Stochastic Dynamics of Financial Markets

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Abstract

This thesis provides a study on stochastic models of financial markets related to problems of asset pricing and hedging, optimal portfolio managing and statistical changepoint detection in trends of asset prices.

Chapter 1 develops a general model of a system of interconnected stochastic markets associated with a directed acyclic graph. The main result of the chapter provides sufficient conditions of hedgeability of contracts in the model. These conditions are expressed in terms of consistent price systems, which generalise the notion of equivalent martingale measures. Using the general results obtained, a particular model of an asset market with transaction costs and portfolio constraints is studied.

In the second chapter the problem of multi-period utility maximisation in the general market model is considered. The aim of the chapter is to establish the existence of systems of supporting prices, which play the role of Lagrange multipliers and allow to decompose a multi-period constrained utility maximisation problem into a family of single-period and unconstrained problems. Their existence is proved under conditions similar to those of Chapter 1.

The last chapter is devoted to applications of statistical sequential methods for detecting trend changes in asset prices. A model where prices are driven by a geometric Gaussian random walk with changing mean and variance is proposed, and the problem of choosing the optimal moment of time to sell an asset is studied. The main theorem of the chapter describes the structure of the optimal selling moments in terms of the Shiryaev–Roberts statistic and the posterior probability process.

Declaration

No portion of the work referred to in the thesis has been submitted in support of an application for another degree or qualification of this or any other university or other institute of learning.

Copyright Statement

Acknowledgements

List of notations

$\mathbb{R}$ the set of real numbers

$\mathbb{R}_+$ the set of non-negative real numbers

L^1 the space of all integrable functions on a measure space

L^∞ the space of all essentially bounded functions on a measure space

X^* the positive dual cone of a set X in a normed space

$\mathrm{P} \mid \mathscr{F}$ the restriction of a measure P to a σ-algebra $\mathscr{F}$

x^+ $\max\{x, 0\}$

x^- $-\min\{x, 0\}$

$\mathbf{I}\{A\}$ the indicator of a statement A ($\mathbf{I}\{A\} = 1$ if A is true, $\mathbf{I}\{A\} = 0$ if A is false)

$\mathcal{T}_{t+}$ the set of all successors of a node t in a graph $\mathcal{T}$

$\mathcal{T}_{t-}$ the set of all predecessors of a node t in a graph $\mathcal{T}$

$\mathcal{T}_-$ the set of all nodes in a graph $\mathcal{T}$ with at least one successor

$\mathcal{T}_+$ the set of all nodes in a graph $\mathcal{T}$ with at least one predecessor

$\mathscr{N}(\mu, \sigma^2)$ Gaussian distribution with mean μ and variance σ^2

Φ the standard Gaussian cumulative distribution function ($\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-y^2/2} dy$)

a.s.   "almost surely" (with probability one)

i.i.d.   independent and identically distributed (random variables)

Introduction

This thesis provides a study on stochastic models of financial markets. Questions of derivatives pricing and hedging, optimal portfolio managing and detection of changes in asset prices trends are considered.

The first range of questions – hedging and pricing of derivative securities – has been studied in the literature since 1960s. The celebrated Black–Scholes formula [3] for the price of a European option was one of the first fundamental results in this direction. *Derivative securities* (or *contingent claims*) play an important role in modern finance as they allow to implement complex trading strategies which reduce the risk from indeterminacy of future asset prices. One of the main questions in derivatives trading consists in determining the fair price of a derivative, which satisfies both the seller and the buyer. The Black–Scholes formula provides an explicit answer to this question in the model when asset prices are modelled by a geometric Brownian motion. Later this result was extended to a wider class of market models. The development of the derivatives pricing theory has resulted in that nowadays the volume of derivatives traded is much higher than the volume of basic assets [41].

The second range of questions considered in the thesis concerns consumption-investment problems, where a trader needs to manage a portfolio of assets choosing how much to consume in order to maximise utility over a period of time. Questions of this type were originally studied in relation to models of economic growth, where the objective is to find a trade-off between goods produced and consumed with the aim of the optimal development of the economy. One of the first results in the financial context was obtained by Merton [42], who provided an explicit solution of the consumption-investment problem for a model when asset prices are driven by a geometric Brownian motion. There have been several extensions of Merton's result which include factors like transaction costs, possibility of bankruptcy, general classes of stochastic processes describing asset prices, etc. (see e.g. [4, 10, 36, 44]).

The third part of the thesis contains applications of sequential methods of mathematical statistics to detecting changes in asset prices trends. The mathematical foundation of the corresponding statistical methods – the theory of *changepoint detection* (or *disorder detection*) – was laid in the papers by W. Shewhart, E. S. Page, S. W. Roberts, A. N. Shiryaev, and others in 1920-1960s, and initially was applied in questions of production quality control and radiolocation. Recently, these methods have gained attention in finance.

The main results of the thesis generalise the classical theory to advanced market models. We obtain results that broaden the models available in the literature and reflect several important features of real markets that have not been studied earlier in the corresponding fields.

The rest of the introduction provides a detailed description of the problems considered in the thesis.

Asset pricing and hedging

Consider the classical model of a stochastic financial market, which operates at discrete moments of time $t = 0, 1, \ldots, T$, and where N assets are traded. The stochastic nature of the market is represented by a filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t=0}^T, \mathrm{P})$, where each σ-algebra $\mathscr{F}_t$ in the filtration $\mathscr{F}_0 \subset \mathscr{F}_1 \subset \ldots \mathscr{F}_T$ describes random factors that might affect the market at time t.

The prices of the assets at time t are given by $\mathscr{F}_t$-measurable strictly positive random variables $S_t^1, S_t^2, \ldots, S_t^N$. Asset 1 is assumed to be riskless (e. g. cash deposited with a bank account) with the price $S_t^1 \equiv 1$ (after being discounted appropriately), while assets $i = 2, \ldots, N$ are risky with random prices.

An investor can trade in the market by means of buying and selling assets. A trading strategy is a sequence $x_0, x_1, \ldots, x_T$ of random N-dimensional vectors, where each $x_t = (x_t^1, \ldots, x_t^N)$ is $\mathscr{F}_t$-measurable and specifies the

portfolio held by the investor between the moments of time t and $t+1$. The coordinate x_t^i ($i = 1, \ldots, N$) is equal to the amount of physical units of asset i in the portfolio.

An important class of trading strategies consists of *self-financing trading strategies*, which have no exogenous inflow or outflow of money. Namely, a trading strategy $(x_t)_{t \leqslant T}$ is called self-financing if

$$x_{t-1} S_t = x_t S_t \text{ for each } t = 1, \ldots, T,$$

where the left-hand side is the value of the old portfolio (established "yesterday"), and the right-hand side is the value of the new portfolio (established "today"). The equality is understood to hold with probability one.

The central question of the derivatives pricing and hedging theory consists in finding fair prices of *derivative securities* (or *contingent claims*). A derivative is a financial instrument which has no intrinsic value in itself, but *derives* its value from underlying basic assets [15]. Derivative securities include options, futures, swaps, and others (see e. g. [31]).

As an example, consider a standard European call option on asset i, which is a contract that allows (but does not oblige) its buyer to buy one unit of asset i at a fixed time T in the future for a fixed price K. The seller incurs a corresponding obligation to fulfil the agreement if the buyer decides to exercise the contract, which she does if the spot price S_T^i is greater than K (thus receiving the gain $S_T^i - K$). In order to obtain the option, the buyer pays the seller some premium, the *price of the option*, at time $t = 0$.

Another example is a European put option, which is a contract giving its buyer the right to sell asset i at a fixed time T for a fixed price K. If the buyer exercises the option (which happens whenever $K > S_T^i$), she receives the gain $K - S_T^i$.

Mathematically, contracts of this type can be identified with random variables X representing the payoff of the seller to the buyer at time T. For example, $X = (S_T - K)^+$ for a European call option, and $X = (K - S_T)^+$ for a European put option.

It is said that a self-financing trading strategy $(x_t)_{t \leqslant T}$ *(super)hedges* a

derivative X if

$$x_T S_T \geqslant X \text{ a.s.,}$$

i. e. the seller who follows the strategy $(x_t)_{t \leqslant T}$ can fulfil the payment associated with the derivative with probability one. The minimal value x of the initial portfolio x_0 is called the (upper) hedging price of X and is denoted by $\mathcal{C}(X)$:

$$\mathcal{C}(X) = \inf\{x : \text{there exists } (x_t)_{t \leqslant T} \text{ superhedging } X \text{ such that } x_0 S_0 = x\}.$$

The price $\mathcal{C}(X)$ is the minimal value of the initial portfolio that allows the seller to fulfil her obligations (provided that the infimum in the definition is attained; see [61, Ch. VI, § 1b-c]). On the other hand, if she could sell the derivative for a higher price $\widetilde{\mathcal{C}} > \mathcal{C}(X)$, it would be possible to find a trading strategy which delivers her a *free-lunch* – a non-negative and non-zero gain by time T, for which the buyer has no incentive to agree.

The central result of the asset pricing and hedging theory states that in a market without *arbitrage opportunities* the price of a contingent claim can be found as the supremum of its expected value with respect to *equivalent martingale measures*.

It is said that a self-financing trading strategy $(x_t)_{t \leqslant T}$ realises an *arbitrage opportunity* in the market if

$$x_0 S_0 = 0, \qquad x_T S_T \geqslant 0 \text{ and } \mathrm{P}(x_T S_T > 0) > 0.$$

A probability measure $\widetilde{\mathrm{P}}$, equivalent to the original measure P $(\widetilde{\mathrm{P}} \sim \mathrm{P})$, is called an *equivalent martingale measure* (EMM), if the price sequence S is a $\widetilde{\mathrm{P}}$-martingale, i. e. $\mathrm{E}^{\widetilde{\mathrm{P}}}(S_t^i \mid \mathscr{F}_{t-1}) = S_{t-1}^i$ for all $t = 1, \ldots, T$, $i = 1, \ldots, N$. The set of all EMMs is denoted by $\mathscr{P}(\mathrm{P})$.

These two notions express some form of market efficiency. The absence of arbitrage opportunities means that there is no trading strategy with zero initial capital, which allows to obtain a non-zero gain without downside risk (a free lunch) at time T. The existence of an equivalent martingale measure allows to change the underlying measure P, preserving the sets of zero probability, in a way that the assets have zero return rates.

Theorem. *Equivalent martingale measures exist in a market if and only if there is no arbitrage opportunities.*

In a market without arbitrage opportunities, the price of a contingent claim X such that $\mathrm{E}^{\widetilde{\mathrm{P}}}|X| < \infty$ for any $\widetilde{\mathrm{P}} \in \mathscr{P}(\mathrm{P})$ can be found as

$$\mathcal{C}(X) = \sup_{\widetilde{\mathrm{P}} \in \mathscr{P}} \mathrm{E}^{\widetilde{\mathrm{P}}} X.$$

Remark. In the case when there is only one equivalent martingale measure (the case of a *complete market*), $\mathcal{C}(X) = \mathrm{E}^{\widetilde{\mathrm{P}}} X$. It turns out that a complete market has a simple structure – the σ-algebra $\mathscr{F}_T$ is purely atomistic with respect to P and consists of no more that N^T atoms. Note that in continuous time, however, there exist examples of complete markets where $\mathscr{F}_T$ is not purely atomistic (e. g. the model of geometric Brownian motion).

The above theorem is referred to as the Fundamental Theorem of Asset Pricing and the Risk-Neutral Pricing Principle (see e. g. [24, 61]). It constitutes the core of the classical derivatives pricing theory. However it does not take into account several important features of real markets, which are necessary to consider when applying the theory in practice. The present thesis addresses these issues and develops a model that includes the following improvements.

1. Transaction costs and portfolio constraints. The act of buying or selling assets in a real market typically reduces the total wealth of a trader (due to broker's commission, differences in bid and offer prices, etc.). As a result, an investor may need to limit the number of trading operations in order not to lose too much money on transaction costs. Real markets also set constraints on admissible portfolios in order to prevent market participants from using too risky trading strategies. For example, such constraints can be expressed in a form of a margin requirement, which obliges investors to choose only those strategies that allow to liquidate their portfolios if prices move unfavourably.

Both transaction costs and portfolio constraints limit investor abilities, and thus, generally, increase hedging prices. These aspects have already been

considered in the literature, but for the most part separately. One can point, e.g., to the monograph by Kabanov and Safarian [33] discussing transaction costs, the papers by Jouini, Kallal [32] and Evstigneev, Schürger, Taksar [22] dealing with portfolio constraints, and the references therein.

