K-THEORY OF THEORIES OF MODULES AND ALGEBRAIC VARIETIES

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***K*-theory of Theories of Modules and Algebraic Varieties**
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Let $(S, \sqcup, \emptyset)$ denote a small symmetric monoidal category whose objects are sets (possibly with some extra structure) and the monoidal operation, $\sqcup$, is disjoint union. Such categories encode cut-and-paste operations of sets. Quillen gave a functorial construction of the abelian groups $(K_n(S))_{n\geq 0}$, known as the K-theory of S, which seek to classify its objects and morphisms. In particular, the group $K_0(S)$, known as the Grothendieck group, is the group completion of the commutative monoid of isomorphism classes of objects of S and it classifies the objects of the category up to 'scissors-congruence'. On the other hand, the group $K_1(S)$ classifies the automorphisms of objects (i.e., maps which cut an object into finitely many pieces which reassemble to give the same object) in the direct limit as the objects become large with respect to $\sqcup$.

In this thesis we consider two classes of symmetric monoidal categories, one from model theory and the other from algebraic geometry. For any language L, the category $\mathcal{S}(M)$ of subsets of finite cartesian powers of a first order L-structure M definable with parameters from M together with definable bijections is symmetric monoidal and thus can be used to define the K-theory of the structure M which is functorial on elementary embeddings. On the other hand, for any field k, the category Var_k of algebraic varieties and rational maps is also symmetric monoidal. In both these cases, the categories carry an additional binary operation induced by the product of objects; this equips the Grothendieck group with a multiplicative structure turning it into a commutative ring known as the Grothendieck ring.

The model-theoretic Grothendieck ring $K_0(M) := K_0(\mathcal{S}(M))$ of a first order structure M was first defined by Krajíček and Scanlon. We compute the ring $K_0(M_\mathcal{R})$ for a right $\mathcal{R}$-module M, where $\mathcal{R}$ is a unital ring, and show that it is a quotient of the monoid ring $\mathbb{Z}[\mathcal{X}]$, where $\mathcal{X}$ is the multiplicative monoid of isomorphism classes of fundamental definable subsets - the *pp*-definable subgroups - of the module, by the ideal that codes indices of pairs of *pp*-definable subgroups. As a corollary we prove a conjecture of Prest that $K_0(M_\mathcal{R})$ is non-trivial, whenever M is non-zero. The main proof uses various techniques from simplicial homology and lattice theory to construct certain counting functions. The K-theory of a module is an invariant of its theory. In the special case of vector spaces we also compute the model-theoretic group K_1.

Let k be an algebraically closed field. Larsen and Lunts asked if two k-varieties having the same class in the Grothendieck ring $K_0(\mathrm{Var}_k)$ are piecewise isomorphic. Gromov asked if a birational self-map of a k-variety can be extended to a piecewise automorphism. We show that these two questions are equivalent over any algebraically closed field. Under the hypothesis of a positive answer to these two questions we prove that the underlying abelian group of the Grothendieck ring is a free abelian group and that the associated graded ring of the Grothendieck ring is the monoid ring $\mathbb{Z}[\mathfrak{B}]$ where $\mathfrak{B}$ is the multiplicative monoid of birational equivalence classes of irreducible k-varieties.

Declaration

No portion of the work referred to in the thesis has been submitted in support of an application for another degree or qualification of this or any other university or other institute of learning.

Important results of this thesis have been written-up in the articles [26] and [27].

Copyright Statement

ती. आईबाबांना समर्पित

To my parents

Acknowledgements

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Chapter 1

Introduction

1.1 Historical background

Grothendieck laid the foundations of K-theory in the mid-1950s as the framework for his far-reaching generalization of the Riemann-Roch theorem in algebraic geometry. (The letter K stands for the German word *Klasse*.) However, special cases of K-groups occur in almost all areas of mathematics, and particular examples of what we now call K_0 (the Grothendieck group) were amongst the earliest studied examples of abelian groups. The Euler-Poincaré characteristic of a topological space - defined as the alternating sum of the dimensions of certain vector spaces - motivated the definition of the Euler characteristic of a chain complex in homological algebra and the latter takes values in a suitably chosen Grothendieck group.

In many cases the group K_0 has a commutative ring structure and is called the Grothendieck ring; it serves the purpose of classifying objects of a (small) category carrying some extra structure. The categories we consider in this thesis are symmetric monoidal categories whose objects are sets and the monoidal operation is disjoint union. The Grothendieck ring $K_0(S)$ of such a category S classifies its objects up to scissors-congruence, i.e., two objects in the category S have the same class in the Grothendieck ring if either they are isomorphic or one object can be cut into finitely many pieces which can be assembled to give the other. Hilbert's 3^{rd} problem is closely related; it asked whether two polyhedra of equal volume are scissors-congruent.

In 1995, Kontsevich invented the concept of motivic integration which, roughly speaking, assigns to subsets of the arc space a volume in the Grothendieck ring $K_0(\mathrm{Var}_k)$ of algebraic varieties over a field k. This branch of algebraic geometry was developed extensively by Denef and Loeser. The Grothendieck ring of varieties first appeared in a letter of Grothendieck in the Serre-Grothendieck correspondence (letter of 16/8/1964) and plays the role of the value ring of the universal motivic measure, where a motivic measure is to motivic integration as an ordinary real-valued measure is to ordinary integration.

Motivated by the motivic integration theory, Krajiček and Scanlon [25] introduced the concept of the model-theoretic Grothendieck ring of a first order structure M. A weak Euler characteristic on the structure M is any function which assigns to each definable set an element of a ring in a way that "preserves" disjoint unions and cartesian products, and thus the Grothendieck ring is the value ring of the universal weak Euler characteristic. In the same paper, the authors proved that such a Grothendieck ring is nontrivial if and only if the definable subsets of the structure satisfy a version of the combinatorial pigeonhole principle, called the "onto pigeonhole principle". Grothendieck rings have been studied for various rings and fields considered as models of a first order theory (see [25], [6], [7], [8] and [9]) and they are found to be trivial in many cases (see [6],[7]).

Prest conjectured that in stark contrast to the case of rings, for any ring $\mathcal{R}$, the Grothendieck ring of a nonzero right $\mathcal{R}$ module $M_{\mathcal{R}}$, denoted $K_0(M_{\mathcal{R}})$, is nontrivial. Perera [31] investigated the problem in his doctoral thesis but found only a partial solution. He showed that elementarily equivalent modules have isomorphic Grothendieck rings, which is not the case for general structures, and he showed that the Grothendieck rings for modules over semisimple rings are polynomial rings in finitely many variables over the ring of integers.

1.2 Contents and connections

Grothendieck ring is arguably the single most important concept in this work. Chapters 4, 5 and 6 deal with the structure and the properties of the model-theoretic Grothendieck rings of modules. Chapter 7 is the only chapter that concerns some questions associated with the Grothendieck ring of varieties. These two situations can be studied under the common name of symmetric monoidal categories. Quillen and Segal's K-theory of such categories is discussed in an earlier chapter, Chapter 3, where we also define the model-theoretic K-theory of a structure as an application. In the same chapter, we compute the groups K_0 and K_1 for vector spaces. The heuristic "K-theory of a free category is free" is a common thread connecting all these chapters which is discussed in detail in the final chapter on conclusions. Each chapter begins with an introductory paragraph briefly describing the contents of its various sections. The remainder of the current chapter follows more or less the same pattern as the arrangement of the chapters; the next three sections comment on various aspects of the proof of the structure theorem for the Grothendieck rings of modules.

The author has written two articles based on this work: 'Grothendieck rings of theories of modules' [26] and 'on the Grothendieck ring of varieties' [27].

Figure 1.1 shows the interdependency of the sections.

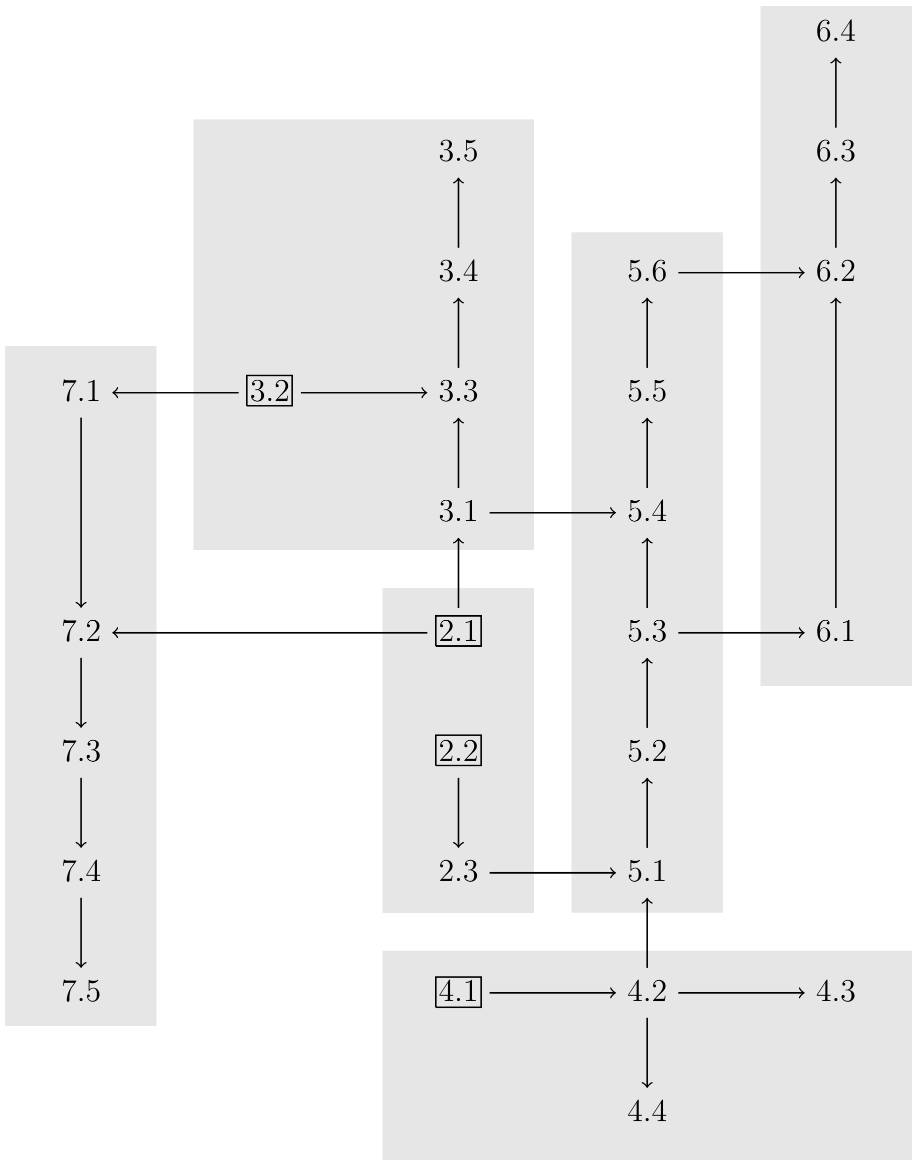

Figure 1.1: Section dependency chart

This work uses ideas from many areas of mathematics. If a result is needed in our proofs, then we state it along with the definitions of the necessary terminology and include a reference. On the other hand, appropriate references are provided for the results not used directly in the work.

Basic terminology in model theory and category theory is assumed throughout the thesis. This background is covered in almost all books in these areas; specific references are [33] and [22] respectively. More specific material on the model theory of modules can be found in [33], [34] and [31]; a condensed version of the required material is presented in Section 4.1. Construction of the Grothendieck ring is functorial on certain maps between modules, called pure maps, which fit with model theory. This is proved in Section 6.3 with applications to the Grothendieck ring of the module category. Symmetric monoidal categories and their K-theory are discussed in Section 3.2 and appropriate references are provided in the text.

The construction of the Grothendieck ring of a semiring is described at a leisurely pace in Section 2.1. The background in algebra and algebraic geometry can be obtained from [19] and [15] respectively, whereas the construction of the group homology can be found in [35, §4.1]. Semidirect products and wreath products of non-abelian groups are required only in Section 3.4 and 3.5 where they are defined and treated in detail. The work is self-contained for the use of lattice-theoretic language. Standard references for algebraic topology are [16] and [38]. Application of the homology theory of simplicial complexes is the content of Sections 2.2 and 2.3; this material can be found in [11].

1.3 Grothendieck rings of theories of modules

In Chapters 5 and 6, we compute the Grothendieck ring for an arbitrary module over an arbitrary ring and show that it is a quotient of a monoid ring $\mathbb{Z}[\mathcal{X}]$, where $\mathcal{X}$ is the multiplicative monoid of isomorphism classes of fundamental definable subsets of the module - the *pp*-definable subgroups. This is the content of the main theorem, Theorem 6.2.3, which also describes the 'invariants ideal' - the ideal of the monoid ring that codes indices of pairs of *pp*-definable subgroups. It should be noted that the proof gives an explicit description of the class $[D] \in K_0(M)$ of a definable set D. We further show (Corollary 6.2.11) that there is a split embedding $\mathbb{Z} \to K_0(M)$, whenever the module M is nonzero, proving Prest's conjecture.

The proof of the main theorem uses inputs from various mathematical areas such as model theory, algebra, combinatorics and algebraic topology. Careful analysis of the meet-semilattice of *pp*-sets using Euler characteristics of abstract simplicial complexes yields a family of counting measures on definable sets; the Grothendieck ring bundles up such measures.

A special case of the main theorem (Theorem 5.4.2) is proved at the end of Section 5.6, where we assume that the theory T of the module M satisfies the model theoretic condition $T = T^{\aleph_0}$. This condition is equivalent to the statement that the invariants ideal is trivial. The reader should note that the proof of the general case of the main theorem is not given in full detail since it develops along lines similar to the special case and uses only a few modifications which are indicated to incorporate the invariants ideal.

1.4 Canonical forms for definable sets

Holly [17] described a canonical form for definable subsets of algebraically closed valued fields by forming swiss cheeses - discs or balls with finitely many holes removed. She used the canonical form to prove the elimination of imaginaries in one dimension. Adler used the swiss cheeses to study definable sets in some classes of VC-minimal theories in [2]. Flenner and Guingona [12] extracted the notion of a directed family of sets from Adler's work and used it to obtain uniqueness results on representations of sets constructible in a directed family. A directed family of sets is a meet-semilattice in which any two elements with non-trivial intersections are comparable. They introduced the notion of packability as the dividing line between absolute uniqueness and optimal uniqueness of representations.

The fundamental theorem of the model theory of modules (Theorem 4.1.5) states that every definable set is a boolean combination of *pp*-definable sets, but such a boolean combination is far from being unique. Under certain extra conditions on the theory of the module, we achieve a 'uniqueness' result as a by-product of the theory we develop. We call this result the 'cell decomposition theorem' (Theorem 4.3.4) which states that definable sets can be represented uniquely using the meet-semilattice of *pp*-definable sets provided the theory T of the module satisfies $T = T^{\aleph_0}$. This model-theoretic condition is analogous to the notion of unpackability in [12, §2] and Theorem 4.3.4 generalizes [12, Corollary 2.3].

Though the cell decomposition theorem is not used directly in any other proof, its underlying idea is one of the most important ingredients of the main proof. Based on this idea, we define various classes of definable sets of increasing complexity, namely *pp*-sets, convex sets, blocks and cells. The terms ball and swiss cheese in [12] correspond to the terms *pp*-set and block in our setting. Chapter 4 is devoted to the formulation and proofs of these conditions. This chapter forms the lattice-theoretic basis for the proof of the structure of the Grothendieck ring of modules. Our strategy to prove every result about a general definable set is to prove it first for convex sets, then blocks and then cells. We deal with the packable case in Section 6.1 and obtain an "optimal" unique representation theorem for definable sets in terms of compatible families.

1.5 Geometric and topological ideas

An important theme of Chapters 4, 5 and 6 is the use of geometric and topological ideas in the setting of definable sets. We use the idea of a 'neighbourhood' and 'localization' to understand the structure of definable sets. We develop a notion of 'connectedness' of a definable set in 4.4 and prove Theorem 4.4.6 which clearly shows the analogy with its topological counterpart.

The proof of the structure theorem for the Grothendieck rings of modules takes place at two different levels, which we name 'local' and 'global' following geometric intuition. We try to describe the "shape" of each definable set in terms of integer valued functions called 'local characteristics'. These numbers are computed using Euler characteristics of various abstract simplicial complexes which code the "local geometry" of the given set. The local data is combined to get a family of integer valued functions, each of which is called a 'global characteristic'. The global characteristics enjoy the property of being preserved under definable bijections. The family of such functions is indexed by the elements of the monoid $\mathcal{X}$ (introduced in a previous section) and the functions collate to give the necessary monoid ring.

Klain and Rota defined valuations on an abstract simplicial complex in [23, §3.2] and proved the existence of the Euler characteristic - the unique invariant valuation on the distributive lattice of abstract simplicial complexes on a finite set. Theorem 5.1.6 and, more generally, Theorem 6.1.6 show that the local characteristics, indexed by the semi-lattice of *pp*-sets, are in fact valuations. In our set-up, we work with finite simplicial complexes on an infinite meet-semilattice.

1.6 The Grothendieck ring of varieties

Let k denote an algebraically closed field. The main object of study in Chapter 7 is the Grothendieck ring of algebraic varieties over k which encodes cut-and-paste operations of (Zariski) closed subvarieties. We consider two weaker equivalences of varieties than variety isomorphisms, namely birational equivalence and cut-and-paste equivalence.

Larsen and Lunts asked if the cut-and-paste equivalence of two varieties is a necessary condition for them to have the same class in the Grothendieck ring (Question 7.1.2). A positive answer to this question would settle a question of Gromov (Question 7.1.3) in the affirmative where the latter that asks if every birational map $X \dashrightarrow X$ can be extended to a cut-and-paste equivalence $X \to X$. In fact it is possible to set up an induction to answer the former question in the affirmative whose base case is a positive answer to the latter; this proves the equivalence of the two questions (Theorem 7.3.3). The proof of this result is essentially combinatorial and involves simultaneous modifications of two bijections between two finite sets of irreducible varieties.

We obtain two results on the structure of the Grothendieck ring of varieties under the hypothesis of a positive answer to both these questions. We show that its underlying abelian group is a free abelian group (Theorem 7.4.1) and that its associated graded ring is the integral monoid ring $\mathbb{Z}[\mathfrak{B}]$ where $\mathfrak{B}$ denotes the multiplicative monoid of birational equivalence classes of irreducible k-varieties (Theorem 7.5.1).

Section 8.4 discusses the similarities and differences of this case with the modules case.

1.7 Higher K-theory with definable sets

The Grothendieck group of a commutative monoid is usually termed its group completion. A symmetric monoidal category is the category-theoretic analogue of a commutative monoid; definable sets and algebraic varieties form symmetric monoidal categories under disjoint union. Quillen described the construction of the "group completion" of a symmetric monoidal category S. The K-theory groups of the category S are defined to be the homotopy groups of its group completion. The definition of the group $K_0(S)$ agrees with the Grothendieck group of the monoid of isomorphism classes of objects of S. This construction is explained in detail in Section 3.2.

In Section 3.3, we define the model-theoretic K-theory of a structure M as the K-theory of the symmetric monoidal category of sets definable (with parameters) in M and compute the lower K-theory (i.e., groups K_0 and K_1) of finite structures.

For a structure M, the group $K_1(M)$ classifies definable bijections of definable sets. We use iterated semidirect products of certain wreath products of matrix groups and finitary permutation groups to compute the groups of definable self-bijections of definable subsets of vector spaces. The direct limit of the abelianization of these groups gives the group K_1 of a vector space. These computations are shown in Sections 3.4 and 3.5.

1.8 Notations and Conventions

Below are some standard notations and conventions used throughout the thesis.

The notation $\mathbb{N}$ denotes the set of natural numbers and we assume that $0 \in \mathbb{N}$. For each $n \geq 1$, the notation $[n]$ denotes the set $\{1, 2, \ldots, n\}$.

The notations $\mathbb{R}$ and $\mathbb{Z}$ denote the field of reals and the ring of integers respectively. For any $n \geq 1$, the notation $\mathbb{Z}_n$ denotes the quotient ring $\mathbb{Z}/n\mathbb{Z}$.

All rings in the thesis are unital and all ring homomorphisms preserve units. The notation $\mathcal{R}$ will always denote a ring and the notation $\mathrm{GL}_n(\mathcal{R})$ denotes the group of invertible $n \times n$ matrices with entries in $\mathcal{R}$.

The notation $\sqcup$ denotes disjoint union.

A triple $(S, +, 0)$ is a commutative monoid whenever $+$ is an associative and commutative binary operation on S with identity 0. A monoid homomorphism is a map respecting the binary operation.

If $X, Y \subseteq S$ and $X, Y \neq \emptyset$, then we use the Minkowski sum notation $X + Y$ to denote the set $\{x + y : x \in X, y \in Y\}$. In case $X = \{x\}$, we use $x + Y$ to denote $X + Y$.

A meet-semilattice is a pair $(\mathcal{L}, \wedge)$ where $\wedge$ is an associative, commutative and idempotent binary operation. Sometimes the meet-semilattice $\mathcal{L}$ contains a zero element, 0, which satisfies the additional condition that $a \wedge 0 = 0$ for all $a \in \mathcal{L}$.

A lattice is a triple $(\mathcal{A}, \wedge, \vee)$ where the binary operations $\wedge$ and $\vee$ are associative, commutative and idempotent and the following absorption laws are satisfied for all $a, b \in \mathcal{A}$:

$$a \wedge (a \vee b) = a = a \vee (a \wedge b).$$

A (semi)lattice morphism is a map compatible with the binary operations. In particular, a sub-(semi)lattice is a subset closed under the induced operations. Each (semi)lattice $\mathcal{A}$ can naturally be treated as a poset where $a \leq b$ in $\mathcal{A}$ if and only if $a \wedge b = a$.

A lattice $\mathcal{A}$ is said to be distributive if additionally the following distributive law holds for all $a, b, c \in \mathcal{L}$:

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c).$$

A morphism of distributive lattices is simply a lattice morphism.

The forgetful functor which takes a distributive lattice to the underlying meet-semilattice has a left adjoint. This left adjoint assigns to a meet-semilattice $\mathcal{L}$ the free distributive lattice $\mathcal{A}$ of finite antichains in $\mathcal{L}$. The details of the construction can be found in [18, Lemma C1.1.3]; we discuss it in a particular situation in Section 5.1.

Chapter 2

Background

Section 2.1 describes the construction of the Grothendieck ring of a semiring and the remaining two sections contain essential background in the homology theory of simplicial complexes that is necessary in Chapter 5.

The material in this chapter is contained in [26, §2.1,2.3,2.4].

2.1 Grothendieck rings of semirings

We recall the notion of a semiring and how to construct a ring in a canonical fashion from a given semiring. A detailed exposition on this material can be found in [21]. Note that all definitions and constructions in this section apply to a commutative monoid simply by forgetting the clauses involving multiplication and replacing the words 'semiring' and 'ring' with 'commutative monoid' and 'abelian group' respectively. In particular, we get the notion of a cancellative monoid and the construction of the Grothendieck group $K_0(S)$ of a commutative monoid S.

Let $L_{ring} = \langle 0, 1, +, \cdot \rangle$ be the language of rings without a symbol for subtraction.

Definitions 2.1.1. *Any L_{ring}-structure S satisfying the following conditions is a commutative* ***semiring*** *with unity.*

- $(S, +, 0)$ *is a commutative monoid.*
- $(S, \cdot, 1)$ *is a commutative monoid.*
- $a \cdot 0 = 0$ *for all* $a \in S$.
- *Multiplication* $(\cdot)$ *distributes over addition* $(+)$.

A ***semiring homomorphism*** *is an L_{ring}-homomorphism.*

A semiring S is said to be ***cancellative*** *if* $a + c = b + c \Rightarrow a = b$ *for all* $a, b, c \in S$.

All the semirings considered here are commutative semirings with unity, allowing the possibility $0 = 1$.

Definition 2.1.2. *A binary relation $\sim$ on a semiring S is said to be a* ***congruence relation*** *if the following properties hold.*

- *$\sim$ is an equivalence relation.*
- *For all $a, b, c, d \in S$, $a \sim b, c \sim d \Rightarrow (a + c) \sim (b + d), a \cdot c \sim b \cdot d$.*

There is a canonical way of constructing a cancellative semiring from any semiring S as stated in the following theorem.

Theorem 2.1.3 (**Quotient construction**)**.** *Let S be a semiring and let $\sim$ be the binary relation defined as follows.*

$$For\ a, b \in S,\ a \sim b \ \Leftrightarrow\ \exists c \in S,\ a + c = b + c \tag{2.1}$$

Then $\sim$ is a congruence relation. If $\tilde{a}$ denotes the $\sim$ equivalence class of $a \in S$, then $\tilde{S} := \{\tilde{a} : a \in S\}$ is a cancellative semiring with respect to the induced addition and multiplication operations. There is a surjective semiring homomorphism $q : S \to \tilde{S}$ given by $a \mapsto \tilde{a}$. Furthermore, given any cancellative semiring T and a semiring homomorphism $f : S \to T$, there exists a unique semiring homomorphism $\tilde{f} : \tilde{S} \to T$ such that the diagram

$$\begin{array}{ccc} S & \xrightarrow{q} & \tilde{S} \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle \exists! \tilde{f}} & \\ & T & \end{array}$$

commutes.

One can embed a cancellative semiring in a ring in a canonical fashion. For this reason, a cancellative semiring is called a *halfring* in [21]. The following theorem imitates the construction of the ring of integers from the cancellative semiring $\mathbb{N}$ of natural numbers.

Theorem 2.1.4 (**Ring of differences for a cancellative semiring**)**.** *Let R denote a cancellative semiring and let E denote the binary relation on the set $R \times R$ of ordered pairs of elements from R defined as follows.*

$$For\ (a, b), (c, d) \in R \times R,\ (a, b)E(c, d) \ \Leftrightarrow\ a + d = b + c \tag{2.2}$$

Then R is an equivalence relation. If $(a, b)_E$ denotes the E-equivalence class of (a, b), then the quotient structure $(R \times R)/E := \{(a, b)_E : (a, b) \in R \times R\}$ is a ring with respect to the operations given by

$$(a, b)_E + (c, d)_E \ := \ (a + c, b + d)_E \tag{2.3}$$

$$(a, b)_E \cdot (c, d)_E \ := \ (a \cdot c + b \cdot d, a \cdot d + b \cdot c)_E \tag{2.4}$$

$$-(a, b)_E \ := \ (b, a)_E \tag{2.5}$$

for $(a,b)_E, (c,d)_E \in (R \times R)/E$. We denote the ring $(R \times R)/E$ by $K_0(R)$ following the conventions of K-theory. The semiring R can be embedded into the ring $K_0(R)$ by the semiring homomorphism i given by $a \mapsto (a, 0)$. Furthermore, given any ring T and a semiring homomorphism $g : R \to T$, there exists a unique ring homomorphism $\overline{g} : K_0(R) \to T$ such that the diagram

$$\begin{array}{ccc} R & \xrightarrow{\ i\ } & K_0(R) \\ & {}_{g}\searrow \quad \swarrow_{\exists!\overline{g}} & \\ & T & \end{array}$$

commutes.

For a semiring S, we say that the ring $K_0(\tilde{S})$ is its **Grothendieck ring** where $\tilde{S}$ is the cancellative semiring obtained from S as stated in the Theorem 2.1.3. We denote the Grothendieck ring by $K_0(S)$ for simplicity and the canonical map $S \to K_0(S)$ by η_S. We finally note the following result which combines the previous two theorems.

Corollary 2.1.5. *A semiring S can be embedded in a ring if and only if S is cancellative. Given any ring T and a semiring homomorphism $g : S \to T$, there exists a unique ring homomorphism $\overline{g} : K_0(S) \to T$ such that the diagram*

$$\begin{array}{ccc} S & \xrightarrow{\ \eta_S\ } & K_0(S) \\ & {}_{g}\searrow \quad \swarrow_{\exists!\overline{g}} & \\ & T & \end{array}$$

commutes.

This result can be stated in category theoretic language as follows. Let CSemiRing denote the category of commutative semirings with unity and semiring homomorphisms preserving unity. Let CRing denote its full subcategory consisting of commutative rings with unity and let I : CRing $\to$ CSemiRing be the inclusion functor. Then I admits a left adjoint, namely K_0 : CSemiRing $\to$ CRing. If η is the unit of the adjunction, the diagram in the above corollary represents the universal property of the adjunction.

2.2 Euler characteristic of simplicial complexes

We introduce the concept of an abstract simplicial complex and a couple of ways to calculate its Euler characteristic. We also state some important results in the homology theory of simplicial complexes. The material on homology and relative homology presented in this section is taken from [11, §II.4]. This theory provides the basis for the analysis of 'local characteristics' in 5.1.

Definition 2.2.1. *An **abstract simplicial complex** is a pair $(X, \mathcal{K})$ where X is a finite set and $\mathcal{K}$ is a collection of subsets of X satisfying the following properties:*

- $\emptyset \notin \mathcal{K}$;

- *$\{x\} \in \mathcal{K}$ for each $x \in X$;*
- *if $F \in \mathcal{K}$ and $\emptyset \neq F' \subsetneq F$, then $F' \in \mathcal{K}$.*

We usually identify the simplicial complex $(X, \mathcal{K})$ with $\mathcal{K}$. The elements $F \in \mathcal{K}$ are called the **faces** of the complex and the singleton faces are called the **vertices** of the complex. We use $\mathcal{V}(\mathcal{K})$ to denote the set of vertices of $\mathcal{K}$.

For each $k \geq 0$, let $\Delta^k := \mathbb{P}([k+1]) \setminus \{\emptyset\}$ denote the **standard k-simplex**, where $\mathbb{P}$ denotes the power set operator. We define the **geometric realization** of the standard k-simplex, denoted $|\Delta^k|$, to be the set of all points of $\mathbb{R}^{k+1}$ which can be expressed as a convex linear combination of the standard basis vectors of $\mathbb{R}^{k+1}$. In fact we can associate to every abstract simplicial complex a topological space $|\mathcal{K}|$, called its geometric realization. This topological space is constructed by 'gluing together' the geometric realizations of its simplices.

We assign dimension to every face $F \in \mathcal{K}$ by stating $\dim F := |F| - 1$ and we say that the **dimension of the complex** is the maximum of the dimensions of its faces.

Definition 2.2.2. *We define the **Euler characteristic** of the complex $\mathcal{K}$, denoted $\chi(\mathcal{K})$, to be the integer $\Sigma_{n=0}^{\dim \mathcal{K}}(-1)^n v_n$ where v_n is the number of faces in $\mathcal{K}$ with dimension n.*

It is easy to check that $\chi(\Delta^k) = 1$ for each $k \geq 0$. Since we also allow our complex to be empty, we define $\chi(\emptyset) := 0$ though $\dim \emptyset$ is undefined.

There is another way to obtain the Euler characteristics of simplicial complexes, via homology. In the context of simplicial complexes, the word homology will always mean simplicial homology with integer coefficients. Let $C_*(\mathcal{K}) := (C_n(\mathcal{K}))_{n \geq 0}$ and $H_*(\mathcal{K}) := (H_n(\mathcal{K}))_{n \geq 0}$ be the chain complexes associated with the simplicial complex $\mathcal{K}$, where $C_n(\mathcal{K})$ is the free abelian group generated by the set of n-simplices in $\mathcal{K}$ and $H_n(\mathcal{K})$ is the n^{th} homology group of the chain complex $C_*(\mathcal{K})$. If b_n denotes the n^{th} Betti number of the simplicial complex $\mathcal{K}$ (i.e., the rank of the group $H_n(\mathcal{K})$), then we have the identity $\chi(\mathcal{K}) = \Sigma_{n=0}^{\infty}(-1)^n b_n$ where the sum on the right hand side is finite.

The following result states that homology is a homotopy invariant. It will be useful in proving a key result (Proposition 5.1.8).

Theorem 2.2.3. *If $\mathcal{K}_1$ and $\mathcal{K}_2$ (meaning, their geometric realizations) are homotopy equivalent, then $H_*(\mathcal{K}_1) \cong H_*(\mathcal{K}_2)$.*

The definition of Euler characteristic in terms of Betti numbers gives the following corollary.

Corollary 2.2.4. *If $\mathcal{K}_1$ and $\mathcal{K}_2$ are homotopy equivalent, then $\chi(\mathcal{K}_1) = \chi(\mathcal{K}_2)$.*

The homology groups $H_n(\mathcal{K})$, for $n \geq 1$, calculate the number of "n-dimensional holes" in the geometric realization of the complex $\mathcal{K}$. Sometimes it is important to ignore the data present in a smaller part of the given structure. This can be done in two ways, viz. using the cone construction for a subcomplex or by using relative homology. Given a complex $\mathcal{K}$ and a subcomplex $\mathcal{Q} \subseteq \mathcal{K}$, we write $\mathcal{K} \cup \text{Cone}(\mathcal{Q})$ for the simplicial complex whose vertex set is $\mathcal{V}(\mathcal{K}) \cup \{x\}$, where $x \notin \mathcal{V}(\mathcal{K})$, and the faces are $\mathcal{K} \cup \{\{x\} \cup F : F \in \mathcal{Q}\}$. We say that x is the **apex** of the cone. In the same situation, we use the notation $H_n(\mathcal{K}; \mathcal{Q})$ to denote the n^{th} homology of $\mathcal{K}$ relative to $\mathcal{Q}$.

The following theorem connects the relative homologies with the homologies of the original complexes.

Theorem 2.2.5 (see [16, Theorem 2.16])**.** *Given a pair of simplicial complexes $\mathcal{Q} \subset \mathcal{K}$, we have the following long exact sequence of homologies.*

$$\cdots \to H_n(\mathcal{Q}) \to H_n(\mathcal{K}) \to H_n(\mathcal{K}; \mathcal{Q}) \to H_{n-1}(\mathcal{Q}) \to \cdots \to H_0(\mathcal{K}; \mathcal{Q}) \to 0$$

We shall also make use of the following result.

Theorem 2.2.6 (see [11, Theorem II.4.7])**.** *Given a pair of simplicial complexes $\mathcal{Q} \subseteq \mathcal{K}$, we have $H_n(\mathcal{K}; \mathcal{Q}) \cong H_n(\mathcal{K} \cup \text{Cone}(\mathcal{Q}))$ for $n \geq 1$ and $H_0(\mathcal{K} \cup \text{Cone}(\mathcal{Q})) \cong H_0(\mathcal{K}; \mathcal{Q}) \oplus \mathbb{Z}$.*

Illustration 2.2.7. Let $\mathcal{K} = \{\{1\}, \{2\}, \{3\}, \{1,2\}, \{2,3\}\}$ and $\mathcal{Q}$ denote the subcomplex $\{\{1\}, \{3\}\}$. Then

$$\begin{aligned}
H_n(\mathcal{K}) &= \begin{cases} \mathbb{Z}, & \text{if } n = 0, \\ 0, & \text{otherwise,} \end{cases} \\
H_n(\mathcal{Q}) &= \begin{cases} \mathbb{Z} \oplus \mathbb{Z}, & \text{if } n = 0, \\ 0, & \text{otherwise,} \end{cases} \\
H_n(\mathcal{K}; \mathcal{Q}) &= \begin{cases} \mathbb{Z}, & \text{if } n = 1, \\ 0, & \text{otherwise,} \end{cases} \\
H_n(\mathcal{K} \cup \text{Cone}(\mathcal{Q})) &= \begin{cases} \mathbb{Z}, & \text{if } n = 0, 1, \\ 0, & \text{otherwise.} \end{cases}
\end{aligned}$$

Combining the above two results with the definition of Euler characteristic, we get

Corollary 2.2.8. *For a pair of simplicial complexes $\mathcal{Q} \subseteq \mathcal{K}$, $\chi(\mathcal{K} \cup \text{Cone}(\mathcal{Q})) + \chi(\mathcal{Q}) = \chi(\mathcal{K}) + 1$.*

2.3 Products of simplicial complexes

We define various products of simplicial complexes and study their interrelations. The inclusion-exclusion principle stated in Lemma 2.3.4 is equivalent to the statement that 'local characteristics are multiplicative' (Lemma 5.5.1).

Let $\mathcal{K}$ and $\mathcal{Q}$ be two simplicial complexes with vertex sets $\mathcal{V}(\mathcal{K})$ and $\mathcal{V}(\mathcal{Q})$ respectively and let $\pi_1 : \mathcal{V}(\mathcal{K}) \times \mathcal{V}(\mathcal{Q}) \to \mathcal{V}(\mathcal{K})$ and $\pi_2 : \mathcal{V}(\mathcal{K}) \times \mathcal{V}(\mathcal{Q}) \to \mathcal{V}(\mathcal{Q})$ denote the projection maps. We define two simplicial complexes with the vertex set $\mathcal{V}(\mathcal{K}) \times \mathcal{V}(\mathcal{Q})$. The following product is defined in [10, §3].

Definition 2.3.1. *The **simplicial product** $\mathcal{K} \vartriangle \mathcal{Q}$ of two simplicial complexes $\mathcal{K}$ and $\mathcal{Q}$ is a simplicial complex with vertex set $\mathcal{V}(\mathcal{K}) \times \mathcal{V}(\mathcal{Q})$ where a nonempty set $F \subseteq \mathcal{V}(\mathcal{K}) \times \mathcal{V}(\mathcal{Q})$ is a face of $\mathcal{K} \vartriangle \mathcal{Q}$ if and only if $\pi_1(F) \in \mathcal{K}$ and $\pi_2(F) \in \mathcal{Q}$.*

Definition 2.3.2. *The **disjunctive product** $\mathcal{K} \boxtimes \mathcal{Q}$ of two simplicial complexes $\mathcal{K}$ and $\mathcal{Q}$ is a simplicial complex with vertex set $\mathcal{V}(\mathcal{K}) \times \mathcal{V}(\mathcal{Q})$ where a nonempty set $F \subseteq \mathcal{V}(\mathcal{K}) \times \mathcal{V}(\mathcal{Q})$ is a face of $\mathcal{K} \boxtimes \mathcal{Q}$ if and only if $\pi_1(F) \in \mathcal{K}$ or $\pi_2(F) \in \mathcal{Q}$.*

Observe that the previous two definitions are identical except for the word 'and' in the former is replaced by the word 'or' in the latter. Thus the simplicial product $\mathcal{K} \vartriangle \mathcal{Q}$ is always contained in the disjunctive product $\mathcal{K} \boxtimes \mathcal{Q}$.

