parts of modern stochastic calculus. The origin of this theory goes back to Wald's sequential analysis [72] in 1943. Later Snell [67] in 1952 formulated a general optimal stopping problem for discrete-time case and characterised the value function as the smallest supermartingale which dominates the gain process (so-called Snell's envelope). Snell's work refers to martingale methods and the solutions to the optimal stopping problems were in the form of conditional expectations with respect to natural filtration of the process and they are hard to compute explicitly unless the underlying process is Markov in which case the conditional expectations are significantly simplified. Dynkin [19] in 1963 discovered the key principle of the optimal stopping theory for Markov processes called the superharmonic characterisation and which states that the value function of optimal stopping problem is the smallest superharmonic function (with respect to underlying Markov process) which dominates the gain function. This principle has a clear geometric interpretation for the case of a Wiener process and states that the value function of sup (inf) problem is the smallest concave (convex) function dominating the obstacle. The latter is nothing but the Legendre transform in convex theory and goes back to Mandelbrot and Fenchel (see [56] for a detailed explanation of this connection) and this fact allows us to find a closed form for the value function using the known expression for the Legendre transform of the gain function.

Another fascinating feature of the optimal stopping theory is that it is the place where probabilists and PDE specialists meet together, since discovered by Mikhalevich [49] in 1958 and McKean in 1965 [47] an optimal stopping problems can be reduced to free-boundary problems from mathematical analysis. The resulting system contains a PDE (with differential operator associated to underlying diffusion process) of the value function in some open set with an unknown (free) boundary which is the optimal stopping boundary of the original problem. A systematic research of optimal stopping problems by their reduction to free-boundary problems was started by Grigelionis and Shiryaev [25] in 1966.

Most of these theoretical developments above were driven by solving particular examples with real applications which can be formulated or reformulated as the optimal stopping problems. Some of the examples are the following: 1) statistics of stochastic processes: quickest detection and sequential testing problems (see [65]); 2) sharp inequalities in stochastic analysis, e.g. Wald inequalities, Doob inequalities and Burkholder-Davis-Gundy inequalities etc; 3) mathematical finance problems, particularly, American options pricing; 4) financial engineering issues such as optimal asset selling and optimal prediction problems. All these examples above were thoroughly exposed and studied in [58].

This thesis deals with the application of optimal stopping theory to the arbitrage-free pricing of American style options and a development of the methodology to tackle arising problems. The classical example of an optimal stopping problem in mathematical finance is the American put option

$$V = \sup_{0 \le \tau \le T} \mathsf{E}\, e^{-r\tau} \Big(K - X_\tau \Big)^+ \tag{1.1}$$

where V is the arbitrage-free price, the process X is a geometric Brownian motion, $K > 0$ is the strike price, $0 < T \le \infty$ is the maturity time, $r > 0$ is the interest rate, and the supremum is taken over all stopping times τ with respect to natural filtration of X and the discount price $(e^{-rs}X_s)_{s\ge 0}$ is a martingale under P. When $T = \infty$ the problem (1.1) becomes one-dimensional and can be tackled by using a free-boundary approach and the so-called smooth-fit property. The value function V then can be found explicitly along with the optimal stopping threshold b. On the other hand when T is finite the optimal stopping problem becomes two-dimensional as now the time left to expiry plays a significant role and b is not constant but a function of time. Therefore the mathematical analysis of this problem is more challenging and has been developed gradually. Firstly, McKean [47] in 1965 expressed V in terms of boundary b and the latter itself was solution to a countable system of nonlinear integral equations. This approach goes back to Kolodner [38] in 1956 who applied this train of thought to the Stefan problem in mathematical physics. Van Moerbeke [69] in 1976 then continued the work of McKean and obtained single integral equation by connecting the American put option problem to the physical problem. In both works [47] and [69] the integral equations did not have any financial interpretations and had purely mathematical tractability.

Eventually Kim [32] in 1990, Carr, Jarrow, Myneni [12] in 1992 and Jacka [30] in 1997 independently derived a nonlinear integral equation for b that had appeared from the early exercise premium representation for V having a financial meaning. However the question of the uniqueness of the solution to the integral equation was left open. Finally, Peskir [53] in 2005 provided the early exercise representation for V and integral equation for b using the local time-space formula [52]. Moreover and most crucially, he proved the uniqueness of solution. This work [53] opened the door for solving other finite horizon optimal stopping problems, e.g. Russian option [54], Asian option etc.

In this thesis we mainly deal with some exotic American options by using local time-space calculus and provide their theoretical and financial analysis. Chapter 2 has appeared as journal publication in *Stochastics* and Chapters 3 and 4 have been submitted to *Applied Mathematical Finance* and *Finance and Stochastics*, respectively, and currently are under review. Chapter 5 is a preprint [35] and Chapter 6 is based on work which is under progression now. All chapters are self-contained and have detailed introductions however below we will highlight the main contribution and novelty of the research presented in this thesis.

Chapter 2 is based on [33] and considers the American lookback option with fixed strike

$$V = \sup_{0 \le \tau \le T} \mathsf{E}\, e^{-r\tau} \Big(\max_{0 \le s \le \tau} S_s - K \Big)^+ \tag{1.2}$$

where τ is a stopping time of the geometric Brownian motion S solving

$$dS_t = rS_t\, dt + \sigma S_t\, dB_t \quad (S_0 = s) \tag{1.3}$$

and where B is a standard Brownian motion started at zero, $T > 0$ is the expiration date (maturity), $K > 0$ is the strike, $r > 0$ is the interest rate, and σ is the volatility coefficient.

Optimal stopping problems of type (1.2) have been solved for different cases in a number of papers. The main difficulty arising in this problem is that the maximum process is not Markov itself and we have to add the stock price process to achieve this which increases the dimension of the underlying process. First, in the case $T = \infty$ Shepp and Shiryaev [62] solved the problem when the strike K equals zero (the Russian option) using a two–dimensional Markov setting (stock price and its running maximum). Later in [63] they noticed that this optimal stopping problem can be reduced to a one-dimensional Markov setting by a Girsanov change-of-measure theorem. Pedersen [50] then solved the problem (1.2) when $K > 0$ in the case of infinite horizon using the Peskir's maximality principle [51]. As we mentioned above the optimal stopping problems with finite horizon include time as an extra dimension and thus are analytically more difficult than those with an infinite horizon. After applying the change of measure [63] and reducing the problem (1.2) with finite horizon and zero strike (the Russian option) to two dimensions (for a time-space Markov process), the resulting optimal stopping problem was solved by Peskir [54] using the change-of-variable formula with local time on curves [52]. Finally extending this method to the problem (1.2) with finite horizon and non-zero strike (without applying the change of measure), the resulting optimal stopping problem in three dimensions was solved by Gapeev [21] using the change-of-variable formula with local time on surfaces [55].

The main contribution and novelty of Chapter 2 is the illustration of another approach for solving this three-dimensional problem when $K > 0$ and $T < \infty$ using the Girsanov theorem. We show that the arbitrage-free price and optimal stopping set in (1.2) can be expressed by the value function and the optimal stopping boundary of a two-dimensional problem with a scaling strike. We then provide some analysis and prove of all technical conditions in two-dimensional setting. However it is important to emphasize that we first fix strike $\widetilde{K}$ and solve the two-dimensional problem, and then to determine the option price and rational exercise boundary we vary the strike, thus the problem inherently remains three-dimensional. This method was also used to derive a solution for the British lookback option with fixed strike (cf. Chapter 3). This approach simplifies the discussion and expressions for the arbitrage-free price and the rational exercise boundary rather than solving (1.2) straightforwardly in three-dimensional setting. The dimension of optimal stopping problems often plays a crucial role in finding their solutions, therefore the idea of this approach could be useful in reducing dimension of related optimal stopping problems as well.

Chapter 3 is based on preprint [34] and studies the British lookback option with fixed strike. Recently, Peskir and Samee (see [59], [60]) introduced a new type of options called 'British'. The main idea of this protection option is to give a holder the early exercise feature of American options whereupon his payoff (deliverable immediately) is the 'best prediction' of the European payoff under the hypothesis that the true drift of the stock price equals a contract drift which agreed initially. Financial analysis of the returns of the British put option showed that with the contract drift properly selected this put option becomes a very attractive alternative to the classic American put. Then following the rationale of the British put and call options, this type of options was extended to the Russian option in [23]. Chapter 3 addresses the British lookback option with fixed strike (non-zero) of call type as we believe that this is the most interesting case from a mathematical point of view and actually this problem motivated the development of the method of a scaling strike in Chapter 2.

Chapter 3 includes two parts: analytical solution and financial analysis. The theoretical solution is based on the method of a scaling strike which allows us to reduce the three-dimensional problem to two-dimensional one with a scaling strike. Using a local time-space calculus on curves [52] we derive a closed form expression for the arbitrage-free price in terms of the optimal stopping boundary of the two-dimensional optimal stopping problem and show that the rational exercise boundary of the option can be characterised via the unique solution to a nonlinear integral equation. We also show the remarkable numerical example where the rational exercise boundary exhibits a *discontinuity* with respect to space variable (see Figure 3.4), therefore it was not possible to apply a change-of-variable formula with local time on surfaces (as e.g. in [21]) in order to solve the three-dimensional stopping problem directly. This is another advantage of dimension reduction by the method of a scaling strike. The solution of the zero-strike case $K = 0$ (the British Russian option) is fully embedded into the present problem and can be considered as a particular case. Second part of Chapter 3 provides the financial analysis of the returns of British lookback option with fixed strike in comparison with its the American and the European counterparts. In line with [59], [60] and [23] this option has been shown to be a very attractive financial instrument for investors.

Chapter 4 is a preprint [17] and applies the local time-space calculus for pricing so-called swing options. These contracts are financial products designed primarily to allow for flexibility on purchase, sale and delivery of commodities in the energy market. They have features of American-type options with multiple early exercise rights and in many relevant cases are mathematically described in terms of multiple optimal stopping problems. Mathematical formulations of such problems in the economic-financial literature date back to the early 1980's and an exhaustive survey of them may be found in [31, Sec. 1 and 2] and references therein. Theoretical and numerical aspects of pricing and hedging swing contracts have received increasing attention in the last decade with many contributions from a number of authors developing in parallel several methods of solution (see e.g. [45] for an extensive review of recent results). Main examples are Monte-Carlo methods, variational approach in Markov setting, BSDE techniques, martingale methods and Snell envelope. On the other hand despite the general interest towards the theoretical aspects of swing options it seems that the problem of characterising analytically the optimal exercise boundaries has not been thoroughly studied yet. For perpetual options such boundaries have been provided for a put payoff in the Black-Scholes framework in [11], whereas more general dynamics and payoffs were studied in [10]. For the case of finite horizon the problem is still widely open and the question of finding analytical equations for the optimal stopping boundaries remains unanswered. In Chapter 4 we address this issue in a setting described below.

We consider the case of a swing option with a put payoff, finite maturity $T > 0$, strike price $K > 0$ and $n \in \mathbb{N}$ exercise rights. The underlying price follows a geometric Brownian motion and we consider an option whose structure was described in [29] and [31]. In particular the holder can only exercise one right at a time and then must wait at least for a so-called refracting period of length $\delta > 0$ between two consecutive exercises. If the holder has not used the first of the n rights by time $T-(n-1)\delta$ then at that time she must exercise it and remains with a portfolio of $n-1$ European put options with different maturities up to time T. This corresponds to the case of a swing option with a constrained minimum number of exercise rights equal to n. We first perform an analysis using probabilistic arguments of the option with $n = 2$ and prove the existence of two continuous, monotone, bounded optimal stopping boundaries. It turns out that the continuation set is between these two boundaries. We provide an early exercise premium representation for the price of the option in terms of the optimal stopping boundaries and adapting arguments of [18] (see also [53]) we show that such boundaries uniquely solve a system of coupled integral equations of Volterra type. Finally we extend the result to the general case of n exercise rights by induction.

Chapter 5 is a preprint [35] and studies the shout put option. This option belongs to the class of contracts with reset feature, i.e. the holder can change the structure of the European option at some point. There are two groups of options of this type: 1) shout (call or put) option which allows the holder to lock the profit at some favourable time τ (if there is such) and then at time T take the maximum between two payoffs at τ and T; 2) reset (call or put) option gives to investor the right to reset the strike

K to current price, i.e. to substitute the current out-of-the money option to the at-the-money one. We study the first group and we note that they have both European (since the payoff is known at T only) and American features (due to early 'shouting' opportunity). Therefore we formulate the pricing problem as an optimal stopping problem, or more precisely an optimal prediction problem since the payoff is claimed and known only at T and thus the gain function is non-adapted. We then reduce it to a standard optimal stopping problem with adapted payoff and study the associated free-boundary problem.

The main contribution of this Chapter 5 is that we exploit probabilistic arguments including local time-space calculus ([52]) and as a result we characterise the optimal shouting boundary as the unique solution to a nonlinear integral equation. Then we derive a shouting premium presentation for the option's price via optimal shouting boundary. These results have been proven for some case, since in the opposite case the proof of the monotonicity of the boundary is currently an open problem. However, numerical analysis shows that the optimal shouting boundary seems to be increasing. In the literature the numerical methods such as binomial tree, Monte-Carlo and analytical approaches such as PDE, variational inequalities, series expansions, Laplace transform have been applied. We note that the technique we used can be applied to solve pricing problems for shout call and reset call and put options. Moreover, the shout put option is equivalent to reset call option in the sense that their optimal strategies coincide and the same fact is true for shout call and reset put options. We conclude the paper by financial analysis of the shout put option and particularly its financial returns compare to its American, European and British (see [59]) put counterparts. In the numerical example it has been shown that the British option generally outperform others and that there is a large region below K where the shout option's returns are greater than American put option's returns. This fact is pleasant for an investor who wishes to lock the profit in that region while enjoying the possibility to increase his payoff from a favourable future movement at the maturity T.

Finally in Chapter 6 we discuss a problem about the smooth-fit property for the

American put option in an exponential Lévy model with dividends. This principle has been proved in a classical Black-Scholes model for both infinite and finite horizons (see e.g. Section 25 in [58]) and helps to solve the corresponding optimal stopping and free-boundary problems. However for exponential Lévy model the picture changes and the smooth-fit property may not hold, e.g. in [13] authors showed an example of the $CGMY$ model where the principle fails. Alili and Kyprianou [2] studied perpetual case and delivered the necessary and sufficient condition (namely the regularity of the logarithm of stock price with respect to negative half-line) in the exponential Lévy model without dividends. Recently Lamberton and Mikou [40] proved several results for ab exponential Lévy model with dividends on finite horizon. Firstly they showed that the condition derived in [2] is also sufficient for the finite horizon case. Then without this condition, i.e. when logarithm of the stock price is of finite variation and has positive drift, Lamberton and Mikou showed absence of the smooth-fit at least for large maturities. Finally, under a stronger condition they proved that the smooth-fit fails irrespective of the size of maturity.

The contribution of Chapter 6 is to provide an example showing that the necessary and sufficient condition for the infinite horizon case is not applicable for the finite horizon case and it is caused by the fact that the optimal stopping boundary is strictly increasing unlike in the perpetual case. Namely, we take the logarithm of the stock price as a Lévy process of finite variation with zero drift and finitely many jumps, and prove that one has the smooth-fit property without regularity. Secondly, we provide some analysis and calculations for the another case uncovered in [40] where the drift is positive but for all maturities and removing the additional condition they used. We then propose open questions and finding answers to them could help to resolve this problem.

Chapter 2

The American lookback option with fixed strike

2.1. Introduction

According to theory of modern finance (see e.g. [66]) the arbitrage-free price of the lookback option with fixed strike coincides with the value function of the optimal stopping problem (2.1) below. In the case of infinite horizon T Shepp and Shiryaev [62] solved the problem when the strike K equals zero (the Russian option) using a two–dimensional Markov setting (stock price and its running maximum). Then in [63] they noticed that this optimal stopping problem can be reduced to a one-dimensional Markov setting by a Girsanov change-of-measure theorem. Pedersen [50] solved the problem (2.1) when $K > 0$ in the case of infinite horizon using the maximality principle [51] (for recent extensions to Lévy processes see [39]).

The optimal stopping problems for the maximum processes with finite horizon are inherently three-dimensional (time-process-maximum) and thus analytically more difficult than those with infinite horizon. After applying the change of measure [63] and reducing the problem (2.1) with finite horizon and zero strike (the Russian option) to two dimensions (for a time-space Markov process), the resulting optimal stopping problem was solved by Peskir [54] using the change-of-variable formula with local time on curves [52]. The optimal stopping boundary was determined as the unique solution of the nonlinear integral equation arising from the formula. Extending this method to the problem (2.1) with finite horizon and non-zero strike (without applying the change of measure), the resulting optimal stopping problem in three dimensions was solved by Gapeev [21] using the change-of-variable formula with local time on surfaces [55].

The main purpose of this paper is to illustrate another approach for solving this three-dimensional problem using the Girsanov theorem. We show that the arbitrage-free price and optimal stopping set in (2.1) can be expressed by the value function and the optimal stopping boundary of a two-dimensional problem with a scaling strike. Hence we prove all technical conditions in two-dimensional setting. However it is important to emphasize that we first fix strike K and solve the two-dimensional problem, and then to determine the option price and rational exercise boundary we vary the strike, thus the problem inherently remains three-dimensional. This method can also be used to derive solution for the British lookback option with fixed strike (Chapter 3). Another feature of this method is that closed form expression for the value function in (2.1) and nonlinear integral equations for optimal stopping boundary are simpler than in [21]. Dimension of optimal stopping problems often plays a crucial point in finding their solutions, therefore the idea of this approach could be useful in reducing dimension of related optimal stopping problems as well.

In Section 2.2 we formulate the lookback option with fixed strike in the case of finite horizon and present reduction of the initial problem to a two-dimensional optimal stopping problem using a change of measure. In Section 2.3 we solve the two-dimensional problem and in Section 2.4 we apply that solution to the initial problem. In Section 2.5 we make a conclusion and propose a programme for future research using this approach.

2.2. Formulation of the problem and its reduction

The arbitrage-free price of the lookback option with fixed strike is given by

$$V = \sup_{0 \le \tau \le T} \mathsf{E}\, e^{-r\tau} \Big(\max_{0 \le s \le \tau} S_s - K \Big)^+ \tag{2.1}$$

where τ is a stopping time of the geometric Brownian motion S solving

$$dS_t = rS_t\,dt + \sigma S_t\,dB_t \quad (S_0 = s). \tag{2.2}$$

We recall that B is a standard Brownian motion started at zero, $T > 0$ is the expiration date (maturity), $K > 0$ is the strike, $r > 0$ is the interest rate, and σ is the volatility coefficient.

Let us consider the Markovian extension using the structures of processes $\max S$ and S, which leads to the following value function

$$V(t,m,s) = \sup_{0\le\tau\le T-t} \mathsf{E}\,\mathrm{e}^{-r\tau}\Big(m \vee \max_{0\le u\le\tau} sS_u - K\Big)^+ \tag{2.3}$$

with $S_t = \mathrm{e}^{\sigma B_t + (r-\frac{\sigma^2}{2})t}$ starting at 1. It is well known that process $M_t = \max_{0\le s\le t} S_s$ is not Markov, but the pair (S, M) forms a Markov process. Hence this problem is three-dimensional with the Markov process $(t, M_t, S_t)_{t\ge 0}$ due to presence of the finite horizon. Since the dimension of the problem is very important, any possibility of reducing the dimension becomes significant. When $K = 0$ in [63] and [54] the following reduction was used

$$\mathsf{E}e^{-r\tau}M_\tau = \mathsf{E}e^{-r\tau}S_\tau\frac{M_\tau}{S_\tau} = \widetilde{\mathsf{E}}\frac{M_\tau}{S_\tau} \tag{2.4}$$

where the expectation $\widetilde{\mathsf{E}}$ is taken under a new measure $\widetilde{\mathsf{P}}$ and the process M/S is a one-dimensional Markov process under this measure. In the case of non-zero strike one cannot make reduction in the same way straightforwardly. Gapeev [21] solved (2.3) in a three-dimensional setting using the local time-space calculus on surfaces [55]. Current paper illustrates a different approach to the solution of (2.3).

Now we will discuss how to reduce problem (2.3) to a two-dimensional problem with a scaling strike. By the change of measure we have

$$\begin{aligned}\mathsf{E}\,\mathrm{e}^{-r\tau}\Big(m \vee \max_{0\le u\le\tau} sS_u - K\Big)^+ &= s\,\widetilde{\mathsf{E}}\left(\frac{m\vee\max_{0\le u\le\tau} sS_u}{sS_\tau} - \frac{K}{sS_\tau}\right)^+ \\ &= s\,\widetilde{\mathsf{E}}\left(\frac{m\vee K\vee\max_{0\le u\le\tau} sS_u}{sS_\tau} - \frac{K}{sS_\tau}\right)\end{aligned} \tag{2.5}$$

where $d\widetilde{\mathsf{P}} = \mathrm{e}^{-rT}S_T d\mathsf{P}$ so that $\widehat{B_t} = B_t - \sigma t$ is a standard Brownian motion under $\widetilde{\mathsf{P}}$ for $0\le t\le T$ and in the second equality we used fact that $(x-y)^+ = x\vee y - y$ for

$x, y \in I\!R$. The strong solution of (2.2) is given by

$$S_t = s \exp\Big(\sigma B_t + (r-\sigma^2/2)t\Big) = s \exp\Big(\sigma \widehat{B_t} + (r+\sigma^2/2)t\Big). \tag{2.6}$$

Hence it is easily seen that $\widetilde{\mathsf{E}}\frac{1}{S_\tau} = \widetilde{\mathsf{E}}\mathrm{e}^{-r\tau}$ and it follows from (2.5) that we have

$$\begin{aligned}\mathsf{E}\,\mathrm{e}^{-r\tau}\Big(m \vee \max_{0\le u\le \tau} sS_u - K\Big)^+ &= s\,\widetilde{\mathsf{E}}\left(\frac{\frac{m\vee K}{s} \vee M_\tau}{S_\tau} - \tfrac{K}{s}\mathrm{e}^{-r\tau}\right) \\ &= s\,\widetilde{\mathsf{E}}\left(\frac{\frac{m\vee K}{s} \vee M_\tau}{S_\tau} - \tfrac{Ke^{rt}}{s}\mathrm{e}^{-r(t+\tau)}\right)\end{aligned} \tag{2.7}$$

where $M_t = \max_{0\le u\le t} S_u$. This motivates us to fix $\widetilde{K} = Ke^{rt}/s$ and consider the following two-dimensional (time-space) optimal stopping problem:

$$W(t,x) = W(t,x;\widetilde{K}) = \sup_{0\le\tau\le T-t} \widetilde{\mathsf{E}}\left(X_\tau^x - \widetilde{K}\mathrm{e}^{-r(t+\tau)}\right) \tag{2.8}$$

where $X_t^x = \frac{x\vee M_t}{S_t}$ is a Markov process. By Ito's formula one finds that

$$dX_t = -rX_t dt + \sigma X_t d\widetilde{B}_t + dR_t \tag{2.9}$$

under $\widetilde{\mathsf{P}}$ where $\widetilde{B} = -\widehat{B}$ is a standard Brownian motion, and we set

$$R_t = \int_0^t I(X_s = 1)\,\frac{dM_s}{S_s}. \tag{2.10}$$

It is clear from (2.7) that the initial value (2.3) can be expressed as

$$V(t,m,s) = s\,W\Big(t, \tfrac{m\vee K}{s}; \tfrac{Ke^{rt}}{s}\Big) \tag{2.11}$$

and the optimal stopping set in (2.3) is given by

$$D = \{\,(t,m,s) : s\,W\Big(t, \tfrac{m\vee K}{s}; \tfrac{Ke^{rt}}{s}\Big) = (m-K)^+\,\}. \tag{2.12}$$

In the next section we solve the two-dimensional problem (2.8).

2.3. The two-dimensional problem

Let us consider the optimal stopping problem

$$W(t,x) = \sup_{0\le\tau\le T-t} \widetilde{\mathsf{E}}\,G(t+\tau, X_\tau^x) \tag{2.13}$$

where the process X from (2.9) with $X_0^x = x$ under $\widetilde{\mathsf{P}}$, $0 \le t \le T$, $x \ge 1$ and the gain function is given by $G(t,x) = x - \widetilde{K}e^{-rt}$. This section parallels the derivation of the solution [54] when $G(t,x) = x$ for $x \ge 1$ and presents needed modifications since in (2.13) the gain function depends on time in a nonlinear way.

Standard Markovian arguments (see e.g. [58]) indicate that W solves the following free-boundary problem:

$$W_t + \mathbb{L}_X W = 0 \qquad \text{in } C \tag{2.14}$$

$$W(t,x) = G(t,x) \qquad \text{for } x = b(t) \tag{2.15}$$

$$W_x(t,x) = 1 \qquad \text{for } x = b(t) \tag{2.16}$$

$$W_x(t,1+) = 0 \qquad \text{(normal reflection)} \tag{2.17}$$

$$W(t,x) > G(t,x) \qquad \text{in } C \tag{2.18}$$

$$W(t,x) = G(t,x) \qquad \text{in } D \tag{2.19}$$

where the continuation set C and the stopping set D are defined by

$$C = \{ (t,x) \in [0,T) \times [1,\infty) : x < b(t) \} \tag{2.20}$$

$$D = \{ (t,x) \in [0,T) \times [1,\infty) : x \ge b(t) \} \tag{2.21}$$

and $b : [0,T] \to R$ is the unknown optimal stopping boundary, i.e the stopping time

$$\tau_b = \inf \{ 0 \le s \le T-t : X_s^x \ge b(t+s) \} \tag{2.22}$$

is optimal in the problem (2.13).

Our main aim is to follow the train of thought where W is first expressed in terms of b, and b itself is shown to satisfy a nonlinear integral equation. We will moreover see that the nonlinear equation derived for b cannot have other solutions. (We also note that in the Section 2.4 we will consider the value function $W(t,x) = W(t,x;\widetilde{K})$ and the optimal stopping boundary $b(t) = b(t;\widetilde{K})$ as the functions of strike $\widetilde{K}$ as well.) Below we will use the following functions:

$$F(t,x) = \widetilde{\mathsf{E}}\,(X_t^x) \tag{2.23}$$

$$H(t,u,x,y) = \widetilde{\mathsf{E}}_{t,x}(G(u,X_u)I(X_u \geq y)) = \int_y^\infty (z - \widetilde{K}\mathrm{e}^{-ru}) f(u{-}t,x,z)\, dz \quad (2.24)$$

for $t \in [0,T]$, $x \geq 1$, $u \in (t,T]$, $y \geq 1$, where $z \mapsto f(u{-}t,x,z)$ is the probability density function of X^x_{u-t} under $\widetilde{\mathsf{P}}$ and $\widetilde{\mathsf{P}}_{t,x}(\cdot) = \widetilde{\mathsf{P}}(\,\cdot\,|X_t = x\,)$.

The main result of present section may now be stated as follows.

Theorem 2.3.1. *The optimal stopping boundary in the problem* (2.13) *can be characterised as the unique continuous decreasing solution* $b : [0,T] \to I\!R$ *of the nonlinear integral equation*

$$b(t) = \widetilde{K}(e^{-rt} - e^{-rT}) + F(T{-}t, b(t)) + r\int_t^T H(t,u,b(t),b(u))\, du \qquad (2.25)$$

satisfying $b(t) > (\widetilde{K}e^{-rt} \vee 1)$ *for* $0 < t < T$. *The solution* b *satisfies* $b(T-) = (\widetilde{K}e^{-rT} \vee 1)$ *and the stopping time* τ_b *from* (2.22) *is optimal in* (2.13) *(see Figure 2.1).*

The value function (2.13) *admits the following representation:*

$$W(t,x) = -\widetilde{K}e^{-rT} + F(T{-}t,x) + r\int_t^T H(t,u,x,b(u))\, du \qquad (2.26)$$

for all $(t,x) \in [0,T] \times [1,\infty)$.

Proof. The proof will be carried out in several steps. We start by stating some general remarks.

We see that W admits the following representation:

$$W(t,x) = \sup_{0 \leq \tau \leq T-t} \widetilde{\mathsf{E}}\Big(\frac{(x - M_\tau)^+ + M_\tau}{S_\tau} - \widetilde{K}\mathrm{e}^{-r(t+\tau)}\Big) \qquad (2.27)$$

for $(t,x) \in [0,T] \times [1,\infty)$. It follows that

$$x \mapsto W(t,x) \text{ is increasing and convex on } [1,\infty) \qquad (2.28)$$

for each $t \geq 0$ fixed.

1. We show that $W : [0,T] \times [1,\infty) \to I\!R$ is continuous. For this, using $\sup(f) - \sup(g) \leq \sup(f-g)$ and $(y-z)^+ - (x-z)^+ \leq (y-x)^+$ for $x,y,z \in I\!R$, it follows that

$$W(t,y) - W(t,x) \leq (y-x) \sup_{0 \leq \tau \leq T-t} \widetilde{\mathsf{E}}\Big(\frac{1}{S_\tau}\Big) \leq y - x \qquad (2.29)$$

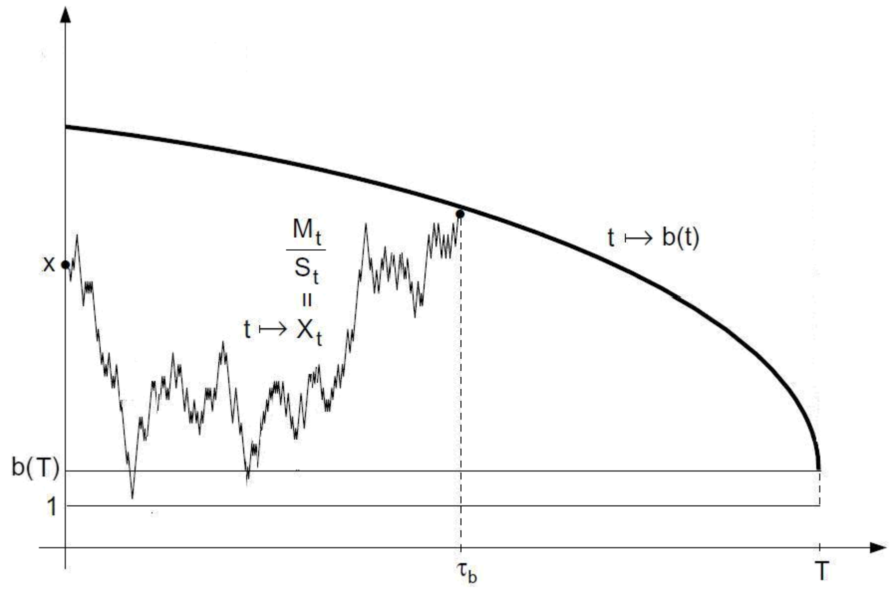

Figure 2.1: A computer drawing of the optimal stopping boundary b for the problem (2.13) in the case $\widetilde{K} = 1.5, T = 1, r = 0.1, \sigma = 0.4$ with the boundary condition $b(T) = \widetilde{K}\mathrm{e}^{-rT} > 1$.

for $1 \leq x < y$ and all $t \geq 0$. From (2.28) and (2.29) we see that $x \mapsto W(t,x)$ is continuous uniformly over $t \in [0,T]$. Thus to prove that W is continuous on $[0,T] \times [1,\infty)$ it is enough to show that $t \mapsto W(t,x)$ is continuous on $[0,T]$ for each $x \geq 1$ given and fixed. For this, take any $t_1 < t_2$ in $[0,T]$ and $\varepsilon > 0$, and let τ_1^ε be a stopping time such that $\widetilde{\mathsf{E}}\,(X^x_{\tau_1^\varepsilon} - \widetilde{K}\mathrm{e}^{-r(t_1+\tau_1^\varepsilon)}) \geq W(t_1,x) - \varepsilon$. Setting $\tau_2^\varepsilon = \tau_1^\varepsilon \wedge (T-t_2)$ we see that $W(t_2,x) \geq \widetilde{\mathsf{E}}\,(X^x_{\tau_2^\varepsilon} - \widetilde{K}\mathrm{e}^{-r(t_2+\tau_2^\varepsilon)})$. Hence, we get

$$\begin{aligned} 0 &\leq |W(t_1,x) - W(t_2,x)| \qquad (2.30)\\ &\leq |\widetilde{\mathsf{E}}\,(X^x_{\tau_1^\varepsilon} - \widetilde{K}\mathrm{e}^{-r(t_1+\tau_1^\varepsilon)} - X^x_{\tau_2^\varepsilon} + \widetilde{K}\mathrm{e}^{-r(t_2+\tau_2^\varepsilon)})| + \varepsilon. \end{aligned}$$

Letting first $t_2 - t_1 \to 0$ using $\tau_1^\varepsilon - \tau_2^\varepsilon \to 0$ and then $\varepsilon \to 0$ we see that $|W(t_1,x) - W(t_2,x)| \to 0$ by dominated convergence. This shows that $t \mapsto W(t,x)$ is continuous on $[0,T]$, and the proof of the initial claim is complete.

Introduce the continuation set $C = \{\,(t,x) \in [0,T] \times [1,\infty) : V(t,x) > G(t,x)\,\}$ and the stopping set $D = \{\,(t,x) \in [0,T] \times [1,\infty) : V(t,x) = G(t,x)\,\}$. Since V and G are continuous, we see that C is open and D is closed in $[0,T] \times [1,\infty)$. Standard arguments based on the strong Markov property (see [58]) show that the first hitting time $\tau_D = \inf \{ \, 0 \le s \le T-t : (t{+}s, X_s^x) \in D \, \}$ is optimal in (2.13).

2. We show that the continuation set C just defined is given by (2.20) for some function $b : [0,T) \to (1,\infty)$. It follows in particular that the stopping set coincides with the set D in (2.21) as claimed. To verify the initial claim, note that by Ito's formula and (2.9) we have

$$X_s^x = x - r \int_0^s X_u^x \, du + \int_0^s \frac{dM_u}{S_u} + N_s \tag{2.31}$$

where $N_s = \sigma \int_0^s X_u^x \, d\widetilde{B}_u$ is a martingale for $0 \le s \le T$. We will first show that $(t,x) \in C$ implies that $(t,y) \in C$ for $x > y \ge 1$ be given and fixed. For this, let $\tau_* = \tau_*(t,x)$ denote the optimal stopping time for $W(t,x)$. Using (2.31) and the optimal sampling theorem, we find

$$\begin{aligned}
W(t,y) - y + \widetilde{K}\mathrm{e}^{-rt} &\ge \widetilde{\mathsf{E}}\,[X_{\tau_*}^y - \widetilde{K}\mathrm{e}^{-r(t+\tau_*)}] - y + \widetilde{K}\mathrm{e}^{-rt} \\
&= -r\widetilde{\mathsf{E}} \int_0^{\tau_*} X_u^y \, du + \widetilde{\mathsf{E}} \int_0^{\tau_*} \frac{dM_u}{S_u} - \widetilde{K}\widetilde{\mathsf{E}}\mathrm{e}^{-r(t+\tau_*)} + \widetilde{K}\mathrm{e}^{-rt} \\
&\ge -r\widetilde{\mathsf{E}} \int_0^{\tau_*} X_u^x \, du + \widetilde{\mathsf{E}} \int_0^{\tau_*} \frac{dM_u}{S_u} - \widetilde{K}\widetilde{\mathsf{E}}\mathrm{e}^{-r(t+\tau_*)} + \widetilde{K}\mathrm{e}^{-rt} \\
&= \widetilde{\mathsf{E}}\,[X_{\tau_*}^x - \widetilde{K}\mathrm{e}^{-r(t+\tau_*)}] - x + \widetilde{K}\mathrm{e}^{-rt} \\
&= W(t,x) - x + \widetilde{K}\mathrm{e}^{-rt} > 0
\end{aligned}$$

proving the claim. The fact just proved establishes the existence of a function $b : [0,T] \to [1,\infty]$ such that the continuation set C is given by (2.20) above.

To gain a deeper insight into the solution, let us apply Ito's formula for G using (2.14),(2.17) and that $G_t + \mathbb{L}_X G = -rG$:

$$G(t{+}s, X_s^x) = G(t,x) - r \int_0^s G(t{+}u, X_u^x) \, du + N_s + \int_0^s dR_u. \tag{2.32}$$

Thus the optional sampling theorem yield

$$\widetilde{\mathsf{E}}\, G(t{+}\tau, X_\tau^x) = G(t,x) - r\, \widetilde{\mathsf{E}} \int_0^\tau G(t{+}u, X_u^x) \, du + \widetilde{\mathsf{E}} \int_0^\tau dR_u \tag{2.33}$$

for all stopping times τ of X with values in $[0, T-t]$ with $t \in [0,T)$ and $x \ge 1$ given and fixed.

It can be seen from (2.33) and the structure of G that no point (t,x) in $[0,T) \times [1,\infty)$ with $x < \big(\widetilde{K}\mathrm{e}^{-rt} \vee 1\big)$ is a stopping point (for this one can make use of the first exit time from a sufficiently small time-space ball centred at the point). Likewise, it is also clear and can be verified that if $x > \big(\widetilde{K}e^{-rt} \vee 1\big)$ and $t < T$ is sufficiently close to T then it is optimal to stop immediately (since the gain obtained from being below G cannot offset the cost of getting there due to the lack of time). This shows that the optimal stopping boundary b satisfies $b(T) = \big(\widetilde{K}e^{-rT} \vee 1\big)$. It is also clear and can be verified that if the initial point $x \geq 1$ of the process X is sufficiently large then it is optimal to stop immediately (since the gain obtained from being below G cannot offset the cost of getting there due to the shortage of time). This shows that the optimal stopping boundary b is finite valued.

In [54] it was shown that the optimal stopping boundary $b(t;0) > 1$ for $t \in [0,T)$ when $\widetilde{K} = 0$. It is easily seen that $b(t;\widetilde{K}) \geq b(t;0)$ for $t \in [0,T)$ and $\widetilde{K} > 0$. Thus we have that $b(t) > 1$ for every $t \in [0,T)$.

3. We show that the smooth-fit condition (2.16) holds. For this, let $t \in [0,T)$ be given and fixed and set $x = b(t)$. We know that $x > 1$ so that there exists $\varepsilon > 0$ such that $x - \varepsilon > 1$ too. Since $W(t,x) = G(t,x)$ and $W(t,x-\varepsilon) > G(t,x-\varepsilon)$, we have:

$$\frac{W(t,x) - W(t,x-\varepsilon)}{\varepsilon} \leq \frac{G(t,x) - G(t,x-\varepsilon)}{\varepsilon} \leq 1 \tag{2.34}$$

Then, let $\tau_\varepsilon = \tau_\varepsilon^*(t,x-\varepsilon)$ denote the optimal stopping time for $W(t,x-\varepsilon)$. We have

$$\begin{aligned}
&\frac{W(t,x) - W(t,x-\varepsilon)}{\varepsilon} \\
&\quad \geq \frac{1}{\varepsilon}\,\widetilde{\mathsf{E}}\Big(\frac{(x-M_{\tau_\varepsilon})^+ + M_{\tau_\varepsilon}}{S_{\tau_\varepsilon}} - \frac{(x-\varepsilon-M_{\tau_\varepsilon})^+ + M_{\tau_\varepsilon}}{S_{\tau_\varepsilon}}\Big) \\
&\quad = \frac{1}{\varepsilon}\,\widetilde{\mathsf{E}}\Big(\frac{(x-M_{\tau_\varepsilon})^+}{S_{\tau_\varepsilon}} - \frac{(x-\varepsilon-M_{\tau_\varepsilon})^+}{S_{\tau_\varepsilon}}\Big) \\
&\quad \geq \frac{1}{\varepsilon}\,\widetilde{\mathsf{E}}\Big(\frac{1}{S_{\tau_\varepsilon}}((x-M_{\tau_\varepsilon})^+ - (x-\varepsilon-M_{\tau_\varepsilon})^+)I(M_{\tau_\varepsilon} \leq x-\varepsilon)\Big) \\
&\quad = \widetilde{\mathsf{E}}\Big(\frac{1}{S_{\tau_\varepsilon}}I(M_{\tau_\varepsilon} \leq x-\varepsilon)\Big) \to 1
\end{aligned} \tag{2.35}$$

as $\varepsilon \downarrow 0$ by bounded convergence, since $\tau_\varepsilon \to 0$ so that $M_{\tau_\varepsilon} \to 1$ with $1 < x-\varepsilon$ and likewise $S_{\tau_\varepsilon} \to 1$. It thus follows from (2.28),(2.34) and (2.35) that $W_x^-(t,x) \geq 1$ and $W_x^-(t,x) \leq 1$. Thus we have that $W_x^-(t,x) = 1$. Since $W(t,y) = G(t,y)$ for $y > x$, it is clear that $W_x^+(t,x) = 1$. We may thus conclude that $y \mapsto W(t,y)$ is C^1 at $b(t)$ and

$W_x(t, b(t)) = 1$ as stated in (2.16).

4. We show that inequality is satisfied:

$$W_t(t, x) \leq G_t(t, x) \tag{2.36}$$

for all $0 < t < T$ and $x \geq 1$.