It turns out that under the presence of transaction costs and portfolio constraints, the problem of pricing contingent claims has a solution similar to the classical model. Namely, the price of a contingent claim can be found as the supremum of its expected value with respect to *consistent price systems*, which are vector analogues of equivalent martingale measures (see Chapter 1 for details). However, the existence of consistent price systems in a market with transaction costs and portfolio constraints becomes a considerably difficult question, and, generally, requires conditions stronger than the absence of arbitrage opportunities. Several stronger conditions have been introduced in the literature which guarantee the existence of consistent price systems, thus allowing to price contingent claims. Their formulations can be found in e.g. [22, 33].

2. Hedging with risk. The classical superhedging condition requires that the seller of a derivative chooses a trading strategy that covers the payment with probability one, i.e. without any risk of non-fulfilling her obligation. However, it may be acceptable for the seller to guarantee the required amount of payment only with some (high) level of confidence – for example, in unfavourable outcomes she may use exogenous funds, which is compensated by a higher gain in favourable outcomes.

This is especially important when the volume of trading is large, so not the result of every single deal is important, but only the average result of a large number of them. Weakening the superhedging criterion can possibly reduce derivatives prices and lead to a potential gain while maintaining an acceptable level of risk of unfavourable situations.

An approach of hedging with risk, used in the thesis, is based on replacing the superhedging condition with a general principle requiring that the difference between the required payment and the portfolio used to cover it belongs to a certain set of acceptable portfolios (the exact formulation will

be given in Chapter 1). The superhedging condition is a particular case of this model.

This approach has already been used in the literature. Especially, much attention has been devoted to hedging with respect to coherent and convex risk measures (see e. g. the papers [7, 9], where it is called the *no good deals pricing principle*).

3. Multimarket trading. The framework developed in the thesis is capable of modelling a *system of interconnected markets* associated with nodes of a given acyclic directed graph. The nodes of the graph represent different trading sessions that may be related to different moments of time and/or different asset markets. In particular, the standard model of a single market can be represented by the graph being the linearly ordered set of moments of time, while the general case allows to consider the problem of distributing assets between several markets, which may operate at the same or different moments of time.

We also consider contracts with payments at arbitrary trading sessions (not only the terminal ones), which broadens the range of possible financial instruments. This also makes the model potentially applicable not only in finance, but in other areas, e.g. it can be used in insurance, where an insurer receives a premium at time $t = 0$ and needs to manage a portfolio in order to be able to cover claims occurring randomly.

The above features of real markets have been studied in the literature for the most part separately. In the thesis a new general model is proposed, which incorporates all of them. The central result of the first chapter for the new model is the hedging criterion formulated in terms of consistent price systems – direct analogues of equivalent martingale measures. We prove their existence and show that a contract is hedgeable if and only if its value with respect to any consistent price system is non-negative. In order to obtain the result, we systematically use the idea of margin requirements which limit allowed leverage of admissible portfolios. This differs from the standard approach based on the absence of arbitrage. However, margin requirements always present in one form or another in any real market, which makes our

approach fully justified from the applied point of view.

The model is based on the framework of von Neumann – Gale dynamical systems introduced by von Neumann [68] and Gale [25] for deterministic models of a growing economy, and later extended by Dynkin, Radner and their research groups to the stochastic case. A model of a financial market based on von Neumann – Gale systems was also proposed in the paper [11] for the case of a discrete probability space $(\Omega, \mathscr{F}, \mathrm{P})$ and a linearly ordered set of moments of time.

In the thesis this framework is extended to financial market models, which have several important distinctions from models of growing economies.

Optimal trading strategies

In the second chapter the problem of finding trading strategies that maximise the utility function of an investor over a period of time is studied.

In the financial literature, problems of this type are commonly referred to as consumption-investment problems, and there exists a large number of results in this area. The subject of our research is the optimal investment problem for the general model proposed in Chapter 1. The goal is to obtain conditions for the existence of *supporting prices*, which allow to reduce a multi-period constrained maximisation problem of investor's utility function to a family of single-stage unconstrained problems. Supporting prices play the role similar to that of Lagrange multipliers.

The problem of utility maximisation and existence of supporting prices plays a central role in the von Neumann – Gale framework of economic growth. The setting of the problem and the main results there consist in the following.

A von Neumann – Gale system is a sequence of pairs of random vectors (x_t, y_t) $t = 0, 1, \ldots, T$, such that $(x_t, y_t) \in \mathcal{Z}_t$ for some given sets $\mathcal{Z}_t$, and $x_t \leqslant y_{t-1}$. In the financial interpretation, a vector x_t can be regarded as a portfolio of assets held before a trading session t, and y_t as a portfolio

obtained during the session. The inequality $x_t \leqslant y_{t-1}$ means that there is no exogenous infusion of assets (but disposal of assets is allowed). The sets $\mathcal{Z}_t$ can describe the self-financing condition, portfolio constraints, etc.

In models of economic growth vectors x_t describe amounts of input commodities for a production process with output commodities y_t. The sets $\mathcal{Z}_t$ consist of all possible production processes and the condition $x_t \leqslant y_{t-1}$ reflects the requirement that the input of any production process should not exceed the output of the previous one.

Suppose that with each set $\mathcal{Z}_t$ a real-valued utility function $u_t(x_t, y_t)$ is associated and interpreted as the utility from the production process with input commodities x_t and output commodities y_t. Let $\overline{x}_0$ be a given vector of initial resources. Then the problem consists in finding a production process $\zeta = (x_t, y_t)_{t \leqslant T}$, represented by a von Neumann – Gale system, such that $x_0 = \overline{x}_0$ and which maximises the utility

$$u(\zeta) := \sum_{t=0}^{T} u_t(x_t, y_t).$$

This is a constrained maximisation problem of the function $u(\zeta)$ over all sequences $\zeta = (x_t, y_t)_{t \leqslant T}$ with $(x_t, y_t) \in \mathcal{Z}_t$ satisfying the constraints $x_t \leqslant y_{t-1}$. Under some assumptions the solution ζ^* of the problem exists, and, moreover, there exist random vectors p_t, $t = 0, \ldots, T+1$ such that

$$u(\zeta^*) \geqslant \sum_{t=0}^{T} \big(u_t(x_t, y_t) + \mathrm{E}[y_t p_{t+1} - x_t p_t] \big) + \overline{\mathrm{E}} x_0 p_0$$

for any sequence $(x_t, y_t)_{t \leqslant T}$ with $(x_t, y_t) \in \mathcal{Z}_t$. Thus ζ^* solves the unconstrained problem of maximising the right hand side of the above inequality.

Moreover, in order to maximise the right-hand size it is sufficient to maximise each term $u_t(x_t, y_t) + \mathrm{E}[y_t p_{t+1} - x_t p_t]$ independently. Gale [26] noted the great importance of results of this type by saying that it is "the single most important tool in modern economic analysis both from the theoretical and computational point of view".

The aim of the second chapter is to obtain similar results for our model of interconnected financial markets. The main mathematical difficulty here

consists in that in our model portfolios (x_t, y_t) may have negative coordinates (corresponding to *short sales*), unlike commodities vectors in models of economic growth. The key role in establishing the main results of the second chapter will be played by the assumption of margin requirements. The existence of supporting prices will be proved under a condition on the size of the margin.

Detection of trend changes in asset prices

The third part of the thesis studies statistical methods of change detection in trends of asset prices. We consider a model, where prices initially rise, but may start falling at a random (and unknown) moment of time. The aim of an investor is to detect this change in the prices trend and to sell the asset as close as possible to its highest price.

It will be assumed that the price of an asset is represented by a geometric Gaussian random walk $S = (S_t)_{t=0}^T$ defined on a probability space $(\Omega, \mathscr{F}, \mathrm{P})$, whose drift and volatility coefficients may change at an unknown time θ:

$$S_0 > 0, \qquad \log \frac{S_t}{S_{t-1}} = \begin{cases} \mu_1 + \sigma_1 \xi_t, & t < \theta, \\ \mu_2 + \sigma_2 \xi_t, & t \geqslant \theta, \end{cases} \quad \text{for } t = 1, 2, \ldots, T,$$

where $\mu_1, \mu_2, \sigma_1, \sigma_2$ are known parameters, $\xi_t \sim \mathscr{N}(0,1)$ are i.i.d. normal random variables with zero mean and unit variance, and θ is the moment of time when the probabilistic character of the price sequence changes.

In order to model the uncertainty of the moment θ, it will be assumed that θ is a random variable defined on $(\Omega, \mathscr{F}, \mathrm{P})$, but an investor can observe only the information included in the filtration $\mathbb{F} = (\mathscr{F}_t)_{t=0}^T$, $\mathscr{F}_t = \sigma(S_u; u \leqslant t)$, generated by the price sequence, and cannot observe θ directly. The distribution law of θ is known, and θ takes values $1, 2, \ldots, T$ with known probabilities $p_t \geqslant 0$, so that $\sum_{t=1}^T p_t \leqslant 1$. The quantity $p_{T+1} = 1 - \sum_{t=1}^T p_t$ is the probability that the change of the parameters does not occur until the final

time T and p_1 is the probability that the logarithmic returns already follow $\mathscr{N}(\mu_2, \sigma_2)$ since the initial moment of time.

By definition, a moment τ when one can sell the asset should be a *stopping time* of the filtration $\mathbb{F}$, which means that $\{\omega : \tau(\omega) \leqslant t\} \in \mathscr{F}_t$ for any $0 \leqslant t \leqslant T$. The notion of a stopping time expresses the idea that the decision to sell the asset at time t should be based only on the information available from the price history up to time t, and does not rely on future prices. The class of all stopping times $\tau \leqslant T$ of the filtration $\mathbb{F}$ is denoted by $\mathfrak{M}$.

The problem we consider consists in maximising the power or logarithmic utility from selling the asset. Namely, let $U_\alpha(x) = \alpha x^\alpha$ for $\alpha \neq 0$ and $U_0(x) = \log x$. For an arbitrary $\alpha \in \mathbb{R}$ we consider the *optimal stopping problem*

$$V_\alpha = \sup_{\tau \in \mathfrak{M}} \mathrm{E} U_\alpha(S_\tau).$$

The problem consists in finding the value V_α, which is the maximum expected utility one can obtain from selling the asset, and finding the stopping time τ_α^* at which the supremum is attained (we show that it exists).

The study of methods of detecting changes in probabilistic structure of random sequences and processes (called *disorder detection problems* or *changepoint detection problems*) began in the 1950-1960s in the papers by E. Page, S. Roberts, A. N. Shiryaev and others (see [45, 46, 49, 57–59]); the method of control charts proposed by W. A. Shewhart in the 1920s [55] is also worth mentioning.

A financial application of changepoint detection methods was considered in the paper [2] by M. Beibel and H. R. Lerche, who studied the problem of choosing the optimal time to sell the asset in continuous time, when the asset price process $S = (S_t)_{t \geqslant 0}$ is modelled by a geometric Brownian motion whose drift changes at time θ:

$$dS_t = S_t[\mu_1 \mathbf{I}(t < \theta) + \mu_2 \mathbf{I}(t \geqslant \theta)]dt + \sigma S_t dB_t, \qquad S_0 > 0,$$

where $B = (B_t)_{t \geqslant 0}$ is a standard Brownian motion on a probability space $(\Omega, \mathscr{F}, \mathrm{P})$, and μ_1, μ_2, σ are known real parameters. The paper [2] assumes

that θ is an exponentially distributed random variable with a known parameter $\lambda > 0$ and is independent of B. An investor looks for the stopping time τ^* of the filtration generated by the process S that maximises the expected gain $\mathrm{E}S_\tau$ (the time horizon in the problem is $T = \infty$, i.e. τ can be unbounded). By changing the parameters μ_1, μ_2, σ one can solve the problem of maximising $\mathrm{E}S_\tau^\alpha$ for any α except $\alpha = 0$.

Beibel and Lerche show that if μ_1, μ_2, σ satisfy some relation, then the optimal stopping time τ^* can be found as the first moment of time when the posterior probability process $\pi = (\pi_t)_{t\geqslant 0}$, $\pi_t = \mathrm{P}(\theta \leqslant t \mid \mathscr{F}_t)$, exceeds some level $A = A(\mu_1, \mu_2, \sigma, \lambda)$:

$$\tau^* = \inf\{t \geqslant 0 : \pi_t \geqslant A\}.$$

In other words, the optimal stopping time has a very clear interpretation: one needs to sell the asset as soon as the posterior probability that the change has happened exceeds a certain threshold. An explicit representation of π_t through the observable process S_t is available (see e.g. [47, Section 22]).