Illustration 2.3.3. Let $\mathcal{K} = \{\{1\}, \{2\}\}$ denote the complex consisting precisely of two vertices. Then $\mathcal{K} \vartriangle \mathcal{K}$ contains only the vertices of the 'square' $\mathcal{K} \boxtimes \mathcal{K}$ given by $\{\{(1,1)\}, \{(1,2)\}, \{(2,1)\}, \{(2,2)\}, \{(1,1),(1,2)\}, \{(2,1),(2,2)\}, \{(1,1),(2,1)\}, \{(1,2),(2,2)\}\}$. For each $k \geq 0$ the complex $\mathcal{K} \vartriangle \Delta^k$ is the union of two disjoint copies of Δ^k, whereas the complex $\mathcal{K} \boxtimes \Delta^k$ is a copy of Δ^{2k+1}.

The main aim of this section is to prove the following lemma about the Euler characteristic of the disjunctive product.

Lemma 2.3.4. *The Euler characteristics of two simplicial complexes $\mathcal{K}$ and $\mathcal{Q}$ satisfy*

$$\chi(\mathcal{K} \boxtimes \mathcal{Q}) = \chi(\mathcal{K}) + \chi(\mathcal{Q}) - \chi(\mathcal{K})\chi(\mathcal{Q}). \tag{2.6}$$

Illustration 2.3.5. Let $\mathcal{K}$ be as defined in 2.3.3. Then we observe that $\chi(\mathcal{K}) = 2$. Since $\mathcal{K} \boxtimes \mathcal{K}$ contains 4 vertices and 4 edges, we get $\chi(\mathcal{K} \boxtimes \mathcal{K}) = 0 = 2\chi(\mathcal{K}) - \chi(\mathcal{K})^2$ verifying Equation (2.6) in this case.

The proof of the lemma uses tensor products of chain complexes.

Definition 2.3.6. *Let $C_* = \{C_n, \partial_n\}_{n \geq 0}$ and $D_* = \{D_n, \delta_n\}_{n \geq 0}$ denote two bounded chain complexes of abelian groups. The **tensor product complex** $C_* \otimes D_* = \{(C_* \otimes$*

$D_*)_n, d_n\}_{n\geq 0}$ *is defined by*

$$\begin{aligned} (C_* \otimes D_*)_n &= \bigoplus_{i+j=n} C_i \otimes D_j, \\ d_n(a_i \otimes b_j) &= \partial_i(a_i) \otimes b_j + (-1)^i a_i \otimes \delta_j(b_j). \end{aligned}$$

Illustration 2.3.7. We compute the tensor product $C_*(\partial\Delta^2) \otimes C_*(\Delta^1)$ as an example, where $\partial\Delta^2$ denotes the boundary of Δ^2.

$$C_n(\partial\Delta^2) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}, & \text{if } n = 0, 1, \\ 0, & \text{otherwise} \end{cases}$$

$$C_n(\Delta^1) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z}, & \text{if } n = 0, \\ \mathbb{Z}, & \text{if } n = 1, \\ 0, & \text{otherwise} \end{cases}$$

$$(C_*(\partial\Delta^2) \otimes C_*(\Delta^1))_n = \begin{cases} \oplus_{i=1}^6 \mathbb{Z}, & \text{if } n = 0, \\ \oplus_{i=1}^9 \mathbb{Z}, & \text{if } n = 1, \\ \oplus_{i=1}^3 \mathbb{Z}, & \text{if } n = 2, \\ 0, & \text{otherwise} \end{cases}$$

There is yet one more product of simplicial complexes, viz., the cartesian product, defined in the literature (see [10]). We avoid its use by dealing with the product of geometric realizations (with the product topology). The homology of such (finite) product spaces is easily computed using triangulation. We first note that the Euler characteristic is multiplicative.

Proposition 2.3.8 (see [38, p.205, Ex. B.4])**.** *Let $\mathcal{K}$ and $\mathcal{Q}$ be any simplicial complexes. Then*

$$\chi(|\mathcal{K}| \times |\mathcal{Q}|) = \chi(\mathcal{K})\chi(\mathcal{Q}).$$

A famous theorem of Eilenberg and Zilber (see [10]) connects the homologies of two semi-simplicial complexes (a term used in 1950 that includes the class of simplicial complexes) with that of their cartesian product. We state this result below using the cartesian product of their geometric realizations. More details can be found in [16, §2.1] and [11, §III.6].

Theorem 2.3.9 (see [11, §III.6.2])**.** *Let $\mathcal{K}$ and $\mathcal{Q}$ be any two simplicial complexes. Then we have $H_*(|\mathcal{K}| \times |\mathcal{Q}|) \cong H_*(C_*(\mathcal{K}) \otimes C_*(\mathcal{Q}))$.*

Furthermore, Eilenberg and Zilber state the following corollary of the above theorem in [10].

Corollary 2.3.10. *Let $\mathcal{K}$ and $\mathcal{Q}$ be any two simplicial complexes. Then* $H_*(\mathcal{K} \vartriangle \mathcal{Q}) \cong H_*(C_*(\mathcal{K}) \otimes C_*(\mathcal{Q}))$.

Illustration 2.3.11. We continue the example in 2.3.7. The computation of the boundary operators yields

$$H_n(C_*(\partial\Delta^2) \otimes C_*(\Delta^1)) = \begin{cases} \mathbb{Z}, & \text{if } n = 0, 1, \\ 0, & \text{otherwise} \end{cases}$$

The space $|\partial\Delta^2| \times |\Delta^1|$ is a cylinder which is homotopy equivalent to S^1. Hence $H_n(|\partial\Delta^2| \times |\Delta^1|) = \mathbb{Z}$ for $n = 0, 1$ and is zero for other values of n. This completes the illustration of Theorem 2.3.9.

Furthermore the complex $\partial\Delta^2 \vartriangle \Delta^1$ is the union of three copies of Δ^3 each of which shares exactly one edge (i.e., a copy of Δ^1) with every other copy and these three edges are pairwise disjoint. It can be easily see that this complex (i.e., its geometric realization) is homotopy equivalent to the circle and hence the conclusions of the Corollary 2.3.10 hold.

Proof. (Lemma 2.3.4) We first observe that there is an embedding of simplicial complexes $\iota_1 : \mathcal{K} \vartriangle (\Delta^{|\mathcal{V}(\mathcal{Q})|-1}) \to \mathcal{K} \boxtimes \mathcal{Q}$ induced by some fixed enumeration of $\mathcal{V}(\mathcal{Q})$. Similarly there is an embedding $\iota_2 : (\Delta^{|\mathcal{V}(\mathcal{K})|-1}) \vartriangle \mathcal{Q} \to \mathcal{K} \boxtimes \mathcal{Q}$ induced by some fixed enumeration of $\mathcal{V}(\mathcal{K})$. Furthermore, the intersection $\iota_1(\mathcal{K} \vartriangle (\Delta^{|\mathcal{V}(\mathcal{Q})|-1})) \cap \iota_2((\Delta^{|\mathcal{V}(\mathcal{K})|-1}) \vartriangle \mathcal{Q})$ is precisely the complex $\mathcal{K} \vartriangle \mathcal{Q}$.

This gives us, using the counting definition of the Euler characteristics, that the identity

$$\chi(\mathcal{K} \boxtimes \mathcal{Q}) = \chi(\mathcal{K} \vartriangle (\Delta^{|\mathcal{V}(\mathcal{Q})|-1})) + \chi((\Delta^{|\mathcal{V}(\mathcal{K})|-1}) \vartriangle \mathcal{Q}) - \chi(\mathcal{K} \vartriangle \mathcal{Q}) \tag{2.7}$$

holds.

Note that it is sufficient to prove that $\chi(\mathcal{K} \vartriangle \mathcal{Q}) = \chi(\mathcal{K})\chi(\mathcal{Q})$ for all simplicial complexes $\mathcal{K}$ and $\mathcal{Q}$ because, in that case, (2.6) follows from (2.7) and the identity $\chi(\Delta^k) = 1$ for each $k \geq 0$.

Now we have $H_*(\mathcal{K} \vartriangle \mathcal{Q}) \cong H_*(C_*(\mathcal{K}) \otimes C_*(\mathcal{Q})) \cong H_*(C_*(|\mathcal{K}| \times |\mathcal{Q}|))$, where the first isomorphism is by 2.3.10 and the second by 2.3.9.

Hence we have $\chi(\mathcal{K} \vartriangle \mathcal{Q}) = \chi(|\mathcal{K}| \times |\mathcal{Q}|) = \chi(\mathcal{K})\chi(\mathcal{Q})$ by 2.3.8 as required. This completes the proof. □

Chapter 3

K-Theory of Model-theoretic Structures

Let L denote any language and M denote a first order L-structure. In this thesis, the term definable will always mean definable with parameters from M. Following [25], we introduce the notion of the model-theoretic Grothendieck ring of M, denoted $K_0(M)$, in Section 3.1. Definable sets with definable bijections form a symmetric monoidal category, $\mathcal{S}(M)$, under disjoint union. Discussion of the K-theory of symmetric monoidal categories is the content of Section 3.2. The K-theory group $K_0(\mathcal{S}(M))$, defined in Section 3.2, agrees with the underlying group of the Grothendieck ring $K_0(M)$. Based on this observation, we define the K-theory of the structure M in Section 3.3 as the K-theory of the category $\mathcal{S}(M)$. The final two sections are devoted to the computation of the group K_1 of a vector space.

The material in Section 3.1 is contained in [26, §2.2].

3.1 Model-theoretic Grothendieck rings

After setting some background in model theory, we state how to construct the semiring of definable isomorphism classes of definable subsets of finite cartesian powers of the structure M. Following the method described in Section 2.1 we then construct the Grothendieck ring $K_0(M)$.

Definitions 3.1.1. *For each $n \geq 1$, we define* $\mathrm{Def}(M^n)$ *to be the collection of all definable subsets of M^n. We also define* $\overline{\mathrm{Def}}(M) := \bigcup_{n\geq 1} \mathrm{Def}(M^n)$.

Definition 3.1.2. *We say that two definable sets $A, B \in \overline{\mathrm{Def}}(M)$ are **definably isomorphic** if there exists a definable bijection between them, i.e., a bijection $f : A \to B$ such that the graph $Graph(f) \in \overline{\mathrm{Def}}(M)$. This is an equivalence relation on*

$\overline{\mathrm{Def}}(M)$ *and the equivalence class of a set* A *is denoted by* $[A]$. *We use* $\widetilde{\mathrm{Def}}(M)$ *to denote the set of all equivalence classes with respect to this relation.*

The assignment $A \mapsto [A]$ defines a surjective map $[-] : \overline{\mathrm{Def}}(M) \to \widetilde{\mathrm{Def}}(M)$. We can regard $\widetilde{\mathrm{Def}}(M)$ as an L_{ring}-structure. In fact, it is a semiring with respect to the operations defined as follows:

- $0 := [\emptyset]$;
- $1 := [\{*\}]$ for any singleton subset $\{*\}$ of M;
- $[A] + [B] := [A' \sqcup B']$ for $A' \in [A], B' \in [B]$ such that $A' \cap B' = \emptyset$;
- $[A] \cdot [B] := [A \times B]$.

Now we are ready to give an important definition.

Definition 3.1.3. *We define the* ***model-theoretic Grothendieck ring of the first order structure*** M, *denoted by* $K_0(M)$, *to be the ring* $K_0(\widetilde{\mathrm{Def}}(M))$ *obtained from Corollary 2.1.5, where the semiring structure on* $\widetilde{\mathrm{Def}}(M)$ *is as defined above.*

We are interested to know whether the ring $K_0(M)$ - that captures some aspects of the definable combinatorics of the structure M - is trivial. The motivation behind the study of this ring lies in the question which asks what of elementary combinatorics holds true in a class of first order structures if sets, relations, and maps must be definable?

Definition 3.1.4. *We say that an infinite structure* M *satisfies the* ***pigeonhole principle*** *if for each* $A \in \overline{\mathrm{Def}}(M)$, *each definable injection* $f : A \rightarrowtail A$ *is an isomorphism.*

This condition is very strong to be true for many structures. As an example, consider the additive group of integers $\mathbb{Z}$ in the language of abelian groups. The function $\mathbb{Z} \xrightarrow{(-)\times 2} \mathbb{Z}$ is a definable injection but not an isomorphism. Hence it is useful to consider some weaker forms. Though there are several of them (see [25]), we note the one important for us.

Definition 3.1.5. *We say that an infinite structure* M *satisfies the* ***onto pigeonhole principle*** *if for each* $A \in \overline{\mathrm{Def}}(M)$ *and each definable injection* $f : A \rightarrowtail A$, *we have* $f(A) \neq A \setminus \{a\}$ *for any* $a \in A$.

The following proposition gives the necessary and sufficient condition for $K_0(M)$ to be nontrivial (i.e., $0 \neq 1$ in $K_0(M)$). We include a proof for the sake of completeness.

Proposition 3.1.6. *Given any infinite structure M, $K_0(M) \neq \{0\}$ if and only if M satisfies the onto pigeonhole principle.*

Proof. Recall the construction of the cancellative semiring from (2.1). The condition $0 = 1$ in $K_0(M)$ is thus equivalent to the statement that for some $A \in \overline{\mathrm{Def}}(M)$, we have $0 + [A] = 1 + [A]$. This is precisely the statement that M satisfies the onto pigeonhole principle. □

Grothendieck rings behave very well with respect to elementary embeddings, but elementary equivalence between structures gives only a weak type of equivalence between their Grothendieck rings.

Proposition 3.1.7 ([25, Theorem 7.3]). *If M, N are L-structures and $M \preceq N$ then $K_0(M) \leq K_0(N)$. If $M \equiv N$, then $\widetilde{\mathrm{Def}}(M) \equiv_{\exists_1} \widetilde{\mathrm{Def}}(N)$ in L_{ring}. As the Grothendieck ring $K_0(M)$ is existentially interpretable in $\widetilde{\mathrm{Def}}(M)$, we have $K_0(M) \equiv_{\exists_1} K_0(N)$.*

A brief survey of known Grothendieck Rings: Only a few examples of Grothendieck rings are known. If M is a finite structure, then $K_0(M) \cong \mathbb{Z}$. Krajiček and Scanlon showed in [25, Ex. 3.6] that $K_0(\mathbf{R}) \cong \mathbb{Z}$ using dimension theory and cell decomposition theorem for o-minimal structures, where $\mathbf{R}$ denotes a real closed field. Cluckers and Haskell [6], [7] proved that the fields of p-adic numbers and $F_q((t))$, the field of formal Laurent series, both have trivial Grothendieck rings, by constructing definable bijections from a set to the same set minus a point. Denef and Loeser [8], [9] found that the Grothendieck ring $K_0(\mathbb{C})$ of the field $\mathbb{C}$ of complex numbers regarded as an L_{ring}-structure admits the ring $\mathbb{Z}[X, Y]$ as a quotient. Krajiček and Scanlon obtained a strong result that $K_0(\mathbb{C})$ contains an algebraically independent set of size $\mathfrak{c}$ of the continuum, and hence the ring $\mathbb{Z}[X_i : i \in \mathfrak{c}]$ embeds into $K_0(\mathbb{C})$. Perera showed in [31, Theorem 4.3.1] that $K_0(M) \cong \mathbb{Z}[X]$ whenever M is an infinite module over an infinite division ring. Prest conjectured [31, Ch. 8, Conjecture A] that $K_0(M)$ is nontrivial for all nonzero right $\mathcal{R}$-modules M. We prove that $K_0(M)$ is actually a quotient of a monoid ring and, furthermore, it is nontrivial. Chapters 5 and 6 are devoted to the proof of this statement.

3.2 K-theory of symmetric monoidal categories

Algebraic K-theory seeks to classify finitely generated projective (right) modules (i.e., the modules which are direct summands of finitely generated free modules) over a unital ring R. In fact, for each $n \geq 0$, there is a functor $K_n : \mathrm{Ring} \to \mathrm{Ab}$. These groups fit together nicely in a long exact sequence. In particular the group $K_0(R)$ classifies isomorphism classes of projective R-modules under direct sum, whereas the group $K_1(R)$ classifies automorphisms of projective modules in the direct limit. The idea of classification can be extended to more general settings than just projective modules; one can obtain a (functorial) sequence of K-groups for categories with some extra structure. Classes of such categories include exact categories, Waldhausen categories and symmetric monoidal categories. This section contains a summary of important definitions and theorems from the K-theory of symmetric monoidal categories. More details about the algebraic K-theory of rings can be found in [35] whereas detailed treatment of the K-theory of categories can be found in the excellent "K-book" [40] by Weibel. Some background from category theory and algebraic topology is assumed; a reader less familiar with this material can refer to standard texts like [22] and [16].

A symmetric monoidal category is a category-theoretic analogue of a commutative monoid.

Definition 3.2.1. *A triple $(S, *, e)$ is a* ***symmetric monoidal category*** *if the category S is equipped with a bifunctor $* : S \times S \to S$, and a distinguished object e such that, for all objects $s, t, u \in S$, there are natural coherent isomorphisms*

$$e * s \cong s \cong s * e, \quad s * (t * u) \cong (s * t) * u, \quad s * t \cong t * s,$$

that satisfy certain obvious commutative diagrams.

A ***(strict) monoidal functor*** *$F : (S, *, e) \to (S', *', e')$ is a functor $F : S \to S'$ such that $F(e) = e'$ and, for all objects $s, t \in S$, $F(s*t) = F(s)*' F(t)$ and F preserves the coherence isomorphisms.*

More details can be found in [22, §VII,XI]. The symmetric monoidal categories we consider in this chapter are (skeletally) small, i.e., the set of isomorphism classes of objects of S form a set.

Definition 3.2.2. *Suppose S is symmetric monoidal, and suppose S^{iso} denotes the set of isomorphism classes of objects of S. Then S^{iso} is a commutative monoid under the product induced by $*$ and has e as the identity element. The Grothendieck group of this commutative monoid is denoted by $K_0^*(S)$ (or just $K_0(S)$ if the product is clear from the context).*

Examples 3.2.3. The category $(\text{FinSets}, \sqcup, \emptyset)$ is a symmetric monoidal category, where FinSets is the category of finite sets and functions between them. The additive monoid $\mathbb{N}$ of natural numbers is isomorphic to the monoid FinSets^{iso} and hence $K_0(\text{FinSets}) = \mathbb{Z}$.

Given a unital ring R, let Proj-R denote the category of finitely generated projective right R-modules with R-module homomorphisms. This is a symmetric monoidal category under direct sum, $\oplus$, and is the key example in algebraic K-theory. The Grothendieck group of the ring R is defined to be the group $K_0(\text{Proj-}R)$.

Let k be a field and Var_k denote the category of k-varieties (i.e., reduced separated k-schemes of finite type - note that we do not require varieties to be irreducible). Then Var_k is a (large, but skeletally small) symmetric monoidal category under disjoint union of varieties (see [39, Ex. 9.1A]). The group $K_0(\text{Var}_k)$ is the Grothendieck group of k-varieties. Chapter 7 is devoted to the study of questions about the structure of this group.

Remark 3.2.4. For a first order structure M, let $\mathcal{S}(M)$ denote the category of definable bijections between definable sets in $\overline{\text{Def}}(M)$. Then $(\mathcal{S}(M), \sqcup, \emptyset)$ is a symmetric monoidal category and $K_0(\mathcal{S}(M))$ is the underlying abelian group of the Grothendieck ring $K_0(M)$.

Given a small category C, its geometric realization (or classifying space) BC is a topological space (constructed as the geometric realization of its nerve) and gives a way of attaching certain topological adjectives to C. Moreover, this construction is functorial. More details about the geometric realization can be found in [40, §IV.3].

Category theoretic properties of C match well with the homotopy theoretic properties of BC.

Proposition 3.2.5 ([40, §IV.4.3.2])**.** *Any natural transformation $\eta : F_1 \to F_2$ between functors $F_1, F_2 : C \to D$ induces a homotopy $B\eta : BC \times [0,1] \to BD$ between the maps BF_1 and BF_2. Hence any adjoint pair of functors $F : C \to D$ and $G : D \to C$ induce a homotopy equivalence between BC and BD. In particular, any equivalence of categories induces a homotopy equivalence between their geometric realizations, and every category with a terminal (or initial) object is contractible.*

The geometric realization of a symmetric monoidal category has an extra structure induced from the monoidal operation.

Definition 3.2.6. *An H-**space** is a topological space X with a continuous binary operation $\mu : X \times X \to X$ such that there is a point $e \in X$ for which the functions $x \mapsto \mu(x, e)$ and $x \mapsto \mu(e, x)$ are homotopic to the identity on X, through homotopies preserving the point e.*

For a symmetric monoidal category S, the space BS is an H-space with a homotopy-commutative and homotopy-associative product. In many cases, the unit object e of S is an initial object and hence BS is contractible. Therefore, we focus our attention to the subcategory $iso\, S$ of isomorphisms in S whose realization $B(iso\, S)$ is an H-space as well. Restricting to isomorphisms does not change the group K_0 as it only depends on the commutative monoid $\pi_0(iso\, S)$ of isomorphism classes of objects of S.

A **groupoid** is a category in which all morphisms are isomorphisms. Henceforth the underlying categories of all our symmetric monoidal categories will be groupoids. Any groupoid S is equivalent to $\coprod_{s\in S^{iso}} \mathrm{Aut}_S(s)$ and hence BS is homotopy equivalent to the disjoint union of the classifying spaces $B\mathrm{Aut}_S(s)$ as s ranges over S^{iso}, where $\mathrm{Aut}_S(s)$ is the group of automorphisms (= endomorphisms, since S is a groupoid) of the object $s \in S$.

Examples 3.2.7. The space $B(iso\,\mathrm{FinSets})$ is homotopy equivalent to $\coprod_{n\geq 0} B\Sigma_n$, where Σ_n is the permutation group on the set of n elements.

The space $B(iso\,\mathrm{Proj}\text{-}R)$ is homotopy equivalent to $\coprod B\mathrm{Aut}(P)$, where P runs over isomorphism classes.

If $\mathrm{F}(R)$ is the category whose morphisms are $\coprod_{n\geq 1} \mathrm{GL}_n(R)$ and objects are based free R-modules, then the space $B\mathrm{F}(R)$ is equivalent to $\coprod_n B\,\mathrm{GL}_n(R)$.

We say that **translations are faithful** in S if, for all $s, t \in S$, the translation $\mathrm{Aut}_S(s) \to \mathrm{Aut}_S(s * t)$ defined by $f \mapsto f * id_t$ is an injective map. All examples of (symmetric monoidal) categories we consider in this and the next sections have faithful translations. If translations are faithful in S, then the following construction of the category $S^{-1}S$ gives a "group completion" $B(S^{-1}S)$ of BS. The motivation comes from the group completion of a commutative monoid and the term group completion is a well-defined notion for homotopy-associative H-spaces.

Definition 3.2.8 (Quillen's $S^{-1}S$-construction)**.** *If S is any symmetric monoidal groupoid, then we define a new category $S^{-1}S$ as follows. The objects are pairs of objects of S and a morphism is an equivalence class of composites $(m_1, m_2) \xrightarrow{s*} (s * m_1, s * m_2) \xrightarrow{(f,g)} (n_1, n_2)$, where this composite is equivalent to $(f', g') \circ (t*)$ whenever there is an isomorphism $\alpha : s \to t$ satisfying $(f', g') \circ (\alpha * m_i) = (f, g)$. This assignment is functorial for strict monoidal functors.*

The category $S^{-1}S$ is a symmetric monoidal category and the natural map $BS \to B(S^{-1}S)$ taking $s \in S$ to $(s, e) \in S^{-1}S$ is an H-space map. It induces a map of monoids $\pi_0(S) \to \pi_0(S^{-1}S)$, where the target is an abelian group owing to the existence of morphisms $\eta : (e, e) \to (m, n) * (n, m)$ in $S^{-1}S$.

Definition 3.2.9. *If S is any symmetric monoidal category, then we define the K-**theory space** $K^*(S)$ of S as the geometric realization of $S^{-1}S$. The K-groups of S are defined as $K^*_n(S) := \pi_n K^*(S)$.*

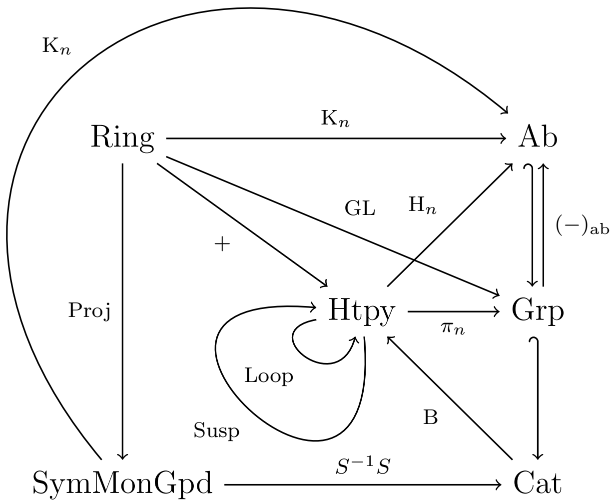

Figure 3.1: K-theory of symmetric monoidal groupoids

This assignment is functorial and the definition of $K_0^*(S)$ agrees with the earlier definition of $K_0(S)$ i.e., the group completion of $\pi_0(S^{iso})$. For each $n \geq 1$, the algebraic K-theory groups for rings are defined by $K_n(R) := K_n(iso\,\text{Proj-}R)$.

See Figure 3.1 for the complete description of the categories involved in the definitions of the K-theory, where Htpy is the homotopy category of topological spaces with two endofunctors, the loop-space functor Loop and the suspension functor Susp. Note that we do not expect that all diagrams commute.

The group $K_0(S)$ is, in some sense, "orthogonal" to the groups $K_n(S)$ for $n \geq 1$. This is because $K_0(S)$ identifies isomorphic objects and, in turn, disregards the automorphism groups $\text{Aut}_S(s)$, whereas the higher K-groups - especially $K_1(S)$ - seek to classify automorphisms.

The following is a very deep theorem connecting the K-theory of the apparently simple combinatorial category of finite sets and bijections with the stable homotopy theory of spheres. The statement uses the language of infinite loop spaces (= spectra = topological spaces with a delooping, which again has a delooping and so on), but a reader not familiar with these terms can think about ordinary topological spaces.

Theorem 3.2.10 (Barratt-Priddy-Quillen-Segal, [40, Theorem IV.4.9.3])**.** *The following three infinite loop spaces are the same:*

(a) The group completion $K(iso\,\text{FinSets})$ of $B(iso\,\text{FinSets})$;

(b) $\mathbb{Z} \times B\Sigma^\infty$ *where* Σ^∞ *is the finitary permutation group on a countable set;*

(c) The infinite loop space $\mathrm{Loop}^\infty S^\infty = \lim_{n\to\infty} \mathrm{Loop}^n S^n$*, where* S^n *is the sphere of dimension* n*.*

Hence the groups $K_n(\mathrm{FinSets})$ *are the stable homotopy groups of spheres,* π_n^s*.*

The groups $K_0(iso\,\mathrm{FinSets}) = \pi_0^s = \mathbb{Z}$ and $K_1(iso\,\mathrm{FinSets}) = \pi_1^s = \mathbb{Z}_2$ are of particular interest. The free R-module functor $\mathrm{FinSets} \to \mathrm{F}(R)$ induces maps $\pi_n^s \to K_n(R)$.

For many nice examples of symmetric monoidal categories, the group $K_0(S)$ carries an extra multiplicative structure turning it into a ring. This is a special case of a more general phenomenon. Let S_1, S_2, S be any symmetric monoidal categories. A **pairing** of S_1 and S_2 is a monoidal functor $\otimes : S_1 \times S_2 \to S$ such that $s \otimes e = e \otimes s = e$, and there is a coherent natural bi-distributivity law

$$(s_1 * t_1) \otimes (s_2 * t_2) \cong (s_1 \otimes s_2) * (s_1 \otimes t_2) * (t_1 \otimes s_2) * (t_1 \otimes t_2).$$

Examples include the cartesian product, *times*, on FinSets and the tensor product, $\otimes$, on $\mathrm{F}(R)$.

Theorem 3.2.11 (May, [40, Theorem IV.4.6])**.** *A pairing* $S_1 \times S_2 \to S$ *of symmetric monoidal categories determines a natural pairing* $K(S_1) \wedge K(S_2) \to K(S)$ *of infinite loop spaces, which in turn induces bilinear products* $K_p(S_1) \otimes K_q(S_2) \to K_{p+q}(S)$*. In particular, if there is a pairing* $S \times S \to S$*, then* $K_0(S)$ *is a ring.*

Computation of the K-theory of a given category is in general a very hard problem. We collect some tools below that will facilitate the task.

Sometimes a subcategory of a symmetric monoidal category contains enough information about the K-theory of the category.

Definition 3.2.12. *Let* S' *be a full subcategory of the symmetric monoidal category* S*. If* S' *contains the identity object* e *of* S *and is closed under finite products, then* S' *is symmetric monoidal. We say that* S' *is* ***cofinal*** *in* S *if for every object* s *in* S *there is* t *in* S *such that* $s * t$ *is isomorphic to an object in* S'*.*

Theorem 3.2.13. *Let* S' *be cofinal in the symmetric monoidal category* S*. Then:*

1. $K_0(S')$ *is a subgroup of* $K_0(S)$*;*

2. *Every element of* $K_0(S)$ *is of the form* $[s] - [t]$ *for* $s \in S$ *and* $t \in S'$*;*

3. *If* $[s] = [t]$ *in* $K_0(S)$*, then* $s * u \cong t * u$ *for some* u *in* S'*;*

4. For each $n \geq 1$, $K_n(S) = K_n(S')$.

Example 3.2.14. The category of finite rank (based) free modules over a ring R, $F(R)$, is cofinal in the category of finitely generated projective modules, Proj-R and hence $K_n(R) := K_n(F(R))$ for each $n \geq 1$.

Bass defined and studied lower K-groups (for $n = 0, 1, 2$) of symmetric monoidal categories before Quillen. It turns out that, under certain conditions, Quillen's definitions agree with Bass' definitions.

Theorem 3.2.15 ([40, Cor. IV.4.8.1])**.** *If S is a symmetric monoidal groupoid whose translations are faithful, then*

$$\begin{aligned} K_1(S) &= \varinjlim_{s \in S} H_1(\mathrm{Aut}_S(s); \mathbb{Z}), \\ K_2(S) &= \varinjlim_{s \in S} H_2([\mathrm{Aut}_S(s), \mathrm{Aut}_S(s)]; \mathbb{Z}), \end{aligned}$$

where $H_i(G; \mathbb{Z})$ denotes the i^{th} integral homology group of the group G for $i = 1, 2$ and $[G, G]$ denotes the commutator subgroup of the group G.

The following remark will be useful for the computation of K_1.

Remark 3.2.16. Suppose that $(S, *, e)$ is a symmetric monoidal groupoid whose translations are faithful. Further suppose that S has a countable sequence of objects $s_1, s_2, \ldots$ such that $s_{n+1} = s_n * a_n$ for some $a_n \in S$, and satisfying the cofinality condition that for every $s \in S$ there is an s' and an n such that $s * s' \cong s_n$. In this case we can form the group $\mathrm{Aut}(S) = \mathrm{colim}_{n \to \infty} \mathrm{Aut}_S(s_n)$. Since the functor $H_1(-, \mathbb{Z})$ commutes with colimits, we obtain $K_1(S) = H_1(\mathrm{Aut}(S); \mathbb{Z})$.

3.3 Defining $K_n(M)$ for a structure M

We use the notation of Section 3.1 in this section. Let L denote a language and M denote an L-structure. Recall that definability always means with parameters.

In Remark 3.2.4 we defined a symmetric monoidal category $(\mathcal{S}(M), \sqcup, \emptyset)$ and observed that $K_0(\mathcal{S}(M))$ is the underlying abelian group of the Grothendieck ring $K_0(M)$. Translations are clearly faithful in $\mathcal{S}(M)$. The cartesian product of definable sets in $\overline{\mathrm{Def}}(M)$ is a pairing on $\mathcal{S}(M)$, which turns $K_0(\mathcal{S}(M))$ into a ring (Theorem 3.2.11) which is isomorphic to the Grothendieck ring $K_0(M)$. This motivates the following definition.

Definition 3.3.1. *If M is a first order L-structure, then define $K_n(M) := K_n(\mathcal{S}(M))$ for each $n \geq 0$, where the groups $K_n(\mathcal{S}(M))$ are as defined in Definition 3.2.9.*

If M is an elementary substructure of N and $\phi(\overline{x}, \overline{a})$ is any formula with parameters from M, then $\phi(N, \overline{a})$ defines a subset of $N^{|\overline{x}|}$. This defines a strict monoidal functor $\mathcal{S}(M) \to \mathcal{S}(N)$. Hence K-theory is functorial on elementary embeddings.

We will write the category $\mathcal{S}(M)$ as $\mathcal{S}$ whenever the structure M is clear from the context.

Example 3.3.2. Let M be a finite structure. Then every subset of M^n is definable with parameters. Thus the symmetric monoidal groupoid $(\mathcal{S}, \sqcup, \emptyset)$ is equivalent to the groupoid $(iso\,\mathrm{FinSets}, \sqcup, \emptyset)$ of finite sets and bijections. Hence by the Barratt-Priddy-Quillen-Segal theorem (Theorem 3.2.10), $K_n(M)$ is the n^{th} stable homotopy group of spheres. In particular, $K_0(M) = \mathbb{Z}$ and $K_1(M) = \mathbb{Z}_2$.

We will mainly be interested in the groups K_0 and K_1 for modules as structures in the language of right R-modules. Bass's definition (Theorem 3.2.15) allows us to compute $K_1(M)$ as $\operatorname{colim}_{n\to\infty} H_1(\mathrm{Aut}_{\mathcal{S}}(M^n); \mathbb{Z})$. Computation and the study of the properties of the Grothendieck rings of modules forms the basis of the next three chapters. We tackle the special case of vector spaces in the next two sections.

3.4 Definable bijections in vector spaces

Let F denote an infinite field. Every F-vector space V_F is a first-order structure for the language $L_F = \langle 0, +, -, m_r : r \in F\rangle$, where each m_r is a unary function symbol representing the scalar multiplication by the element r. When we are working in a fixed vector space V_F, we usually write the element $m_r(a)$ in formulas as ar for each $a \in V$. Instead of working with formulas, we fix an infinite F-vector space V_F and work with the definable subsets of its finite cartesian powers, i.e., objects of the category $\mathcal{S} = \mathcal{S}(V_F)$. The theory of V eliminates quantifiers in this language ([34, Theorem 2.3.24]) and hence every object of $\mathcal{S}$ is a finite boolean combination of the basic definable subsets, viz., $\{0\}, V, V^2, \ldots$ and their cosets in higher dimensional spaces.

The following theorem describes the Grothendieck ring of the vector space V_F. It follows from the structure theorem for the Grothendieck rings of modules by noting that the multiplicative monoid of (the isomorphism classes of) the fundamental definable sets is isomorphic to the additive monoid $\mathbb{N}$ of natural numbers.

Theorem 3.4.1 (see Theorem 6.2.3)**.** *The semiring $\widetilde{\mathrm{Def}}(V_F)$ of (definable) isomorphism classes of objects of $\mathcal{S}(V_F)$ is isomorphic to the sub-semiring of the polynomial ring $\mathbb{Z}[X]$ consisting of polynomials with non-negative leading coefficients. Hence $K_0(V_F) = \mathbb{Z}[X]$.*

The assignment $\dim(D) := \deg([D])$ is a well-defined dimension function on the objects of $\mathcal{S}$ which also agrees with the Morley rank of D. The aim of the rest of this section is to study the structure the groups $\mathrm{Aut}_{\mathcal{S}}(D)$ for $D \in \mathcal{S}$. The elements of this group are definable bijections $D \to D$, but we will refer to them as automorphisms of the definable set D since they are automorphisms in the category $\mathcal{S}$. The reader is warned not be confused with this slightly unusual use of the term which otherwise in model theory, in the case $D = V$, refers to structure-preserving maps $V \to V$.

We begin with some definitions and constructions in group theory.

Definition 3.4.2. *Let $(G, \cdot, e)$ be a group, $N \lhd G$ and $H \leq G$. If $G = NH$ and $N \cap H = \{e\}$, then we say that G is an **(inner) semidirect product** of N and H and write $G = H \ltimes N$.*

*Given any two groups N and H (not necessarily subgroups of a given group) and a group homomorphism $\varphi : H \to \mathrm{Aut}(N)$, the **(outer) semidirect product** of N and H with respect to φ is defined as follows. As a set, $H \ltimes_{\varphi} N$ is the cartesian product $H \times N$. Multiplication of elements in $H \ltimes_{\varphi} N$ is defined by $(h_1, n_1)(h_2, n_2) = (h_1h_2, n_1\varphi_{h_1}(n_2))$ for $n_1, n_2 \in N$ and $h_1, h_2 \in H$. The homomorphism φ is usually suppressed in the notation when it is clear from the context.*

*Let K, L be groups and T be a set with L acting on it. Let $B := \bigoplus_{x \in T} K$. The **(restricted) wreath product** $K \mathrm{wr}_T L$ is defined to be the group $L \ltimes B$ where the action of L on the element $(k_x)_{x \in T} \in B$ is defined by $l((k_x)_{x\in T}) := (k_{l^{-1}x})_{x \in T}$. In this case the group B is said to be the **base** of the wreath product.*

For each $0 \leq m < n$, let Σ^n_m denote the finitary permutation group on a countable set of cosets of an m-dimensional subspace of V^n. If $n, p > m$, then it is easy to see that $\Sigma^n_m \cong \Sigma^p_m$. For a finitary permutation group Σ on a set T and a group G, the notation $G \wr \Sigma$ will always denote the restricted wreath product $G \mathrm{wr}_T \Sigma$.

Definition 3.4.3. *Let $D \in \mathcal{S}$ and $f \in \mathrm{Aut}_{\mathcal{S}}(D)$. The **support of** f is the (definable) set* $\mathrm{Supp}(f) := \{a \in D : f(a) \neq a\}$.