To prove (2.36) fix $0 < t < t+h < T$ and $x \geq 1$. Let τ be the optimal stopping time for $W(t+h, x)$. Since $\tau \in [0, T-t-h] \subseteq [0, T-t]$ we see that $W(t,x) \geq \widetilde{\mathsf{E}}(X_\tau^x - \widetilde{K}\mathrm{e}^{-r(t+\tau)})$ and we get:

$$\begin{aligned} &W(t+h, x) - W(t,x) - (G(t+h,x) - G(t,x)) \\ &\leq \widetilde{\mathsf{E}}\,(-\widetilde{K}e^{-r(t+h+\tau)} + \widetilde{K}e^{-r(t+\tau)}) + \widetilde{K}e^{-r(t+h)} - \widetilde{K}e^{-rt} \\ &= \widetilde{K}e^{-rt}\,\widetilde{\mathsf{E}}\,(-e^{-r(h+\tau)} + e^{-r\tau} + e^{-rh} - 1) \\ &= \widetilde{K}e^{-rt}\,\widetilde{\mathsf{E}}\,(e^{-rh}(-e^{-r\tau} + 1) + e^{-r\tau} - 1) \\ &= \widetilde{K}e^{-rt}\,\widetilde{\mathsf{E}}\,(e^{-rh} - 1)(-e^{-r\tau} + 1) \leq 0. \end{aligned} \tag{2.37}$$

Dividing initial expression in (2.37) by h and letting $h \downarrow 0$ we obtain (2.36) for all (t, x).

5. We show that b is decreasing on $[0, T]$. This is an immediate consequence of (2.37). Indeed, if (t_2, x) belongs to C and t_1 from $(0, T)$ satisfies $t_1 < t_2$, then by (2.37) we have that $V(t_1, x) - G(t_1, x) \geq V(t_2, x) - G(t_2, x) > 0$ so that (t_1, x) must belong to C. It follows that b is decreasing thus proving the claim.

6. We show that b is continuous. Note that the same proof also shows that $b(T-) = \big(\widetilde{K}\mathrm{e}^{-rT} \vee 1\big)$ as already established above.

Since the stopping set equals $D = \{\, (t,x) \in [0,T) \times [1, \infty) : V(t,x) = G(t,x) \,\}$ and b is decreasing, it is easily seen that b is right-continuous on $[0, T)$.

Then note that since the supremum in (2.13) is attained at the first exit time τ_b from the open set C, standard arguments based on the strong Markov property (cf. [58]) imply that W is $C^{1,2}$ on C and satisfies (2.14). Suppose that there exists $t \in (0, T)$ such that $b(t-) > b(t)$ and fix any $x \in [b(t), b(t-))$. Note that by (2.16) we have:

$$W(s, x) - x + \widetilde{K}\mathrm{e}^{-rs} = \int_x^{b(s)} \int_y^{b(s)} W_{xx}(s, z)\, dz\, dy \tag{2.38}$$

for each $s < t$. If $W_{xx} \geq c$ on $P = \{\, (u,y) \in C \,|\, s \leq u < t$ and $x \leq y < b(u) \,\}$ for some $c > 0$ (for all $s < t$ close enough to t and some $x < b(t-)$ close enough to $b(t-)$) then by letting $s \uparrow t$ in (2.38) we get:

$$W(t,x) - x + \widetilde{K}\mathrm{e}^{-rt} \geq c\frac{(b(t-)-x)^2}{2} > 0 \tag{2.39}$$

contradicting the fact that (t,x) belongs to D and thus is an optimal stopping point. Hence the proof reduces to showing that $W_{xx} \geq c$ on small enough P for some $c > 0$.

To derive the latter fact we may first note from (2.14) upon using (2.36) that $\frac{\sigma^2}{2}x^2 W_{xx} = -W_t + rxW_x \geq -r\widetilde{K}\mathrm{e}^{-rt} + rxW_x$ in C. Suppose now that for each $\delta > 0$ there is $s < t$ close enough to t and there is $x < b(t-)$ close enough to $b(t-)$ such that $W_x(u,y) \geq 1-\delta$ for all $(u,y) \in P$ (where we recall that $1 = G_x(u,y)$ for all $(u,y) \in P$). Then from the previous inequality we find that $W_{xx}(u,y) \geq (2/(\sigma^2 y^2))(r(y - \widetilde{K}\mathrm{e}^{-ru}) - ry\delta) \geq c > 0$ for $\delta > 0$ small enough since $y > \widetilde{K}\mathrm{e}^{-ru}$ and for all $(u,y) \in P$. Hence the proof reduces to showing that $W_x(u,y) \geq 1-\delta$ for all $(u,y) \in P$ with P small enough when $\delta > 0$ is given and fixed.

To derive latter inequality we can make use of the estimate (2.35) to conclude that

$$\frac{W(u,y) - W(u,y-\varepsilon)}{\varepsilon} \geq \widetilde{\mathsf{E}}\Big(\frac{1}{S_{\sigma_\varepsilon}} I(M_{\sigma_\varepsilon} \leq x-\varepsilon)\Big) \tag{2.40}$$

where $\sigma_\varepsilon = \inf \{\ 0 \leq v \leq T-u : X_v^{y-\varepsilon} = b(u+v)\ \}$ and $M_t = \max_{0 \leq s \leq t} S_s$. Using the fact that b is decreasing and letting $\varepsilon \to 0$ in (2.40) we get

$$W_x(u,y) \geq \widetilde{\mathsf{E}}\Big(\frac{1}{S_\sigma} I(M_\sigma \leq x)\Big) \tag{2.41}$$

for all $(u,y) \in P$ where $\sigma = \inf \{\ 0 \leq v \leq T-s : X_v^x = b(s+v)\ \}$. Since by regularity of X we find that $\sigma \downarrow 0 \quad \widetilde{\mathsf{P}}$−a.s. as $s \uparrow t$ and $x \downarrow b(t-)$, it follows from (2.41) that

$$W_x(u,y) \geq 1 - \delta \tag{2.42}$$

for all $s < t$ close enough to t and some $x > b(t-)$ close enough to $b(t-)$. This completes the proof of the claim.

7. We show that the normal reflection condition (2.17) holds. For this, note first that since $x \mapsto W(t,x)$ is increasing (and convex) on $[1,\infty)$ it follows that

$W_x(t,1+) \geq 0$ for all $t \in [0,T)$. Suppose that there exists $t \in [0,T)$ such that $W_x(t,1+) > 0$. Recalling that W is $C^{1,2}$ on C so that $t \mapsto W_x(t,1+)$ is continuous on $[0,T)$, we see that there exists $\delta > 0$ such that $W_x(s,1+) \geq \varepsilon > 0$ for all $s \in [t, t+\delta]$ with $t+\delta < T$. Setting $\tau_\delta = \tau_b \wedge \delta$ it follows by Ito's formula that

$$\widetilde{\mathsf{E}}\, W(t+\tau_\delta, X^1_{\tau_\delta}) = W(t,1) + \widetilde{\mathsf{E}} \int_0^{\tau_\delta} W_x(t+u, X^1_u)\, dR_u \tag{2.43}$$

using (2.14) and the optional sampling theorem since W_x is bounded.

Since $(W(t+(s\wedge\tau_b), X^1_{s\wedge\tau_b}))_{0\leq s\leq T-t}$ is a martingale under $\widetilde{\mathsf{P}}$, we find that the expression on the left-hand side in (2.43) equals the first term on the right-hand side, and thus

$$\widetilde{\mathsf{E}} \int_0^{\tau_\delta} W_x(t+u, X^1_u)\, dR_u = 0. \tag{2.44}$$

On the other hand, since $W_x(t+u, X^1_u)\ dR_u = W_x(t+u, 1+)\ dR_u$ by (2.10), and $W_x(t+u,1+) \geq \varepsilon > 0$ for all $u \in [0,\tau_\delta]$, we see that (2.44) implies that

$$\widetilde{\mathsf{E}} \int_0^{\tau_\delta} dR_u = 0. \tag{2.45}$$

By (2.9) and the optional sampling theorem we see that (2.45) is equivalent to

$$\widetilde{\mathsf{E}}\,(X^1_{\tau_\delta}) - 1 + r\,\widetilde{\mathsf{E}} \int_0^{\tau_\delta} X^1_u\, du = 0. \tag{2.46}$$

Since $X_s \geq 1$ for all $s \in [0,T]$ we see that (2.46) implies that $\tau_\delta = 0$ $\widetilde{\mathsf{P}}$-a.s. As clearly this is impossible, we see that $W_x(t,1+) = 0$ for all $t \in [0,T)$ as claimed (2.17).

8. It is clear that $W(t,x)$ is a function satisfying the following conditions:

$$W \text{ is } C^{1,2} \text{ on } C \cup D, \tag{2.47}$$

$$W_t + \mathbb{L}_X W \text{ is locally bounded}, \tag{2.48}$$

$$x \mapsto W(t,x) \text{ is convex}, \tag{2.49}$$

$$t \mapsto W_x(t, b(t)\pm) \text{ is continuous}. \tag{2.50}$$

It follows that we can use the change-of-variable formula [52] for $W(t+s, X^x_s)$:

$$W(t+s, X^x_s) = W(t,x) + \int_0^s (W_t + \mathbb{L}_X W)(t+u, X^x_u) I(X^x_u \neq b(t+u))\, du \tag{2.51}$$

$$
\begin{aligned}
&+M_s+\int_0^s W_x(t{+}u,X_u^x)I(X_u^x\neq b(t{+}u))\,dR_u\\
&+\frac{1}{2}\int_0^s (W_x(t{+}u,X_u^x{+})-W_x(t{+}u,X_u^x{-}))\\
&\qquad\times I(X_u^x=b(t{+}u))\,d\ell_u^b(X)\\
&=W(t,x)+\int_0^s (G_t{+}\mathbb{L}_X G)(t{+}u,X_u^x)I(X_u^x\geq b(t{+}u))\,du+M_s\\
&=W(t,x)-r\int_0^s G(t{+}u,X_u^x)I(X_u^x\geq b(t{+}u))\,du+M_s
\end{aligned}
$$

where $M_s=\int_0^s W_x(u,X_x^u)\sigma X_u^x\,d\widetilde{B}_u$ is a martingale, $\ell_u^b(X)$ is the local time of X at the curve b and we used that $G_t{+}\mathbb{L}_X G={-}rG$ and (2.10)+(2.14)−(2.17). Let us also note that the condition (2.49) can further be relaxed to the form where $W_{xx}=W_1+W_2$ on $C\cup D$ where W_1 is non-negative and W_2 is continuous on $[0,T)\times[1,\infty)$. This will be referred to below as the relaxed form of (2.47)-(2.50).

Inserting $s=T{-}t$ in (2.51), using that $W(T,x)=x{-}\widetilde{K}e^{-rT}$, the sample stopping theorem and taking $\widetilde{\mathsf{P}}$-expectation we get

$$
W(t,x)=\widetilde{\mathsf{E}}(X_{T-t}^x){-}\widetilde{K}\mathrm{e}^{-rT}{+}r\int_0^{T-t}\widetilde{\mathsf{E}}\,G(t{+}u,X_u^x)I(X_u^x\geq b(t{+}u))\,du. \tag{2.52}
$$

Inserting $x=b(t)$ in (2.52) and using (2.15) we have nonlinear integral equation for b with boundary condition $b(T)=\big(\widetilde{K}\mathrm{e}^{-rT}\vee 1\big)$:

$$
b(t)=\widetilde{K}(\mathrm{e}^{-rt}{-}\mathrm{e}^{-rT})+F(T{-}t,b(t))+r\int_t^T H(t,u,b(t),b(u))\,du. \tag{2.53}
$$

Thus we have proved (2.25) and (2.26) as claimed. It remains now to show that equation (2.53) has the unique solution in the class of continuous decreasing functions satisfying $b(t)>(\widetilde{K}e^{-rt}\vee 1)$ for $0<t<T$.

In order to prove the uniqueness we will follow the approach which originally was devised by Peskir for the American put option [53] and then applied to the Russian option [54].

9. We show that b is the unique solution of the equation (2.53) in the class of continuous decreasing functions $c:[0,T]\to I\!R$ satisfying $c(t)>\big(\widetilde{K}\mathrm{e}^{-rt}\vee 1\big)$ for $0\leq t<T$. Let us assume that a function c belonging to the class described above solves (2.53), and let us show that this c must then coincide with the optimal stopping boundary b.

For this, in view of (2.52), let us introduce the function

$$U^c(t,x) = \widetilde{\mathsf{E}}(X^x_{T-t}) - \widetilde{K}\mathrm{e}^{-rT} + r\int_0^{T-t} \widetilde{\mathsf{E}}\, G(t{+}u, X^x_u) I(X^x_u \geq c(t{+}u))\, du \quad (2.54)$$

for $(t,x) \in [0,T] \times [1,\infty)$. A direct inspection of the expression in (2.54) using (2.23) and (2.24) shows that U^c_x is continuous on $[0,T] \times [1,\infty)$.

10. Let us define a function $W^c : [0,T]\times[1,\infty) \to I\!R$ by setting $W^c(t,x) = U^c(t,x)$ for $x < c(t)$ and $W^c(t,x) = G(t,x)$ for $x \geq c(t)$ when $0 \leq t < T$. Note that since c solves (2.53) we have that W^c is continuous on $[0,T]\times[1,\infty)$, i.e $W^c(t,x) = U^c(t,x) = G(t,x)$ for $x = c(t)$ when $0 \leq t < T$. Let C and D be defined by means of c as in (2.20) and (2.21) respectively.

Standard arguments based on the Markov property (or a direct verification) show that W^c i.e. U^c is $C^{1,2}$ on C and that

$$W^c_t + \mathbb{L}_X W^c = 0 \qquad \text{in } C, \quad (2.55)$$

$$W^c_x(t, 1+) = 0 \quad (2.56)$$

for all $t \in [0,T)$. Moreover, since U^c_x is continuous on $[0,T) \times [1,\infty)$ we see that W^c_x is continuous on $\overline{C}$. Finally, it is obvious that W^c i.e. G is $C^{1,2}$ on D.

11. Summarizing the preceding conclusions one can easily verify that the function W^c satisfies (2.47)-(2.50) (in the relaxed form) so that the change-of-variable formula [52] can be applied. Using (2.56) we get

$$\begin{aligned} W^c(t{+}s, X^x_s) = W^c(t,x) &+ \int_0^s (W^c_t{+}\mathbb{L}_X W^c)(t{+}u, X^x_u) I(X^x_u \neq c(t{+}u))\, du \quad (2.57) \\ &+ M^c_s + \frac{1}{2}\int_0^s (W^c_x(t{+}u, X^x_u{+}) - W^c_x(t{+}u, X^x_u{-})) \\ &\qquad \times I(X^x_u = c(t{+}u))\, d\ell^c_u(X) \end{aligned}$$

where M^c is a martingale under $\widetilde{\mathsf{P}}$.

12. If we know that

$$U^c(t,x) = G(t,x) \text{ for all } x \geq c(t) \quad (2.58)$$

holds using the general fact

$$\frac{\partial}{\partial x}(U^c(t,x) - G(t,x))\big|_{x=c(t)} = W^c_x(t, c(t)-) - W^c_x(t, c(t)+) \quad (2.59)$$

for all $0 \le t < T$ we see that

$$x \mapsto W^c(t,x) \text{ is } C^1 \text{ at } c(t) \text{ for each } 0 \le t < T \tag{2.60}$$

holds too (since U^c_x is continuous).

13. To derive (2.58) first note that standard arguments based on the Markov property (or a direct verification) show that U^c is $C^{1,2}$ on D and that

$$U^c_t + \mathbb{L}_X U^c = -rG \quad \text{in } D. \tag{2.61}$$

Since the function U^c is continuous and satisfies (2.47)-(2.50) (in the relaxed form), we see that (2.51) can be applied just like in (2.57) with U^c instead of W^c, and this yields

$$U^c(t+s, X^x_s) = U^c(t,x) - r \int_0^s G(t+u, X^x_u) I(X^x_u \ge c(t+u))\, du + L^c_s \tag{2.62}$$

upon using (2.55)-(2.56), (2.61) and fact that U^c_x is continuous. L^c is a martingale under $\widetilde{\mathsf{P}}$.

Next note that Ito's formula implies

$$G(t+s, X^x_s) = G(t,x) - r \int_0^s G(t+u, X^x_u)\, du + M_s + \int_0^s dR_u \tag{2.63}$$

upon using that $G_t + \mathbb{L}_X G = -r\, G$ as well as that $G_x = 1$. M is a martingale under $\widetilde{\mathsf{P}}$.

For $x \ge c(t)$ consider the stopping time

$$\sigma_c = \inf \{\ 0 \le s \le T-t : X^x_s \le c(t+s)\ \}. \tag{2.64}$$

Then using that $U^c(t, c(t)) = G(t, c(t))$ for all $0 \le t < T$ since c solves (2.53), and that$U^c(T,x) = G(T,x)$ for all $x \ge 1$ by (2.54), we see that $U^c(t+\sigma_c, X^x_{\sigma_c}) = G(t+\sigma_c, X^x_{\sigma_c})$. Hence from (2.62) and (2.63) using the optional sampling theorem

$$\begin{aligned} U^c(t,x) &= \widetilde{\mathsf{E}}\, U^c(t+\sigma_c, X^x_{\sigma_c}) + r \int_0^{\sigma_c} \widetilde{\mathsf{E}}\, G(t+u, X^x_u) I(X^x_u \ge c(t+u))\, du \qquad (2.65)\\ &= \widetilde{\mathsf{E}}\, G(t+\sigma_c, X^x_{\sigma_c}) + r \int_0^{\sigma_c} \widetilde{\mathsf{E}}\, G(t+u, X^x_u) I(X^x_u \ge c(t+u))\, du \\ &= G(t,x) - r \int_0^{\sigma_c} \widetilde{\mathsf{E}}\, G(t+u, X^x_u)\, du \end{aligned}$$

$$
\begin{aligned}
&+r\int_0^{\sigma_c} \widetilde{\mathsf{E}}\, G(t{+}u, X_u^x) I(X_u^x \geq c(t{+}u))\, du \\
&= G(t,x)
\end{aligned}
$$

since $X_u^x \geq c(t{+}u) > 1$ for all $0 \leq u \leq \sigma_c$. This establishes (2.58) and thus (2.60) holds too.

14. Consider the stopping time

$$\tau_c = \inf \{\ 0 \leq s \leq T{-}t : X_s^x \geq c(t{+}s)\ \}. \tag{2.66}$$

Note that (2.57) using (2.55) and (2.60) reads

$$W^c(t{+}s, X_s^x) = W^c(t,x) - r\int_0^s G(t{+}u, X_u^x) I(X_u^x \geq c(t{+}u))\, du + M_s^c. \tag{2.67}$$

Using that M^c is a martingale under $\widetilde{\mathsf{P}}$ and inserting τ_c in place of s in (2.67), it follows upon taking the $\widetilde{\mathsf{P}}$-expectation that

$$W^c(t,x) = \widetilde{\mathsf{E}}\, G(t{+}\tau_c, X_{\tau_c}^x) \tag{2.68}$$

for $(t,x) \in [0,T] \times [1,\infty)$ where we use that $W^c(t,x) = G(t,x)$ for $x \geq c(t)$ or $t = T$. Comparing (2.68) and (2.13) we see that

$$W^c(t,x) \leq W(t,x) \tag{2.69}$$

for all $(t,x) \in [0,T] \times [1,\infty)$.

15. Let us now show that $b \geq c$ on $[0,T]$. For this, recall that by the same arguments as for W^c we also have

$$W(t{+}s, X_s^x) = W(t,x) - r\int_0^s G(t{+}u, X_u^x) I(X_u^x \geq b(t{+}u))\, du + M_s^b \tag{2.70}$$

where M^b is a martingale $\widetilde{\mathsf{P}}$. Fix $(t,x) \in [0,T) \times [1,\infty)$ such that $x > b(t) \vee c(t)$ and consider the stopping time

$$\sigma_b = \inf \{\ 0 \leq s \leq T - t : X_s^x \leq b(t{+}s)\ \}. \tag{2.71}$$

Inserting σ_b in place of s in (2.67) and (2.70) and taking the $\widetilde{\mathsf{P}}$-expectation, we get

$$\widetilde{\mathsf{E}}\, W^c(t{+}\sigma_b, X_{\sigma_b}^x) = x - \widetilde{K} e^{-rt} - r\, \widetilde{\mathsf{E}}\Big(\int_0^{\sigma_b} G(t{+}u, X_u^x) I(X_u^x \geq c(t{+}u))\, du \Big) \tag{2.72}$$

$$\widetilde{\mathsf{E}}\, W(t+\sigma_b, X^x_{\sigma_b}) = x - \widetilde{K} e^{-rt} - r\, \widetilde{\mathsf{E}}\Big(\int_0^{\sigma_b} G(t+u, X^x_u)\, du \Big). \tag{2.73}$$

Hence by (2.69) we see that

$$\widetilde{\mathsf{E}}\Big(\int_0^{\sigma_b} G(t+u, X^x_u) I(X^x_u \geq c(t+u))\, du \Big) \geq \widetilde{\mathsf{E}}\Big(\int_0^{\sigma_b} G(t+u, X^x_u)\, du \Big) \tag{2.74}$$

from where it follows by the continuity of b and c, using $G(t,x) > 0$ for $x > b(t)$, that $b(t) \geq c(t)$ for all $t \in [0,T]$.

16. Finally, let us show that c must be equal to b. For this, assume that there is $t \in (0,T)$ such that $b(t) > c(t)$, and pick $x \in (c(t), b(t))$. Under $\widetilde{\mathsf{P}}$ consider the stopping time τ_b from (2.22). Inserting τ_b in place of s in (2.67) and (2.70) and taking the $\widetilde{\mathsf{P}}$-expectation, we get

$$\widetilde{\mathsf{E}}\, G(t+\tau_b, X^x_{\tau_b}) = W^c(t,x) - r\, \widetilde{\mathsf{E}}\Big(\int_0^{\tau_b} G(t+u, X^x_u) I(X^x_u \geq c(t+u))\, du \Big) \tag{2.75}$$

$$\widetilde{\mathsf{E}}\, G(t+\tau_b, X^x_{\tau_b}) = W(t,x). \tag{2.76}$$

Hence by (2.69) we see that

$$\widetilde{\mathsf{E}}\Big(\int_0^{\tau_b} G(t+u, X^x_u) I(X^x_u \geq c(t+u))\, du \Big) \leq 0 \tag{2.77}$$

from where it follows by the continuity of c and b using $G(t,x) > 0$ for $x > c(t)$that such a point x cannot exist. Thus c must be equal to b, and the proof is complete. □

2.4. The arbitrage-free price and stopping region

This section derives a representation of the arbitrage-free price for the option and the integral equation for the rational exercise boundary. We make use of the results from Sections 2.2 and 2.3, as well as the relationship (2.11) between the value functions (2.3) and (2.26).

1. After determining W and the stopping boundary b in (2.13) we can now solve the initial problem (2.3). Indeed, from (2.11) and (2.52) we have

$$V(t,m,s) \quad = \quad s\, W\Big(t, \tfrac{m \vee K}{s}; \tfrac{K e^{rt}}{s}\Big) \tag{2.78}$$

$$\begin{aligned} = \;& \widetilde{\mathsf{E}}\Big(sX_{T-t}^{\frac{m\vee K}{s}}\Big) - K\mathrm{e}^{r(t-T)} \\ & + r\int_0^{T-t} \widetilde{\mathsf{E}}\Big(sX_u^{\frac{m\vee K}{s}} - K\mathrm{e}^{-ru}\Big) I\Big(X_u^{\frac{m\vee K}{s}} \geq b(t+u; K\mathrm{e}^{rt}/s)\Big)\, du. \end{aligned}$$

2. Using that $V(t,m,s) > 0$ for every $t \in [0,T)$ and $0 \leq s \leq m$ we have that

$$K < b(t; K\mathrm{e}^{rt}) \tag{2.79}$$

since otherwise $V(t,1,1) = W(t,K;K\mathrm{e}^{rt}) = K - K\mathrm{e}^{rt}\mathrm{e}^{-rt} = 0$.

Standard arguments based on the strong Markov property (see [58]) show that the stopping set in (2.3) is given by

$$\begin{aligned} D &= \{\ (t,m,s) : V(t,m,s) = (m-K)^+\} \qquad (2.80)\\ &= \{\ (t,m,s) : sW\Big(t, \tfrac{m\vee K}{s}; \tfrac{K\mathrm{e}^{rt}}{s}\Big) = (m-K)^+\ \} \\ &= \{\ (t,m,s) : W\Big(t, \tfrac{m\vee K}{s}; \tfrac{K\mathrm{e}^{rt}}{s}\Big) = \tfrac{m\vee K}{s} - \tfrac{K\mathrm{e}^{rt}}{s}\mathrm{e}^{-rt}\ \} \\ &= \{\ (t,m,s) : \tfrac{m\vee K}{s} \geq b\left(t; \tfrac{K\mathrm{e}^{rt}}{s}\right)\ \} \\ &= \{\ (t,m,s) : m \geq s\, b\left(t; \tfrac{K\mathrm{e}^{rt}}{s}\right)\ \} \end{aligned}$$

for $0 \leq t < T$ and $0 < s \leq m$ where in final equality we used (2.79). Thus the optimal stopping time in (2.3) is given by

$$\tau_g = \inf\ \{\ 0 \leq u < T-t : M_u \geq g(t+u, S_u)\ \} \tag{2.81}$$

where the function g is given by

$$g(t,s) = s\, b\Big(t; \tfrac{K\mathrm{e}^{rt}}{s}\Big). \tag{2.82}$$

3. We now show some properties of the optimal stopping boundary g. From (2.82) one can see that in order to compute $s \to g(t,s)$ for fixed $t \in [0,T)$ we need to calculate the optimal stopping boundary $b(\,\cdot\,; \frac{K\mathrm{e}^{rt}}{s})$ with a scaling strike as a solution to (2.53) for every $s > 0$. Using that the problem (2.3) time-homogeneous, in the sense that the gain function does not depend on time, it follows that $t \mapsto g(t,s)$ is decreasing on $[0,T)$ for each $s > 0$ fixed. It is also clear from (2.3) that $s \mapsto g(t,s)$ is increasing on $[0,\infty)$ for $t \in [0,T)$ fixed. Since $b(T-;\widetilde{K}) = \widetilde{K}e^{-rT} \vee 1$ we have $g(T-,s) = s(\frac{K}{s} \vee 1) = K \vee s$ for any $s > 0$. From (2.53) we see that $K \mapsto b(t;K)$

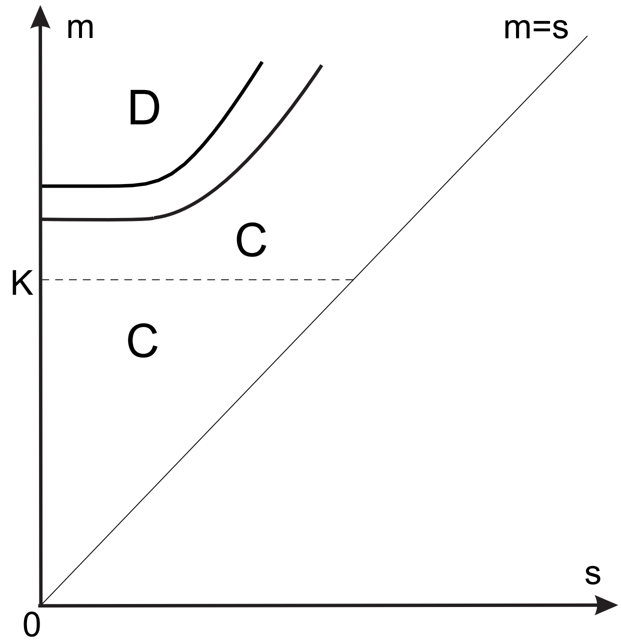

Figure 2.2: A computer drawing of the optimal stopping boundary $s \mapsto g(t,s)$ for 1) $t = 0$ (upper) and 2) $t = 0.3$ (lower) in the case $K = 1.2, T = 1, r = 0.1, \sigma = 0.4$. The limit of $g(t,\,\cdot\,)$ at zero is greater than K for every t.

is continuous on $(0,\infty)$ for each $t \in [0,T)$ fixed. Hence we have that $s \mapsto g(t,s)$ is continuous on $(0,\infty)$ for $t \in [0,T)$ fixed. From (2.82) it is easily to seen that $g(t,s) \uparrow s\, b_0(t)$ as $s \to \infty$ for fixed $t \in [0,T)$ where $b_0(t) = b(t;0)$ is the optimal stopping boundary for the Russian option [54]. The Figure 2.2 illustrates computer drawing of the optimal stopping boundaries $s \mapsto g(u,s)$ and $s \mapsto g(v,s)$ for $0 < u < v < T$. The boundaries shift to the right as time goes to T and the optimal stopping time is the first hitting time of M over $g(\,\cdot, S)$.

2.5. Conclusion

In this section we make a conclusion of the results obtained in this paper and propose the future research program.

The main idea of the paper is to show a reduction of the three-dimensional optimal stopping problem (2.3) to the two-dimensional optimal stopping problem (2.13). However, after solving the two-dimensional problem and coming back to the initial problem we vary parameter in (2.13), and thus the problem remains inherently three-dimensional. We determined expressions for the arbitrage-free price (2.78) and the rational exercise boundary (2.82).

As remarked in the introduction, the reduction of the three-dimensional problem to a two-dimensional one with a scaling strike can be used to tackle the optimal stopping problem for the British lookback option with fixed strike. Performing the financial analysis of American and British versions of this option as in [60] leads to an extensive programme of research which we present in Chapter 3 below.

The method of scaling strike also allows to examine the problem (2.3) in exponential Lévy models and then after reducing dimensions and applying the local time-space calculus for general semimartingales (see [55]) the nonlinear integral equations for optimal stopping boundaries can be obtained.

Chapter 3

The British lookback option with fixed strike

3.1. Introduction

The aim of this paper is to examine the British payoff mechanism in the context of the lookback option with fixed strike and continue the research proposed in [23]. This mechanism provides its holder with a protection against unfavourable scenarios for stock prices and is intrinsically built into the option contract using the concept of optimal prediction (see e.g. [18]) and we refer to such contracts as 'British' for the reasons outlined in [59] and [60], where the British put and call options were introduced. The main idea of the 'British' feature is to substitute the true drift by a contract drift in the Black-Scholes model and then its payoff is the 'best prediction' of the European lookback payoff. The most interesting feature of this protection mechanism as not only is the option buyer offered a protection against unfavourable stock movements but also when the price movements are favourable he will generally receive high returns (see [59] and [60] for details).

Following the rationale of the British put and call options, this type of options was extended to the path-dependant options in [22] and [23]. Particularly, the British Russian option was introduced and studied in [23]. Herein we use terminology the *Russian* option for the lookback option with zero strike which was identified in [62] (see last paragraph of introduction in [23] for detailed explanation of this terminology and its history). According to the financial theory (see e.g. [66]) the arbitrage-free price of the option is a solution to an optimal stopping problem with the gain function as the payoff of the option. The corresponding optimal stopping problem for the British Russian option [23] was originally a three-dimensional (time-process-running maximum) and was reduced fully to two dimensions using Girsanov theorem as in [63] and [54]. It was shown that exercising of the British version provides very attractive returns compared with exercising of the American option and selling the European option (the latter can be considered only in a liquid market). The final section in [23] proposed to research different types of lookback options: (i) calls and puts; (ii) those with fixed (non-zero) or floating strike; (iii) those based on the maximum or minimum; (iv) the weighting in the maximum or minimum may be equal or flexible.

In this paper we study the British lookback option with fixed strike (non-zero) of call type as we believe that this is the most interesting case from a mathematical point of view. The paper includes two parts: analytical solution and financial analysis. The theoretical solution is based on the method of a scaling strike. It was remarked above that the optimal stopping problems for the lookback option are three-dimensional and in the case of non-zero strike no *full* reduction to dimension two appears to be possible. However we will illustrate the reduction to two-dimensional problem with a scaling strike which originally was used in [33] for the American lookback option with fixed strike. This approach simplifies the discussion and expressions for the arbitrage-free price and it allows to decrease a dimension of the integral equation for a rational exercise boundary. Using a local time-space calculus on curves [52] we derive a closed form expression for the arbitrage-free price in terms of the optimal stopping boundary of the two-dimensional optimal stopping problem and show that the rational exercise boundary of the option can be characterised via the unique solution to a nonlinear integral equation. We also show the remarkable numerical example where the rational exercise boundary exhibits a *discontinuity* with respect to space variable, hence it was not possible to apply a change-of-variable formula with local time on surfaces in order to solve the three-dimensional stopping problem directly. This is another advantage of reduction by the method of a scaling strike. The solution of the zero-strike case $K = 0$ (the British Russian option) is fully embedded into the present problem and can be considered as a particular case.

We perform the analysis of returns of the British lookback option with fixed strike in comparison with its the American and the European counterparts. After observing the returns upon exercising or selling options we conclude the remarkable features of the British lookback option: (i) the option provides an effective protection against unfavourable stock movements unlike the American version which gives zero returns in this case; (ii) the British option holder receives very high returns also when stock movements are favourable (much better than the American option and comparable with upon selling the European option); (iii) the holder enjoys two features above in both liquid and illiquid markets. The latter fact is very fruitful, since lookback options are usually traded in illiquid markets and their selling can be very problematic. We believe that these properties of the British lookback option with fixed strike make it a very attractive financial instrument.

The paper is organised as follows. In Section 3.2 we present a basic motivation for the British lookback option with fixed strike. In Section 3.3 we formally define the British lookback option with fixed strike and show some of its basic properties. Then in Section 3.4 where we derive a closed form expression for the arbitrage-free price in terms of the optimal stopping boundary of the two-dimensional problem and show that the rational exercise boundary of the option can be characterised via unique the unique solution to a nonlinear integral equation. Using these results in Section 3.5 we present a financial analysis of the British lookback option with fixed strike (making comparisons with the European/American lookback options).

3.2. Basic motivation for the British lookback option with fixed strike

The basic economic motivation for the British lookback option with fixed strike is parallel to that of the British put, call, Asian and Russian options (see [59], [60],

[22] and [23]). In this section we briefly review key elements of this motivation. We remark that the full financial scope of the British lookback option with fixed strike goes beyond these initial considerations (see Section 3.5 below for further details).

1. Consider the financial market consisting of a risky stock S and a riskless bond B whose prices respectively evolve as

$$dS_t = \mu S_t \, dt + \sigma S_t \, dW_t \quad (S_0 = s) \tag{3.1}$$

$$dB_t = r B_t \, dt \quad (B_0 = 1) \tag{3.2}$$

where $\mu \in I\!R$ is the stock drift, $\sigma > 0$ is the volatility coefficient, $W = (W_t)_{t \geq 0}$ is a standard Wiener process defined on a probability space $(\Omega, \mathcal{F}, \mathsf{P})$, and $r > 0$ is the interest rate. Recall that the lookback option with fixed strike of European type is a financial contract between a seller/hedger and a buyer/holder entitling the latter to exercise the option at a specified maturity time $T > 0$ and receive the payoff

$$\Big(M_T - K \Big)^+ = \Big(\max_{0 \leq t \leq T} S_t - K \Big)^+ \tag{3.3}$$

from the seller. Standard hedging arguments based on self-financing portfolios imply that the arbitrage-free price of the option is given by

$$V = \widetilde{\mathsf{E}} \, [e^{-rT} (M_T - K)^+] \tag{3.4}$$

where the expectation $\widetilde{\mathsf{E}}$ is taken with respect to the (unique) equivalent martingale measure $\widetilde{\mathsf{P}}$ (see e.g. [66]). In this section (as in [59], [60] and [23]) we will analyse the option from the standpoint of a true buyer. By 'true buyer' we mean a buyer who has no ability or desire to sell the option nor to hedge his own position. Thus every true buyer will exercise the option at time T in accordance with the rational performance. For more details on the motivation and interest for considering a true buyer in this context we refer to [59].

2. With this in mind we now return to the holder of the lookback option whose payoff is given by (3.3) above. Recall that the unique strong solution to (3.1) is given by

$$S_t = S_t(\mu) = s \exp \Big(\sigma W_t + (\mu - \tfrac{\sigma^2}{2}) t \Big) \tag{3.5}$$

under P for $t \in [0,T]$ where $\mu \in I\!R$ is the actual drift. Inserting (3.5) into (3.3) we find that the expected value of the buyer's payoff equals

$$P = P(\mu) = \mathsf{E}\,[e^{-rT}(M_T(\mu) - K)^+]. \tag{3.6}$$

Moreover, it is well known that $\text{Law}(S(\mu)\,|\,\widetilde{\mathsf{P}})$ is the same as $\text{Law}(S(r)\,|\,\mathsf{P})$ so that the arbitrage-free price of the option equals

$$V = P(r) = \mathsf{E}\,[e^{-rT}(M_T(r) - K)^+]. \tag{3.7}$$

A direct comparison of (3.6) and (3.7) shows that if $\mu = r$ then the return is 'fair' for the buyer, in the sense that $V = P$, where V represents the value of his investment and P represents the expected value of his payoff. On the other hand, if $\mu > r$ then the return is 'favourable' for the buyer, in the sense that $V < P$, and if $\mu < r$ then the return is 'unfavourable' for the buyer, in the sense that $V > P$ with the same interpretations as above. Exactly the same analysis can be performed for the lookback option of American type and as the conclusions are the same we omit the details. We recall that the actual drift μ is unknown at time $t = 0$ and also difficult to estimate at later times $t \in (0,T]$ unless T is unrealistically large.

3. The brief analysis above shows that whilst the actual drift μ of the underlying stock price is irrelevant in determining the arbitrage-free price of the option, to a (true) buyer it is crucial, and he will buy the option if he believes that $\mu > r$. If this appears to be a true then on average he will make a profit. Thus, after purchasing the option, the holder will be happy if the observed stock price movements confirm his belief that $\mu > r$.

The British lookback option with fixed strike seeks to address the opposite scenario: What if the option holder observes stock price movements which change his belief regarding the actual drift and cause him to believe that $\mu < r$ instead? In this contingency the British lookback holder is effectively able to substitute this unfavourable drift with a contract drift and minimise his losses. In this way he is endogenously protected from any stock price drift smaller than the contract drift. The value of the contract drift is therefore selected to represent the buyer's level of tolerance for the deviation of the actual drift from his original belief.

It will be shown below (similarly to [59], [60] and [23]) that the practical implications of this protection feature are most remarkable as not only is the option holder afforded an unique protection against unfavourable stock price movements (covering the ability to sell in a liquid option market completely endogenously) but also when the stock price movements are favourable he will generally receive high returns. We refer to the final paragraph of Section 2 in [60] for further comments regarding the option holder's ability to sell his contract (releasing the true buyer's perspective) and its connection with option market liquidity. This translates into the present setting, since lookback options are not so popularly traded as plain vanilla call and put options.

3.3. The British lookback option with fixed strike: Definition and basic properties

We begin this section by presenting a formal definition of the British lookback option with fixed strike. This is then followed by a brief analysis of the optimal stopping problem and the free-boundary problem characterising the arbitrage-free price and the rational exercise strategy. These considerations are continued in Sections 3.4 and 3.5 below.

1. Consider the financial market consisting of a risky stock S and a riskless bond B whose prices evolve as (3.1) and (3.2) respectively, where $\mu \in I\!R$ is the appreciation rate (drift), $\sigma > 0$ is the volatility coefficient, $K > 0$ is a fixed strike, $W = (W_t)_{t \geq 0}$ is a standard Wiener process defined on a probability space $(\Omega, \mathcal{F}, \mathsf{P})$, and $r > 0$ is the interest rate. Let a maturity time $T > 0$ be given and fixed, and let M_T denote the maximum stock price given by (3.3) above.

Definition 3.3.1. *The British lookback option with fixed strike is a financial contract between a seller/hedger and a buyer/holder entitling the latter to exercise at any (stopping) time τ prior to T whereupon his payoff (deliverable immediately) is the 'best prediction' of the European payoff $(M_T - K)^+$ given all the information up to* *time τ under the hypothesis that the true drift of the stock price equals μ_c.*

The quantity μ_c is defined in the option contract and we refer to it as the 'contract drift'. We will show below that the contract drift should satisfy the following inequality

$$\mathsf{E}^{\mu_c}\big[r\tfrac{K}{S_0} - \mu_c M_T\big] I(M_T > \tfrac{K}{S_0}) > 0 \tag{3.8}$$

where the expectation E^{μ_c} is taken under assumption that the drift in (3.1) equals to μ_c. If (3.8) does not hold, we are not able to guarantee that it is not optimal to exercise immediately, i.e. the buyer would not be overprotected (see Remark 3.3.2 below for details). Condition (3.8) gives for us relationship between μ_c, the interest rate r, and the strike K when $K > 0$, since in the case of $K = 0$ (Russian option) we have simply $\mu_c < 0$ as in [23]. We denote by $\mu_c^* = \mu_c^*(r, K)$ the unique solution (clearly it exists and $0 < \mu_c^* < r$) to equation

$$\mathsf{E}^{\mu_c}\big[r\tfrac{K}{S_0} - \mu_c M_T\big] I(M_T > \tfrac{K}{S_0}) = 0 \tag{3.9}$$

and hence the condition (3.8) is equivalent to $\mu_c < \mu_c^*$. Recall from Section 3.2 above that the value of the contract drift is selected to represent the buyer's level of tolerance for the deviation of the true drift μ from his original belief.