In the paper [14] the conditions on μ_1, μ_2, σ were relaxed and it was shown that the result holds for all possible values of the parameters (except some trivial cases). Also, the optimal threshold was found explicitly as $A = A'/(1 + A')$, where the constant $A' = A'(\mu_1, \mu_2, \sigma, \lambda)$ is the unique positive root of the (algebraic) equation

$$2 \int_0^\infty e^{-at} t^{(b+\gamma-3)/2}(1 + A't)^{\gamma-b+1)/2} dt$$
$$= (\gamma - b + 1)(1 + A') \int_0^\infty e^{-at} t^{(b+\gamma-1)/2}(1 + A't)^{(\gamma-b-1)/2} dt$$

with the parameters

$$a = \frac{2\lambda}{\nu^2}, \qquad b = \frac{2}{\nu}\left(\frac{\lambda}{\nu} - \sigma\right), \qquad \gamma = \sqrt{(b-1)^2 + 4c},$$

where

$$\nu = \frac{\mu_2 - \mu_1}{\sigma}, \qquad c = \frac{2(\lambda - \mu_2)}{\nu^2}.$$

The paper [56] studied the problem of maximising the logarithmic utility

from selling the asset with finite time horizon T (but still assuming that θ is exponentially distributed, so the parameters do not change until the end of the time horizon with positive probability). The solution was based on an earlier result of the paper [28]. It was shown that the optimal stopping time can be expressed as the first moment of time when π_t exceeds some *time-dependent* threshold:

$$\pi_t = \inf\{t \geqslant 0 : \pi_t \geqslant a^*(t)\},$$

where $a^*(t)$ is a function on $[0, T]$, dependent on λ, μ_1, μ_2, σ. The authors showed that it can be found as a solution of some nonlinear integral equation. They also briefly discussed the optimal stopping problem for the linear utility function with a finite time horizon, and reduced it to some two-dimensional optimal stopping problem for the process π_t, but did not provide its explicit solution.

The problem on a finite time horizon was solved in the paper [69] by M. V. Zhitlukhin and A. N. Shiryaev for the both logarithmic and linear utility functions, provided that θ is *uniformly distributed* on $[0, T]$ (however, the solution can be generalised to a wide class of prior distributions of θ). In each problem, the optimal stopping time can be expressed as the first time when the value of π_t exceeds some function $\widetilde{a}^*(t)$ characterised by a certain integral equation. These equations can be solved numerically by "backward induction" as demonstrated in the paper.

This result was used by A. N. Shiryaev, M. V. Zhitlukhin and W. T. Ziemba in the research [66] on stock prices bubbles of Internet related companies. The method of changepoint detection was applied to the daily closing prices of Apple Inc. in 2009–2012 and the daily closing values of NASDAQ-100 index in 1994–2002. These two assets had spectacular runs from their bottom values and dramatic falls after reaching the top values, thus being good candidates to be modelled by processes with changepoints in trends. For specific dates of entering the market, the method provided exit points at approximately 75% of the maximum value of the NASDAQ-100 index, and 90% of the maximum price of Apple Inc. stock.

The aim of the third chapter of the thesis is to solve the optimal stopping problem for a geometric Gaussian random walk with a changepoint in discrete time and a finite time horizon for an arbitrary prior distribution of θ. It will be shown that the optimal stopping time can be expressed as the first moment of time when the sequence of the Shiryaev–Roberts statistic (which is obtained from the posterior probability sequence by a simple transformation) exceeds some time-dependent level. This result is similar to the results available in the literature for the case of continuous time. However it allows to consider any prior distribution of θ (not only exponential or uniform) and can be used in models where also the volatility coefficient σ changes.

A backward induction algorithm for computing the optimal stopping level is described in the chapter. Using it, we present numerical simulations of random sequences with changepoints and obtain the corresponding optimal stopping times.

Chapter 1

Multimarket hedging with risk

The results of this chapter extend the classical theory of asset pricing and hedging in several directions. We develop a general model including transaction costs and portfolio constraints and consider hedging with risk, which is "softer" than the classical superreplication approach. These aspects of the modelling of asset markets have already been considered in the literature, but for the most part separately. One can point, e.g., to the monograph by Kabanov and Safarian [33] discussing transaction costs, the papers by Jouini and Kallal [32] and Evstigneev, Schürger and Taksar [22] dealing with portfolio constraints and the studies by Cochrane and Saa-Requejo [9] and Cherny [7] involving hedging with risk. However, up to now no general model reflecting *all* these features of real financial markets has been proposed.

Another novel aspect of this study is that, in contrast with the conventional theory, we consider asset pricing and hedging in a *system of interconnected markets*. These markets (functioning at certain moments of discrete time) are associated with nodes of a given acyclic directed graph. The model involves *stochastic control of random fields* on directed graphs. Control problems of this kind were considered in the context of modelling economies with locally interacting agents in the series of papers by Evstigneev and Taksar [16–20].

In the case of a single market – when the graph is a linearly ordered set of moments of time – the model extends the one proposed by Dempster, Evstigneev and Taksar [11]. The approach of [11] was inspired by a parallelism between dynamic securities market models and models of economic growth. The underlying mathematical structures in both modelling frameworks are related to *von Neumann-Gale dynamical systems* (von Neumann [68], Gale [25]) characterised by certain properties of convexity and homo-

geneity. This parallelism served as a conceptual guideline for developing the model and obtaining the results.

The main results of the chapter provide general hedging criteria stated in terms of *consistent price systems*, generalising the notion of an equivalent martingale measure. Existence theorems for such price systems are counterparts of various versions of the well-known Fundamental Theorem of Asset Pricing (Harrison, Kreps, Pliska and others). However, the assumptions we impose to obtain hedging criteria are substantially distinct from the standard ones. We systematically use the idea of *margin requirements* on admissible portfolios, setting limits for the allowed *leverage*. Such requirements are present in one form or another in all real financial markets. Being fully justified from the applied point of view, they make it possible to substantially broaden the frontiers of the theory.

The chapter is organised as follows. In Section 1.1 we introduce the general model. In Section 1.2 we state and prove the main results. Section 1.3 contains examples of general hedging conditions, and Section 1.4 explains the connection between consistent price systems and equivalent martingale measures. By using the general results obtained, we study a specialised model of a stock market in Section 1.5. Several auxiliary results from functional analysis are assembled in Sections 1.6 and 1.7.

A shortened version of this chapter was published in the paper [23].

1.1 The model of interconnected markets and the hedging principle

Let $(\Omega, \mathscr{F}, \mathrm{P})$ be a probability space and G a directed acyclic graph with a finite set of nodes $\mathcal{T}$. The nodes of the graph represent different *trading sessions* that may be related to different moments of time and/or different asset markets. With each $t \in \mathcal{T}$ a σ-algebra $\mathscr{F}_t \subset \mathscr{F}$ is associated describing random factors that might affect the trading session t.

A measurable space $(\Theta, \mathscr{J}, \mu)$ with a finite measure μ is given, whose

points represent all available assets. If Θ is infinite, the model reflects the idea of a "large" asset market (cf. Hildenbrand [30]). For each $t \in \mathcal{T}$, a real-valued $(\mathrm{P} \otimes \mu)$-integrable $(\mathscr{F}_t \otimes \mathscr{J})$-measurable function $x : \Omega \times \Theta \to \mathbb{R}$ is interpreted as a *portfolio of assets* that can be bought or sold in the trading session t. The space of all (equivalence classes of) such functions with the norm $\|x\| = \int |x| d(\mathrm{P} \otimes \mu)$ is denoted by $L_t^1(\Omega \times \Theta)$ or simply L_t^1. The value $x(\omega, \theta)$ of a function x represents the number of "physical units" of asset θ in the portfolio x. Positive values $x(\omega, \theta)$ are referred to as *long positions*, while negative ones as *short positions* of the portfolio.

If ξ is an integrable real-valued measurable function defined on some measure space, by $\mathrm{E}\xi$ we denote the value of its integral over the whole space (when the measure is a probability measure, $\mathrm{E}\xi$ is equal to the expectation of ξ).

For a subset $T \subset \mathcal{T}$ we denote by $L_T^1(\Omega \times \Theta)$, or simply L_T^1, the space of functions $y \colon T \times \Omega \times \Theta \to R$ with finite norm $\|y\| = \sum_{t \in T} \int |y(t, \omega, \theta)| d(\mathrm{P} \otimes \mu)$. Where it is convenient, we represent such functions as families $y = (y_t)_{t \in T}$ of functions $y_t(\omega, \theta) = y(t, \omega, \theta)$.

The symbols $\mathcal{T}_{t-}$ and $\mathcal{T}_{t+}$ will be used to denote, respectively, the sets of all direct predecessors and successors of a node $t \in \mathcal{T}$, and $\mathcal{T}_-$, $\mathcal{T}_+$ will stand, respectively, for the set of all nodes having at least one successor and the set of all nodes having at least one predecessor (so that $\mathcal{T}_- = \bigcup_{t \in \mathcal{T}} \mathcal{T}_{t-}$ and $\mathcal{T}_+ = \bigcup_{t \in \mathcal{T}} \mathcal{T}_{t+}$). For the convenience of further notation we define $L_{t+}^1 = L_{\mathcal{T}_{t+}}^1$ for each $t \in \mathcal{T}_-$, and $L_{t-}^1 = L_{\mathcal{T}_{t-}}^1$ for each $t \in \mathcal{T}_+$.

In a trading session $t \in \mathcal{T}_-$ one can buy and sell assets and distribute them between trading sessions $u \in \mathcal{T}_{t+}$. This distribution is specified by a set of portfolios $y_t = (y_{t,u})_{u \in \mathcal{T}_{t+}} \in L_{t+}^1$, where $y_{t,u}$ is the portfolio delivered to the session u.

Trading constraints in the model are defined by some given (convex) cones $\mathcal{Z}_t \subset L_t^1 \otimes L_{t+}^1$, $t \in \mathcal{T}_-$. A *trading strategy* ζ is a family of functions $\zeta = (x_t, y_t)_{t \in \mathcal{T}_-}$ such that

$$(x_t, y_t) \in \mathcal{Z}_t \text{ for each } t \in \mathcal{T}_-.$$

Each function x_t represents the portfolio held before one buys and sells assets in the session t. For $t \in \mathcal{T}_-$, the function $y_t = (y_{t,u})_{u \in \mathcal{T}_{t+}}$ specifies the distribution of assets to the sessions $u \in \mathcal{T}_{t+}$.

Let $y_{t-} = \sum_{u \in \mathcal{T}_{t-}} y_{u,t}$ denote the portfolio of assets delivered to a trading session t from other sessions ($y_{t-} := 0$ if t is a *source*, i.e. has no predecessors). In each session $t \in \mathcal{T}$, the portfolio $y_{t-} - x_t$ can be used for the *hedging of a contract* (we define $x_t = 0$ if t is a *sink*, i.e. $t \notin \mathcal{T}_-$). By definition, a *contract* γ is a family of portfolios

$$\gamma = (c_t)_{t \in \mathcal{T}}, \qquad c_t \in L^1_t,$$

where c_t stands for the portfolio which has to be delivered – according to the contract – at the trading session t. The value of $c_t(\omega, \theta)$ can be negative; in this case the corresponding amount of asset θ is received rather than delivered. The notion of a contract encompasses contingent claims, derivative securities, insurance contracts, etc.

Assume that a non-empty closed cone $\mathcal{A} \subset L^1_{\mathcal{T}}$ is given. We say that a trading strategy $\zeta = (x_t, y_t)_{t \in \mathcal{T}_-}$ *hedges* a contract $\gamma = (c_t)_{t \in \mathcal{T}}$ if

$$(a_t)_{t \in \mathcal{T}} \in \mathcal{A}, \text{ where } a_t = y_{t-} - x_t - c_t. \tag{1.1}$$

Each a_t represents the difference between the portfolio $y_{t-} - x_t$ delivered at the session t by the strategy ζ and the portfolio c_t that must be delivered according to the contract γ. The cone $\mathcal{A}$ is interpreted as the set of all *risk-acceptable* families of portfolios. If $\mathcal{A}$ is the cone of all non-negative functions, then ζ is said to *superhedge (superreplicate)* γ. General cones $\mathcal{A}$ make it possible to consider *hedging with risk*.