Proposition 3.4.4. *For $D \in \mathcal{S}$, let $\Omega_m(D) := \{f \in \mathrm{Aut}_{\mathcal{S}}(D) : \dim(\mathrm{Supp}(f)) \leq m\}$ be the subgroup of $\mathrm{Aut}_{\mathcal{S}}(D)$ of elements fixing all automorphisms of D outside a subset of dimension at most m. If $D_1, D_2 \in \mathcal{S}$ have dimension strictly greater than m, then $\Omega_m(D_1) \cong \Omega_m(D_2)$.*

Proof. Since $\dim D_1, \dim D_2 > m$, it is always possible to find a set D with an arrow $g : D_2 \to D$ in $\mathcal{S}$ such that $\dim(D_1 \cap D) > m$. The definable bijection g induces an isomorphism between $\Omega_m(D_2)$ and $\Omega_m(D)$. Therefore it is sufficient to prove the result when $D_1 \subseteq D_2$.

For each $i = 1, 2$, consider the full subcategory $\mathcal{S}_m(D_i)$ of $\mathcal{S}$ containing definable subsets of D_i of dimension at most m. The restriction of $\sqcup$ to $\mathcal{S}_m(D_i)$ equips it with a symmetric monoidal structure. Then $\Omega_m(D_i) \cong \mathrm{Aut}(\mathcal{S}_m(D_i))$, where the groups on the right hand side of the equation can be constructed as follows.

Let $S_1 \subset S_2 \subset \cdots$ be a sequence of objects of $\mathcal{S}_m(D_1)$, where S_1 is a copy of V^m in D_1 and S_{k+1} is obtained by adding a disjoint copy of V^m to S_k for each $k \geq 1$. This sequence is cofinal in $\mathcal{S}_m(D_1)$ and thus, using (3.2.16) for this sequence, we construct $\mathrm{Aut}(\mathcal{S}_m(D_1))$ as $\mathrm{colim}_{k\to\infty} \mathrm{Aut}_{\mathcal{S}}(S_k)$. When $D_1 \subseteq D_2$, the same colimit can be used to construct the group $\Omega_m(D_2) \cong \mathrm{Aut}(\mathcal{S}_m(D_2))$. Hence $\Omega_m(D_1)$ is isomorphic to $\Omega_m(D_2)$. □

For each $0 \leq m < n$, let $\Omega^n_m := \Omega_m(V^n)$. To construct these groups, we construct a sequence $S_{m,1} \subset S_{m,2} \subset \ldots$ of objects of $\mathcal{S}_m(V^n)$ where $S_{m,k}$ is a disjoint union of k copies of V^m in V^n as described in the above proposition. Note that $\Omega^n_0 = \Sigma^n_0$. For simplicity of notation, for each $n \geq 1$, we also set $\Omega^n_n := \mathrm{Aut}_{\mathcal{S}}(V^n)$ to get a chain of normal subgroups of Ω^n_n:

$$\Omega^n_0 \lhd \Omega^n_1 \lhd \cdots \lhd \Omega^n_{n-1} \lhd \Omega^n_n. \tag{3.1}$$

For each $n \geq 1$, let Υ^n denote the subgroup of $\mathrm{Aut}_{\mathcal{S}}(V^n)$ consisting only of definable linear (i.e., *pp*-definable - a term that will be defined in Section 4.1) bijections. In other words, Υ^n is the group $\mathrm{GL}_n(F) \ltimes V^n$, where the action of $\mathrm{GL}_n(F)$ on V^n is given by matrix multiplication.

The group Υ^n acts on Ω^n_{n-1} by conjugation and, in fact, $\Omega^n_n = \Upsilon^n \ltimes \Omega^n_{n-1}$.

For $0 < m < n$, we want to find a subgroup Υ^n_m of Ω^n_m such that $\Omega^n_m = \Upsilon^n_m \ltimes \Omega^n_{m-1}$. To do this, we look at the construction of the colimit in (3.2.16).

Note that

$$\mathrm{Aut}_{\mathcal{S}}(S_{m,1}) \cong \Omega^m_m \cong \Upsilon^m \ltimes \Omega^m_{m-1} \cong \Upsilon^m \ltimes \Omega^n_{m-1},$$

where the action of Υ^m on Ω^n_{m-1} is induced by the isomorphism $\Omega^m_{m-1} \cong \Omega^n_{m-1}$ given by Proposition 3.4.4. For similar reasons, we also have

$$\mathrm{Aut}_{\mathcal{S}}(S_{m,k}) \cong (\Upsilon^m \wr \Sigma_k) \ltimes \Omega^m_{m-1} \cong (\Upsilon^m \wr \Sigma_k) \ltimes \Omega^n_{m-1},$$

where Σ_k is the permutation group on k elements, the group $(\Upsilon^m \wr \Sigma_k)$ acts on Ω^m_{m-1} by conjugation and permutes lower dimensional subsets of $S_{m,k} \subset V^n$. Thus

$$\begin{aligned}
\Omega^n_m &\cong \mathrm{colim}_{k\to\infty} \mathrm{Aut}_{\mathcal{S}}(S_{m,k}) \\
&\cong \mathrm{colim}_{k\to\infty} \big((\Upsilon^m \wr \Sigma_k) \ltimes \Omega^n_{m-1}\big) \\
&\cong (\mathrm{colim}_{k\to\infty}(\Upsilon^m \wr \Sigma_k)) \ltimes \Omega^n_{m-1} \\
&\cong (\Upsilon^m \wr \Sigma^n_m) \ltimes \Omega^n_{m-1}.
\end{aligned}$$

Define $\Upsilon^n_m := \Upsilon^m \wr \Sigma^n_m$ which acts on Ω^n_{m-1} by conjugation. Thus each Ω^n_n is an iterated semidirect product of certain wreath products.

$$\begin{aligned} \Omega^n_n &= \Upsilon^n \ltimes \Omega^n_{n-1} \\ &= \Upsilon^n \ltimes (\Upsilon^n_{n-1} \ltimes \Omega^n_{n-2}) \\ &= \Upsilon^n \ltimes (\Upsilon^n_{n-1} \ltimes (\Upsilon^n_{n-2} \ltimes \Omega^n_{n-3})) \\ &= \Upsilon^n \ltimes (\Upsilon^n_{n-1} \ltimes (\Upsilon^n_{n-2} \ltimes (\cdots (\Upsilon^n_1 \ltimes \Omega^n_0) \cdots))). \end{aligned} \tag{3.2}$$

3.5 K_1 of vector spaces

We continue to use V_F to denote an infinite vector space over an infinite field F. In this section, we write all groups multiplicatively unless otherwise stated.

The aim of this section is to prove the following theorem.

Theorem 3.5.1. *Suppose V_F is an infinite vector space over an infinite field F and $F^\times$ is the group of units in F. Then*

$$K_1(V_F) = \left(\bigoplus_{i=1}^{\infty} (F^\times \oplus \mathbb{Z}_2) \right) \oplus \mathbb{Z}_2. \tag{3.3}$$

Remark 3.5.2. Note that, since F is infinite, the (lower) K-theory of the vector space V_F is the same as that of the 1-dimensional vector space F_F. In other words, the (lower) K-theory "does not see" V.

The groups $K_0(V_F)$ and $K_1(V_F)$ are both graded by dimensions owing to the fact that the multiplicative monoid of (the isomorphism classes of) the fundamental definable sets in $\mathcal{S}(V_F)$ is isomorphic to the additive monoid $(\mathbb{N}, 0, +)$ of dimensions. In view of the Barratt-Priddy-Quillen-Segal Theorem (Theorem 3.2.10), the existence of the group $K_0(iso\,\mathrm{FinSets}) = \mathbb{Z}$ in $K_0(V_F)$ and the group $K_1(iso\,\mathrm{FinSets}) = \mathbb{Z}_2$ in $K_1(V_F)$ in each dimension can be attributed to the embedding of the groupoid $iso\,\mathrm{FinSets}$ in each dimension that was (indirectly) used to construct sequences in the proof of Proposition 3.4.4.

Theorem 3.2.15 is our main tool to prove the above theorem which can be stated as

$$K_1(V_F) = \mathrm{colim}_{n\to\infty} (\Omega^n_n)^{ab}, \tag{3.4}$$

since the first integral homology of a group G is its abelianization G^{ab}.

We compute $(\Omega^n_n)^{ab}$ in several steps. First we note a result on the abelianization of finitary permutation groups.

Proposition 3.5.3 ([4, §6.1]). *If T is an infinite set, then the finitary alternating group* $\mathrm{Alt}(T)$ *is the commutator subgroup of the finitary permutation group* $\mathrm{FSym}(T)$*. In particular,* $\mathrm{FSym}(T)^{ab} = \mathbb{Z}_2$.

The following proposition gives a way to compute the abelianization of a semidirect product.

Proposition 3.5.4 (see [13, Proposition 3.3]). *Let G be a group acting on H. Then $(G \ltimes H)^{ab} = G^{ab} \times (H^{ab})_G$; here $(H^{ab})_G$ is the quotient of H^{ab} by the subgroup generated by elements of the form $h^g h^{-1}$, where h^g denotes the action of $g \in G$ on $h \in H^{ab}$ induced by the action of G on H.*

Sketch Proof. The commutator subgroup $[G \ltimes H, G \ltimes H]$ is generated by $[H,H] \cup [G,H] \cup [G,G]$. Therefore $(G \ltimes H)^{ab} = (G \ltimes H)/\langle [H,H] \cup [G,H] \cup [G,G] \rangle$.

Applying the relators $[H,H]$ gives $G \ltimes H^{ab}$, then applying the relators $[G,H]$ gives $G \times (H^{ab})_G$. Finally applying $[G,G]$, we get the desired group $G^{ab} \times (H^{ab})_G$. □

The following lemma is crucial for computation.

Lemma 3.5.5. *Let G be a group and $\Sigma = \mathrm{FSym}(T)$ for an infinite set T. Then*

$$(G \wr \Sigma)^{ab} = G^{ab} \times \mathbb{Z}_2. \tag{3.5}$$

Proof. Let $\mathbf{G} := (\bigoplus_{x \in T} G_x)$, where G_x is a copy of G for each $x \in T$. Then $G \wr \Sigma = \mathbf{G} \rtimes \Sigma$, where Σ acts on the indices of elements of $\mathbf{G}$. Clearly $\mathbf{G}^{ab} = \bigoplus_{x \in T} G_x^{ab}$. Define a map $\varepsilon : \mathbf{G}^{ab} \to G^{ab}$ by $(g_x)_{x \in T} \mapsto \prod_{x \in T} g_x$. It can be easily seen that ε is a homomorphism.

The action of $\sigma \in \Sigma$ on $\mathbf{g} := (g_x)_{x \in T} \in \mathbf{G}^{ab}$, denoted $\mathbf{g}^\sigma$, is given by $(g_{\sigma^{-1}x})_{x \in T}$. Let H denote the subgroup of $\mathbf{G}^{ab}$ generated by $\{\mathbf{g}^\sigma \mathbf{g}^{-1} : \mathbf{g} \in \mathbf{G}^{ab}, \sigma \in \Sigma\}$.

We claim that $H = \ker \varepsilon$.

Note that $\varepsilon \mathbf{g} = \varepsilon \mathbf{g}^\sigma$ for each $\mathbf{g} := (g_x)_{x \in T} \in \mathbf{G}^{ab}, \sigma \in \Sigma$. Hence $H \subseteq \ker \varepsilon$.

On the other hand, consider $\mathbf{g} := (g_x)_{x \in T} \in \ker \varepsilon$. Since Σ consists of only finitary permutations on T, there are only finitely many $x \in T$ such that $g_x \neq 1$, say $x_1, x_2, \ldots, x_n$. We will use induction on n to show that $\mathbf{g} \in H$.

The case when $n = 0$ is trivial. If $n > 0$, the identity $\prod_{i=1}^n g_{x_i} = 1$ gives $n \geq 2$.

Assume for induction that the result holds for all values of n strictly less than $k > 0$.

Suppose $n = k$. Let σ be the transposition $(x_1, x_2) \in \Sigma$ and $\mathbf{g}' := (g'_x)_{x \in T}$ be the element of $\mathbf{G}^{ab}$ whose only nontrivial component is $g'_{x_1} = g_{x_1}^{-1}$. Let $\mathbf{g}'' := \mathbf{g}\mathbf{g}'((\mathbf{g}')^\sigma)^{-1}$. Then $g''_{x_1} = 1$ and $\varepsilon \mathbf{g}'' = 1$. The number of non-identity components of $\mathbf{g}''$ is strictly less than k and thus, using induction hypothesis, $\mathbf{g}'' \in H$. Therefore $\mathbf{g} = \mathbf{g}''(\mathbf{g}')^{\sigma}(\mathbf{g}')^{-1} \in H$ proving the claim.

Now $(\mathbf{G}^{ab})_{\Sigma} = \mathbf{G}^{ab}/H = \mathbf{G}^{ab}/\ker\varepsilon = G^{ab}$. We also have $\Sigma^{ab} = \mathbb{Z}_2$ from Proposition 3.5.3. Thus Proposition 3.5.4 gives $(G \wr \Sigma)^{ab} = (\mathbf{G}^{ab})_{\Sigma} \times \Sigma^{ab} = G^{ab} \times \mathbb{Z}_2$. □

Corollary 3.5.6. *For each $n \geq 1$, we have*

$$(\Omega_n^n)^{ab} = F^{\times} \oplus \bigoplus_{i=1}^{n-1}(F^{\times} \oplus \mathbb{Z}_2) \oplus \mathbb{Z}_2. \tag{3.6}$$

Proof. Fix $n \geq 1$ and $0 \leq m < n$. We use the presentation of Ω_n^n given by (3.2).

We know that $\Upsilon^n = \mathrm{GL}_n(F) \ltimes V^n$. Since the additive group V^n is abelian, Proposition 3.5.4 gives $(\Upsilon^n)^{ab} = (\mathrm{GL}_n(F))^{ab} \times (V^n)_{\mathrm{GL}_n(F)}$. For any $a(\neq 1) \in F^{\times}$ (which exists since the field F is infinite), we have $aI_n \in \mathrm{GL}_n(F)$ where I_n is the identity matrix. Now each $v \in V^n$ can be expressed as $(aI_n)v' - v'$ for $v' = (a-1)^{-1}v$. Thus the quotient $(V^n)_{\mathrm{GL}_n(F)}$ of V^n is trivial which, in turn, gives $(\Upsilon^n)^{ab} = (\mathrm{GL}_n(F))^{ab} = F^{\times}$.

Recall from Proposition 3.5.3 that $(\Sigma_m^n)^{ab} = \mathbb{Z}_2$. Lemma 3.5.5 applied to $\Upsilon_m^n = \Upsilon^m \wr \Sigma_m^n$ gives $(\Upsilon_m^n)^{ab} = (\Upsilon^m)^{ab} \times \mathbb{Z}_2 = F^{\times} \oplus \mathbb{Z}_2$.

The group Υ_m^n acts on Ω_{m-1}^n by conjugation. Recall that the action of Υ_m^n preserves the determinant of a matrix in $\mathrm{GL}_n(F)$ and the parity of a permutation. Therefore repeated use of Proposition 3.5.4 gives

$$\begin{aligned}
(\Omega_n^n)^{ab} &\cong (\Upsilon^n \ltimes (\Upsilon_{n-1}^n \ltimes (\cdots(\Upsilon_1^n \ltimes \Omega_0^n)\cdots)))^{ab} \\
&\cong (\Upsilon^n)^{ab} \oplus ((\Upsilon_{n-1}^n \ltimes (\cdots(\Upsilon_1^n \ltimes \Omega_0^n)\cdots))^{ab})_{\Upsilon^n} \\
&\cong (\Upsilon^n)^{ab} \oplus ((\Upsilon_{n-1}^n)^{ab} \oplus ((\cdots(\Upsilon_1^n \ltimes \Omega_0^n)\cdots)^{ab})_{\Upsilon_{n-1}^n})_{\Upsilon^n} \\
&\cong (\Upsilon^n)^{ab} \oplus ((\Upsilon_{n-1}^n)^{ab} \oplus (\cdots((\Upsilon_1^n)^{ab} \oplus ((\Omega_0^n)^{ab})_{\Upsilon_1^n})_{\Upsilon_2^n}\cdots)_{\Upsilon_{n-1}^n})_{\Upsilon^n} \\
&\cong F^{\times} \oplus ((F^{\times} \oplus \mathbb{Z}_2) \oplus (\cdots((F^{\times} \oplus \mathbb{Z}_2) \oplus (\mathbb{Z}_2)_{\Upsilon_1^n})_{\Upsilon_2^n}\cdots)_{\Upsilon_{n-1}^n})_{\Upsilon^n} \\
&\cong F^{\times} \oplus ((F^{\times} \oplus \mathbb{Z}_2) \oplus (\cdots((F^{\times} \oplus \mathbb{Z}_2) \oplus \mathbb{Z}_2)\cdots)) \\
&\cong F^{\times} \oplus \bigoplus_{i=1}^{n-1}(F^{\times} \oplus \mathbb{Z}_2) \oplus \mathbb{Z}_2.
\end{aligned}$$

□

The presentation of the group $(\Omega_n^n)^{ab}$ in the above corollary clearly shows its decomposition in different dimensions - a copy of $\mathbb{Z}_2$ in dimension 0, a copy of $F^{\times}$ in the highest dimension and a copy of $(F^{\times} \oplus \mathbb{Z}_2)$ in each other dimension.

In the construction of the sequence $S_{n,1} \subset S_{n,2} \subset \cdots$ to compute $\mathrm{Aut}(\mathcal{S}_n(V^{n+1}))$, we can choose the copy $V^n \times \{0\}$ as $S_{n,1}$. This induces an embedding of Ω_n^n into $\Omega_n^{n+1} \lhd \Omega_{n+1}^{n+1}$. This further induces the dimension preserving inclusion of $(\Omega_n^n)^{ab}$ into

$(\Omega_{n+1}^{n+1})^{ab}$. Hence

$$\begin{aligned} K_1(V_F) &= \operatorname{colim}_{n\to\infty}(\Omega_n^n)^{ab} \\ &= \operatorname{colim}_{n\to\infty}\left(F^\times \oplus \bigoplus_{i=1}^{n-1}(F^\times \oplus \mathbb{Z}_2) \oplus \mathbb{Z}_2\right) \\ &= \bigoplus_{i=1}^{\infty}(F^\times \oplus \mathbb{Z}_2) \oplus \mathbb{Z}_2. \end{aligned}$$

This completes the proof of Theorem 3.5.1.

More discussion on the significance of different components of the groups $K_0(V_F)$ and $K_1(V_F)$, and a conjecture on the structure of $K_1(M_{\mathcal{R}})$ for a right $\mathcal{R}$-module $M_{\mathcal{R}}$ can be found in Chapter 8.

Chapter 4

Definable Subsets of Modules

After setting some notations and terminology in the model theory of modules in Section 4.1, we describe four special classes of definable subsets of (finite powers of) a module M namely *pp*-sets, *pp*-convex sets, blocks and cells. Under certain hypotheses on the theory of the module M (stated in Section 4.2), definable sets admit unique representations in terms of fundamental definable sets i.e., *pp*-sets. This chapter forms the lattice-theoretic basis for the analysis in the following two chapters; Corollary 4.2.12 and Corollary 4.3.5 are the highlights. We also define the notion of a connected definable set in Section 4.4 as a byproduct of the theory we develop.

The material in this chapter is contained in [26, §2.5,3.1,6.3,6.4].

4.1 Model theory of modules

We introduce the terminology and some basic results in the model theory of modules in this section. A detailed exposition can be found in [33]. Instead of working with formulas all the time, we fix a structure and work with the definable subsets of its finite cartesian powers.

Let $\mathcal{R}$ be a fixed ring with unity. Then every right $\mathcal{R}$-module M is a structure for the first order language $L_{\mathcal{R}} = \langle 0, +, -, m_r : r \in \mathcal{R} \rangle$, where each m_r is a unary function symbol representing the action of right multiplication by the element r. When we are working in a fixed module M, we usually write the element $m_r(a)$ in formulas as ar for each $a \in M$.

First we note a result of Perera which states that the Grothendieck ring of a module is an invariant of its theory. A proof of this proposition can be found at the end of Section 6.2 as a corollary of Theorem 6.2.3.

Proposition 4.1.1 (see [31, Cor. 5.3.2])**.** *Let M and N be two right $\mathcal{R}$-modules such that $M \equiv N$, then $K_0(M) \cong K_0(N)$.*

Let us fix a right $\mathcal{R}$-module M. Then, for each $n \geq 1$, every definable subset of M^n can be expressed as a boolean combination of certain fundamental definable subsets of M^n. In order to state this partial quantifier elimination result, we first define the formulas which define these fundamental subsets.

Definition 4.1.2. *A* ***positive primitive formula (pp-formula*** *for short) is a formula in the language $L_{\mathcal{R}}$ which is equivalent to one of the form*

$$\phi(x_1, x_2, \ldots, x_n) = \exists y_1 \exists y_2 \ldots \exists y_m \bigwedge_{i=1}^{t} \left(\sum_{j=1}^{n} x_j r_{ij} + \sum_{k=1}^{m} y_k s_{ik} + c_i = 0 \right),$$

where $r_{ij}, s_{ik} \in \mathcal{R}$ and the c_i are parameters from M.

A subset of M^n which is defined by a *pp*-formula (with parameters) will be called a *pp*-set. If a subgroup of M^n is *pp*-definable, then its cosets are also *pp*-definable. The following lemma is well known and a proof can be found in [33, Cor. 2.2].

Lemma 4.1.3. *Every parameter-free pp-formula $\phi(\overline{x})$ defines a subgroup of M^n, where n is the length of $\overline{x}$. If $\phi(\overline{x})$ contains parameters from M, then it defines either the empty set or a coset of a pp-definable subgroup of M^n. Furthermore, the conjunction of two pp-formulas is (equivalent to) a pp-formula.*

Let $\mathcal{L}_n$ denote the meet-semilattice of all *pp*-subsets of M^n ordered by the inclusion relation $\subseteq$. We will use the notation $\mathcal{L}_n(M_{\mathcal{R}})$, specifying the module, when we work with more than one module at a time.

Definition 4.1.4. *Let M be a right $\mathcal{R}$-module and let $A, B \in \mathcal{L}_n$ be subgroups. We define the invariant* $\mathrm{Inv}(M; A, B)$ *to be the index $[A : A \cap B]$ if this if finite or ∞ otherwise.*

An **invariants condition** is a statement that a given invariant is greater than or equal to or less than a certain number. These invariant conditions can be expressed as sentences in $L_{\mathcal{R}}$. An **invariants statement** is a finite boolean combination of invariants conditions.

We are now ready to state the promised fundamental theorem of the model theory of modules.

Theorem 4.1.5 (see [3])**.** *Let T be the theory of right $\mathcal{R}$-modules and $\phi(\overline{x})$ be an $L_{\mathcal{R}}$ formula (possibly with parameters). Then we have*

$$T \vDash \forall \overline{x} (\phi(\overline{x}) \leftrightarrow \left(\bigvee_{i=1}^{m} \left(\psi_i(\overline{x}) \wedge \bigwedge_{j=1}^{l_i} \neg \chi_{ij}(\overline{x}) \right) \wedge I \right)),$$

where I is an invariants statement and $\psi_i(\overline{x}), \chi_{ij}(\overline{x})$ are pp-formulas.

We may assume that $\chi_{ij}(M) \subseteq \psi_i(M)$ for each value of i and j, otherwise we redefine χ_{ij} as $\chi_{ij} \wedge \psi_i$. When we work in a complete theory, the invariants statements will vanish and hence we get the following form.

Theorem 4.1.6. *For each $n \geq 1$, every definable subset of M^n can be expressed as a finite boolean combination of pp-subsets of M^n.*

Using this result together with the meet-semilattice structure of $\mathcal{L}_n$, we can express each definable subset of M^n in a "disjunctive normal form" of *pp*-sets. Expressing a definable set as a disjoint union helps to break it down to certain low complexity fragments, each of which has a specific shape given by the normal form. A proof of this result can be found in [31, Lemma 3.2.1].

Lemma 4.1.7. *Every definable subset of M^n can be written as $\bigsqcup_{i=1}^{t}(A_i \setminus (\bigcup_{j=1}^{s_i} B_{ij}))$ for some $A_i, B_{ij} \in \mathcal{L}_n$.*

The following lemma is one of the important tools in our analysis.

Lemma 4.1.8 (**Neumann's Lemma**, see [33, Theorem 2.12])**.** *If H and $\{G_i\}_{i\in I}$ are subgroups of an abelian group K and a coset of H is covered by a finite union of cosets of the G_i, then this coset of H is in fact covered by the union of just those cosets of G_i where G_i is of finite index in H, i.e., where $[H : G_i] := [H : H \cap G_i]$ is finite.*

$$c + H \subseteq \bigcup_{i\in I} c_i + G_i \quad \Rightarrow \quad c + H \subseteq \bigcup_{i\in I_0} c_i + G_i,$$

where $I_0 = \{i \in I : [H : G_i] < \infty\}$.

4.2 The condition $\mathrm{T} = \mathrm{T}^{\aleph_0}$

Let M be a fixed right $\mathcal{R}$-module. For brevity we denote $Th(M)$ by T. We work with this fixed module throughout this section.

Proposition 4.2.1 (see [33, p.34, Exercise 2(i)])**.** *The following conditions are equivalent for a module M:*

1. *for each $n \geq 1$ and for each $A, B \in \mathcal{L}_n$ such that $0 \in A \cap B$, $\mathrm{Inv}(M; A, B)$ is either equal to 1 or ∞;*

2. *$M \equiv M \oplus M$;*

3. *$M \equiv M^{(\aleph_0)}$.*

Definition 4.2.2. *The theory $T = Th(M)$ is said to satisfy the condition $T = T^{\aleph_0}$ if either (and hence all) of the conditions of Proposition 4.2.1 hold.*

We wish to add yet one more condition to the list. The rest of this section is devoted to formulating the condition and deriving its consequences.

We need to introduce some new notation to do this. Let us denote the set of all finite subsets of $\mathcal{L}_n \setminus \{\emptyset\}$ by $\mathcal{P}_n$ and the set of all finite antichains in $\mathcal{L}_n \setminus \{\emptyset\}$ by $\mathcal{A}_n$. Clearly $\mathcal{A}_n \subseteq \mathcal{P}_n$ for each $n \geq 1$. We use the lowercase Greek letters $\alpha, \beta, \ldots$ to denote the elements of $\mathcal{A}_n$ and $\mathcal{P}_n$.

Definition 4.2.3. *A definable subset A of M^n will be called pp-**convex** if there is some $\alpha \in \mathcal{P}_n$ such that $A = \bigcup \alpha$.*

Neumann's lemma (Lemma 4.1.8) takes the following simple form if we add the equivalent conditions of 4.2.1 to our hypotheses.

Corollary 4.2.4. *Suppose that $T = T^{\aleph_0}$ holds. If $A \in \mathcal{L}_n$ and $\mathcal{F} \in \mathcal{P}_n$ such that $A \subseteq \bigcup \mathcal{F}$, then $A \subseteq F$ for at least one $F \in \mathcal{F}$.*

Under the same hypotheses, we show that for every $\alpha \in \mathcal{P}_n$ the *pp*-convex set $\bigcup \alpha$ uniquely determines the antichain $\beta \subseteq \alpha$ of all maximal elements in α.

Proposition 4.2.5. *Suppose that $T = T^{\aleph_0}$ holds. Let $A \subseteq M^n$ be a pp-convex set for some $n \geq 1$. Then there is a unique $\beta \in \mathcal{A}_n$ such that $A = \bigcup \beta$.*

Proof. Let $\alpha_1, \alpha_2 \in \mathcal{P}_n$ be such that $A = \bigcup \alpha_1 = \bigcup \alpha_2$. Without loss of generality we may assume $\alpha_1, \alpha_2 \in \mathcal{A}_n$. Let $\alpha_1 = \{C_1, C_2, \ldots, C_l\}$ and $\alpha_2 = \{D_1, D_2, \ldots, D_m\}$.

We have $D_j \subseteq \bigcup_{i=1}^{l} C_i$ for each $1 \leq j \leq m$. Then by 4.2.4, we have $D_j \subseteq C_i$ for at least one i. By symmetry we also get that each C_i is contained in a D_j. Using that both α_1 and α_2 are antichains with the same union, the proof is complete. □

This proposition shows that under the hypothesis $T = T^{\aleph_0}$ the set of *pp*-convex subsets of M^n is in bijection with $\mathcal{A}_n$ for each $n \geq 1$. We shall often use this correspondence without mention. For $\alpha \in \mathcal{A}_n$, we define the **rank** of the *pp*-convex set $\bigcup \alpha$ to be the integer $|\alpha|$.

The set $\mathcal{A}_n$ can be given the structure of a poset by introducing the relation $\prec_n$ defined by $\beta \prec_n \alpha$ if and only if for each $B \in \beta$, there is some $A \in \alpha$ such that $B \subsetneq A$.

Definition and Lemma 4.2.6. *Assume that $T = T^{\aleph_0}$. We say that a definable subset C of M^n is a **cell** if there are $\alpha, \beta \in \mathcal{A}_n$ with $\beta \prec_n \alpha$ such that $C = \bigcup \alpha \setminus \bigcup \beta$. We denote the set of all cells contained in M^n by $\mathcal{C}_n$. The antichains α and β, denoted by $P(C)$ and $N(C)$ respectively, are uniquely determined by the cell C. In other words,* *there is a bijection between the set $\mathcal{C}_n$ and the set of pairs of antichains strictly related by $\prec_n$. In case $|P(C)| = 1$, we say that C is a* ***block****. We denote the set of all blocks in $\mathcal{C}_n$ by $\mathcal{B}_n$.*

Proof. Given any $\alpha, \beta \in \mathcal{A}_n$ such that $\beta \prec_n \alpha$ and $C = \bigcup \alpha \setminus \bigcup \beta$, the *pp*-convex set $\bigcup(\alpha \cup \beta)$ is determined by C. But this set is uniquely determined by the set of maximal elements in $\alpha \cup \beta$ by 4.2.5. Since $\beta \prec_n \alpha$, the required set of maximal elements is precisely α. Furthermore, the set $\bigcup \alpha \setminus C = \bigcup \beta$ is *pp*-convex and thus is uniquely determined by β by 4.2.5 and this finishes the proof. □

See Figure 4.1 for a sketch of a block and a cell.

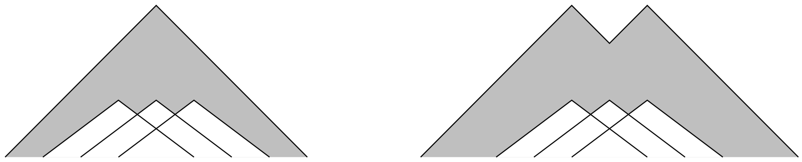

Figure 4.1: A block and a cell (shaded regions)

We know from Lemma 4.1.7 that any non-empty definable subset of M^n can be represented as a disjoint union of blocks. So it will be important for us to understand the structure of blocks in detail. For each $B \in \mathcal{B}_n$, we use the notation $\overline{B}$ to denote the unique element of $P(B)$.

Remark 4.2.7. A block is always nonempty since any finite union of proper *pp*-subsets cannot cover the given *pp*-set by 4.2.5.

Definition 4.2.8. *Let $\mathcal{D}$ be a finite subset of $\mathcal{L}_n \setminus \{\emptyset\}$. The smallest sub-meet-semilattice of $\mathcal{L}_n$ containing $\mathcal{D}$ will be called the pp-**nest** (or simply nest) corresponding to $\mathcal{D}$ and will be denoted by $\hat{\mathcal{D}}$. Note that $\hat{\mathcal{D}}$ is finite. In general, any finite sub-meet-semilattice of $\mathcal{L}_n$ will also be referred to as a pp-nest.*

Definition 4.2.9. *For each finite subset $\mathcal{F}$ of $\mathcal{L}_n \setminus \{\emptyset\}$ and $F \in \mathcal{F}$, we define the* ***$\mathcal{F}$-core*** *of F to be the block $\mathrm{Core}_{\mathcal{F}}(F) := F \setminus \bigcup\{G \in \mathcal{F} : G \cap F \subsetneq F\}$.*

Let $D \subseteq M^n$ be definable. Then $D = \bigsqcup_{i=1}^m B_i$ for some $B_i \in \mathcal{B}_n$ by 4.1.7. We say that $\mathcal{D}$ is the nest corresponding to this partition of D if it is the nest corresponding to the finite family $\bigcup_{i=1}^m (P(B_i) \cup N(B_i))$. Every definable set can be partitioned canonically given a suitable nest, which is the content of the following lemma whose straightforward proof is omitted.

Definition and Lemma 4.2.10. *Suppose $D \subseteq M^n$ is definable and $\mathcal{D}$ is the nest corresponding to a given partition $D = \bigsqcup_{i=1}^m B_i$. For every nonempty $F \in \mathcal{D}$, the* *set* $\mathrm{Core}_{\mathcal{D}}(F) \cap D$ *is nonempty if and only if* $\mathrm{Core}_{\mathcal{D}}(F) \subseteq D$. *We define the* ***characteristic function*** *of the nest* $\mathcal{D}$, $\delta_{\mathcal{D}} : \mathcal{D} \to \{0,1\}$, *by* $\delta_{\mathcal{D}}(F) = 1$ *if and only if* $F \neq \emptyset$ *and* $\mathrm{Core}_{\mathcal{D}}(F) \subseteq D$. *We define* $\mathcal{D}^+ := \delta_{\mathcal{D}}^{-1}(1)$ *and* $\mathcal{D}^- := \delta_{\mathcal{D}}^{-1}(0)$. *Then* $D = \bigcup_{F \in \mathcal{D}^+} \mathrm{Core}_{\mathcal{D}}(F)$.

See Figure 4.2 for an illustration of the terms.

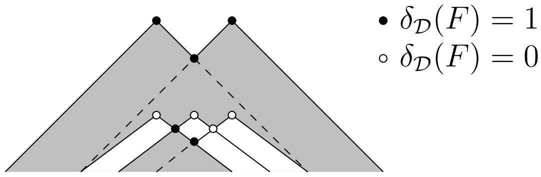

Figure 4.2: Nest, cores and the characteristic function

Now we explore the relation between a block B and the *pp*-set $\overline{B}$.

Theorem 4.2.11. *Let* M *be an* $\mathcal{R}$*-module. Then* $Th(M) = Th(M)^{\aleph_0}$ *if and only if for each* $B \in \mathcal{B}_n$, $n \geq 1$, *we have* $B + B - B = \overline{B}$. *Under these conditions, we also get* $B - B = \overline{B}$ *whenever* $\overline{B}$ *is a subgroup.*

Proof. Assume that $Th(M) = Th(M)^{\aleph_0}$ holds. Let $B \in \mathcal{B}_n$ be defined by $N(B) = \{D_1, D_2, \ldots, D_l\} \prec P(B) = \{A\}$. Let $D = \bigcup N(B)$. We want to show that $B + B - B = A$. But clearly $B \subseteq B + B - B$. So it suffices to show that $D \subseteq B + B - B$.

First assume that A is a subgroup of M^n. Let $d \in D$. Since $A \setminus (D - d)$ is a block, we can choose some $x \in A \setminus (D - d)$ by 4.2.7. Then $x + d \in (A + d) \setminus D = A \setminus D = B$, since A is a subgroup. Again choose some $y \in A \setminus ((D - d) \cup (D - d - x))$. Then $y + d \in A \setminus (D \cup (D - x))$ for similar reasons. Thus $y + d, y + x + d \in A \setminus D = B$. Now $d = (d + x) + (d + y) - (d + x + y) \in B + B - B$ and hence the conclusion follows.

In the case when A is a coset of a *pp*-definable subgroup G, say $A = a + G$, let $C = D - a$. Then, by the first case, $G = C + C - C$. Now if $d \in A$, then $d - a \in G$. Hence there are $x, y, z \in C$ such that $d - a = x + y - z$. Thus $d = (x + a) + (y + a) - (z + a) \in B + B - B$ and this completes the proof in one direction.

For the converse, suppose that $Th(M) \neq Th(M)^{\aleph_0}$. Then there are two *pp*-definable subgroups $H \leq G$ of M^n for some $n \geq 1$ such that$1 < [G : H] < \infty$. Let $[G : H] = k$ and let $H_1, H_2, \cdots, H_k$ be the distinct cosets of H in G. Since H is a *pp*-set, all the cosets H_i are *pp*-sets as well. Now let $B = H_k = G \setminus \bigcup_{i=1}^{k-1} H_i$. Then B is a nonempty block since $k > 1$. But, since B is a coset, $B + B - B = B \neq G$ which proves the result in the other direction.

Now we prove the last statement under the hypothesis $Th(M) = Th(M)^{\aleph_0}$. Let B, A, D be as defined in the first paragraph of the proof and assume that A is a subgroup of M^n. Given any $a \in A$, we can choose $x \in A \setminus (D \cup (D - a))$ by 4.2.7. Then $x, x + a \in B$ and hence $a = (x + a) - x \in (B - B)$. This shows the inclusion $A \subseteq B - B$. We clearly have $(B - B) \subseteq (A - A)$ and $A - A = A$ since A is a subgroup. This completes the proof. □

A map $f : B \to M^n$ is said to be **linear** if $f(x + y - z) = f(x) + f(y) - f(z)$ for all $x, y, z \in B$ such that $x + y - z \in B$. It is said to be a *pp*-**definable** map if there is a *pp*-formula (with parameters from M) $\rho(\overline{x}, \overline{y})$ which defines a function on $\overline{B}$ that extends f. A *pp*-definable map is clearly linear.

We can use the above theorem to show that any injective linear map on B can be extended uniquely to an injective linear map on $\overline{B}$. The following corollary proves this assertion only for injective *pp*-definable maps.