2. Denoting by $(\mathcal{F}_t)_{0 \le t \le T}$ the natural filtration generated by S (possibly augmented by null sets or in some other way of interest) the payoff of the British lookback option with fixed strike at a given stopping time τ with values in $[0, T]$ can be formally written as

$$\mathsf{E}^{\mu_c}[(M_T - K)^+ | \mathcal{F}_\tau] \tag{3.10}$$

where the conditional expectation is taken with respect to a new probability measure P^{μ_c} under which the stock price S evolves as

$$dS_t = \mu_c\, S_t\, dt + \sigma S_t\, dW_t \tag{3.11}$$

with $S_0 = s$ in $(0, \infty)$. Comparing (3.1) and (3.11) we see that the effect of exercising the British lookback option with fixed strike is to substitute the true (unknown) drift of the stock price with the contract drift for the remaining term of the contract.

3. Setting that $M_t = \max_{0 \le s \le t} S_s$ for $t \in [0,T]$ and using stationary and independent increments of W governing S we find that

$$\mathsf{E}^{\mu_c}\Big[(M_T - K)^+ | \mathcal{F}_t\Big] = S_t \, \mathsf{E}^{\mu_c}\Big[\big(\tfrac{M_t}{S_t} \vee \max_{t \le s \le T} \tfrac{S_s}{S_t} - \tfrac{K}{S_t}\big)^+ | \mathcal{F}_t\Big] \tag{3.12}$$

$$= S_t \, Z^{\mu_c}(t, M_t, S_t)$$

where the function Z^{μ_c} can be expressed as

$$Z^{\mu_c}(t,m,s) = \mathsf{E}^{\mu_c}\big(\tfrac{m}{s} \vee M_{T-t} - \tfrac{K}{s}\big)^+ \tag{3.13}$$

$$= \mathsf{E}^{\mu_c}\big(\tfrac{m \vee K}{s} \vee M_{T-t} - \tfrac{K}{s}\big) = G^{\mu_c}\big(t, \tfrac{m \vee K}{s}\big) - \tfrac{K}{s}$$

where $G^{\mu_c}(t,x) = \mathsf{E}^{\mu_c}(x \vee M_{T-t})$ for $t \in [0,T]$, $m \ge s > 0$, $x \in [1,\infty)$ and $M_0 = 1$. A lengthy calculation based on the known law of M_{T-t} under P^{μ_c} (see e.g. [66, Lemma 1, p. 759]) shows that

$$G^{\mu_c}(t,x) = x\, \Phi\Big(\tfrac{1}{\sigma\sqrt{T-t}}\Big[\log x - (\mu_c - \tfrac{\sigma^2}{2})(T-t)\Big]\Big) \tag{3.14}$$

$$- \tfrac{\sigma^2}{2\mu_c} x^{2\mu_c/\sigma^2} \Phi\Big(-\tfrac{1}{\sigma\sqrt{T-t}}\Big[\log x + (\mu_c - \tfrac{\sigma^2}{2})(T-t)\Big]\Big)$$

$$+ \Big(1 + \tfrac{\sigma^2}{2\mu_c}\Big) e^{\mu_c(T-t)} \Phi\Big(-\tfrac{1}{\sigma\sqrt{T-t}}\Big[\log x - (\mu_c + \tfrac{\sigma^2}{2})(T-t)\Big]\Big)$$

for $t \in [0,T]$ and $x \in [1,\infty)$, where Φ is the standard normal distribution function given by $\Phi(x) = (1/\sqrt{2\pi}) \int_{-\infty}^{x} e^{-y^2/2} dy$ for $x \in I\!R$. The function G^{μ_c} appeared in [23] and its properties were studied there. We will recall and make use of them below.

Standard hedging arguments based on self-financing portfolios (with consumption) imply that the arbitrage-free price of the British lookback option with fixed strike is given by

$$V = \sup_{0 \le \tau \le T} \widetilde{\mathsf{E}}\left[e^{-r\tau} \mathsf{E}^{\mu_c}\big((M_T - K)^+ | \mathcal{F}_\tau\big) \right] \tag{3.15}$$

where the supremum is taken over all stopping times τ of S with values in $[0,T]$ and $\widetilde{\mathsf{E}}$ is taken with respect to the (unique) equivalent martingale measure $\widetilde{\mathsf{P}}$. From (3.12) we see that the underlying Markov process in the optimal stopping problem (3.15) equals $(t, S_t, M_t)_{0 \le t \le T}$ for $t \in [0,T]$ thus making it three-dimensional.

4. Since $\mathrm{Law}(S(\mu)\,|\,\widetilde{\mathsf{P}})$ is the same as $\mathrm{Law}(S(r)\,|\,\mathsf{P})$, it follows from the well-known ladder structure of M and multiplicative structure of S that (3.15) extends as follows

$$V(t,m,s) = \sup_{0\le\tau\le T-t} \mathsf{E}\, e^{-r\tau} sS_\tau \Big[G^{\mu_c}\Big(t{+}\tau, \tfrac{K\vee m\vee \max_{0\le u\le\tau} sS_u}{sS_\tau}\Big) - \tfrac{K}{sS_\tau}\Big] \quad (3.16)$$

for $t\in[0,T]$ and $m\ge s$ in $(0,\infty)$ where the supremum is taken as in (3.15) above and the process $S = S(r)$ under P solves

$$dS_t = rS_t\,dt + \sigma S_t\,dW_t \quad (3.17)$$

with $S_0 = 1$. By the Girsanov theorem it follows that

$$\begin{aligned}
&\mathsf{E}\, e^{-r\tau} sS_\tau \Big[G^{\mu_c}\Big(t{+}\tau, \tfrac{K\vee m\vee \max_{0\le u\le\tau} sS_u}{sS_\tau}\Big) - \tfrac{K}{sS_\tau}\Big] \\
&= s\, \widehat{\mathsf{E}}\Big[G^{\mu_c}\Big(t{+}\tau, \tfrac{\frac{m\vee K}{s}\vee M_\tau}{S_\tau}\Big) - \tfrac{K}{sS_\tau}\Big] \\
&= s\, \widehat{\mathsf{E}}\Big[G^{\mu_c}\Big(t{+}\tau, X_\tau^x\Big) - \tfrac{K}{s}\, e^{-r\tau}\Big] \\
&= s\, \widehat{\mathsf{E}}\Big[G^{\mu_c}\Big(t{+}\tau, X_\tau^x\Big) - \tfrac{Ke^{rt}}{s}\, e^{-r(t+\tau)}\Big]
\end{aligned} \quad (3.18)$$

for every stopping time τ of S and where we used fact that $\widehat{\mathsf{E}}S_\tau^{-1} = \widehat{\mathsf{E}}e^{-r\tau}$ and we set

$$X_t^x = \frac{x\vee M_t}{S_t} \quad (3.19)$$

with $x = (m\vee K)/s$ and $\widehat{\mathsf{P}}$ is defined by $d\widehat{\mathsf{P}} = \exp\left(\sigma W_T - (\sigma^2/2)T\right)d\mathsf{P}$ so that $\widehat{W}_t = W_t - \sigma t$ is a standard Wiener process under $\widehat{\mathsf{P}}$ for $t\in[0,T]$. By Ito's formula one finds that

$$dX_t = -rX_t\,dt + \sigma X_t\,d\widehat{W}_t + dR_t \quad (3.20)$$

with $X_0 = x$ under $\widehat{\mathsf{P}}$ and we set

$$R_t = \int_0^t I(X_s = 1)\,\tfrac{dM_s}{S_s} \quad (3.21)$$

for $t\in[0,T]$ and $x\in[1,\infty)$. Note that the state space of the Markov process X equals $[1,\infty)$ where 1 is an instantaneously reflecting boundary point. Thus (3.18) motivates us to consider the following optimal stopping problem

$$\widetilde{V}(t,x) = \widetilde{V}(t,x;\widetilde{K}) = \sup_{0\le\tau\le T-t} \mathsf{E}\left[G^{\mu_c}(t{+}\tau, X_\tau^x) - \widetilde{K}e^{-r(t+\tau)}\right] \quad (3.22)$$

for $t \in [0,T]$ and $x \in [1,\infty)$ where the supremum is taken over all stopping times τ of X with values in $[0,T-t]$ and E stands for $\widehat{\mathsf{E}}$ to simplify the notation. It follows from (3.16) and (3.22) using (3.18) that

$$V(t,m,s) = s\,\widetilde{V}(t, \tfrac{m\vee K}{s}; \tfrac{Ke^{rt}}{s}) \tag{3.23}$$

for $t \in [0,T]$, $m \geq s > 0$ and using the established probabilistic techniques (see e.g. [58]) one can verify that the optimal stopping set in (3.16) is given by

$$D = \big\{ (t,m,s) \in [0,T) \times \mathcal{S} : s\,\widetilde{V}(t, \tfrac{m\vee K}{s}; \tfrac{Ke^{rt}}{s}) = s\,G^{\mu_c}(t, \tfrac{m\vee K}{s}) - K \big\} \tag{3.24}$$

where $\mathcal{S} := \{ (m,s) : m \geq s > 0 \}$. As in Chapter 2 we have reduced the three-dimensional problem (3.16) to the two-dimensional problem (3.22), but with a scaling strike $\widetilde{K}$, since we will vary $\widetilde{K}$ to determine the solution of (3.16) using (3.23) and (3.24). Also we note that the solution to the British Russian option problem is fully embedded into the present solution when $K = 0$.

5. The analysis below is parallel to that of [23]. Let us now make use of Ito's formula (combined with the fact that $G_x^{\mu_c}(t, 1+) = 0$ for all $t \in [0,T)$) and the optional sampling theorem yield

$$\begin{aligned} &\mathsf{E}\Big[G^{\mu_c}(t+\tau, X_\tau^x) - \widetilde{K}e^{-r(t+\tau)}\Big] \\ &\quad = G^{\mu_c}(t,x) + \mathsf{E}\Big[\int_0^\tau H^{\mu_c}(t+s, X_s^x)\,ds - \widetilde{K}e^{-r(t+\tau)}\Big] \\ &\quad = G^{\mu_c}(t,x) - \widetilde{K}e^{-rt} + \mathsf{E}\Big[\int_0^\tau \big(H^{\mu_c}(t+s, X_s^x) + r\widetilde{K}e^{-r(t+s)}\big)\,ds\Big] \end{aligned} \tag{3.25}$$

for all stopping times τ of X with values in $[0,T-t]$ with $t \in [0,T)$ and $x \in [1,\infty)$ given and fixed, where the function $H^{\mu_c} = H^{\mu_c}(t,x)$ is given by

$$H^{\mu_c} = G_t^{\mu_c} - rx\,G_x^{\mu_c} + \tfrac{\sigma^2}{2}\,x^2 G_{xx}^{\mu_c}. \tag{3.26}$$

To simplify this expression note that by the Girsanov theorem we find

$$G^{\mu_c}(t,x) = \mathsf{E}^{\mu_c}(x \vee M_{T-t}) = \mathsf{E}^{\mu_c}\Big[S_{T-t}\big(\tfrac{x\vee M_{T-t}}{S_{T-t}}\big)\Big] = e^{\mu_c(T-t)}\,\widehat{\mathsf{E}}^{\mu_c}\Big(X_{T-t}^x\Big) \tag{3.27}$$

where X under $\widehat{\mathsf{P}}^{\mu_c}$ solves (3.20) with μ_c in place of r. This shows that $G^{\mu_c} = G^{\mu_c}(t,x)$ solves the 'killed' version of the Kolmogorov backward equation (see e.g. [58, Section

7])

$$G^{\mu_c}_t - \mu_c\, x\, G^{\mu_c}_x + \tfrac{\sigma^2}{2}\, x^2 G^{\mu_c}_{xx} + \mu_c G^{\mu_c} = 0. \tag{3.28}$$

Inserting (3.28) into (3.26) we find that

$$H^{\mu_c} = -\mu_c\, G^{\mu_c} + (\mu_c - r) x\, G^{\mu_c}_x. \tag{3.29}$$

A direct use of (3.14) in (3.29) leads to a complicated expression and for this reason we proceed by deriving a probabilistic interpretation of the right-hand side in (3.29). To this end note that $G^{\mu_c}(t,x) = \mathsf{E}^{\mu_c}(x \vee M_{T-t}) = x\, \mathsf{P}^{\mu_c}(M_{T-t} \le x) + \mathsf{E}^{\mu_c}[M_{T-t} I(M_{T-t} > x)]$ as well as $G^{\mu_c}(t,x) = \int_0^\infty \mathsf{P}^{\mu_c}(x \vee M_{T-t} > z)\, dz = x + \int_x^\infty \mathsf{P}^{\mu_c}(M_{T-t} > z)\, dz$ so that $G^{\mu_c}_x(t,x) = 1 - \mathsf{P}^{\mu_c}(M_{T-t} > x) = \mathsf{P}^{\mu_c}(M_{T-t} \le x)$. Inserting these expressions into (3.29) we find that

$$H^{\mu_c}(t,x) = -\mu_c\, \mathsf{E}^{\mu_c}[M_{T-t} I(M_{T-t} > x)] - rx\, \mathsf{P}^{\mu_c}(M_{T-t} \le x) \tag{3.30}$$

for $t \in [0,T]$ and $x \in [1,\infty)$. A lengthy calculation based on the known law of M_{T-t} under P^{μ_c} (recall (3.14) above) then shows that

$$\begin{aligned} H^{\mu_c}(t,x) = & - rx\, \Phi\Big(\tfrac{1}{\sigma\sqrt{T-t}} \Big[\log x - (\mu_c - \sigma^2/2)(T-t) \Big] \Big) \\ & + (r - \mu_c + \tfrac{\sigma^2}{2}) x^{2\mu_c/\sigma^2} \Phi\Big(-\tfrac{1}{\sigma\sqrt{T-t}} \Big[\log x + (\mu_c - \tfrac{\sigma^2}{2})(T-t) \Big] \Big) \\ & - (\mu_c + \tfrac{\sigma^2}{2}) e^{\mu_c (T-t)} \Phi\Big(-\tfrac{1}{\sigma\sqrt{T-t}} \Big[\log x - (\mu_c + \tfrac{\sigma^2}{2})(T-t) \Big] \Big) \end{aligned} \tag{3.31}$$

for $t \in [0,T]$ and $x \in [1,\infty)$. Now we denote by $\widehat{H}^{\mu_c}$ the integrand in (3.25)

$$\widehat{H}^{\mu_c}(t,x) = H^{\mu_c}(t,x) + r\widetilde{K} e^{-rt} \tag{3.32}$$

for $t \in [0,T]$ and $x \in [1,\infty)$. Then the expression (3.25) reads

$$\begin{aligned} & \mathsf{E}\Big[G^{\mu_c}(t+\tau, X^x_\tau) - \widetilde{K} e^{-r(t+\tau)} \Big] \\ & \quad = G^{\mu_c}(t,x) - \widetilde{K} e^{-rt} + \mathsf{E}\Big[\int_0^\tau \widehat{H}^{\mu_c}(t+s, X^x_s)\, ds \Big]. \end{aligned} \tag{3.33}$$

From now on the analysis differs from that of [23] due to presence of non-zero strike $K \neq 0$ and thus the integrand $\widehat{H}^{\mu_c}$ is not equal to the function H^{μ_c}. The expression (3.33) is useful for getting some insight into the structure of stopping and continuation sets: if $\widehat{H}^{\mu_c}(t,x) > 0$ then a point (t,x) belongs to continuation set, however if $\widehat{H}^{\mu_c}(t,x) < 0$ then it is not sufficient that at point (t,x) it is optimal to stop at once. The important property of the British options is an admissible set of values for contract drift, since if this is not properly selected, the buyer of option becomes overprotected at the beginning when $t=0$, i.e. he should exercise at once and the option becomes meaningless. Using probabilistic representation (3.30) it was shown in [23] that when $K=0$ the buyer is not overprotected if and only if $\mu_c < 0$. Indeed, if $\mu_c \geq 0$ then H is always negative and thus it is optimal to stop at once, but if $\mu_c < 0$ then at an initial point $(0,1)$ the function $H^{\mu_c}(0,1) > 0$ is positive and thus it is optimal to continue and the buyer is not overprotected. However in the case $K \neq 0$ the analysis becomes more delicate. Clearly when $\mu_c < 0$ the buyer again is not overprotected at $t=0$, but there are also positive admissible values for contract drift. For this we need to consider $\widehat{H}^{\mu_c}$ at financial initial point, i.e. originally to solve option pricing problem (3.15) we insert $t=0$, $m=s=S_0$ in (3.16) and assume that $S_0 < K$ (this is the usual assumption for lookback options and does not simplify analysis), thus using (3.23) we have that $(t,x;\widetilde{K}) = (0,\frac{K}{S_0};\frac{K}{S_0})$ and the function $\widehat{H}^{\mu_c}$ at the initial point is given by

$$\begin{aligned}\widehat{H}^{\mu_c}(0,\tfrac{K}{S_0}) &= H^{\mu_c}(0,\tfrac{K}{S_0}) + r\tfrac{K}{S_0} \\ &= -\mu_c\,\mathsf{E}^{\mu_c}[M_T I(M_T > \tfrac{K}{S_0})] + r\tfrac{K}{S_0}\,\mathsf{P}^{\mu_c}(M_T > \tfrac{K}{S_0}) \\ &= \mathsf{E}^{\mu_c}\big[r\tfrac{K}{S_0} - \mu_c M_T\big] I(M_T > \tfrac{K}{S_0})\end{aligned} \tag{3.34}$$

where we used (3.30). It follows from (3.34) that there exists a unique root $\mu_c = \mu_c^*$ of equation $\widehat{H}^{\mu_c}(0,\frac{K}{S_0}) = 0$ such that $0 < \mu_c^* < r$ and $\widehat{H}^{\mu_c}(0,\frac{K}{S_0}) > 0$ if and only if $\mu_c < \mu_c^*$. As we said above it is not certain that if $\widehat{H}^{\mu_c} < 0$ at initial point then it is optimal to stop at once and we cannot determine it analytically, hence the best we can do is to reassure that the buyer is not overprotected and we will require the condition $\mu_c < \mu_c^*$ so that $\widehat{H}^{\mu_c}(0,\frac{K}{S_0}) < 0$.

Remark 3.3.2. *It is important to note that the condition $\mu_c < \mu_c^*$ does not fully describe all admissible values for contract drift and there are values for contract drift* *greater than μ_c^* such that the holder is still not overprotected, but due to reasons outlined above, we are not to able to determine the exact threshold. Indeed from the analysis in Section 3.2 above we have that the contract drift is smaller than r. It will be proven below that if $\mu_c < r$ then there exists an optimal stopping boundary b separating the continuation set from the stopping set and thus the overprotection of the buyer is equivalent to the condition $b(0) \leq \frac{K}{S_0}$. For $\mu_c = \mu_c^*$ the optimal stopping boundary at zero $b^{\mu_c^*}(0) > \frac{K}{S_0}$ and there are values $\mu_c > \mu_c^*$ such that $b^{\mu_c}(0) > \frac{K}{S_0}$. However since $\mu_c \mapsto b^{\mu_c}(0)$ is decreasing then there exists a threshold $\mu_c^{**} < r$ such that $b^{\mu_c^{**}}(0) = \frac{K}{S_0}$ and the holder is overprotected if and only if $\mu_c > \mu_c^{**}$. It means that in order to determine the real threshold μ_c^{**} for the contract drift one should find the value of boundary $b(0)$ by solving numerically a nonlinear integral equation (see Theorem 3.4.1 below). Thus the equation $\widehat{H}^{\mu_c}(0, \frac{K}{S_0}) = 0$ gives more accessible condition for the contract drift rather than the equation $b^{\mu_c}(0) = \frac{K}{S_0}$ and we will use the threshold μ_c^* further for the financial analysis. Moreover we will show in Section 3.5 that the British option with contract drift $\mu_c < \mu_c^*$ provides attractive returns.*

Fixing $\mu_c < \mu_c^*$ it follows from (3.29) and (3.32) that

$$\widehat{H}_x^{\mu_c}(t,x) = H_x^{\mu_c}(t,x) = -rx\, G_x^{\mu_c}(t,x) + (\mu_c - r)x\, G_{xx}^{\mu_c}(t,x) < 0 \tag{3.35}$$

for any $x \geq 1$ and fixed $t \in [0,T)$ where we used fact that $G_x^{\mu_c} > 0$, $G_{xx}^{\mu_c} > 0$ and $\mu_c < \mu_c^* < r$. Hence it gives to us that

$$x \mapsto H^{\mu_c}(t,x) \text{ is decreasing on } [1,\infty) \tag{3.36}$$

for any given and fixed $t \in [0,T)$ and any choice of $\widetilde{K}$. It follows from (3.30), (3.32) and (3.36) that there exists a continuous (smooth) function $h : [0,T] \to I\!R$ such that

$$\widehat{H}^{\mu_c}(t, h(t)) = 0 \tag{3.37}$$

for $t \in [0,T]$ with $\widehat{H}^{\mu_c}(t,h(t)) > 0$ for $x \in [1, h(t))$ and $\widehat{H}^{\mu_c}(t,h(t)) < 0$ for $x \in (h(t),\infty)$ when $t \in [0,T]$ given and fixed. In view of (3.33) this implies that no point (t,x) in $[0,T) \times [1,\infty)$ with $x < h(t)$ is a stopping point (for this one can make use of the first exit time from a sufficiently small time-space ball centred at the

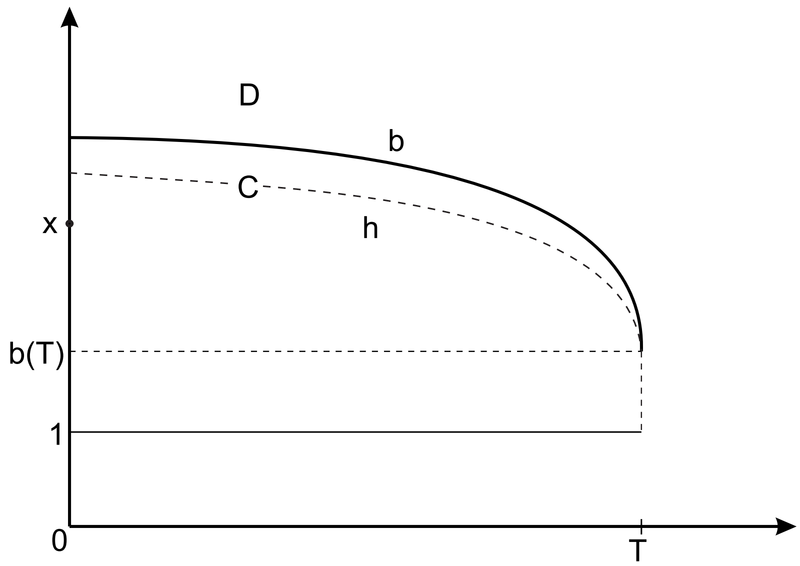

Figure 3.1: A computer drawing of the optimal stopping boundary b for the problem (3.22) in the case $K = 1.2, S_0 = 1, \widetilde{K} = \frac{K}{S_0} = 1.2, T = 1, \mu_c = 0.05 < \mu_c^* \approx 0.075, r = 0.1, \sigma = 0.4$ with the boundary condition $b(T) = \widetilde{K}e^{-rT} > 1$ and the starting point $x = \frac{K}{S_0} < h(0)$.

point). Likewise, it is also clear and can be verified that if $x > h(t)$ and $t < T$ is sufficiently close to T then it is optimal to stop immediately (since the gain obtained from being below h cannot offset the cost of getting there due to the lack of time). This shows that the optimal stopping boundary $b : [0,T] \to [0,\infty]$ separating the continuation set from the stopping set satisfies $b(T) = h(T)$ and this value equals $\big(\widetilde{K}e^{-rT} \vee 1\big)$. Moreover, the fact (3.36) combined with the identity (3.33) implies that the continuation set is given by $C = \{ (t,x) \in [0,T)\times[1,\infty) : x < b(t) \}$ and the stopping set is given by $D = \{ (t,x) \in [0,T)\times[1,\infty) : x \geq b(t) \}$ so that the optimal stopping time in the problem (3.22) is given by (see Figure 3.1)

$$\tau_b = \inf \{ 0 \leq t \leq T : X_t \geq b(t) \}. \tag{3.38}$$

It is also clear and can be verified that if the initial point $x \geq 1$ of the process X is sufficiently large then it is optimal to stop immediately (since the gain obtained from being below h cannot offset the cost of getting there due to the shortage of time). This shows that the optimal stopping boundary b is finite valued.

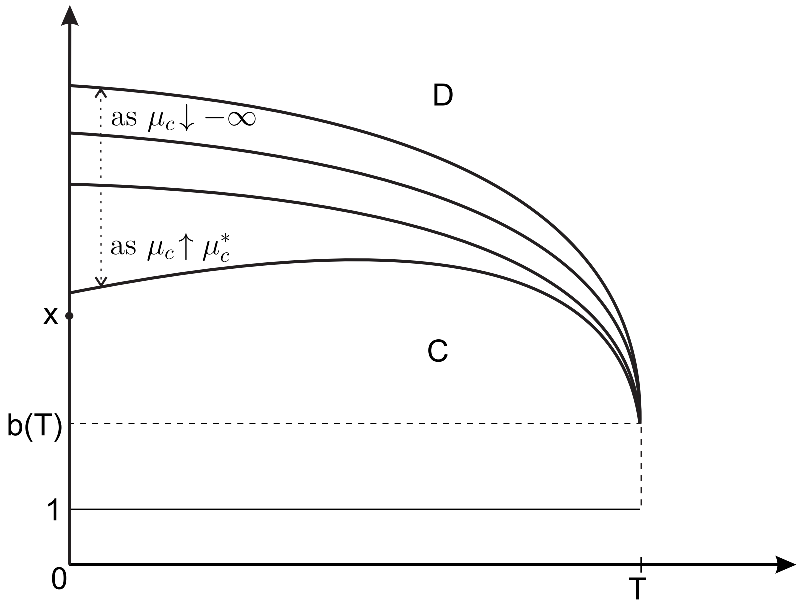

Figure 3.2: A computer drawing showing how the optimal stopping boundary b for the problem (3.22) increases as one decreases the contract drift. There are four different cases: 1)$\mu_c = 0.074$; 2)$\mu_c = 0.07$; 3)$\mu_c = 0.05$; 4)$\mu_c = -\infty$ (the latter corresponds to the American lookback option problem). The set of parameters: $K = 1.2, S_0 = 1, \widetilde{K} = \frac{K}{S_0} = 1.2, T = 1, r = 0.1, \sigma = 0.4$ with the boundary condition $b(T) = \widetilde{K}\mathrm{e}^{-rT} > 1$, the starting point $x = \frac{K}{S_0}$ and the root of (3.9) $\mu_c^* \approx 0.075$.

6. Standard Markovian arguments lead to the following free-boundary problem (for the value function $\widetilde{V} = \widetilde{V}(t,x)$ and the optimal stopping boundary $b = b(t)$ to be determined):

$$\widetilde{V}_t - rx\widetilde{V}_x + \frac{\sigma^2}{2}x^2\widetilde{V}_{xx} = 0 \quad \text{for } x \in (1, b(t)) \text{ and for } t \in [0,T) \tag{3.39}$$

$$\widetilde{V}(t,x) = G^{\mu_c}(t,x) - \widetilde{K}e^{-rt} \quad \text{for } x \geq b(t) \text{ and for } t \in [0,T) \tag{3.40}$$

$$\widetilde{V}_x(t,b(t)) = G_x^{\mu_c}(t,b(t)) \quad \text{for } t \in [0,T) \tag{3.41}$$

$$\widetilde{V}_x(t,1+) = 0 \quad \text{for } t \in [0,T) \tag{3.42}$$

where $b(T) = \widetilde{K}e^{-rT} \vee 1$ and $\widetilde{V}(T,x) = G^{\mu_c}(T,x) - \widetilde{K}e^{-rT} = x - \widetilde{K}e^{-rT}$ for $x \geq 1$. It can be shown that this free-boundary problem has a unique solution $\widetilde{V}$ and b which coincide with the value function (3.22) and the optimal stopping boundary respectively (cf. [58]). Fuller details of the analysis go beyond our aims in this paper and for this reason will be omitted since the fact that unless $\mu_c < 0$ the boundary b is not necessarily a monotone function of time (see Figure 3.2) makes this analysis more complicated (in comparison with the American lookback option in Chapter 2). In the next section we will derive equations which characterise $\widetilde{V}$ and b uniquely and can be used for their calculation.

Note that $x \mapsto \widetilde{V}(t,x)$ is increasing and convex on $[1,\infty)$ for every $t \in [0,T]$ (since G^{μ_c} is so). Note also that if we let μ_c to $-\infty$ then the optimal stopping boundary b goes to a continuous decreasing function $b_{-\infty} : [0,T] \to I\!R$ satisfying $b_{-\infty}(T) = \widetilde{K}e^{-rT} \vee 1$ (see Figure 3.2). The limiting boundary $b_{-\infty}$ is optimal in the problem (3.22) where $G^{-\infty}(t,x) = x$ for $(t,x) \in [0,T] \times [1,\infty)$. This problem corresponds to the American lookback option with fixed strike in the case of finite horizon (see Chapter 2).

7. From (3.14) we see that the volatility parameter appears explicitly in the payoff of the British option and thus should be agreed in the contract. However, since the underlying process is assumed to be a geometric Brownian motion, one may take any of the standard estimators for the volatility (e.g. using the Central Limit Theorem) over an arbitrarily small time period prior to the initiation of the contract. It is important to note that the estimation of the stock drift μ cannot be estimated in practice, therefore it seems natural to provide a true buyer with protection an drift rather than a volatility, at least in current model. For more details about this question we address reader to final paragraph of Section 3 in [59].

3.4. The arbitrage-free price and the rational exercise boundary

In this section we derive a closed form expression for the value function $\widetilde{V}$ for the problem (3.22) in terms of the optimal stopping boundary b and show that the optimal stopping boundary b itself can be characterised as the unique solution to a nonlinear integral equation (Theorem 3.4.1). We will make use of the following functions in Theorem 3.4.1 below:

$$F(t,x) = G^{\mu_c}(t,x) - e^{-r(T-t)}G^r(t,x) \tag{3.43}$$

$$L(t,x,v,z) = -\int_z^\infty \widehat{H}^{\mu_c}(v,y) f(v-t,x,y)\,dy \tag{3.44}$$

for $t \in [0,T)$, $x \geq 1$, $v \in (t,T)$ and $z \geq 1$, where the functions G^{μ_c} and G^r are given by (3.14) above (upon identifying μ_c with r in the latter case), the function $\widehat{H}^{\mu_c}$ is given by (3.32) above, and $y \mapsto f(v-t,x,y)$ is the probability density function of X^x_{v-t} under $\widehat{\mathsf{P}}$ given by

$$\begin{aligned} f(v-t,x,y) = &\frac{1}{\sigma y\sqrt{v-t}}\Big[\varphi\Big(\frac{1}{\sigma y\sqrt{v-t}}\big[\log\frac{x}{y} - (r+\frac{\sigma^2}{2})(v-t)\big]\Big) \\ &+ x^{1+2r/\sigma^2}\varphi\Big(\frac{1}{\sigma y\sqrt{v-t}}\big[\log xy + (r+\frac{\sigma^2}{2})(v-t)\big]\Big)\Big] \\ &+ \frac{1+2r/\sigma^2}{y^{2(1+r/\sigma^2)}}\Phi\Big(-\frac{1}{\sigma y\sqrt{v-t}}\big[\log xy - (r+\frac{\sigma^2}{2})(v-t)\big]\Big) \end{aligned} \tag{3.45}$$

for $y \geq 1$ (with $v-t$ and x as above) where φ is the standard normal density function given by $\varphi(x) = (1/\sqrt{2\pi})e^{-x^2/2}$ for $x \in I\!R$ (and Φ is the standard normal distribution function defined following (3.14) above). It should be noted that $L(t,x,v,b(v)) > 0$ for all $t \in [0,T)$, $x \geq 1$ and $v \in (t,T)$, since $\widehat{H}^{\mu_c}(v,y) < 0$ for all $y > b(v)$ as b lies above h (recall (3.37) above).

1. The main result of this section may now be stated as follows.

Theorem 3.4.1. *The value function* $\widetilde{V}$ *for the problem* (3.22) *admits the following representation*

$$\widetilde{V}(t,x) = e^{-r(T-t)}G^r(t,x) - \widetilde{K}e^{-rT} + \int_t^T L(t,x,v,b(v))\,dv \tag{3.46}$$

for all $(t,x) \in [0,T) \times [1,\infty)$.

The optimal stopping boundary (see Figure 3.1 above) can be characterised as the unique continuous solution $b : [0,T] \to I\!R_+$ *to the nonlinear integral equation*

$$F(t,b(t)) = \widetilde{K}(e^{-rt} - e^{-rT}) + \int_t^T L(t,b(t),v,b(v))\,dv \tag{3.47}$$

satisfying $b(t) \geq h(t)$ *for all* $t \in [0,T]$ *where* h *is defined by* (3.37) *above.*

Proof. We derive (3.46) and show that the rational exercise boundary solves (3.47). We omit the proof of fact that (3.47) cannot have other (continuous) solutions, since it is parallel to similar proofs in [23] and [33].

a) Recall that the value function $\widetilde{V} : [0,T]\times[1,\infty) \to I\!R$ and the rational exercise boundary $b : [0,T] \to I\!R_+$ solve the free-boundary problem (3.39)-(3.42) (where $\widetilde{V}$ extends as G^{μ_c} above b), set $C_b = \{ (t,x) \in [0,T)\times [1,\infty) : x < b(t) \}$ and $D_b = \{ (t,x) \in [0,T)\times [1,\infty) : x \ge b(t) \}$ and let $I\!L_X \widetilde{V}(t,x) = -rx\, \widetilde{V}_x(t,x) + \frac{\sigma^2}{2}\, x^2\, \widetilde{V}_{xx}(t,x)$ for $(t,x) \in C_b \cup D_b$. Then $\widetilde{V}$ and b are continuous functions satisfying the following conditions: (i) $\widetilde{V}$ is $C^{1,2}$ on $C_b \cup D_b$; (ii) b is of bounded variation; (iii) $\mathsf{P}(X_t^x = c) = 0$ for all $c > 0$ whenever $t \in [0,T]$ and $x \ge 1$; (iv) $\widetilde{V}_t + I\!L_X \widetilde{V}$ is locally bounded on $C_b \cup D_b$ (recall that $\widetilde{V}$ satisfies (3.39) on C_b and coincides with G^{μ_c} on D_b); (v) $x \mapsto \widetilde{V}(t,x)$ is convex on $[1,\infty)$ for every $t \in [0,T]$; and (vi) $t \mapsto \widetilde{V}_x(t,b(t)\pm) = G_x^{\mu_c}(t,b(t))$ is continuous on $[0,T]$ (recall that $\widetilde{V}$ satisfies the smooth-fit condition (3.41) at b). From these conditions we see that the local time-space formula [52] is applicable to $(s,y) \mapsto \widetilde{V}(t+s,y)$ with $t \in [0,T)$ given and fixed. Fixing an arbitrary $x \ge 1$ and making use of (3.42) this yields

$$
\begin{aligned}
\widetilde{V}(t+s, X_s^x) &= \widetilde{V}(t,x) && (3.48)\\
&+ \int_0^s (\widetilde{V}_t + I\!L_X \widetilde{V})(t+v, X_v^x) I(X_v^x \neq b(t+v))\, dv + M_s^b \\
&+ \frac{1}{2} \int_0^s (\widetilde{V}_x(t+v, X_v^x+) - \widetilde{V}_x(t+v, X_v^x-))\, I(X_v^x = b(t+v))\, d\ell_v^b(X^x)
\end{aligned}
$$

where $M_s^b = \sigma \int_0^s X_v^x \widetilde{V}_x(t+v, X_v^x)\, I(X_v^x \neq b(t+v))\, dW_v$ is a martingale for $s \in [0, T-t]$ and $\ell^b(X^x) = (\ell_v^b(X^x))_{0 \le v \le s}$ is the local time of $X^x = (X_v^x)_{0\le v\le s}$ on the curve b for $s \in [0,T-t]$. Moreover, since $\widetilde{V}$ satisfies (3.39) on C_b and equals $G^{\mu_c} - \widetilde{K}e^{-rt}$ on D_b, and the smooth-fit condition (3.41) holds at b, we see that (3.48) simplifies to

$$
\widetilde{V}(t+s, X_s^x) = \widetilde{V}(t,x) + \int_0^s \widehat{H}^{\mu_c}(t+v, X_v^x)\, I(X_v^x > b(t+v))\, dv + M_s^b \quad (3.49)
$$

for $s \in [0,T-t]$ and $(t,x) \in [0,T)\times[1,\infty)$.

b) Replacing s by $T-t$ in (3.49), using that $\widetilde{V}(T,x) = G^{\mu_c}(T,x) - \widetilde{K}e^{-rT} = x - \widetilde{K}e^{-rT}$ for $x \ge 1$, taking E on both sides and applying the optional sampling theorem, we get

$$\mathsf{E}(X^x_{T-t}) - \widetilde{K}e^{-rT} = \widetilde{V}(t,x) + \int_0^{T-t} \mathsf{E}\left[\widehat{H}^{\mu_c}(t+v, X^x_v)\, I(X^x_v > b(t+v))\right] dv \tag{3.50}$$
$$= \widetilde{V}(t,x) - \int_t^T L(t,x,v,b(v))\, dv$$

for all $(t,x) \in [0,T) \times [1,\infty)$ where L is defined in (3.44) above. We see that (3.50) yields the representation (3.46). Moreover, since $\widetilde{V}(t,b(t)) = G^{\mu_c}(t,b(t)) - \widetilde{K}e^{-rt}$ for all $t \in [0,T]$ we see from (3.46) with (3.43) that b solves (3.47). This establishes the existence of the solution to (3.47). □

2. Now we can determine the arbitrage-free price (3.16) of the British lookback option with fixed strike K. Indeed, from (3.23) and (3.46) we have

$$V(t,m,s) = s\, e^{-r(T-t)} G^r\big(t,x\big) - Ke^{r(t-T)} \tag{3.51}$$
$$+ \int_t^T \mathsf{E}\left[\big(H^{\mu_c}(v, X^x_{v-t}) + \tfrac{rK}{s}\big)\, I\big(X^x_{v-t} > b(v; \tfrac{Ke^{rt}}{s})\big)\right] dv$$

for $t \in [0,T]$, $m \geq s > 0$ where $x = \frac{m \vee K}{s}$ and the optimal stopping boundary b is computed under assumption $\widetilde{K} = \frac{Ke^{rt}}{s}$ in (3.47).