A contract is called *hedgeable* if there exists a trading strategy hedging it. The main aim of our study is to characterise the class of hedgeable contracts.

Let $L^\infty_t = L^\infty_t(\Omega \times \Theta)$ denote the space of all essentially bounded $\mathscr{F}_t \otimes \mathscr{J}$-measurable functions $p \colon \Omega \times \Theta \to \mathbb{R}$, and $\mathcal{A}^+ = \mathcal{A} \setminus (-\mathcal{A})$ stand for the set of *strictly* risk-acceptable families of portfolios (if $\alpha \in \mathcal{A}^+$, then α is acceptable but $-\alpha$ is not acceptable).

The characterisation of hedgeable contracts will be given in terms of *(mar-*

ket) consistent price systems that are, by definition, families $\pi = (p_t)_{t \in \mathcal{T}}$ of functions $p_t \in L_t^\infty(\Omega \times \Theta)$ satisfying the properties

$$\mathrm{E}x_t p_t \geqslant \sum_{u \in \mathcal{T}_{t+}} \mathrm{E}y_{t,u}p_u \text{ for each } (x_t, y_t) \in \mathcal{Z}_t \text{ and } t \in \mathcal{T}_-, \qquad (1.2)$$

$$\sum_{t \in \mathcal{T}} \mathrm{E}a_t p_t > 0 \text{ for each } \alpha = (a_t)_{t \in \mathcal{T}} \in \mathcal{A}^+. \qquad (1.3)$$

Property (1.2) means that it is impossible to obtain strictly positive expected profit $\mathrm{E}(\sum_{u \in \mathcal{T}_{t+}} y_{t,u}p_u - x_t p_t)$, which is computed in terms of the prices p_t, in the course of trading, as long as the trading constraints are satisfied. Condition (1.3) is a non-degeneracy assumption, saying that the expected value of any strictly risk-acceptable family of portfolios is strictly positive.

Throughout the chapter, we suppose that the cone $\mathcal{A}$ satisfies the following assumption.

Assumption (A). $\mathcal{A}^+ \neq \emptyset$ *and there exists* $\pi = (p_t)_{t \in \mathcal{T}}$, $p_t \in L_t^\infty$, *satisfying* (1.3).

The assumption states that there exists an $L_{\mathcal{T}}^1$-continuous linear functional strictly positive on $\mathcal{A}^+ \neq \emptyset$. Its existence for any closed $\mathcal{A}$ with $\mathcal{A}^+ \neq \emptyset$ can be established, e.g., when $L_{\mathcal{T}}^1(\Omega \times \Theta)$ is separable (see Remark 1.6 in Section 1.6).

Our main results, given in the next section, provide conditions guaranteeing that the following principle holds.

Hedging principle. *The class of consistent price systems is non-empty, and a contract* $(c_t)_{t \in \mathcal{T}}$ *is hedgeable if and only if* $\sum_{t \in \mathcal{T}} \mathrm{E}c_t p_t \leqslant 0$ *for all consistent price systems* $(p_t)_{t \in \mathcal{T}}$.

The hedging principle states that a contract is hedgeable if and only if its value in any consistent price system is non-positive. This principle extends various hedging (and pricing) results available in the literature (cf. e.g. [61, Ch. V.5, VI.1], [33, Ch. 2.1, 3.1–3.3]).

Note that unlike the classical frictionless asset pricing and hedging theory, we do not aim to find the *price* of a contract – in the general model it may be unclear what can be called the price of a contract (for example, there may be

no "basic" asset in terms of which the price can be expressed, or, due to the possible presence of transaction costs, a portfolio may have different bid and offer prices). However, if one chooses a particular method of computing the price of a portfolio, then the price of a contract can be found as the infimum over the set of the prices of the initial portfolios of trading strategies hedging this contract. Consequently, the problem of establishing the validity of the hedging principle can be considered to be more general than the problem of finding contracts prices.

We conclude this section by several remarks about the properties of consistent price systems.

Remark 1.1. It is useful to observe that condition (1.3) implies that the price of any risk-acceptable family of portfolios is non-negative with respect to any consistent price system, i.e. $E\alpha\pi := \sum_{t\in\mathcal{T}} Ea_t p_t \geqslant 0$ for any consistent price system $\pi = (p_t)_{t\in\mathcal{T}}$ and any $\alpha = (a_t)_{t\in\mathcal{T}} \in \mathcal{A}$. Indeed, suppose the contrary: $E\alpha\pi < 0$ for some π and α. Consider any $\alpha' \in \mathcal{A}^+$. Then $r\alpha + \alpha' \in \mathcal{A}^+$ for any real $r \geqslant 0$, while $E(r\alpha + \alpha')\pi < 0$ for all r large enough, which contradicts condition (1.3).

Remark 1.2. The interpretation of a consistent price system $(p_t)_{t\in\mathcal{T}}$ as a system of *prices* is justified if each p_t is strictly positive ($p_t > 0$, $P\otimes\mu$-a.s.). A simple sufficient condition for that is when the cone $\mathcal{A}$ contains all sequences $(a_t)_{t\in\mathcal{T}}$ of non-negative functions a_t and does not contain any sequence of non-positive a_t except the zero one. Another mild condition is provided by the following proposition.

Proposition 1.1. *Each p_t is strictly positive if the following two conditions hold:*

(a) *for any $t \in \mathcal{T}_-$ and $0 \leqslant x_t \in L^1_t$, $P(x_t \neq 0) > 0$, there exists $0 \leqslant y_t \in L^1_{\mathcal{T}_{t+}}$, $P(y_t \neq 0) > 0$, such that $(x_t, y_t) \in \mathcal{Z}_t$;*

(b) *for any $t \in \mathcal{T} \setminus \mathcal{T}_-$ and $0 \leqslant a_t \in L^1_t$, $P(a_t \neq 0) > 0$, we have $(a'_u)_{u\in\mathcal{T}} \in \mathcal{A}^+$ if $a'_t = a_t$ and $a'_u = 0$ for $u \neq t$.*

In other words, (a) means that it is possible to distribute a non-negative non-zero portfolio x_t into non-negative portfolios $y_{t,u}$ at least one of which

is non-zero; and (b) means that a sequence of portfolios with only one non-zero portfolio a_t, where t is a sink node, is strictly risk-acceptable if a_t is non-negative.

Proof. Denote by $\mathcal{T}^k$ the set $\{t \in \mathcal{T} : \kappa(t) = K - k\}$, where K is the maximal length of a directed path in the graph and $\kappa(t)$ is the maximal length of a directed path emanating from a node t. The sets $\mathcal{T}^0, \mathcal{T}^1, \ldots, \mathcal{T}^K$ form a partition of $\mathcal{T}$ such that if there is a path from $t \in \mathcal{T}^k$ to $u \in \mathcal{T}^n$, then $k < n$. Also, note that $\mathcal{T}^- = \mathcal{T}^0 \cup \mathcal{T}^1 \cup \ldots \cup \mathcal{T}^{K-1}$.

The proposition is proved by induction over $k = K, K-1, \ldots, 0$. For any $t \in \mathcal{T}^K$ and a_t as in (b), from (1.3) we have $\mathrm{E}a_t p_t > 0$, so $p_t > 0$. Suppose $p_s > 0$ for any $s \in \bigcup_{n \geqslant k} \mathcal{T}^n$. Then for arbitrary $t \in \mathcal{T}^{k-1}$, according to (a), for any $0 \leqslant x_t \in L^1_t$, $\mathrm{P}(x_t \neq 0) > 0$, we can find $0 \leqslant y_t \in L^1_{\mathcal{T}_{t+}}$ such that $\mathrm{P}(y_t \neq 0) > 0$ and $(x_t, y_t) \in \mathcal{Z}_t$. Inequality (1.2) implies $\mathrm{E}x_t p_t \geqslant \sum_{u \in \mathcal{T}_{t+}} \mathrm{E}y_{t,u} p_u > 0$, and hence $p_t > 0$. $\qquad\square$

Remark 1.3. Some comments on the relations between the above framework and the von Neumann – Gale model of economic growth [1, 21, 25, 68] are in order.

In the latter, elements (x_t, y_t) in the cones $\mathcal{Z}_t$, $t = 1, 2, \ldots$, are interpreted as feasible *production processes*, with *input* x_t and *output* y_t. Coordinates of x_t, y_t represent amounts of *commodities*. The cones $\mathcal{Z}_t$ are termed *technology sets*. The counterparts of contracts in that context are *consumption plans* $(c_t)_t$. Sequences of $z_t = (x_t, y_t) \in \mathcal{Z}_t$, $t = 1, 2, \ldots$, are called *production plans*. The inequalities $y_{t-1} - x_t \geqslant c_t$, $t = 1, 2, \ldots$ (analogous to the hedging condition (1.1)) mean that the production plan (z_t) guarantees the consumption of c_t at each date t. Consistent price systems are analogues of sequences of *competitive prices* in the von Neumann – Gale model.

1.2 Conditions for the validity of the hedging principle

This section contains the formulations and the proofs of the general results related to the model described in the previous section.

We will use the notation $x^+ = \max\{x, 0\}$, $x^- = -\min\{x, 0\}$. We write "a.s." if some property holds for $P \otimes \mu$-almost all (ω, θ). We say that a set $\mathcal{A} \subset L^1$ is closed with respect to L^1-bounded a.s. convergence if for any sequence $\alpha^i = (a_t^i)_{t \in \mathcal{T}} \in \mathcal{A}$ such that $\sup_i \|\alpha^i\| < \infty$ and $\alpha^i \to \alpha$ a.s., we have $\alpha \in \mathcal{A}$. In particular, this implies the closedness of $\mathcal{A}$ in L^1 because from any sequence converging in L^1 it is possible to extract a subsequence converging with probability one.

Theorem 1.1. *The hedging principle holds if the cones $\mathcal{A}$ and $\mathcal{Z}_t$, $t \in \mathcal{T}_-$, are closed with respect to L^1-bounded a.s. convergence and there exist functions $s_t^1 \in L_t^\infty$, $s_{t,u}^2 \in L_t^\infty$, $t \in \mathcal{T}_-$, $u \in \mathcal{T}_{t+}$, with values in $[\underline{s}, \overline{s}]$, where $\underline{s} > 0$, $\overline{s} \geqslant 1$, and a constant $0 \leqslant m < 1$ such that for all $t \in \mathcal{T}_-$, $u \in \mathcal{T}_{t+}$, $(x_t, y_t) \in \mathcal{Z}_t$, and $(a_r)_{r \in \mathcal{T}} \in \mathcal{A}$, the following conditions are satisfied:*

(a) $\mathrm{E}x_t s_t^1 \geqslant \mathrm{E}y_{t,u} s_{t,u}^2$;

(b) $m\mathrm{E}y_{t,u}^+ s_{t,u}^2 \geqslant \mathrm{E}y_{t,u}^- s_{t,u}^2$;

(c) $m\mathrm{E}x_t^+ s_t^1 \geqslant \mathrm{E}x_t^- s_t^1$;

(d) $\mathrm{E}a_t s_t^1 \geqslant 0$.

The functions $s_t^1(\omega, \theta)$ and $s_{t,u}^2(\omega, \theta)$ can be interpreted as some systems of asset prices. Condition (a) means that in the course of trading the portfolio value cannot increase "too much", at least on average. In specific examples, this assumption follows from the condition of self-financing. Conditions (b) and (c) express a *margin requirement*, saying that the total short position of any admissible portfolio should not exceed on average m times the total long position (cf. e.g. [29]). Condition (d) states that the expectation of the value (in terms of the price system s_t^1) of any portfolio in a risk-acceptable family is non-negative.

The proof of the theorem is based on a lemma. Below we denote by $\mathcal{H}$ the set of all hedgeable contracts.

Lemma 1.1. *For any* $\pi = (p_t)_{t \in \mathcal{T}}$, $p_t \in L_t^\infty$, *such that* $\mathrm{E}\alpha\pi > 0$ *for every* $\alpha \in \mathcal{A}^+$, *the following conditions are equivalent:*

(i) π *is a consistent price system;*

(ii) $\mathrm{E}\gamma\pi \leqslant 0$ *for any contract* $\gamma \in \mathcal{H}$.