Corollary 4.2.12. *Suppose that $T = T^{\aleph_0}$ holds. Then for each $n \geq 1$, each $B \in \mathcal{B}_n$ and each injective pp-definable map $f : B \rightarrowtail M^n$, there exists a unique injective pp-definable extension $\overline{f} : \overline{B} \rightarrowtail M^n$.*

Proof. Let $\phi(\overline{x}) \wedge \neg(\vee_{i=1}^{k} \psi_i(\overline{x}))$ be a formula defining the block B where $\phi(\overline{x}), \psi_i(\overline{x})$ are *pp*-formulas with parameters. Suppose $\rho(\overline{x}, \overline{y})$ is a *pp*-formula (with parameters) such that $\exists \overline{y} \rho(\overline{x}, \overline{y}) \equiv \phi(\overline{x})$. Further suppose that $\rho(\overline{x}, \overline{y})$ defines a function on $\overline{B}$ which extends the injective function $f : B \rightarrowtail M^n$.

Now the graph of f is the block defined by $\rho(\overline{x}, \overline{y}) \wedge \neg(\vee_{i=1}^{k} \psi_i(\overline{x}))$. Theorem 4.2.11 says that the *pp*-formula $\rho(\overline{x}, \overline{y})$ is uniquely determined by the graph of f.

Now we need to show that the function $\overline{f} : \overline{B} \to M^n$ defined by the formula $\rho(\overline{x}, \overline{y})$ is injective. Let $D := \overline{B} \setminus B$. Without loss we can assume that $\overline{B}$ is a subgroup of M^n, $0 \in B$ and, by translating the map $\overline{f}$ if necessary, that $\overline{f}(0) = 0$.

Now suppose there is $a \in \overline{B}$ such that $\overline{f}(a) = 0$. Then certainly $a \in D$ otherwise it contradicts the hypothesis that f is injective. Now the set $\overline{B} \setminus (D \cup (D - a))$ is nonempty. In other words, there is some $c \in B$ such that $c + a \in B$. Then $f(c + a) = \overline{f}(c + a) = \overline{f}(c) + \overline{f}(a) = f(c)$. This contradicts the hypothesis that f is injective on B and completes the proof. □

4.3 Representing definable sets uniquely

We fix some $\mathcal{R}$-module M whose theory T satisfies the condition $T = T^{\aleph_0}$ and some $n \geq 1$. We drop all the subscripts n and write $\mathcal{L} \setminus \{\emptyset\}, \mathcal{A} \setminus \{\emptyset\}, \ldots$ as $\mathcal{L}^*, \mathcal{A}^*, \ldots$ respectively.

The aim of this section is to generalize the uniqueness of representation result of [12, Cor. 2.3] to modules which satisfy unpackability condition ($T = T^{\aleph_0}$).

The *pp*-elimination theorem for the model theory of modules (Theorem 4.1.5) states that every definable set can be written as a finite disjoint union of blocks. But this representation is far from being unique in any sense. On the other hand we have unique representations for *pp*-convex sets (Proposition 4.2.5) and cells (Lemma 4.2.6). We exploit these ideas to achieve a unique representation for every definable set - an expression as a disjoint union of cells. This result will be called the 'cell decomposition theorem'.

Proposition 4.3.1. *Let* $\{A_i\}_{i=1}^m \in \mathcal{P}$ *and* $B \in \mathcal{B}$ *be such that* $B \subseteq \bigcup_{i=1}^m A_i$*. Then* $\overline{B} \subseteq \bigcup_{i=1}^m A_i$*.*

Proof. We have $\overline{B} = B \cup \bigcup N(B)$. Hence $\overline{B} \subseteq \bigcup_{i=1}^m A_i \cup \bigcup N(B)$. By 4.2.4, $\overline{B} \subseteq A_i$ for some i, or $\overline{B} \subseteq D$ for some $D \in N(B)$. The latter case is not possible since $N(B) \prec P(B) = \{\overline{B}\}$. Hence the result. □

Lemma 4.3.2. *Let* $D \in \mathrm{Def}(M^n)$*. Then there is a unique pp-convex set* $\overline{D}$ *which satisfies* $D \subseteq \bigcup \alpha \;\Rightarrow\; \overline{D} \subseteq \bigcup \alpha$ *for every* $\alpha \in \mathcal{A}$*.*

Proof. Let $D = \bigsqcup_{i=1}^m B_i = \bigsqcup_{j=1}^l B'_j$ be any two representations of D as disjoint unions of blocks.

Claim: $\bigcup_{i=1}^m \overline{B_i} = \bigcup_{j=1}^l \overline{B'_j}$

Proof of the claim: We have $B_i \subseteq \bigsqcup_{i=1}^m B_i = \bigsqcup_{j=1}^l B'_j \subseteq \bigcup_{j=1}^l \overline{B'_j}$ for each i. Hence $\overline{B_i} \subseteq \bigcup_{j=1}^l \overline{B'_j}$ by the above proposition. Therefore $\bigcup_{i=1}^m \overline{B_i} \subseteq \bigcup_{j=1}^l \overline{B'_j}$. The reverse containment is by symmetry and hence the claim.

Now we define $\overline{D} = \bigcup_{i=1}^m \overline{B_i}$. By the claim, this *pp*-convex set is uniquely defined.

Let $\alpha \in \mathcal{A}$ be such that $D \subseteq \bigcup \alpha$. But $D = \bigsqcup_{i=1}^m B_i$. Hence $B_i \subseteq \bigcup \alpha$ for each i. By arguments similar to the proof of the claim, we get $\bigcup_{i=1}^m \overline{B_i} \subseteq \bigcup \alpha$ i.e., $\overline{D} \subseteq \bigcup \alpha$. □

The assignment $D \mapsto \overline{D}$, where $\overline{D}$ is the *pp*-convex set obtained from the lemma, defines a closure operator $\mathrm{Def}(M^n) \to \mathcal{A}_n$. This closure operation is the key in proving the cell decomposition theorem.

The relation $\prec$ on $\mathcal{A}$ induces a partial order on the class $\mathcal{C}$ of cells.

Definition 4.3.3. *Given* $C_1, C_2 \in \mathcal{C}$*, we say that* $C_1 \prec C_2$ *if* $P(C_1) \prec N(C_2)$ *in* $\mathcal{A}$*. A* ***tower of cells*** *is a finite subset* $\mathcal{F}$ *of* $\mathcal{C}$ *that is linearly ordered by* $\prec$*. We denote the set of all finite towers of cells by* $\mathcal{T}$*.*

Theorem 4.3.4 (Cell Decomposition Theorem). *For each* $n \geq 1$*, there is a bijection between the set* $\mathrm{Def}(M^n)$ *of all definable subsets of* M^n *and the set* $\mathcal{T}_n$ *of towers of cells.*

Proof. Let $D \in \mathrm{Def}(M^n)$. We construct a tower $\mathcal{F}$ of cells by defining a nested sequence $\{D_j\}_{j\geq 0}$ of definable subsets of D as follows.

We set $D_0 := D$ and, for each $j > 0$, we set $D_j := D_{j-1} \setminus C_j$, where $C_j := \overline{D_{j-1}} \setminus (\overline{\overline{D_{j-1}} \setminus D_{j-1}})$ is a cell. We stop this process when we obtain $D_j = \emptyset$ for the first time. This process must terminate because the elements of the antichains involved in this process belong to some finite nest containing a fixed decomposition of D into blocks.

In the converse direction, we assign $\bigcup \mathcal{F} \in \mathrm{Def}(M^n)$ to $\mathcal{F} \in \mathcal{T}$.

It is easy to verify that the two assignments defined above are actually inverses of each other. □

A chain $\alpha_{2k} \prec \alpha_{2k-1} \prec \ldots \prec \alpha_1$ in $\mathcal{A}$ naturally corresponds to a tower $C_k \prec \ldots \prec C_1$ where $C_i := \bigcup \alpha_{2i-1} \setminus \bigcup \alpha_{2i}$ for $1 \leq i \leq k$. Furthermore, every $\prec$-chain in $\mathcal{A}^*$ can be uniquely thought of as a $\prec$-chain of even length (i.e., even number of elements) in $\mathcal{A}$ by adding the empty antichain at the bottom of the chain if necessary. This sets up a bijection between the set of non-empty $\prec$-chains in $\mathcal{A}^*$ and non-empty towers of cells.

Suppose $\alpha \in \mathcal{P}$ and α_1 is the set of maximal elements of α. For each $j \geq 1$, we set α_{j+1} to be the set of maximal elements of $\alpha \setminus \bigcup_{i=1}^{j} \alpha_i$ (if non-empty). This produces a finite $\prec$-chain in $\mathcal{A}^*$. In the other direction, the union of antichains in a finite $\prec$-chain gives an element of $\mathcal{P}$.

We summarize this discussion in the following corollary to Theorem 4.3.4 which gives a combinatorial representation theorem for $\mathrm{Def}(M^n)$.

Corollary 4.3.5. *For each $n \geq 1$, the set $\mathrm{Def}(M^n)$ is in bijection with the set $\mathcal{P}_n$ of finite subsets of $\mathcal{L}_n$.*

This bijection imparts a boolean algebra structure to $\mathcal{P}_n$ owing to the presence of such structure on $\mathrm{Def}(M^n)$. The boolean algebra $\mathcal{P}_n$ can be rightly termed as the 'free boolean algebra' on the meet-semilattice $\mathcal{L}_n$ (equivalently, on the free distributive lattice $\mathcal{A}_n$).

4.4 Connected definable sets

We fix a right $\mathcal{R}$-module M satisfying $Th(M) = Th(M)^{\aleph_0}$ and some $n \geq 1$. We drop all the subscripts n as usual.

In this section we describe what we mean by the statement that a definable subset of a (finite power of a) module is connected. The property of being connected is not preserved under definable isomorphisms. We prove a (topological) property of connected sets which states that a definable connected set A contained in another definable set B is in fact contained in a connected component of B.

Let $\mathcal{F}, \mathcal{F}' \subseteq \mathcal{B}$ be two finite families of disjoint blocks such that $\bigcup \mathcal{F} = \bigcup \mathcal{F}'$. Then we say that $\mathcal{F}'$ is a **refinement** of $\mathcal{F}$ if for each $F' \in \mathcal{F}'$, there is a unique $F \in \mathcal{F}$ such that $F' \subseteq F$. Recall from 4.2.10 that if $\bigcup \mathcal{F} \in \mathcal{B}$ and if $\mathcal{D}$ is the corresponding nest, then $\{\mathrm{Core}_{\mathcal{D}}(D)\}_{D \in \mathcal{D}^+}$ is a refinement of $\mathcal{F}$, where $\mathcal{D}^+$ is the set $\delta_{\mathcal{D}}^{-1}\{1\}$. We use this property of nests to attach a digraph with each of them.

Definition 4.4.1. *Let $\mathcal{D}$ be a nest corresponding to a fixed finite family of pairwise disjoint blocks. We define a* ***digraph structure*** *$\mathcal{H}(\mathcal{D}^+)$ on the set $\mathcal{D}^+$. The pair (F_1, F_2) of elements of $\mathcal{D}^+$ will be said to constitute an arrow in the digraph if $F_1 \subsetneq F_2$ and $F_1 \subseteq F \subseteq F_2$ for some $F \in \mathcal{D}^+$ if and only if $F = F_1$ or $F = F_2$.*

If $\bigcup_{F \in \mathcal{D}^+} \mathrm{Core}_{\mathcal{D}}(F) \in \mathcal{B}$, then $\mathcal{D}^+$ is an upper set and in particular $\mathcal{H}(\mathcal{D}^+)$ is **weakly connected** i.e., its underlying undirected graph is connected. It seems natural to use this property to define the connectedness of a definable set.

Definition 4.4.2. *Let $D \in \mathrm{Def}(M^n)$ be represented as $D = \bigcup \mathcal{F}$, where $\mathcal{F} \subseteq \mathcal{B}$ is a finite family of pairwise disjoint blocks and let $\mathcal{D}$ denote the nest corresponding to $\mathcal{F}$. We say that D is* ***connected*** *if and only if the digraph $\mathcal{H}(\mathcal{D}'^+)$ is weakly connected for some nest $\mathcal{D}'$ containing $\mathcal{D}$.*

Note the existential clause in this definition. Let $\mathcal{F}, \mathcal{F}'$ be two finite families of pairwise disjoint blocks with $\bigcup \mathcal{F} = \bigcup \mathcal{F}'$ and let $\mathcal{D}, \mathcal{D}'$ denote the nests corresponding to them. If $\mathcal{F}'$ refines $\mathcal{F}$, then the number of weakly connected components of $\mathcal{H}(\mathcal{D}'^+)$ is bounded between 0 and the number of weakly connected components of $\mathcal{H}(\mathcal{D}^+)$. This observation allows us to define the following invariant.

Definition 4.4.3. *We define the* ***number of connected components*** *of D, denoted $\lambda(D)$, for every nonempty definable set D to be the least number of weakly connected components of $\mathcal{H}(\mathcal{D}^+)$, where $\mathcal{D}$ varies over nests refining a fixed partition of D into disjoint blocks. We set $\lambda(\emptyset) = 0$.*

Note that a definable set D is connected if and only if $\lambda(D) = 1$. We denote the set of all connected definable subsets of M^n by $\mathbf{Con}_n$. In the discussion on connectedness, we have treated blocks as if they are the basic connected sets. We have $\mathcal{B}_n \subseteq \mathbf{Con}_n$ as expected.

Illustration 4.4.4. Consider the vector space $\mathbb{R}_{\mathbb{R}}$. The *pp*-definable subsets of the plane, $\mathbb{R}^2$, are points, lines and the plane.

Note that if a definable subset of $\mathbb{R}^2$ is topologically connected, then it is connected according to Definition 4.4.2. The converse is not true in general. The set $B = \{(x, 0) : x \neq 0\}$ is not topologically connected, but $B \in \mathbf{Con}_2$ since B is a block.

If D denotes the union of two coordinate axes with the origin removed, then the number of topologically connected components of D is 4, whereas $\lambda(D) = 2$.

Remark 4.4.5. If X is a 'nice' topological space (e.g., a simplicial complex), then the rank b_0 of the homology group $H_0(X)$ is the number of (path-)connected components of X. In section 5.3 we will associate a simplicial complex $\mathcal{K}^P(\alpha)$ to each pair (P, α), where $P \in \mathcal{L}_n$ and $\alpha \in \mathcal{A}_{(P)}$. Now if $\alpha \neq \emptyset$, then $b_0(\mathcal{K}^P(\alpha)) = \lambda(\bigcup \alpha \setminus P)$. Note that the 'deleted neighbourhood' of P in α, i.e., the set $\bigcup \alpha \setminus P$, occurs in this correspondence since the clause $\alpha \in \mathcal{A}_{(P)}$ makes sure that the 'non-deleted neighbourhood' $\bigcup \alpha$ is connected.

Topologically connected sets satisfy the following property. If a connected set A is contained in another set B, then A is actually contained in a connected component of B. We have a similar result here.

Theorem 4.4.6. *Let $A, B_i \in \mathbf{Con}_n$ for $1 \leq i \leq m$ be such that $\lambda(\bigcup_{i=1}^m B_i) = m$. If $A \subseteq \bigcup_{i=1}^m B_i$, then $A \subseteq B_i$ for a unique i.*

Proof. Let $\mathcal{D}$ be a nest containing the nests corresponding to some fixed families of blocks partitioning A and all the B_i. The restriction of the digraph $\mathcal{H}(\mathcal{D}^+)$ to A is a subdigraph of $\mathcal{H}(\mathcal{D}^+)$. Since the former is weakly connected, it is a sub-digraph of exactly one of the m weakly connected components of the latter. $\square$

Chapter 5

Grothendieck Rings of Modules: Case $\mathrm{T} = \mathrm{T}^{\aleph_0}$

As the title of the chapter suggests, we compute the Grothendieck ring of a fixed right $\mathcal{R}$-module M whose theory T satisfies the model-theoretic condition $T = T^{\aleph_0}$ and show that it is a monoid ring (Theorem 5.4.2). In the first three sections we fix some $n \geq 1$ and drop the subscripts n from $\mathcal{L}_n, \mathcal{A}_n, \ldots$ and, for brevity, we denote the sets $\mathcal{L} \setminus \{\emptyset\}, \mathcal{A} \setminus \{\emptyset\}, \ldots$ by $\mathcal{L}^*, \mathcal{A}^*, \ldots$ respectively.

Local characteristics (Section 5.1) are a family of integer valued functions indexed by *pp*-sets imparting shape to definable sets. Theorems 5.1.6 and 5.5.1 - which state that the local characteristics are additive and multiplicative respectively - are the two key ingredients of our recipe to prove the main theorem. Using *pp*-definable isomorphisms of *pp*-sets, which we call colours, we bundle the local characteristics into coloured global characteristics in Section 5.3. The coloured global characteristics are families of integer valued functions on definable isomorphism classes of definable sets (Theorem 5.3.9). Given any definable set D, the data stored in the global characteristics associated with (the definable isomorphism class of) D gives its class $[D]$ in the Grothendieck ring $K_0(M)$.

The material in this chapter is contained in [26, §3.2,3.3,3.4,3.5,4.1,4.2,4.3].

5.1 Local characteristics

The aim of this section is to define functions κ_a, for each $a \in M^n$, on the distributive lattice of finite abstract simplicial complexes on the meet-semilattice $\mathcal{L}$ and prove that they are valuations in the sense of [23].

In Section 4.2, we defined the characteristic function associated with a nest. In this section, we will use another family of characteristic functions defined below.

Definition 5.1.1. *Given any $C \in \mathcal{C}$, we define the* ***characteristic function*** *of the cell C, $\delta(C) : \mathcal{L}^* \to \{0,1\}$, as $\delta(C)(P) = 1$ if $\delta_{\mathcal{D}}(P) = 1$ for the nest $\mathcal{D}$ corresponding to $P \cup P(C) \cup N(C)$, and $\delta(C)(P) = 0$ otherwise, for each $P \in \mathcal{L}^*$. When $P = \{a\}$, we write the expression $\delta(C)(a)$ instead of $\delta(C)(\{a\})$.*

The set $\mathcal{A}$ of antichains is ordered by the relation $\prec$ but can also be considered as a poset with respect to the natural inclusion ordering on the set of all *pp*-convex sets. For $\alpha, \beta \in \mathcal{A}$, we define $\alpha \wedge \beta$ to be the antichain corresponding (in the sense of 4.2.5) to $(\bigcup \alpha) \cap (\bigcup \beta)$ and $\alpha \vee \beta$ to be the antichain corresponding to $(\bigcup \alpha) \cup (\bigcup \beta)$. Since the intersection and union of two *pp*-convex sets are again *pp*-convex, the binary operations $\wedge, \vee : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ are well-defined. It can be easily seen that $\mathcal{A}$ is a distributive lattice with respect to these operations.

We want to understand the structure of a definable set "locally" in a neighbourhood of a point in M^n. The following lemma defines a class of sub-lattices of $\mathcal{A}$ which provides the necessary framework to define the concept of localization. The proof is an easy verification of an adjunction and is not given here.

Definition and Lemma 5.1.2. *Fix some $a \in M^n$. Let $\mathcal{L}_a := \{A \in \mathcal{L} : a \in A\}$ and $\mathcal{A}_a$ denote the set of all antichains in the meet-semilattice $\mathcal{L}_a$. Then $\mathcal{A}_a$ is a sub-lattice of $\mathcal{A}$. We denote the inclusion $\mathcal{A}_a \to \mathcal{A}$ by $\mathcal{I}_a$. We also consider the map $\mathcal{N}_a : \mathcal{A} \to \mathcal{A}_a$ defined by $\alpha \mapsto \alpha \cap \mathcal{L}_a$. We call the antichain $\mathcal{N}_a(\alpha)$ the* ***localization*** *of α at a. Then $\mathcal{N}_a$ is a right adjoint to $\mathcal{I}_a$ if we consider the posets $\mathcal{A}$ and $\mathcal{A}_a$ as categories in the usual way, and the composite $\mathcal{N}_a \circ \mathcal{I}_a$ is the identity on $\mathcal{A}_a$. This in particular means that $\mathcal{A}_a$ is a reflective subcategory of $\mathcal{A}$. Furthermore, the map $\mathcal{N}_a$ not only preserves the meets of antichains, being a right adjoint, but it also preserves the joins of antichains.*

Fix some $a \in M^n$. Let us denote the set of all finite subsets of $\mathcal{L}_a$ by $\mathcal{P}_a$ and let $\alpha \in \mathcal{P}_a$. We construct a simplicial complex $\mathcal{K}^a(\alpha)$ which determines the "geometry" of the intersection of elements of α around a. This construction is similar to the construction of the nerve of an open cover, except for the meaning of the "triviality" of the intersection. We know that a *pp*-set is finite if and only if it has at most 1 element. We also know that $\bigcap \alpha \supseteq \{a\}$.

Definition 5.1.3. *We associate an abstract simplicial complex $\mathcal{K}^a(\alpha)$ to each $\alpha \in \mathcal{P}_a$ by taking the vertex set $\mathcal{V}(\mathcal{K}^a(\alpha)) := \alpha \setminus \{a\}$. We say that a nonempty set $\beta \subseteq \alpha$ is a face of $\mathcal{K}^a(\alpha)$ if and only if $\bigcap \beta$ is infinite (i.e., strictly contains a). If the only element of α is $\{a\}$ or if $\alpha = \emptyset$, then we set $\mathcal{K}^a(\alpha) = \emptyset$, the empty complex.*

Illustration 5.1.4. Consider the real vector space $\mathbb{R}_{\mathbb{R}}$. The theory of this vector space satisfies the condition $\mathrm{T} = \mathrm{T}^{\aleph_0}$. We consider subsets of $\mathbb{R}^3$. If α denotes the antichain corresponding to the union of 3 coordinate planes and a is the origin, then $\mathcal{K}^a(\alpha)$ is a copy of $\partial\Delta^2$. The 2-dimensional face of Δ^2 is absent since the intersection of the coordinate planes does not contain the origin properly.

Since $\beta_1 \subseteq \beta_2 \Rightarrow \bigcap\beta_2 \subseteq \bigcap\beta_1$, $\mathcal{K}^a(\alpha)$ is indeed a simplicial complex. We tend to drop the superscript a when it is clear from the context. To extend this definition to arbitrary elements of $\mathcal{P}$, we extend the notion of localization operator (at a) to $\mathcal{P}$ by setting $\mathcal{N}_a(\alpha) = \alpha \cap \mathcal{L}_a$ for each $\alpha \in \mathcal{P}$. Now we are ready to define a family of numerical invariants for convex subsets of M^n, which we call "local characteristics".

Definition 5.1.5. *We define the function $\kappa_a : \mathcal{P} \to \mathbb{Z}$ by setting*

$$\kappa_a(\alpha) := \chi(\mathcal{K}(\mathcal{N}_a(\alpha))) - \delta(\alpha)(a),$$

*where $\chi(\mathcal{K})$ denotes the Euler characteristic of the complex $\mathcal{K}$ as defined in 2.2.2 and $\delta(\alpha)$ is the characteristic function of the set $\bigcup\alpha$ as defined in 5.1.1. The value $\kappa_a(\alpha)$ will be called the **local characteristic** of the antichain α at a.*

If we view the local characteristic $\kappa_a(\alpha)$ as a function of a for a fixed antichain α, the correction term $\delta(\alpha)(a)$ makes sure that $\kappa_a(\alpha) = 0$ for all but finitely many values of a. This fact will be useful in the next section.

We want to show that each local characteristic is a valuation in the sense of [23] i.e., it satisfies the inclusion-exclusion principle for antichains.

Theorem 5.1.6. *For each $a \in M^n$ and $\alpha, \beta \in \mathcal{A}_a$ the following identity holds:*

$$\kappa_a(\alpha \vee \beta) + \kappa_a(\alpha \wedge \beta) = \kappa_a(\alpha) + \kappa_a(\beta).$$

The rest of this section is devoted to the proof of this theorem. First we observe that it is sufficient to prove this theorem for $\alpha, \beta \in \mathcal{A}_a$. We also observe that it is sufficient to prove this theorem in the case when κ_a is replaced by the function $\chi(\mathcal{K}(-))$ because $\kappa_a(\alpha) = \chi(\mathcal{K}(\alpha)) - 1$ whenever $a \in \bigcup\alpha$ and the cases when either $a \notin \bigcup\alpha$ or $a \notin \bigcup\beta$ are trivial. We write κ_a as κ for simplicity of notation.

Illustration 5.1.7. Consider the real vector space $\mathbb{R}_\mathbb{R}$. The theory of this vector space satisfies the condition $\mathrm{T} = \mathrm{T}^{\aleph_0}$. We consider subsets of $\mathbb{R}^3$. Suppose α is the antichain corresponding to the formula $(X = 0) \vee (Y = 0)$ and β is the antichain corresponding to the formula $Z = 0$. Then $\alpha \vee \beta$ is the antichain corresponding to the formula $(X = 0) \vee (Y = 0) \vee (Z = 0)$ and $\alpha \wedge \beta$ is the antichain corresponding to the formula $(X = Z = 0) \vee (Y = Z = 0)$. Hence $\mathcal{K}(\alpha), \mathcal{K}(\beta), \mathcal{K}(\alpha \vee \beta)$ and $\mathcal{K}(\alpha \wedge \beta)$ are copies of $\Delta^1, \Delta^0, \partial\Delta^2$ and $\Delta^0 \sqcup \Delta^0$ respectively. Their Euler characteristics can be readily seen to satisfy the required inclusion-exclusion principle.

The following proposition is the first step in this direction, which states that $\kappa(\alpha)$ is actually determined by the *pp*-convex set $\bigcup \alpha$.

Proposition 5.1.8. *Let $\alpha \in \mathcal{A}_a$ and $\beta \in \mathcal{P}_a$. If $\bigcup \alpha = \bigcup \beta$, then $\kappa(\alpha) = \kappa(\beta)$.*

Proof. It is clear that $\beta \supseteq \alpha$ since β is finite. Hence $\mathcal{K}(\alpha)$ is a full sub-complex of $\mathcal{K}(\beta)$ (i.e., if $\beta' \in \mathcal{K}(\beta)$ and $\beta' \subseteq \alpha$, then $\beta' \in \mathcal{K}(\alpha)$). We can also assume that $\{a\} \notin \beta$. Note that every element $\beta \setminus \alpha$ is properly contained in at least one element of α. Now we use induction on the size of $\beta \setminus \alpha$ to prove this result.

If $\beta \setminus \alpha = \emptyset$, then the conclusion is trivially true. For the inductive case, suppose $\alpha \subseteq \beta' \subsetneq \beta$ and the result has been proved for β'. Let $B \in \beta \setminus \beta'$. Since α is the set of maximal elements of β, there is some $A \in \alpha$ such that $A \supsetneq B$.

Consider the complex $\mathcal{K}_1 = \{F \in \mathcal{K}(\beta') : (F \cup \{B\}) \in \mathcal{K}(\beta' \cup \{B\})\}$ as a full sub-complex of $\mathcal{K}(\beta')$. Observe that whenever $B \in F \in \mathcal{K}(\beta' \cup \{B\})$, we have $(F \cup \{A\}) \setminus \{B\} \in \mathcal{K}(\beta')$. As a consequence, $\mathcal{K}_1 = \mathrm{Cone}(\mathcal{K}(\beta' \setminus \{A\}))$ where the apex of the cone is A. In particular, $\mathcal{K}_1$ is contractible.

Also note that $\mathcal{K}(\beta' \cup \{B\}) = \mathcal{K}(\beta') \cup \mathrm{Cone}(\mathcal{K}_1)$, where the apex of the cone is B. Now we compare the pair $\mathcal{K}_1 \subseteq \mathcal{K}(\beta')$ with another pair $\mathrm{Cone}(\mathcal{K}_1) \subseteq \mathcal{K}(\beta' \cup \{B\})$ of simplicial complexes. Observe the set equality $\mathcal{K}(\beta') \setminus \mathcal{K}_1 = \mathcal{K}(\beta' \cup \{B\}) \setminus \mathrm{Cone}(\mathcal{K}_1)$. Also both $\mathcal{K}_1$ and $\mathrm{Cone}(\mathcal{K}_1)$ are contractible. Thus we conclude that $\mathcal{K}(\beta' \cup \{B\})$ and $\mathcal{K}(\beta')$ are homotopy equivalent. Finally, an application of 2.2.4 completes the proof. □

Note that this result is very helpful for the computation of local characteristics as we get the equalities $\kappa(\alpha \vee \beta) = \kappa(\alpha \cup \beta)$ and $\kappa(\alpha \wedge \beta) = \kappa(\alpha \circ \beta)$ for all $\alpha, \beta \in \mathcal{A}_a$, where $\alpha \circ \beta = \{A \cap B : A \in \alpha, B \in \beta\}$. The vertices of $\mathcal{K}(\alpha \circ \beta)$ will be denoted by the elements from $\alpha \times \beta$.

We use induction twice, first on $|\beta|$ and then on $|\alpha|$, to prove the main theorem of this section. The following lemma is the first step of this induction.

Lemma 5.1.9. *For $\alpha, \beta \in \mathcal{A}_a$ and $|\alpha| \leq 1$, we have $\kappa(\alpha \vee \beta) + \kappa(\alpha \wedge \beta) = \kappa(\alpha) + \kappa(\beta)$.*

Proof. The cases $|\alpha| = 0$ and $\alpha = \{\{a\}\}$ are trivial. So we assume that $\alpha = \{A\}$ where A is infinite. We can make similar non-triviality assumptions on β, namely there is at least one element in β and all the elements of β are infinite.

There are only two possible cases when $|\beta| = 1$ and the conclusion holds true in both these cases. For example when $\beta = \{B\}$ and $A \cap B = \{a\}$, we have $\mathcal{K}(\alpha) \cong \mathcal{K}(\beta) \cong \Delta^0$, $\mathcal{K}(\alpha \circ \beta)$ is empty and $\mathcal{K}(\alpha \cup \beta)$ is disjoint union of two copies of Δ^0. Hence the identity in the statement of the lemma takes the form $1 + (-1) = 0 + 0$.

Suppose for the inductive case that the result is true for β i.e., $\kappa(\alpha \vee \beta) + \kappa(\alpha \wedge \beta) = \kappa(\alpha) + \kappa(\beta)$ holds. We want to show that the result holds for $\beta \cup \{B\}$ i.e., $\kappa(\alpha \vee (\beta \cup \{B\})) + \kappa(\alpha \wedge (\beta \cup \{B\})) = \kappa(\alpha) + \kappa(\beta \cup \{B\})$.

We introduce some superscript and subscript notations to denote new simplicial complexes obtained from the original. The following list describes them and also explains the rules to handle two or more scripts at a time.

- Let $\mathcal{K}_0$ denote the complex $\mathcal{K}(\alpha)$, i.e., the complex consisting of only one vertex and $\mathcal{K}$ denote the complex $\mathcal{K}(\beta)$.
- Let $\mathcal{K}^S$ denote the complex $\mathcal{K}(\beta \cup S)$ for any finite $S \subseteq \mathcal{L}_a$ which contains only infinite elements. Also, $\mathcal{K}^{A,B}$ is a short hand for $\mathcal{K}^{\{A,B\}}$.
- Whenever C is a vertex of $\mathcal{Q}$, the notation $\mathcal{Q}_C$ denotes the sub-complex $\{F \in \mathcal{Q} : C \notin F, F \cup \{C\} \in \mathcal{Q}\}$ of $\mathcal{Q}$.
- If $\mathcal{Q} = \mathcal{K}(\gamma)$ for some antichain γ and $A \notin \gamma$, then the notation ${}^A\mathcal{Q}$ denotes the complex $\mathcal{K}(\{A\} \circ \gamma)$.
- The notation ${}^C\mathcal{K}^S_B$ means ${}^C((\mathcal{K}^S)_B)$. This describes the order of the scripts.
- The Euler characteristic of ${}^C\mathcal{K}^S_B$ will be denoted by ${}^C\chi^S_B$.

Using this notation, the inductive hypothesis is

$$\chi^A + {}^A\chi = \chi_0 + \chi \tag{5.1}$$

and our claim is

$$\chi^{B,A} + {}^A\chi^B = \chi_0 + \chi^B. \tag{5.2}$$

Case I: $(A \cap B) = \{a\}$. In this case, the faces of $\mathcal{K}^{A,B}$ not present in $\mathcal{K}^A$ are precisely the faces of $\mathcal{K}^B$ not present in $\mathcal{K}$. Thus $H_*(\mathcal{K}^B; \mathcal{K}) = H_*(\mathcal{K}^{A,B}; \mathcal{K}^A)$. An application of 2.2.5 gives

$$\chi^{B,A} - \chi^A = \chi^B - \chi$$

Also note that the hypothesis $(A \cap B) = \{a\}$ yields $H_*({}^A\mathcal{K}) = H_*({}^A\mathcal{K}^B)$ since only infinite elements matter for the computations. It follows that Equation (5.2) holds in this case.

Case II: $A \cap B \supsetneq \{a\}$. Note that whenever C is not a vertex of $\mathcal{Q}$, we have $\mathcal{Q}^C_C \subseteq \mathcal{Q}$ and $\mathcal{Q} \cup \text{Cone}(\mathcal{Q}^C_C) = \mathcal{Q}^C$, where the apex of the cone is C. Hence Corollary 2.2.8 can be restated in this notation as the following identity.

$$\chi(\mathcal{Q}) + 1 = \chi(\mathcal{Q}^C) + \chi(\mathcal{Q}^C_C) \tag{5.3}$$

As particular cases of (5.3), we get the following equalities.

$$\chi + 1 = \chi^B + \chi^B_B. \tag{5.4}$$

$$\chi^A + 1 = \chi^{A,B} + \chi^{A,B}_B \tag{5.5}$$

$$\chi^B_B + 1 = \chi^{A,B}_B + \chi^{A,B}_{A,B} \tag{5.6}$$

It can be checked that $\mathcal{K}^{A,B}_{A,B} \cong {}^A\mathcal{K}^B_B$ via the map $F \mapsto \{\{C, A\} : C \in F\}$. This gives us the following equation.

$$\chi^{A,B}_{A,B} = {}^A\chi^B_B \tag{5.7}$$

If we combine Equations (5.1), (5.4), (5.5), (5.6) and (5.7), it remains to prove the following to get Equation (5.2) in the claim.

$$ {}^A\chi + 1 = {}^A\chi^B + {}^A\chi^B_B \tag{5.8}$$

Observe that the natural inclusion maps $i_1 : \mathcal{K}_0 \to \mathcal{K}^A$ and $i_2 : \mathcal{K} \to \mathcal{K}^A$ are inclusions of sub-complexes and their images are disjoint. Furthermore, the set theoretic map $g : \mathcal{K}^A \setminus (Im(i_1) \sqcup Im(i_2)) \to {}^A\mathcal{K}$ defined by $F \mapsto \{\sigma \subseteq F : A \in \sigma, |\sigma| = 2\}$ is a bijection. Now consider the composition ${}^A\mathcal{K}^B \cong \mathcal{K}^{A,B} \setminus (i_1(\mathcal{K}_0) \sqcup i_2(\mathcal{K}^B)) \xrightarrow{\pi_B} \mathcal{K}^A \setminus (i_1(\mathcal{K}_0) \sqcup i_2(\mathcal{K})) \cong {}^A\mathcal{K}$, where $\pi_B(F) = F \setminus \{B\}$. The union of images (under this composition of maps) of those faces in ${}^A\mathcal{K}^B$ which contain $A \cap B$ is the sub-complex ${}^A\mathcal{K}^B_B$ of ${}^A\mathcal{K}$. Hence $({}^A\mathcal{K} \cup \mathrm{Cone}({}^A\mathcal{K}^B_B)) \cong {}^A\mathcal{K}^B$, where the apex of the cone is $\{A, B\}$. An application of 2.2.8 gives the required identity in Equation (5.8). □

We use Definition 2.2.2 of Euler characteristic to prove the second step in the proof of the main theorem since we do not have a proof using homological techniques. In this step, we allow the size of β to be an arbitrary but fixed positive integer and we use induction on the size of α. The lemma just proved is the base case for this induction. Let A be a new element of $\mathcal{L}_a$ to be added to α and assume the result is true for α. Again we may assume that A is infinite.

We construct the complex $\mathcal{K}(\alpha\cup\beta\cup\{A\})$ in steps starting with the complex $\mathcal{K}(\alpha\cup\beta)$ and the conclusion of the theorem holds for the latter by the inductive hypothesis. We do this in such a way that at each step $\mathcal{K}_1$ of the construction, the following identity is satisfied.

$$\begin{aligned}\chi(\mathcal{K}(\alpha \cup \{A\} \cup \beta) \cap \mathcal{K}_1) + \chi(\mathcal{K}((\alpha \cup \{A\}) \circ \beta) \cap \mathcal{K}_1) = \\ \chi(\mathcal{K}(\alpha \cup \{A\}) \cap \mathcal{K}_1) + \chi(\mathcal{K}(\beta))\end{aligned} \tag{5.9}$$

In this expression, $\mathcal{K}((\alpha \cup \{A\}) \circ \beta) \cap \mathcal{K}_1$ denotes the subcomplex of $\mathcal{K}((\alpha \cup \{A\}) \circ \beta)$ whose faces are appropriate projections of the faces of $\mathcal{K}_1$.

For the first step, we construct all the elements in $\mathcal{K}(\alpha \cup \{A\})$ not in $\mathcal{K}(\alpha)$. Let $\mathcal{K}_1'$ denote the resulting complex. No new faces of the complex $\mathcal{K}((\alpha \cup \{A\}) \circ \beta)$ are constructed in this process. Hence, for each $n \geq 0$, we have

$$v_n(\mathcal{K}_1') - v_n(\mathcal{K}(\alpha \cup \beta \cup \{A\})) = v_n(\mathcal{K}_1' \cap \mathcal{K}(\alpha \cup \{A\})) - v_n(\mathcal{K}(\alpha \cup \{A\})),$$

where $v_n(\mathcal{Q})$ denotes the number n-dimensional faces of $\mathcal{Q}$. Hence Equation (5.9) is satisfied for $\mathcal{K}_1'$.

For the second step, we further construct all the faces corresponding to $\{A\} \circ \beta$. The conclusion in this case follows from the previous lemma.

Finally we inductively construct the faces containing A and intersecting both $\alpha \cup \{A\}$ and β whenever all its proper sub-faces have already been constructed. We construct a face F of size $(m + k)$ - where $|F \cap (\alpha \cup \{A\})| = m \geq 1$, $|F \cap \beta| = k \geq 1$.