Standard arguments based on the strong Markov property (see [58]) show that the stopping region in (3.16) has the following form:

$$D = \left\{\, (t,m,s) \in [0,T) \times \mathcal{S} : V(t,m,s) = s\, G^{\mu_c}\big(t, \tfrac{m \vee K}{s}\big) - K \right\} \tag{3.52}$$
$$= \left\{\, (t,m,s) \in [0,T) \times \mathcal{S} : s\, \widetilde{V}\Big(t, \tfrac{m \vee K}{s}; \tfrac{Ke^{rt}}{s}\Big) = s\, G^{\mu_c}\big(t, \tfrac{m \vee K}{s}\big) - K \,\right\}$$
$$= \left\{\, (t,m,s) \in [0,T) \times \mathcal{S} : \widetilde{V}\big(t, \tfrac{m \vee K}{s}; \tfrac{Ke^{rt}}{s}\big) = G^{\mu_c}\big(t, \tfrac{m \vee K}{s}\big) - \tfrac{Ke^{rt}}{s}\, e^{-rt} \,\right\}$$
$$= \left\{\, (t,m,s) \in [0,T) \times \mathcal{S} : \tfrac{m \vee K}{s} \geq b\,\big(t; \tfrac{Ke^{rt}}{s}\big) \,\right\}$$
$$= \left\{\, (t,m,s) \in [0,T) \times \mathcal{S} : m \vee K \geq s\, b\,\big(t; \tfrac{Ke^{rt}}{s}\big) \,\right\}$$

where $\mathcal{S} = \{(m,s) : m \geq s > 0\}$ and we used (3.23) and (3.40). Thus the optimal stopping time in (3.16) is given by

$$\tau_g = \inf\,\{\, 0 \leq u < T-t : M_u \geq g(t+u, S_u) \,\} \tag{3.53}$$

where the rational exercise boundary g reads

$$g(t,s) = \begin{cases} s\, b\big(t; \frac{Ke^{rt}}{s}\big), & \text{if } K < s\, b\big(t; \frac{Ke^{rt}}{s}\big) \\ s, & \text{if } K \geq s\, b\big(t; \frac{Ke^{rt}}{s}\big) \end{cases} \tag{3.54}$$

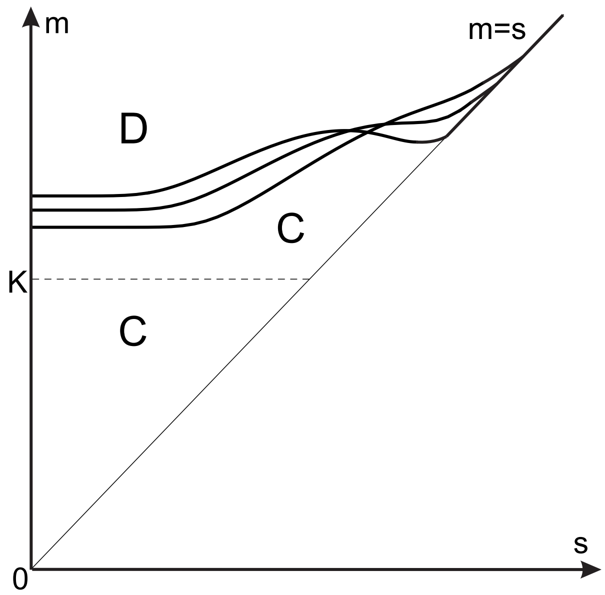

Figure 3.3: A computer drawing of the rational exercise boundary $s \mapsto g(t,s)$ for 1)$t=0$ (top at $s=0$); 2)$t=0.3$; 3)$t=0.6$ (bottom at $s=0$) in the case $K=1.2, S_0=1, \widetilde{K}=\frac{K}{S_0}=1.2, T=1, \mu_c=0.05<\mu_c^*\approx 0.075, r=0.1, \sigma=0.4$. The limit of $g(t,\,\cdot\,)$ at zero is greater than K for every t.

for $t\in[0,T)$, $s>0$. Hence if there exist $t\in[0,T)$, $s^*\in(0,K)$ and $\varepsilon>0$ small enough such that $K=s^*\,b\big(t;\frac{K\mathrm{e}^{rt}}{s^*}\big)$ and $K>s\,b\big(t;\frac{K\mathrm{e}^{rt}}{s}\big)$ for $s\in(s^*,s^*+\varepsilon)$ then the boundary $s\mapsto g(t,s)$ exhibits a left-discontinuity at $s=s^*$, which is a quite rare case in the optimal stopping theory. Below we show a numerical example where indeed the boundary has a jump down.

3. We now provide the numerical analysis and computer drawing of the rational exercise boundaries for (3.16). It follows from (3.54) that in order to determine $s\mapsto g(t_0,s)$ for given and fixed $t_0\in[0,T)$ we need to calculate the optimal stopping boundary $b(\,\cdot\,;\widetilde{K})$ as a solution to (3.47) with $\widetilde{K}=\frac{Ke^{rt_0}}{s}$ for every $s>0$ and then $g(t_0,s)=s\,b(t_0;\widetilde{K})$. For computer drawing of the boundaries we assume that the initial stock price equals 1, the strike price $K=1.2$, the maturity time $T=1$ year, the contract drift $\mu_c=0.05<\mu_c^*\approx 0.075$, the interest rate $r=0.1$, the volatility $\sigma=0.4$, i.e. we consider the option out-of-the money. We discretise the interval $(0,2)$ with step $h=0.05$ and for every $0\le s\le 2$ of this grid we solve numerically the

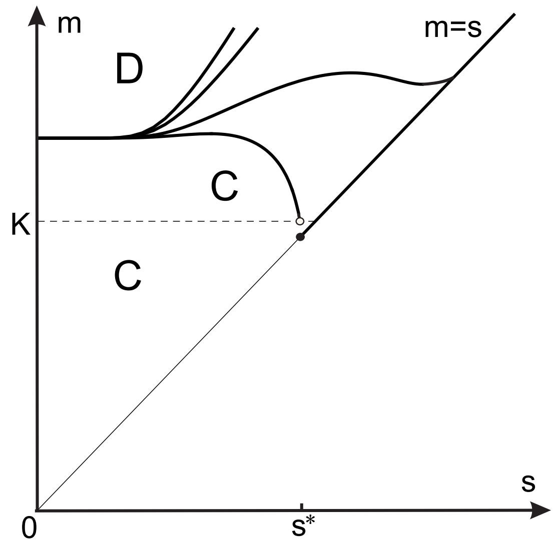

Figure 3.4: A computer drawing showing how the rational exercise boundary g for the problem (3.16) increases as one decreases the contract drift for fixed $t = 0$. There are four different cases: 1)$\mu_c = 0.074$; 2)$\mu_c = 0.05$; 3)$\mu_c = -0.05$; 4)$\mu_c = -\infty$ (the latter corresponds to the American lookback option problem). All boundaries have the same limit at $s = 0$. The set of parameters: $K = 1.2, S_0 = 1, T = 1, \mu_c^* \approx 0.075, r = 0.1, \sigma = 0.4$. The rational exercise boundary in the case $\mu_c = 0.074$ is discontinuous at $s^* \approx 1.14 < K$.

integral equation (3.47) with $\widetilde{K} = \frac{Ke^{rt_0}}{s}$ and then put $g(t_0, s) = s\, b(t_0; \widetilde{K})$. In Figure 3.3 we draw the rational exercise boundary $s \mapsto g(t, s)$ for different values of t in order to gain insight how the boundary evolves over the time. The Figure 3.3 shows that the rational exercise boundary of the British version is not a monotone function of both variables unlike the American counterparty, since it was shown in Chapter 2 that for the American lookback option with fixed strike the rational exercise boundary has the following pattern: $t \mapsto g^A(t, s)$ is decreasing on $[0, T)$ for each $s > 0$ fixed and $s \mapsto g^A(t, s)$ is increasing on $(0, \infty)$ for $t \in [0, T)$ fixed. From (3.54) it is easily to seen that $g(t, s) \uparrow s\, b_0(t)$ as $s \to \infty$ for fixed $t \in [0, T)$ where $b_0(t) = b(t; 0)$ is the optimal stopping boundary for the British Russian option [23]. Since $\mu_c > 0$ we have that $b_0 \equiv 1$ (see [23]) so that $g(t, s) = s$ for $s \gg 0$ large enough, i.e. the rational exercise boundary becomes a diagonal (see Figure 3).

The Figure 3.4 shows how the rational exercise boundary changes as one varies the contract drift for fixed $t = 0$. It can be seen that stronger the protection (i.e. the contract drift increases) the larger the stopping region (i.e. the rational exercise boundary decreases). In the case $\mu_c = 0.074$ we observe a remarkable feature: the rational exercise boundary is *discontinuous* at point $s^* \approx 1.14 < K$. Hence it was not possible to apply a change-of-variable formula with local time on surfaces [55] in order to solve the three-dimensional stopping problem directly. This is another advantage of reduction by the method of a scaling strike. Also using the remarks from previous section we note that if we let μ_c to $-\infty$ then the optimal stopping boundary g goes increasingly to a continuous function $g_{-\infty}$ (see Figure 3.4). The limiting boundary $g_{-\infty}$ is optimal in the problem (3.16) where $G^{-\infty}(t,x)=x$ for $(t,x) \in [0,T] \times [1,\infty)$. This problem corresponds to the American lookback option with fixed strike in the case of finite horizon (see [33]).

3.5. The financial analysis

In this section we present the analysis of financial returns of the British lookback option with fixed strike and highlight the practical features of the option. We perform comparisons with both the American lookback option with fixed strike and the European lookback option with fixed strike since the former option has been the subject of much research activity in recent years (see e.g. [21], [33]) whilst the latter is commonly traded and well understood. The so-called 'skeleton analysis' was applied to analyse financial returns of options in [59], [60], [22] and [23], where the main question was addressed as to what the return would be if the underlying process enters the given region at a given time (i.e. the probability of the latter event was not discussed nor do we account for any risk associated with its occurrence). Such a 'skeleton analysis' is both natural and practical since it places the question of probabilities and risk under the subjective assessment of the option holder (irrespective of whether the stock price model is correct or not). In the present setting an analysis of option performance based on returns seems especially insightful since lookback options are most often used exclusively for speculation and thus for achieving high returns.

1. In the end of Section 3.4 above we saw that the rational exercise strategy (3.54) of the British lookback option with fixed strike in the problem (3.16) above changes as one varies the contract drift μ_c. This is illustrated in Figure 3.4 above. We recall that the contract drift must satisfy $\mu_c < \mu_c^*$, since otherwise we are not able to reassure that buyer is not overprotected. On the other hand, when $\mu_c \downarrow -\infty$ then g tends to the American lookback boundary $g_{-\infty}$ and the British lookback option effectively reduces to the American lookback option. In the latter case a contract drift represents an infinite tolerance of unfavourable drifts and the British lookback holder will exercise the option rationally in the limit at the same time as the American lookback holder.

2. In the numerical example below (see Tables 3.1 and 3.2) the parameter values have been chosen to present the practical features of the British lookback option with fixed strike in a fair and representative way. We assume that the initial stock price equals 1, the strike price $K = 1.2$, the maturity time $T = 1$ year, the interest rate $r = 0.1$, the volatility coefficient $\sigma = 0.4$, i.e. we consider the option out-of-the money. We choose the contract drift $\mu_c = 0.05$, which satisfies the condition

$$\mathsf{E}^{\mu_c}\big[r\tfrac{K}{S_0} - \mu_c M_T\big] I(M_T > \tfrac{K}{S_0}) \approx 0.026 > 0. \tag{3.55}$$

For this set of parameters the arbitrage-free price of the British lookback option with fixed strike is 0.254, the price of the American lookback option is 0.251, and the price of the European lookback option is 0.245. Observe that the closer the contract drift gets to μ_c^*, the stronger the protection feature provided (with generally better returns), and the more expensive the British lookback option becomes. Recall also that when $\mu_c \downarrow -\infty$ then the British lookback option effectively reduces to the American lookback option and the price of the former option converges to the price of the latter. The fact that the price of the British lookback option is close to the price of the European (and American) lookback option in situations of interest for trading is of considerable practical value.

Exercise time (months)	0	2	4	6	8	10	12
$\mathsf{S} = 0.6$							
Exercise at $\mathsf{M} \leq 1.2$ with $\mu_c = 0.05$	6%	4%	2%	1%	0%	0%	0%
Exercise at $\mathsf{M} \leq 1.2$ (American)	0%	0%	0%	0%	0%	0%	0%
Exercise at $\mathsf{M} = 1.4$ with $\mu_c = 0.05$	81%	80%	79%	79%	79%	79%	79%
Exercise at $\mathsf{M} = 1.4$ (American)	80%	80%	80%	80%	80%	80%	80%
$\mathsf{S} = 1.0$							
Exercise at $\mathsf{M} \leq 1.2$ with $\mu_c = 0.05$	93%	78%	62%	46%	28%	10%	0%
Exercise at $\mathsf{M} \leq 1.2$ (American)	0%	0%	0%	0%	0%	0%	0%
Exercise at $\mathsf{M} = 1.4$ with $\mu_c = 0.05$	133%	121%	109%	97%	87%	80%	79%
Exercise at $\mathsf{M} = 1.4$ (American)	80%	80%	80%	80%	80%	80%	80%
Exercise at $\mathsf{M} = 1.6$ with $\mu_c = 0.05$	189%	180%	171%	164%	160%	157%	157%
Exercise at $\mathsf{M} = 1.6$ (American)	159%	159%	159%	159%	159%	159%	159%
$\mathsf{S} = 1.4$							
Exercise at $\mathsf{M} = 1.4$ with $\mu_c = 0.05$	297%	274%	250%	223%	193%	157%	79%
Exercise at $\mathsf{M} = 1.4$ (American)	80%	80%	80%	80%	80%	80%	80%
Exercise at $\mathsf{M} = 1.6$ with $\mu_c = 0.05$	309%	287%	264%	239%	212%	182%	157%
Exercise at $\mathsf{M} = 1.6$ (American)	159%	159%	159%	159%	159%	159%	159%
Exercise at $\mathsf{M} = 1.8$ with $\mu_c = 0.05$	340%	320%	300%	280%	260%	242%	236%
Exercise at $\mathsf{M} = 1.8$ (American)	239%	239%	239%	239%	239%	239%	239%
$\mathsf{S} = 1.8$							
Exercise at $\mathsf{M} = 1.8$ with $\mu_c = 0.05$	517%	487%	456%	422%	383%	337%	237%
Exercise at $\mathsf{M} = 1.8$ (American)	239%	239%	239%	239%	239%	239%	239%
Exercise at $\mathsf{M} = 2.0$ with $\mu_c = 0.05$	526%	497%	467%	434%	398%	357%	315%
Exercise at $\mathsf{M} = 2.0$ (American)	318%	318%	318%	318%	318%	318%	318%

Table 3.1: Returns observed upon exercising the British lookback option with fixed strike compared with returns observed upon exercising the American lookback option with fixed strike. The returns are expressed as a percentage of the original option price paid by the buyer (rounded to the nearest integer), i.e. $R(t,m,s)/100 = (s\,G^{\mu_c}(t,\frac{m\vee K}{s})-K)/V(0,1,1)$ and $R_A(t,m,s)/100 = (m-K)^+/V_A(0,1,1)$. The parameter set is $\mu_c = 0.05,\ K = 1.2,\ T = 1,\ r = 0.1,\ \sigma = 0.4$ and the initial stock price equals 1.

3. Tables 3.1 and 3.2 below provide the analysis of comparison between the British lookback option with fixed strike and its American and European versions. We consider the set of parameters above, the arbitrage-free price of the British option in this setting can be computed using (3.51) so that $V(0,1,1) = 0.254$. We exploit the same method to find the price of the American option $V_A(0,1,1) = 0.251$ (see [33]). The European option price $V_E(0,1,1) = 0.245$ can be easily evaluated using the following manipulations

$$\begin{aligned} V_E(t,m,s) &= e^{-r(T-t)}\mathsf{E}^r\big(m \vee \max_{u\leq T-t} sS_u - K\big)^+ \qquad (3.56)\\ &= s\,e^{-r(T-t)}\mathsf{E}^r\big(\tfrac{m\vee K}{s} \vee \max_{u\leq T-t} S_u - \tfrac{K}{s}\big) \end{aligned}$$

Exercise time (months)	0	2	4	6	8	10	12
S = 0.6							
Exercise at M ≤ 1.2 with $\mu_c = 0.05$	6%	4%	2%	1%	0%	0%	0%
Selling at M ≤ 1.2 (European)	8%	5%	2%	1%	0%	0%	0%
Exercise at M = 1.4 with $\mu_c = 0.05$	81%	80%	79%	79%	79%	79%	79%
Selling at M = 1.4 (European)	77%	77%	77%	78%	79%	80%	82%
S = 1.0							
Exercise at M ≤ 1.2 with $\mu_c = 0.05$	93%	78%	62%	46%	28%	10%	0%
Selling at M ≤ 1.2 (European)	100%	84%	68%	50%	31%	12%	0%
Exercise at M = 1.4 with $\mu_c = 0.05$	133%	121%	109%	97%	87%	80%	79%
Selling at M = 1.4 (European)	134%	122%	110%	99%	88%	82%	82%
Exercise at M = 1.6 with $\mu_c = 0.05$	189%	180%	171%	164%	160%	157%	157%
Selling at M = 1.6 (European)	184%	176%	169%	164%	160%	160%	163%
S = 1.4							
Exercise at M = 1.4 with $\mu_c = 0.05$	297%	274%	250%	223%	193%	157%	79%
Selling at M = 1.4 (European)	299%	278%	255%	229%	200%	163%	82%
Exercise at M = 1.6 with $\mu_c = 0.05$	309%	287%	264%	239%	212%	182%	157%
Selling at M = 1.6 (European)	308%	288%	267%	243%	217%	188%	163%
Exercise at M = 1.8 with $\mu_c = 0.05$	340%	320%	300%	280%	260%	242%	236%
Selling at M = 1.8 (European)	335%	318%	300%	282%	263%	248%	245%
S = 1.8							
Exercise at M = 1.8 with $\mu_c = 0.05$	517%	487%	456%	422%	383%	337%	237%
Selling at M = 1.8 (European)	511%	486%	458%	428%	392%	346%	245%
Exercise at M = 2.0 with $\mu_c = 0.05$	526%	497%	467%	434%	398%	357%	315%
Selling at M = 2.0 (European)	518%	494%	468%	439%	406%	366%	326%

Table 3.2: Returns observed upon exercising the British lookback option with fixed strike compared with returns observed upon selling the European lookback option with fixed strike. The returns are expressed as a percentage of the original option price paid by the buyer (rounded to the nearest integer), i.e. $R(t,m,s)/100 = (s\,G^{\mu_c}(t,\frac{m\vee K}{s}) - K)/V(0,1,1)$ and $R_E(t,m,s)/100 = V_E(t,m,s)/V_E(0,1,1)$. The parameter set is $\mu_c = 0.05$, $K = 1.2$, $T = 1$, $r = 0.1$, $\sigma = 0.4$ and the initial stock price equals 1.

$$= e^{-r(T-t)}\big(s\,G^r\big(t,\tfrac{m\vee K}{s}\big) - K\big).$$

We compare the returns that the British lookback holder can obtain upon exercising his option with the returns observed upon (i) exercising the American lookback option in the same contingency (Table 3.1) and (ii) selling the European lookback option in the same contingency (Table 3.2). The latter is motivated by the fact that in practice the European option holder may choose to sell his option at any time during the term of the contract, and in this case one may view his ‘payoff’ as the price he receives upon selling. It is important to note that the payoff of the American option depends only on the maximum process, but both the British payoff and the price of the European option depend on the three-dimensional process (maximum-stock price-time), hence in Tables 3.1 and 3.2 we fix four different values of the current stock price (0.6, 1.0, 1.4, 1.8) and then compare returns for the range of the maximum process and the time. From (3.13) and (3.56) we see that the payoff of the British and the European options does not depend on the maximum process M when $M \leq K = 1.2$. From Tables 3.1 and 3.2 we see that (i) exercising the British lookback option provides generally much better returns than exercising the American lookback option: it is more appreciable for favourable stock movements rather than unfavourable; (ii) exercising the British lookback option provides very comparable returns to selling the European lookback option: the British returns are generally better away from expiry and the European returns are better near maturity. However, as remarked in [59], [60] and [23] in a real financial market the option holder's ability to sell his contract will depend upon a number of exogenous factors. These include his ability to access the option market, the transaction costs and/or taxes involved in selling, and in particular the liquidity of the option market itself. For lookback options the latter factor is especially important, since they generally trade in over-the-counter markets which have no organised exchange and as such these markets can be illiquid and thus the selling of the European option can be problematic. Crucially, the protection feature of the British lookback option is intrinsic to it, that is, it is completely endogenous. It is inherent in the payoff function itself (obtained as a consequence of optimal prediction), and as such it is independent of any exogenous factors. From this point of view the British lookback option is a particularly attractive financial instrument.

Chapter 4

The American swing put option

4.1. Introduction

Swing contracts are financial products designed primarily to allow for flexibility on purchase, sale and delivery of commodities in the energy market. They have features of American-type options with multiple early exercise rights and in many relevant cases may be mathematically described in terms of multiple optimal stopping problems combined with control problems. The stopping part of the contract accounts for the choice of the optimal times to exercise the flexibility and the control part describes the kind of flexibility entailed by the contract. Mathematical formulations of such problems in the economic-financial literature date back to the early 1980's and an exhaustive survey of them may be found in [31, Sec. 1 and 2] and references therein. Theoretical and numerical aspects of pricing and hedging swing contracts have received increasing attention in the last decade with many contributions from a number of authors developing in parallel several methods of solution (see e.g. [45] for an extensive survey of recent results).

Amongst the earliest contributions on the numerical study of swing options we find for instance [31] where a pricing algorithm based mainly on trinomial trees was developed, and [29] where Monte Carlo methods were used to compute the option prices and optimal exercise boundaries. Contracts analysed in those papers included constraints on the volumes of traded commodities and the number of trades at each exercise date. Lately a wide range of diverse numerical methods has been employed to deal with general models for both the dynamics of the underlying commodity price (including for example jump processes) and the structure of the options (including regime switching opportunities). Some of those results may be found for instance in [4], [5], [27], [71] amongst others. To the best of our knowledge a first theoretical analysis of the optimal stopping theory underpinning swing contracts was given in [11] and it was based on martingale methods and Snell envelope. Later on [36] provided a systematic study of martingale methods for multiple stopping time problems for càdlàg positive processes. A characterisation of the value function of multiple stopping time problems in terms of excessive functions was given in [10] in the case of one-dimensional linear diffusions. Duality methods instead were studied from both theoretical and numerical point of view in [48], [1] and [7], amongst others.

In the Markovian setting variational methods and BSDEs techniques have been widely employed. In [8] for instance the HJB equation for a swing option with volume constraint is analysed both theoretically and numerically. Variational inequalities for multiple optimal stopping problems have been studied for instance in [46] in the (slightly different) context of evaluation of stock options and in [44] in an extension of results of [11] to one-dimensional diffusions with jumps. A study of BSDEs with jumps related to swing options may be found instead in [9].

Numerical characterisations of the optimal exercise boundaries of swing options are available in a variety of settings in both the perpetual case and the finite maturity one (cf. for instance [8], [11], [29] and [46]). On the other hand despite the general interest towards theoretical aspects of swing options it seems that the problem of characterising analytically optimal exercise boundaries has not been thoroughly studied yet. For perpetual options such boundaries have been provided for a put payoff in the Black & Scholes framework by [11], whereas more general dynamics and payoffs were studied in [10]. For the case of finite maturity instead the problem is still widely open and the question of finding analytical equations for the optimal boundaries remains unanswered. In this paper we address this issue in a setting described below.

We consider the case of a swing option with a put payoff, finite maturity $T > 0$, strike price $K > 0$ and $n \in \mathbb{N}$ exercise rights. The underlying price follows a geometric Brownian motion according to the Black & Scholes model and we consider an option whose structure was described in [29] and [31]. In particular the holder can only exercise one right per time and must wait a so-called refracting period of length $\delta > 0$ between two consecutive exercises. If the holder has not used the first of the n rights by time $T-(n-1)\delta$ then at that time she must exercise it and remains with a portfolio of $n-1$ European put options with different maturities up to time T. This corresponds to the case of a swing option with a constrained minimum number of exercise rights equal to n.

We first perform using probabilistic arguments an analysis of the price function of the option with $n = 2$ and prove an existence of two continuous, monotone, bounded optimal stopping boundaries denoted $b^{(2)}$ and $c^{(2)}$ such that $b^{(2)}(t) < K < c^{(2)}(t)$ for $t \in [0, T-\delta)$. It turns out that it is optimal to exercise the first right of the swing option as soon as the underlying price falls below $b^{(2)}$ or exceeds $c^{(2)}$. We provide an early exercise premium (EEP) representation for the price of the option in terms of the optimal stopping boundaries and adapting arguments of [18] (see also [53]) we show that such boundaries uniquely solve a system of coupled integral equations of Volterra type. Finally we extend the result to the general case of n exercise rights by an induction.

The paper is organised as follows. In Section 4.2 we introduce the financial problem and provide its mathematical formulation. Section 4.3 is devoted to the detailed analysis of the case of a swing option with two exercise rights. In Section 4.4 we extend the results of Section 4.3 to the case of swing options with arbitrary many rights. The paper is completed by a technical appendix.

4.2. Formulation of the swing put option problem

Here we formulate the valuation problem for a swing put option on an underlying asset with price X as a sequence of optimal stopping problems defined recursively. On a complete probability space $(\Omega, \mathcal{F}, \mathsf{P})$ we consider the Black and Scholes model for the asset price dynamics

$$dX_t = rX_t\,dt + \sigma X_t\,dB_t\,, \quad X_0 = x > 0 \tag{4.1}$$

where B is a standard Brownian motion started at zero, $r > 0$ is the interest rate, and $\sigma > 0$ is the volatility coefficient. We denote by $(\mathcal{F}_t)_{t\geq 0}$ the natural filtration generated by $(B_t)_{t\geq 0}$ completed with the P-null sets and by $(X_t^x)_{t\geq 0}$ the unique strong solution to (4.1). It is well known that for any $t \geq 0$ and $x > 0$ it holds

$$X_t^x = x\,e^{\sigma B_t + (r-\frac{1}{2}\sigma^2)t}. \tag{4.2}$$

Now for this model let us $V^{(n)}$ denote the price of a swing option with a put payoff $(K-x)^+$, strike $K > 0$, maturity $T > 0$, n exercise rights and refracting period $\delta > 0$. The latter is the minimum amount of time that the option holder must wait between two consecutive exercises of the option, therefore we have that $T \geq (n-1)\delta$.

An important parameter of the option is the minimal number n_0 of rights the holder must exercise. Due to existence of the refracting period δ and finite horizon T the holder may desire to miss some of his rights in order to benefit from a potential better future exercises. Therefore the price of the option $V^{(n)}$ is decreasing with respect to $1 \leq n_0 \leq n$. In our work we consider the case $n_0 = n$ and hence the holder must exercise all rights up to T and the structure of the contract is specified according to examples analysed for instance in [29, Sec. 3] and [31, Sec. 2.3.1] and it is the following:

i) if at time $t = T-(n-1)\delta$ the first right has not been exercised yet the holder gets the payoff of a put option and remains with a portfolio of $n-1$ European put options with maturity dates

$$\big\{T-(n-2)\delta\,, T-(n-3)\delta\,, \dots\, T-\delta\,, T\big\}$$

ii) if the holder exercises the first right at any time $t < T-(n-1)\delta$ then he receives a put payoff and uses remaining $n-1$ rights after a refracting period δ. That means that after an inaction period of length δ the holder has again a swing option with $n-1$ exercise rights.

We would like to remark that this formulation differs from on for instance in [11] where $n_0 = 1$ and at time $t = T-(n-1)\delta$ the holder can decide to whether exercise the first right or not. There if the right is given up the holder remains with a swing put option with $n-1$ exercise rights available immediately. The value of the contract in [11] is larger than the one of that considered here since the holder in [11] does not lose the early exercise opportunity of future rights beyond time $T-(n-1)\delta$ if the first right is not used. In our case instead (as in e.g. [29] and [31]) the holder has a binding constraint of making a decision prior to time $T-(n-1)\delta$ in order to be entitled to use future early exercise rights.

We now define the payoff of immediate exercise of the first right and the option's value recursively (see e.g. [11] and [36] for a full justification and will not repeat it here). Trivially for $n = 0$ the value $V^{(0)}$ is the value of a European put option with maturity $T > 0$ and strike price $K > 0$. Similarly for $n = 1$ the swing contract reduces to a standard American put option, again with maturity $T > 0$ and strike price $K > 0$. We recall for completeness that in our Markovian framework if at time $t \in [0, T]$ the underlying asset price is $x > 0$ the value of the European and American put options are respectively

$$V^{(0)}(t, x) = \mathsf{E}\Big[e^{-r(T-t)}(K - X^x_{T-t})^+\Big] \tag{4.3}$$

and

$$V^{(1)}(t, x) = \sup_{0 \le \tau \le T-t} \mathsf{E}\Big[e^{-r\tau}(K - X^x_\tau)^+\Big]. \tag{4.4}$$

Remark 4.2.1. *Notice that in order to take into account for different maturities one should specify them in the definition of the value function, i.e. for instance denoting $V^{(n)}(t, x; T)$, $n = 0, 1$, for the European/American put option with maturity T. However this notation is unnecessarily complex since what effectively matters in pricing put options is the time-to-maturity. In fact for fixed $x \in (0, \infty)$ and $\lambda > 0$ the value at time $t \in [0, T]$ of a European/American put option with maturity T is the same as the value of the option with maturity $T + \lambda$ but considered at time $t + \lambda$, i.e. $V^{(n)}(t, x; T) = V^{(n)}(t + \lambda, x; T + \lambda)$, $n = 0, 1$. In this work we mainly deal with a single maturity T and simplify our notation by setting $V^{(n)}(t, x) := V^{(n)}(t, x; T)$.*

According to *i)* above the early exercise feature of the contract can only be exploited by the holder until $t < T-(n-1)\delta$. In particular using Remark 4.2.1 we observe that the option price is given by

$$\begin{aligned} V^{(n)}(T-(n-1)\delta, x) =& (K-x)^+ + \sum_{j=0}^{n-2} V^{(0)}(t, x; T-j\delta) \\ =& (K-x)^+ + \sum_{j=0}^{n-2} V^{(0)}(t+j\delta, x) \end{aligned} \tag{4.5}$$

for every $x > 0$. We now denote by $G^{(n)}$ the payoff of immediate exercise of the first right and it equals

$$G^{(n)}(t, x) = (K-x)^+ + R^{(n)}(t, x) \tag{4.6}$$

for any $t \in \big[0, T-(n-1)\delta\big]$ and $x > 0$ and where

$$R^{(n)}(t, x) = \mathsf{E}\Big[e^{-r\delta} V^{(n-1)}(t+\delta, X_\delta^x)\Big] \tag{4.7}$$

is the expected discounted value of a swing option with $n-1$ exercise rights, available to the option holder after the refracting time δ and it accounts for the opportunity of future exercises. It is easy to verify that $V^{(n)}(T-(n-1)\delta, x) = G^{(n)}(T-(n-1)\delta, x)$ for all $x > 0$.

The option holder aims to maximise the payoff of the swing option by using its multiple early exercise rights. The above discussion regarding $i)$ and $ii)$ shows that the choice of the first early exercise is crucial to determine the successive structure of the contract. Pricing the option and finding the optimal multiple-exercise strategy then reduces to solving the optimal stopping problem

$$V^{(n)}(t, x) = \sup_{0 \le \tau \le T-(n-1)\delta} \mathsf{E}\Big[e^{-r\tau} G^{(n)}(t+\tau, X_\tau^x)\Big] \tag{4.8}$$

for $t \in [0, T-(n-1)\delta]$ and $x > 0$. Since $G^{(n)}$ is defined recursively through $V^{(n-1)}$, it turns out that in order to price a swing option with n exercise rights one must first price the options with $2, 3, \dots\ n-1$ rights. It is then natural to begin with analysing the simplest case of $n = 2$ and this will be accomplished in the next section.

In our study we will rely on known results about the American put option problem (see e.g. [58, Sec. 25]) and we define following sets

$$C^{(1)} := \{ (t,x) \in [0,T) \times (0,\infty) : V^{(1)}(t,x) > (K-x)^+ \} \tag{4.9}$$

$$D^{(1)} := \{ (t,x) \in [0,T) \times (0,\infty) : V^{(1)}(t,x) = (K-x)^+ \} \tag{4.10}$$

and recall that the first entry time of X into $D^{(1)}$ is an optimal stopping time in (4.3). Moreover, it is well known that there exists an unique continuous boundary $b^{(1)}$ separating $C^{(1)}$ from $D^{(1)}$ and such that $0 < b^{(1)}(t) < K$ for $t \in [0,T)$. The stopping time

$$\tau_{b^{(1)}} = \inf \left\{ 0 \le s \le T-t \,:\, X_s^x \le b^{(1)}(t+s) \right\} \tag{4.11}$$

is therefore optimal in (4.3). It is also well known that $V^{(1)} \in C^{1,2}$ in $C^{(1)}$ and it solves there following PDE

$$V_t^{(1)} + I\!L_X V^{(1)} - rV^{(1)} = 0 \tag{4.12}$$

where $I\!L_X = rxd/dx + (\sigma^2/2)x^2 d^2/dx^2$ the infinitesimal generator of X.

The map $x \mapsto V_x^{(1)}(t,x)$ is continuous across the optimal stopping boundary $b^{(1)}$ for all $t \in [0,T)$ (so-called *smooth-fit* condition) and $\left|V_x\right| \le 1$ on $[0,T] \times (0,\infty)$. A change-of-variable formula ([52]) then gives following representation

$$e^{-rs} V^{(1)}(t+s, X_s^x) = V^{(1)}(t,x) - rK \int_0^s e^{-ru} I(X_u^x < b^{(1)}(t+u))du + M_s \tag{4.13}$$

for $s \in [0, T-t]$ and $x > 0$ where $(M_s)_{s \in [0,T-t]}$ is a continuous martingale and where we have used that in $D^{(1)}$

$$V_t^{(1)} + I\!L_X V^{(1)} - rV^{(1)} = -rK. \tag{4.14}$$

4.3. Free-boundary analysis of the swing option with $n = 2$

In this section we study the optimal stopping problem associated to a swing option with two exercise rights and optimal stopping strategy for the first of them. Our main aim is to provide an early-exercise premium (EEP) representation formula for the value function $V^{(2)}$ and characterisation of its optimal stopping region.

To simplify notation we denote $T_\delta = T-\delta$, $G = G^{(2)}$ and $R = R^{(2)}$, then for $t \in [0, T_\delta]$ and $x > 0$ we have

$$G(t,x) = (K-x)^+ + R(t,x) = (K-x)^+ + e^{-r\delta}\mathsf{E}V^{(1)}(t+\delta, X_\delta^x) \tag{4.15}$$

and

$$V^{(2)}(t,x) = \sup_{0\le\tau\le T_\delta - t} \mathsf{E}e^{-r\tau}G(t+\tau, X_\tau^x). \tag{4.16}$$

1. We now provide the expression for the function H defined as

$$H(t,x) := (G_t + I\!\!L_X G - rG)(t,x) \tag{4.17}$$

for $t \in [0, T_\delta]$ and $x \in (0,K) \cup (K,\infty)$. By straightforward calculations and using that $R(t,x) = V^{(1)}(t,x) - rK\int_0^\delta e^{-rs}\mathsf{P}\big(X_s^x \le b^{(1)}(t+s)\big)ds$ we have that

$$\big(R_t + I\!\!L_X R - rR\big)(t,x) = -rKf(t,x) \tag{4.18}$$

for $(t,x) \in (0,T_\delta) \times (0,\infty)$ and

$$H(t,x) = -rK\big(I(x<K) + f(t,x)\big) \tag{4.19}$$

for $(t,x) \in (0,T_\delta) \times \big[(0,K)\cup(K,\infty)\big]$ and where

$$f(t,x) := e^{-r\delta}\mathsf{P}\big(X_\delta^x \le b^{(1)}(t+\delta)\big) \tag{4.20}$$

for $(t,x) \in (0,T_\delta) \times (0,\infty)$. A key feature of H that we will make of use in the rest of our analysis is that

$$t \mapsto H(t,x) \;\text{ is decreasing for all } x>0$$

since $t \mapsto b^{(1)}(t)$ is increasing.

2. Now applying of Ito-Tanaka's formula and optional sampling theorem we have that

$$\begin{aligned}\mathsf{E}e^{-r\tau}G(t+\tau, X_\tau^x) =& G(t,x) + \mathsf{E}\int_0^\tau e^{-ru}H(t+u, X_u^x)du \\ &+ \frac{1}{2}\mathsf{E}\int_0^\tau e^{-ru}d\ell_u^K(X^x)\end{aligned} \tag{4.21}$$

for $(t,x) \in [0,T_\delta] \times (0,\infty)$ and any stopping time $\tau \in [0, T_\delta - t]$ where $\big(\ell^K_u(X^x)\big)_{u\geq 0}$ is the local time process of X^x at level K and we have used that $H(t{+}u, X^x_u)I(X^x_u \neq K) = H(t{+}u, X^x_u)$ P-a.s. for all $u \in [0, T_\delta - t]$.

The continuation and stopping sets of problem (4.16) are given respectively by

$$C^{(2)} := \{\, (t,x) \in [0,T_\delta)\times(0,\infty) : V^{(2)}(t,x) > G(t,x) \,\} \tag{4.22}$$

$$D^{(2)} := \{\, (t,x) \in [0,T_\delta)\times(0,\infty) : V^{(2)}(t,x) = G(t,x) \,\}. \tag{4.23}$$

Since the gain function G is continuous on $[0,T_\delta] \times (0,\infty)$ the standard arguments of optimal stopping stopping theory (see. e.g. [58, Corollary 2.9, Sec. 2]) guarantees that the optimal stopping time in (4.16) is given by

$$\tau^* = \inf \,\{\; 0 \leq s \leq T_\delta{-}t : (t{+}s, X^x_s) \in D^{(2)} \;\}. \tag{4.24}$$

3. Below we prove the continuity of $V^{(2)}$.

Proposition 4.3.1. *The value function $V^{(2)}$ is continuous on $[0,T_\delta]\times(0,\infty)$. In particular $x \mapsto V^{(2)}(t,x)$ is convex and Lipschitz continuous with constant independent of $t \in [0,T_\delta]$.*

Proof. (i) It follows from convexity of $x \mapsto V^{(1)}(t,x)$ and (4.15) that the map $x \mapsto G(t,x)$ is convex on $(0,\infty)$ for every $t \in [0,T_\delta]$ fixed. Hence the function $x \mapsto V^{(2)}(t,x)$ is convex on $(0,\infty)$ as well and thus $x \mapsto V^{(2)}(t,x)$ is continuous on $(0,\infty)$ for every given and fixed $t \in [0,T_\delta]$. Moreover $x \mapsto G(t,x)$ is also decreasing and Lipschitz uniformly with respect to $t \in [0,T_\delta]$. Indeed, since $-1 \leq V^{(1)}_x \leq 0$ and $x \mapsto (K-x)^+$ is Lipschitz, we have

$$\begin{aligned} 0 \leq G(t,x_1) - G(t,x_2) \leq &|x_2 - x_1| + e^{-r\delta}\mathsf{E}\big|X^{x_2}_\delta - X^{x_1}_\delta\big| \\ =&\big(x_2 - x_1\big)\big(1 + \mathsf{E}e^{-r\delta}X^1_\delta\big) = 2\big(x_2 - x_1\big) \end{aligned} \tag{4.25}$$

for $t \in [0,T_\delta]$ and $0 < x_1 < x_2 < \infty$. It then follows by (4.2), (4.25) and optional sampling theorem that

$$0 \leq V^{(2)}(t,x_1) - V^{(2)}(t,x_2) \leq \sup_{0\leq\tau\leq T_\delta - t} \mathsf{E}e^{-r\tau}\big[G(t{+}\tau, X^{x_1}_\tau) - G(t{+}\tau, X^{x_2}_\tau)\big] \tag{4.26}$$

$$\leq 2(x_2 - x_1) \sup_{0\leq\tau\leq T_\delta - t} \mathsf{E}e^{-r\tau}X^1_\tau = 2(x_2 - x_1)$$

for $t \in [0, T_\delta]$ and $0 < x_1 < x_2 < \infty$ and hence $x \mapsto V^{(2)}(t, x)$ is Lipschitz continuous with constant 2.

(ii) It remains to prove that $t \mapsto V^{(2)}(t, x)$ is continuous on $[0, T_\delta]$ for $x \in (0, \infty)$. We first notice that $t \mapsto G(t, x)$ is decreasing for $x > 0$ fixed since $t \mapsto V^{(1)}(t, x)$ is such and therefore $t \mapsto V^{(2)}(t, x)$ is decreasing as well by simple comparison. Now let us take $0 \leq t_1 < t_2 \leq T_\delta$ and $x \in (0, \infty)$, denote $\tau_1 = \tau^*(t_1, x)$ optimal for $V^{(1)}(t_1, x)$ and set $\tau_2 := \tau_1 \wedge (T_\delta - t_2)$. Then using (4.18), the fact that $\tau_1 \geq \tau_2$ and the inequality $(K-y)^+ - (K-z)^+ \leq (z-y)^+$ for $y, z \in I\!R$, we find

$$\begin{aligned}
0 &\leq V^{(2)}(t_1, x) - V^{(2)}(t_2, x) \qquad (4.27)\\
&\leq \mathsf{E}e^{-r\tau_1}G(t_1+\tau_1, X^x_{\tau_1}) - \mathsf{E}e^{-r\tau_2}G(t_2+\tau_2, X^x_{\tau_2})\\
&\leq \mathsf{E}e^{-r\tau_1}(X^x_{\tau_2} - X^x_{\tau_1})^+ + \mathsf{E}\Big[e^{-r\tau_1}R(t_1+\tau_1, X^x_{\tau_1}) - e^{-r\tau_1}R(t_2+\tau_2, X^x_{\tau_2})\Big]\\
&\leq \mathsf{E}e^{-r\tau_1}(X^x_{\tau_2} - X^x_{\tau_1})^+ + R(t_1, x) - R(t_2, x)\\
&\quad - rK\mathsf{E}\int_0^{\tau_2} e^{-rs}\big[f(t_1+s, X^x_s) - f(t_2+s, X^x_s)\big]ds.
\end{aligned}$$

Taking now $t_2 - t_1 \to 0$ one has that the first term of the last expression in (4.27) goes to zero by standard arguments (see (25.2.12)–(25.2.14), p.381 of [58]), the second one goes to zero by continuity of $V^{(1)}$ and $b^{(1)}$ and the third term goes to zero by dominated convergence theorem and continuity of f.

The continuity of $V^{(2)}$ on $[0, T_\delta] \times (0, \infty)$ follows by combining *(i)* and *(ii)* above.