Proof. (i)$\Rightarrow$(ii). Suppose $\gamma = (c_t)_{t \in \mathcal{T}} \in \mathcal{H}$ and consider a trading strategy $(x_t, y_t)_{t \in \mathcal{T}^-}$ hedging the contract γ.

Then $(a_t)_{t \in \mathcal{T}} \in \mathcal{A}$, where $a_t = y_{t-} - x_t - c_t$ (recall that $y_{t-} := 0$ if t is a source and $x_t := 0$ if t is a sink), and so the following formula is valid:

$$0 \leqslant \sum_{t \in \mathcal{T}} \mathrm{E}a_t p_t = \sum_{t \in \mathcal{T}} \mathrm{E}\left[\sum_{u \in \mathcal{T}_{t-}} y_{u,t} p_t - x_t p_t\right] - \sum_{t \in \mathcal{T}} c_t p_t$$

$$= \sum_{t \in \mathcal{T}_-} \mathrm{E}\left[\sum_{u \in \mathcal{T}_{t+}} y_{t,u} p_u - x_t p_t\right] - \sum_{t \in \mathcal{T}} c_t p_t \leqslant - \sum_{t \in \mathcal{T}} \mathrm{E}c_t p_t,$$

where we put $y_{u,t} := 0$ if $\mathcal{T}_{t-} = \emptyset$, i.e. t is a source node. In the above chain of relations, the first inequality holds by virtue of (1.3) and Remark 1.1, the second equality obtains by changing the order of summation (and interchanging "t" and "u"),

$$\sum_{t \in \mathcal{T}} \sum_{u \in \mathcal{T}_{t-}} \mathrm{E}y_{u,t} p_t = \sum_{u \in \mathcal{T}_-} \sum_{t \in \mathcal{T}_{u+}} \mathrm{E}y_{u,t} p_t = \sum_{t \in \mathcal{T}_-} \sum_{u \in \mathcal{T}_{t+}} \mathrm{E}y_{t,u} p_u,$$

and the last inequality follows from (1.2). Consequently, (ii) holds.

(ii)$\Rightarrow$(i). Fix $t \in \mathcal{T}_-$ and suppose $(x_t, y_t) \in \mathcal{Z}_t$ for some arbitrary (x_t, y_t), where $y_t = (y_{t,u})_{u \in \mathcal{T}_{t+}}$.

Consider the trading strategy $\zeta' = (x_t', y_t')_{t \in \mathcal{T}_-}$ with $x_u' = y_u' = 0$ for $u \neq t$ and $x_t' = x_t$, $y_t' = y_t$. Define a contract $\gamma = (c_t)_{t \in \mathcal{T}}$ by

$$c_t = -x_t, \qquad c_u = y_{t,u} \text{ for } u \in \mathcal{T}_{t+}, \qquad c_u = 0 \text{ for } u \notin \{t\} \cup \mathcal{T}_{t+}.$$

Then we have $y_{v-}' - x_v' - c_v = 0$ for all $v \in \mathcal{T}$. Indeed, for $v = t$, we have $y_{t-}' - x_v' - c_v = 0 - x_t + x_t = 0$. If $v = u \in \mathcal{T}_{t+}$, then $y_{v-}' - x_v' - c_v = y_{t,u} - 0 - y_{t,u} = 0$. If $v \neq t$ and $v \notin \mathcal{T}_{t+}$, then $y_{v-}' = x_v' = c_v = 0$.

Consequently, ζ' hedges γ, and so $\gamma \in \mathcal{H}$. By virtue of (ii), we have

$$0 \geqslant \sum_{v \in \mathcal{T}} \mathrm{E}c_v p_v = -\mathrm{E}x_t p_t + \sum_{u \in \mathcal{T}_{t+}} \mathrm{E}y_{t,u} p_u,$$

which implies that π is a consistent price system. $\qquad\qquad\qquad\square$

Proof of Theorem 1.1. In order to prove the theorem, we first show that $\mathcal{H}$ is closed in $L^1_{\mathcal{T}}$ and $\mathcal{H} \cap \mathcal{A}^+ = \emptyset$, and then apply a version of the Kreps-Yan theorem and its corollary (Propositions 1.4 and 1.5 in Section 1.6) to the cones $\mathcal{H}$ and $\mathcal{A}$.

Step 1. Let us show that there exists a constant C such that for any $(x_t, y_t) \in \mathcal{Z}_t$, $t \in \mathcal{T}_-$, $u \in \mathcal{T}_{t+}$, it holds that

$$\|x_t\| \leqslant C\mathrm{E}x_t s_t^1 \quad \text{and} \quad \|y_{t,u}\| \leqslant C\|x_t\|.$$

Define $\widetilde{C} = (1+m)/(1-m)$. Then we have

$$\begin{aligned}
\widetilde{C}\mathrm{E}x_t s_t^1 &= \widetilde{C}\left[\frac{1}{\widetilde{C}} \cdot \mathrm{E}x_t^+ s_t^1 + \left(1 - \frac{1}{\widetilde{C}}\right) \cdot \mathrm{E}x_t^+ s_t^1 - \mathrm{E}x_t^- s_t^1\right] \\
&\geqslant \widetilde{C}\left[\frac{1}{\widetilde{C}} \cdot \mathrm{E}x_t^+ s_t^1 + \left(\frac{1}{m}\left(1 - \frac{1}{\widetilde{C}}\right) - 1\right) \cdot \mathrm{E}x_t^- s_t^1\right] \qquad (1.4) \\
&\geqslant \mathrm{E}x_t^+ s_t^1 + \mathrm{E}x_t^- s_t^1 = \|x_t s_t^1\| = \mathbf{E}|x_t||s_t^1| \\
&\geqslant \|x_t\|\underline{s},
\end{aligned}$$

where the first inequality follows from the fact that $\mathrm{E}x_t^+ s_t^1 \geq m^{-1}\mathrm{E}x_t^- s_t^1$ according to (c), the second holds because $\widetilde{C}((1 - 1/\widetilde{C})/m - 1) = 1$, and the last is valid because $s_t^1 \geq \underline{s}$.

Further, we have

$$\underline{s}\|y_{t,u}\| \leqslant \widetilde{C}\mathrm{E}y_{t,u} s_{t,u}^2 \leqslant \widetilde{C}\mathrm{E}x_t s_t^1 \leq \widetilde{C}\|x_t s_t^1\| \leq \widetilde{C}\|x_t\|\overline{s},$$

where the first inequality is proved similarly to the one for x_t (replace in the above argument x_t by $y_{t,u}$, s_t^1 by $s_{t,u}^2$ and use (b) instead of (c)), and the second inequality follows from (a). Consequently, the sought-for constant C can be defined as

$$C = \frac{\widetilde{C} \cdot \overline{s}}{\underline{s}}.$$

Step 2. Let us prove that $\mathcal{H}$ is closed in $L_{\mathcal{T}}^1$. Consider a sequence of hedgeable contracts $\gamma^i = (c_t^i)_{t \in \mathcal{T}} \in \mathcal{H}$, $i = 1, 2, \ldots$, such that $\gamma^i \to \gamma$ in $L_{\mathcal{T}}^1$ as $i \to \infty$, where $\gamma = (c_t)_{t \in \mathcal{T}}$. We have to show $\gamma \in \mathcal{H}$.

Let $\mathcal{T}^k = \{t \in \mathcal{T} : \kappa(t) = K - k\}$ be the sets introduced in the proof of Proposition 1.1 (i.e. K denotes the maximal length of a directed path in the graph and $\kappa(t)$ is the maximal length of a directed path emanating from t).

Let $\zeta^i = (x_t^i, y_t^i)_{t \in \mathcal{T}_-}$ be trading strategies hedging γ^i. We will prove the following assertion:

$$\sup_i \|x_t^i\| < \infty \text{ and } \sup_i \|y_{t,u}^i\| < \infty, \ u \in \mathcal{T}_{t+}, \tag{1.5}$$

for each $t \in \mathcal{T}_-$.

To this end we will prove by induction with respect to $k = 0, \ldots, K - 1$ that (1.5) is valid for all $t \in \mathcal{T}^k$. We first note that if $\sup_i \|y_{t-}^i\| < \infty$ for some node t of the graph, then (1.5) is true for this node. Indeed, put $a_t^i = y_{t-}^i - c_t^i - x_t^i$. Then $(a_t^i)_{t \in \mathcal{T}} \in \mathcal{A}$ because ζ^i hedges γ^i, and so $\mathrm{E} a_t^i s_t^1 \geqslant 0$ by virtue of (d). Therefore

$$\|x_t^i\| \leqslant C \mathrm{E} x_t^i s_t^1 = C \mathrm{E} (y_{t-}^i - c_t^i) s_t^1 - C \mathrm{E} a_t^i s_t^1 \leqslant C \overline{s} \cdot (\|y_{t-}^i\| + \|c_t^i\|),$$

and so $\sup_i \|x_t^i\| < \infty$ and $\sup_i \|y_{t,u}^i\| \leqslant C \sup_i \|x_t^i\| < \infty$.

Having this in mind, we proceed by induction. For $t \in \mathcal{T}^0$ we have $y_{t-}^i = 0$ since any $t \in \mathcal{T}^0$ is a source. Thus (1.5) is valid for all $t \in \mathcal{T}^0$. Suppose we have established (1.5) for all t in each of the sets $\mathcal{T}^0, \mathcal{T}^1, \ldots, \mathcal{T}^k$. Consider any $t \in \mathcal{T}^{k+1}$, where $k + 1 < K$. We have $\sup_i \|y_{t-}^i\| < \infty$ because all the predecessors of the node t belong to one of the sets $\mathcal{T}^0, \mathcal{T}^1, \ldots, \mathcal{T}^k$. This implies, as we have demonstrated, the validity of (1.5) for the node t. Thus (1.5) holds for all $t \in \mathcal{T}_-$.

Step 3. By the Komlós theorem (Proposition 1.6), there exists a subsequence $\zeta^{i_1}, \zeta^{i_2}, \ldots$ Cesàro-convergent a.s. to some $\zeta = (x_t, y_t)_{t \in \mathcal{T}_-}$, i.e. for each $t \in \mathcal{T}_-$, $u \in \mathcal{T}_{t+}$ we have $\widetilde{x}_t^j := j^{-1}(x_t^{i_1} + \ldots + x_t^{i_j}) \to x_t \in L_t^1$ a.s. and $\widetilde{y}_{t,u}^j := j^{-1}(y_{t,u}^{i_1} + \ldots + y_{t,u}^{i_j}) \to y_{t,u} \in L_u^1$ a.s. Then ζ is trading strategy since $\mathcal{Z}_t$ are closed with respect to L^1-bounded a.s. convergence.

Moreover, ζ hedges γ because $\widetilde{a}_t^j := \widetilde{y}_{t-}^j - \widetilde{x}_t^j - \widetilde{c}_t^j$ (with $\widetilde{c}_t^j = j^{-1}(c_1^{i_1} + \ldots +$

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$c_t^{i_j}$)) converge a.s. to $a_t = y_{t-} - x_t - c_t$, and $\sup_j \|\widetilde{a}_t^j\| < \infty$ for each $t \in \mathcal{T}$, which implies $(a_t)_{t \in \mathcal{T}} \in \mathcal{A}$ since $\mathcal{A}$ is closed with respect to L^1-bounded a.s. convergence. Thus $\mathcal{H}$ is a closed cone in $L_{\mathcal{T}}^1$.

Step 4. Let us show that

$$\mathcal{H} \cap \mathcal{A}^+ = \emptyset,$$

which can be interpreted as the absence of arbitrage opportunities.

Suppose there exists $\gamma = (c_t)_{t \in \mathcal{T}} \in \mathcal{H} \cap \mathcal{A}$. Consider a trading strategy $\zeta = (x_t, y_t)_{t \in \mathcal{T}_-}$ hedging the contract γ. We claim that in this case $x_t = y_{t,u} = 0$ for all $t \in \mathcal{T}_-$ and $u \in \mathcal{T}_{t+}$.