Let $\mathcal{K}_2$ denote the sub-complex of $\mathcal{K}(\alpha \cup \{A\} \cup \beta)$ just before the construction of F and assume, for induction, that Equation (5.9) is true for $\mathcal{K}_2$. It is clear that $\chi(\mathcal{K}(\alpha \cup \{A\}) \cap (\mathcal{K}_2 \cup \{F\})) = \chi(\mathcal{K}(\alpha \cup \{A\}) \cap \mathcal{K}_2)$. We also have the identity $\chi(\mathcal{K}_2 \cup \{F\}) - \chi(\mathcal{K}_2) = (-1)^{\dim F} = (-1)^{m+k-1}$.

Whenever Y is a finite set, we use the notation Δ^Y to denote the simplex $\mathbb{P}(Y) \setminus \{\emptyset\}$ which is a copy of $\Delta^{|Y|}$. Let $\mathcal{F}$ denote the set of all maximal proper subsets of F.

For $F' \subsetneq F$, define $\phi(F') := \{\sigma \subseteq F' : |\sigma \cap (\alpha \cup \{A\})| = 1, |\sigma \cap \beta| = 1\}$. Let $\mathcal{K}_3 = \bigcup_{F' \in \mathcal{F}} \Delta^{\phi(F')}$. Note that $\mathcal{V}(\mathcal{K}_3) \subseteq (\alpha \cup \{A\}) \times \beta$ and $\mathcal{K}_3 \subseteq \mathcal{K}((\alpha \cup \{A\}) \circ \beta)$.

Observe that $\chi(\mathcal{K}((\alpha \cup \{A\}) \circ \beta) \cap (\mathcal{K}_2 \cup \{F\})) - \chi(\mathcal{K}((\alpha \cup \{A\}) \circ \beta) \cap \mathcal{K}_2) = \chi(\Delta^{\phi(F)}) - \chi(\mathcal{K}_3)$. The complex $\Delta^{\phi(F)}$ is contractible and hence $\chi(\phi(F)) = 1$ by 2.2.4. Therefore, to obtain Equation (5.9) for $\mathcal{K}_2 \cup \{F\}$, it remains to show that $\chi(\mathcal{K}_3) = (-1)^{m+k+1} + 1$.

Using the inclusion-exclusion principle for Euler characteristic, we obtain

$$\chi(\mathcal{K}_3) = \sum_{\emptyset \neq S \subseteq \mathcal{F}} (-1)^{|S|-1} \, \chi\left(\bigcap_{F' \in S} \Delta^{\phi(F')}\right)$$

It can be readily checked that ϕ preserves intersections, and so does the operator $\Delta^{\phi(-)}$. Thus the right hand side of the above equation becomes

$$\sum_{\emptyset \neq S \subseteq \mathcal{F}} (-1)^{|S|-1} \, \chi\left(\Delta^{\phi(\bigcap S)}\right).$$

The Euler characteristic of $\Delta^{\phi(\bigcap S)}$ is either 1 or 0 depending on whether $\bigcap S$ intersects both $\alpha \cup \{A\}$ and β or not. Therefore the above sum is equal to $\sum (-1)^{|S|-1}$ where S ranges over all nonempty subset of $\mathcal{F}$ such that $\bigcap S$ intersects both $\alpha \cup \{A\}$ and β.

Note that the map $\mathbb{P}(\mathcal{F}) \setminus \{\emptyset\} \to \mathbb{P}(F) \setminus \{F\}$ defined by $S \mapsto \bigcap S$ is a bijection. Hence the above sum is equal to $\sum (-1)^{|F|-|F'|-1}$ where F' ranges over all proper subsets of F which intersect both $\alpha \cup \{A\}$ and β.

Let w_n denote the number of subsets of F of size n which intersect both $\alpha \cup \{A\}$ and β. Then clearly $w_n = \Sigma_{j=1}^{n-1}\binom{m}{j}\binom{k}{n-j}$. This number can be easily shown to be equal to $\binom{m+k}{n} - \binom{m}{n} - \binom{k}{n}$. Hence

$$\chi(\mathcal{K}_3) = \sum_{n=2}^{m+k-1} (-1)^{m+k-1-n} w_n = \sum_{n=2}^{m+k-1} (-1)^{m+k-1-n} \left[\binom{m+k}{n} - \binom{m}{n} - \binom{k}{n}\right]$$

But we know that $\Sigma_{n=0}^{m+k}(-1)^{m+k-1-n}\left[\binom{m+k}{n} - \binom{m}{n} - \binom{k}{n}\right] = 0$ since each of the three alternating sums is zero. This equation rearranges to give the required identity and completes the proof.

5.2 Global characteristic

In this section, we extend the definition of the valuations κ_a from the distributive lattice of finite antichains in $\mathcal{L}$ to the boolean algebra of definable subsets of M^n. We also describe a way to combine the information in the local characteristics of a definable set to a single invariant called the global characteristic.

Let the function $\kappa : \mathcal{A} \times M^n \to \mathbb{Z}$ be defined by $\kappa(\alpha, a) = \kappa_a(\alpha)$. Suppose $\alpha = \{A\}$. If A is infinite, then $\kappa(\alpha, -)$ is the constant 0 function and if $A = \{a\}$, then $\kappa(\alpha, b) = 0$ for all $b \neq a$ and $\kappa(\alpha, a) = -1$. For an arbitrary $\alpha \in \mathcal{A}$, if $a \notin \bigcup \alpha$, then $\kappa(\alpha, a) = 0$.

Definitions 5.2.1. *For $\alpha \in \mathcal{A}$, we define the* ***set of singular points*** *of α to be the set* $\mathrm{Sing}(\alpha) := \{a \in M^n : \kappa(\alpha, a) \neq 0\}$. $\mathrm{Sing}(\alpha)$ *is always finite since all the singular points appear as singletons in the nest corresponding to α. We define the* ***global characteristic*** *of α to be the sum $\Lambda(\alpha) := -\Sigma_{a\in M^n}\kappa(\alpha, a)$, which in fact is equal to the finite sum $\Lambda(\alpha) = -\Sigma_{a\in \mathrm{Sing}(\alpha)}\kappa(\alpha, a)$.*

Fix some $a \in M^n$. Let $\alpha, \beta \in \mathcal{A}$ be such that $\beta \prec \alpha$. Then either $\mathcal{N}_a(\alpha) = \mathcal{N}_a(\beta) = \emptyset$ or $\mathcal{N}_a(\beta) \prec \mathcal{N}_a(\alpha)$. If $C := \bigcup \alpha \setminus \bigcup \beta$ is a cell, we define the homology $H_*(C)$ to be the relative homology $H_*(\mathcal{K}(\mathcal{N}_a(\alpha \cup \beta)); \mathcal{K}(\mathcal{N}_a(\beta)))$. In particular, the alternating sum of the Betti numbers of $H_*(C)$, denoted by $\chi_a(C)$, is equal to the difference $\chi(\mathcal{K}(\mathcal{N}_a(\alpha))) - \chi(\mathcal{K}(\mathcal{N}_a(\beta)))$ by 5.1.8 and 2.2.5. We also have the equation $\delta(C) = \delta(\alpha) - \delta(\beta)$. Hence if we define the local characteristic of C as $\kappa_a(C) := \chi_a(C) - \delta(C)(a)$, we get the identity $\kappa_a(C) = \kappa_a(P(C)) - \kappa_a(N(C))$. We define the extension of the function κ to $\mathcal{C} \times M^n$ by setting $\kappa(C, a) := \kappa_a(C)$ for $a \in M^n, C \in \mathcal{C}$.

Definitions 5.2.2. *We define the set of singular points* $\mathrm{Sing}(C)$ *for $C \in \mathcal{C}$ analogously by setting* $\mathrm{Sing}(C) := \{a \in M^n : \kappa_a(C) \neq 0\}$. *This set is finite since* $\mathrm{Sing}(C) \subseteq \mathrm{Sing}(P(C)) \cup \mathrm{Sing}(N(C))$. *We also extend the definition of global characteristic for cells by setting $\Lambda(C) := -\Sigma_{a\in M^n}\kappa(C, a)$.*

It is immediate that $\Lambda(C) = \Lambda(P(C)) - \Lambda(N(C))$ for every $C \in \mathcal{C}$.

Illustration 5.2.3. The definition of a singular point agrees with the geometrical intuition. Consider the cell $C \in \mathcal{C}_2(\mathbb{R}_\mathbb{R})$ defined by $P(C) = (X = 0) \cup (Y = 0) \cup (X + Y = 1)$ and $N(C) = \{(0,0)\}$. Then $\kappa_{(0,0)}(C) = 2, \kappa_{(0,1)}(C) = 1$ and $\kappa_{(1,0)}(C) = 1$. Hence $\mathrm{Sing}(C) = \{(0,0), (0,1), (1,0)\}$ and $\Lambda(C) = -4$.

The main aim of this section is to prove that the global characteristic is additive in the following sense.

Theorem 5.2.4. *If $\{B_i : 1 \leq i \leq l\}, \{B'_j : 1 \leq j \leq m\}$ are two finite families of pairwise disjoint blocks such that $\bigsqcup_{i=1}^{l} B_i = \bigsqcup_{j=1}^{m} B'_j$, then $\Sigma_{i=1}^{l}\Lambda(B_i) = \Sigma_{j=1}^{m}\Lambda(B'_j)$.*

The proof of this theorem follows at once from the following local version.

Lemma 5.2.5. *If $a \in M^n$ and $\{B_i : 1 \leq i \leq l\}, \{B'_j : 1 \leq j \leq m\}$ are two finite families of pairwise disjoint blocks such that $\bigsqcup_{i=1}^{l} B_i = \bigsqcup_{j=1}^{m} B'_j$, then $\Sigma_{i=1}^{l}\kappa_a(B_i) = \Sigma_{j=1}^{m}\kappa_a(B'_j)$.*

Proof. It will be sufficient to show that both these numbers are equal to the sum $\Sigma_{B \in \mathcal{F}}\, \kappa_a(B)$ where $\mathcal{F}$ is any finite family of blocks finer than both the given families. We can in particular choose a finite *pp*-nest $\mathcal{D}$ containing all the elements in $\bigcup_{i=1}^{l}(P(B_i) \cup N(B_i)) \cup \bigcup_{j=1}^{m}(P(B_j) \cup N(B_j))$ and set $\mathcal{F} = \{\mathrm{Core}_{\mathcal{D}}(D) : D \in \mathcal{D}^+\}$. This involves partitioning every B_i and B'_j into smaller blocks of the form $\mathrm{Core}_{\mathcal{D}}(D)$ for $D \in \mathcal{D}^+$.

Thus it will be sufficient to show that if $\mathcal{F}$ is a finite family of blocks corresponding to cores of a *pp*-nest $\mathcal{D}$ such that $B = \bigcup \mathcal{F} \in \mathcal{B}$, then $\kappa_a(B) = \Sigma_{F \in \mathcal{F}}\kappa_a(F)$. Consider the sub-poset $\mathcal{H}$ of $\mathcal{L}$ containing all the elements of $\bigcup_{F \in \mathcal{F}}(P(F) \cup N(F))$. Then we construct the antichains $\{\alpha_s\}_{s \geq 0}$ in such a way that α_s is the set of all minimal elements of $\mathcal{H} \setminus \bigcup_{0 \leq t < s} \alpha_t$. Then this process stops, say α_v is $P(B)$. Then we have a chain of antichains $\alpha_0 \prec \alpha_1 \prec \cdots \prec \alpha_v$. Now $\kappa_a(B) = \kappa_a(\alpha_v) - \kappa_a(\alpha_0) = \Sigma_{t=1}^{v}\kappa_a(\alpha_t) - \kappa_a(\alpha_{t-1})$. In other words, if C_t denotes the cell $\bigcup \alpha_t \setminus \bigcup \alpha_{t-1}$ for $1 \leq t \leq v$, then $\kappa_a(B) = \Sigma_{t=1}^{v}\kappa_a(C_t)$.

Now it remains to show that for each $1 \leq t \leq v$, $\kappa_a(C_t) = \Sigma_{F \in \alpha_t}\kappa_a(\mathrm{Core}_{\mathcal{D}}(F))$. This follows from the following proposition by first choosing A_j to consist of elements of α_t and then choosing A_j to consist of elements of α_{t-1}. Then by our construction of the chain and the definition of $\kappa_a(C_t)$, we get the required result. □

Proposition 5.2.6. *Suppose $k \geq 2$. If, for each $j \in [k]$, $\alpha_j \in \mathcal{A}$ and $A_j = \bigcup \alpha_j$, then $\kappa_a(\bigcup_{j \in [k]} A_j) = \Sigma_{S \subseteq [k], S \neq \emptyset}\kappa_a(\bigcap_{s \in S} A_s \setminus \bigcup_{t \notin S} A_t)$.*

Proof. We observe that all the arguments on the right hand side of the above expression are cells or possibly empty sets and they form a partition of the cell in the argument of the left hand side. Then we restate Theorem 5.1.6 as $\kappa_a((\bigcup \alpha) \cup (\bigcup \beta)) = \kappa_a((\bigcup \alpha) \setminus (\bigcup \beta)) + \kappa_a((\bigcup \beta) \setminus (\bigcup \alpha)) + \kappa_a((\bigcup \alpha) \cap (\bigcup \beta))$. Since the set of *pp*-convex sets is closed under taking unions and intersections, a simple induction proves the proposition with 5.1.6 being the base case. □

Theorem 5.2.4 allows us to define the global characteristic for arbitrary definable sets.

Definition 5.2.7. *Let $D \subseteq M^n$ be definable. We define the* ***global characteristic*** *$\Lambda(D)$ as the sum of global characteristics of any finite family of blocks partitioning D.*

The global characteristic is preserved under definable isomorphisms.

Theorem 5.2.8. *Suppose $D \in \mathrm{Def}(M^n)$ and $f : D \to M^n$ is a definable injection. Then $\Lambda(D) = \Lambda(f(D))$.*

Proof. We first prove the local version which states that, for any $a \in M^n$ and $B \in \mathcal{B}$, if $g : B \to M$ is a *pp*-definable injection, then $\kappa_a(B) = \kappa_{g(a)}(g(B))$. We observe that $\delta(B)(a) = \delta(g(B))(g(a))$. Corollary 4.2.12 gives that the complex $\mathcal{K}(\mathcal{N}_a(\alpha))$ is isomorphic to the complex $\mathcal{K}(\mathcal{N}_{g(a)}(g[\alpha]))$ where $g[\alpha] = \{g(A) : A \in \alpha\}$ and α is either $P(B)$ or $N(B)$. We conclude that $g(\mathrm{Sing}(B)) = \mathrm{Sing}(g(B))$. Hence $\Lambda(B) = \Sigma_{a \in \mathrm{Sing}(B)} \kappa_a(B) = \Sigma_{a \in \mathrm{Sing}(B)} \kappa_{g(a)}(g(B)) = \Sigma_{a \in \mathrm{Sing}(g(B))} \kappa_a(g(B)) = \Lambda(g(B))$.

To prove the theorem, we consider any partition of D into finitely many blocks $B_i, 1 \leq i \leq m$ such that $f \restriction B_i$ is *pp*-definable. This is possible by an application of Lemma 4.1.7 to the set *Graph*(f) followed by projection of the finitely many blocks onto the first n coordinates. Note that $D = \bigsqcup_{i=1}^m B_i \Rightarrow f(D) = \bigsqcup_{i=1}^m f(B_i)$ since f is injective. Hence $\Lambda(f(D)) = \Sigma_{i=1}^m \Lambda(f(B_i)) = \Sigma_{i=1}^m \Lambda(B_i) = \Lambda(D)$, where the first and the third equality follows from Theorem 5.2.4 and the second equality follows from the previous paragraph. □

Illustration 5.2.9. We continue to work with the cell $C \in \mathcal{C}_2(\mathbb{R}_\mathbb{R})$ defined in Illustration 5.2.3. Now we define a function $f : C \to \mathbb{R}^2$ as follows:

$$f(X,Y) = \begin{cases} (X,1) & \text{if } (X+Y=1) \text{ and } (X,Y) \neq (1,0), \\ (X,0) & \text{if } (Y=0) \text{ and } X \neq 0,1, \\ (Y,-1) & \text{if } (X=0) \text{ and } Y \neq 0. \end{cases}$$

This function is clearly injective and definable. It is readily seen that $\mathrm{Sing}(f(C))$ is the set $\{(0,1),(0,0),(1,0),(0,-1)\}$ and $\kappa_a(f(C)) = 1$ for each $a \in \mathrm{Sing}(f(C))$. Hence $\Lambda(C) = -4 = \Lambda(f(C))$.

Now we are ready to prove a special case of the result promised at the end of Section 3.1.

Corollary 5.2.10. *Let M be a non-zero right $\mathcal{R}$-module whose theory* T *satisfies* $\mathrm{T} = \mathrm{T}^{\aleph_0}$. *Suppose $D \subseteq M^n$ is definable and $f : D \rightarrowtail D$ is a definable injection whose image is cofinite in the codomain, then f is an isomorphism. In particular $K_0(M)$ is nontrivial.*

Proof. We extend the function f to an injective function $g : M^n \rightarrowtail M^n$ by setting $g(a) = f(a)$ if $a \in D$ and $g(a) = a$ otherwise. Now $F := M^n \setminus Im(g)$ is finite; say it has p elements. Further $\Lambda(Im(g)) = \Lambda(M^n \setminus F) = \Lambda(M^n) - \Lambda(F) = -p$.

By Theorem 5.2.8, we get $\Lambda(M^n) = \Lambda(Im(g))$ since g is definable injective. Hence $p = 0$ and thus g is an isomorphism. Since g is the identity function outside D, we conclude that f is a definable isomorphism.

This shows that the classes of two finite sets in the Grothendieck ring are equal if and only if the two sets are in bijection. Hence the underlying abelian group of $K_0(M)$ contains $\mathbb{Z}$ as a subgroup. □

5.3 Coloured global characteristics

Let $P \in \mathcal{L}^*$ be fixed for this section. We develop the notion of localization at P and local characteristic at P; we have developed these ideas earlier when P is a singleton. After stating what we mean by a colour, we define the notion of a "coloured global characteristic" and outline the proof that these invariants are preserved under definable isomorphisms.

Definition 5.3.1. *We use $\mathcal{L}_P$ to denote the meet-semilattice of all upper bounds of P in $\mathcal{L}$, i.e., $\mathcal{L}_P := \{A \in \mathcal{L} : A \supseteq P\}$. As usual, we denote the set of all finite antichains in this semilattice by $\mathcal{A}_P$.*

Since every element of $\mathcal{L}_P$ contains P, we may as well quotient out P from each such element. Such a process is consistent with our earlier definition of localization since taking quotient with respect to a singleton set gives an isomorphic copy of the original set.

Definitions 5.3.2. *We define the operator $\mathcal{Q}_P$ on the elements of $\mathcal{L}_P$ by setting $\mathcal{Q}_P(A) := p + \frac{A-p}{P-p} = \{a + (P - p) : a \in A\}$ for any $p \in P$. Note that this definition is independent of the choice of $p \in P$. We can clearly extend this operator to finite subsets of $\mathcal{L}_P$. Now let $\mathcal{L}_{(P)} := \mathcal{Q}_P[\mathcal{L}_P]$. We use $\mathcal{A}_{(P)}$ to denote the set of all finite antichains in this semilattice.*

It is easy to see that $\mathcal{A}_{(P)} = \mathcal{Q}_P[\mathcal{A}_P]$.

The appropriate analogue of the localization operator $\mathcal{N}_a : \mathcal{A} \to \mathcal{A}_a$ is a function $\mathcal{N}_P : \mathcal{A} \to \mathcal{A}_{(P)}$.

Definition and Lemma 5.3.3. *For $\alpha \in \mathcal{A}$, we define $\mathcal{N}_P(\alpha) := \mathcal{Q}_P(\alpha \cap \mathcal{L}_P)$. As an operator on pp-convex sets, $\mathcal{N}_P$ preserves both unions and intersections.*

The proof is easy and thus omitted.

Recall from Definition 5.1.3 of $\mathcal{K}^a(\alpha)$ that the "trivial intersections" were precisely those which were empty or a singleton. On the other hand, "nontrivial intersections" were precisely those which contained the *pp*-set $\{a\}$ properly. As $\mathcal{N}_P$ takes values in $\mathcal{A}_{(P)}$, we get the correct notion of non-trivial intersections followed by the quotient operation so that the techniques developed for a singleton P still remain valid. Now we are ready to state the analogue of Definition 5.1.3.

Definition 5.3.4. *For $\alpha \in \mathcal{A}$, we define the **simplicial complex** of α **in the neighbourhood of** P as the complex $\mathcal{K}(\mathcal{N}_P(\alpha)) = \{\beta \subseteq \mathcal{N}_P(\alpha) : \beta \neq \emptyset, |\bigcap \beta| = \infty\}$. For simplicity of notation, we denote this complex by $\mathcal{K}^P(\alpha)$.*

We can easily extend the notion of local characteristic at P as follows.

Definition 5.3.5. *We define the **local characteristic** of α at P by*

$$\kappa_P(\alpha) := \chi(\mathcal{K}^P(\alpha)) - \delta(\alpha)(P).$$

It can be observed that we recover the definition of the local characteristic at a point $a \in M$ by choosing $P = \{a\}$. The proofs of Theorem 5.1.6 and Lemma 5.2.5 go through if we replace κ_a by κ_P. Thus we can define $\kappa_P(D)$ for arbitrary definable sets $D \subseteq M^n$.

We define the function $\kappa : \mathrm{Def}(M^n) \times \mathcal{L}^* \to \mathbb{Z}$ by setting $\kappa(D, P) := \kappa_P(D)$.

Definition 5.3.6. *The **set of $\mathcal{L}$-singular elements** of a definable set $D \subseteq M^n$ is defined as the set $\mathrm{Sing}_{\mathcal{L}}(D) := \{P \in \mathcal{L} : \kappa(D, P) \neq 0\}$.*

Fixing any partition of D into blocks, it can be checked that the set $\mathrm{Sing}_{\mathcal{L}}(D)$ is contained in the nest corresponding to that partition and hence is finite. This finiteness will be used to define analogues of the global characteristic, which we call "coloured global characteristics".

Definition 5.3.7. *For a given $P \in \mathcal{L}$, we define the **colour** of P to be the set $\{A \in L :$ there is a bijection $f : A \cong P$ such that $Graph(f)$ is pp-definable $\}$. We denote the colour of P by $[[P]]$.*

Note the significance of this definition. Theorem 4.1.5 describes the *pp*-sets as fundamental definable sets and we are trying to classify definable sets up to definable isomorphism (Definition 3.1.2). In fact it is sufficient to classify *pp*-sets up to *pp*-definable isomorphisms, which is the motivation behind the definition of a colour.

Let $\mathcal{X}$ denote the set of colours of elements from $\mathcal{L}$. We use letters $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}$ etc. to denote the colours. It can be observed that $[[\emptyset]]$ is a singleton. We use $\mathcal{X}^*$ to denote $\mathcal{X} \setminus \{[[\emptyset]]\}$. We denote the colour of any singleton by $\mathfrak{U}$.

The global characteristic $\Lambda(D)$ is equal to $-\Sigma_{P\in\mathfrak{U}}\kappa_P(D)$ for each definable set D. This observation can be used to extend the notion of global characteristic.

Definition 5.3.8. *For $\mathfrak{A} \in \mathcal{X}^*$, we define the **coloured global characteristic** with respect to $\mathfrak{A}$ for a definable set D to be the integer $\Lambda_{\mathfrak{A}}(D) := -\Sigma_{P\in\mathfrak{A}}\kappa_P(D)$. This integer is well defined as it is equal to the finite sum $-\Sigma_{P\in(\mathfrak{A}\cap\mathrm{Sing}_{\mathcal{L}}(D))}\kappa_P(D)$.*

The property of coloured global characteristics that we are looking for is stated in the following analogue of Theorem 5.2.8. The proof is analogous to that of 5.2.8 and thus is omitted.

Theorem 5.3.9. *If $f : D \to D'$ is a definable bijection between definable sets D, D', then $\Lambda_{\mathfrak{A}}(D) = \Lambda_{\mathfrak{A}}(D')$ for each $\mathfrak{A} \in \mathcal{X}^*$.*

5.4 Monoid rings

We need the notion of an algebraic structure called a *monoid ring*.

Definition 5.4.1. *Let $(A, \star, 1)$ be a commutative monoid and S be a commutative ring with unity. Then we define an L_{ring}-structure $(S[A], 0, 1, +, \cdot)$ called a **monoid ring** as follows:*

- $S[A] := \{\phi : A \to S :$ *the set* $\mathrm{Supp}(\phi) = \{a : \phi(a) \neq 0\}$ *is finite*$\}$;
- $(\phi + \psi)(a) := \phi(a) + \psi(a)$ *for* $a \in A$;
- $(\phi \cdot \psi)(a) := \Sigma_{b\star c=a}\phi(b)\psi(c)$ *for* $a \in A$.

An element ϕ of $S[A]$ can be represented as a formal sum $\Sigma_{a\in A}s_a a$ where $s_a = \phi(a)$.

As an example, let $A = \mathbb{N}$, equivalently the monoid $\{X^n\}_{n\geq 0}$ considered multiplicatively. Then the monoid ring $S[A] = S[\mathbb{N}] \cong S[X]$, the polynomial ring in one variable with coefficients from S.

Let $\overline{\mathcal{L}}, \overline{\mathcal{A}}, \overline{\mathcal{X}}, \ldots$ denote the ascending unions $\bigcup_{n=1}^{\infty} \mathcal{L}_n$, $\bigcup_{n=1}^{\infty} \mathcal{A}_n$, $\bigcup_{n=1}^{\infty} \mathcal{X}_n, \ldots$ respectively. We shall be especially concerned with the sets $\overline{\mathcal{L}}^* := \overline{\mathcal{L}} \setminus \{\emptyset\}$ and $\overline{\mathcal{X}}^* := \overline{\mathcal{X}} \setminus \{[[\emptyset]]\}$.

There is a binary operation $\times : \overline{\mathcal{L}}^* \times \overline{\mathcal{L}}^* \to \overline{\mathcal{L}}^*$ which maps a pair (A, B) to the cartesian product $A \times B$. This map commutes with the operation $[[-]]$ of taking colour i.e., whenever $[[A_1]] = [[A_2]]$ and $[[B_1]] = [[B_2]]$, we have $[[A_1 \times B_1]] = [[A_2 \times B_2]]$. This allows us to define a binary operation $\star : \overline{\mathcal{X}}^* \times \overline{\mathcal{X}}^* \to \overline{\mathcal{X}}^*$ which takes a pair of colours $(\mathfrak{A}, \mathfrak{B})$ to $[[A \times B]]$ for any $A \in \mathfrak{A}, B \in \mathfrak{B}$. The colour $\mathfrak{U}$ of singletons acts as the identity element for the operation $\star$. Hence $(\overline{\mathcal{X}}^*, \star, \mathfrak{U})$ is a monoid.

Consider the maps $\Lambda_{\mathfrak{A}} : \widetilde{\mathrm{Def}}(M) \to \mathbb{Z}$ for $\mathfrak{A} \in \overline{\mathcal{X}}^*$ defined by $[D] \mapsto \Lambda_{\mathfrak{A}}(D')$ for any $D' \in [D]$. These maps are well defined thanks to Theorem 5.3.9. We can fix some $[D] \in \widetilde{\mathrm{Def}}(M)$ and look at the set $\mathrm{Supp}([D]) := \{\mathfrak{A} \in \overline{\mathcal{X}}^* : \Lambda_{\mathfrak{A}}(D) \neq 0\}$. This set is finite since it is contained in the finite set $\{[[P]] : P \in \mathrm{Sing}_{\overline{\mathcal{L}}}(D)\}$. This shows that the evaluation map $ev_{[D]} : \overline{\mathcal{X}}^* \to \mathbb{Z}$ defined by $\mathfrak{A} \mapsto \Lambda_{\mathfrak{A}}([D])$ for each $[D] \in \widetilde{\mathrm{Def}}(M)$ is an element of the monoid ring $\mathbb{Z}[\overline{\mathcal{X}}^*]$.

Let us consider an example. We take $\mathcal{R}$ to be an infinite skew-field (i.e., a (possibly non-commutative) ring in which every nonzero element has two-sided multiplicative inverse) and M to be any nonzero $\mathcal{R}$-vector space. This example has been studied in detail in [31]. In this case, we have $Th(M) = Th(M)^{\aleph_0}$. Using the notion of affine dimension, it can be shown that $\overline{\mathcal{X}}^* \cong \mathbb{N}$. It has been shown that $K_0(M) \cong \mathbb{Z}[X] \cong \mathbb{Z}[\mathbb{N}]$. The proof in [31] explicitly shows that the semiring $\widetilde{\mathrm{Def}}(M)$ is cancellative and is isomorphic to the semiring of polynomials in $\mathbb{Z}[X]$ with non-negative leading coefficients.

We will prove that a similar fact holds for an arbitrary module M, i.e., the structure of the Grothendieck ring $K_0(M)$ is entirely determined by the monoid $\overline{\mathcal{X}}^*$.

Theorem 5.4.2. *Let M be a right $\mathcal{R}$-module satisfying $Th(M) = Th(M)^{\aleph_0}$. Then $K_0(M) \cong \mathbb{Z}[\overline{\mathcal{X}}^*]$. In particular, $K_0(M)$ is nontrivial for every nonzero module M.*

The proof of this theorem will occupy the next two sections.

5.5 Multiplicative structure of $\widetilde{\mathrm{Def}}(M)$

Given $D_1 \in \mathrm{Def}(M^n)$ and $D_2 \in \mathrm{Def}(M^m)$, the cartesian product $D_1 \times D_2 \in \mathrm{Def}(M^{n+m})$. This shows that $\overline{\mathrm{Def}}(M)$ is closed under cartesian products. We want to show that the sets $\overline{\mathcal{L}}$, $\overline{\mathcal{A}}$, $\overline{\mathcal{B}}$ and $\overline{\mathcal{C}}$ are all closed under multiplication.

Let $P \in \mathcal{L}_n$ and $Q \in \mathcal{L}_m$. Then there are *pp* formulas $\phi(\overline{x})$ and $\psi(\overline{y})$ defining those sets respectively. Without loss, we may assume that $\overline{x} \cap \overline{y} = \emptyset$. Now the formula

$\rho(\overline{x}, \overline{y}) = \phi(\overline{x}) \wedge \psi(\overline{y})$ is again a *pp*-formula and it defines the set $P \times Q \in \mathcal{L}_{n+m}$. This shows that the set $\overline{\mathcal{L}}$ is closed under multiplication.

Now we want to show that the product of two antichains $\alpha \in \mathcal{A}_n$ and $\beta \in \mathcal{A}_m$ is again an antichain in $\mathcal{A}_{n+m}$. We have natural projection maps $\pi_1 : M^{n+m} \to M^n$ and $\pi_2 : M^{n+m} \to M^m$ which project onto the first n and the last m coordinates respectively. First we observe that $(\bigcup \alpha) \times (\bigcup \beta) = \bigcup_{A \in \alpha} \bigcup_{B \in \beta} A \times B$. If either $A_1, A_2 \in \alpha$ are distinct or $B_1, B_2 \in \beta$ are distinct, then all the distinct elements from $\{A_i \times B_j\}_{i,j=1}^2$ are incomparable with respect to the inclusion ordering since at least one of their projections is so. Hence $\bigcup \alpha \times \bigcup \beta$ is indeed an antichain of rank $|\alpha| \times |\beta|$. We will denote this antichain by $\alpha \times \beta$.

Given $C_1, C_2 \in \overline{\mathcal{C}}$, we have $C_1 \times C_2 = \bigcup(\alpha_1 \times \alpha_2) \setminus (\bigcup(\alpha_1 \times \beta_2) \cup \bigcup(\beta_1 \times \alpha_2))$ where $\alpha_i = P(C_i)$ and $\beta_i = N(C_i)$ for $i = 1, 2$. This shows that $C_1 \times C_2 \in \overline{\mathcal{C}}$ since $\overline{\mathcal{A}}$ is closed under both products and unions. Furthermore, we observe that $P(C_1 \times C_2) = P(C_1) \times P(C_2)$. This in particular shows that the set $\overline{\mathcal{B}}$ of blocks is also closed under products.

Lemma 5.5.1. *Let $P, Q \in \overline{\mathcal{L}}$ and $\alpha, \beta \in \overline{\mathcal{A}}$. Then $\kappa_{P \times Q}(\alpha \times \beta) = -\kappa_P(\alpha)\kappa_Q(\beta)$.*

Proof. First assume that $\delta(\alpha)(P) = \delta(\beta)(Q) = 1$. Then observe that

$$\mathcal{K}^{P \times Q}(\alpha \times \beta) \cong \mathcal{K}^P(\alpha) \boxtimes \mathcal{K}^Q(\beta). \tag{5.10}$$

Hence we have

$$\begin{aligned}
\kappa_{P \times Q}(\alpha \times \beta) &= \chi(\mathcal{K}^{P \times Q}(\alpha \times \beta)) - 1 \\
&= \chi(\mathcal{K}^P(\alpha)) + \chi(\mathcal{K}^Q(\beta)) - \chi(\mathcal{K}^P(\alpha))\chi(\mathcal{K}^Q(\beta)) - 1 \\
&= (\kappa_P(\alpha) + 1) + (\kappa_Q(\beta) + 1) - (\kappa_P(\alpha) + 1)(\kappa_Q(\beta) + 1) - 1 \\
&= -\kappa_P(\alpha)\kappa_Q(\beta)
\end{aligned}$$

The first and the third equality is by definition of the local characteristic and the second is by Equation (2.6) of Lemma 2.3.4 applied to (5.10).

In the remaining case when either $\delta(\alpha)(P)$ or $\delta(\beta)(Q)$ is 0, we have $\delta(\alpha \times \beta)(P \times Q) = 0$. Hence $\kappa_{P \times Q}(\alpha \times \beta) = 0$ and either $\kappa_P(\alpha)$ or $\kappa_Q(\beta)$ is 0. This gives the necessary identity and thus completes the proof in all cases. □

The aim of this section is to prove the following theorem.

Theorem 5.5.2. *The map $ev : \widetilde{\mathrm{Def}}(M) \to \mathbb{Z}[\overline{\mathcal{X}}^*]$ defined by $[D] \mapsto ev_{[D]}$ is a semiring homomorphism.*

Proof. We have already seen that ev is additive, since each $\Lambda_{\mathfrak{A}}$ is. So it remains to show that it is multiplicative.

We have observed that the set $[\overline{\mathcal{A}}]$ is a monoid with respect to cartesian product, the isomorphism class of a singleton being the identity for the multiplication. So we will first show that $ev : [\overline{\mathcal{A}}] \to \mathbb{Z}[\overline{\mathcal{X}}^*]$ is a multiplicative monoid homomorphism.

Let $\alpha, \beta \in \overline{\mathcal{A}}$ be fixed. Note that

$$S := \mathrm{Sing}_{\overline{\mathcal{L}}}(\alpha \times \beta) \subseteq \{P \times Q : P \in \mathrm{Sing}_{\overline{\mathcal{L}}}(\alpha), Q \in \mathrm{Sing}_{\overline{\mathcal{L}}}(\beta)\}. \tag{5.11}$$

We need to show that $ev_{[\alpha]} \cdot ev_{[\beta]} = ev_{[\alpha\times\beta]}$ as maps on $\overline{\mathcal{X}}^*$. This is equivalent to $ev_{[\alpha\times\beta]}(\mathfrak{C}) = \sum_{\mathfrak{A}\star\mathfrak{B}=\mathfrak{C}} ev_{[\alpha]}(\mathfrak{A}) ev_{[\beta]}(\mathfrak{B})$ for each $\mathfrak{C} \in \overline{\mathcal{X}}^*$. Using the definition of the evaluation map, it is enough to check that $\Lambda_{\mathfrak{C}}([\alpha \times \beta]) = \sum_{\mathfrak{A}\star\mathfrak{B}=\mathfrak{C}} \Lambda_{\mathfrak{A}}([\alpha])\Lambda_{\mathfrak{B}}([\beta])$ for each $\mathfrak{C} \in \overline{\mathcal{X}}^*$.

The left hand side of the above equation is

$$\begin{aligned}\Lambda_{\mathfrak{C}}([\alpha \times \beta]) &= -\sum_{R\in\mathfrak{C}} \kappa_R(\alpha \times \beta) \\ &= -\sum_{R\in(\mathfrak{C}\cap S)} \kappa_R(\alpha \times \beta) \\ &= \sum_{R\in(\mathfrak{C}\cap S)} \kappa_{\pi_1(R)}(\alpha)\kappa_{\pi_2(R)}(\beta)\end{aligned}$$

The last equality is given by the Lemma 5.5.1 since, by (5.11), every $R \in \mathfrak{C} \cap S$ can be written as $R = \pi_1(R) \times \pi_2(R)$. The right hand side is

$$\begin{aligned}\sum_{\mathfrak{A}\star\mathfrak{B}=\mathfrak{C}} \Lambda_{\mathfrak{A}}([\alpha])\Lambda_{\mathfrak{B}}([\beta]) &= \sum_{\mathfrak{A}\star\mathfrak{B}=\mathfrak{C}} \left(-\sum_{P\in\mathfrak{A}} \kappa_P(\alpha)\right)\left(-\sum_{Q\in\mathfrak{B}} \kappa_Q(\beta)\right) \\ &= \sum_{\mathfrak{A}\star\mathfrak{B}=\mathfrak{C}} \sum_{P\in\mathfrak{A}, Q\in\mathfrak{B}} \kappa_P(\alpha)\kappa_Q(\beta)\end{aligned}$$

Using the definition of $\mathrm{Sing}_{\overline{\mathcal{L}}}(-)$, we observe that the final expressions on both sides are equal. This completes the proof that ev is a multiplicative monoid homomorphism on $[\overline{\mathcal{A}}]$.

Now we will show that ev is also multiplicative on the monoid $[\overline{\mathcal{C}}]$. Let C_1, C_2 be cells with $\alpha_i = P(C_i)$ and $\beta_i = N(C_i)$ for each $i = 1, 2$. Then $C_1 \times C_2 = \bigcup(\alpha_1 \times \alpha_2) \setminus (\bigcup(\alpha_1 \times \beta_2) \cup \bigcup(\beta_1 \times \alpha_2))$. We also know that $ev_{[C]} = ev_{P(C)} - ev_{N(C)}$ for each cell C.