□

4. We now notice that since $V^{(2)}$ and G are continuous we have that $C^{(2)}$ is an open set and $D^{(2)}$ is a closed set. In the next proposition we obtain an initial insight on the structure of the set $D^{(2)}$ in terms of the set $D^{(1)}$ (see (4.10)).

Proposition 4.3.2. *The restriction to $[0, T_\delta]$ of the stopping set $D^{(1)}$ is contained in the stopping set $D^{(2)}$, i.e.*

$$D^{(1)} \cap \big([0, T_\delta] \times (0, \infty)\big) \subseteq D^{(2)}. \qquad (4.28)$$

Proof. Take any point $(t,x) \in [0,T_\delta] \times (0,\infty)$ and let $\tau = \tau^*(t,x)$ denote the optimal stopping time for $V^{(2)}(t,x)$, then by using (4.15), (4.18) and recalling that $f \geq 0$ we have

$$V^{(2)}(t,x) - V^{(1)}(t,x) \leq \mathsf{E} e^{-r\tau} G(t{+}\tau, X_\tau^x) - \mathsf{E} e^{-r\tau}(K - X_\tau^x)^+ \tag{4.29}$$
$$= \mathsf{E} e^{-r\tau} R(t{+}\tau, X_\tau^x)$$
$$= R(t,x) - rK\mathsf{E}\int_0^\tau e^{-rs} f(t{+}s, X_s^x)ds \leq G(t,x) - (K{-}x)^+.$$

It then follows that for any $(t,x) \in D^{(1)}$ with $t \in [0,T_\delta]$, i.e. such that $V^{(1)}(t,x) = (K{-}x)^+$, it must be $V^{(2)}(t,x) = G(t,x)$ and thus $(t,x) \in D^{(2)}$. □

We now define the t-sections of the continuation and stopping sets of problem (4.16) by

$$C_t^{(2)} := \{\, x \in (0,\infty) : V^{(2)}(t,x) > G(t,x) \,\} \tag{4.30}$$

$$D_t^{(2)} := \{\, x \in (0,\infty) : V^{(2)}(t,x) = G(t,x) \,\} \tag{4.31}$$

for $t \in [0,T_\delta]$.

Proposition 4.3.3. *We have that $C_{t_2}^{(2)} \subseteq C_{t_1}^{(2)}$ (equivalently $D_{t_2}^{(2)} \supseteq D_{t_1}^{(2)}$) for any $0 \leq t_1 < t_2 \leq T_\delta$, i.e. the family $\{C_t^{(2)}\, ,\, t \in [0,T_\delta]\}$ is decreasing in t (equivalently the family $\{D_t^{(2)}\, ,\, t \in [0,T_\delta]\}$ is increasing in t).*

Proof. Fix $0 \leq t_1 < t_2 < T_\delta$ and $x \in (0,\infty)$, and set $\tau = \tau^*(t_2,x)$ optimal for $V^{(2)}(t_2,x)$. Then we have

$$V^{(2)}(t_1,x) - V^{(2)}(t_2,x) \tag{4.32}$$
$$\geq \mathsf{E} e^{-r\tau} G(t_1{+}\tau, X_\tau^x) - \mathsf{E} e^{-r\tau} G(t_2{+}\tau, X_\tau^x)$$
$$= \mathsf{E} e^{-r\tau}\big(R(t_1{+}\tau, X_\tau^x) - R(t_2{+}\tau, X_\tau^x)\big)$$
$$= R(t_1,x) - R(t_2,x) - rK\mathsf{E}\int_0^\tau e^{-rs}\big[f(t_1{+}s, X_s^x) - f(t_2{+}s, X_s^x)\big]ds$$
$$\geq R(t_1,x) - R(t_2,x) = G(t_1,x) - G(t_2,x)$$

where in the last inequality we used that $t \mapsto f(t,x)$ is increasing on $[0,T_\delta]$. It follows from (4.32) that $(t_2,x) \in C^{(2)}$ implies $(t_1,x) \in C^{(2)}$ and thus the proof is complete. □

5. So far the analysis of the swing option has produced results which are somehow similar to those found in the standard American put option problem. In what follows instead we will establish that the structure of $C^{(2)}$ is radically different from the one of $C^{(1)}$. The optimal exercise of the swing option then requires to take into account for features that were not observed in the case of American put options. In the rest of the paper we will require the next simple result that is obtained by an application of Itô-Tanaka formula, optional sampling theorem and observing that the process X has independent increments.

Lemma 4.3.4. *For any $\sigma \le \tau$ stopping times in $[0, T_\delta]$ we have*

$$\begin{aligned} &\mathsf{E}\Big[\int_\sigma^\tau e^{-rt} dL_t^K(X^x)\Big|\mathcal{F}_\sigma\Big] \qquad (4.33)\\ &= \mathsf{E}\Big[e^{-r\tau}\big|X_\tau^x - K\big|\Big|\mathcal{F}_\sigma\Big] - e^{-r\sigma}\big|X_\sigma^x - K\big| - rK\mathsf{E}\Big[\int_\sigma^\tau e^{-rt} sign(X_t^x - K)dt\Big|\mathcal{F}_\sigma\Big]. \end{aligned}$$

Now we characterise the structure of the continuation region $C^{(2)}$.

Theorem 4.3.5. *There exist two functions $b^{(2)}$, $c^{(2)}$: $[0, T_\delta] \to (0,\infty]$ such that $0 < b^{(2)}(t) < K < c^{(2)}(t) \le \infty$ and $C_t^{(2)} = (b^{(2)}(t), c^{(2)}(t))$ for all $t \in [0, T_\delta]$ (See Figure 4.1). Moreover $b^{(2)}(t) \ge b^{(1)}(t)$ for all $t \in [0, T_\delta]$ (See Figure 4.2), $t \mapsto b^{(2)}(t)$ is increasing and $t \mapsto c^{(2)}(t)$ is decreasing on $[0, T_\delta]$ with*

$$\lim_{t\uparrow T_\delta} b^{(2)}(t) = \lim_{t\uparrow T_\delta} c^{(2)}(t) = K. \qquad (4.34)$$

Proof. The proof of existence is provided in 3 steps.

(i) First we show that it is not optimal to stop at $x = K$. To accomplish that we use arguments inspired by [70]. Let us fix $\varepsilon > 0$, set $\tau_\varepsilon = \inf\{t \ge 0 : X_t^K \in (K-\varepsilon, K+\varepsilon)\}$, take $t \in [0, T_\delta]$ and denote $s = T_\delta - t$ then by (4.19) and (4.21) we have that

$$\begin{aligned} &V^{(2)}(t, K) - G(t, K) \qquad (4.35)\\ &\ge \mathsf{E}e^{-r\tau_\varepsilon\wedge s}G(t+\tau_\varepsilon \wedge s, X^K_{\tau_\varepsilon\wedge s}) - G(t, K)\\ &= \frac{1}{2}\mathsf{E}\int_0^{\tau_\varepsilon\wedge s} e^{-ru} d\ell_u^K(X^K) - rK\mathsf{E}\int_0^{\tau_\varepsilon\wedge s} e^{-ru}\big(I(X_u^K \le K) + f(t+u, X_u^K)\big)du\\ &\ge \frac{1}{2}\mathsf{E}\int_0^{\tau_\varepsilon\wedge s} e^{-ru} d\ell_u^K(X^K) - C_1\mathsf{E}(\tau_\varepsilon \wedge s) \end{aligned}$$

for some constant $C_1 > 0$. The integral involving the local time can be estimated by using Itô-Tanaka's formula as follows

$$\begin{aligned}
&\mathsf{E}\int_0^{\tau_\varepsilon\wedge s} e^{-ru} d\ell_u^K(X^K) \qquad (4.36)\\
&\quad = \mathsf{E} e^{-r\tau_\varepsilon\wedge s}|X^K_{\tau_\varepsilon\wedge s} - K| - rK\,\mathsf{E}\int_0^{\tau_\varepsilon\wedge s} e^{-ru}\text{sign}(X_u^K - K)du\\
&\quad \geq \mathsf{E} e^{-r\tau_\varepsilon\wedge s}|X^K_{\tau_\varepsilon\wedge s} - K| - C_2\,\mathsf{E}(\tau_\varepsilon\wedge s)
\end{aligned}$$

for some constant $C_2 = C_2(\varepsilon) > 0$ where we used that the process X^K is bounded prior to τ_ε. Since $e^{-r(\tau_\varepsilon\wedge s)}|X^K_{\tau_\varepsilon\wedge s} - K| \leq \varepsilon$ it is not hard to see that for any $0 < p < 1$ we have

$$e^{-r(\tau_\varepsilon\wedge s)}|X^K_{\tau_\varepsilon\wedge s} - K| \geq e^{-rp(\tau_\varepsilon\wedge s)}\frac{|X^K_{\tau_\varepsilon\wedge s} - K|^p}{\varepsilon^p}e^{-r(\tau_\varepsilon\wedge s)}|X^K_{\tau_\varepsilon\wedge s} - K|$$

then by taking the expectation and using the integral version of (4.1) we get

$$\begin{aligned}
\mathsf{E} e^{-r\tau_\varepsilon\wedge s}|X^K_{\tau_\varepsilon\wedge s} - K| &\geq \frac{1}{\varepsilon^p}\mathsf{E}\Big|e^{-r\tau_\varepsilon\wedge s}(X^K_{\tau_\varepsilon\wedge s} - K)\Big|^{1+p} \qquad (4.37)\\
&= \frac{1}{\varepsilon^p}\mathsf{E}\Big|rK\int_0^{\tau_\varepsilon\wedge s} e^{-ru}du + \sigma\int_0^{\tau_\varepsilon\wedge s} e^{-ru}X_u^K dB_u\Big|^{1+p}.
\end{aligned}$$

We now use the standard inequality $|a+b|^{p+1} \geq \frac{1}{2^{p+1}}|a|^{p+1} - |b|^{p+1}$ for any $a, b \in I\!R$ (see e.g. Ex. 5 in [37, Ch. 8, Sec. 50, p. 83]) and Burkholder-Davis-Gundy (BDG) inequality (see e.g. [58, p. 63]) to obtain

$$\begin{aligned}
\mathsf{E} e^{-r\tau_\varepsilon\wedge s}|X^K_{\tau_\varepsilon\wedge s} - K| &\geq \frac{1}{\varepsilon^p 2^{p+1}}\mathsf{E}\Big|\sigma\int_0^{\tau_\varepsilon\wedge s} e^{-ru}X_u^K dB_u\Big|^{1+p} \qquad (4.38)\\
&\quad - \frac{1}{\varepsilon^p}\mathsf{E}\Big|rK\int_0^{\tau_\varepsilon\wedge s} e^{-ru}du\Big|^{1+p}\\
&\geq C_4\,\mathsf{E}\Big|\sigma^2\int_0^{\tau_\varepsilon\wedge s} e^{-2ru}(X_u^K)^2 du\Big|^{(1+p)/2} - C_3\mathsf{E}(\tau_\varepsilon\wedge s)^{1+p}\\
&\geq C_4\,C_5\,\mathsf{E}(\tau_\varepsilon\wedge s)^{(1+p)/2} - C_3\mathsf{E}(\tau_\varepsilon\wedge s)^{1+p}
\end{aligned}$$

for some constants $C_3 = C_3(\varepsilon, p)$, $C_4 = C_4(\varepsilon, p)$, $C_5 = C_5(\varepsilon . p) > 0$. Since we are interested in the limit as $T_\delta - t \to 0$ we take $s < 1$, and combining (4.35), (4.36) and (4.38) we get

$$V^{(2)}(t,K) - G(t,K) \geq C_4\,C_5\,\mathsf{E}(\tau_\varepsilon\wedge s)^{(1+p)/2} - (C_1 + C_2 + C_3)\mathsf{E}(\tau_\varepsilon\wedge s) \qquad (4.39)$$

for any $t \in [0, T_\delta)$ such that $s = T_\delta - t < 1$. Since $p+1 < 2$ it follows from (4.39) by letting $s \downarrow 0$ that there exists $t^* < T_\delta$ such that $V^{(2)}(t, K) > G(t, K)$ for all $t \in (t^*, T_\delta)$. Therefore $(t, K) \in C_t^{(2)}$ for all $t \in (t^*, T_\delta)$ and since $t \mapsto C_t^{(2)}$ is decreasing (see Proposition 4.3.3) this implies $(t, K) \in C_t^{(2)}$ for all $t \in [0, T_\delta)$, i.e. it is never optimal to stop when the underlying price X equals the strike K.

(ii) Now we study the portion of $D^{(2)}$ above the strike K and show that it is not empty unlike in well known American put option problem. For that we prove by contradiction and we assume that there are no points in the stopping region above K. Then we take $\varepsilon > 0$, $t \in [0, T_\delta)$ and $x \geq K + 2\varepsilon$, denote $\tau = \tau(t, x)$ the optimal stopping time for $V^{(2)}(t, x)$, set $s = T_\delta - t$ and define $\sigma_\varepsilon := \inf\{u \geq 0 \, : \, X_u^x \leq K{+}\varepsilon\}$. Then by (4.19) and (4.21) we get

$$
\begin{aligned}
&V^{(2)}(t,x) - G(t,x) \qquad (4.40)\\
&= \mathsf{E} e^{-r\tau} G(t{+}\tau, X_\tau^x) - G(t,x)\\
&\leq - rK\mathsf{E} \int_0^\tau e^{-ru} f(t{+}u, X_u^x) du + \frac{1}{2} \mathsf{E} \int_0^\tau e^{-ru} d\ell_u^K(X^x)\\
&\leq - rK\mathsf{E}\Big[I(\tau < s) \int_0^\tau e^{-ru} f(t{+}u, X_u^x) du \Big]\\
&\quad - rK\mathsf{E}\Big[I(\tau = s) \int_0^s e^{-ru} f(t{+}u, X_u^x) du \Big] + \frac{1}{2} \mathsf{E}\Big[I(\sigma_\varepsilon < \tau) \int_{\sigma_\varepsilon}^\tau e^{-ru} d\ell_u^K(X^x) \Big]\\
&= - rK\mathsf{E}\Big[\int_0^s e^{-ru} f(t{+}u, X_u^x) du \Big] + rK\mathsf{E}\Big[I(\tau < s) \int_\tau^s e^{-ru} f(t{+}u, X_u^x) du \Big]\\
&\quad + \frac{1}{2} \mathsf{E}\Big[I(\sigma_\varepsilon < \tau) \int_{\sigma_\varepsilon}^\tau e^{-ru} d\ell_u^K(X^x) \Big]
\end{aligned}
$$

where we have used the fact that for $u \leq \sigma_\varepsilon$ the local time $\ell_u^K(X^x)$ is zero. Since we are assuming that it is never optimal to stop above K then it must be $\big\{\tau < s\big\} \subset \big\{\sigma_\varepsilon < s\big\}$. Obviously we also have $\big\{\sigma_\varepsilon < \tau\big\} \subset \big\{\sigma_\varepsilon < s\big\}$ and hence

$$
\begin{aligned}
&V^{(2)}(t,x) - G(t,x) \qquad (4.41)\\
&\leq - rK\mathsf{E}\Big[\int_0^s e^{-ru} f(t{+}u, X_u^x) du \Big]\\
&\quad + \mathsf{E}\Big[I(\sigma_\varepsilon < s) \Big(rK \int_\tau^s e^{-ru} f(t{+}u, X_u^x) du + \frac{1}{2} \int_{\sigma_\varepsilon}^s e^{-ru} d\ell_u^K(X^x) \Big) \Big]\\
&\leq - rK\mathsf{E}\Big[\int_0^s e^{-ru} f(t{+}u, X_u^x) du \Big]
\end{aligned}
$$

$$+\,rKs\mathsf{P}(\sigma_\varepsilon < s) + \frac{1}{2}\mathsf{E}\left[I(\sigma_\varepsilon < s)\mathsf{E}\Big(\int_{\sigma_\varepsilon}^{\sigma_\varepsilon\vee s} e^{-ru} d\ell^K_u(X^x)\Big|\mathcal{F}_{\sigma_\varepsilon}\Big)\right]$$

where we have used $0 \le f \le 1$ and the fact that $I(\sigma_\varepsilon < s)$ is $\mathcal{F}_{\sigma_\varepsilon}$-measurable. From Lemma 4.3.4 with $\sigma = \sigma_\varepsilon$ and $\tau = \sigma_\varepsilon \vee s$ and by the martingale property of $(e^{-rt}X^x_t)_{t\ge 0}$ we get

$$\begin{aligned}\mathsf{E}\Big[\int_{\sigma_\varepsilon}^{\sigma_\varepsilon\vee s} e^{-ru} d\ell^K_u(X^x)\Big|\mathcal{F}_{\sigma_\varepsilon}\Big] \le& 2K + \mathsf{E}\big[e^{-r(\sigma_\varepsilon\vee s)}X^x_{\sigma_\varepsilon\vee s}\big|\mathcal{F}_{\sigma_\varepsilon}\big] - e^{-r\sigma_\varepsilon}X^x_{\sigma_\varepsilon} \\ &+ rK\mathsf{E}\Big[I(\sigma_\varepsilon < s)\int_{\sigma_\varepsilon}^{s} e^{-rt}dt\Big]du \le 3K.\end{aligned} \tag{4.42}$$

Combining (4.41) and (4.42) we obtain

$$\begin{aligned}V^{(2)}(t,x) - G(t,x) \le& -rK\mathsf{E}\Big[\int_0^s e^{-ru} f(t+u, X^x_u)du\Big] \\ &+ K\big(\frac{3}{2} + rs\big)\mathsf{P}(\sigma_\varepsilon < s).\end{aligned} \tag{4.43}$$

To estimate $\mathsf{P}(\sigma_\varepsilon < s)$ we set $\alpha := \ln\big(\frac{x}{K+\varepsilon}\big)$, $Y_t := \sigma B_t + (r - \sigma^2/2)t$ and $Z_t := -\sigma B_t + c\,t$ with $c := r + \sigma^2/2$. Notice that $Y_t \ge -Z_t$ for $t \in [0, T_\delta]$ and hence

$$\begin{aligned}\mathsf{P}(\sigma_\varepsilon < s) =& \mathsf{P}\Big(\inf_{0\le u\le s} X^x_u \le K+\varepsilon\Big) = \mathsf{P}\Big(\inf_{0\le u\le s} Y_u \le -\alpha\Big) \\ \le& \mathsf{P}\Big(\inf_{0\le u\le s} -Z_u \le -\alpha\Big) = \mathsf{P}\Big(\sup_{0\le u\le s} Z_u \ge \alpha\Big) \le \mathsf{P}\Big(\sup_{0\le u\le s} \big|Z_u\big| \ge \alpha\Big)\end{aligned} \tag{4.44}$$

where we also recall that $x \ge K+2\varepsilon$ and hence $\alpha > 0$. We now use Markov inequality, Doob's inequality and BDG inequality to estimate the last expression in (4.44) and it follows that for any $p > 1$

$$\begin{aligned}\mathsf{P}\Big(\sup_{0\le u\le s} \big|Z_u\big| \ge \alpha\Big) \le& \frac{1}{\alpha^p}\mathsf{E}\sup_{0\le u\le s}\big|Z_u\big|^p \\ \le& \frac{2^{p-1}}{\alpha^p}\Big(cs^p + \sigma^p\mathsf{E}\sup_{0\le u\le s}\big|B_u\big|^p\Big) \le C_1\big(s^p + s^{p/2}\big)\end{aligned} \tag{4.45}$$

with suitable $C_1 = C_1(p,\varepsilon,x) > 0$. Collecting (4.43) and (4.45) we get

$$\begin{aligned}V^{(2)}(t,x) - G(t,x) \le s\Big(& C_2(s^p + s^{p/2}) + C_3\big(s^{p-1} + s^{p/2-1}\big) \\ &- rK\mathsf{E}\Big[\frac{1}{s}\int_0^s e^{-ru} f(t+u, X^x_u)du\Big]\Big)\end{aligned} \tag{4.46}$$

for some $C_2 = C_2(p,\varepsilon,x) >$ and $C_3 = C_3(p,\varepsilon,x) > 0$. We take $p > 2$ and observe that in the limit as $s \downarrow 0$ we get

$$-rK\mathsf{E}\Big[\frac{1}{s}\int_0^s e^{-ru} f(t+u, X^x_u)du\Big] + C_2(s^p + s^{p/2}) \tag{4.47}$$

$$+ C_3\big(s^{p-1}+s^{p/2-1}\big) \to -rKf(T_\delta, x)$$

and therefore the negative term in (4.46) dominates since $f(T_\delta, x) > 0$ for all $x \in (0,\infty)$. From (4.46) and (4.47) we get a contradiction and by arbitrariness of ε we conclude that for any $x > K$ there must be $t < T_\delta$ large enough and such that $(t, x) \in D^{(2)}$.

Now we show that $(t, x) \in D^{(2)}$ with $x > K$ implies $(t, y) \in D^{(2)}$ for any $y > x$. Let us take $y > x > K$ and assume $(t, y) \in C^{(2)}$. Set $\tau = \tau^*(t, y)$ optimal for $V^{(2)}(t, y)$ defined as in (4.24) and notice that the horizontal segment $[t, T_\delta] \times \{x\}$ belongs to $D^{(2)}$ by Proposition 4.3.3. Then due to the continuity of the process $(t+s, X^y_s)_{s\in[0,T_\delta-t]}$ cannot hit the horizontal segment $[t, T_\delta] \times \{K\}$ without entering into the stopping set. Hence by (4.19) and (4.21) we have

$$\begin{aligned} V^{(2)}(t, y) =& \mathsf{E}e^{-r\tau}G(t+\tau, X^y_\tau) \\ =& G(t, y) - rK\mathsf{E}\Big[\int_0^\tau e^{-rs}f(t+s, X^y_s)ds\Big] \le G(t, y) \end{aligned} \tag{4.48}$$

which means that it is optimal to stop at once at (t, y) and therefore we get a contradiction. We then conclude that for each $t \in [0, T_\delta)$ there exists the unique point $c^{(2)}(t) > K$ such that $D^{(2)}_t \cap (K,\infty) = [c^{(2)}(t),\infty)$ with the convention that if $c^{(2)}(t) = +\infty$ the set is empty. We remark that for now we have only proven that $c^{(2)}(t) < +\infty$ for t close enough to T_δ and finiteness of $c^{(2)}$ will be provided in Proposition 4.3.6 below.

(iii) Now let us consider the set $\{(t, x) \in [0, T_\delta) \times (0, K]\}$. From Proposition 4.3.2 it follows that for each $t \in [0, T_\delta)$ the set $D^{(2)}_t \cap (0, K)$ is not empty. However using (4.48) and arguments as in the last paragraph of (ii) above one can also prove that if $x \in D^{(2)}_t \cap (0, K)$ and $0 < y \le x$, then $y \in D^{(2)}_t \cap (0, K)$. The latter implies that for each $t \in [0, T_\delta)$ there exists the unique point $b^{(2)}(t) \in (0, K)$ such that $D^{(2)}_t \cap (0, K) = (0, b^{(2)}(t)]$.

We can conclude that (i), (ii) and (iii) above imply that $C^{(2)}_t = \big(b^{(2)}(t), c^{(2)}(t)\big)$ for all $t \in [0, T_\delta]$ and for some functions $b^{(2)}$ and $c^{(2)} : [0, T_\delta] \to (0,\infty]$. The fact that $b^{(2)}(t) \ge b^{(1)}(t)$ is an obvious consequence of Proposition 4.3.2. On the other hand

Proposition 4.3.3 implies that $t \mapsto b^{(2)}(t)$ is increasing and $t \mapsto c^{(2)}(t)$ is decreasing so that their left-limits always exist. It is clear from (ii) above that $\lim_{t\to T_\delta} c^{(2)}(t) = K$ and similar arguments can also be used to prove that $\lim_{t\to T_\delta} b^{(2)}(t) = K$. □

In Theorem 4.3.5 we have proven that $c^{(2)}(t) < \infty$ for $[t^*, T_\delta]$ with some $t^* < T_\delta$. In fact the following proposition holds.

Proposition 4.3.6. *For all $t \in [0, T_\delta]$ the upper boundary $c^{(2)}$ is finite, i.e.*

$$\sup_{t\in[0,T_\delta]} c^{(2)}(t) < +\infty. \tag{4.49}$$

Proof. The proof is provided in two steps.

(i) Let us assume that (4.49) is violated and denote $t_0 := \sup\{t \in [0, T_\delta] : c^{(2)}(t) = +\infty\}$. Consider for now the case $t_0 > 0$ and note that since $t \mapsto c^{(2)}(t)$ is decreasing by Theorem 4.3.5 then $c^{(2)}(t) = +\infty$ for all $t \in [0, t_0)$. The function $c^{(2)}$ is right-continuous on $[t_0, T_\delta]$, in fact for any $t \in [t_0, T_\delta]$ we take $t_n \downarrow t$ as $n \to \infty$ and the sequence $(t_n, c^{(2)}(t_n)) \in D^{(2)}$ converges to $(t, c^{(2)}(t+))$ with $c^{(2)}(t+) := \lim_{s\downarrow t} c^{(2)}(s)$. Since $D^{(2)}$ is closed it must also be $(t, c^{(2)}(t)) \in D^{(2)}$ and $c^{(2)}(t+) \geq c^{(2)}(t)$ by Theorem 4.3.5, hence $c^{(2)}(t+) = c^{(2)}(t)$ by monotonicity.

We define the left-continuous inverse of $c^{(2)}$ by $t_c(x) := \sup\big\{t \in [0, T_\delta] : c^{(2)}(t) > x\big\}$ and observe that $t_c(x) \geq t_0$ for $x \in (K, +\infty)$. Fix $\varepsilon > 0$ such that $\varepsilon < \delta \wedge t_0$, then there exists $\overline{x} = \overline{x}(\varepsilon) > K$ such that $t_c(x) - t_0 \leq \varepsilon/2$ for all $x \geq \overline{x}$ and we denote $\theta = \theta(x) := \inf\big\{s \geq 0 : X^x_s \leq \overline{x}\big\}$. In particular we note that if $c^{(2)}(t_0+) = c^{(2)}(t_0) < +\infty$ we have $t_c(x) = t_0$ for all $x > c^{(2)}(t_0)$. We fix $t = t_0 - \varepsilon/2$, take $x > \overline{x}$ and set $\tau = \tau^*(t, x)$ the optimal stopping time for $V^{(2)}(t, x)$ (cf. (4.24)). Since we assume that $c^{(2)}(t) = +\infty$ for $t \in [t_0 - \varepsilon/2, t_0)$ and the boundary is decreasing then it must be $\{\tau \leq \theta\} \subseteq \{\tau \geq \varepsilon/2\}$.

Using (4.21) gives

$$\begin{aligned} &V^{(2)}(t,x) - G(t,x) \\ &\quad= \mathsf{E} e^{-r\tau} G(t+\tau, X^x_\tau) - G(t,x) \\ &\quad\leq \mathsf{E}\Big[-rK \int_0^\tau e^{-rs} f(t+s, X^x_s) ds + \frac{1}{2}\int_0^\tau e^{-rs} d\ell^K_s(X^x)\Big] \end{aligned} \tag{4.50}$$

$$
\begin{aligned}
&\leq -rK\mathsf{E}\Big[I(\tau\leq\theta)\int_0^{\tau}e^{-rs}f(t{+}s,X_s^x)ds\Big]+\mathsf{E}\Big[I(\tau>\theta)\frac{1}{2}\int_\theta^{\tau}e^{-rs}d\ell_s^K(X^x)\Big]\\
&\leq -rK\mathsf{E}\Big[\int_0^{\varepsilon/2}e^{-rs}f(t{+}s,X_s^x)ds\Big]\\
&\quad+\mathsf{E}\Big[I(\tau>\theta)\Big(\frac{1}{2}\int_\theta^{\tau}e^{-rs}d\ell_s^K(X^x)+rK\int_0^{\varepsilon/2}e^{-rs}f(t{+}s,X_s^x)ds\Big)\Big]\\
&\leq -rK\mathsf{E}\Big[\int_0^{\varepsilon/2}e^{-rs}f(t{+}s,X_s^x)ds\Big]+\frac{1}{2}\mathsf{E}\left[I(\tau>\theta)\mathsf{E}\Big[\int_\theta^{\tau\vee\theta}e^{-rs}d\ell_s^K(X^x)\Big|\mathcal{F}_\theta\Big]\right]\\
&\quad+rK\frac{\varepsilon}{2}\mathsf{P}(\tau>\theta)
\end{aligned}
$$

where we have used that $\ell_s^K(X^x)=0$ for $s\leq\theta$ and in the last inequality we have also used that $0\leq f\leq 1$ on $[0,T_\delta]\times(0,\infty)$. We now estimate separately the two positive terms in the last expression of (4.50). For the one involving the local time we argue as in (4.42), i.e. we use Lemma 4.3.4 and the martingale property of the discounted price to get

$$
I(\tau>\theta)\mathsf{E}\Big(\int_\theta^{\tau\vee\theta}e^{-rs}d\ell_s^K(X^x)\Big|\mathcal{F}_\theta\Big)\leq 3KI(\tau>\theta). \tag{4.51}
$$

Then for a suitable constant $C_1>0$ independent of x we get

$$
\mathsf{E}\left[I(\tau>\theta)\mathsf{E}\Big(\int_\theta^{\tau\vee\theta}e^{-rs}dL_s^K(X^x)\Big|\mathcal{F}_\theta\Big)\right]+rK\frac{\varepsilon}{2}\mathsf{P}(\tau>\theta)\leq C_1\mathsf{P}(\tau>\theta). \tag{4.52}
$$

Observe now that on $\big\{\tau>\theta\big\}$ the process X started at time $t=t_0-\varepsilon/2$ from $x>\overline{x}$ must hit $\overline{x}$ prior to time $t_0+\varepsilon/2$, hence, for $c=r+\sigma^2/2$, we obtain

$$
\mathsf{P}(\tau>\theta)\leq\mathsf{P}\big(\inf_{0\leq t\leq\varepsilon}X_t^x<\overline{x}\big)\leq\mathsf{P}\left(\inf_{0\leq t\leq\varepsilon}B_t<\tfrac{1}{\sigma}\Big(\ln(\overline{x}/x)+c\,\varepsilon\Big)\right). \tag{4.53}
$$

We now introduce another Brownian motion by taking $W:=-B$, then from (4.53) and the *reflection principle* we find

$$
\begin{aligned}
\mathsf{P}(\tau>\theta)\leq&\mathsf{P}\left(\sup_{0\leq t\leq\varepsilon}W_t>-\tfrac{1}{\sigma}\Big(\ln(\overline{x}/x)+c\,\varepsilon\Big)\right) \qquad (4.54)\\
=&2\mathsf{P}\left(W_\varepsilon>-\tfrac{1}{\sigma}\Big(\ln(\overline{x}/x)+c\,\varepsilon\Big)\right)\\
=&2\Big[1-\Phi\Big(\tfrac{1}{\sigma\sqrt{\varepsilon}}\big(\ln(x/\overline{x})-c\,\varepsilon\big)\Big)\Big]=2\Phi\Big(\tfrac{1}{\sigma\sqrt{\varepsilon}}\big(\ln(\overline{x}/x)+c\,\varepsilon\big)\Big)
\end{aligned}
$$

with $\Phi(y)=1/\sqrt{2\pi}\int_{-\infty}^{y}e^{-z^2/2}dz$ for $y\in I\!R$ and where we have used $\Phi(y)=1-\Phi(-y)$ for $y\in I\!R$.

Going back to (4.50) we aim to estimate the first term in the last expression. For that we use Markov property to obtain

$$\begin{aligned}\mathsf{E}f(t+s,X_s^x) =& e^{-r\delta}\mathsf{E}\big[\mathsf{P}\big(X_{s+\delta}^x \le b^{(1)}(t+s+\delta)\big|\mathcal{F}_s\big)\,\big] \\ =& e^{-r\delta}\mathsf{P}\big[X_{s+\delta}^x \le b^{(1)}(t+s+\delta)\big]\end{aligned} \tag{4.55}$$

for $s \in [0,\varepsilon/2]$. Now we denote $\alpha := b^{(1)}(t+\delta)$ and have that for all $s \in [0,\varepsilon/2]$ and $x > \overline{x}$ the expectation in (4.55) is bounded from below by recalling that $b^{(1)}$ is increasing, namely

$$\begin{aligned}\mathsf{E}f(t+s,X_s^x) \ge& e^{-r\delta}\mathsf{P}(X_{s+\delta}^x \le \alpha) \\ \ge& e^{-r\delta}\mathsf{P}\Big(B_{s+\delta} \le \frac{1}{\sigma}\big[\ln(\alpha/x) - c(\delta+\varepsilon/2)\big]\Big) \\ =& e^{-r\delta}\Phi\Big(\tfrac{1}{\sigma\sqrt{\delta+s}}\big[\ln(\alpha/\,x) - c\,(\delta+\varepsilon/2)\big]\Big) \\ \ge& e^{-r\delta}\Phi\Big(\tfrac{1}{\sigma\sqrt{\delta}}\big[\ln(\alpha/\,x) - c\,(\delta+\varepsilon/2)\big]\Big) =: \hat{F}(x)\end{aligned} \tag{4.56}$$

where in the last inequality we have used that $\ln(\alpha/\,x) < 0$ and Φ is increasing. From (4.56) and using Fubini's theorem we get

$$\mathsf{E}\Big[\int_0^{\varepsilon/2} e^{-rs}f(t+s,X_s^x)ds\Big] = \int_0^{\varepsilon/2} e^{-rs}\mathsf{E}f(t+s,X_s^x)\,ds \ge \frac{\varepsilon}{2}e^{-r\varepsilon/2}\hat{F}(x) \tag{4.57}$$

for $x > \overline{x}$. By combining (4.50), (4.52), (4.54) and (4.57) we now obtain

$$\begin{aligned}&V^{(2)}(t,x) - G(t,x) \\ &\le 2C_1\Phi\Big(\tfrac{1}{\sigma\sqrt{\varepsilon}}\big(\ln(\overline{x}/x) + c\,\varepsilon\big)\Big) - C_2\Phi\Big(\tfrac{1}{\sigma\sqrt{\delta}}\big[\ln\big(\alpha/\,x\big) - c\,(\delta+\varepsilon/2)\big]\Big)\end{aligned} \tag{4.58}$$

where $C_2 = C_2(\varepsilon) > 0$ and independent of x. Since $t, \overline{x}, \varepsilon$ are fixed with $\delta > \varepsilon$, we take the limit as $x \to \infty$ and it is easy to verify by L'Hôpital's rule that

$$\begin{aligned}&\lim_{x\to\infty}\frac{\Phi\Big(\tfrac{1}{\sigma\sqrt{\varepsilon}}\big(\ln(\overline{x}/x) + c\,\varepsilon\big)\Big)}{\Phi\Big(\tfrac{1}{\sigma\sqrt{\delta}}\big[\ln\big(\alpha/\,x\big) - c\,(\delta+\varepsilon/2)\big]\Big)} \\ &= C_3\lim_{x\to\infty}\frac{\varphi\Big(\tfrac{1}{\sigma\sqrt{\varepsilon}}\big(\ln(\overline{x}/x) + c\,\varepsilon\big)\Big)}{\varphi\Big(\tfrac{1}{\sigma\sqrt{\delta}}\big[\ln\big(\alpha/\,x\big) - c\,(\delta+\varepsilon/2)\big]\Big)} \\ &= C_4\lim_{x\to\infty} x^{\beta}\exp\Big(\tfrac{1}{\sigma^2}\big(1/\delta - 1/\varepsilon\big)\big(\ln\,x\big)^2\Big) = 0\end{aligned} \tag{4.59}$$

for some constants $\beta > 0$, C_3 and C_4 and with $\varphi := \Phi'$ the standard normal density function. Hence the negative term in (4.58) dominates for large values of x and we reach a contradiction so that it must be $c^{(2)}(t) < +\infty$ for all $t \in (0, T_\delta]$ by arbitrariness of t_0.

(ii) It remains to show that $c^{(2)}(0) < +\infty$ as well. In order to do so we recall Remark 4.2.1 and notice that since $V^{(1)}(t+\lambda, x; T+\lambda) = V^{(1)}(t, x; T)$ for all $(t, x) \in [0, T] \times (0, \infty)$ and $\lambda > 0$, then (with the same notation for the maturity in the function G) we have

$$\begin{aligned} G(t+\lambda, x; T+\lambda) =&(K-x)^+ + e^{-r\delta}\mathsf{E}V^{(1)}(t+\lambda+\delta, X^x_\delta; T+\lambda) \\ =&(K-x)^+ + e^{-r\delta}\mathsf{E}V^{(1)}(t+\delta, X^x_\delta; T) = G(t, x; T) \end{aligned} \tag{4.60}$$

for $(t, x) \in [0, T_\delta] \times (0, \infty)$. It easily follows that by denoting $V^{(2)}(\,\cdot\,, \cdot\,; T_\delta)$ the value function of problem (4.16) with maturity at T_δ one has $V^{(2)}(t, x; T_\delta) = V^{(2)}(t+\lambda, x; T_\delta+\lambda)$ for $(t, x) \in [0, T_\delta] \times (0, \infty)$ and $\lambda > 0$. Hence, assuming that $c^{(2)}(0) = +\infty$ would imply $V^{(2)}(0, x; T_\delta) > G(0, x; T)$ for all $x > 0$. However by taking $\lambda > 0$ that would also imply $V^{(2)}(\lambda, x; T_\delta+\lambda) > G(\lambda, x; T+\lambda)$ for all $x > 0$. The latter is impossible by (i) above with $t_0 = \lambda$ since all the arguments used there can be repeated with T_δ replaced by $T_\delta+\lambda$. □

6. We will show in the next proposition that the value function $V^{(2)}$ also fulfills the so-called *smooth-fit* condition at the optimal stopping boundaries $b^{(2)}$ and $c^{(2)}$.

Proposition 4.3.7. *For all $t \in [0, T_\delta)$ the map $x \mapsto V^{(2)}(t, x)$ is C^1 across the optimal boundaries, i.e.*

$$V^{(2)}_x(t, b^{(2)}(t)+) = G_x(t, b^{(2)}(t)) \tag{4.61}$$

$$V^{(2)}_x(t, c^{(2)}(t)-) = G_x(t, c^{(2)}(t)). \tag{4.62}$$

Proof. We provide a full proof only for (4.62) as the case of $b^{(2)}$ can be treated in a similar way. Let us fix $0 \le t < T_\delta$ and set $x_0 := c^{(2)}(t)$. It is clear that for arbitrary $\varepsilon > 0$ it holds

$$\frac{V^{(2)}(t, x_0) - V^{(2)}(t, x_0-\varepsilon)}{\varepsilon} \le \frac{G(t, x_0) - G(t, x_0-\varepsilon)}{\varepsilon} \tag{4.63}$$

and hence

$$\limsup_{\varepsilon\to 0} \frac{V^{(2)}(t,x_0)-V^{(2)}(t,x_0-\varepsilon)}{\varepsilon} \le G_x(t,x_0). \tag{4.64}$$

To prove the reverse inequality, we denote $\tau_\varepsilon = \tau^*(t,x_0-\varepsilon)$ which is the optimal stopping time for $V^{(2)}(t,x_0-\varepsilon)$. Then using the law of iterated logarithm at zero for Brownian motion and the fact that $t\mapsto c^{(2)}(t)$ is decreasing we obtain $\tau_\varepsilon \to 0$ as $\varepsilon\to 0$ P-a.s. By using the mean value theorem we have

$$\begin{aligned} &\frac{1}{\varepsilon}\Big(V^{(2)}(t,x_0)-V^{(2)}(t,x_0-\varepsilon)\Big) \\ &\quad\ge \frac{1}{\varepsilon}\mathsf{E}\Big[e^{-r\tau^\varepsilon}\Big(G(t+\tau_\varepsilon,X^{x_0}_{\tau_\varepsilon})-G(t+\tau_\varepsilon,X^{x_0-\varepsilon}_{\tau_\varepsilon})\Big)\Big] \\ &\quad\ge \frac{1}{\varepsilon}\mathsf{E}\Big[e^{-r\tau_\varepsilon}G_x(t+\tau_\varepsilon,\xi)\big(X^{x_0}_{\tau_\varepsilon}-X^{x_0-\varepsilon}_{\tau_\varepsilon}\big)\Big] = \mathsf{E}\Big[e^{-r\tau_\varepsilon}G_x(t+\tau_\varepsilon,\xi)X^1_{\tau_\varepsilon}\Big] \end{aligned} \tag{4.65}$$

with $\xi(\omega)\in[X^{x_0-\varepsilon}_{\tau_\varepsilon}(\omega),X^{x_0}_{\tau_\varepsilon}(\omega)]$ for all $\omega\in\Omega$. Thus recalling that G_x is bounded and $X^1_{\tau_\varepsilon}\to 1$ P-a.s. as $\varepsilon\to 0$, using dominated convergence theorem we obtain

$$\liminf_{\varepsilon\to 0} \frac{V^{(2)}(t,x_0)-V^{(2)}(t,x_0-\varepsilon)}{\varepsilon} \ge G_x(t,x_0). \tag{4.66}$$

Finally combining (4.64) and (4.66) we obtain (4.62). □

7. Standard arguments based on the strong Markov property and continuity of $V^{(2)}$(cf. [58, Sec. 7]) together with the results that we have proved so far lead to the following free-boundary problem for the value function $V^{(2)}$ and unknown boundaries $b^{(2)}$ and $c^{(2)}$:

$$V^{(2)}_t+I\!L_X V^{(2)}-rV^{(2)}=0 \qquad \text{in } C^{(2)} \tag{4.67}$$

$$V^{(2)}(t,b^{(2)}(t))=G(t,b^{(2)}(t)) \qquad \text{for } t\in[0,T_\delta] \tag{4.68}$$

$$V^{(2)}(t,c^{(2)}(t))=G(t,c^{(2)}(t)) \qquad \text{for } t\in[0,T_\delta] \tag{4.69}$$

$$V^{(2)}_x(t,b^{(2)}(t)+)=G_x(t,b^{(2)}(t)) \qquad \text{for } t\in[0,T_\delta) \tag{4.70}$$

$$V^{(2)}_x(t,c^{(2)}(t)-)=G_x(t,c^{(2)}(t)) \qquad \text{for } t\in[0,T_\delta) \tag{4.71}$$

$$V^{(2)}(t,x)>G(t,x) \qquad \text{in } C^{(2)} \tag{4.72}$$

$$V^{(2)}(t,x)=G(t,x) \qquad \text{in } D^{(2)} \tag{4.73}$$

where $V^{(2)} \in C^{1,2}$ in $C^{(2)}$ and the continuation set $C^{(2)}$ and the stopping set $D^{(2)}$ are given by

$$C^{(2)} = \{\, (t,x) \in [0,T_\delta) \times (0,\infty) : b^{(2)}(t) < x < c^{(2)}(t) \,\} \tag{4.74}$$

$$D^{(2)} = \{\, (t,x) \in [0,T_\delta) \times (0,\infty) : x \le b^{(2)}(t) \text{ or } x \ge c^{(2)}(t) \,\}. \tag{4.75}$$

We now proceed to prove that the boundaries $b^{(2)}$ and $c^{(2)}$ are indeed continuous functions of time and we follow an approach proposed in [16].