Indeed, proceeding by induction over $\mathcal{T}^0, \dots, \mathcal{T}^{K-1}$, suppose $y_{t-} = 0$ for each $t \in \mathcal{T}^k$. By virtue of Step 1,

$$\|x_t\| \leqslant C\mathrm{E}x_t s_t^1 \leqslant C\mathrm{E}(y_{t-} - c_t)s_t^1 = -C\mathrm{E}c_t s_t^1 \leq 0$$

for any $t \in \mathcal{T}^k$. Hence $\|x_t\| = 0$ for each $t \in \mathcal{T}^k$. Furthermore, $\|y_{t,u}\| \leqslant C\|x_t\| = 0$ for each $u \in \mathcal{T}_{t+}$, and so $y_{t-} = 0$ for each $t \in \mathcal{T}^{k+1}$. Therefore, $x_t = y_{t,u} = 0$, which implies $\gamma \in (-\mathcal{A})$ by the definition of hedging, meaning that $\mathcal{H} \cap \mathcal{A}^+ = \emptyset$.

Step 5. Applying Proposition 1.4 to the cones $\mathcal{H}$ and $\mathcal{A}$ in the space $L_{\mathcal{T}}^1$, we obtain the existence of a family $\pi = (p_t)_{t \in \mathcal{T}}$, $p_t \in L_t^\infty$, such that $\mathrm{E}\alpha\pi \leq 0$ for any $\alpha \in \mathcal{H}$ and $\mathrm{E}\gamma\pi > 0$ for any $\gamma \in \mathcal{A}^+$. According to Lemma 1.1, π is a consistent price system, so the class of such price systems is non-empty.

Step 6. If a contract γ is hedgeable, we have $\mathrm{E}\gamma\pi \leqslant 0$ for any consistent price system π according to the implication (i)$\Rightarrow$(ii) in Lemma 1.1.

To prove the converse, suppose $\mathrm{E}\gamma\pi \leqslant 0$ for some contract γ and any consistent price system π. Observe that any $\pi = (p_t)_{t \in \mathcal{T}}$, $p_t \in L_t^\infty$, such that $\mathrm{E}\alpha\pi > 0$ for any $\alpha \in \mathcal{A}^+$ and $\mathrm{E}\gamma'\pi \leqslant 0$ for any $\gamma' \in \mathcal{H}$ is a consistent price system according to the implication (ii)$\Rightarrow$(i) in Lemma 1.1. Therefore, $\mathrm{E}\gamma\pi \leq 0$ for any such π and the given γ. By virtue of Proposition 1.5, this implies $\gamma \in \mathcal{H}$. $\qquad\qquad\square$

The next result provides a version of Theorem 3.1 with other assumptions.

Theorem 1.2. *The hedging principle holds if the cones $\mathcal{A}$ and $\mathcal{Z}_t$, $t \in \mathcal{T}_-$, are closed in L^1, for any $(a_t)_{t \in \mathcal{T}} \in \mathcal{A}$ we have $a_u = 0$, $u \in \mathcal{T}_-$, and there exist functions $s_t^1 \in L_t^\infty$, $s_{t,u}^2 \in L_t^\infty$, $t \in \mathcal{T}_-$, $u \in \mathcal{T}_{t+}$, with values in $[\underline{s}, \overline{s}]$ (where $\underline{s} > 0$, $\overline{s} \geqslant 1$), and a constant $0 \leqslant m < 1$ satisfying conditions*

(a′) $x_t s_t^1 \geqslant y_{t,u} s_{t,u}^2$ a.s.;

(b′) $m y_{t,u}^+ s_{t,u}^2 \geqslant y_{t,u}^- s_{t,u}^2$ a.s.

Remark 1.4. The condition $a_u = 0$, $u \in \mathcal{T}_-$, for any $(a_t)_{t \in \mathcal{T}} \in \mathcal{A}$, means that a contract should be hedged *exactly* at all the intermediate trading sessions and it should be hedged with risk (in a risk-acceptable manner) at all the terminal trading sessions – those trading sessions that are represented by sink nodes of the graph.

Proof of Theorem 1.2. The proof is conducted along the lines of the proof of Theorem 1.1. First we show that $|y_{t,u}| \leqslant C|x_t|$ a.s. for all $t \in \mathcal{T}_-$, $u \in \mathcal{T}_{t+}$, where the constant $C = \widetilde{C} \cdot (\overline{s}/\underline{s})$ with $\widetilde{C} = (1 + m)/(1 - m)$.

Indeed, with probability one it holds that

$$\underline{s}|y_{t,u}| \leqslant \widetilde{C} y_{t,u} s_{t,u}^2 \leqslant \widetilde{C} x_t s_t^1 \leqslant \widetilde{C}|x_t s_t^1| \leqslant \widetilde{C}|x_t|\overline{s},$$

where the first inequality is proved similarly to (1.4), and the second inequality follows from (a′).

In order to prove the closedness of $\mathcal{H}$, consider any $\gamma^i = (c_t^i)_{t \in \mathcal{T}} \in \mathcal{H}$ converging in $L_\mathcal{T}^1$ to $\gamma = (c_t)_{t \in \mathcal{T}}$. For trading strategies $\zeta^i = (x_t^i, y_t^i)_{t \in \mathcal{T}_-}$ hedging γ_i, using the induction over $\mathcal{T}^k$, we prove that $\sup_i \|x_t^i\| < \infty$, $\sup_i \|y_{t,u}^i\| < \infty$ and the sequences x_t^i, $y_{t,u}^i$ are uniformly integrable.

Indeed, if $\sup_i \|y_{t-}^i\| < \infty$ and y_{t-}^i is uniformly integrable for any $t \in \mathcal{T}^k$ then $x_t^i = y_{t-}^i - c_t^i$ is uniformly integrable and $\sup_i \|x_t^i\| < \infty$. Consequently, $\sup_i \|y_{t-}^i\| < \infty$ and y_{t-}^i is uniformly integrable for any $t \in \mathcal{T}^{k+1}$, because $|y_{t,u}^i| \leqslant C|x_t^i|$ a.s.

By using the Komlós Theorem, we find a subsequence ζ^{i_j} Cesàro-convergent a.s. to some $\zeta = (x_t, y_t)_{t \in \mathcal{T}_-}$, and hence convergent in L^1 because x_t^i and $y_{t,u}^i$ are uniformly integrable. Since $\mathcal{Z}_t$ are closed in L_t, ζ is a trading strategy. It hedges γ because $\widetilde{a}_t^j := \widetilde{y}_{t-}^j - \widetilde{x}_t^j - \widetilde{c}_t^j$ converge in L^1 to $a_t = y_{t-} - x_t - c_t$ as

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they are uniformly integrable and a.s. convergent, and the cone $\mathcal{A}$ is closed in $L^1_{\mathcal{T}}$. Thus $\mathcal{H}$ is closed in $L^1_{\mathcal{T}}$.

Furthermore, $\mathcal{H} \cap \mathcal{A}^+ = \emptyset$ because if there exists $\gamma = (c_t)_{t \in \mathcal{T}} \in \mathcal{H} \cap \mathcal{A}$ then for a trading strategy $(x_t, y_t)_{t \in \mathcal{T}_-}$ hedging γ, we have $x_t = y_{t,u} = 0$ for all $t \in \mathcal{T}_-$, $u \in \mathcal{T}_t^+$ (this is proved by induction over $\mathcal{T}^k$ using that $x_t = y_{t-} - c_t$, $c_t = 0$ for each $t \in \mathcal{T}_-$, and $|y_{t,u}| \leqslant C|x_t|$ a.s.). This implies $\gamma \in (-\mathcal{A})$ by the definition of hedging. To complete the proof it remains to apply Propositions 1.4 and 1.5 to the cones $\mathcal{H}$ and $\mathcal{A}$. $\qquad\square$

Remark 1.5 (on the no-arbitrage hypothesis)**.** The central role in the classical theory of asset pricing and hedging is played by the *no-arbitrage hypothesis*, which postulates that arbitrage opportunities do not exist. This assumption is rather natural since it is thought that in real markets arbitrage opportunities are quickly eliminated by market forces. Perhaps, it is even more important from the point of view of constructing *mathematical* models of financial markets, as the absence of arbitrage is equivalent to the existence of equivalent martingale measures (in the frictionless framework), which allows to apply the risk-neutral principle for pricing contingent claims.

In the frictionless model, the validity of the no-arbitrage hypothesis depends on the probabilistic structure of the random process (or sequence) which describes the evolution of asset prices. Consequently, processes that allow arbitrage opportunities are usually considered as inadequate models of a market.

However, as follows from the results of this chapter, the presence of arbitrage opportunities can also be caused by inadequate trading constraints in a model. Indeed, the classical frictionless approach allows unlimited short sales and borrowings from the bank account, which is certainly impossible in real trading. On the other hand, the introduction of margin requirements eliminates arbitrage opportunities under the conditions of Theorems 1.1 and 1.3 (see Step 4 in the proof of Theorem 1.1) and implies the validity of the hedging principle.

This fact has important implications for modelling financial markets since the introduction of margin requirements allows to consider price processes

that describe important market features, but allow arbitrage.

One example of such a process is fractional geometric Brownian motion. Unlike the standard geometric Brownian motion, it exhibits long-range dependence – a property typically observed in real asset prices. However, it is well-known that this process admits arbitrage opportunities in the frictionless model [51, 60]. Moreover, even its approximation by binary random walks constructed on a finite probability space is not arbitrage-free [67].

Thus, consideration of market models with margin requirements can possibly allow to use a wider class of stochastic processes describing asset prices and broaden the frontiers of the theory.

1.3 Risk-acceptable portfolios: examples

In this section we provide examples of cones $\mathcal{A}$ of risk-acceptable families of portfolios defined in terms of their *liquidation values*.

Suppose for each $t \in \mathcal{T}$ there is an operator $V_t \colon L^1_t(\Omega \times \Theta) \to L^1(\Omega)$, where $V_t(a_t)(\omega)$ is interpreted as the liquidation value of the portfolio a_t in the trading session t, and a closed cone $A_t \subset L^1_t(\Omega)$ interpreted as a cone of *risk-acceptable liquidation values*.

Define
$$\mathcal{A} = \{(a_t)_{t \in \mathcal{T}} \in L^1_{\mathcal{T}} : V_t(a_t) \in A_t \text{ for all } t \in \mathcal{T}\}.$$

According to this definition, a family of portfolios is acceptable if the liquidation value of each portfolio is acceptable. In order to guarantee that $\mathcal{A}$ is a closed cone in L^1, it is sufficient to assume that for each $t \in \mathcal{T}$ the following conditions are satisfied:

(i) $V_t(ra_t) = rV_t(a_t)$, $V_t(a_t + a'_t) \geqslant V_t(a_t) + V_t(a'_t)$ for any real $r \geqslant 0$ and $a_t, a'_t \in L^1_t$,

(ii) the cone A_t contains all non-negative $\mathscr{F}_t$-measurable integrable random variables,

(iii) there exists a constant c such that $\|V_t(a_t) - V_t(a'_t)\| \leqslant c\|a_t - a'_t\|$ for any $a_t, a'_t \in L^1_t$.

The cone $\mathcal{A}$ is closed with respect to $L^1_{\mathcal{T}}(\Omega \times \Theta)$-bounded a.s. convergence, if, additionally, the following two conditions hold:

(iv) $V_t(a^n_t) \to V_t(a_t)$ a.s. whenever $a^n_t \to a_t$ a.s., $n \to \infty$, and $a^n_t, a_t \in L^1_t$;

(v) the cone A_t is closed with respect to $L^1_t(\Omega)$-bounded a.s. convergence.

A natural example of a liquidation value is

$$V_t(a_t)(\omega) = \int_\Theta a_t(\omega, \theta) S_t(\omega, \theta) \mu(d\theta),$$

where the function $S_t \in L^\infty_t(\Omega \times \Theta)$ represents asset prices at the trading session t. In other words, the liquidation value of a portfolio is equal to its value in terms of the prices S_t.

More generally, one can define

$$V_t(a_t)(\omega) = \int_\Theta \left[a^+_t(\omega, \theta) \underline{S}_t(\omega, \theta) - a^-_t(\omega, \theta) \overline{S}_t(\omega, \theta) \right] \mu(d\theta),$$

where $\underline{S}_t, \overline{S}_t \in L^\infty_t(\Omega \times \Theta)$, $\underline{S}_t \leqslant \overline{S}_t$, are *bid* and *ask* asset prices. It is easy to see that this liquidation value operator satisfies above conditions (i) and (iii). If the asset space $(\Theta, \mathscr{J}, \mu)$ is finite, it also satisfies (iv).

We provide three examples of cones A_t of risk-acceptable liquidation values and show that they are closed with respect to $L^1(\Omega)$-bounded a.s. convergence or in L^1, and $\mathrm{E}v > 0$ for each $v \in A_t \setminus \{0\}$, i.e. any non-zero risk-acceptable liquidation value is strictly positive on average.