We need to show that $\Lambda_{\mathfrak{C}}(C_1 \times C_2) = \sum_{\mathfrak{A}\star\mathfrak{B}=\mathfrak{C}} \Lambda_{\mathfrak{A}}([C_1])\Lambda_{\mathfrak{B}}([C_2])$ for each $\mathfrak{C} \in \overline{\mathcal{X}}^*$. Now we have

$$\Lambda_{\mathfrak{C}}(C_1 \times C_2) = \Lambda_{\mathfrak{C}}(\alpha_1 \times \alpha_2) - \Lambda_{\mathfrak{C}}((\alpha_1 \times \beta_2) \vee (\beta_1 \times \alpha_2))$$

and we also have

$$\begin{aligned}\sum_{\mathfrak{A}\star\mathfrak{B}=\mathfrak{C}} \Lambda_{\mathfrak{A}}([C_1])\Lambda_{\mathfrak{B}}([C_2]) &= \sum_{\mathfrak{A}\star\mathfrak{B}=\mathfrak{C}} (\Lambda_{\mathfrak{A}}([\alpha_1]) - \Lambda_{\mathfrak{A}}([\beta_1]))(\Lambda_{\mathfrak{B}}([\alpha_2]) - \Lambda_{\mathfrak{B}}([\beta_2]))\\ &= \Lambda_{\mathfrak{C}}(\alpha_1 \times \alpha_2) + \Lambda_{\mathfrak{C}}(\beta_1 \times \beta_2) - \Lambda_{\mathfrak{C}}(\beta_1 \times \alpha_2) - \Lambda_{\mathfrak{C}}(\alpha_1 \times \beta_2)\end{aligned}$$

Therefore we need to show

$$\Lambda_{\mathfrak{C}}((\alpha_1 \times \beta_2) \vee (\beta_1 \times \alpha_2)) + \Lambda_{\mathfrak{C}}(\beta_1 \times \beta_2) = \Lambda_{\mathfrak{C}}(\alpha_1 \times \beta_2) + \Lambda_{\mathfrak{C}}(\beta_1 \times \alpha_2).$$

This is true by Theorem 5.1.6 since we have $(\alpha_1 \times \beta_2) \wedge (\beta_1 \times \alpha_2) = (\beta_1 \times \beta_2)$.

In the last step, we show that $ev_{[D_1\times D_2]} = ev_{[D_1]}\cdot ev_{[D_2]}$ for arbitrary definable sets D_1, D_2. Let $[D_1] = \sum_{i=1}^k [B_{1i}]$ and $[D_2] = \sum_{j=1}^l [B_{2j}]$ be obtained from any decompositions of D_1 and D_2 into blocks. Then $[D_1 \times D_2] = \sum_{i=1}^k \sum_{j=1}^l [B_{1i} \times B_{2j}])$. For each $\mathfrak{C} \in \overline{\mathcal{X}}^*$, we have

$$\begin{aligned}ev_{[D_1]}\cdot ev_{[D_2]}(\mathfrak{C}) &= \sum_{\mathfrak{A}\star\mathfrak{B}=\mathfrak{C}} \Lambda_{\mathfrak{A}}([D_1])\Lambda_{\mathfrak{B}}([D_2])\\ &= \sum_{\mathfrak{A}\star\mathfrak{B}=\mathfrak{C}} \left(\sum_{i=1}^k \Lambda_{\mathfrak{A}}([B_{1i}])\right)\left(\sum_{j=1}^l \Lambda_{\mathfrak{B}}([B_{2j}])\right)\\ &= \sum_{i=1}^k \sum_{j=1}^l \sum_{\mathfrak{A}\star\mathfrak{B}=\mathfrak{C}} \Lambda_{\mathfrak{A}}([B_{1i}])\Lambda_{\mathfrak{B}}([B_{2j}])\\ &= \sum_{i=1}^k \sum_{j=1}^l \Lambda_{\mathfrak{C}}([B_{1i} \times B_{2j}])\\ &= \Lambda_{\mathfrak{C}}(\sum_{i=1}^k \sum_{j=1}^l [B_{1i} \times B_{2j}])\\ &= ev_{[D_1\times D_2]}(\mathfrak{C}).\end{aligned}$$

This completes the proof showing ev is a semiring homomorphism. □

5.6 Computation of the Grothendieck ring

In the previous section, we showed that $ev : \widetilde{\mathrm{Def}}(M) \to \mathbb{Z}[\overline{\mathcal{X}}^*]$ is a semiring homomorphism. Since the codomain of this map is a ring, it factorizes through the unique homomorphism of cancellative semirings $\widetilde{ev} : \widetilde{\widetilde{\mathrm{Def}}}(M) \to \mathbb{Z}[\overline{\mathcal{X}}^*]$ where $\widetilde{\widetilde{\mathrm{Def}}}(M)$ is the quotient semiring of $\widetilde{\mathrm{Def}}(M)$ obtained as in Theorem 2.1.3. Our next aim is to prove the following lemma.

Lemma 5.6.1. *The map $\widetilde{ev} : \widetilde{\widetilde{\mathrm{Def}}}(M) \to \mathbb{Z}[\overline{\mathcal{X}}^*]$ is injective.*

Proof. We will prove this lemma in several steps. First we will identify a subset of $\overline{\mathrm{Def}}(M)$ where the restriction of the evaluation function is injective.

Let $\mathcal{U} = \{\alpha \in \overline{\mathcal{A}} : A_1 \cap A_2 = \emptyset$ for all distinct $A_1, A_2 \in \alpha\}$. Then it can be easily checked that $\Lambda_{\mathfrak{A}}(\alpha) = |\alpha \cap \mathfrak{A}|$ for each $\mathfrak{A} \in \overline{\mathcal{X}}^*$ and $\alpha \in \mathcal{U}$. Hence if $ev_{[\alpha]} = ev_{[\beta]}$ for any $\alpha, \beta \in \mathcal{U}$, then we have $[\alpha] = [\beta]$. This proves that the map ev is itself injective on $\mathcal{U}$.

Given any $[D_1], [D_2] \in \widetilde{\mathrm{Def}}(M)$ such that $ev_{[D_1]} = ev_{[D_2]}$, we will find some $[X] \in \widetilde{\mathrm{Def}}(M)$ such that $[D_1] + [X] = [\alpha']$ and $[D_2] + [X] = [\beta']$ for some $\alpha', \beta' \in \mathcal{U}$. Then we get $ev_{[\alpha']} = ev_{[D_1]} + ev_{[X]} = ev_{[D_2]} + ev_{[X]} = ev_{[\beta']}$ and hence we will be done by the previous paragraph.

Claim: It is sufficient to assume $[D_1], [D_2] \in [\overline{\mathcal{A}}]$.

Let $[D_1] = \sum_{i=1}^{k}[B_{1i}]$ and $[D_2] = \sum_{j=1}^{l}[B_{2j}]$ be obtained from any decompositions of D_1 and D_2 into blocks. We have $[P(B)] = [B] + [N(B)]$ for any $B \in \overline{\mathcal{B}}$. Therefore if we choose $[Y] = \sum_{i=1}^{k}[N(B_{1i})] + \sum_{j=1}^{l}[N(B_{2j})]$, we get $[D_1] + [Y] = \sum_{i=1}^{l}[P(B_{1i})] + \sum_{j=1}^{l}[N(B_{2j})]$ and $[D_2] + [Y] = \sum_{i=1}^{l}[N(B_{1i})] + \sum_{j=1}^{l}[P(B_{2j})]$. Hence both $[D_1] + [Y], [D_2] + [Y] \in [\overline{\mathcal{A}}]$. This finishes the proof of the claim.

Now let $\alpha, \beta \in \overline{\mathcal{A}}$ be such that $ev_{[\alpha]} = ev_{[\beta]}$. We describe an algorithm which terminates in finitely many steps and yields some $[X]$ such that $[\alpha]+[X], [\beta]+[X] \in [\mathcal{U}]$. Before stating the algorithm, we define a **complexity function** $\Gamma : \overline{\mathcal{A}} \to \mathbb{N}$. For each antichain α, the complexity $\Gamma(\alpha)$ is defined to be the maximum of the lengths of chains in the smallest nest corresponding to α, where the length of a chain is the number of elements in it. Note that $\Gamma(\alpha) \leq 1$ if and only if $\alpha \in \mathcal{U}$.

Let $\alpha = \{A_1, A_2, \ldots, A_k\}$ be any enumeration and let $\alpha_i = \{A_1, A_2, \ldots, A_i\}$ for each $1 \leq i \leq k$. Similarly choosing an enumeration $\beta = \{B_1, B_2, \ldots, B_l\}$, we define β_j for each $1 \leq j \leq l$. Then we observe that $\bigcup \alpha = \bigsqcup_{i=1}^{k} \mathrm{Core}_{\alpha_i}(A_i)$ and $\bigcup \beta = \bigsqcup_{j=1}^{l} \mathrm{Core}_{\beta_j}(B_j)$. Now each $\mathrm{Core}_{\alpha_i}(A_i)$ is a block, which can be completed to a *pp*-set if we take its (disjoint) union with $N(\mathrm{Core}_{\alpha_i}(A_i))$. This can be written as the equation $[A_i] = [\mathrm{Core}_{\alpha_i}(A_i)] + [N(\mathrm{Core}_{\alpha_i}(A_i))]$. If $\bigcup \alpha \subseteq M^n$, we consider M^{nk} and inject $\mathrm{Core}_{\alpha_i}(A_i)$ in the obvious way into the i^{th} copy of M^n in M^{nk} for each i. This gives us a definable set definably isomorphic to $\bigcup \alpha$. The advantage of this decomposition is that we can also add an isomorphic copy of $N(\mathrm{Core}_{\alpha_i}(A_i))$ at the appropriate place for each i and obtain a new antichain representing $\sum_{i=1}^{k}[A_i]$.

Repeating the same procedure for β yields a representative of $\sum_{j=1}^{l}[B_j]$. In order to maintain the evaluation function on both sides, we add disjoint copies of the antichains $N(\mathrm{Core}_{\alpha_i}(A_i))$, $N(\mathrm{Core}_{\beta_j}(B_j))$ to both sides. So we choose $[W] = \sum_{i=1}^{k}[N(\mathrm{Core}_{\alpha_i}(A_i))] + \sum_{j=1}^{l}[N(\mathrm{Core}_{\beta_j}(B_j))]$, hence $[\alpha] + [W], [\beta] + [W]$ are both in

$[\overline{\mathcal{A}}]$ so that the particular antichains α', β' in these classes we constructed above satisfy $\Gamma((\bigcup \alpha') \sqcup (\bigcup \beta')) < \Gamma((\bigcup \alpha) \sqcup (\bigcup \beta))$. The inequality holds since we isolate the maximal elements of the nest corresponding to $(\bigcup \alpha) \cup (\bigcup \beta)$ in the process.

We repeat this process, inducting on the complexity of the antichains, till the disjoint union of the pair of antichains in the output lies in $\mathcal{U}$. Since the complexity decreases at each step, this algorithm terminates in finitely many steps. The required $[X]$ is the sum of the $[W]$'s obtained at each step. This finishes the proof of the injectivity of the map $\widetilde{ev}$. □

Finally we are ready to prove Theorem 5.4.2 regarding the structure of the ring $K_0(M)$.

Proof. (Theorem 5.4.2) It is easy to observe that the image of $\mathcal{U}$ under the evaluation map is the monoid semiring $\mathbb{N}[\overline{\mathcal{X}}^*]$. The Grothendieck ring $K_0(\mathbb{N}[\overline{\mathcal{X}}^*])$ is clearly isomorphic to the monoid ring $\mathbb{Z}[\overline{\mathcal{X}}^*]$.

Since the map $\widetilde{ev}$ is injective by Lemma 5.6.1 and $\mathbb{N}[\overline{\mathcal{X}}^*] \subseteq Im(\widetilde{ev}) \subseteq \mathbb{Z}[\overline{\mathcal{X}}^*]$, we have $K_0(M) = K_0(Im(\widetilde{ev})) \cong \mathbb{Z}[\overline{\mathcal{X}}^*]$ by the universal property of K_0 in Theorem 2.1.4. □

Chapter 6

Grothendieck Rings of Modules: Case $\mathrm{T} \neq \mathrm{T}^{\aleph_0}$

In the last chapter we considered the Grothendieck ring of a right $\mathcal{R}$-module M whose theory $T := Th(M)$ satisfies $T = T^{\aleph_0}$; henceforth we will refer to this case as the special case. In this chapter we remove this condition and work with a fixed arbitrary right $\mathcal{R}$-module M. We continue to use the notations $\mathcal{L}_n, \mathcal{P}_n, \mathcal{A}_n, \mathcal{X}_n$ to denote the set of all *pp*-subsets of M^n, the set of all finite subsets of $\mathcal{L}_n$, the set of all finite antichains in $\mathcal{L}_n$ and the set of all *pp*-isomorphism classes (colours) in $\mathcal{L}_n$ respectively.

Section 6.1 introduces new terminology and the modifications to the proofs of Theorems 5.1.6 and 5.5.1 necessary to handle the general case i.e., $T \neq T^{\aleph_0}$. Theorem 6.2.3 subsumes Theorem 5.4.2 and describes the Grothendieck ring of the module M as the quotient of a monoid ring by the 'invariants ideal' that codes non-trivial indices of *pp*-pairs of subgroups. The maps between modules which fit with model theory are called pure embeddings. We study their relation with the Grothendieck rings in 6.3. We also show the existence of Grothendieck rings containing nontrivial torsion elements in 6.4.

The material in this chapter is contained in [26, §5.1,5.2,6.1,6.2].

6.1 Finite indices of *pp*-pairs

Since $\mathrm{T} \neq \mathrm{T}^{\aleph_0}$, Lemma 4.2.4 is unavailable to obtain the uniqueness result (Proposition 4.2.5) but we can still use the representation theorem (Theorem 4.1.7). As a result we do not have a bijection between the set of all *pp*-convex sets, which we denote by $\mathcal{O}_n$, and the set $\mathcal{A}_n$. The elements of the set $\mathcal{C}_n := \{(\bigcup \alpha) \setminus (\bigcup \beta) | \alpha, \beta \in \mathcal{A}_n,\ \bigcup \beta \subsetneq \bigcup \alpha\}$ will be called cells. The cells allowing a representation of the form $P \setminus \bigcup \beta$ for some $P \in \mathcal{L}_n$ and $\beta \in \mathcal{A}$ such that $P \subsetneq \bigcup \beta$ will be called blocks and the set of all blocks in $\mathcal{C}_n$ is denoted by $\mathcal{B}_n$.

Let $(-)^\circ : \mathcal{L}_n \to \mathcal{L}_n$ denote the function which takes a coset P to the subgroup $P^\circ := P - p$, where $p \in P$ is any element. We use $\mathcal{L}_n^\circ$ to denote the image of this function, i.e., the set of all *pp*-definable subgroups. Let $\sim_n$ denote a relation on $\mathcal{L}_n^\circ$ defined by $P \sim_n Q$ if and only if $[P : P \cap Q] + [Q : P \cap Q] < \infty$. This is the **commensurability relation** and it can be easily checked to be an equivalence relation. We can extend this relation to all elements of $\mathcal{L}_n$ using the same definition if we set the index $[P : Q] := [P^\circ : P^\circ \cap Q^\circ]$ for all $P, Q \in \mathcal{L}_n$. Let $\mathcal{Y}_n$ denote the set of all commensurability equivalence classes of $\mathcal{L}_n$ (**bands** for short). We use capital bold letters $\mathbf{P}, \mathbf{Q}, \cdots$ etc. to denote bands. The equivalence class (band) of P will be denoted by the corresponding bold letter $\mathbf{P}$.

Now we fix some $n \geq 1$ and drop all the subscripts as usual. Note that, in the special case, a band is just the collection of all cosets of a *pp*-subgroup. In particular any two distinct elements of a band are disjoint. This 'discreteness' has been exploited heavily in all the proofs for the special case. We need to work hard to set up the technical machinery for defining the local characteristics in the general case; the proofs will be similar to those for the special case once we obtain the required discreteness condition.

Let $\mathbf{P} \in \mathcal{Y}$. It can be easily checked that if $P, Q \in \mathbf{P}$ and $P \cap Q \neq \emptyset$ then $P \cap Q \in \mathbf{P}$ i.e., $\mathbf{P}$ is closed under intersections which are nonempty. It is also clear that if $P \in \mathbf{P}$ and $a \in M^n$, then $a + P \in \mathbf{P}$. Let $\mathcal{A}(\mathbf{P}), \mathcal{P}(\mathbf{P})$ and $\mathcal{O}(\mathbf{P})$ denote the sets of all finite antichains in $\mathbf{P}$, finite subsets of $\mathbf{P}$ and unions of finite subsets of $\mathbf{P}$ respectively.

We have the following analogue of Proposition 4.2.5 for *pp*-convex sets. The proof is omitted as it is similar to the $\mathrm{T} = \mathrm{T}^{\aleph_0}$ case.

Proposition 6.1.1. *Let $X \in \mathcal{O}$. The set $S(X) := \{\mathbf{P} \in \mathcal{Y} : \exists P \in \mathbf{P}, \alpha \in \mathcal{A}\ (P \in \alpha, \bigcup \alpha = X)\}$ is finite. Furthermore, for $\alpha, \beta \in \mathcal{A}$ and $\mathbf{P} \in S(X)$, if $\bigcup \alpha = \bigcup \beta = X$ then $\bigcup (\alpha \cap \mathbf{P}) = \bigcup (\beta \cap \mathbf{P})$.*

In other words, if $\alpha \in \mathcal{A}$ such that $\bigcup \alpha = X$ and, for each $\mathbf{P} \in S(X)$, the set $X_{\mathbf{P}}$ denotes $\bigcup (\alpha \cap \mathbf{P}) \in \mathcal{O}(\mathbf{P})$, then X determines the family $\{X_{\mathbf{P}} \mid \mathbf{P} \in S(X)\}$ independent of the choice of α.

Given some $X \in \mathcal{O}(\mathbf{P})$ there could be two different $\alpha, \beta \in \mathcal{A}(\mathbf{P})$ such that $\bigcup \alpha = \bigcup \beta = X$. The nests corresponding to such antichains could have entirely different (semilattice) structures. The following proposition gives us a way to obtain an antichain α representing X such that if $A, B \in \alpha$ and $A \neq B$, then $A \cap B = \emptyset$.

Proposition 6.1.2. *Let $X \in \mathcal{O}(\mathbf{P})$. Then for any $\alpha \in \mathcal{A}(\mathbf{P})$ such that $\bigcup \alpha = X$, there is some $\mathbf{P}(\alpha) \in \mathbf{P}^\circ$ such that X is a finite union of distinct cosets of $\mathbf{P}(\alpha)$.*

Proof. Choose $\mathbf{P}(\alpha) = \bigcap\{Q^\circ : Q \in \alpha\}$ and observe that $\mathbf{P}(\alpha) \in \mathbf{P}$ since $\mathbf{P}$ is closed under finite nonempty intersections. □

The previous two propositions together imply that we can always find a 'nice' antichain representing the given *pp*-convex set. The following definition describes what we mean by this.

Definition 6.1.3. *A finite set $\alpha \in \mathcal{P}$ is said to be in* ***discrete form*** *if for each $\mathbf{P} \in \mathcal{Y}$, $\alpha \cap \mathbf{P}$ consists of finitely many cosets of a fixed element of $\mathbf{P}^\circ$, denoted $\mathbf{P}(\alpha)$. The set of all finite sets $\alpha \in \mathcal{P}$ in discrete form will be denoted by $\mathcal{P}^d$ and the set of all antichains in discrete form will be denoted by $\mathcal{A}^d$.*

We would like to define the local characteristics for the elements of $\mathcal{P}^d$ as before and show that they satisfy the conclusion of Theorem 5.1.6. We will restrict our attention only to those $\alpha \in \mathcal{P}^d$ such that $\alpha = \hat{\alpha}$ (i.e., the nest corresponding to α is α itself). We denote the set of all such finite sets by $\hat{\mathcal{P}}^d$. Since we will deal with finite index subgroup pairs in $\mathcal{L}^\circ$, we will need more conditions on compatibility of P and α as stated in the following definition.

Definition 6.1.4. *A finite family $\mathcal{F}$ of elements of $\mathcal{P}$ is called* ***compatible*** *if $\mathcal{F} \subseteq \hat{\mathcal{P}}^d$ and for all $\alpha, \beta \in \mathcal{F}$ and $\mathbf{P} \in \mathcal{Y}$, we have $\mathbf{P}(\alpha) = \mathbf{P}(\beta)$ whenever $\mathbf{P} \cap \alpha, \mathbf{P} \cap \beta \neq \emptyset$. Furthermore, we say that $P \in \mathcal{L}$ is* ***compatible with*** *a finite family $\mathcal{F}$ of elements of $\mathcal{P}$ if $\mathcal{F}$ is compatible and $P \in \bigcup \mathcal{F}$.*

It is easy to observe that given any finite family $\{X_1, X_2, \dots, X_k\}$ of *pp*-convex sets, we can obtain a compatible family $\{\alpha_1, \alpha_2, \dots, \alpha_k\}$ of antichains such that $\bigcup \alpha_i = X_i$ for each i. Finally we are ready to define the local characteristics in this set-up.

Definition 6.1.5. *Let $P \in \mathcal{L}$ be compatible with a family $\mathcal{F}$ and let $\alpha \in \mathcal{F}$. We associate an abstract simplicial complex $\mathcal{K}^P(\alpha)$ with the pair (α, P) by setting $\mathcal{K}^P(\alpha) := \{\beta \subseteq \alpha : \beta \neq \emptyset, \bigcap \beta \supsetneq P\}$. We define the* ***local characteristic*** *κ_P by the formula $\kappa_P(\alpha) := \chi(\mathcal{K}^P(\alpha)) - \delta(\alpha)(P)$.*

Now we are ready to state the analogue of Theorem 5.1.6 and it has essentially the same proof. The previous statement is justified because we have carefully developed the idea of a compatible family to avoid finite index pairs of *pp*-subgroups. Since we achieve discreteness simultaneously for any finite family of antichains, no changes in the proof of Theorem 5.1.6 are necessary.

Theorem 6.1.6. *Let $X, Y \in \mathcal{O}$. Then $X \cup Y, X \cap Y \in \mathcal{O}$. For any compatible family $\mathcal{F} := \{\alpha_1, \alpha_2, \beta_1, \beta_2\}$ such that $\bigcup \alpha_1 = X$, $\bigcup \alpha_2 = Y$, $\bigcup \beta_1 = X \cup Y$ and $\bigcup \beta_2 = X \cap Y$ and any $P \in \mathcal{L}$ compatible with $\mathcal{F}$, we have*

$$\kappa_P(\alpha_1) + \kappa_P(\alpha_2) = \kappa_P(\beta_1) + \kappa_P(\beta_2).$$

We observe that the set $\overline{\mathcal{A}^d}$ is closed under cartesian products and thus we have the following analogue of Lemma 5.5.1 with the same proof.

Lemma 6.1.7. *Let $P, Q \in \overline{\mathcal{L}}$ be compatible with $\{\alpha, \beta\} \subseteq \overline{\mathcal{A}^d}$. Then*

$$\kappa_{P\times Q}(\alpha \times \beta) = -\kappa_P(\alpha)\kappa_Q(\beta).$$

6.2 The invariants ideal

Once again, we use the notations $\overline{\mathcal{L}}, \overline{\mathcal{X}}$ to denote the unions $\bigcup_{n=1}^{\infty} \mathcal{L}_n$, $\bigcup_{n=1}^{\infty} \mathcal{X}_n$ and set $\overline{\mathcal{L}}^* = \overline{\mathcal{L}} \setminus \{\emptyset\}, \overline{\mathcal{X}}^* = \overline{\mathcal{X}} \setminus \{[[\emptyset]]\}$ where $[[-]] : \overline{\mathcal{L}} \to \overline{\mathcal{X}}$ is the map taking a *pp*-set to its colour. Now $\overline{\mathcal{X}}^*$ is a multiplicative monoid and we consider the monoid ring $\mathbb{Z}[\overline{\mathcal{X}}^*]$.

In the case when $\mathrm{T} \neq \mathrm{T}^{\aleph_0}$, there are $P, Q \in \mathcal{L}_n$ such that $1 < \mathrm{Inv}(M; P, Q) < \infty$ for each $n \geq 1$. We can assume without loss that $0 \in Q \subseteq P$. Now we define an ideal of the monoid ring, called the invariants ideal, which encodes these invariants. The following proposition is the motivation.

Proposition 6.2.1. *Let $\mathbf{P} \in \mathcal{Y}_n$ and $X \in \mathcal{O}(\mathbf{P})$. For any $\alpha, \beta \in \mathcal{A}_n^d$ with $\bigcup \alpha = \bigcup \beta = X$, we have*

$$[\mathbf{P}(\alpha) : \mathbf{P}(\beta)]|\alpha \cap \mathbf{P}| = [\mathbf{P}(\beta) : \mathbf{P}(\alpha)]|\beta \cap \mathbf{P}|$$

Proof. Partition those cosets of both $\mathbf{P}(\alpha)$ and of $\mathbf{P}(\beta)$ which are contained in X into cosets of $\mathbf{P}(\alpha) \cap \mathbf{P}(\beta)$ to get the required equality. □

Definition 6.2.2. *Let $\delta_{\mathfrak{A}} : \overline{\mathcal{X}}^* \to \mathbb{Z}$ denote the characteristic function of the colour $\mathfrak{A}$ for each $\mathfrak{A} \in \overline{\mathcal{X}}^*$. We define* ***the invariants ideal*** *$\mathcal{J}$ of the monoid ring $\mathbb{Z}[\overline{\mathcal{X}}^*]$ to be the ideal generated by the set*

$$\{\delta_{[[P]]} = [P : Q]\delta_{[[Q]]} : P, Q \in \overline{\mathcal{L}},\ P \supseteq Q,\ \mathrm{Inv}(M; P, Q) < \infty\}.$$

The main aim of this section is to prove the following theorem.

Theorem 6.2.3. *For every right $\mathcal{R}$-module M, we have*

$$K_0(M) \cong \mathbb{Z}[\overline{\mathcal{X}}^*]/\mathcal{J}.$$

Recall that we have proved this theorem when $\mathrm{T} = \mathrm{T}^{\aleph_0}$ since the invariants ideal is trivial in that case. As a corollary of the theorem we can give a proof that the Grothendieck ring of a module is an invariant of its theory.

Proof. (Proposition 4.1.1) Elementarily equivalent modules have isomorphic lattices of *pp*-sets and they also satisfy the same invariant conditions (see [33, Cor. 2.18]). Hence Theorem 6.2.3 yields the result. □

Let $\overline{\mathcal{Y}} = \bigcup_{n=1}^{\infty} \mathcal{Y}_n$. Given $\mathfrak{A} \in \overline{\mathcal{X}}^*$, we define $\mathcal{Y}(\mathfrak{A}) := \{\mathbf{P} \in \overline{\mathcal{Y}} : \mathbf{P} \cap \mathfrak{A} \neq \emptyset\}$. In order to define the global characteristics in this case, we need to find the set over which they vary. Let $\mathfrak{A}, \mathfrak{B} \in \overline{\mathcal{X}}^*$. We say that $\mathfrak{A} \approx \mathfrak{B}$ if $\mathcal{Y}(\mathfrak{A}) \cap \mathcal{Y}(\mathfrak{B}) \neq \emptyset$. This relation is reflexive and symmetric. We use $\approx$ again to denote its transitive closure. The $\approx$-equivalence class of $\mathfrak{A}$ will be denoted by $\widetilde{\mathfrak{A}}$.

Definition 6.2.4. *Let $\mathfrak{A} \in \overline{\mathcal{X}}^*$. Define the* ***colour class group*** *$\mathcal{R}(\widetilde{\mathfrak{A}})$ as the quotient of the free abelian group $\mathbb{Z}\langle \delta_{\mathfrak{A}} : \mathfrak{A} \in \widetilde{\mathfrak{A}} \rangle$ by the subgroup $\mathcal{J}(\widetilde{\mathfrak{A}})$ generated by the relations $\{\delta_{[[P]]} = [P : Q]\delta_{[[Q]]} : P, Q \in \bigcup \widetilde{\mathfrak{A}},\ P \supseteq Q\}$.*

It can be observed that the underlying abelian group of the monoid ring $\mathbb{Z}[\overline{\mathcal{X}}^*]$ is formed by taking the quotient of the direct sum of the free abelian groups $\mathbb{Z}\langle \delta_{\mathfrak{A}} : \mathfrak{A} \in \widetilde{\mathfrak{A}} \rangle$, one for each equivalence class of colours, by the multiplicative relations of the monoid $\overline{\mathcal{X}}^*$. Furthermore, the set $\bigcup \{\mathcal{J}(\widetilde{\mathfrak{A}}) : \widetilde{\mathfrak{A}} \in \overline{\mathcal{X}}^*\}$ generates the ideal $\mathcal{J}$ in this ring.

The discussion in the previous paragraph suggests to us to isolate the information in the evaluation map into different global characteristics, one for each colour class. These maps take values in the corresponding colour class group. We define the **global characteristic** $\Lambda_{\widetilde{\mathfrak{A}}}$ corresponding to $\widetilde{\mathfrak{A}}$ as the function $\overline{\hat{\mathcal{P}}^d} \to \mathcal{R}(\widetilde{\mathfrak{A}})$ given by $\alpha \mapsto -\sum_{\mathfrak{A} \in \widetilde{\mathfrak{A}}} \left(\sum_{P \in \mathfrak{A}} \kappa_P(\alpha) \right) \delta_{\mathfrak{A}}$.

The following result is an easy corollary of Proposition 6.2.1. It states that the global characteristics depend only on the *pp*-convex sets and not on their representations as antichains.

Corollary 6.2.5. *Let $X \in \overline{\mathcal{O}}$ and $\alpha, \beta \in \overline{\hat{\mathcal{P}}^d}$ be such that $\bigcup \alpha = \bigcup \beta = X$. Then $\Lambda_{\widetilde{\mathfrak{A}}}(\alpha) = \Lambda_{\widetilde{\mathfrak{A}}}(\beta)$ for each $\mathfrak{A} \in \overline{\mathcal{X}}^*$.*

This finishes the technical setup for the general case when the theory T of the module M does not necessarily satisfy $T = T^{\aleph_0}$. The antichains in discrete form behave as if the theory satisfies $T = T^{\aleph_0}$, the bands allow us to go down (via intersections) so that any finite family can be converted to a compatible family and the notion of compatibility allows us to do appropriate local analysis. The local data can be pasted together using the information coded in the colour class groups.

Now we give some important definitions and state results from the special case $\mathrm{T} = \mathrm{T}^{\aleph_0}$ in a form compatible with the general case. The proofs of these results are omitted since they are similar to their special counterparts; the basic ingredients are provided by Lemma 4.2.4, Theorem 6.1.6, Lemma 6.1.7 and Corollary 6.2.5. The necessary change is to deal only with antichains which are in discrete form.

Since cells are the difference sets of two *pp*-convex sets, we can obtain a compatible family $\{\alpha, \beta\}$ for any $C \in \overline{C}$ such that $C = \bigcup \alpha \setminus \bigcup \beta$.

Definition 6.2.6. *Let $C \in \overline{\mathcal{C}}$ and $\mathfrak{A}$ be a colour. We define the global characteristic $\Lambda_{\widetilde{\mathfrak{A}}}(C) := \Lambda_{\widetilde{\mathfrak{A}}}(\alpha) - \Lambda_{\widetilde{\mathfrak{A}}}(\beta) \in \mathcal{R}(\widetilde{\mathfrak{A}})$ for any compatible family $\{\alpha, \beta\}$ representing C.*

The following theorem is the analogue of Theorem 5.2.4 and uses the inductive version of 6.1.6 in its proof.

Theorem 6.2.7. *If $\{B_i : 1 \leq i \leq l\}, \{B'_j : 1 \leq j \leq m\}$ are two finite families of pairwise disjoint blocks such that $\bigsqcup_{i=1}^{l} B_i = \bigsqcup_{j=1}^{m} B'_j$, then $\Sigma_{i=1}^{l} \Lambda_{\widetilde{\mathfrak{A}}}(B_i) = \Sigma_{j=1}^{m} \Lambda_{\widetilde{\mathfrak{A}}}(B'_j)$ for every $\mathfrak{A} \in \overline{\mathcal{X}}^*$.*

This theorem allows us to extend the definition of global characteristics to all sets in $\overline{\mathrm{Def}}(M)$. Moreover the following theorem, the proof of which is an easy adaptation of that of Theorem 5.2.8, states that each of them is preserved under definable bijections.

Theorem 6.2.8. *Suppose $D \in \mathrm{Def}(M^n)$ and $f : D \to M^n$ is a definable injection. Then $\Lambda_{\widetilde{\mathfrak{A}}}(D) = \Lambda_{\widetilde{\mathfrak{A}}}(f(D))$ for each colour class $\widetilde{\mathfrak{A}}$.*

Let $ev : \overline{\mathrm{Def}}(M) \to \mathbb{Z}[\overline{\mathcal{X}}^*]/\mathcal{J}$ be the map defined by $D \mapsto \sum\{\Lambda_{\widetilde{\mathfrak{A}}}(D) : \widetilde{\mathfrak{A}} \in (\overline{\mathcal{X}}^*/\approx)\}$. This map is well defined since the sum is finite for every D for reasons similar to those for the special case. Furthermore $ev_{D_1} = ev_{D_2}$ whenever D_1 and D_2 are definably isomorphic since $\Lambda_{\widetilde{\mathfrak{A}}}(D_1) = \Lambda_{\widetilde{\mathfrak{A}}}(D_2)$ for each colour class $\widetilde{\mathfrak{A}}$. In fact ev is a semiring homomorphism. The proof of the following theorem is analogous to that of Theorem 5.5.2.

Theorem 6.2.9. *The map $ev : \widetilde{\mathrm{Def}}(M) \to \mathbb{Z}[\overline{\mathcal{X}}^*]/\mathcal{J}$ defined by $[D] \mapsto ev_{[D]}$ is a semiring homomorphism.*

The final step in the proof of 6.2.3 is the following analogue of Lemma 5.6.1.

Lemma 6.2.10. *The map $\widetilde{ev} : \widetilde{\widetilde{\mathrm{Def}}(M)} \to \mathbb{Z}[\overline{\mathcal{X}}^*]/\mathcal{J}$ is injective.*

Proof. The proof of this lemma needs some modification of the first paragraph of the proof of Lemma 5.6.1 in order to incorporate the invariants ideal. Let $\mathcal{U} := \{\alpha \in \overline{\mathcal{A}^d} : A_1 \cap A_2 = \emptyset$ for all distinct $A_1, A_2 \in \alpha\}$.

If $ev_{[\alpha]} = ev_{[\beta]}$ for some $\alpha, \beta \in \mathcal{U}$, then we can obtain antichains $\alpha' \in [\alpha] \cap \mathcal{U}$, $\beta' \in [\beta] \cap \mathcal{U}$ such that $\bigcup \alpha = \bigcup \alpha', \bigcup \beta = \bigcup \beta'$ and $\{\alpha', \beta'\}$ is compatible. Hence we have $\Lambda_{\widetilde{\mathfrak{A}}}(\alpha) = \Lambda_{\widetilde{\mathfrak{A}}}(\alpha')$, $\Lambda_{\widetilde{\mathfrak{A}}}(\beta) = \Lambda_{\widetilde{\mathfrak{A}}}(\beta')$ for each colour class $\widetilde{\mathfrak{A}}$. Observe that the equalities, if considered in the codomain ring, are modulo the invariants ideal. Now $\Lambda_{\widetilde{\mathfrak{A}}}(\alpha') = |\alpha' \cap (\bigcup \widetilde{\mathfrak{A}})| \delta_{[[\mathbf{P}(\alpha')]]}$, where $\mathbf{P}$ is the only band (if exists) such that $\mathbf{P} \cap \alpha' \cap (\bigcup \widetilde{\mathfrak{A}}) \neq \emptyset$. Since $\mathbf{P}(\alpha') = \mathbf{P}(\beta')$ for each such colour class by the definition of compatibility, we get $|\alpha \cap (\bigcup \widetilde{\mathfrak{A}})| = |\beta \cap (\bigcup \widetilde{\mathfrak{A}})|$ for each colour class $\widetilde{\mathfrak{A}}$.

A definable isomorphism can be easily constructed between the *pp*-convex sets represented by α' and β', which are the sets represented by α and β respectively. The rest of the proof is similar to the proof of 5.6.1. □

Proof. (Theorem 6.2.3) We have shown that the map $\widetilde{ev}$ is injective in the above lemma. Then we observe that the sets of the form $\bigcup \alpha$ for some $\alpha \in \mathcal{U}$ are capable of producing every element of the quotient ring $\mathbb{Z}[\overline{\mathcal{X}}^*]/\mathcal{J}$ of the form $\sum n_{\mathfrak{A}}\delta_{\mathfrak{A}} + \mathcal{J}$, where the nonzero coefficients are positive. This completes the proof by an argument similar to the proof of Theorem 5.4.2. □

Since the Grothendieck ring is a quotient ring, we do not necessarily know if it is nontrivial. But the following corollary of Theorem 6.2.3 shows this result, proving Prest's conjecture in full generality.

Corollary 6.2.11. *If M is a nonzero right $\mathcal{R}$-module, then there is a split embedding $\mathbb{Z} \rightarrowtail K_0(M)$.*

Proof. Consider the colour class $\widetilde{\mathfrak{U}}$, where $\mathfrak{U}$ is the identity element of the monoid $\overline{\mathcal{X}}^*$. A *pp*-set P is an element of $\bigcup \widetilde{\mathfrak{U}}$ if and only if P is finite. Finite sets enjoy the special property that two finite sets are isomorphic to each other if and only their cardinalities are equal. Furthermore, every such isomorphism is definable. In particular, $\mathcal{R}(\widetilde{\mathfrak{U}}) \cong \mathbb{Z}$ if M is a nonzero module. Next we observe that the set $\bigcup \widetilde{\mathfrak{U}}$ is closed under multiplication and hence the colour class group $\mathcal{R}(\widetilde{\mathfrak{U}})$ can be given the structure of a quotient of the monoid ring $\mathbb{Z}[\bigcup \widetilde{\mathfrak{U}}]$ with certain relations, where the multiplicative relations of the monoid ring are finitary and hence already present in the relations for $\mathcal{R}(\widetilde{\mathfrak{U}})$. We have thus described the ring structure of $\mathcal{R}(\widetilde{\mathfrak{U}})$ and this ring is naturally a subring of $K_0(M)$.