Theorem 4.3.8. *The optimal boundaries $b^{(2)}$ and $c^{(2)}$ are continuous on $[0,T_\delta]$.*

Proof. The proof is provided in 3 steps.

(i) We first show that $b^{(2)}$ and $c^{(2)}$ are right-continuous. Let us consider $b^{(2)}$, fix $t \in [0,T_\delta)$ and take a sequence $t_n \downarrow t$ as $n \to \infty$. Since $b^{(2)}$ is increasing, the right-limit $b^{(2)}(t+)$ exists and $(t_n, b^{(2)}(t_n))$ belongs to $D^{(2)}$ for all $n \ge 1$. Recall that $D^{(2)}$ is closed so that $(t_n, b^{(2)}(t_n)) \to (t, b^{(2)}(t+)) \in D^{(2)}$ as $n \to \infty$ and we may conclude that $b^{(2)}(t+) \le b^{(2)}(t)$. The fact that $b^{(2)}$ is increasing gives the reverse inequality thus $b^{(2)}$ is right-continuous as claimed. We can argue in analogous way to obtain that $c^{(2)}$ is right-continuous.

(ii) Now we prove that $b^{(2)}$ is also left-continuous. Assume that there exists $t_0 \in (0,T)$ such that $b^{(2)}(t_0-) < b^{(2)}(t_0)$ where $b^{(2)}(t_0-)$ denotes the left-limit of $b^{(2)}$ at t_0. Take $x_1 < x_2$ such that $b^{(2)}(t_0-) < x_1 < x_2 < b^{(2)}(t_0)$ and $h > 0$ such that $t_0 > h$, then by defining $u := V^{(2)} - G$ and using (4.17), (4.19), (4.67), (4.73) we have

$$u_t + I\!\!L_X u - ru = -H \quad \text{on } C^{(2)} \text{ and below } K \tag{4.76}$$

$$u(t_0, x) = 0 \quad \text{for } x \in (x_1, x_2). \tag{4.77}$$

Denote by $C_c^\infty(a,b)$ the set of continuous functions which are differentiable infinitely many times with continuous derivatives and compact support on (a,b). Take $\varphi \in C_c^\infty(x_1,x_2)$ such that $\varphi \ge 0$ and $\int_{x_1}^{x_2} \varphi(x)dx = 1$. Multiplying (5.41) by φ and integrating by parts we obtain

$$\int_{x_1}^{x_2} \varphi(x) u_t(t,x)dx = -\int_{x_1}^{x_2} u(t,x) \left(I\!\!L_X^* \varphi(x) - r\varphi(x) \right) dx \tag{4.78}$$

$$- \int_{x_1}^{x_2} H(t,x)\varphi(x)dx$$

for $t \in (t_0 - h, t_0)$ and with $\mathbb{L}_X^*$ denoting the formal adjoint of $\mathbb{L}_X$. Since $u_t \le 0$ in $C^{(2)}$ below K by (4.32) in the proof of Proposition 4.3.3, the left-hand side of (4.78) is negative. Then taking limits as $t \to t_0$ and by using dominated convergence theorem we find

$$\begin{aligned} 0 &\ge - \int_{x_1}^{x_2} u(t_0,x)\left(\mathbb{L}_X^*\varphi(x) - r\varphi(x)\right)dx - \int_{x_1}^{x_2} H(t_0,x)\varphi(x)dx \qquad (4.79) \\ &= - \int_{x_1}^{x_2} H(t_0,x)\varphi(x)dx \end{aligned}$$

where we have used that $u(t_0,x) = 0$ for $x \in (x_1,x_2)$ by (4.77). We now observe that $H(t_0,x) < -c$ for $x \in (x_1,x_2)$ and a suitable $c > 0$ by (4.19), therefore (4.79) leads to a contradiction and it must be $b^{(2)}(t_0-) = b^{(2)}(t_0)$.

(iii) To prove that $c^{(2)}$ is left-continuous we can use arguments that follow the very same lines as those in *(ii)* above and therefore we omit them for brevity. □

8. Finally we are able to find an early-exercise premium (EEP) representation for $V^{(2)}$ of the problem (4.16) and a coupled system of integral equations for the free-boundaries $b^{(2)}$ and $c^{(2)}$.

Theorem 4.3.9. *The value function $V^{(2)}$ of (4.16) has the following representation*

$$\begin{aligned} V^{(2)}(t,x) &= e^{-r(T_\delta - t)}\mathsf{E}G(T_\delta, X^x_{T_\delta - t}) \qquad (4.80) \\ &\quad + rK\int_0^{T_\delta - t} e^{-rs}\Big[\mathsf{P}\big(X^x_s \le b^{(2)}(t+s)\big) \\ &\qquad + e^{-r\delta}\mathsf{P}\big(X^x_s \le b^{(2)}(t+s), X^x_{s+\delta} \le b^{(1)}(t+s+\delta)\big) \\ &\qquad + e^{-r\delta}\mathsf{P}\big(X^x_s \ge c^{(2)}(t+s), X^x_{s+\delta} \le b^{(1)}(t+s+\delta)\big)\Big]ds \end{aligned}$$

for $t \in [0,T_\delta]$ and $x \in (0,\infty)$. The optimal stopping boundaries $b^{(2)}$ and $c^{(2)}$ of (4.74) *and* (4.75) *are the unique couple of functions solving the system of nonlinear integral equations (see Figure 4.1)*

$$G(t,b^{(2)}(t)) = e^{-r(T_\delta - t)}\mathsf{E}G(T_\delta, X^{b^{(2)}(t)}_{T_\delta - t}) \qquad (4.81)$$

$$+ rK\int_0^{T_\delta - t} e^{-rs}\Big[\mathsf{P}\big(X_s^{b^{(2)}(t)} \le b^{(2)}(t+s)\big) + e^{-r\delta}\mathsf{P}\big(X_s^{b^{(2)}(t)} \le b^{(2)}(t+s), X_{s+\delta}^{b^{(2)}(t)} \le b^{(1)}(t+s+\delta)\big) + e^{-r\delta}\mathsf{P}\big(X_s^{b^{(2)}(t)} \ge c^{(2)}(t+s), X_{s+\delta}^{b^{(2)}(t)} \le b^{(1)}(t+s+\delta)\big)\Big]ds$$

$$G(t, c^{(2)}(t)) = e^{-r(T_\delta - t)}\mathsf{E}G(T_\delta, X_{T_\delta - t}^{c^{(2)}(t)}) + rK\int_0^{T_\delta - t} e^{-rs}\Big[\mathsf{P}\big(X_s^{c^{(2)}(t)} \le b^{(2)}(t+s)\big) + e^{-r\delta}\mathsf{P}\big(X_s^{c^{(2)}(t)} \le b^{(2)}(t+s), X_{s+\delta}^{b^{(2)}(t)} \le b^{(1)}(t+s+\delta)\big) + e^{-r\delta}\mathsf{P}\big(X_s^{c^{(2)}(t)} \ge c^{(2)}(t+s), X_{s+\delta}^{b^{(2)}(t)} \le b^{(1)}(t+s+\delta)\big)\Big]ds \tag{4.82}$$

in the class of continuous increasing functions $t \mapsto b^{(2)}(t)$ *and continuous decreasing functions* $t \mapsto c^{(2)}(t)$ *on* $[0, T_\delta]$ *with* $b^{(2)}(T_\delta) = c^{(2)}(T_\delta) = K$.

Proof. (*A*) We start by recalling that the following conditions hold: *(i)* $V^{(2)}$ is $C^{1,2}$ on $C^{(2)}$ and on $D^{(2)}$ and $V_t^{(2)} + I\!L_X V^{(2)} - rV^{(2)}$ is locally bounded on $C^{(2)} \cup D^{(2)}$ (see (4.67)–(4.73) and (4.19)); *(ii)* $b^{(2)}$ and $c^{(2)}$ are of bounded variation due to monotonicity; *(iii)* $x \mapsto V^{(2)}(t, x)$ is convex (recall proof of Proposition 4.3.1); *(iv)* $t \mapsto V_x^{(2)}(t, b^{(2)}(t)\pm)$ and $t \mapsto V_x^{(2)}(t, c^{(2)}(t)\pm)$ are continuous for $t \in [0, T_\delta)$ by (4.70) and (4.71). Hence for any $(t, x) \in [0, T_\delta] \times (0, \infty)$ and $s \in [0, T_\delta - t]$ we can apply the local time-space formula on curves [52] to obtain

$$\begin{aligned} e^{-rs}V^{(2)}(t+s, X_s^x) &= V^{(2)}(t, x) + M_u \\ &\quad + \int_0^s e^{-ru}(V_t^{(2)} + I\!L_X V^{(2)} - rV^{(2)})(t+u, X_u^x) I\big(X_u^x \ne \{b^{(2)}(t+u),\, c^{(2)}(t+u)\}\big)du \\ &= V^{(2)}(t, x) + M_u + \int_0^s e^{-ru}(G_t + I\!L_X G - rG)(t+u, X_u^x) I\big(X_u^x < b^{(2)}(t+u)\big)du \\ &\quad + \int_0^s e^{-ru}(G_t + I\!L_X G - rG)(t+u, X_u^x) I\big(X_u^x > c^{(2)}(t+u)\big)du \\ &= V^{(2)}(t, x) + M_u - rK\int_0^s e^{-ru}\big(1 + f(t+u, X_u^x)\big) I\big(X_u^x < b^{(2)}(t+u)\big)du \\ &\quad - rK\int_0^s e^{-ru} f(t+u, X_u^x) I\big(X_u^x > c^{(2)}(t+u)\big)du \end{aligned} \tag{4.83}$$

where we used (4.17), (4.19), (4.67) and smooth-fit conditions (4.70)-(4.71) and where $M = (M_u)_{u \ge 0}$ is a martingale. Recall that the law of X_u^x is absolutely continuous with respect to the Lebesgue measure for all $u > 0$, then from strong Markov property and (4.20) we deduce

$$f(t{+}u, X_u^x) I\big(X_u^x < b^{(2)}(t{+}u)\big) \tag{4.84}$$
$$= e^{-r\delta} \mathsf{P}\big(X_{u+\delta}^x \le b^{(1)}(t{+}u{+}\delta) \big| \mathcal{F}_u\big) I\big(X_u^x < b^{(2)}(t{+}u)\big)$$
$$= e^{-r\delta} \mathsf{P}\big(X_{u+\delta}^x \le b^{(1)}(t{+}u{+}\delta), X_u^x \le b^{(2)}(t{+}u) \big| \mathcal{F}_u\big)$$

and analogously we have that

$$f(t{+}u, X_u^x) I\big(X_u^x > c^{(2)}(t{+}u)\big) \tag{4.85}$$
$$= e^{-r\delta} \mathsf{P}\big(X_{u+\delta}^x \le b^{(1)}(t{+}u{+}\delta), X_u^x \ge c^{(2)}(t{+}u) \big| \mathcal{F}_u\big).$$

In (4.83) we let $s = T_\delta - t$, take the expectation E, use (4.84)-(4.85) and the optional sampling theorem for M, then after rearranging terms and noting that $V^{(2)}(T_\delta, x) = G(T_\delta, x)$ for all $x > 0$, we get (4.80). The coupled system of integral equations (4.81)-(4.82) is obtained by simply putting $x = b^{(2)}(t)$ and $x = c^{(2)}(t)$ into (4.80) and using (4.68)-(4.69).

(B) Now we show that $b^{(2)}$ and $c^{(2)}$ are the unique solution pair to the system (4.81)-(4.82) in the class of continuous functions $t \mapsto b(t)$, $t \mapsto c(t)$ with terminal value K and such that b is increasing and c is decreasing. Note that there is no need to assume that b is increasing and c is decreasing as established above as long as $b(t) \neq K$ and $c(t) \neq K$ for all $t \in [0, T_\delta)$. The proof is divided in few steps and it is based on arguments similar to those employed in [18] and originally derived in [53].

$(B.1)$ Let $b : [0, T_\delta] \to (0, \infty)$ and $c : [0, T_\delta] \to (0, \infty)$ be another solution pair to the system (4.81)-(4.82) such that b and c are continuous and $b(t) \le c(t)$ for all $t \in [0, T_\delta]$. We will show that these b and c must be equal to the optimal stopping boundaries $b^{(2)}$ and $c^{(2)}$, respectively.

We define a function $U^{b,c} : [0, T_\delta) \to I\!R$ by

$$U^{b,c}(t,x) := e^{-r(T_\delta - t)} \mathsf{E} G(T_\delta, X_{T_\delta - t}^x) \tag{4.86}$$
$$- \mathsf{E} \int_0^{T_\delta - t} e^{-ru} H(t{+}u, X_u^x) I(X_u^x \le b(t{+}u) \text{ or } X_u^x \ge c(t{+}u)) du$$

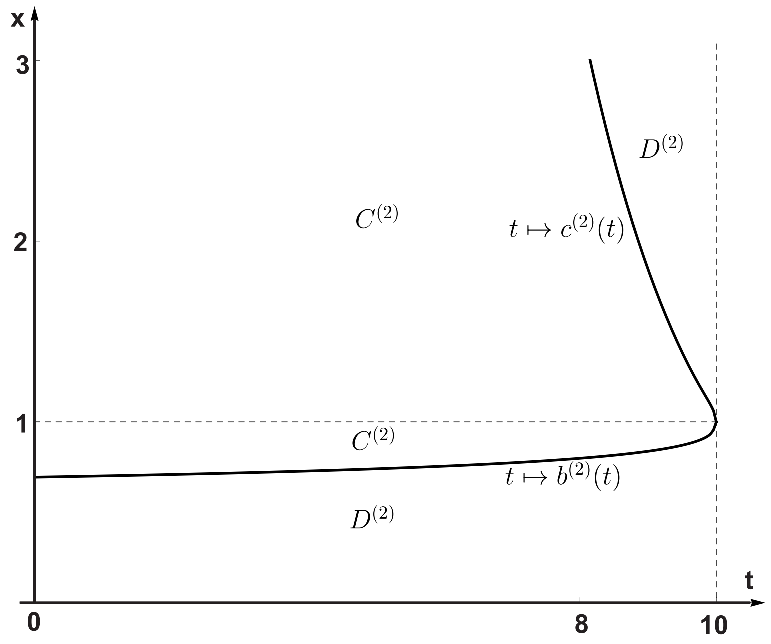

Figure 4.1: A computer drawing of the optimal exercise boundaries $t \mapsto b^{(2)}(t)$ and $t \mapsto c^{(2)}(t)$ for the problem (4.16) in the case $K = 1$, $r = 0.1$ (annual), $\sigma = 0.4$ (annual), $T = 11$ months, $\delta = 1$ month. The decreasing boundary $c^{(2)}$ is finite on $[0, T_\delta]$ but it takes values much larger than those of $b^{(2)}$ on $[0, 8]$ and therefore in order to present the structure of the continuation set in a clear way we only plot the vertical axis up to $x = 3$.

for $(t, x) \in [0, T_\delta] \times (0, \infty)$. Observe that since b and c solve the system (4.81)-(4.82) then $U^{b,c}(t, b(t)) = G(t, b(t))$ and $U^{b,c}(t, c(t)) = G(t, c(t))$ for all $t \in [0, T_\delta]$. Notice also that the Markov property of X gives

$$e^{-rs}U^{b,c}(t+s, X_s^x) - \int_0^s e^{-ru}H(t+u, X_u^x)I(X_u^x \le b(t+u) \text{ or } X_u^x \ge c(t+u))du \quad (4.87)$$
$$= U^{b,c}(t, x) + N_s$$

for $s \in [0, T_\delta - t]$ and where $(N_s)_{0 \le s \le T_\delta - t}$ is a P-martingale.

$(B.2)$ We now show that $U^{b,c}(t, x) = G(t, x)$ for $x \in (0, b(t)] \cup [c(t), \infty)$ and $t \in [0, T_\delta]$. For $x \in (0, b(t)] \cup [c(t), \infty)$ with $t \in [0, T_\delta]$ given and fixed, consider the stopping time

$$\sigma_{b,c} = \sigma_{b,c}(t, x) = \inf \{ \ 0 \le s \le T_\delta - t : b(t+s) \le X_s^x \le c(t+s) \ \}. \quad (4.88)$$

Using that $U^{b,c}(t, b(t)) = G(t, b(t))$ and $U^{b,c}(t, c(t)) = G(t, c(t))$ for all $t \in [0, T_\delta)$ and

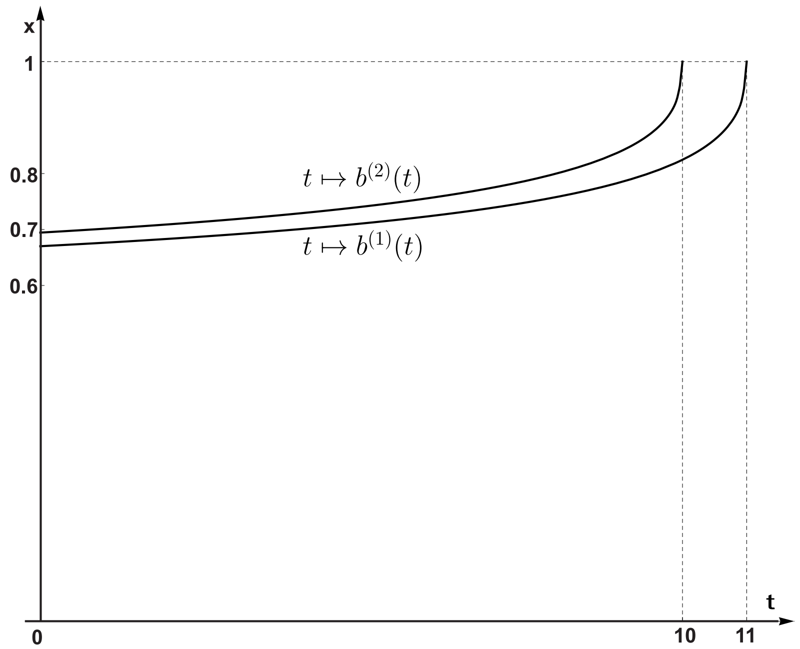

Figure 4.2: A computer drawing of the lower optimal exercise boundary $t \mapsto b^{(2)}(t)$ of problem (4.16) and the optimal exercise boundary $t \mapsto b^{(1)}(t)$ of problem (4.4) (American put) in the case $K = 1$, $r = 0.1$ (annual), $\sigma = 0.4$ (annual), $T = 11$ months, $\delta = 1$ month.

$U^{b,c}(T_\delta, x) = G(T_\delta, x)$ for all $x > 0$, we get $U^{b,c}(t{+}\sigma_{b,c}, X^x_{\sigma_{b,c}}) = G(t{+}\sigma_{b,c}, X^x_{\sigma_{b,c}})$ P-a.s. Hence from (4.21) and (4.87) using the optional sampling theorem and noting that $L^K_u(X^x) = 0$ for $u \le \sigma_{b,c}$ we find

$$\begin{aligned} U^{b,c}(t,x) &= \mathsf{E} e^{-r\sigma_{b,c}} U^{b,c}(t{+}\sigma_{b,c}, X^x_{\sigma_{b,c}}) \\ &\quad - \mathsf{E} \int_0^{\sigma_{b,c}} e^{-ru} H(t{+}u, X^x_u) I(X^x_u \le b(t{+}u) \text{ or } X^x_u \ge c(t{+}u))du \\ &= \mathsf{E} e^{-r\sigma_{b,c}} G(t{+}\sigma_{b,c}, X^x_{\sigma_{b,c}}) - \mathsf{E} \int_0^{\sigma_{b,c}} e^{-ru} H(t{+}u, X^x_u)du = G(t,x) \end{aligned} \tag{4.89}$$

since $X^x_u \in (0, b(t{+}u)) \cup (c(t{+}u), \infty)$ for all $u \in [0, \sigma_{b,c})$.

$(B.3)$ Next we prove that $U^{b,c}(t,x) \le V^{(2)}(t,x)$ for all $(t,x) \in [0,T_\delta] \times (0,\infty)$. For this consider the stopping time

$$\tau_{b,c} = \tau_{b,c}(t,x) = \inf \{\ 0 \le s \le T_\delta {-} t : X^x_s \le b(t{+}s) \text{ or } X^x_s \ge c(t{+}s)\ \} \tag{4.90}$$

with $(t,x) \in [0,T_\delta] \times (0,\infty)$ given and fixed. Again arguments as those following (4.88) above show that $U^{b,c}(t{+}\tau_{b,c}, X^x_{\tau_{b,c}}) = G(t{+}\tau_{b,c}, X^x_{\tau_{b,c}})$ P-a.s. Then taking $s = \tau_{b,c}$ in

(4.87) and using the optional sampling theorem, we get

$$U^{b,c}(t,x) = \mathsf{E}e^{-r\tau_{b,c}}U^{b,c}(t+\tau_{b,c}, X^x_{\tau_{b,c}}) = \mathsf{E}e^{-r\tau_{b,c}}G(t+\tau_{b,c}, X^x_{\tau_{b,c}}) \le V^{(2)}(t,x). \tag{4.91}$$

$(B.4)$ In order to compare the couples (b,c) and $(b^{(2)}, c^{(2)})$ we initially prove that $b(t) \ge b^{(2)}(t)$ and $c(t) \le c^{(2)}(t)$ for $t \in [0, T_\delta]$. For this, suppose that there exists $t \in [0, T_\delta)$ such that $c(t) > c^{(2)}(t)$, take a point $x \in [c(t), \infty)$ and consider the stopping time

$$\sigma = \sigma(t,x) = \inf \{ \ 0 \le s \le T_\delta - t : b^{(2)}(t+s) \le X^x_s \le c^{(2)}(t+s) \ \}. \tag{4.92}$$

Setting $s = \sigma$ in (4.83) and (4.87) and using the optional sampling theorem, we get

$$\mathsf{E}e^{-r\sigma}V^{(2)}(t+\sigma, X^x_\sigma) = V^{(2)}(t,x) + \mathsf{E}\int_0^\sigma e^{-ru}H(t+u, X^x_u)du \tag{4.93}$$

$$\mathsf{E}e^{-r\sigma}U^{b,c}(t+\sigma, X^x_\sigma) = U^{b,c}(t,x) + \mathsf{E}\int_0^\sigma e^{-ru}H(t+u, X^x_u)I\big(X^x_u \le b(t+u) \text{ or } X^x_u \ge c(t+u)\big)du. \tag{4.94}$$

Since $U^{b,c} \le V^{(2)}$ and $V^{(2)}(t,x) = U^{b,c}(t,x) = G(t,x)$ for $x \in [c(t), \infty)$ with $t \in [0, T_\delta]$, it follows by subtracting (4.94) from (4.93) that

$$\mathsf{E}\int_0^\sigma e^{-ru}H(t+u, X^x_u)I\big(b(t+u) \le X^x_u \le c(t+u)\big)du \ge 0. \tag{4.95}$$

The function H is always strictly negative and by the continuity of $c^{(2)}$ and c it must be $\mathsf{P}(\sigma(t,x) > 0) = 1$, hence (4.95) leads to a contradiction and we can conclude that $c(t) \le c^{(2)}(t)$ for all $t \in [0, T_\delta]$. Arguing in a similar way one can also derive that $b(t) \ge b^{(2)}(t)$ for all $t \in [0, T_\delta]$ as claimed.

$(B.5)$ To conclude the proof we show that $b = b^{(2)}$ and $c = c^{(2)}$ on $[0, T_\delta]$. For that, let us assume that there exists $t \in [0, T_\delta)$ such that $b(t) > b^{(2)}(t)$ or $c(t) < c^{(2)}(t)$. Choose an arbitrary point $x \in (b^{(2)}(t), b(t))$ or alternatively $x \in (c(t), c^{(2)}(t))$ and consider the optimal stopping time τ^* of (4.24) with $D^{(2)}$ as in (4.75). Take $s = \tau^*$ in (4.83) and (4.87) and use the optional sampling theorem to get

$$\mathsf{E}e^{-r\tau^*}G(t+\tau^*, X^x_{\tau^*}) = V^{(2)}(t,x) \tag{4.96}$$

$$\mathsf{E}e^{-r\tau^*}G(t+\tau^*, X^x_{\tau^*}) = U^{b,c}(t,x) \tag{4.97}$$
$$+\,\mathsf{E}\int_0^{\tau^*} e^{-ru}H(t+u, X^x_u)I\big(X^x_u \le b(t+u) \text{ or } X^x_u \ge c(t+u)\big)du$$

where we use that $V^{(2)}(t+\tau^*, X^x_{\tau^*}) = G(t+\tau^*, X^x_{\tau^*}) = U^{b,c}(t+\tau^*, X^x_{\tau^*})$ P-a.s. upon recalling that $b \ge b^{(2)}$ and $c \le c^{(2)}$, and $U^{b,c} = G$ either below b and above c (see $(B.2)$ above) or at T_δ. Since $U^{b,c} \le V^{(2)}$ then subtracting (4.96) from (4.97) we get

$$\mathsf{E}\int_0^{\tau^*} e^{-ru}H(t+u, X^x_u)I\big(X^x_u \le b(t+u) \text{ or } X^x_u \ge c(t+u)\big)du \ge 0. \tag{4.98}$$

Again we recall that H is always strictly negative and by continuity of $b^{(2)}$, $c^{(2)}$, b and c we have $\mathsf{P}(\tau^*(t,x) > 0) = 1$ and the process $(X^x_u)_{u\in[0,T_\delta-t]}$ spends a strictly positive amount of time either below $b(t+\,\cdot\,)$ if it starts from $x \in \big(b^{(2)}(t), b(t)\big)$ or above $c(t+\,\cdot\,)$ if it starts from $x \in \big(c(t), c^{(2)}(t)\big)$ with probability one. Therefore we reach a contradiction and thus $b = b^{(2)}$ and $c = c^{(2)}$. □

It is worth observing that the pricing formula (4.80) is consistent with the economic intuition behind the structure of the swing contract and it includes a European part plus three integral terms accounting for the early exercise premium (EEP). The first of such terms is the analogous of the EEP in the American put price formula and it accounts for the value produced by exercising the put option once and getting the usual payoff $(K-x)^+$. Along with that we find two other terms, related to the extra value produced by the second exercise right, which are weighted with the discounted probability of exercising the second option once the refracting period is elapsed. Since we have shown that for the swing put option it is sometimes profitable to exercise the first right even if the immediate put payoff is zero, the second and third terms of the EEP account for both the cases when the first right has been exercised below the strike K or above it, respectively. In fact when the underlying price is above a critical value (namely the optimal boundary $c^{(2)}$) it is convenient to "give up" the first put payoff in order to gain the opportunity of holding the American put option after the refracting period.

4.4. Solution of the swing option with n rights

In this section we complete our study of the swing put option by dealing with the general case of n exercise rights. The results follow by induction and we will only sketch their proofs as they are obtained by repeating step by step arguments as those presented in Section 4.3. Let us start by introducing some notation. For $n \geq 2$ we denote $C^{(n)}$ and $D^{(n)}$ the continuation and stopping region, respectively, for the problem (4.8) with value function $V^{(n)}$. Similarly we denote their t-sections by $C_t^{(n)}$ and $D_t^{(n)}$ and to simplify notation we set $T_\delta^{(j)} = T - j\delta$. From now on we fix $n \geq 2$ and since we prove by induction let us introduce

Assumption 4.4.1. *For $j \in \{2, 3, \dots n\}$ and $t \in [0, T_\delta^{(j-1)}]$ one has $D_t^{(j)} = (0, b^{(j)}(t)] \cup [c^{(j)}(t), \infty)$ where*

i) $t \mapsto b^{(j)}(t)$ is continuous, bounded and increasing with $b^{(j)}\big(T_\delta^{(j-1)}\big) = K$,

ii) $t \mapsto c^{(j)}(t)$ is continuous, bounded and decreasing with $c^{(j)}\big(T_\delta^{(j-1)}\big) = K$,

iii) $b^{(j-1)}(t) \leq b^{(j)}(t) < K < c^{(j)}(t) \leq c^{(j-1)}(t)$ for $t \in [0, T_\delta^{(j-1)})$ with the convention $c^{(1)} \equiv +\infty$.

In Section 4.3 we have shown indeed that Assumption 4.4.1 holds for $n = 2$. Now for $1 \leq j \leq n-1$ we also define

$$p_j^{(n)}(t,x,s) := \mathsf{P}\big(X_s^x \in D_{t+s}^{(n)}, X_{s+\delta}^x \in D_{t+s+\delta}^{(n-1)}, \dots \\ \dots, X_{s+(j-1)\delta}^x \in D_{t+s+(j-1)\delta}^{(n-(j-1))}, X_{s+j\delta}^x < b^{(n-j)}(t+s+j\delta)\big) \tag{4.99}$$

$$p_0^{(n)}(t,x,s) := \mathsf{P}\big(X_s^x < b^{(n)}(t+s)\big). \tag{4.100}$$

Under Assumption 4.4.1 one has

$$p_j^{(n)}(t,x,s) - p_j^{(n-1)}(t,x,s) \geq 0 \tag{4.101}$$

for $(t,x,s) \in [0, T_\delta^{(n-1)}] \times (0,\infty) \times [0, T_\delta^{(n-1)} - t]$ and $j = 0, \dots, n-2$ since $D^{(1)} \subseteq D^{(2)} \subseteq \dots \subseteq D^{(n)}$. Let us recall definition (4.6) in order to introduce the next

Assumption 4.4.2. *One has $G^{(n)} \in C^{1,2}$ in $(0, T_\delta^{(n-1)}) \times [(0,K) \cup (K,\infty)]$ with*

$$\big(G_t^{(n)} + I\!L_X G^{(n)} - rG^{(n)}\big)(t,x) = -rK\Big(I(x<K) + \sum_{j=0}^{n-2} e^{-r(j+1)\delta} p_j^{(n-1)}(t,x,\delta)\Big) \quad (4.102)$$

for $t \in (0, T_\delta^{(n-1)})$ and $x \in (0,K) \cup (K,\infty)$. Moreover $V^{(n)}$ is continuous on $[0, T_\delta^{(n-1)}] \times (0,\infty)$, $V^{(n)} \in C^{1,2}$ in $C^{(n)}$ and it solves there

$$V_t^{(n)} + I\!L_X V^{(n)} - rV^{(n)} = 0 \quad (4.103)$$

and for $s \in [0, T_\delta^{(n-1)}]$ and $x \in (0,\infty)$ it holds that

$$\mathsf{E} e^{-rs} V^{(n)}(t+s, X_s^x) = V^{(n)}(t,x) - rK \sum_{j=0}^{n-1} \int_0^s e^{-r(u+j\delta)} p_j^{(n)}(t,x,u)\,du. \quad (4.104)$$

Notice that for $n = 2$ Assumption 4.4.2 holds since (4.19) is equivalent to (4.102) and by taking expectation in (4.83), using (4.84) and (4.85) one obtains (4.104).

Proposition 4.4.3. *Under Assumptions 4.4.1 and 4.4.2 the equation* (4.102) *also holds with n replaced by $n+1$.*

Proof. Observe that by (4.104) one has

$$\begin{aligned} G^{(n+1)}(t,x) =& (K-x)^+ + \mathsf{E} e^{-r\delta} V^{(n)}(t+\delta, X_\delta^x) \quad (4.105) \\ =& (K-x)^+ + V^{(n)}(t,x) - rK \sum_{j=0}^{n-1} \int_0^\delta e^{-r(u+j\delta)} p_j^{(n)}(t,x,u)\,du \end{aligned}$$

for $t \in (0, T_\delta^{(n)})$ and $x \in (0,K) \cup (K,\infty)$. Then by (4.102) it holds (at least formally for now)

$$\begin{aligned} &\big(G_t^{(n+1)} + I\!L_X G^{(n+1)} - rG^{(n+1)}\big)(t,x) \quad (4.106) \\ &= -rK I(x<K) + \big(G_t^{(n)} + I\!L_X G^{(n)} - rG^{(n)}\big)(t,x) I(x \in D_t^{(n)}) \\ &\quad - rK \sum_{j=0}^{n-1} \big(\partial_t + I\!L_X - r\big) g_j^{(n)}(t,x) \end{aligned}$$

for $t \in (0, T_\delta^{(n)})$ and $x \in (0,K) \cup (K,\infty)$, where we have set

$$g_j^{(n)}(t,x) := \int_0^\delta e^{-r(u+j\delta)} p_j^{(n)}(t,x,u)\,du.$$

Using the strong Markov property and by straightforward calculations one can show that, for $j \in \{0, 1, \dots, n-1\}$ and $(t, x) \in (0, T_\delta^{(n)}) \times (0, \infty)$, it holds

$$\lim_{u \to 0} \frac{\mathsf{E} e^{-ru} g_j^{(n)}(t+u, X_u^x) - g_j^{(n)}(t, x)}{u} \tag{4.107}$$
$$= -e^{-rj\delta} p_j^{(n)}(t, x, 0) + e^{-r(j+1)\delta} p_j^{(n)}(t, x, \delta).$$

Hence if we calculate as in (4.19), we obtain $g_j^{(n)}(t, \cdot\,) \in C^1(0, \infty)$ for $t \in (0, T_\delta^{(n)})$ and

$$\big(\partial_t + I\!\!L_X - r\big) g_j^{(n)}(t, x) = -e^{-rj\delta} p_j^{(n)}(t, x, 0) + e^{-r(j+1)\delta} p_j^{(n)}(t, x, \delta) \tag{4.108}$$

for a.e. $(t, x) \in (0, T_\delta^{(n)}) \times (0, \infty)$ and in particular $(g_j^{(n)})_t$ and $(g_j^{(n)})_{xx}$ are only undefined across $b^{(n)}$ and $c^{(n)}$. Recalling (4.102) one gets from (4.106) and (4.108)

$$\big(G_t^{(n+1)} + I\!\!L_X G^{(n+1)} - rG^{(n+1)}\big)(t, x) \tag{4.109}$$
$$\begin{aligned} = &- rKI(x < K) - rK\big[I(x < K) \\ &+ \sum_{j=0}^{n-2} e^{-r(j+1)\delta} p_j^{(n-1)}(t, x, \delta)\big] I(x \in D_t^{(n)}) \\ &+ rK \sum_{j=0}^{n-1} \big(e^{-rj\delta} p_j^{(n)}(t, x, 0) - e^{-r(j+1)\delta} p_j^{(n)}(t, x, \delta)\big) \end{aligned}$$

for a.e. $(t, x) \in (0, T_\delta^{(n)}) \times (0, \infty)$ and using (4.99) and (4.100) we have that

$$I(x < K) I(x \in D_t^{(n)}) = I(x < b^{(n)}(t)) = p_0^{(n)}(t, x, 0) \tag{4.110}$$

$$I(x \in D_t^{(n)}) p_j^{(n-1)}(t, x, \delta) = p_{j+1}^{(n)}(t, x, 0) \tag{4.111}$$

which allow us to finally conclude that

$$\big(G_t^{(n+1)} + I\!\!L_X G^{(n+1)} - rG^{(n+1)}\big)(t, x) = -rK\big(I(x < K) + \sum_{j=0}^{n-1} p_j^{(n)}(t, x, \delta)\big) \tag{4.112}$$

for $t \in (0, T_\delta^{(n)})$ and $x \in (0, K) \cup (K, \infty)$ and the claim is proved. Notice that as already observed in Remark 4.2.1, in (4.112) we appreciate the mollifying effect of the log-normal distribution of X and $R^{(n)}(t, x) := \mathsf{E} e^{-r\delta} V^{(n)}(t+\delta, X_\delta^x)$ turns out to be $C^{1,2}$ on $(0, T_\delta^{(n-1)}) \times (0, \infty)$. $\square$

We now define

$$H^{(n)}(t,x) := \big(G^{(n)}_t + I\!L_X G^{(n)} - rG^{(n)}\big)(t,x) \tag{4.113}$$

for $t \in (0, T^{(n)}_\delta)$, $x \in (0,K) \cup (K,\infty)$ and observe that under Assumption 4.4.1 the map $t \mapsto H^{(n)}(t,x)$ is decreasing for all $x > 0$. This was also the case for $H = H^{(2)}$ in (4.19) and it was the key property needed to prove most of our results in Section 4.3. We are now ready to provide the EEP representation formula of $V^{(n)}$ for $n > 2$ and to characterise the corresponding stopping sets $D^{(n)}$.

Theorem 4.4.4. *For all $n \geq 2$ Assumptions 4.4.1 and 4.4.2 hold true and the value function $V^{(n)}$ of (4.8) has the following representation*

$$\begin{aligned} V^{(n)}(t,x) = {} & e^{-r(T^{(n-1)}_\delta - t)} \mathsf{E} G^{(n)}\big(T^{(n-1)}_\delta, X^x_{T^{(n-1)}_\delta - t}\big) \\ & + rK \sum_{j=0}^{n-1} \int_0^{T^{(n-1)}_\delta - t} e^{-r(u+j\delta)} p^{(n)}_j(t,x,u)\,du \end{aligned} \tag{4.114}$$

for $t \in [0, T^{(n-1)}_\delta]$ and $x \in (0,\infty)$.

Proof. The result is true for $n = 2$, then we argue by induction and assume that it holds for n. By Proposition 4.4.3 we obtain that (4.102) holds with n replaced by $n+1$ and $H^{(n+1)}$ is well defined (cf. (4.113)). Now we repeat step by step (with obvious modifications) the arguments used in Section 4.3 to obtain generalisations of Propositions 4.3.1, 4.3.2, 4.3.3, 4.3.6 and Theorems 4.3.5 and 4.3.8 to the case $n > 2$. We observe that some proofs simplify as the generalisation of Proposition 4.3.2 (which uses (4.101)) immediately implies finiteness of $c^{(n+1)}$ due to finiteness of $c^{(2)}$ and hence $D^{(n+1)}_t \cap (K,\infty) \neq \emptyset$ for $t \in [0, T^{(n)}_\delta]$. Then for the swing option problem with $n+1$ exercise rights there exist two optimal stopping boundaries $b^{(n+1)}$ and $c^{(n+1)}$ which fulfil Assumption 4.4.1 with $n+1$ instead of n (notice that the proof of Theorem 4.3.8 does not rely on the smooth-fit property).

It remains to prove Assumption 4.4.2 and the EEP representation formula for $V^{(n+1)}$. Following the same arguments as in the proof of Proposition 4.3.7 it is possible to show that $V^{(n+1)}_x(t,\,\cdot\,)$ is continuous across $b^{(n+1)}(t)$ and $c^{(n+1)}(t)$ for all $t \in (0, T^{(n)}_\delta)$.