1. Superhedging. Define

$$A^1 = \{v \in L^1 : v \geq 0 \text{ a.s.}\},$$

i. e. each risk-acceptable value is non-negative with probability one, which is the classical approach to hedging (see e. g. [24, 61]). Clearly, A^1 is closed under L^1-bounded a.s.-convegence and $\mathrm{E}v > 0$ for any $v \in A^1 \setminus \{0\}$.

Superhedging is a comparatively strong assumption, which requires to fulfil contract obligations with probability one. In the next two examples we deal with *weaker* approaches to hedging: the cones A^2 and A^3 introduced below are larger than the cone A^1.

2. Acceptable Sharpe ratio. The *Sharpe ratio* of a non-constant random variable $v \in L^2$ is defined by $\mathrm{E}v/\sqrt{\mathrm{Var}\ v}$. For a given number $\lambda > 0$ define the cone

$$A^2 = \{v \in L^1 : \exists\, u \in L^2,\ u \leqslant v,\ \mathrm{E}u \geqslant \lambda\sqrt{\mathrm{Var}\ u}\}.$$

In other words, a liquidation value is acceptable if it exceeds a random variable with the Sharpe ratio not less than λ.

Clearly $\mathrm{E}v > 0$ for any $v \in A^2 \setminus \{0\}$. Let us show that A^2 is closed with respect to L^1-bounded a.s. convergence. Suppose $v^n \in A^2$, $\mathrm{E}|v^n| \leqslant \alpha < \infty$ and $v^n \to v$ a.s. Take $u^n \in L^2$ such that $u^n \leqslant v^n$, $\mathrm{E}u^n \geqslant \lambda\sqrt{\mathrm{Var}\ u^n}$. Then $\lambda\sqrt{\mathrm{Var}\ u^n} \leqslant \alpha$ and the sequence u^n is bounded in the L^2-norm. Since the ball in L^2 is weakly compact, we can find a subsequence $u^{n_k} \to u$ weakly in L^2. Then we obtain

$$\mathrm{E}u = \lim_{k \to \infty} \mathrm{E}u^{n_k} \geqslant \lambda \liminf_{k \to \infty} \sqrt{\mathrm{Var}\ u^{n_k}} \geqslant \lambda\sqrt{\mathrm{Var}\ u},$$

where the last inequality follows from the weak lower semi-continuity of the norm. Using the Komlós theorem, we find a subsequence $u^{n_{k_l}}$ such that $\widetilde{u}^l \equiv l^{-1}(u^{n_{k_1}} + \ldots + u^{n_{k_l}}) \to \widetilde{u}$ a.s. Then $v \geqslant \widetilde{u}$ and $\widetilde{u} = u$, since $\mathrm{E}\mathbf{I}_\Gamma u = \lim_l \mathrm{E}\mathbf{I}_\Gamma \widetilde{u}^l = \mathrm{E}\mathbf{I}_\Gamma \widetilde{u}$ for each measurable set Γ, where the former is valid because $u^{n_k} \to u$ weakly and the latter is true because the sequence $\widetilde{u}^l$ is uniformly integrable (bounded in L^2) and converges to $\widetilde{u}$ a.s. Thus $v \in A^2$.

3. Acceptable average value at risk. The *average value at risk* at a level λ of a random variable $v \in L^1$ is defined by the formula

$$\mathrm{AV@R}_\lambda(v) = \sup_{q \in Q_\lambda} \mathrm{E}(-qv),$$

where Q_λ is the set of all random variables $0 \leqslant q \leqslant 1/\lambda$ such that $\mathrm{E}q = 1$. In other words, $\mathrm{AV@R}_\lambda(v)$ is the maximal expected value of $(-v)$ under all probability measures absolutely continuous with respect to the original measure such that the density does not exceed $1/\lambda$ (see [24, Ch. 4] for details).

Consider the cone

$$A^3 = \{v \in L^1 :\ \mathrm{AV@R}_\lambda(v) \leqslant 0\},$$

where $\lambda \in (0,1)$ is a given number. According to this definition of A^3, the average value at risk at the level λ of each acceptable liquidation value is non-positive (i.e. such a liquidation value is *not risky* in terms of AV@R_λ).

To show that $\text{E}v > 0$ for any $v \in A^3$ we use the representation

$$\text{AV@R}_\lambda(v) = -\int_0^1 q_v(\lambda s)ds,$$

where $q_v(s) = \inf\{q : \text{P}(v \le q) > s\}$ is the quantile function of v (see [24, Ch. 4]). If v is non-constant, we have

$$\text{AV@R}_\lambda(v) > -\int_0^1 q_v(s)ds = -\text{E}v,$$

and so $\text{E}v > 0$ for a non-constant $v \in A^3 \setminus \{0\}$. But if $v \in A^3 \setminus \{0\}$ is constant, then necessarily $v > 0$, and so $\text{E}v > 0$.

Finally observe that the cone A^3 is closed in L^1. Indeed, if $v^n \in A^3$ and $v^n \to v$ in L^1, then $\text{E}(-qv) = \lim_n \text{E}(-qv^n) \le 0$ for any random variable $0 \le q \le 1/\lambda$ such that $\text{E}q = 1$, consequently, $\text{AV@R}_\lambda(v) \le 0$.

1.4 Connections between consistent price systems and equivalent martingale measures

The notion of a consistent price system generalises the notion of *an equivalent martingale measure* – the cornerstone of the classical stochastic finance.

To see this, consider the model of a *frictionless* market of N assets represented by a linear graph G with nodes $t = 0, 1, \ldots, T$, where asset 1 is a riskless asset with the discounted price $S_t^1 \equiv 1$ at each trading session t, and assets $i = 2, \ldots, N$ are risky with the discounted prices $S_t^i > 0$ being $\mathscr{F}_t$-measurable random variables (the space of assets $(\Theta, \mathscr{J}, \mu)$ here is simply $\Theta = \{1, 2, \ldots, N\}$, $\mathscr{J} = 2^\Theta$, $\mu\{\theta\} = 1$ for each $\theta \in \Theta$).

It is assumed that the σ-algebras $\mathscr{F}_t$ form a filtration, i.e. $\mathscr{F}_t \subset \mathscr{F}_{t+1}$ for each $t = 0, \ldots, T-1$, and each $\mathscr{F}_t$ is completed by all $\mathscr{F}$-measurable sets of measure 0.

Denoting $S_t = (S_t^1, \ldots, S_t^N)$, introduce the cones

$$\mathcal{Z}_t = \left\{ (x_t, y_t) \in L_t^1 \otimes L_t^1 \; : \; \sum_{i=1}^N x_t^i S_t^i \geqslant \sum_{i=1}^N y_t^i S_t^i \text{ a.s.} \right\},$$

$$\mathcal{A} = \left\{ (0, \ldots, 0, a_T) \; : \; a_T \in L_T^1, \; \sum_{i=1}^N a_T^i S_T^i \geqslant 0 \text{ a.s.} \right\}.$$

The cones $\mathcal{Z}_t$ define *self-financing* trading strategies (with free disposal), i.e. the values of their portfolios do not increase in the course of trading. According to the definition of $\mathcal{A}$, a contract $(c_0, \ldots, c_T)$ is hedgeable, if there exists a trading strategy that pays exactly c_t at each trading session $t < T$ and pays not less than c_T at the session T (the *superhedging* approach).

Since it is assumed that $\mathscr{F}_t$ form a filtration and are complete, we have $L_t \subset L_{t+1}$ for each $t = 0, \ldots, T-1$, so this model is a particular case of the general model (in the general model $y_t \equiv y_{t,t+1} \in L_{t+1}$).

Recall that a probability measure $\widetilde{P}$ defined on $(\Omega, \mathscr{F})$ and equivalent to the original probability measure P ($\widetilde{P} \sim P$) is called an *equivalent martingale measure* if the sequence $S_0, S_1, \ldots, S_T$ is a $\widetilde{P}$-martingale, which means that $E^{\widetilde{P}}(S_t^i \mid \mathscr{F}_{t-1}) = S_{t-1}^i$ a.s. for all $t = 1, \ldots, T$, $i = 1, \ldots, N$.

Consistent price systems and equivalent martingale measures in the model at hand are connected by the following properties.

Proposition 1.2. *1) If $\widetilde{P}$ is an equivalent martingale measure, then the sequence of $p_t = \lambda_t S_t$, where $\lambda_t = E(d\widetilde{P}/dP \mid \mathscr{F}_t)$, is a consistent price system, provided that $p_t \in L_t^\infty$.*

2) If $(p_0, \ldots, p_t)$ is a consistent price system, then the measure $\widetilde{P}$ defined by $d\widetilde{P} = (p_T^1/E p_T^1) dP$ is an equivalent martingale measure.

Proof. To keep the notation concise, let us denote by $x_t S_t$ and $y_t S_t$ the corresponding scalar products of the vectors x_t, y_t, S_t.

1) If $(x_t, y_t) \in \mathcal{Z}_t$ then $x_t S_t \geqslant y_t S_t$ and consequently $E^P p_t x_t \geqslant E^P p_t y_t$. Also $E^P p_t y_t = E^P p_{t+1} y_t$, because the sequence p_t is a P-martingale as follows from the formula (see e.g. [63, Chapter II, §7])

$$E^P(\lambda_t S_t \mid \mathscr{F}_{t-1}) = \lambda_{t-1} E^{\widetilde{P}}(S_t \mid \mathscr{F}_{t-1}). \tag{1.6}$$

Thus $\mathrm{E}^{\mathrm{P}} p_t x_t \geqslant \mathrm{E}^{\mathrm{P}} p_{t+1} y_t$. It is also clear that $\mathrm{E} p_T a_T > 0$ for any a_T such that $a_T S_T \geqslant 0$ and $\mathrm{P}(a_T S_T \neq 0) > 0$, so $(p_0, \ldots, p_T)$ satisfies the definition of a consistent price system.

2) Observe that the following relations hold:

$$\mathrm{E}(p_{t+1} \mid \mathscr{F}_t) = p_t \text{ for } t = 0, \ldots, T-1, \qquad p_t = p_t^1 S_t \text{ for } t = 0 \ldots, T.$$

The first one follows from (1.2) with $(x_t, x_t) \in \mathcal{Z}_t$ for $x_t = (0, \ldots, \pm \mathbf{I}_\Gamma, \ldots, 0)$ and an arbitrary $\Gamma \in \mathscr{F}_t$. The second one for $t = 0, \ldots, T-1$ follows from (1.2) taking $(x_t, 0) \in \mathcal{Z}_t$, $x_t = (\pm S_t^i \mathbf{I}_\Gamma, 0, \ldots, \mp \mathbf{I}_\Gamma, \ldots, 0)$ with an arbitrary $\Gamma \in \mathscr{F}_t$, and for $t = T$ it follows from (1.3) taking $x_T \in L_T^1$ of the same form such that $(0, \ldots, x_T) \in \mathcal{A}$. Using (1.6) with $\lambda_t = p_t^1$, we obtain that S_t is a $\widetilde{\mathrm{P}}$-martingale, so $\widetilde{\mathrm{P}}$ is an equivalent martingale measure. $\square$

1.5 A model of an asset market with transaction costs and portfolio constraints

We consider a market where N assets are traded at trading sessions $t \in \mathcal{T}$. Assets $i = 2, \ldots, N$ represent stock and asset $i = 1$ cash (deposited with a bank account). The model we deal with in this section is a special case of the general one where $\Theta = \{1, 2, \ldots, N\}$ and $\mu(i) = 1$ for each $i \in \Theta$. We assume that the σ-algebras $\mathscr{F}_t$ associated with the nodes of the graph are such that $\mathscr{F}_t \subset \mathscr{F}_u$ for any $t \in \mathcal{T}_-$ and $u \in \mathcal{T}_{t+}$ (i.e. the "information" does not decrease along the paths in the graph), and each $\mathscr{F}_t$, $t \in \mathcal{T}$, is completed by all $\mathscr{F}$-measurable sets of measure 0.