To complete the proof, we show that the map $\pi_0 : K_0(M) \to \mathcal{R}(\widetilde{\mathfrak{U}})$ given by $\sum_{\widetilde{\mathfrak{A}} \in (\overline{\mathcal{X}}^*/\approx)} n_{\widetilde{\mathfrak{A}}}\delta_{\widetilde{\mathfrak{A}}} \mapsto n_{\widetilde{\mathfrak{U}}}\delta_{\widetilde{\mathfrak{U}}}$ is a surjective ring homomorphism.

The map π_0 is clearly an additive group homomorphism. Note that the multiplicative monoid $\bigcup \widetilde{\mathfrak{U}}$ is a sub-monoid of $\overline{\mathcal{X}}^*$. Also note that $\mathcal{J}(\widetilde{\mathfrak{A}}) \cap \mathcal{J}(\widetilde{\mathfrak{B}}) = \emptyset$ if $\widetilde{\mathfrak{A}} \neq \widetilde{\mathfrak{B}}$. Furthermore, $\mathfrak{A} \star \mathfrak{B} \in \bigcup \widetilde{\mathfrak{U}}$ if and only if $\mathfrak{A}, \mathfrak{B} \in \bigcup \widetilde{\mathfrak{U}}$. Thus the coefficient of $\delta_{\widetilde{\mathfrak{U}}}$ in the product of two elements of $K_0(M)$ is determined by the coefficient of $\delta_{\widetilde{\mathfrak{U}}}$ of the individual elements. Hence π_0 is also multiplicative. The surjectivity is clear. This completes the proof. □

6.3 Pure embeddings and Grothendieck rings

We will investigate some categorical properties of the Grothendieck rings of modules in this section. The main aim is to prove the following theorem.

Theorem 6.3.1. *Let $i : N \to M$ be a pure embedding of right $\mathcal{R}$-modules such that the theory of M satisfies $Th(M) = Th(M)^{\aleph_0}$. Then i induces a surjective ring homomorphism $I : K_0(M) \twoheadrightarrow K_0(N)$.*

This theorem will be proved using a series of results of functorial nature. We begin with the definition of a pure embedding.

Definition 6.3.2. *Let M be a right $\mathcal{R}$-module. A submodule $N \leq M$ is called a* ***pure submodule*** *if, for each n, $A \cap N^n \in \mathcal{L}_n^\circ(N)$ for every $A \in \mathcal{L}_n^\circ(M)$.*
A monomorphism $i : N \to M$ is said to be a ***pure monomorphism*** *if iN is a pure submodule of M.*

The following lemma states that a pure embedding induces a map of semilattices of *pp*-formulas.

Lemma 6.3.3 (see [34, Lemma 3.2.2])**.** *If $i : N \to M$ is a pure embedding then, for each n, the natural map $\bar{i} : \mathcal{L}_n^\circ(M) \to \mathcal{L}_n^\circ(N)$ given by $\bar{i}(A) = A \cap N^n$ is surjective morphism of semilattices.*

Now we state the following result about integral monoid rings.

Proposition 6.3.4 (see [19, II, Proposition 3.1])**.** *Let $\Phi : A \to B$ be a homomorphism of monoids. Then there exists a unique homomorphism $h : \mathbb{Z}[A] \to \mathbb{Z}[B]$ such that $h(x) = \Phi(x)$ for all $x \in A$ and $h(1) = 1$. Furthermore, h is surjective if Φ is so.*

Corollary 6.3.5. *A pure embedding $i : N \to M$ induces a surjective homomorphism $\mathfrak{i} : \mathbb{Z}[\overline{\mathcal{X}}^*(M)] \twoheadrightarrow \mathbb{Z}[\overline{\mathcal{X}}^*(N)]$ of rings.*

Proof. Observe that every colour $\mathfrak{A} \in \overline{\mathcal{X}}^*$ has a representative in $\overline{\mathcal{L}}^\circ := \bigcup_{n=1}^\infty \mathcal{L}_n^\circ$. Thus we get an induced surjective homomorphism $\overline{\mathcal{X}}^*(M) \twoheadrightarrow \overline{\mathcal{X}}^*(N)$ of the colour monoids using Lemma 6.3.3. Then Proposition 6.3.4 yields the required surjective map of the integral monoid rings. □

Proof. (Theorem 6.3.1) Observe that since $Th(M) = Th(M)^{\aleph_0}$ holds, Theorem 5.4.2 gives $K_0(M) \cong \mathbb{Z}[\overline{\mathcal{X}}^*(M)]$. By Theorem 6.2.3, we have $K_0(N) \cong \mathbb{Z}[\overline{\mathcal{X}}^*(N)]/\mathcal{J}(N)$. Let $\pi : \mathbb{Z}[\overline{\mathcal{X}}^*(N)] \twoheadrightarrow K_0(N)$ denote the natural quotient map. Take $I = \pi \circ \mathfrak{i}$, where $\mathfrak{i}$ is the map from the previous corollary, to finish the proof. □

We will see an example at the end of the next section to see that Theorem 6.3.1 fails if $Th(M) \neq Th(M)^{\aleph_0}$.

Recall that the notation $M^{(\aleph_0)}$ denotes the direct sum of countably many copies of a module M. It follows immediately from [33, Lemma 2.23(c)] that the lattices $\mathcal{L}_1(M)$ and $\mathcal{L}_1(M^{(\aleph_0)})$ are isomorphic and $T := Th(M^{(\aleph_0)})$ satisfies $T = T^{\aleph_0}$. We summarize these observations in the following corollary of Theorem 6.3.1.

Corollary 6.3.6. *Let $i_n : M \to M^{(\aleph_0)}$ denote the natural embedding of M onto the n^{th} component of $M^{(\aleph_0)}$. Then i_n induces the natural quotient map $K_0(M^{(\aleph_0)}) = \mathbb{Z}[\overline{\mathcal{X}}^*(M)] \twoheadrightarrow \mathbb{Z}[\overline{\mathcal{X}}^*(M)]/\mathcal{J}(M) = K_0(M)$.*

For a ring $\mathcal{R}$, let Mod-$\mathcal{R}$ denote the category of right $\mathcal{R}$-modules. The theory $Th(\text{Mod-}\mathcal{R})$ is not a complete theory. But we may take a canonical complete theory extending it as follows. Recall that Grothendieck rings of elementarily equivalent modules are isomorphic by Proposition 4.1.1. Equivalently, $K_0(M)$ is determined by $Th(M)$ which, in turn, is determined by its invariants conditions (Theorem 6.2.3).

Definition 6.3.7. *Let P be a direct sum of one model of each complete theory of right $\mathcal{R}$-modules. Then $T^* = Th(P)$ is referred to as* ***the largest complete theory of right $\mathcal{R}$-modules****.*

Thus every right $\mathcal{R}$-module is elementarily equivalent to a direct summand of some model of $Th(P)$. Now we note the following result without proof and define the Grothendieck ring of the module category.

Definition and Lemma 6.3.8 (see [31, 6.1.1, 6.1.2])**.** *Let T^* denote the largest complete theory of right $\mathcal{R}$-modules. Then $T^* = (T^*)^{\aleph_0}$. Furthermore if P_1 and P_2 are both direct sums of one model of each complete theory of right $\mathcal{R}$-modules, then $K_0(P_1) \cong K_0(P_2)$. We define the* ***Grothendieck ring of the module category****, denoted $K_0(\text{Mod-}\mathcal{R})$, to be the Grothendieck ring of the largest complete theory of right $\mathcal{R}$-modules.*

As a consequence of Theorem 6.3.1, we state a result connecting Grothendieck rings of individual modules with that of the module category.

Corollary 6.3.9. *Let $M \in \text{Mod-}\mathcal{R}$. Then $K_0(M)$ is a quotient of $K_0(\text{Mod-}\mathcal{R})$.*

Proof. Let T^* be the largest complete theory of right $\mathcal{R}$-modules. Then Lemma 6.3.8 gives that, for any $P \models T^*$, $Th(P) = T^*$ satisfies $T^* = (T^*)^{\aleph_0}$ and we also have $K_0(P) \cong K_0(\text{Mod-}\mathcal{R})$.

By the definition of T^*, there is a module M' elementarily equivalent to M such that M' is a direct summand of P. Since the embedding $M' \rightarrowtail P$ is pure, we get a surjective homomorphism $K_0(P) \twoheadrightarrow K_0(M')$. Thus the required quotient map is the composite $K_0(\text{Mod-}\mathcal{R}) \cong K_0(P) \twoheadrightarrow K_0(M') \cong K_0(M)$, where the last isomorphism is obtained from Proposition 4.1.1. □

6.4 Torsion in Grothendieck rings

As an application of the structure theorem for Grothendieck rings, Theorem 6.2.3, we provide an example of a module whose Grothendieck ring contains a nonzero torsion element (i.e., a nonzero element a such that $na = 0$ for some $n \geq 1$). We also calculate the Grothendieck ring $K_0(\mathbb{Z}_{\mathbb{Z}})$.

Definition 6.4.1. *The **ring of p-adic integers**, denoted $\widehat{\mathbb{Z}_p}$, is the inverse limit of the system $\ldots \twoheadrightarrow \mathbb{Z}_{p^n} \twoheadrightarrow \ldots \twoheadrightarrow \mathbb{Z}_{p^2} \twoheadrightarrow \mathbb{Z}_p \twoheadrightarrow 0$.*

The ring $\widehat{\mathbb{Z}_p}$ is a commutative local PID with the ideal structure given by

$$\widehat{\mathbb{Z}_p} \supsetneq p\widehat{\mathbb{Z}_p} \supsetneq \ldots \supsetneq p^n\widehat{\mathbb{Z}_p} \supsetneq \ldots \supsetneq 0.$$

In particular, $\widehat{\mathbb{Z}_p}$ is a commutative noetherian ring and hence satisfies the hypothesis of the following proposition.

Proposition 6.4.2 (see [33, p.19, Ex. 2(ii)])**.** *If $\mathcal{R}$ is a commutative noetherian ring then the pp-definable subgroups of the module $\mathcal{R}_{\mathcal{R}}$ are precisely the finitely generated ideals of $\mathcal{R}$.*

It can be observed that the maps $t_n : \widehat{\mathbb{Z}_p} \to p^n\widehat{\mathbb{Z}_p}$ which are 'multiplication by p^n' are *pp*-definable isomorphisms for each $n \geq 1$. Thus a simple computation shows that the monoid of colours, $\overline{\mathcal{X}}^*(\widehat{\mathbb{Z}_p})$, is isomorphic to the monoid $\mathbb{N}$.

If X denotes the class of $\widehat{\mathbb{Z}_p}$ in $K_0(\widehat{\mathbb{Z}_p})$, then the invariants ideal $\mathcal{J}(\widehat{\mathbb{Z}_p})$ is generated by the relations $\{X = p^n X : n \geq 1\}$. The relation $(p^n - 1)X = 0$ is an integral multiple of the relation $(p-1)X = 0$ for each $n \geq 1$. Thus $\mathcal{J}(\widehat{\mathbb{Z}_p})$ is principal and generated by the single relation $(p-1)X = 0$. We summarize this discussion as the following corollary to Theorem 6.2.3.

Corollary 6.4.3. *Let $\widehat{\mathbb{Z}_p}$ denote the ring p-adic integers. Then*

$$K_0(\widehat{\mathbb{Z}_p}) \cong \mathbb{Z}[X]/\langle (p-1)X \rangle.$$

Consider the split (hence pure) embedding $i : \widehat{\mathbb{Z}_p}^{(2)} \rightarrowtail \widehat{\mathbb{Z}_p}^{(3)}$ of $\widehat{\mathbb{Z}_p}$-modules given by $(a, b) \mapsto (a, b, 0)$, where $M^{(k)}$ denotes the direct sum of k copies of M. We want to show that this embedding witnesses the failure of Theorem 6.3.1 since the theory $T := Th(\widehat{\mathbb{Z}_p}^{(3)})$ of the target module doesn't satisfy the condition $T = T^{\aleph_0}$. The following proposition is helpful for the calculation of Grothendieck rings.

Proposition 6.4.4 (see [33, Lemma 2.23])**.** *If $\phi(x)$ and $\psi(x)$ denote pp-formulas, then the following hold.*

1. $\phi(M \oplus N) = \phi(M) \oplus \phi(N)$.

2. $\mathrm{Inv}(M \oplus N; \phi, \psi) = \mathrm{Inv}(M; \phi, \psi)\mathrm{Inv}(N; \phi, \psi)$.

It is clear that the induced map $\mathfrak{i} : \mathbb{Z}[\overline{\mathcal{X}}^*(\widehat{\mathbb{Z}_p}^{(3)})] \to \mathbb{Z}[\overline{\mathcal{X}}^*(\widehat{\mathbb{Z}_p}^{(2)})]$ is the identity map on $\mathbb{Z}[X]$ since $\mathbb{Z}[\overline{\mathcal{X}}^*(\widehat{\mathbb{Z}_p}^{(k)})] \cong K_0(\widehat{\mathbb{Z}_p}^{(\aleph_0)}) \cong \mathbb{Z}[X]$ for each $k \geq 1$. Further the previous proposition shows that $\mathcal{J}(\widehat{\mathbb{Z}_p}^{(k)}) = \langle (p^k - 1)X \rangle$ for each $k \geq 1$. Since $\mathcal{J}(\widehat{\mathbb{Z}_p}^{(3)}) \nsubseteq \mathcal{J}(\widehat{\mathbb{Z}_p}^{(2)})$, there is no surjective map $K_0(\widehat{\mathbb{Z}_p}^{(3)}) \twoheadrightarrow K_0(\widehat{\mathbb{Z}_p}^{(2)})$.

The abelian group of integers: Since the ring $\mathbb{Z}$ is a commutative PID, the *pp*-definable subgroups of the module $\mathbb{Z}_\mathbb{Z}$ are precisely the ideals $n\mathbb{Z}$ for $n \geq 0$. Thus the monoid $\overline{\mathcal{X}}^*(\mathbb{Z})$ is isomorphic to $\mathbb{N}$. Furthermore if X denotes the class of $\mathbb{Z}$ in $K_0(\mathbb{Z})$, the invariants ideal is generated by the relations $X = nX$ for each $n \geq 1$. This forces $\mathcal{J}(\mathbb{Z}) = \langle X \rangle$ and thus $K_0(\mathbb{Z}_\mathbb{Z}) \cong \mathbb{Z}$.

Chapter 7

The Grothendieck Ring of Varieties

The category Var_k of (not necessarily irreducible) algebraic varieties over a field k carries a natural symmetric monoidal structure of disjoint union (coproduct) of varieties. The Grothendieck ring of varieties is historically presented in terms of generators and relations, but in case when k is algebraically closed, this presentation produces the same ring as the Grothendieck ring construction of Section 2.1 applied to the semiring of cut-and-paste (piecewise) isomorphism classes of Var_k (Section 7.2).

The aim of this chapter is to show, in case of an algebraically closed field, the equivalence of two statements (Theorem 7.3.3) regarding different equivalences of varieties (Question 7.1.2 and Conjecture 7.1.4) stated in Section 7.1. Under the hypothesis of Conjecture 7.1.4 we obtain particularly nice results about the Grothendieck ring, namely that the Grothendieck group of varieties is a free abelian group (Section 7.4) and that the associated graded ring of the Grothendieck ring of varieties with respect to the dimension grading is a monoid ring (Section 7.5).

The material in this chapter is contained in [27].

7.1 Questions under consideration

Let k be a field. A variety over k is a reduced separated scheme of finite type. A subvariety of a variety X is said to be locally closed if it can be written as the intersection of an open subvariety with a closed subvariety. Let Var_k denote the category of k-varieties and rational morphisms. Disjoint union and reduced product of varieties are respectively the coproduct and product in this category. Thus $(\mathrm{Var}_k, \sqcup, \emptyset, \times)$ is a (skeletally small) symmetric monoidal category with pairing. We present below the classical definition of the Grothendieck ring of varieties and defer the proof of the fact that, whenever k is algebraically closed, then this definition agrees with the definition of $K_0(\mathrm{Var}_k)$ in Section 3.2 until next section.

The Grothendieck group $K_0^+(\mathrm{Var}_k)$ is the quotient of the free abelian group generated by the isomorphism classes of k-varieties by the following relations.

$$[X] - [Y] = [X \setminus Y] \text{ whenever } Y \subseteq X \text{ is a closed subvariety.} \tag{7.1}$$

It can be given a ring structure by taking the reduced product $(X \times_{\mathrm{Spec}\, k} Y)_{red}$ of varieties. Recall that if the field k is algebraically closed, then we can simply talk about the product $X \times_{\mathrm{Spec}\, k} Y$. We denote the Grothendieck ring of varieties by $K_0(\mathrm{Var}_k)$.

Recall that if R is a commutative ring, then an R-valued motivic measure is a ring homomorphism $K_0(\mathrm{Var}_k) \to R$. The Grothendieck ring plays an important role in the theory of motivic integration being the value ring of the universal motivic measure on k-varieties. But very little is known about this ring. Poonen [32, Theorem 1] and Kollár [24, Ex. 6] show that this ring is not a domain when k has characteristic 0.

Characterizing equality in the Grothendieck ring is an important issue. In order to state this problem precisely, we need the notion of 'cut-and-paste equivalence' of varieties.

Definition 7.1.1. *Two varieties X and Y are said to be* ***piecewise isomorphic****, written $X \doteqdot Y$, if there are partitions $X = \bigsqcup_{i\in[n]} X_i$ and $Y = \bigsqcup_{j\in[n]} Y_j$ of X and Y into locally closed subvarieties such that there is a permutation σ of $[n]$ with X_i isomorphic to $Y_{\sigma(i)}$ as a variety.*

If $X \doteqdot Y$, then clearly $[X] = [Y]$ in $K_0(\mathrm{Var}_k)$. Larsen and Lunts asked whether the converse is true.

Question 7.1.2 ([29, Question 1.2])**.** Suppose X and Y are two k-varieties such that $[X] = [Y]$ in $K_0(\mathrm{Var}_k)$. Is it true that $X \doteqdot Y$?

In the case when k is algebraically closed, we reformulate this question as the cancellative property of the Grothendieck semiring $\mathcal{S}_k$ of piecewise isomorphic classes of k-varieties in Question 7.2.2.

Liu and Sebag answered this question over an algebraically closed field of characteristic 0 for varieties with dimension at most one [30, Propositions 5, 6] and for some classes of dimension two varieties [30, Theorems 4, 5]. Sebag [37, Theorem 3.3] extended this result further.

This question is quite natural and has many important applications to birational geometry. Consider the following question asked by Gromov as an example.

Question 7.1.3 ([14, §3.G‴])**.** Let X and Y be algebraic varieties which admit an embedding into a third one, say $X \hookrightarrow Z$ and $Y \hookrightarrow Z$, such that the complements $Z \setminus X$

and $Z \setminus Y$ are biregularly isomorphic. How far are X and Y from being birationally equivalent? Under what conditions are X and Y piecewise isomorphic?

Lamy and Sebag [28] studied the following conjectural reformulation of this question in characteristic 0.

Conjecture 7.1.4 (see [28, Conjecture 1]). Let k be an algebraically closed field and let X be a k-variety. Let $\phi : X \dashrightarrow X$ be a birational map. Then it is possible to extend the map ϕ to a piecewise automorphism of X.

It is known (see [28]) that a positive answer to Question 7.1.2 will settle this conjecture in the affirmative. We prove the converse in Theorem 7.3.3 showing that the two statements are in fact equivalent. Zakharevich [41, Theorem 6.5(1)] has also recently proved this result using advanced techniques in K-theory and spectral sequences.

Larsen and Lunts [29] obtained an important motivic measure described in the following theorem.

Theorem 7.1.5 ([29, Theorem 2.3]). *Suppose k is an algebraically closed field of characteristic 0. Let $\mathfrak{sb}$ denote the multiplicative monoid of stable birational equivalence classes of irreducible varieties. There exists a unique surjective ring homomorphism $\Psi : K_0(\mathrm{Var}_k) \to \mathbb{Z}[\mathfrak{sb}]$ that assigns to the class in $K_0(\mathrm{Var}_k)$ of a smooth irreducible proper variety its stable birational equivalence class in $\mathbb{Z}[\mathfrak{sb}]$.*

Bittner [5] obtained the following presentations of the Grothendieck group. Larsen and Lunts mention that Bittner's presentation subsumes the theorem above [29, Remark 2.4] and this assertion has been proved in detail by Sahasrabudhe in [36].

Theorem 7.1.6 ([5, Theorem 3.1]). *Suppose k is a field of characteristic 0. The Grothendieck group $K_0^+(\mathrm{Var}_k)$ has the following presentations:*

- *(sm) as the abelian group generated by the isomorphism classes of smooth varieties over k subject to the relations $[X] = [Y] + [X \setminus Y]$, where X is smooth and $Y \subseteq X$ is a smooth closed subvariety;*
- *(bl) as the abelian group generated by the isomorphism classes of smooth projective k-varieties subject to the relations $[\emptyset] = 0$ and $[\mathrm{Bl}_Y X] - [E] = [X] - [Y]$, where X is smooth and complete, $Y \subseteq X$ is a smooth closed subvariety, $\mathrm{Bl}_Y X$ is the blow-up of X along Y and E is the exceptional divisor of this blow-up.*

In Theorem 7.4.1, we show that if Question 7.1.2 admits a positive answer over an algebraically closed field k then the Grothendieck group $K_0^+(\mathrm{Var}_k)$ is a free abelian group. Further if k has characteristic 0, then this result subsumes Bittner's presentation in view of Hironaka's theorem on resolution of singularities.

Conventions: In the rest of this chapter, k denotes an algebraically closed field unless otherwise mentioned. If X is a variety, we use $\dim X$ to denote its dimension and $d(X)$ to denote the number of its irreducible components of maximal dimension.

7.2 The Grothendieck semiring of varieties

Recall the construction of the Grothendieck ring, $K_0(S)$, of a semiring S from Section 2.1.

Suppose $\{A\}$ denotes the piecewise isomorphism class of a variety A. The set $\mathcal{S}_k$ of piecewise isomorphism classes of k-varieties carries a natural semiring structure:

$$\begin{aligned} 0 &:= \{\emptyset\}; \\ \{A\}+\{B\} &:= \{A' \sqcup B'\} \text{ where } A' \in \{A\},\ B' \in \{B\},\ A' \cap B' = \emptyset. \end{aligned}$$

The product of the classes of varieties is defined by $\{A\}\cdot\{B\} := \{(A \times_{\mathrm{Spec}\,k} B)_{red}\}$, where $\{\mathrm{Spec}\,k\}$ is the multiplicative identity.

A general element of $\mathrm{K}_0(\mathcal{S}_k)$ can be written as $\{A\}-\{B\}$ for some varieties A, B. Furthermore, $\{A_1\}-\{B_1\}=\{A_2\}-\{B_2\}$ if and only if there is some variety C such that $A_1' \sqcup B_2' \sqcup C \doteqdot A_2' \sqcup B_1' \sqcup C$ for some $A_j' \in \{A_j\}, B_j' \in \{B_j\}$ for $j=1,2$ such that A_1', B_2', C and A_2', B_1', C are families of pairwise disjoint varieties.

On the other hand, a general element of $K_0(\mathrm{Var}_k)$ can be expressed as a finite linear combination $\sum_i a_i[A_i]-\sum_j b_j[B_j]$ with $a_i, b_j \in \mathbb{Z}^+$ and $A_i, B_j \in \mathrm{Var}_k$. We can choose some $A_{i1}', A_{i2}', \ldots, A_{ia_i}' \in [A_i]$ and $B_{j1}', B_{j2}', \ldots, B_{jb_j}' \in [B_j]$ for each i, j such that every two distinct A_{ik}' and B_{jl}' are disjoint. Let $A := \bigsqcup_{i,k} A_{ik}'$ and $B := \bigsqcup_{j,l} B_{jl}'$. Then the identities $[A]=\sum_i a_i[A_i]$ and $[B]=\sum_j b_j[B_j]$ are clearly true in $K_0(\mathrm{Var}_k)$. Therefore a general element of $K_0(\mathrm{Var}_k)$ can be expressed as $[A]-[B]$ for some varieties A, B.

Proposition 7.2.1. *Let k be an algebraically closed field. Then the natural map $\psi : K_0(\mathrm{Var}_k) \to \mathrm{K}_0(\mathcal{S}_k)$ defined by $\psi([A]-[B]) := \{A\}-\{B\}$ is an isomorphism of rings.*

Proof. Recall that piecewise isomorphic varieties have the same class in $K_0(\mathrm{Var}_k)$. Note as a consequence of (7.1) that if Z_1 and Z_2 are two disjoint varieties, then $[Z_1]=[Z_2]$ if and only if there is some W disjoint from both Z_1 and Z_2 such that $Z_1 \sqcup W \doteqdot Z_2 \sqcup W$.

Let A_1, B_1, A_2 and B_2 be pairwise disjoint varieties.

Now $[A_1]-[B_1]=[A_2]-[B_2]$ in $K_0(\mathrm{Var}_k)$

$\iff$ $[A_1] + [B_2] = [A_2] + [B_1]$ in $K_0(\mathrm{Var}_k)$
$\iff$ there is a variety C disjoint from all A_i and B_j
such that $A_1 \sqcup B_2 \sqcup C \doteqdot A_2 \sqcup B_1 \sqcup C$
$\iff$ $\{A_1\} + \{B_2\} = \{A_2\} + \{B_1\}$ in $\mathcal{S}_k$
$\iff$ $\{A_1\} - \{B_1\} = \{A_2\} - \{B_2\}$ in $\mathrm{K}_0(\mathcal{S}_k)$.

Thus the map ψ is both well-defined and injective. It is clearly surjective and preserves addition.

Finally we note that ψ also preserves multiplication. Observe that for any two varieties X and Y, we have $[X]\cdot[Y] = [X \times_{\mathrm{Spec\,k}} Y]$ in $K_0(\mathrm{Var}_k)$ and $\{X\}\cdot\{Y\} = \{X \times_{\mathrm{Spec\,k}} Y\}$ in $\mathrm{K}_0(\mathcal{S}_k)$. Hence ψ preserves multiplication of varieties. Using the distributivity of multiplication over addition completes the proof. □

The following question is natural.

Question 7.2.2. Let k be an algebraically closed field. Is the semiring $\mathcal{S}_k$ of piecewise isomorphism classes of k-varieties cancellative?

A positive answer to this question is equivalent to injectivity of the natural map $q : \mathcal{S}_k \to \mathrm{K}_0(\mathcal{S}_k)$. In view of Proposition 7.2.1, it is also equivalent to injectivity of the map $\psi^{-1} \circ q : \mathcal{S}_k \to K_0(\mathrm{Var}_k)$. Hence a positive answer to Question 7.2.2 is equivalent to a positive answer to Question 7.1.2.

7.3 Question 7.1.2 ≡ Conjecture 7.1.4

We note a consequence of equality in the Grothendieck ring. The proof here has been provided by Kollár. A special case has been proved in [30, Cor. 5] when k is an algebraically closed field of characteristic 0. Scanlon has pointed out another proof of the special case using counting function methods from [25].

Proposition 7.3.1. *Let k be a field and let A and B be two varieties with $[A] = [B]$ in $K_0(\mathrm{Var}_k)$. Then $\dim A = \dim B$ and $d(A) = d(B)$.*

Proof. Let A and B be two varieties with $[A] = [B]$ in $K_0(\mathrm{Var}_k)$. Then, by Proposition 7.2.1, there is a variety C disjoint from both A and B such that $A \sqcup C \doteqdot B \sqcup C$. Let G be the graph of such an isomorphism. Then the diagram

$$A \sqcup C \xleftarrow{\pi_1} G \xrightarrow{\pi_2} B \sqcup C \tag{7.2}$$

can be reduced to any finitely generated subring R of the field k. We can further pass the diagram over the finite field $R/\mathfrak{m}$, where $\mathfrak{m}$ is a maximal ideal of the ring R.

If R is a subring of k containing all the elements of k necessary to define the varieties in the Diagram (7.2), then counting in the residue fields $R/\mathfrak{m}$ yields the required equalities. □

Given two varieties V and W such that $[V] = [W]$ in $K_0(\mathrm{Var}_k)$, Proposition 7.2.1 states that there is some variety Z disjoint from V and W such that $V \sqcup Z \div W \sqcup Z$. Under the hypothesis of Conjecture 7.1.4, we develop a technique in Proposition 7.3.2 to remove a dense subset of Z from both $V \sqcup Z$ and $W \sqcup Z$ to leave piecewise isomorphic complements. In fact the following proposition is a reformulation of Conjecture 7.1.4.

Proposition 7.3.2. *Suppose Conjecture 7.1.4 holds for an algebraically closed field k.*

Let V, W and Z be k-varieties such that Z is disjoint from both V and W, $\dim V \leq \dim W \leq t = \dim Z$ and $d(Z) = e$. Assume that $d(V) = d(W)$ if $\dim V = \dim W$. Further let $d = \begin{cases} d(V) & \text{if } \dim V = \dim W = t, \\ 0 & \text{otherwise.} \end{cases}$

Let $S_1, S_2, \ldots, S_{d+e}$ and $T_1, T_2, \ldots, T_{d+e}$ be families of pairwise disjoint irreducible subvarieties of $V \sqcup Z$ and $W \sqcup Z$ respectively such that $\dim S_l = \dim T_l = t$ for each $l \in [d+e]$. Assume that the varieties S_l and T_l are either disjoint from or contained in Z for each $l \in [d+e]$. Furthermore assume that τ is a permutation of $[d+e]$ such that $f_l : S_l \cong T_{\tau(l)}$ is a variety isomorphism for each $l \in [d+e]$.

Then there are subsets $P, Q \subseteq [d+e]$ of size e, a bijection $\lambda : Q \to P$ and dense subvarieties $S'_l \subseteq S_l$, $T'_l \subseteq T_l$ for $l \in [d+e]$ such that the following hold:

- $S'_{\lambda(l)} = T'_l \subseteq Z$ *for each* $l \in Q$*;*
- $\bigsqcup_{m \notin P} S'_m \div \bigsqcup_{l \notin Q} T'_l$*;*
- $\bigsqcup_{l \in [d+e]} (S_l \setminus S'_l) \div \bigsqcup_{l \in [d+e]} (T_l \setminus T'_l)$*.*

Proof. We have $S_l \subseteq Z$ and $T_m \subseteq Z$ for exactly e values of both l and m. Let $Q, P \subseteq [d+e]$ be the sets of such m and l respectively. For each $l \in P$, there is a unique $m \in Q$ such that $\dim(S_l \cap T_m) = t$. Let $\lambda : Q \to P$ define this correspondence.

Case I: Suppose that $\lambda(l) = \tau^{-1}(l)$ for each $l \in Q$.

In this case we set $S'_{\lambda(l)} = T'_l := S_{\lambda(l)} \cap T_l$ for each $l \in Q$. The isomorphism $f_{\lambda(l)} : S_{\lambda(l)} \to T_l$ can be seen as a birational self-map of $S_{\lambda(l)} \cup T_l$. Since Conjecture 7.1.4 holds, this birational map can be extended to obtain a piecewise automorphism of $S_{\lambda(l)} \cup T_l$. In particular, one gets a piecewise isomorphism $T_l \setminus S_{\lambda(l)} \div S_{\lambda(l)} \setminus T_l$ of lower dimensional subvarieties.

Case II: Suppose that $\lambda(i) \neq \tau^{-1}(i)$ for some $i \in Q$. Fix such i and let $j := \lambda(i)$.

The idea of the proof is to find subvarieties $S^1_l \subseteq S_l$ and $T^1_l \subseteq T_l$ for each $l \in [d+e]$ and a permutation τ_1 of $[d+e]$ such that the following properties are satisfied:

(i) $\{l \in Q : \lambda(l) = \tau_1^{-1}(l)\} \supsetneq \{l \in Q : \lambda(l) = \tau^{-1}(l)\}$;

(ii) $f_l : S_l \setminus S_l^1 \cong T_{\tau(l)} \setminus T^1_{\tau(l)}$ is an isomorphism for each $l \in [d+e]$;

and then continue inductively.

Since $j \neq \tau^{-1}(i)$, we make the following assignments.

$$S_l^1 := \begin{cases} T_i \cap S_j & \text{if } l = j, \\ f^{-1}_{\tau^{-1}(i)}(T_i \cap S_j) & \text{if } l = \tau^{-1}(i), \\ S_l & \text{otherwise.} \end{cases}$$

$$T_l^1 := \begin{cases} f_j(T_i \cap S_j) & \text{if } l = \tau(j), \\ T_i \cap S_j, & \text{if } l = i \\ T_l & \text{otherwise.} \end{cases}$$

The maps $f_{\tau^{-1}(i)}$ and f_j clearly restrict to isomorphisms $S_{\tau^{-1}(i)} \setminus S^1_{\tau^{-1}(i)} \cong T_i \setminus T_i^1$ and $S_j \setminus S_j^1 \cong T_{\tau(j)} \setminus T^1_{\tau(j)}$ of lower dimensional subvarieties. This takes care of property *(ii)*.

Now we define $\tau_1 : [d+e] \to [d+e]$ as follows.

$$\tau_1(l) := \begin{cases} i & \text{if } l = j, \\ \tau(j) & \text{if } l = \tau^{-1}(i) \\ \tau(l) & \text{otherwise.} \end{cases}$$

Note that $\lambda(i) \neq \tau^{-1}(i)$, but $\lambda(i) = \tau_1^{-1}(i)$. This shows that *(i)* holds.

Furthermore, $f^1_{\tau^{-1}(i)} := f_j \circ f_{\tau^{-1}(i)} : S^1_{\tau^{-1}(i)} \to T^1_{\tau_1(\tau^{-1}(i))}$ and $f_j^1 := id : S_j^1 \to T^1_{\tau_1(j)}$ are isomorphisms. For the remaining $l \in [d+e]$, we set $f_l^1 := f_l$.

Thus $f_l^1 : S_l^1 \to T^1_{\tau_1(l)}$ is an isomorphism for each $l \in [d+e]$. If λ does not agree with τ_1^{-1} on Q, we iterate the process with varieties S_l^1, T_l^1, functions f_l^1 and permutation τ_1 until some $(\tau_n)^{-1}$ agrees with λ on Q.

We set $T_l' := T_l^n$ for each $l \notin Q$ and $S_l' := S_l^n$ for each $l \notin P$.

The varieties $S^n_{\lambda(l)}$, T_l^n, for $l \in Q$, together with the function $\lambda = \tau_n^{-1} \restriction_Q$ is the set-up for the first case. A construction similar to that case gives the required varieties $S'_{\lambda(l)}$, T_l'.

In both cases, it is clear that the construction guarantees the final two conditions in the statement of the proposition. □

Theorem 7.3.3 (cf. [41, Theroem 6.5(1)]). *Let k be an algebraically closed field. If Conjecture 7.1.4 holds for k, then Question 7.1.2 admits a positive answer over k.*

Proof. Let V and W be two varieties with $[V] = [W]$ in $K_0(\mathrm{Var}_k)$. Then Proposition 7.2.1 states that there is a variety Z of dimension t and $d(Z) = e$, say, disjoint from both V and W, which witnesses this equality, i.e., $V \sqcup Z \doteqdot W \sqcup Z$. Proposition 7.3.1 then gives $\dim V = \dim W =: s$ and $d(V) = d(W)$.

If $s = 0$, then $d(V) = d(W)$ implies $V \doteqdot W$.

If $s > 0$, then we describe a procedure to reduce the sum $s + t$ in two different cases.

Case I: Suppose $s \leq t$. Let $d = \begin{cases} d(V) & \text{if } s = t, \\ 0 & \text{otherwise.} \end{cases}$

The piecewise isomorphism $V \sqcup Z \doteqdot W \sqcup Z$ gives two families $S_1, S_2, \ldots, S_{d+e}$ and $T_1, T_2, \ldots, T_{d+e}$ of irreducible subvarieties of $V \sqcup Z$ and $W \sqcup Z$ respectively of maximal dimension satisfying the hypotheses of Proposition 7.3.2. Furthermore, we also get that $(V \sqcup Z) \setminus \left(\bigsqcup_{i\in[d+e]} S_i\right) \doteqdot (W \sqcup Z) \setminus \left(\bigsqcup_{i\in[d+e]} T_i\right)$.

If Conjecture 7.1.4 holds, we can apply Proposition 7.3.2 to obtain a dense subvariety $Z_1 := \bigsqcup_{i\in Q} T_i' \subseteq Z$. The other conclusions of the proposition give the following properties:

- the variety $Z' := Z \setminus Z_1$ witnesses $[V] = [W]$, i.e., $V \sqcup Z' \doteqdot W \sqcup Z'$;
- $\dim Z' < \dim Z$.

The use of Proposition 7.3.2 can be repeated if $\dim Z' \geq s$. Hence the equality $[V] = [W]$ in $K_0(\mathrm{Var}_k)$ is witnessed by some variety Z'' of dimension less than s.

Case II: Suppose $s > t$. In this case, the piecewise isomorphism $V \sqcup Z \doteqdot W \sqcup Z$ gives $V' \subset V$, $W' \subset W$ with $\dim V' = \dim W' < s$ such that $V \setminus V' \doteqdot W \setminus W'$ and $V' \sqcup Z \doteqdot W' \sqcup Z$.

The two cases complete the proof that $V \doteqdot W$. □

7.4 Freeness of $K_0^+(\mathrm{Var}_k)$ under Conjecture 7.1.4

In this section, we assume that Conjecture 7.1.4 holds (equivalently, in view of Theorem 7.3.3, Question 7.1.2 admits a positive answer) for an algebraically closed field k.