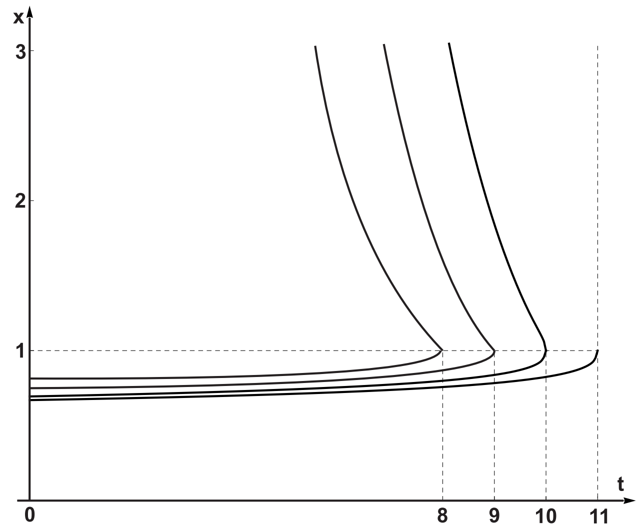

Figure 4.3: Structure of the optimal exercise boundaries $t \mapsto b^{(n)}(t)$ (lower boundary) and $t \mapsto c^{(n)}(t)$ (upper boundary) of problem (4.8) with $n = 2, 3, 4$ and $t \mapsto b^{(1)}(t)$ of problem (4.4) (American put) in the case $K = 1$, $r = 0.1$ (annual), $\sigma = 0.4$ (annual), $T = 11$ months, $\delta = 1$ month.

Then $V^{(n+1)}$ solves a free-boundary problem analogous to (4.67)–(4.73) but with $V^{(2)}$, $G^{(2)}$, $b^{(2)}$, $c^{(2)}$ and T_δ replaced by $V^{(n+1)}$, $G^{(n+1)}$, $b^{(n+1)}$, $c^{(n+1)}$ and $T_\delta^{(n)}$ respectively. Now $V^{(n+1)}$, $b^{(n+1)}$ and $c^{(n+1)}$ satisfy all the conditions needed to apply the local time-space formula of [52] (cf. also proof of Theorem 4.3.9 above), hence by using (4.110) and (4.111) with n replaced by $n+1$ and the fact that $\mathsf{E}p_j^{(n+1)}(t+u, X_u^x, 0) = p_j^{(n+1)}(t, x, u)$, we obtain

$$\begin{aligned} &\mathsf{E}e^{-rs}V^{(n+1)}(t+s, X_s^x) \qquad (4.115)\\ &=V^{(n+1)}(t,x) + \mathsf{E}\int_0^s e^{-ru} H^{(n+1)}(t+u, X_u^x) I(X_u^x \in D_{t+u}^{(n+1)})du \\ &=V^{(n+1)}(t,x) - rK\sum_{j=0}^{n}\int_0^s e^{-r(u+j\,\delta)} p_j^{(n+1)}(t,x,u)du. \end{aligned}$$

Hence $V^{(n+1)}$ satisfies (4.104) and taking $s = T_\delta^{(n)} - t$ and rearranging terms we obtain the EEP representation for the value of the swing option with $n+1$ exercise rights.

Since Assumptions 4.4.1 and 4.4.2 hold for $V^{(n+1)}$, $G^{(n+1)}$, $b^{(n+1)}$ and $c^{(n+1)}$ the proof can be completed by iterating the arguments above. □

It is now matter of routine to substitute $b^{(n)}(t)$ and $c^{(n)}(t)$ into (4.114) to find the integral equations that characterise the optimal boundaries. Arguments analogous to those employed in part 3 of the proof of Theorem 4.3.9 allow us to show that $b^{(n)}$ and $c^{(n)}$ uniquely solve such equations. For completeness we provide the theorem but we omit its proof.

Theorem 4.4.5. *For every $n \geq 2$ the optimal stopping boundaries $b^{(n)}$ and $c^{(n)}$ of Theorem 4.4.4 are the unique couple of functions solving the system of nonlinear integral equations (see Figure 4.3)*

$$G(t, b^{(n)}(t)) = e^{-r(T_\delta^{(n-1)}-t)}\mathsf{E}G^{(n)}\big(T_\delta^{(n-1)}, X^{b^{(n)}(t)}_{T_\delta^{(n-1)}-t}\big) + rK\sum_{j=0}^{n-1}\int_0^{T_\delta^{(n-1)}-t} e^{-r(u+j\delta)}p_j^{(n)}(t, b^{(n)}(t), u)du \tag{4.116}$$

$$G(t, c^{(n)}(t)) = e^{-r(T_\delta^{(n-1)}-t)}\mathsf{E}G^{(n)}\big(T_\delta^{(n-1)}, X^{c^{(n)}(t)}_{T_\delta^{(n-1)}-t}\big) + rK\sum_{j=0}^{n-1}\int_0^{T_\delta^{(n-1)}-t} e^{-r(u+j\delta)}p_j^{(n)}(t, c^{(n)}(t), u)du \tag{4.117}$$

in the class of continuous increasing functions $t \mapsto b^{(n)}(t)$ and continuous decreasing functions $t \mapsto c^{(n)}(t)$ on $[0, T_\delta^{(n-1)}]$ with $b^{(n)}(T_\delta^{(n-1)}) = c^{(n)}(T_\delta^{(n-1)}) = K$.

Chapter 5

Shout put option

5.1. Introduction

Let us imagine an investor who holds a standard European call (or put) option with strike price K and maturity date T. Then there are at least two possible scenarios when the holder feels regret: 1) there is period of time before T where stock movements are favourable for him however he cannot early exercise his option and then the stock price will turn and at time T he gets small or zero payoff; 2) near maturity T the stock price is below K for call (above K for put) option and most likely he gets zero payoff at time T. In last two decades options with reset feature have been introduced and studied and which address these unfavourable scenarios and they can be divided into two groups of options: 1) shout (call or put) option which allows the holder to lock the profit at some favourable time τ (if there is such) and then at time T take the maximum between two payoffs at τ and T; 2) reset (call or put) option gives to investor the right to reset the strike K to current price, i.e. to substitute the current out-of-the money option to the at-the-money one. The first group, i.e. shout options, allows the investor to lock the profit while having the opportunity to increase his payoff at T. The pricing problem for both type of options can be formulated as optimal stopping problems where stopping times represent shouting or reset strategies. They have both European (since the payoff is known at T only) and American features (due to early 'shouting' or 'reset' opportunity).

Below we provide literature review on the shout and reset options. The origin of the shout option goes to the paper [68] and brief analysis can be found in textbooks [73] and [28] where binomial tree method is offered for pricing the option. There are numbers of papers where these options were thoroughly studied from both theoretical and numerical points of view. In series of works [74]-[76] several sophisticated numerical schemes have been developed to price shout options. Then in [15] the reset put option was studied and integral equation for optimal shouting boundary was obtained heuristically and without addressing the question of uniqueness of solution to the integral equation. Theoretical analysis in paper [78] applies PDE and variational approaches to show the existence and uniqueness of solution to the free-boundary problem associated to the reset put option pricing, also monotonicity and regularity of the optimal shouting boundary have been shown in some cases, however no explicit expressions for the price and shouting boundaries were given. Then in [3] using a Laplace transform, the Fredholm integro-differential equations for optimal shouting boundaries of shout call and put options were obtained, monotonicity of the boundaries was claimed without the proof. Finally, in [24] the formal series expansions were discovered for the price and optimal shouting boundaries of the reset put and shout call options, but which have not been proven to either converge or diverge.

In this paper we study the shout put option and formulate the pricing problem as an optimal stopping problem. However it has non-adapted gain function, since the payoff is claimed and known only at T and therefore the problem falls into the class of optimal prediction problems (see e.g. [18]). We reduce it to standard optimal stopping problem with adapted payoff and then reformulate it as a free-boundary problem. The latter we solve by probabilistic arguments including local time-space calculus ([52]). We characterise the optimal shouting boundary as the unique solution to nonlinear integral equation which can be easily solved numerically. Then we derive a shouting premium presentation for the option's price via optimal shouting boundary. These results have been proven for some values of parameters, since in the remaining case the proof of monotonicity of the boundary currently is an open problem. However, numerical drawing of the boundaries shows that they seem to be increasing for all values of the parameters.

We conclude the paper by financial analysis of the shout put option and particularly its financial returns compare to American, European and British (see [59]) put counterparts. The shout put option is more expensive than the American option, however in the numerical example it has been shown that there is a curve between optimal shouting boundary and optimal American put boundary such that above this curve and below K the shout option's returns are greater, which is pleasant for an investor who wishes to lock the profit in that region while enjoying the possibility to earn also from a favourable future movement at the maturity T. On the other hand we can see that the British option generally outperforms both counterparts. The British and shout options both have 'optimal prediction' feature because it is intrinsically built into the former option and the decision of shouting the latter option depends on prediction of the price at T. We note that the technique used in this paper can be applied to pricing shout call and reset call and put options. Moreover, shout put option is equivalent to reset call one in the sense that their optimal strategies coincide and the same fact is true for shout call and reset put options. This was also observed in [24]. We believe also that the approach we used in this paper and ideas from Chapter 4 can be applied for options with multiple shout or reset rights (see e.g. [14] where numerical analysis has been provided using binomial tree method).

The paper is organised as follows. In Section 5.2 we formulate the shout put option problem as an optimal prediction problem, which we reduce to a standard optimal stopping problem. In Section 5.3 we study a free-boundary problem and then in Section 5.4 we derive shouting premium representation for the price of the option and characterise the optimal shouting boundary as the unique solution to a nonlinear integral equation. Using these results in Section 5.5 we present a financial analysis of the shout put option comparing it to American, British and European put options.

5.2. Formulation of the problem

We study the shout put option problem in geometric Brownian motion model

$$dX_t = rX_t\,dt + \sigma X_t\,dB_t \quad (X_0 = x) \tag{5.1}$$

where B is a standard Brownian motion started at zero, $r > 0$ is the interest rate, and σ is the volatility coefficient. The solution X to the stochastic differential equation (5.1) is given by

$$X_t^x = x\exp\Big(\sigma B_t + (r - \sigma^2/2)t\Big) \tag{5.2}$$

for $t \geq 0$, $x > 0$.

By definition of shout put option the payoff at maturity time T is following: if the buyer 'shouts' at time $\tau < T$ he gets $\max(K - X_\tau, K - X_T)$ and if the buyer does not 'shout' until T his payoff equals $(K - X_T)^+$, where $K > 0$ is the strike price. Hence the shout option allows to fix minimal payoff $K - X_\tau$ by shouting at time τ. Clearly one should shout only when $X_\tau < K$. This option of an European type since the payoff is delivered at T, however it has an American type feature of early exercising (or shouting).

If the holder shouts at stopping time $0 \leq \tau \leq T$ with respect to natural filtration of X, then the expected payoff at maturity T under risk-neutral measure P is given by

$$\mathsf{E}\max(K - X_T, K - X_\tau, 0). \tag{5.3}$$

Thus we can associate the arbitrage-free price of the shout put option at $t = 0$ as a value function of the following optimal stopping problem

$$V = e^{-rT} \sup_{0 \leq \tau \leq T} \mathsf{E}\max(K - X_T, K - X_\tau, 0) \tag{5.4}$$

where we include the discount factor e^{-rT} since the payoff is delivered at $t = T$. It is important to note that since the gain function in (5.4) is not adapted to the natural filtration of X, the optimal stopping problem (5.4) falls into the class of optimal prediction problems (see e.g. [18]). Hence we first need to reduce this problem to a standard optimal stopping problem with adapted gain function. For this we prove the following lemma

Lemma 5.2.1. . *We have following identity*

$$\mathsf{E}\big[\max(K-X_T, K-X_t, 0)|\mathcal{F}_t^X\big] = (K-X_t)^+ + \widetilde{G}(t, X_t) \tag{5.5}$$

where the function

$$\begin{aligned}\widetilde{G}(t,x) &= E(\min(x,K)-X_{T-t}^x)^+ \\ &= \min(x,K)\Phi\Big(\tfrac{1}{\sigma\sqrt{T-t}}\big[\log(\tfrac{\min(x,K)}{x})-(r-\tfrac{\sigma^2}{2})(T-t)\big]\Big) \\ &\quad - xe^{r(T-t)}\Phi\Big(\tfrac{1}{\sigma\sqrt{T-t}}\big[\log(\tfrac{\min(x,K)}{x})-(r+\tfrac{\sigma^2}{2})(T-t)\big]\Big)\end{aligned} \tag{5.6}$$

for $0 \le t \le T$ and $x > 0$ is the price of the European put option without discount factor at time t with the stock price x, the strike price $\min(x, K)$ *and maturity time T.*

Proof. By stationary independent increments of B and we have

$$\mathsf{E}\big[\max(K-X_T, K-X_t, 0)|\mathcal{F}_t^X\big] = \big(\mathsf{E}\max(K-xZ_{T-t}, K-x, 0)\big)|_{x=X_t} \tag{5.7}$$

for t fixed and where Z is the solution to (5.1) with $Z_0 = 1$. Straightforward calculations give

$$\begin{aligned}\mathsf{E}\max(K-xZ_{T-t}, K-x, 0) &= \mathsf{E}\max(K-xZ_{T-t} - (K-x)^+), 0) + (K-x)^+ \\ &= \mathsf{E}(\min(x,K)-xZ_{T-t})^+ + (K-x)^+ \\ &= \widetilde{G}(t,x) + (K-x)^+\end{aligned} \tag{5.8}$$

for all $x > 0$. Combining (5.7) and (5.8) we obtain (5.5). □

Standard arguments based on the fact that each stopping time can be written as the limit of a decreasing sequence of discrete stopping times (see e.g. [20, Ch. 2, Sec. 1]) imply that (5.5) can be extended to for all stopping times τ of X taking values in $[0, T]$ and taking the supremum on both sides over all such stopping times we can rewrite now the problem (5.4) in the following form

$$V = e^{-rT} \sup_{0\le\tau\le T} \mathsf{E}\big[(K-X_\tau)^+ + \widetilde{G}(\tau, X_\tau)\big]. \tag{5.9}$$

We will study the problem (5.9) in the Markovian setting and thus we introduce dependance on time t and initial points of X:

$$V(t,x) = \sup_{0 \le \tau \le T-t} \mathsf{E} G(t+\tau, X_\tau^x) \tag{5.10}$$

for $0 \le t \le T$ and $x > 0$ where the gain function reads

$$G(t,x) := (K-x)^+ + \widetilde{G}(t,x). \tag{5.11}$$

We note that the value function (5.10) represents the undiscounted option's price at time t so that we multiply $V(t,x)$ by $e^{-r(T-t)}$ to get the discounted price.

5.3. Free-boundary problem

In this section we will reduce the problem (5.10) into a free-boundary problem and the latter will be tackled in the next section using local time-space calculus ([52]). First using that the gain function $G(t,x)$ is continuous and standard arguments (see e.g. Corollary 2.9 (Finite horizon) with Remark 2.10 in [58]) we have that continuation and stopping sets read

$$C = \{\, (t,x) \in [0,T) \times [0,\infty) : V(t,x) > G(t,x) \,\} \tag{5.12}$$

$$D = \{\, (t,x) \in [0,T) \times [0,\infty) : V(t,x) = G(t,x) \,\} \tag{5.13}$$

and the optimal stopping time in (5.10) is given by

$$\tau_b = \inf \{ \; 0 \le s \le T-t : (t+s, X_s^x) \in D \; \}. \tag{5.14}$$

Throughout this paper we need to make an assumptions on parameters which are though financially reasonable:

$$1) r \le \sigma/\sqrt{T} \quad \& \quad 2) r \ge \sigma^2/2. \tag{5.15}$$

For instance if we consider annual values of parameters and we take an option with $T = 1$ then the condition 1) becomes very natural: $r \le \sigma$. The condition 2) holds usually for not very 'volatile' assets.

1. We will show now that the functions G and V are convex with respect to x for any fixed $t \in [0,T)$. The gain function G reads

$$G(t,x) = K + x\big[\Phi\big(-(\tfrac{r}{\sigma}-\tfrac{\sigma}{2})\sqrt{T-t}\big) - e^{r(T-t)}\Phi\big(-(\tfrac{r}{\sigma}+\tfrac{\sigma}{2})\sqrt{T-t}\big) - 1\big] \quad (5.16)$$

for $0 < x < K$. On other hand G equals to

$$\begin{aligned} G(t,x) = {} & K\Phi\Big(\tfrac{1}{\sigma\sqrt{T-t}}\big[\log(\tfrac{K}{x}) - (r-\tfrac{\sigma^2}{2})(T-t)\big]\Big) \\ & - xe^{r(T-t)}\Phi\Big(\tfrac{1}{\sigma\sqrt{T-t}}\big[\log(\tfrac{K}{x}) - (r+\tfrac{\sigma^2}{2})(T-t)\big]\Big) \end{aligned} \quad (5.17)$$

for $x \geq K$ and this is exactly the price $P(t,x)$ of the European put option multiplied by $e^{r(T-t)}$. We know that $x \mapsto P(t,x)$ is convex on $(0,\infty)$ for any $0 \leq t < T$ fixed so that $x \mapsto G(t,x)$ is convex on (K,∞) for every t fixed. Since G is linear in x on $(0,K)$ in order to prove that $x \mapsto G(t,x)$ is convex on $(0,\infty)$ for every t fixed we need to show that $G_x(t,K+) \geq G_x(t,K-)$. Using (5.16), (5.17) and well-known expression for 'delta' coefficient of the European put option $\Delta = \frac{\partial P}{\partial x} = -\Phi\big(\frac{1}{\sigma\sqrt{T-t}}\big[\log(\frac{K}{x}) - (r+\frac{\sigma^2}{2})(T-t)\big]\big)$ we have that

$$G_x(t,K+) = -e^{r(T-t)}\Phi\big(-(\tfrac{r}{\sigma}+\tfrac{\sigma}{2})\sqrt{T-t}\big) \quad (5.18)$$

$$G_x(t,K-) = \Phi\big(-(\tfrac{r}{\sigma}-\tfrac{\sigma}{2})\sqrt{T-t}\big) - e^{r(T-t)}\Phi\big(-(\tfrac{r}{\sigma}+\tfrac{\sigma}{2})\sqrt{T-t}\big) - 1 \quad (5.19)$$

and thus it is clear that $G_x(t,K+) \geq G_x(t,K-)$. Hence the function $x \mapsto G(t,x)$ is convex on $(0,\infty)$ for every t fixed and thus using (5.10) $x \mapsto V(t,x)$ is convex on $(0,\infty)$ as well.

2. Below we calculate the expression $\widetilde{H} := \widetilde{G}_t + I\!L_X\widetilde{G}$ for $(t,x) \in [0,T) \times (0,\infty)$ where $I\!L_X = rxd/dx + (\sigma^2/2)x^2 d^2/dx^2$ the infinitesimal generator of X. Since we have that $\widetilde{G}(t,x) = e^{r(T-t)}P(t,x)$ for $(t,x) \in [0,T) \times [K,\infty)$ and it is well-known that $P_t + I\!L_X P - rP = 0$ for all $(t,x) \in [0,T) \times (0,\infty)$ then

$$\widetilde{H}(t,x) = 0 \quad \text{on} \quad [0,T) \times [K,\infty). \quad (5.20)$$

Now we consider set $\{(t,x) \in [0,T) \times (0,K)\}$ and there $\widetilde{G}$ reads

$$\widetilde{G}(t,x) = x\big[\Phi\big(-(\tfrac{r}{\sigma}-\tfrac{\sigma}{2})\sqrt{T-t}\big) - e^{r(T-t)}\Phi\big(-(\tfrac{r}{\sigma}+\tfrac{\sigma}{2})\sqrt{T-t}\big)\big] \quad (5.21)$$

so that we have

$$\widetilde{G}_t(t,x) = x\big[re^{r(T-t)}\Phi\big(-(\tfrac{r}{\sigma}+\tfrac{\sigma}{2})\sqrt{T-t}\big) - \tfrac{\sigma}{2\sqrt{T-t}}\varphi\big(-(\tfrac{r}{\sigma}-\tfrac{\sigma}{2})\sqrt{T-t}\big)\big] \quad (5.22)$$

$$I\!\!L_X\widetilde{G}(t,x) = r\widetilde{G}(t,x) \quad (5.23)$$

which gives

$$\widetilde{H}(t,x) = x\big[r\Phi\big(-(\tfrac{r}{\sigma}-\tfrac{\sigma}{2})\sqrt{T-t}\big) - \tfrac{\sigma}{2\sqrt{T-t}}\varphi\big(-(\tfrac{r}{\sigma}-\tfrac{\sigma}{2})\sqrt{T-t}\big)\big] \quad (5.24)$$

for $(t,x)\in[0,T)\times(0,K)$. For further analysis the following function is useful

$$\begin{aligned} H(t,x) :=&(G_t+I\!\!L_X G)(t,x) = -rx+\widetilde{H}(t,x) \quad (5.25)\\ =&x\big[r\Phi\big(-(\tfrac{r}{\sigma}-\tfrac{\sigma}{2})\sqrt{T-t}\big) - \tfrac{\sigma}{2\sqrt{T-t}}\varphi\big(-(\tfrac{r}{\sigma}-\tfrac{\sigma}{2})\sqrt{T-t}\big) - r\big]\\ =&xf(t)\end{aligned}$$

for $(t,x)\in[0,T)\times(0,K)$ where we used definitions of G and $\widetilde{H}$, the expression (5.24) and we define the function $f:[0,T)\to(-\infty,0)$

$$f(t) = r\Phi\big(-(\tfrac{r}{\sigma}-\tfrac{\sigma}{2})\sqrt{T-t}\big) - \tfrac{\sigma}{2\sqrt{T-t}}\varphi\big(-(\tfrac{r}{\sigma}-\tfrac{\sigma}{2})\sqrt{T-t}\big) - r \quad (5.26)$$

for $t\in[0,T)$.

Now we show that the function $t\mapsto\widetilde{H}(t,x)$ is decreasing on $[0,T]$ for any given and fixed $x\in(0,K)$. Indeed taking the derivative with respect to t in (5.24) we have

$$\begin{aligned}\widetilde{H}_t(t,x) &= x\varphi\big(-(\tfrac{r}{\sigma}-\tfrac{\sigma}{2})\sqrt{T-t}\big)\big[(\tfrac{r}{\sigma}-\tfrac{\sigma}{2})\tfrac{r}{2\sqrt{T-t}} - (\tfrac{r}{\sigma}-\tfrac{\sigma}{2})^2\tfrac{\sigma}{4\sqrt{T-t}} - \tfrac{\sigma}{4(T-t)^{3/2}}\big] \quad (5.27)\\ &= \tfrac{1}{2\sigma\sqrt{T-t}}x\varphi\big(-(\tfrac{r}{\sigma}-\tfrac{\sigma}{2})\sqrt{T-t}\big)\Big[(r-\tfrac{\sigma^2}{2})r - \tfrac{1}{2}(r-\tfrac{\sigma^2}{2})^2 - \tfrac{\sigma^2}{2(T-t)}\Big]\\ &= \tfrac{1}{2\sigma\sqrt{T-t}}x\varphi\big(-(\tfrac{r}{\sigma}-\tfrac{\sigma}{2})\sqrt{T-t}\big)\big[(r-\tfrac{\sigma^2}{2})(r-\tfrac{1}{2}(r-\tfrac{\sigma^2}{2})) - \tfrac{\sigma^2}{2(T-t)}\big]\\ &= \tfrac{1}{4\sigma\sqrt{T-t}}x\varphi\big(-(\tfrac{r}{\sigma}-\tfrac{\sigma}{2})\sqrt{T-t}\big)\big[(r^2-\tfrac{\sigma^4}{4}) - \tfrac{\sigma^2}{T-t}\big]\\ &\le \tfrac{1}{4\sigma\sqrt{T-t}}x\varphi\big(-(\tfrac{r}{\sigma}-\tfrac{\sigma}{2})\sqrt{T-t}\big)\big[r^2-\tfrac{\sigma^2}{T}\big] \le 0\end{aligned}$$

for $(t,x)\in[0,T)\times(0,K)$. It follows from (5.27) that condition 1) in (5.15) we have that $\widetilde{H}_t<0$ and thus $t\mapsto\widetilde{H}(t,x)$ is decreasing on $[0,T]$ for any $x\in(0,K)$.

Using Ito-Tanaka's formula and (5.25) with (5.18)-(5.19) we have

$$\mathsf{E}G(t+\tau,X_\tau^x) = G(t,x) + \mathsf{E}\int_0^\tau H(t+s,X_s^x)I(X_s^x\le K)ds \quad (5.28)$$

$$+\frac{1}{2}\mathsf{E}\int_0^\tau \Phi\big((\tfrac{r}{\sigma}-\tfrac{\sigma}{2})\sqrt{T-t_2-s}\big)d\ell_s^K(X^x)$$

and

$$\mathsf{E}\widetilde{G}(t+\tau, X_\tau^x) = \widetilde{G}(t,x) + \mathsf{E}\int_0^\tau \widetilde{H}(t+s, X_s^x)I(X_s^x \le K)ds \tag{5.29}$$
$$+\frac{1}{2}\mathsf{E}\int_0^\tau (\Phi\big((\tfrac{r}{\sigma}-\tfrac{\sigma}{2})\sqrt{T-t_2-s}\big)-1)d\ell_s^K(X^x)$$

for $(t,x) \in [0,T) \times (0,\infty)$ where $(\ell_s^K(X))_{s\ge 0}$ is the local time process of X at level K and we used $\Phi(-x) = 1-\Phi(x)$ for any $x \in I\!R$.

3. Below we will describe the structure of the stopping set D. Namely, from the fact that it is not optimal to exercise the shout option about K, (5.12), (5.13), convexity of V and linearity of G below K it follows that there exists an optimal shouting boundary $b : [0,T] \to I\!R$ such that

$$\tau_b = \inf\ \{\ 0 \le s \le T-t : X_s^x \le b(t+s)\} \tag{5.30}$$

is optimal in (5.10) and $b(t) < K$ for $t \in [0,T)$.

4. Now we show that the value function $(t,x) \mapsto V(t,x)$ is continuous on $[0,T] \times (0,\infty)$. For this, it is enough to prove that

$$x \mapsto V(t,x) \quad \text{is continuous at } x_0 \tag{5.31}$$

$$t \mapsto V(t,x) \quad \text{is continuous at } t_0 \text{ uniformly over } x \in [x_0-\delta, x_0-\delta] \tag{5.32}$$

for each $(t_0,x_0) \in [0,T] \times (0,\infty)$ with some $\delta > 0$ small enough, which may depend on x_0.

Since (5.31) follows from the fact that $x \mapsto V(t,x)$ is convex on $(0,\infty)$, it remains to establish (5.32). Let us fix any $0 \le t_1 < t_2 \le T$ and $x \in (0,\infty)$ and let τ_1 be the optimal stopping time for $V(t_1,x)$ and we set $\tau_2 \le \tau_1 \wedge (T-t_2)$ then we have

$$0 \le V(t_1,x) - V(t_2,x) \tag{5.33}$$
$$\le \mathsf{E}(K-X_{\tau_1}^x)^+ - \mathsf{E}(K-X_{\tau_2}^x)^+ + \mathsf{E}\big(\widetilde{G}(t_1+\tau_1, X_{\tau_1}^x) - \widetilde{G}(t_2+\tau_2, X_{\tau_2}^x)\big)$$
$$\le \mathsf{E}(X_{\tau_2}^x - X_{\tau_1}^x)^+ + \mathsf{E}\big(\widetilde{G}(t_1+\tau_1, X_{\tau_1}^x) - \widetilde{G}(t_2+\tau_2, X_{\tau_2}^x)\big)$$

where we used fact that $(K-y)^+ - (K-z)^+ \leq (z-y)^+$ for $y, z \in I\!R$. Now to show the uniform convergence over $x \in [x_0-\delta, x_0+\delta]$ of the first term in (5.33) we use an estimation from [58, p. 381]

$$\mathsf{E}(X^x_{\tau_2} - X^x_{\tau_1})^+ \leq xe^{rT}u(t_2-t_1) \tag{5.34}$$

where function u has property $u(t) \to 0$ as $t \to 0$. For the second term, letting $t_2 - t_1 \to 0$ and thus $\tau_1^\varepsilon - \tau_2^\varepsilon \to 0$ we see that it goes to zero by dominant convergence as the function $\widetilde{G} \leq K$. This shows (5.32) and thus the proof of the continuity of V is complete.

5. We show that b is increasing on $[0, T]$ under assumptions (5.15). Let us fix take $0 \leq t_1 < t_2 < T$ and $x \in (0, K)$, denote by τ the optimal stopping time for $V(t_2, x)$ and we have that

$$\begin{aligned} &V(t_1,x) - V(t_2,x) \\ &\quad \geq \mathsf{E}\widetilde{G}(t_1{+}\tau, X^x_\tau) - \mathsf{E}\widetilde{G}(t_2{+}\tau, X^x_\tau) \\ &\quad = \widetilde{G}(t_1,x) - \widetilde{G}(t_2,x) + \mathsf{E}\int_0^\tau \big(\widetilde{H}(t_1{+}s, X^x_s) - \widetilde{H}(t_2{+}s, X^x_s)\big)I(X^x_s \leq K)ds \\ &\quad\quad + \frac{1}{2}\mathsf{E}\int_0^\tau \big(\Phi\big((\tfrac{r}{\sigma}-\tfrac{\sigma}{2})\sqrt{T-t_1-s}\big) - \Phi\big((\tfrac{r}{\sigma}-\tfrac{\sigma}{2})\sqrt{T-t_2-s}\big)d\ell^K_s(X^x) \\ &\quad \geq \widetilde{G}(t_1,x) - \widetilde{G}(t_2,x) = G(t_1,x) - G(t_2,x) \end{aligned} \tag{5.35}$$

where we used (5.29) and that the map $t \mapsto \widetilde{H}(t,x)$ is decreasing and $r \geq \sigma^2/2$. Hence if a point $(t_2, x) \in C$ then $V(t_1,x) - G(t_1,x) \geq V(t_2,x) - G(t_2,x) > 0$ and $(t_1, x) \in C$ as well which shows that the optimal shouting boundary b is increasing on $[0, T]$ under assumptions (5.15).

Remark 5.3.1. *If in the proof above we preserve the condition 1) of* (5.15) *and consider the case $r < \sigma^2/2$ then the the derivative of $\widetilde{H}$ with respect to time becomes more negative, however the integrand of integral with respect to local time in* (5.35) *turns up to be increasing in time. Unfortunately we are not able to compare two integrals with opposite signs in* (5.35) *and prove that b is monotone in this case, however numerical analysis and computer drawing show that b is still increasing. The similar conclusion has been observed by [78] for the reset put option. Therefore we* *have to assume the condition $r > \sigma^2/2$ since below our proofs of smooth-fit condition, and also of the continuity and bounded variation of b are based on its monotonicity. The proof of these facts above without using monotonicity of the optimal stopping boundaries is open and useful problem, which can help to tackle some optimal stopping problems.*

6. Now we prove that the smooth-fit condition holds

$$V_x(t, b(t)+) = G_x(t, b(t)) \tag{5.36}$$

for all $t \in [0, T)$ under assumptions (5.15). For this, let us fix a point $(t, x) \in [0, T) \times (0, \infty)$ lying on the boundary b so that $x = b(t)$. Then we have

$$\frac{V(t, x+\varepsilon) - V(t, x)}{\varepsilon} \geq \frac{G(t, x+\varepsilon) - G(t, x)}{\varepsilon} \tag{5.37}$$

and hence, taking the limit in (5.37) as $\varepsilon \downarrow 0$, we get

$$V_x(t, x+) \geq G_x(t, x) \tag{5.38}$$

where the right-hand derivative exists by convexity of $x \mapsto V(t, x)$ on $(0, \infty)$ for any fixed $t \in [0, T)$.

To prove the reverse inequality, we set $\tau_\varepsilon = \tau_\varepsilon(t, x+\varepsilon)$ as optimal stopping time for $V(t, x+\varepsilon)$. Using fact that $t \mapsto b(t)$ is increasing under assumptions (5.15) and the law of the iterated logarithm at zero for Brownian motion we have that $\tau_\varepsilon \to 0$ as $\varepsilon \to 0$, P-a.s. Then by the mean value theorem we have

$$\begin{aligned} \frac{1}{\varepsilon}\Big(V(t, x+\varepsilon) - V(t, x)\Big) &\leq \frac{1}{\varepsilon}\mathsf{E}\Big[G(t+\tau_\varepsilon, X^{x+\varepsilon}_{\tau_\varepsilon}) - G(t+\tau_\varepsilon, X^{x}_{\tau_\varepsilon})\Big] \\ &\leq \frac{1}{\varepsilon}\mathsf{E}\Big[G_x(t+\tau_\varepsilon, \xi)\big(X^{x+\varepsilon}_{\tau_\varepsilon} - X^{x}_{\tau_\varepsilon}\big)\Big] \\ &= \mathsf{E}\Big[G_x(t+\tau_\varepsilon, \xi)X^{1}_{\tau_\varepsilon}\Big] \end{aligned} \tag{5.39}$$

with $\xi \in [X^x_{\tau_\varepsilon}, X^{x+\varepsilon}_{\tau_\varepsilon}]$ for all $\omega \in \Omega$. Thus using dominated convergence theorem with the fact that $|G_x(t, x)| \leq 3$ for any $(t, x) \in [0, T) \times (0, \infty)$ by (5.16) and also $\tau_\varepsilon \to 0$ we have that

$$V_x(t, x+) \leq G_x(t, x) \tag{5.40}$$

Thus combining (5.38) and (5.40) we obtain (5.36).

7. Here we prove that the boundary b is continuous on $[0,T]$ and that $b(T)=K$ under assumptions (5.15). The proof is provided in 3 steps.

(i) We first show that b is right-continuous. Let us consider b, fix $t\in[0,T)$ and take a sequence $t_n\downarrow t$ as $n\to\infty$. Since b is increasing, the right-limit $b(t+)$ exists and $(t_n,b(t_n))$ belongs to D for all $n\geq 1$. Recall that D is closed so that $(t_n,b(t_n))\to(t,b(t+))\in D$ as $n\to\infty$ and we may conclude that $b(t+)\leq b(t)$. The fact that b is increasing gives the reverse inequality thus b is right-continuous as claimed.

(ii) Now we prove that b is also left-continuous. Assume that there exists $t_0\in(0,T)$ such that $b(t_0-)<b(t_0)$ where $b(t_0-)$ denotes the left-limit of b at t_0. Take $x_1<x_2$ such that $b(t_0-)<x_1<x_2<b(t_0)$ and $h>0$ such that $t_0>h$, then by defining $u:=V-G$ and using (5.45), (5.25), (5.49) we have

$$u_t+\mathbb{L}_X u=-H \quad \text{on } C \text{ and below } K \tag{5.41}$$

$$u(t_0,x)=0 \quad \text{for } x\in(x_1,x_2). \tag{5.42}$$

Denote by $C_c^\infty(a,b)$ the set of continuous functions which are differentiable infinitely many times with continuous derivatives and compact support on (a,b). Take $\varphi\in C_c^\infty(x_1,x_2)$ such that $\varphi\geq 0$ and $\int_{x_1}^{x_2}\varphi(x)dx=1$. Multiplying (5.41) by φ and integrating by parts we obtain

$$\begin{aligned}\int_{x_1}^{x_2}\varphi(x)u_t(t,x)dx=&-\int_{x_1}^{x_2}u(t,x)\mathbb{L}_X^*\varphi(x)dx\\&-\int_{x_1}^{x_2}H(t,x)\varphi(x)dx\end{aligned} \tag{5.43}$$

for $t\in(t_0-h,t_0)$ and with $\mathbb{L}_X^*$ denoting the formal adjoint of $\mathbb{L}_X$. Since $u_t\leq 0$ in C below K by (5.35), the left-hand side of (5.43) is negative. Then taking limits as $t\to t_0$ and by using dominated convergence theorem we find

$$\begin{aligned}0&\geq-\int_{x_1}^{x_2}u(t_0,x)\mathbb{L}_X^*\varphi(x)dx-\int_{x_1}^{x_2}H(t_0,x)\varphi(x)dx\\&=-\int_{x_1}^{x_2}H(t_0,x)\varphi(x)dx\end{aligned} \tag{5.44}$$

where we have used that $u(t_0, x) = 0$ for $x \in (x_1, x_2)$ by (5.41). We now observe that $H(t_0, x) < -c$ for $x \in (x_1, x_2)$ and suitable $c > 0$ by (5.25), therefore (5.44) leads to a contradiction and it must be $b(t_0-) = b(t_0)$.

(*iii*) To prove that $b(T) = K$ is left-continuous we can use the same arguments as those in (*ii*) above with $t_0 = T$ and suppose that $b(T) < K$.

8. The facts proved in paragraphs 1-7 above and standard arguments based on the strong Markov property lead to the following free-boundary problem for the value function V and unknown boundary b:

$$V_t + I\!\!L_X V = 0 \qquad \text{in } C \tag{5.45}$$

$$V(t, b(t)) = G(t, b(t)) \qquad \text{for } t \in [0, T) \tag{5.46}$$

$$V_x(t, b(t)+) = G_x(t, b(t)) \qquad \text{for } t \in [0, T) \tag{5.47}$$

$$V(t, x) > G(t, x) \qquad \text{in } C \tag{5.48}$$

$$V(t, x) = G(t, x) \qquad \text{in } D \tag{5.49}$$

where the continuation set C and the stopping set D are given by

$$C = \{\, (t, x) \in [0, T) \times (0, \infty) : x > b(t) \,\} \tag{5.50}$$

$$D = \{\, (t, x) \in [0, T) \times (0, \infty) : x \le b(t) \,\}. \tag{5.51}$$

The following properties of V and b were also verified above:

$$V \text{ is continuous on } [0, T] \times (0, \infty) \tag{5.52}$$

$$V \text{ is } C^{1,2} \text{ on } C \tag{5.53}$$

$$x \mapsto V(t, x) \text{ is decreasing and convex} \tag{5.54}$$

$$t \mapsto V(t, x) \text{ is decreasing} \tag{5.55}$$

$$t \mapsto b(t) \text{ is increasing and continuous with } b(T-) = K. \tag{5.56}$$

5.4. The arbitrage-free price of the shout option

We now provide the shouting premium representation formula for the undiscounted arbitrage-free price V which decomposes it into the sum of the undiscounted

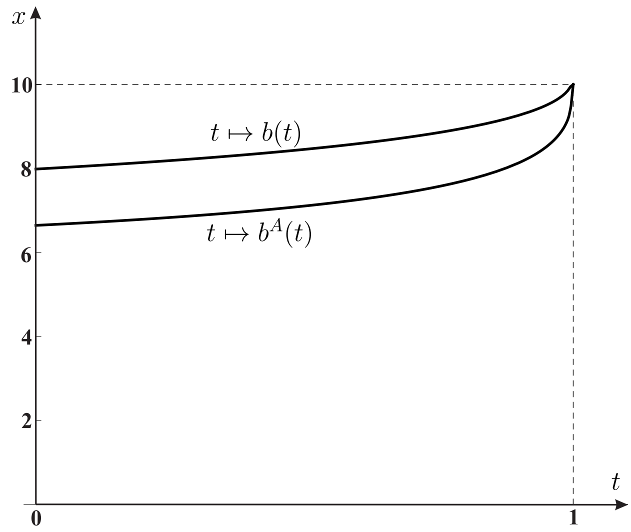

Figure 5.1: A computer drawing of the optimal shouting boundary $t \mapsto b(t)$ (upper) for the shout put option (5.10) and the optimal exercise boundary $t \mapsto b^A(t)$ (lower) for the American put option in the case $K = 10$, $r = 0.1$, $\sigma = 0.4$, $T = 1$.

European put option price and 'shouting' premium. The optimal shouting boundary b will be obtained as the unique solution to the integral equation. We recall that we assume conditions (5.15).

We will make use of the following functions in Theorem 5.4.1 below:

$$L(t,u,x,z) = -f(t+u)\mathsf{E}X_u^x I(X_u^x \le z) = -f(t+u)\int_0^z y\, h(y;x,u)dy \tag{5.57}$$

for $t, u \ge 0$ and $x, z > 0$ and where $h(y) = h(y;x,u)$ is the probability density function of X_u^x under P.

1. The main result of this section may now be stated as follows.

Theorem 5.4.1. *The value function V of* (5.10) *has the following representation*

$$V(t,x) = \mathsf{E}(K - X_{T-t}^x)^+ + \int_0^{T-t} L(t,u,x,b(t+u))du \tag{5.58}$$

for $t \in [0,T]$ and $x \in (0,\infty)$. The optimal shouting boundary b in (5.10) *can be characterised as the unique solution to the nonlinear integral equation*

$$G(t,b(t)) = \mathsf{E}(K - X_{T-t}^{b(t)})^+ + \int_0^{T-t} L(t,u,b(t),b(t+u))du \tag{5.59}$$

for $t \in [0,T]$ in the class of continuous functions $t \mapsto b(t)$ with $b(T) = K$.

Proof. (A) First we clearly have that the following conditions hold: (i) V is $C^{1,2}$ on $C \cup D$; (ii) b is of bounded variation due to monotonicity; (iii) $V_t + I\!L_X V$ is locally bounded; (iv) $x \mapsto V(t,x)$ is convex (recall paragraph 1 above); (v) $t \mapsto V_x(t, b(t)\pm)$ is continuous (recall (5.36)). Hence we can apply the local time-space formula on curves [52] for $V(t+s, X_s^x)$:

$$
\begin{aligned}
V(t+s, X_s^x) &= V(t,x) + M_s \\
&\quad + \int_0^s (V_t + I\!L_X V)(t+u, X_u^x) I(X_u^x \le b(t+u))du \\
&\quad + \frac{1}{2}\int_0^s \big(V_x(t+u, X_u^x+) - V_x(t+u, X_u^x-)\big) I\big(X_u^x = b(t+u)\big) d\ell_u^b(X^x) \\
&= V(t,x) + M_s + \int_0^s (G_t + I\!L_X G)(t+u, X_u^x) I(X_u^x \le b(t+u))du \\
&= V(t,x) + M_s + \int_0^s f(t+u) X_u^x I(X_u^x \le b(t+u))du
\end{aligned}
\tag{5.60}
$$

where we used (5.45), (5.25) and smooth-fit conditions (5.47) and where $M = (M_u)_{u \ge 0}$ is the martingale part, $(\ell_u^b(X^x))_{u \ge 0}$ is the local time process of X^x spending at boundary b. Now upon letting $s = T_\delta - t$, taking the expectation E, the optional sampling theorem for M, rearranging terms and noting that $V(T,x) = G(T,x) = (K-x)^+$ for all $x > 0$, we get (5.58). The integral equation (5.59) one obtains by simply putting $x = b(t)$ into (5.58) and using (5.46).

(B) Now we show that b is the unique solution to the equation (5.59) in the class of continuous functions $t \mapsto b(t)$ with $b(T) = K$. Note that there is no need to assume that b is increasing.

$(B.1)$ Let $c : [0,T] \to I\!R$ be a solution to the equation (5.59) such that c is continuous. We will show that these c must be equal to the optimal shouting boundary b. Now let us consider the function $U^c : [0,T) \to I\!R$ defined as follows

$$
U^c(t,x) = \mathsf{E} G(T, X_{T-t}^x) + \int_0^{T-t} L(t,u,x,c(t+u))du \tag{5.61}
$$

for $(t,x) \in [0,T] \times (0,\infty)$. Observe the fact that c solves the equation (5.59) means exactly that $U^c(t,c(t)) = G(t,c(t))$ for all $t \in [0,T]$. We will moreover show that

$U^c(t,x) = G(t,x)$ for $x \in (0, c(t)]$ with $t \in [0,T]$. This can be derived using martingale property as follows; the Markov property of X implies that

$$U^c(t{+}s, X_s^x) - \int_0^s f(t{+}u) X_u^x I(X_u^x \le c(t{+}u)) du = U^c(t,x) + N_s \tag{5.62}$$

where $(N_s)_{0 \le s \le T-t}$ is a martingale under P. On the other hand, we know from (5.28)

$$\begin{aligned} G(t{+}s, X_s^x) = G(t,x) + \int_0^s f(t{+}u) X_u^x I(X_u^x \le K) du + M_s \\ + \frac{1}{2} \int_0^s \Phi\big((\tfrac{r}{\sigma} - \tfrac{\sigma}{2})\sqrt{T{-}t{-}u}\big) d\ell_u^K(X^x) \end{aligned} \tag{5.63}$$

where $(M_s)_{0 \le s \le T-t}$ is a continuous martingale under P.

For $x \in (0, c(t)]$ with $t \in [0,T]$ given and fixed, consider the stopping time

$$\sigma_c = \inf \{ \ 0 \le s \le T{-}t : c(t{+}s) \le X_s^x \ \} \tag{5.64}$$

under P. Using that $U^c(t, c(t)) = G(t, c(t))$ for all $t \in [0,T]$ and $U^c(T,x) = G(T,x)$ for all $x > 0$, we see that $U^c(t{+}\sigma_c, X_{\sigma_c}^x) = G(t{+}\sigma_c, X_{\sigma_c}^x)$. Hence from (5.63) and (5.64) using the optional sampling theorem we find:

$$\begin{aligned} U^c(t,x) &= \mathsf{E} U^c(t{+}\sigma_c, X_{\sigma_c}^x) - \mathsf{E} \int_0^{\sigma_c} f(t{+}u) X_u^x I(X_u^x \le c(t{+}u)) du \\ &= \mathsf{E} G(t{+}\sigma_c, X_{\sigma_c}^x) - \mathsf{E} \int_0^{\sigma_c} f(t{+}u) X_u^x du = G(t,x) \end{aligned} \tag{5.65}$$

since $X_u^x \in (0, c(t{+}u))$ and $\ell_u^K(X^x) = 0$ for all $u \in [0, \sigma_c)$. This proves that $U^c(t,x) = G(t,x)$ for $x \in (0, c(t)]$ with $t \in [0,T]$ as claimed.

$(B.2)$ We show that $U^c(t,x) \le V(t,x)$ for all $(t,x) \in [0,T] \times (0,\infty)$. For this consider the stopping time

$$\tau_c = \inf \{ \ 0 \le s \le T{-}t : X_s^x \le c(t{+}s) \ \} \tag{5.66}$$

under P with $(t,x) \in [0,T] \times (0,\infty)$ given and fixed. The same arguments as those following (5.64) above show that $U^c(t{+}\tau_c, X_{\tau_c}^x) = G(t{+}\tau_c, X_{\tau_c}^x)$. Inserting τ_c instead of s in (5.62) and using the optional sampling theorem, we get:

$$U^c(t,x) = \mathsf{E} U^c(t{+}\tau_c, X_{\tau_c}^x) = \mathsf{E} G(t{+}\tau_c, X_{\tau_c}^x) \le V(t,x) \tag{5.67}$$

proving the claim.

$(B.3)$ We show that $c \geq b$ on $[0,T]$. For this, suppose that there exists $t \in [0,T)$ such that $b(t) > c(t)$ and choose a point $x \in (0, c(t)]$ and consider the stopping time

$$\sigma = \inf \{ \ 0 \leq s \leq T - t : b(t+s) \leq X_s^x \ \} \tag{5.68}$$

under P. Inserting σ instead of s in (5.60) and (5.62) and using the optional sampling theorem, we get:

$$\mathsf{E}V(t+\sigma, X_\sigma^x) = V(t,x) + \mathsf{E}\int_0^\sigma f(t+u)X_u^x du \tag{5.69}$$

$$\mathsf{E}U^c(t+\sigma, X_\sigma^x) = U^c(t,x) + \mathsf{E}\int_0^\sigma f(t+u)X_u^x I\big(X_u^x \leq c(t+u))\big)du. \tag{5.70}$$

Since $U^c \leq V$ and $V(t,x) = U^c(t,x) = G(t,x)$ for $x \in (0, c(t)]$ with $t \in [0,T]$, it follows from (5.69) and (5.71) that:

$$\mathsf{E}\int_0^\sigma f(t+u)X_u^x I\big(c(t+u) \leq X_u^x\big)du \geq 0. \tag{5.71}$$

Due to the fact that f is always strictly negative we see by the continuity of b and c that (5.71) is not possible so that we arrive at a contradiction. Hence we can conclude that $b(t) \leq c(t)$ for all $t \in [0,T]$.

$(B.4)$ We show that c must be equal to b. For this, let us assume that there exists $t \in [0,T)$ such that $c(t) > b(t)$. Choose an arbitrary point $x \in (b(t), c(t))$ and consider the optimal stopping time τ^* from (5.10) under P. Inserting τ^* instead of s in (5.60) and (5.62), and using the optional sampling theorem, we get:

$$\mathsf{E}G(t+\tau^*, X_{\tau^*}^x) = V(t,x) \tag{5.72}$$

$$\mathsf{E}G(t+\tau^*, X_{\tau^*}^x) = U^c(t,x) + \mathsf{E}\int_0^{\tau^*} f(t+u)X_u^x I\big(X_u^x \leq c(t+u)\big)du \tag{5.73}$$

where we use that $V(t+\tau^*, X_{\tau^*}^x) = G(t+\tau^*, X_{\tau^*}^x) = U^c(t+\tau^*, X_{\tau^*}^x)$ upon recalling that $c \geq b$ and $U^c = G$ either below c or at T. Since $U^c \leq V$ we have from (5.72) and (5.73) that:

$$\mathsf{E}\int_0^{\tau^*} f(t+u)X_u^x I\big(X_u^x \leq c(t+u)\big)du \geq 0. \tag{5.74}$$

Due to the fact that f is always strictly negative we see from (5.74) by continuity of b and c that such a point (t, x) cannot exist. Thus c must be equal to b and the proof of the theorem is complete.

□

5.5. The financial analysis

In this section we present the analysis of financial returns of the shout put option and highlight the practical features of the option. We perform comparisons with the American put option, European put option and the British put option since the first two are standard vanilla options whilst the latter has been introduced by Peskir and Samee in [59] and it was shown there that this option provides a protection mechanism against unfavourable stock movements and also gives high returns with compare to the American option when movements are favourable. The so-called 'skeleton analysis' was applied to analyse financial returns of options in [59] where the main question was addressed as to what the return would be if the underlying process enters the given region at a given time (i.e. the probability of the latter event was not discussed nor do we account for any risk associated with its occurrence). Such a 'skeleton analysis' is both natural and practical since it places the question of probabilities and risk under the subjective assessment of the option holder (irrespective of whether the stock price model is correct or not) and we apply this analysis below.

1. In the numerical example below (see Tables 5.1 and 5.2) the parameter values have been chosen to present the practical features of the shout put option in a fair and representative way and also satisfy (5.15). We assume that the initial stock price equals 10, the strike price $K = 10$, the maturity time $T = 1$ year, the interest rate $r = 0.1$, the volatility coefficient $\sigma = 0.4$, i.e. we consider the option at-the money. For this set of parameters the arbitrage-free price of the shout put option is 1.480, the price of the American put option is 1.196, the price of the European put option is 1.080, and the price of the British put option with the contract drift $\mu_c = 0.13$

Exercise time (months)	0	2	4	6	8	10	12
Shouting at 9	127%	125%	122%	117%	111%	101%	68%
Exercising at 9 (American)	84%	84%	84%	84%	84%	84%	84%
Exercising at 9 (British)	135%	131%	126%	119%	112%	101%	91%
Shouting at 8	181%	180%	178%	176%	171%	164%	135%
Exercising at 8 (American)	167%	167%	167%	167%	167%	167%	167%
Exercising at 8 (British)	182%	180%	178%	176%	174%	173%	182%
Shouting at b	181%	173%	163%	149%	129%	97%	0%
Exercising at b (American)	168%	158%	145%	129%	109%	79%	0%
Exercising at b (British)	183%	174%	163%	148%	128%	97%	0%
Shouting at 7	235%	235%	235%	234%	232%	226%	203%
Exercising at 7 (American)	251%	251%	251%	251%	251%	251%	251%
Exercising at 7 (British)	242%	243%	245%	248%	252%	260%	273%
Shouting at 6	288%	290%	292%	292%	292%	289%	271%
Exercising at 6 (American)	335%	335%	335%	335%	335%	335%	335%
Exercising at 6 (British)	316%	320%	326%	333%	341%	352%	364%
Shouting at 5	342%	346%	348%	351%	352%	351%	388%
Exercising at 5 (American)	418%	418%	418%	418%	418%	418%	418%
Exercising at 5 (British)	402%	409%	417%	426%	435%	445%	455%
Shouting at 4	396%	401%	405%	409%	412%	414%	406%
Exercising at 4 (American)	502%	502%	502%	502%	502%	502%	502%
Exercising at 4 (British)	498%	506%	514%	522%	530%	539%	547%

Table 5.1: Returns observed upon shouting (average discounted payoff at T) the shout put option $R(t,x)/100 = e^{-r(T-t)}G(t,x)/V(0,K)$, exercising the American put option $R_A(t,x)/100 = (K-x)^+/V_A(0,K)$ and exercising the British put option $R_B(t,x)/100 = G^B(t,x))/V_B(0,K)$. The parameter set is $K = 10, T = 1, r = 0.1$, $\sigma = 0.4, \mu_c = 0.13$.

is 1.098. Observe that the 'shouting' premium much more larger than 'exercising' premium of the American put option.

2. Tables 5.1 and 5.2 provide the analysis of comparison between the shout put option and its American, European and British versions. We compare returns upon (i) shouting put option and exercising the American and British options in the same contingency (Table 5.1) and (ii) selling the shout, American British and European options in the same contingency (Table 5.2). The latter is motivated by the fact that in practice the holder may choose to sell his option at any time during the term of the contract, and in this case one may view his 'payoff' as the price he receives upon selling. We also need to note that the return upon shouting at time t means the ratio of discounted average payoff which the holder gets at T over the initial price $V(0,K)$, since he receives claim only at time T and thus we can consider only average return

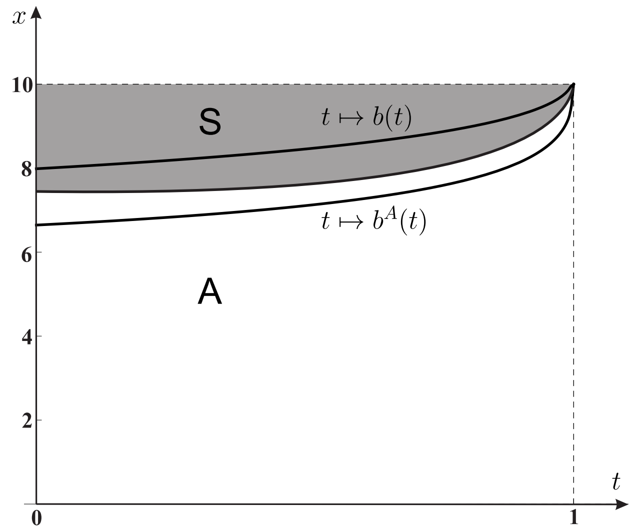

Figure 5.2: A computer drawing showing the (dark grey) region S in which the shout put option outperforms the American put option, and the region A in which the American put option outperforms the shout put option. The parameter set is the same as in Figure 5.1 above ($K = 10$, $r = 0.1$, $\sigma = 0.4$, $T = 1$).

at time t and it depends on chosen model. Figures 5.2 and 5.3 show, respectively: (i) region S in which the shout put option outperforms the American put option, and the region A in which the American put option outperforms the shout put option and (ii) region S in which the shout put option outperforms the British put option, and the surrounding region B in which the British put option outperforms the shout put option.

3. From Table 5.1 and Figures 5.2 and 5.3 we analyse average returns of the shout put option upon shouting along with returns upon exercise of its counterparts. We do not consider the case when stock movements are unfavourable (price is greater or equal than strike) as we know that it is not rational to shout above K. We can point out following observations: (i) there is a curve between optimal shouting boundary and optimal American put boundary such that upon shouting in the region at and above this curve the shout option is much better than the than the American (see Figure 5.2), however below the curve the latter outperforms the former option; (ii) there is a small region S above b when $t \in [2, 8]$ (see Figure 5.3), where the shout

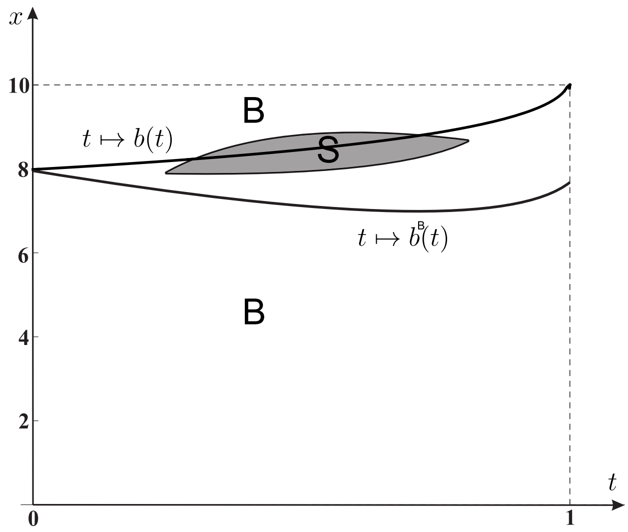

Figure 5.3: A computer drawing showing the (dark grey) region S in which the shout put option outperforms the British put option, and the surrounding region B in which the British put option outperforms the shout put option. The parameter set is the same as in Figure 5.1 above ($K = 10$, $r = 0.1$, $\sigma = 0.4$, $T = 1$).

option has returns greater than the British option's returns and in surrounding large region the British version outperforms the shout one. In order to see comparison between the British and the American options we address to Figure 9 in [59] where it was shown that (*iii*) the British version generally outperforms the American version except within a bounded region corresponding to earlier exercise (before half term). The point (*i*) shows, despite the fact shout option is much more expensive than the American one, that for initially at-the-money shout option its holder enjoys greater returns than American option if he shouts everywhere almost prior to the optimal American put boundary b^A (see Figure 5.2) where the rational investor uses his option. The observations (*ii*) and (*iii*) confirms that the British option is very strong in terms of returns and generally outperforms both counterparts, and it may be explained by the mechanism of optimal prediction which is intrinsically built into the option (for details see [59]).

4. Now we turn to the analysis of Table 5.2 and consider returns of investor upon selling options, where we add also the European option. We can see that

Exercise time (months)	0	2	4	6	8	10	12
Selling at 8 (Shout)	181%	180%	178%	176%	171%	164%	135%
Selling at 8 (American)	186%	183%	179%	175%	171%	167%	167%
Selling at 8 (British)	182%	180%	178%	176%	174%	173%	182%
Selling at 8 (European)	179%	179%	179%	178%	176%	176%	185%
Selling at 9 (Shout)	135%	132%	127%	121%	113%	102%	68%
Selling at 9 (American)	137%	132%	125%	118%	109%	97%	84%
Selling at 9 (British)	135%	131%	127%	121%	113%	102%	91%
Selling at 9 (European)	135%	132%	127%	122%	114%	104%	93%
Selling at 10 (Shout)	100%	95%	88%	80%	68%	52%	0%
Selling at 10 (American)	100%	94%	86%	77%	65%	49%	0%
Selling at 10 (British)	100%	95%	88%	80%	68%	52%	0%
Selling at 10 (European)	100%	95%	89%	81%	69%	52%	0%
Selling at 11 (Shout)	73%	67%	60%	51%	39%	23%	0%
Selling at 11 (American)	73%	66%	58%	49%	37%	21%	0%
Selling at 11 (British)	73%	67%	60%	51%	39%	23%	0%
Selling at 11 (European)	74%	68%	61%	52%	40%	23%	0%
Selling at 12 (Shout)	54%	48%	40%	32%	21%	9%	0%
Selling at 12 (American)	53%	46%	39%	30%	20%	8%	0%
Selling at 12 (British)	54%	47%	40%	32%	21%	9%	0%
Selling at 12 (European)	54%	48%	41%	32%	22%	9%	0%
Selling at 13 (Shout)	39%	33%	27%	19%	11%	3%	0%
Selling at 13 (American)	38%	32%	26%	18%	10%	3%	0%
Selling at 13 (British)	39%	33%	27%	19%	11%	3%	0%
Selling at 13 (European)	39%	34%	27%	20%	11%	3%	0%
Selling at 14 (Shout)	28%	23%	18%	12%	6%	1%	0%
Selling at 14 (American)	28%	22%	17%	11%	5%	1%	0%
Selling at 14 (British)	28%	23%	18%	12%	6%	1%	0%
Selling at 14 (European)	29%	24%	18%	12%	6%	1%	0%

Table 5.2: Returns observed upon selling the shout put option $R(t,x)/100 = e^{-r(T-t)}V(t,x)/V(0,K)$, selling the American put option $R_A(t,x)/100 = V_A(t,x)/V_A(0,K)$, selling the European put option $R_E(t,x)/100 = V_E(t,x)/V_E(0,K)$ and selling the British put option $R_B(t,x)/100 = V_B(t,x)/V_B(0,K)$. The parameter set is $K = 10, T = 1, r = 0.1, \sigma = 0.4, \mu_c = 0.13$.

when the stock price movements are unfavourable (greater than K) and investor decides to liquidate the option, all four options provide comparable returns with only insignificant differences. For prices between 6 and 8 we have that before half terms the American outperforms others and after the European is slightly better than rest. It is important to note that in a real financial market the option holder's ability to sell his contract may depend upon a number of factors such as the access the option market, the transaction costs and the liquidity of the option market itself (which in turn determines the market/liquidation price of the option) so that selling of the options can be problematic. Thus we can only consider liquid markets for calculations in Table 5.2. As it was shown in [59] that exercising the British option in the continuation set produces a remarkably comparable return to selling the contract in a liquid option market, which is not however the case for the shout option.

5. Now we conclude the financial analysis of the shout put option and comparison with its counterparts. In the numerical example the shout option is more expensive than the American one by roughly 23%, however the skeleton analysis shows that there is a curve between optimal shouting boundary b and optimal exercise put boundary b^A such that at and above this curve and below K the shout option's returns greater, which is pleasant for investor who wishes to lock the profit in that region while having the opportunity to gain also from a future price at the maturity T. On the other hand we can see that the British option generally outperforms both counterparts. The British and shout options both have 'optimal prediction' feature because it is intrinsically built into the former option and the decision of shouting the latter option explicitly depends on prediction of the price at T. Another advantage of the British option with compare to the shout option apart from greater returns is that it has generally smaller difference between the option's price and payoff, which is useful in illiquid markets where the selling of the option can be problematic and the British holder may just sell it with good return. The advantage of the shout option that it has clearer definition and structure for the investor.

Chapter 6

Smooth-fit principle for exponential Lévy model

6.1. Introduction

This chapter is devoted to a review of results in the problem of the so-called smooth-fit property for the American put option in an exponential Lévy model with dividends. There are many papers which consider the smooth-fit principle for regular diffusions and its sufficient conditions in terms of differentiability of the gain function and the scale function of the process (see for instance [57] and references therein).

This principle is well-known and has been proved in a classical Black-Scholes model for the American put option for both infinite and finite horizons and helps to solve the corresponding optimal stopping and free-boundary problems. However moving from the geometric Brownian motion to an exponential Lévy model changes the picture and the smooth-fit property may not hold, e.g. in [13] the authors exposed an example of the $CGMY$ model where the principle fails. Alili and Kyprianou [2] studied perpetual case and delivered the necessary and sufficient condition (namely the regularity of the logarithm of stock price with respect to negative half-line) in the exponential Lévy model without dividends. In the finite horizon case this question has been examined in [61], [6] and [79] for a jump-diffusion model.

Recently Lamberton and Mikou [40] proved several results for exponential Lévy model with dividends on finite horizon. Firstly they showed that the condition derived in [2] is also sufficient for finite horizon case. Then without this condition, i.e. when logarithm of the stock price is of finite variation and has positive drift, Lamberton and Mikou showed absence of the smooth-fit at least for large maturities. Finally, under a stronger condition they disproved the smooth-fit for all maturities.

The contribution of this chapter is to provide an example showing that the necessary and sufficient condition for infinite horizon case is not longer applicable for finite horizon and it is caused by the fact that the optimal stopping boundary is strictly increasing unlike in the perpetual case. Namely, we take logarithm of the stock price as a Lévy process of bounded variation with zero drift and finitely many jumps, and prove that one has the smooth-fit property without regularity of Z. Secondly, we attempt to disclose the remaining case in [40] where the drift is positive but removing the additional conditions they used. We provide some analysis and calculations and propose open questions which could help to resolve this problem.

This chapter is organised as follows. In Section 6.2 we set the arbitrage-free model with dividends as exponential Lévy model. In Section 6.3 we recall known results for the American put option problem. Then in Section 6.4 we review existing results in the literature and finally in Section 6.5 we show a counter-example and provide some calculations for the remaining case.

6.2. Model setting

In this Chapter we follow the same setting as in [40]. We consider the probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathsf{P})$ and the general Lévy process $X = (X_t)_{t\geq 0}$ starting at 0 with natural filtration $(\mathcal{F}_t)_{t\geq 0}$ and characteristic triplet (γ, σ, μ), where γ and σ are real numbers and ν is so-called Lévy measure satisfying

$$\int (x^2 \wedge 1)\nu(dx) < \infty. \tag{6.1}$$

Then the characteristic function of X has the following Lévy-Khinchin representation

$$\log(\mathsf{E}e^{izX_t}) = t\big(-\tfrac{1}{2}\sigma^2 z^2 + i\gamma z + \int (e^{izx} - 1 - izxI(|x| \leq 1))\nu(dx)\big) \tag{6.2}$$

for $t \geq 0$ and $z \in \boldsymbol{C}$. Moreover the process X is a Markov with infinitesimal operator given by

$$\begin{aligned} I\!\!L_X f(x) =& (\sigma^2/2)f''(x) + \gamma f'(x) \\ &+ \int \Big(f(x+y) - f(x) - yf'(x)I(|y| \leq 1) \Big) \nu(dy) \end{aligned} \tag{6.3}$$

for every $f \in C_b^2(I\!\!R)$ bounded twice continuously differentiable function with bounded derivatives.

The classical result about Lévy processes states that X is of finite variation if and only if

$$\sigma = 0 \quad \& \quad \int_{|x|\leq 1} |x| \nu(dx) < \infty \tag{6.4}$$

and in this case

$$\frac{X_t}{t} \to \gamma - \int_{-1}^{1} x\, \mu(dx) \quad \text{as } t \to 0 \quad \mathsf{P}\text{-a.s.} \tag{6.5}$$

Now we define our price model based on process X in the following way

$$S_t^x = xe^{(r-\delta)t + X_t} \tag{6.6}$$

where $r > 0$ is the interest rate, $\delta \geq 0$ is the dividend rate. To obtain arbitrage-free prices we require that under P the discounted dividend adjusted stock price $(e^{-(r-\delta)t}S_t)_{t\geq 0}$ is a martingale, which implies two conditions

$$\int_{|x|>1} e^x \mu(dx) < \infty \tag{6.7}$$

$$\frac{\sigma^2}{2} + \gamma + \int \Big(e^x - 1 - xI(|x| \leq 1) \Big) \mu(dx) = 0 \tag{6.8}$$

and therefore the representation (**??**) can be rewritten as

$$\begin{aligned} I\!\!L_X f(x) =& (\tfrac{\sigma^2}{2}(f''(x) - f'(x)) \\ &+ \int \Big(f(x+y) - f(x) - (e^y - 1)f'(x) \Big) \nu(dy). \end{aligned} \tag{6.9}$$

Moreover the stock price S is also a Markov process with infinitesimal operator

$$\begin{aligned} I\!\!L_S f(x) =& (\sigma^2 x^2/2)f''(x) + x(r-\delta)f'(x) \\ &+ \int \Big(f(xe^y) - f(x) - x(e^y - 1)f'(x) \Big) \nu(dy). \end{aligned} \tag{6.10}$$

6.3. American put option on finite horizon

Let us now consider an American put option problem for model described above

$$V(t,x) = \sup_{0\le\tau\le T-t} \mathsf{E}e^{-r\tau}(K - S^x_\tau)^+ \tag{6.11}$$

for $(t,x) \in [0,T) \times (0,\infty)$ where $r > 0$ is the discount rate, $T < \infty$ is the expiration date, $K > 0$ is the strike price, S is given by (6.6), and the supremum is taken over all stopping times τ w.r.t. to $(\mathcal{F}_t)$. As in [40] throughout this Chapter we assume that at least one of the conditions below holds

$$\sigma \neq 0,\ \nu((-\infty,0)) > 0 \ \text{ or } \int_{x>0} (x \wedge 1)\nu(dx) = \infty. \tag{6.12}$$

Under this assumption, we have (as observed in [40]) that $V(t,x) > 0$ for all $(t,x) \in [0,T) \times (0,\infty)$.

We now recall well-known facts and properties in the literature of the value function V and optimal stopping time τ_*:

$$x \mapsto V(t,x) \quad \text{is decreasing and convex on } (0,\infty) \text{ for } t \in [0,T] \tag{6.13}$$

$$t \mapsto V(t,x) \quad \text{is continuous and decreasing on } [0,T] \text{ for } x \in (0,\infty) \tag{6.14}$$

$$V_t + I\!\!L_S V - rV \le 0 \quad \text{on closed set } D \tag{6.15}$$

$$V_t + I\!\!L_S V - rV = 0 \quad \text{on open set } C \tag{6.16}$$

and there exists a function b on $[0,T]$, which we call the optimal stopping boundary, such that

$$\tau_* = \inf\{\, 0 \le s \le T-t |\ S^x_s \le b(t+s)\} \tag{6.17}$$

is optimal in (6.11) and where continuation set C reads

$$\begin{aligned} C &= \{\ (t,x) \in [0,T) \times (0,\infty)|\ V(t,x) > (K-x)^+\} \\ &= \{\ (t,x) \in [0,T) \times (0,\infty)|\ x > b(t)\} \end{aligned} \tag{6.18}$$

and stopping set D

$$D = \{\ (t,x) \in [0,T) \times (0,\infty)|\ V(t,x) = (K-x)^+\} \tag{6.19}$$

$$= \{\ (t,x) \in [0,T) \times (0,\infty) |\ x \leq b(t)\}.$$

It has been proven by Lamberton and Mikou in [41] that $b > 0$ is continuous and increasing and they showed following important result

Proposition 6.3.1. *The limit of the boundary b at T is characterised by*

1) if $d^+ \geq 0$ then $b(T) = K$;

2) if $d^+ < 0$ then $b(T) < K$ and solution to

$$rK - \delta b(T) - \int \big(b(T)e^x - K\big)^+ \mu(dx) = 0 \tag{6.20}$$

where $d^+ := r - \delta - \int \big(e^x - 1\big)^+ \nu(dx)$.

6.4. Smooth-fit principle: review of existing results

The important issue of the American put option problem is so-called smooth-fit property:

$$V_x(t, b(t)+) = -1 \tag{6.21}$$

for all $t \in [0,T)$, where $\{b(t), 0 \leq t \leq T\}$ is the optimal stopping boundary. This condition holds in the case of geometric Brownian motion model. For exponential Lévy model it was proven by Alili and Kyprianou [2] that in the case of infinite horizon the smooth-fit property is equivalent to condition $\mathsf{P}(\tau_0^- = 0) = 1$, where

$$\tau_x^- = \inf\{0 \leq t \leq T : Z_t < x\} \tag{6.22}$$

with $Z_t := (r - \delta)t + X_t$.

Following well-known result can be found in [2]:

Proposition 6.4.1. *Let us define*

$$d := r - \delta - \int \big(e^x - 1\big)\mu(dx) = \lim_{t\to 0} \frac{Z_t}{t} \tag{6.23}$$

due to (6.5) *and* (6.8) *and which is given only for Z with finite variation. Then the process Z is regular to $(-\infty, 0)$, i.e $\mathsf{P}(\tau_0^- = 0) = 1$, if and only if one of the following conditions holds:*

(i) Z has finite variation with $d < 0$

(ii) Z has finite variation with $d = 0$ *and* $\int_{-1}^{0} \frac{|x|\nu(dx)}{\int_0^{|x|}\nu(y,\infty)dy} = \infty$

(iii) Z has infinite variation.

Then Lamberton and Mikou [40] considered the finite horizon case and we review their results below.

Proposition 6.4.2. *Following facts hold:*

(i) if Z is regular then the smooth-fit holds

(ii) if Z has finite variation with $d > 0$ *and under additional assumption* $d^+ = r-\delta-\int \left(e^x-1\right)^+\mu(dx) \geq 0$, *then the smooth-fit breaks down for every* $t \in [0,T)$

(iii) if Z has finite variation with $d > 0$ *and T large enough then there exists* $t \in [0,T)$ *such that smooth-fit does not hold.*

Thus it remains to study cases when Z has finite variation and : $a)$ $d = 0$ and there is no regularity of Z and $b)$ $d^+ < 0 < d$ and for all $T > 0$. We discuss these cases in the next section where we will make of use the following result

Proposition 6.4.3. *If Z has finite variation with* $d > 0$ *and for every* $0 \leq t < T$ *we exhibit the smooth-fit property, then*

$$\limsup_{h\downarrow 0} \frac{b(t+h)-b(t)}{h} \geq b^* d \tag{6.24}$$

where b^* *is the optimal stopping threshold for the perpetual case.*

6.5. Case $d = 0$ and remarks for the case $d^+ < 0 < d$

1. Now we will give some intuition behind the results from previous section. It can be seen from the proofs in [40] that the crucial point for having the smooth-fit property is whether the process S starting at the optimal stopping boundary b enters immediately to the stopping set D or not. In the case of infinite horizon since the boundary b is constant the smooth-fit is equivalent to immediate entry of Z to

negative half-line P-a.s., i.e. $\mathsf{P}(\tau_0^- = 0) = 1$. However in the case of finite horizon the boundary b becomes strictly increasing so now main question is to determine whether

$$\begin{aligned} \tau_h &= \inf\{\, 0 \le s \le T-t |\ S_s^{b(t)+h} \le b(t+s)\} \qquad (6.25)\\ &= \inf\{\, 0 \le s \le T-t |\ Z_s \le \log[b(t+s)/(b(t)+h)]\} \to 0 \quad \text{as } h \to 0 \ \mathsf{P}\text{-a.s.} \end{aligned}$$

Since we can approximate $\log[b(t+s)/b(t)] \approx (b(t+s)-b(t))/b(t)$ we need to estimate a derivative (or slope) of b divided by the value of b itself and compare it with the drift d of the process Z.

Let us consider now conclusions of Proposition 6.4.2 and explain them by arguments from paragraph above. Firstly in (i), if Z is regular, then since b is increasing and thus the slope of b is strictly positive, the process S obviously enter stopping set immediately. Then for (ii), if Z has finite variation and $d \ge d^+ \ge 0$ in [40] an absence of smooth-fit was proved by analytical arguments, but also in more recent paper [42] the asymptotic behaviour of b near maturity T was obtained

$$\lim_{t\uparrow T} \frac{b(T)-b(t))}{T-t} = -b(T)\int_{x<0} \big(e^x - 1\big)\nu(dx) \qquad (6.26)$$

and since $\lim_{t\downarrow 0} \frac{X_t}{t} - \lim_{t\uparrow T} \frac{b(T)-b(t))}{b(T)(T-t)} = d + \int_{x<0} \big(e^x - 1\big)\nu(dx) = d^+ \ge 0$ we can confirm that the process S does not enter into the stopping set D immediately. Finally, in (iii) for large enough T the boundary at $t=0$ has small enough slope so that having $d > 0$ we again do not enter into D. Therefore it remains to consider the behaviour of b at T when $d > 0$ but $d^+ < 0$ and thus by Proposition 6.3.1 we have $b(T) < K$. We discuss this case below in paragraph 3.

2. However first we would like to consider the case $d = 0$ and when Z exhibits finitely many jumps on any interval P-a.s., i.e. we take compound Poisson process. Obviously, the regularity fails $\mathsf{P}(\tau_0^- > 0) = 1$, however we can show that $\tau_h \to 0$ as $h \to 0$. Indeed, let us take arbitrary $\varepsilon > 0$ and then since Z has $d = 0$ and finitely many jumps we can find $\mathsf{P}-a.s.$ $0 < s < \varepsilon$ such that $Z_s = 0$. Now using that b is strictly increasing (see proof in [77]) we can choose $h > 0$ small enough such that $Z_s = 0 \le \log b(t+s)/(b(t)+h)$ and therefore $\tau \le s < \varepsilon$. Hence it follows that $\tau_h \to 0$ as $h \to 0$ $\mathsf{P}-a.s.$ and repeating standard arguments (see e.g. proof of Theorem 4.1

in [40]) we prove that the smooth-fit holds in this case.

This example shows that one can have the smooth-fit property without regularity of Z, i.e. the condition for infinite horizon case is not applicable for finite horizon case due to the strictly increasing boundary b.

3. Now we come back to the case where Z has finite variation and $d^+ < 0 < d$. There are two ways that can be proposed and which are currently under progression.

(i) Following arguments in first paragraph we can study the behaviour of b at T. When $d^+ \geq 0$ the limit at T equals to K and as we mentioned above the linear behaviour of b has been proven by using auxiliary European put boundary. In the case $d^+ < 0$ the limit $b(T) < K$ and one can use that locally the gain function $(K-x)^+$ is smooth near $b(T)$ and apply Ito's formula as in [43] for Black-Scholes model. However in [43] authors used α-stability of the Brownian motion and but our model does not belong to class of stable processes due to (6.7).

Thus in order to determine whether the smooth-fit property holds or not we propose to find asymptotic behaviour of b near T when $d^+ < 0$, which itself is an interesting problem.

(ii) Another approach goes to the proof in [40] for the case $d^+ \geq 0$. The idea is to use that $V_t + \mathbb{L}_S V - rV = 0$ on C and that $\mathbb{L}_S V$ contains first derivative with respect to x and then let $x \downarrow b(t)$. Namely, we have as $x \downarrow b(t)$

$$\begin{aligned} &V_t(t,b(t)+) + b(t)\, d\, V_x(t,b(t)+) \\ &\quad + \int \big(V(t,b(t)e^y) - V(t,b(t))\big)\nu(dy) - rV(t,b(t)) \\ &= V_t(t,b(t)+) + b(t)\, d\, V_x(t,b(t)+) + \int_{-\infty}^{0} b(t)(1-e^y)\nu(dy) \\ &\quad + \int_0^\infty \big(V(t,b(t)e^y) - V(t,b(t))\big)\nu(dy) - r(K-b(t)) = 0 \end{aligned} \tag{6.27}$$

for $t \in [0,T)$ where we used (6.10) and that $\sigma = 0$, $V(t,b(t)e^y) = K - b(t)e^y$ for any $y \leq 0$.

Now after rearranging terms in (6.27) and using definitions of d, d^+ and equation

(6.20) we get

$$
\begin{aligned}
&\big(V_x(t,b(t)+)+1\big)\,d\,b(t) + V_t(t,b(t)+) \qquad (6.28)\\
=&r\big(K-b(t)\big) + d^+b(t) + \int_0^\infty \big(K-b(t)-V(t,b(t)e^x)\big)\mu(dx)\\
=&rK-\delta b(t) - b(t)\int_0^\infty \big(e^x-1\big)\mu(dx) + \int_0^\infty \big(K-b(t)-V(t,b(t)e^x)\big)\mu(dx)\\
=&rK-\delta b(t) + \int_0^\infty \big(K-b(t)e^x-V(t,b(t)e^x)\big)\mu(dx)\\
=&rK-\delta b(t) - rK + \delta b(T) + \int \big(b(T)e^x-K\big)^+\mu(dx)\\
&+\int_0^\infty \big(K-b(t)e^x-V(t,b(t)e^x)\big)\mu(dx)\\
=&\delta\big(b(T)-b(t)\big) + \int_0^A \big(K-b(t)e^x-V(t,b(t)e^x)\big)\mu(dx)\\
&+\int_A^\infty \big((b(T)-b(t))e^x-V(t,b(t)e^x)\big)\mu(dx)\\
=&\delta\big(b(T)-b(t)\big) + \int_0^\infty \Big[(K-b(T)e^x)^+-V(t,b(T)e^x)\Big]\mu(dx)\\
&+\int_0^\infty \Big\{\big[b(T)-b(t)\big]e^x - \big[V(t,b(t)e^x)-V(t,b(T)e^x)\big]\Big\}\mu(dx)\\
=&\delta\big(b(T)-b(t)\big) + \int_0^\infty \Big[\big[b(T)-b(t)\big]e^x + (K-b(T)e^x)^+ - V(t,b(t)e^x)\Big]\mu(dx)
\end{aligned}
$$

where $A := \log(K/b(T)) > 0$. Since $V_t \leq 0$ the if we show that the right-hand side in (6.28) is strictly positive for some t, then it would follow that in this case the smooth-fit breaks down.

Now we show some attempts to disprove the smooth-fit, however we do not have any strong intuition behind this conjecture and one can try to prove the opposite fact. If we consider final expression in (6.28), then the first term is strictly positive if $\delta > 0$, however the integrand of last expression is negative from 0 to some $B > A$ and after B exponentially increases to $+\infty$ (the integral converges due to (6.7)) and thus we do not have clear evidence that this expression is positive.

Also one can examine the limit of penultimate expression in (6.28) as $t \uparrow T$. Then we argue by contradiction and assume that the smooth-fit holds and use (6.24) which gives lower bound with positive linear drift for the first term if $\delta > 0$, then second integrand is obviously positive since $V_x \geq -1$. However the first integrand is always negative and one can divide it by $T-t$ and letting $t \uparrow T$ we have

$$\lim_{t \uparrow T} \frac{(K - b(T)e^x)^+ - V(t, b(T)e^x)}{T - t} = \lim_{t \uparrow T} \frac{(K - b(T)e^x)^+ - V^E(t, b(T)e^x)}{T - t} \quad (6.29)$$

for $x \geq 0$ where V^E is the European put option price and we used that the difference $V - V^E = e$ is early exercise premium for which we have $e(t)/(T-t) = 0$ as $t \uparrow T$.

Thus the question now is to estimate

$$V^E(t, b(T)e^x) - (K - b(T)e^x)^+ \quad (6.30)$$

for $x \geq 0$ as accurately as possible, for instance one can exploit Ito-Tanaka's formula for the European option price.

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