For each $n \in \mathbb{Z}_{\geq 0}$ let Var_k^n denote the proper class of k-varieties of dimension at most n. Then $\{\mathrm{Var}_k^n\}_{n\geq 0}$ is a filtration on the objects of Var_k. Further let S_n denote the monoid, under $\sqcup$, of piecewise isomorphism classes of varieties in Var_k^n and H_n denote the Grothendieck group associated with S_n for each $n \geq 0$. If Conjecture 7.1.4 holds, then H_n is the subgroup of $K_0(\mathrm{Var}_k)$ generated by S_n and thus, for each $n \in \mathbb{Z}_{\geq 0}$, the natural map $H_n \to H_{n+1}$ is injective.

Let $\mathfrak{M}$ denote a set of representatives of birational equivalence classes of irreducible varieties. Then $\mathfrak{M} = \bigsqcup_{n\in\mathbb{Z}_{\geq 0}} \mathfrak{M}_n$, where $\mathfrak{M}_n$ is the set of all dimension n varieties in $\mathfrak{M}$. We use $\mathcal{A}, \mathcal{B}, \ldots$ to denote the elements of $\mathfrak{M}$.

We say that a variety A of dimension n is **$\mathfrak{M}$-admissible** (or just **admissible**) if it can be embedded into some $\mathcal{A} \in \mathfrak{M}_n$. The assignment $A \mapsto \mathcal{A}$ is a well-defined and dimension preserving map on the class of admissible varieties. Note that every admissible variety has a unique irreducible component of maximal dimension. We say that a **partition** $D = \bigsqcup_{i\in[m]} D_i$ of a variety D into locally closed subvarieties **is admissible** if each D_i is admissible. Note that each variety admits an admissible partition.

Theorem 7.4.1. *Suppose Question 7.1.2 admits a positive answer over an algebraically closed field k. Let $\mathfrak{M}$ denote a set of representatives of birational equivalence classes of irreducible k-varieties. Then there is a unique group isomorphism $ev_{\mathfrak{M}} : K_0^+(\mathrm{Var}_k) \to \mathbb{Z}[\mathfrak{M}]$ satisfying $ev_{\mathfrak{M}}([\mathcal{A}]) = \mathcal{A}$ for each $\mathcal{A} \in \mathfrak{M}$.*

Proof. We fix $\mathfrak{M}$ and drop the subscript $\mathfrak{M}$ from $ev_{\mathfrak{M}}$. We inductively define a compatible family of maps $\{ev^n : \mathrm{Var}_k^n \to \mathbb{Z}[\mathfrak{M}]\}_{n\geq 0}$, where ev^n factors through an injective group homomorphism $H_n \to \mathbb{Z}[\mathfrak{M}]$. By an abuse of notation, we also denote the group homomorphism by ev^n.

If $D \in \mathrm{Var}_k^0$ and $d(D) = d$, then the assignment $D \mapsto d\mathcal{U}$ clearly factors through an injective group homomorphism $H_0 \cong \mathbb{Z} \to \mathbb{Z}[\mathfrak{M}]$, where $\mathcal{U}$ is the unique variety in $\mathfrak{M}_0$.

Assume by induction that ev^{n-1} is a well-defined map on Var_k^{n-1} and that it factors through an injective group homomorphism $H_{n-1} \to \mathbb{Z}[\mathfrak{M}]$.

If $D \in \mathrm{Var}_k^{n-1}$, then define $ev^n(D) := ev^{n-1}(D)$ ensuring compatibility.

Let A be an admissible variety of dimension n. Then there is an embedding $f : A \hookrightarrow \mathcal{A}$ for a unique $\mathcal{A} \in \mathfrak{M}_n$. Define $ev^n(A) := \mathcal{A} - ev^{n-1}(\mathcal{A} \setminus f(A))$.

To see that this definition does not depend on the choice of an embedding, let $g : A \hookrightarrow \mathcal{A}$ be another embedding. It will suffice to show that $ev^{n-1}(\mathcal{A} \setminus f(A)) = ev^{n-1}(\mathcal{A} \setminus g(A))$. Note that the following equations hold in H_n.

$$\begin{aligned} [f(A)] &= [g(A)], \\ [f(A)] + [\mathcal{A} \setminus f(A)] &= [g(A)] + [\mathcal{A} \setminus g(A)]. \end{aligned}$$

Hence $[\mathcal{A} \setminus f(A)] = [\mathcal{A} \setminus g(A)]$ in H_n. Under the hypothesis of a positive answer to Question 7.1.2, we conclude the same equation in H_{n-1}.

If $\phi : A' \to A$ is a variety isomorphism, then A' is admissible since A is and both of them embed into the same variety $\mathcal{A} \in \mathfrak{M}_n$. Choosing an embedding f of A into $\mathcal{A}$ gives an embedding $f \circ \phi$ of A' into $\mathcal{A}$. Since $f(A) = f \circ \phi(A')$, we have $ev^n(A) = ev^n(A')$.

If $A = A_1 \sqcup A_2$ is a partition of an admissible variety A of dimension n into locally closed subvarieties and $\dim A_1 = n$, then A_1 is admissible. Further if f is an embedding of A into $\mathcal{A}$, then using that ev^{n-1} is additive we have

$$\begin{aligned} ev^n(A_1) + ev^n(A_2) &= \mathcal{A} - ev^{n-1}(\mathcal{A} \setminus f(A_1)) + ev^{n-1}(A_2) \\ &= \mathcal{A} - ev^{n-1}((\mathcal{A} \setminus f(A)) \sqcup f(A_2)) + ev^{n-1}(A_2) \\ &= \mathcal{A} - ev^{n-1}(\mathcal{A} \setminus f(A)) - ev^{n-1}(f(A_2)) + ev^{n-1}(A_2) \\ &= \mathcal{A} - ev^{n-1}(\mathcal{A} \setminus f(A)) \\ &= ev^n(A). \end{aligned}$$

From the previous two paragraphs it follows that whenever A is admissible and $A \doteqdot B$, then $ev^n(A) = ev^n(B)$.

Now let $D = \bigsqcup_{i \in [m]} D_i$ be an admissible partition of a variety D of dimension n. Define $ev^n(D) := \sum_{i \in [m]} ev^n(D_i)$. Any two admissible partitions of D admit a common admissible refinement and, as shown above, the value of $ev^n(D_i)$ does not change under refinements. Thus $ev^n(D)$ is independent of the choice of an admissible partition and thus is well-defined.

If $D \doteqdot D'$, then we choose partitions $D = \bigsqcup_{i \in [m]} D_i$ and $D' = \bigsqcup_{i \in [m]} D'_i$ such that D_i is isomorphic to D'_i for each i. By further refinements, we may as well assume that these partitions are admissible. Then $ev^n(D) = \sum_{i \in [m]} ev^n(D_i) = \sum_{i \in [m]} ev^n(D'_i) = ev^n(D')$.

This completes the proof that the map ev^n, uniquely determined by $\mathfrak{M}$, factors through an additive map $H_n \to \mathbb{Z}[\mathfrak{M}]$. It remains to show that $ev^n : H_n \to \mathbb{Z}[\mathfrak{M}]$ is injective.

We must show that if $D_1, D_2, D'_1, D'_2 \in \mathrm{Var}_k^n$ satisfy $ev^n([D_1] - [D_2]) = ev^n([D'_1] - [D'_2])$, then $[D_1] - [D_2] = [D'_1] - [D'_2]$. This claim can be restated as: $ev^n([D_1] + [D'_2]) = ev^n([D_2] + [D'_1])$ implies $[D_1] + [D'_2] = [D_2] + [D'_1]$. Therefore it is sufficient to prove that if $D, D' \in \mathrm{Var}_k^n$ satisfy $ev^n([D]) = ev^n([D'])$, then $[D] = [D']$ in H_n.

Let $D, D' \in \mathrm{Var}_k^n$ be such that $ev^n([D]) = ev^n([D'])$. Looking at the "n-dimensional component" of this element of $\mathbb{Z}[\mathfrak{M}]$, we deduce that $\dim D = \dim D'$ and $d(D) = d(D') =: d$.

Suppose $D = \bigsqcup_{i \in [t]} D_i$ is an admissible partition of D. Without loss, we may assume that $\dim D_i = n$ if and only if $i \in [d]$. For each $i \in [d]$, let $f_i : D_i \to \mathcal{A}_i$ be a variety embedding, where $\mathcal{A}_i \in \mathfrak{M}_n$, and set $C_i := \mathcal{A}_i \setminus f_i(D_i)$. Then $\dim C_i < n$ and $D_i \sqcup C_i \doteqdot \mathcal{A}_i$. Hence $ev^n([D_i]) + ev^{n-1}([C_i]) = \mathcal{A}_i$ for each $i \in [d]$.

Similarly starting with an admissible partition $D' = \bigsqcup_{i \in [s]} D'_i$, where $\dim D'_i = n$ if and only if $i \in [d]$, we obtain $\mathcal{A}'_i \in \mathfrak{M}_n$ and C'_i such that $ev^n([D'_i]) + ev^{n-1}([C'_i]) = \mathcal{A}'_i$

for each $i \in [d]$. Now

$$\begin{aligned}\sum_{i\in[d]} \mathcal{A}_i + \sum_{i\in[t]\setminus[d]} ev^{n-1}([D_i]) \ &+ \ \sum_{i\in[d]} ev^{n-1}([C_i']) \\ &= ev^n([D]) + \sum_{i\in[d]} ev^{n-1}([C_i]) + \sum_{i\in[d]} ev^{n-1}([C_i']) \\ &= ev^n([D']) + \sum_{i\in[d]} ev^{n-1}([C_i']) + \sum_{i\in[d]} ev^{n-1}([C_i]) \\ &= \sum_{i\in[d]} \mathcal{A}_i' + \sum_{i\in[s]\setminus[d]} ev^{n-1}([D_i']) + \sum_{i\in[d]} ev^{n-1}([C_i]).\end{aligned}$$

Comparing the components of different dimensions, we get the following equations.

$$\begin{aligned}\sum_{i\in[d]} \mathcal{A}_i &= \sum_{i\in[d]} \mathcal{A}_i', \\ \sum_{i\in[t]\setminus[d]} ev^{n-1}([D_i]) + \sum_{i\in[d]} ev^{n-1}([C_i']) &= \sum_{i\in[s]\setminus[d]} ev^{n-1}([D_i']) + \sum_{i\in[d]} ev^{n-1}([C_i]).\end{aligned}$$

It easily follows from the first equation that the list $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_d$ is the same as the list $\mathcal{A}_1', \mathcal{A}_2', \ldots, \mathcal{A}_d'$. Since the map ev^{n-1} is injective, the second equation gives $\sum_{i\in[t]\setminus[d]}[D_i] + \sum_{i\in[d]}[C_i'] = \sum_{i\in[s]\setminus[d]}[D_i'] + \sum_{i\in[d]}[C_i]$ in H_{n-1}. Combining these, we obtain $[D] + \sum_{i\in[d]}[C_i] + \sum_{i\in[d]}[C_i'] = [D'] + \sum_{i\in[d]}[C_i'] + \sum_{i\in[d]}[C_i]$ in H_n. Since H_n is a group, cancelling common terms from both sides gives $[D] = [D']$. This completes the proof of injectivity of ev^n.

Define the map $ev : \mathrm{Var}_k \to \mathbb{Z}[\mathcal{M}]$ by $ev(D) := ev^n(D)$ whenever $D \in \mathrm{Var}_k^n$. Compatibility of the family $\{ev^n\}$ gives that the map ev is well-defined and factors through $K_0^+(\mathrm{Var}_k)$ to give an injective map $K_0^+(\mathrm{Var}_k) \to \mathbb{Z}[\mathfrak{M}]$.

Since, given $C, D \in \mathrm{Var}_k$, there exists n such that $C, D \in \mathrm{Var}_k^n$, the additivity of $ev^n : H_n \to \mathbb{Z}[\mathfrak{M}]$ for each n implies that $ev : K_0^+(\mathrm{Var}_k) \to \mathbb{Z}[\mathfrak{M}]$ is a group homomorphism. The image of ev generates the group $\mathbb{Z}[\mathfrak{M}]$ since the image of $\mathfrak{M} \subseteq \mathrm{Var}_k$ generates the codomain. Hence $K_0^+(\mathrm{Var}_k) \cong \mathbb{Z}[\mathfrak{M}]$. □

Suppose that k is an algebraically closed field of characteristic 0. Hironaka's theorem on resolution of singularities allows us to choose smooth projective generators of the Grothendieck group. To deduce Theorem 7.1.6 from Theorem 7.4.1, consider the abelian group H freely generated by the isomorphism classes of smooth complete varieties. We only need to find the relations between birational smooth complete varieties in the Grothendieck group. The weak factorization theorem ([1, Theorem 0.1.1]) of Abramovich, Karu, Matsuki and Włodarczyk states that any birational morphism between two smooth complete k-varieties can be factorized as a sequence of blow-ups and blow-downs. As a corollary we obtain that if the subgroup $H' \leq H$ is generated by the relation $[\emptyset] = 0$ together with the "blow-up" relations, then $K_0^+(\mathrm{Var}_k) \cong H/H'$.

7.5 The associated graded ring of $K_0(\mathrm{Var}_k)$

We continue to work under the hypothesis of a positive answer to Question 7.1.2 over an algebraically closed field in this section. Under this hypothesis, the usual dimension function factorizes through the Grothendieck group.

Two varieties X and Y of dimension n are birational if and only if there are open subvarieties $X' \subseteq X$ and $Y' \subseteq Y$ such that $X' \cong Y'$ if and only if $\dim([X] - [Y]) = \dim([X \setminus X'] - [Y \setminus Y']) < n$, where $[X]$ denotes the class of the variety X in $K_0(\mathrm{Var}_k)$.

In general the product of two varieties in $\mathfrak{M}$ is birational (but not necessarily equal) to a variety in $\mathfrak{M}$. This suggests looking at the structure of the associated graded ring of $K_0(\mathrm{Var}_k)$, where the grading on $K_0(\mathrm{Var}_k)$ is induced by dimension. Let $\{F_n\}_{n\geq 0}$ be the filtration on $K_0(\mathrm{Var}_k)$ induced by dimensions and let $\mathfrak{G}$ denote the associated graded ring of $K_0(\mathrm{Var}_k)$ with respect to this filtration.

The construction of $\mathfrak{G}$ is as follows. Set $F_{-1} := \{0\}$ for technical purposes. Let $G_n := F_n/F_{n-1}$ for each $n \geq 0$ and let $\mathfrak{G}$ denote the abelian group $\bigoplus_{n\geq 0} G_n$. There are multiplication maps $G_n \times G_m \to G_{n+m}$ defined by $(x+F_{n-1})(y+F_{m-1}) = xy+F_{n+m-1}$ for each $n, m \geq 0$. These maps combine to give a multiplicative structure on $\mathfrak{G}$.

Let $\mathfrak{B}_n$ denote the set of birational equivalence classes of irreducible varieties of dimension n and let $\mathfrak{B} := \bigsqcup_{n\geq 0} \mathfrak{B}_n$. The set $\mathfrak{B}$ carries a monoid structure induced by the multiplication of varieties, where the class of a singleton acts as the identity. The usual dimension function on varieties factors through $\mathfrak{B}$.

Theorem 7.5.1. *Suppose Question 7.1.2 admits a positive answer over an algebraically closed field k. The associated graded ring $\mathfrak{G}$ of $K_0(\mathrm{Var}_k)$ with respect to the dimension grading is the monoid ring $\mathbb{Z}[\mathfrak{B}]$, where $\mathfrak{B}$ is the multiplicative monoid of birational equivalence classes of irreducible varieties.*

Proof. Since Question 7.1.2 admits a positive answer over k, we can use the group isomorphism of Theorem 7.4.1 induced by the evaluation map to define a multiplicative structure on $\mathbb{Z}[\mathfrak{M}]$. By an abuse of notation, we will say that $\{F_n\}_{n\geq 0}$ is a filtration on $\mathbb{Z}[\mathfrak{M}]$ and $\mathfrak{G}$ is its associated graded ring.

Let $\mathcal{A} \mapsto [[\mathcal{A}]]$ denote the canonical bijection $\mathfrak{M} \to \mathfrak{B}$, which takes an irreducible variety to its birational equivalence class. This clearly extends to a group isomorphism $\Phi : \mathfrak{G} \to \mathbb{Z}[\mathfrak{B}]$. We show that Φ also preserves multiplication.

Given $\mathcal{A} \in \mathfrak{M}_n$ and $\mathcal{B} \in \mathfrak{M}_m$, the product $\mathcal{A} \times_{\mathrm{Spec}\, k} \mathcal{B}$ is irreducible and thus is birational to a unique $\mathcal{C} \in \mathfrak{M}_{n+m}$. In other words, $(\mathcal{A}+F_{n-1})\cdot(\mathcal{B}+F_{m-1}) = \mathcal{C}+F_{n+m-1}$ in $\mathfrak{G}$. We also have $[[\mathcal{C}]] = [[\mathcal{A} \times_{\mathrm{Spec}\, k} \mathcal{B}]] = [[\mathcal{A}]][[\mathcal{B}]]$ in the monoid $\mathfrak{B}$. Hence $\Phi((\mathcal{A}+F_{n-1})\cdot(\mathcal{B}+F_{m-1})) = \Phi(\mathcal{A}+F_{n-1})\cdot\Phi(\mathcal{B}+F_{m-1})$. This shows that Φ preserves multiplication on the image of $\mathfrak{M}$ in $\mathfrak{G}$. It is routine to verify that Φ is multiplicative on the whole of $\mathfrak{G}$. □

7.6 Further Remarks

Recall that an element a of a ring R is said to be **regular** if it is not a zero divisor in R. The following question is important for better understanding of the Grothendieck ring and is open even in the case of algebraically closed fields.

Question 7.6.1. Let k be a field of characteristic 0. Suppose $\mathbb{A}^1_k$ denotes the affine line over k. Is $\mathbb{L} := [\mathbb{A}^1_k]$ a regular element of $K_0(\mathrm{Var}_k)$?

Lemma 4.8 and Proposition 4.9 in [37] connect this question to Question 7.1.2 in special cases, but no further development has been made.

In view of Theorem 7.5.1, one can ask the following question.

Question 7.6.2. Let k be an algebraically closed field. Is the Grothendieck ring $K_0(\mathrm{Var}_k)$ a monoid ring?

The model-theoretic Grothendieck ring $K_0(k)$ of an algebraically closed field k, as defined in [25], is a quotient of $K_0(\mathrm{Var}_k)$. It is natural to ask the following question.

Question 7.6.3. Suppose k is an algebraically close field. Is the model-theoretic Grothendieck ring $K_0(k)$ isomorphic to $K_0(\mathrm{Var}_k)$?

Chapter 8

Conclusions

8.1 Overview

We fix a unital ring $\mathcal{R}$ and an algebraically closed field k in this chapter. The main objects of study in this thesis are certain symmetric monoidal categories whose objects are sets and the monoidal operation is disjoint union, i.e., we study cut-and-paste operations. The objects of the category $\mathcal{S}(M)$ associated with a structure M (Chapter 3), or of the category Var_k associated with a field k (Chapter 7), form a boolean algebra without a top element.

Definition 8.1.1. *A **simplicial poset** is a poset P with a least element 0 such that every interval $[0, p] = \{q \in P : 0 \leq q \leq p\}$ is a boolean algebra.*

One can only talk about relative complements (i.e., the element $p \wedge \neg q$ whenever $q \leq p$ in P) in a simplicial poset.

In this chapter we identify a common thread between different chapters, namely freeness of the simplicial poset of objects of these categories or of the categories themselves. Section 8.2 is based on Chapter 5 and it isolates two key aspects of freeness. One of these aspects is model-theoretic, namely the (partial) elimination of quantifiers whereas the other is of lattice-theoretic nature. We describe how the freeness is transferred to the lower K-theory of such categories in Section 8.3. In the next section, we weaken one of these two aspects at a time and link Chapters 6 and 7 via Chapter 5. Section 8.5 describes miscellaneous questions not covered in the earlier sections.

8.2 Model theory and freeness of groupoids

We focus our attention on the category $\mathcal{S}(M_\mathcal{R})$, where the theory T of the module $M_\mathcal{R}$ satisfies $T = T^{\aleph_0}$. Model theory plays the key role in determining the structure of this category. In this case, the partial elimination of quantifiers (Theorem 4.1.5) is equivalent to the statement that the simplicial poset $\overline{\mathrm{Def}}(M_{\mathcal{R}})$ of the objects of the category $\mathcal{S}(M_{\mathcal{R}})$ is free. To explain the meaning of this last statement we analyse the manipulation of different lattice-like substructures of the simplicial poset, which is another important theme in this thesis.

($\wedge$) The *pp*-elimination of quantifiers for the theories of modules (Theorem 4.1.5) makes, for each $n \geq 1$, the meet-semilattice $\mathcal{L}_n$ of *pp*-definable sets the 'basis' of the boolean algebra $\mathrm{Def}(M^n)$.

($\vee$) The lattice $\mathcal{A}_n$ of antichains is, by definition, the free distributive lattice on $\mathcal{L}_n$. The model-theoretic condition $T = T^{\aleph_0}$ says that the lattice $\mathcal{O}_n$ of *pp*-convex subsets of M^n is isomorphic to the lattice $\mathcal{A}_n$.

($\neg$) The cell decomposition theorem (Corollary 4.3.5) expresses $\mathrm{Def}(M^n)$ as (an isomorphic copy of) the boolean algebra of finite chains of finite antichains in the meet-semilattice $\mathcal{L}_n$. This can be described as the method of freely adjoining relative complements to the free distributive lattice $\mathcal{A}_n$.

Note that the above constructions can be expressed entirely in the language of lattice theory.

Simplicial methods are natural for studying the 'set-theoretic geometry' associated with antichains. The local processes in $\mathrm{Def}(M^n)$ are similar to, but independent from, the local processes in $\mathrm{Def}(M^m)$ when $n \neq m$ and these different 'dimensions' start to interact with each other only when we are concerned with the multiplicative structure. The fact that the *pp*-sets are closed under projections is not directly relevant to the analysis of the local and global characteristics.

The morphisms of the groupoid $\mathcal{S}(M_{\mathcal{R}})$ embed into its objects via the mapping that takes a definable bijection f between two definable subsets to its graph $Graph(f)$. For each $n \geq 1$, let $\mathcal{G}_n$ denote the subcategory of $\mathcal{S}(M_{\mathcal{R}})$ whose objects are from the meet-semilattice $\mathcal{L}_n$ and the morphisms are *pp*-definable bijections. Note that, by construction, the morphisms of the groupoid $\mathcal{G}_n$ also embed into its objects. This suggests that the groupoid $\overline{\mathcal{G}} := \bigcup_{n=1}^{\infty} \mathcal{G}_n$ freely generates the groupoid $\mathcal{S}(M_{\mathcal{R}})$ in the same way as $\overline{\mathcal{L}}$ generates $\overline{\mathrm{Def}}(M)$. Note that the category $\overline{\mathcal{G}}$ is not a symmetric monoidal category. We understand the concept of free generation only in this very special case, which raises the following question.

Question 8.2.1. Under what conditions can one express that a groupoid $\mathcal{S}$ is freely generated by a sub-groupoid $\mathcal{G}$?

8.3 Transferring freeness

Recall that the theory T of an infinite vector space V_F over an infinite field F satisfies the model-theoretic condition $T = T^{\aleph_0}$. Owing to the infinitude of the vector space V_F and the fact that definability is with parameters, logical disjunction ($\vee$) allows one to realize disjoint unions of arbitrarily large finite number of copies of V^n in V^{n+1} for each $n \geq 0$ - a fact that has been heavily exploited in the proof of Theorem 3.5.1. This allows one to embed the groupoid *iso* FinSets in $\mathcal{S}(V_F)$ in each dimension. Moreover, the multiplicative monoid of colours (i.e., *pp*-isomorphism classes of *pp*-definable sets) in $\mathcal{S}(V_F)$ is isomorphic to $(\mathbb{N}, 0, +)$.

The groupoid $\mathcal{S}(V_F)$ can be thought of as the "free boolean semidirect product" of the groupoid *iso* FinSets along the groupoid $\overline{\mathcal{G}}$ of the *pp*-definable bijections of the *pp*-definable sets; the same holds true of the objects. Since the K-theories of equivalent (symmetric monoidal) groupoids are isomorphic, we consider the skeletons of the categories involved in the product. The groupoid $\overline{\mathcal{G}}$ is equivalent to the disjoint union of the groups Υ^n of *pp*-definable automorphisms of the fundamental definable set V^n; the objects of the skeleton of $\overline{\mathcal{G}}$ form the monoid $\mathbb{N}$.

The Grothendieck ring construction is generally thought of as the linearization of a boolean algebra since one converts boolean combinations into linear combinations. Under the new viewpoint of the structure of the category $\mathcal{S}(V_F)$, we can write the integral monoid ring $K_0(V_F) = \mathbb{Z}[\mathbb{N}]$ as the direct sum $\bigoplus_{n\in\mathbb{N}} K_0(iso\,\text{FinSets})$. Similarly, $K_1(V_F) = \bigoplus_{n=1}^{\infty}(F^\times \oplus \mathbb{Z}_2) \oplus \mathbb{Z}_2 = \bigoplus_{n\in\mathbb{N}}(H_1(\Upsilon^n; \mathbb{Z}) \oplus K_1(iso\,\text{FinSets}))$.

The K-theory of the groupoid *iso* FinSets is clearly visible in each dimension, where the monoid $\mathbb{N}$ of dimensions depends only on the groupoid $\overline{\mathcal{G}}$. This gives the complete description of the ring K_0 since it disregards the automorphisms. The explicit dependence of the "automorphism-classifying group" K_1 on the groupoid $\overline{\mathcal{G}}$ can be clearly seen through the homology component. Moreover, the independence of different dimensions can be attributed to the freeness of the groupoid $\mathcal{S}(V_F)$.

The Grothendieck ring, $K_0(M_\mathcal{R})$, of a module $M_\mathcal{R}$ whose theory T satisfies $T = T^{\aleph_0}$ is the integral monoid ring $\mathbb{Z}[\overline{\mathcal{X}}]$ (Theorem 5.4.2) and it can be analyzed in a similar way. This discussion motivates the following conjecture.

Conjecture 8.3.1. Suppose $M_\mathcal{R}$ is an infinite right $\mathcal{R}$-module whose theory T satisfies the condition $T = T^{\aleph_0}$. Then $K_1(M_\mathcal{R}) = \bigoplus_{\mathfrak{A}\in\overline{\mathcal{X}}}(\Upsilon(\mathfrak{A})^{ab} \oplus \mathbb{Z}_2)$, where $\Upsilon(\mathfrak{A})$ is the group of *pp*-definable automorphisms of any (and hence every) *pp*-set in $\mathfrak{A}$.

The method used in the proof of Theorem 3.5.1 - that describes the group $K_1(V_F)$ - depends on the fact that on closing a finite set of colours under the intersections of its elements one obtains only a finite set of colours. This allows one to compute the automorphism groups explicitly. Hence, for general modules, one cannot apply this technique.

The monoidal operation, $\sqcup$, on $\mathcal{S}(M)$, for any structure M, induces a poset structure on the objects. One can use this poset structure to construct other types of categories of definable sets and definable functions whose K-theory can be defined. Zakharevich defined the K-theory of a categorical gadget called an assembler in [41]; definable sets with definable injective functions do form an assembler. The author expects, but did not check, that the K-theories obtained using two different ways coincide. Explicit computation of the K-groups, even K_1, is not easy using either method.

Definable bijections between definable sets in $\overline{\mathrm{Def}}(M)$ also behave like partial finitary definable symmetries of the infinite product $M^{\aleph_0}$; the latter is considered only as a set and is not a model-theoretic structure. This suggests that the definable bijections form an inverse semigroup (see [20]). Some literature is available regarding the K-theory of some special classes of inverse semigroups modelled on the algebraic K-theory of rings, but the author did not find it relevant to the model-theoretic K-theory considered in this thesis.

The following question/conjecture presents the author's viewpoint on the computation of the K-theory in a special class of symmetric monoidal categories.

Question 8.3.2 ('Transfer Principle'). Suppose S is a symmetric monoidal groupoid freely generated by a sub-groupoid G and the objects of G have no non-trivial relations with respect to the monoidal operation of S. Is the K-theory of S freely generated by the data present in G?

The terms and phrases used in the above question are not well-formulated. The next section will provide more examples to justify the author's belief in this principle.

8.4 Two weaker forms of freeness

Consider the category $\mathcal{S}(M_{\mathcal{R}})$ for a right $\mathcal{R}$-module $M_{\mathcal{R}}$. Further let T denote the theory of the module $M_{\mathcal{R}}$.

If the theory T satisfies $T = T^{\aleph_0}$, then, for each $n \geq 1$, the basis set $\mathcal{L}_n$ consisting of the fundamental objects satisfies the following properties which essentially determine the K-theory of such modules as discussed in Section 8.3.

- The set $\mathcal{L}_n$ is closed under meets (intersections).
- The model-theoretic condition $T = T^{\aleph_0}$ is equivalent to the lattice-theoretic statement that every element of $\mathcal{L}_n$ considered as an element of the lattice $\mathcal{A}_n$

is 'join-irreducible', i.e., no $P \in \mathcal{L}_n$ can be expressed as a join (union) of finitely many proper *pp*-subsets of P.

In this section we describe the two situations where exactly one of these two properties hold.

Chapter 6 deals with the case when the theory T of the module $M_{\mathcal{R}}$ does not satisfy the condition $T = T^{\aleph_0}$. In this case, for each $n \geq 1$, the basis set $\mathcal{L}_n$ of the boolean algebra $\mathrm{Def}(M^n)$ is still a meet-semilattice, but it contains some join-reducible elements. We need to take the quotient of the free boolean algebra constructed in the previous section by the non-trivial finite join-relations present in $\mathcal{L}_n$ to obtain the boolean algebra $\mathrm{Def}(M^n)$.

The Grothendieck ring $K_0(M_{\mathcal{R}})$ of such a module is shown to be the integral monoid ring - which one obtains if the non-trivial join-relations were absent - modulo the ideal which encodes these relations (Theorem 6.2.3). One achieves this by modifying the techniques of Chapter 5. This shows that one could apply the 'transfer principle' to the free boolean algebra before taking the quotient to obtain the Grothendieck ring of the quotient of the free groupoid.

Section 3.3 defines the K-theory of a structure M which is functorial on elementary embeddings. In the case of modules, elementary equivalence gives isomorphism of the K-theory but pure maps (Section 6.3) provide an interesting class of maps on which the K-theory is contravariant. It will be interesting to study the long exact sequence of the K-theory in both cases.

Algebraic K-theory associates, for each $n \geq 0$, an abelian group $K_n^{\oplus}(\mathcal{R})$ to a unital ring $\mathcal{R}$ via the monoidal category $(\text{Proj-}\mathcal{R}, 0, \oplus)$. We can also consider $\mathcal{R}$ as an L_{ring}-structure and construct the groups $K_n^{\sqcup}(\mathcal{R})$.

Question 8.4.1. Suppose $\mathcal{R}$ is a unital ring and $M_{\mathcal{R}}$ is a right $\mathcal{R}$-module. What is the relation between the algebraic K-theory $K_*^{\oplus}(\mathcal{R})$, and the model-theoretic K-theory $K_*^{\sqcup}(\mathcal{R})$ of the ring $\mathcal{R}$? More specifically, what is the relation between the rings $K_0^{\oplus}(\mathcal{R})$, $K_0^{\sqcup}(\mathcal{R})$ and the ring $K_0(\text{Mod-}\mathcal{R})$ (Lemma 6.3.8)? What is the relation between either of these with the model-theoretic K-theory $K_*^{\sqcup}(M_{\mathcal{R}})$ of the module $M_{\mathcal{R}}$?

The other situation is dealt with in Chapter 7. The theory of the algebraically closed field k eliminates quantifiers. Every object of the category Var_k is piecewise isomorphic to an object of the category $\mathcal{S}(k)$, but given two objects A, B of $\mathcal{S}(k)$, the set $\mathrm{Var}_k(A, B)$ has fewer morphisms than the set $\mathcal{S}(k)(A, B)$. Nonetheless, both categories are symmetric monoidal categories under disjoint union. The model-theoretic Grothendieck ring $K_0(k)$ is a quotient of $K_0(\mathrm{Var}_k)$. The following question is natural and is expected to admit a positive answer.

Question 8.4.2. Suppose k is an algebraically close field. Is the model-theoretic Grothendieck ring $K_0(k)$ isomorphic to $K_0(\mathrm{Var}_k)$?

Every object of the category $\mathcal{S}(k)$ is a finite boolean combination of irreducible varieties, i.e., the varieties that cannot be expressed as a union of two proper subsets that are closed in the Zariski topology. In fact there is a unique representation theorem for objects of $\mathcal{S}(k)$ in terms of irreducible Zariski closed varieties (see [15]) analogous to Corollary 4.3.5 for definable sets in modules (the proof uses the facts that the Zariski topology is Noetherian and that the irreducible components of a variety are uniquely defined). Thus, for each $n \geq 1$, the set $\mathcal{L}_n$ of irreducible closed subvarieties of k^n can be thought of as the basis of $\mathrm{Def}(k^n)$ and each element of $\mathcal{L}_n$ is join-irreducible as an element of $\mathrm{Def}(k^n)$; this makes the notion of irreducibility an equivalent of the condition $\mathrm{T} = \mathrm{T}^{\aleph_0}$. The set $\mathcal{L}_n$ is not closed under intersections, but such an intersection is a finite union of its elements. This makes the set $\mathcal{A}_n$ of finite antichains in $\mathcal{L}_n$ a distributive lattice. The map assigning the birational equivalence class to an element in $\mathcal{L}_n$ is the appropriate analogue of the map assigning the colour to a *pp*-definable set.

Let us look at how this is reflected in the structure of the Grothendieck ring $K_0(k)$ (conjecturally isomorphic to the ring $K_0(\mathrm{Var}_k)$): Theorem 7.4.1 states that the Grothendieck group $K_0^+(\mathrm{Var}_k)$ is freely generated by the birational equivalence classes of varieties and Theorem 7.5.1 presents the associated graded ring of the Grothendieck ring $K_0(\mathrm{Var}_k)$ as a monoid ring. These presentations are, of course, under the hypothesis of a positive answer to Question 7.1.2 regarding the structure of the Grothendieck semiring. The author's belief in the transfer principle motivates the following.

Conjecture 8.4.3. Suppose k denotes an algebraically closed field. The conclusions of Theorem 7.4.1 and Theorem 7.5.1 hold true irrespective of the answer to Question 7.1.2.

This conjecture is not known to be equivalent to a positive answer to Question 7.1.2. Note the effect of the subtle difference in the structure of the posets $\mathcal{L}_n$ in the two cases. The Grothendieck ring of a module with condition $\mathrm{T} \neq \mathrm{T}^{\aleph_0}$ is a quotient of the integral monoid ring, but we expect the associated graded ring of the Grothendieck ring of varieties to be a monoid ring.

Now we point out some obstacles in the way if we try to follow the techniques of the module case in the case of varieties. Since the intersection of two irreducible closed subvarieties of k^n is not irreducible, one will need to use a geometric/topological notion of localization or intersection theory to define the local characteristics. Preserving the local characteristics under variety isomorphisms/definable isomorphisms is another issue since varieties have curvature.

Suppose $\mathbb{C}$ denotes the complex field. It is known [25, Cor. 5.8] that $K_0(\mathbb{C})$ contains the polynomial ring $\mathbb{Z}[\{x_i : i \in \mathfrak{c}\}]$, where $\mathfrak{c}$ is the cardinality of the continuum. A polynomial ring is naturally a monoid ring over the free monoid of monomials. One can ask the following question in view of Theorem 7.5.1.

Question 8.4.4. Let k be an algebraically closed field. Is the Grothendieck ring $K_0(\mathrm{Var}_k)$ a monoid ring?

Based on the two cases discussed in this section, we can state the following well-formulated instance of the 'transfer principle' for model-theoretic Grothendieck rings.

Question 8.4.5. Are there any other model-theoretic structures admitting some form of elimination of quantifiers whose Grothendieck rings can be computed using the techniques developed in this thesis?

8.5 Further questions

The atomic formulas in the language $L_\mathcal{R}$ of right $\mathcal{R}$-modules are linear equations whereas the atomic formulas in the language of rings (together with a symbol for subtraction) are polynomial equations. Thus the complexity of definable sets in modules is only a "1-dimensional fraction" of the complexity in algebraically closed fields.

For a field k, the Grothendieck ring of varieties $K_0(\mathrm{Var}_k)$ is the value ring for the universal motivic measure on k-varieties and thus is closely related to the theory of motivic integration. We have shown (Theorem 5.1.6) that the local characteristics are valuations in the sense of [23], i.e., finitely additive measures. Local characteristics "measure" the rigid structure of a definable set while global characteristics "measure" the structure of its definable isomorphism class. This motivates the following question which asks for a "1-dimensional fraction" (as discussed in Section 8.4) of the theory of motivic integration for k-varieties.

Question 8.5.1. Is there a reasonable theory of motivic integration on definable subsets of modules?

Flenner and Guingona studied directed families of sets and the unique representation theorem for sets constructible in these families [12]. This result has many interesting model-theoretic consequences including the (1-dimensional) elimination of imaginaries. We ask if Corollary 4.3.5 can be used to generalise these results when a directed family is replaced by a meet-semilattice.

Finally we collect some questions regarding the existence of certain special elements in the Grothendieck rings.

Corollary 6.4.3 demonstrates the existence of modules whose Grothendieck ring contains additive torsion elements. But the author believes that there are no examples with non-trivial multiplicative torsion elements (i.e., elements $a \in K_0(M)$ such that $a^n = 1$ for some $n > 1$).

Conjecture 8.5.2. There are precisely two units (namely ± 1) in the Grothendieck ring $K_0(M_{\mathcal{R}})$ of a nonzero module $M_{\mathcal{R}}$.

Recall that an element a of a ring is said to be **regular** if it is not a zero divisor. The following question is important for better understanding of the Grothendieck ring of varieties and is open even in the case of algebraically closed fields.

Question 8.5.3. Let k be a field of characteristic 0. Suppose $\mathbb{A}^1_k$ denotes the affine line over k. Is $\mathbb{L} := [\mathbb{A}^1_k]$ a regular element of $K_0(\mathrm{Var}_k)$?

Lemma 4.8 and Proposition 4.9 in [37] connect this question to Question 7.1.2 in special cases, but no further development has been made.

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