For each trading session $t \in \mathcal{T}$, we are given $\mathscr{F}_t$-measurable essentially bounded non-negative random variables $\underline{S}_t^i \leqslant \overline{S}_t^i$, $i = 2, \ldots, N$, representing *bid* and *ask stock prices*, and $\underline{D}_t^i \leq \overline{D}_t^i$, $i = 2, \ldots, N$, representing *dividend rates* for long and short stock positions[1]. The bid and ask price vectors are

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[1]It is assumed that a short seller pays the dividends on the amount of stock sold short to the lender of stock. Dividend rates for long and short positions may be different – for example, when some assets pay dividends in a currency different from asset 1 and there is a bid-ask spread in the exchange rates.

denoted by $\underline{S}_t = (1, \underline{S}_t^2, \ldots, \underline{S}_t^N)$ and $\overline{S}_t = (1, \overline{S}_t^2, \ldots, \overline{S}_t^N)$. We assume each $\underline{S}_t^i$ is bounded away from zero (i.e. $\underline{S}_t^i \geq \underline{S}$ for a constant $\underline{S} > 0$).

For each $t \in \mathcal{T}_-$, $\mathscr{F}_t$-measurable essentially bounded non-negative random variables $\underline{R}_{t,u} \leqslant \overline{R}_{t,u}$, $u \in \mathcal{T}_{t+}$, are given, which describe *interest rates* for *lending* and *borrowing* cash between the sessions t and u.

We write $x_t(\omega) = (x_t^1(\omega), \ldots, x_t^N(\omega))$ for portfolios of assets represented by functions $x_t(\omega, i)$ in $L_t^1(\Omega \times \Theta)$, where $x_t^i(\omega) = x_t(\omega, i)$. We define $|x_t| = \sum_i |x_t^i|$, $x_t^+ = ((x_t^1)^+, \ldots, (x_t^N)^+)$ and $x_t^- = ((x^1)^-, \ldots, (x_t^N)^-)$. If x, y are N-dimensional vectors, we denote by xy the scalar product $xy = \sum_i x^i y^i$.

For a trading strategy $(x_t, y_t)_{t \in \mathcal{T}_-}$, the dividends

$$d_t(x_t) = \sum_{i=2}^{N} \left[(x_t^i)^+ \underline{D}_t^i - (x_t^i)^- \overline{D}_t^i \right]$$

are received on each portfolio x_t at each session $t \in \mathcal{T}_-$. Negative values of $d_t(x_t)$ mean that the corresponding amounts should be returned rather than received. After the dividend payments, the portfolio x_t (with the dividends added) is rearranged by buying and selling assets, subject to the self-financing trading constraint, to a family of portfolios $\widetilde{y}_t = (\widetilde{y}_{t,u})_{u \in \mathcal{T}_{t+}}$, where $\widetilde{y}_{t,u} \in L_t^1$. The feasible pairs $(x_t, \widetilde{y}_t)$ form the cone

$$\widetilde{\mathcal{Z}}_t = \left\{ (x_t, \widetilde{y}_t) : x_t \in L_t^1, \ \widetilde{y}_{t,u} \in L_t^1 \text{ and} \right.$$

$$\left. \left(x_t - \sum_{u \in \mathcal{T}_{t+}} \widetilde{y}_{t,u} \right)^+ \cdot \underline{S}_t + d_t(x_t) \geqslant \left(x_t - \sum_{u \in \mathcal{T}_{t+}} \widetilde{y}_{t,u} \right)^- \cdot \overline{S}_t \text{ a.s.} \right\}.$$

The above inequality represents the *self-financing condition*: its left-hand side is equal to the amount of cash obtained from selling assets and receiving dividends, and its right-hand side is the amount of cash paid for buying assets. We denote by $y_t = y_t(\widetilde{y}_t)$ the family of portfolios $(y_{t,u})_{u \in \mathcal{T}_{t+}}$ whose positions are given by

$$y_{t,u}^1 = \widetilde{y}_{t,u}^1 + r_{t,u}(\widetilde{y}_{t,u}) \quad \text{and} \quad y_{t,u}^i = \widetilde{y}_{t,u}^i, \ i = 2, \ldots, N,$$

where

$$r_{t,u}(\widetilde{y}_{t,u}) = (\widetilde{y}_{t,u}^1)^+ \underline{R}_{t,u} - (\widetilde{y}_{t,u}^1)^- \overline{R}_{t,u}$$

is the interest paid on cash held between the trading sessions t and u.

Random $\mathscr{F}_t$-measurable closed cones[2] $Y_{t,u} \colon \Omega \to 2^{\mathbb{R}^N}$, $t \in \mathcal{T}_-$, $u \in \mathcal{T}_{t+}$, specifying *portfolio constraints* are given. It is assumed that each cone $Y_{t,u}(\omega)$ contains all non-negative vectors in $\mathbb{R}^N$. Trading strategies are defined in terms of the cones $(t \in \mathcal{T}_-)$

$$\mathcal{Z}_t = \{(x_t, y_t) : y_t = y_t(\widetilde{y}_t),\ (x_t, \widetilde{y}_t) \in \widetilde{\mathcal{Z}}_t,\ y_{t,u} \in Y_{t,u} \text{ a.s.}\}. \qquad (1.7)$$

Portfolios $y_{t,u}$ are obtained from $\widetilde{y}_{t,u}$ by adding interest to the 1st (bank account) position. Only those portfolios $y_{t,u}$ are admissible that satisfy the constraint $y_{t,u} \in Y_{t,u}$ a.s.

Note that we assume y_t are $\mathscr{F}_t$-measurable, which can be explained by that "the portfolio for tomorrow should be obtained today". Since the σ-algebras $\mathscr{F}_t$ are such that $\mathscr{F}_t \subset \mathscr{F}_u$ whenever $u \in \mathcal{T}_{t+}$ and are completed by all sets of measure 0, we have $L_t^1 \subset L_u^1$ for $u \in \mathcal{T}_{t+}$, and so the model at hand can be included into the framework described in Section 1.1 (where, generally, $y_{t,u} \in L_u^1$).

Also note that the class of trading strategies defined by the cones $\mathcal{Z}_t$ in (1.7) excludes the possibility of bankruptcy, i.e. a trading strategy cannot interrupt at an intermediate node of the graph. Nevertheless, the model allows the modeling of the possibility that a trader does not fulfil the obligations of a contract through the notion of hedging with risk.

We consider cones of risk-acceptable families of portfolios $\mathcal{A}$ given by $\mathcal{A} = \bigotimes\limits_{t \in \mathcal{T}} \mathcal{A}_t$, where for each $t \in \mathcal{T}$

$$\mathcal{A}_t = \{0\} \quad \text{or} \quad \mathcal{A}_t = \left\{ a_t \in L_t^1 : a_t^+ \underline{S}_t - a_t^- \overline{S}_t + d_t(a_t) \in A_t \right\} \qquad (1.8)$$

with some given cones $A_t \subset L_t^1(\Omega)$ interpreted as cones of risk-acceptable liquidation values (see Section 1.3). It is assumed that $\mathcal{A}_t \neq \{0\}$ for all sink nodes $t \notin \mathcal{T}_-$, and for each $t \in \mathcal{T}$, where $\mathcal{A}_t \neq \{0\}$, the cone A_t contains all non-negative integrable random variables and $\mathrm{E}v > 0$ for any $v \in A_t \setminus \{0\}$.

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[2]A random $\mathscr{F}_t$-measurable closed cone in $\mathbb{R}^N$ is a mapping $Y \colon \Omega \to 2^{\mathbb{R}^N}$ such that $Y(\omega)$ is a closed cone in $\mathbb{R}^N$ for each $\omega \in \Omega$, and $\{\omega : Y(\omega) \cap K \neq \emptyset\} \in \mathscr{F}_t$ for any open set $K \subset \mathbb{R}^N$. See Appendix 1 for details.

Let us prove an auxiliary proposition regarding the structure of the cones $\mathcal{A}_t$ that will be used below.

Proposition 1.3. *Suppose $\mathcal{A}_t \neq \{0\}$ is a cone of form (1.8) and $p_t \in L_t^\infty$ is a random vector such that*

(a) $p_t^1 > 0$,

(b) $\underline{S}_t^i + \underline{D}_t^i \leqslant p_t^i/p_t^1 \leqslant \overline{S}_t^i + \overline{D}_t^i$ *a.s. and* $\underline{S}_t^i + \underline{D}_t^i < p_t^i/p_t^1 < \overline{S}_t^i + \overline{D}_t^i$ *a.s. on the set* $\{\omega : \underline{S}_t^i + \underline{D}_t^i < \overline{S}_t^i + \overline{D}_t^i\}$, $i = 2, \dots, N$,

(c) $\mathrm{E}v_t p_t^1 > 0$ *for any* $v_t \in A_t \setminus \{0\}$.

Then $\mathrm{E}a_t p_t \geq 0$ for each $a_t \in \mathcal{A}_t$ and $\mathrm{E}a_t p_t > 0$ for each $a_t \in \mathcal{A}_t^+$.

Proof. For any $a_t \in \mathcal{A}_t \neq \{0\}$ we have $a_t p_t/p_t^1 \geqslant a_t^+ \underline{S}_t - a_t^- \overline{S}_t + d_t(a_t)$, consequently, $a_t p_t/p_t^1 \in A_t$, and so $\mathrm{E}a_t p_t = \mathrm{E}(a_t p_t/p_t^1)p_t^1 \geqslant 0$.

Suppose $\mathrm{E}a_t p_t = 0$. Then $a_t p_t/p_t^1 = a_t^+ \underline{S}_t - a_t^- \overline{S}_t + d_t(a_t) = 0$, which implies that $a_t^i = 0$ a.s. on the set $\{\omega : \underline{S}_t^i + \underline{D}_t^i < \overline{S}_t^i + \overline{D}_t^i\}$, $i = 2, \dots, N$. Then in this case $(-a_t) \in \mathcal{A}_t$, and so $\mathrm{E}a_t p_t > 0$ for any $a_t \in \mathcal{A}_t^+$. $\qquad\square$

The above proposition guarantees that the cone $\mathcal{A}$ in the model at hand satisfies assumption (A) with $\pi = (p_t)_{t \leq T}$,

$$p_t^1 = 1, \quad p_t^i = (\underline{S}_t^i + \underline{D}_t^i + \overline{S}_t^i + \overline{D}_t^i)/2 \text{ for } i = 2, \dots N.$$

Let R be a constant such that $R \leq (1 + \underline{R}_{t,u})/(1 + \overline{R}_{t,u})$ a.s. for all $t \in \mathcal{T}_-$, $u \in \mathcal{T}_{t+}$. We say that the market model at hand satisfies a *margin requirement* if there exists a constant M such that $0 \leqslant M < R$ and for almost each $\omega \in \Omega$ we have (cf. conditions (b) and (c) in Theorem 1.1)

$$M\widetilde{y}_{t,u}^+ \underline{S}_t(\omega) \geqslant \widetilde{y}_{t,u}^- \overline{S}_t(\omega) \text{ for any } y_{t,u} \in Y_{t,u}(\omega), \ t \in \mathcal{T}_-, \ u \in \mathcal{T}_{t+}, \qquad (1.9)$$

where

$$\widetilde{y}_{t,u}^1 = \frac{(y_{t,u}^1)^+}{1 + \underline{R}_{t,u}(\omega)} - \frac{(y_{t,u}^1)^-}{1 + \overline{R}_{t,u}(\omega)}, \qquad \widetilde{y}_{t,u}^i = y_{t,u}^i \text{ for } i = 2, \dots N.$$

Inequality (1.9) requires that the total short position of any portfolio $\widetilde{y}_{t,u}$ does not exceed on average M times the total long position, and can be interpreted as a weak form of a margin requirement. Note that margin

requirements in real markets have a more complex structure, and, in particular, differentiate initial margin (the amount that should be collateralized in order to open a position) and maintenance margin (the amount required to be kept in collateral until the position is closed). Such additional limitations typically imply (1.9) with some constant M and can be modeled by appropriate cones $Y_{t,u}$.

The following theorem establishes the validity of the hedging principle for a stock market with a margin requirement.

Theorem 1.3. *The hedging principle is valid for a stock market with a margin requirement if at least one of the following conditions is satisfied:*

(i) for each $t \in \mathcal{T}$ such that $\mathcal{A}_t \neq \{0\}$, the cone A_t is closed with respect to $L_t^1(\Omega)$-bounded a.s. convergence, and $\underline{S}_t^i + \varepsilon < \overline{S}_t^i$ a.s. for some $\varepsilon > 0$ and all $t \in \mathcal{T}_-$, $i = 2, \ldots, N$;

(ii) $\mathcal{A}_t = \{0\}$ for all $t \in \mathcal{T}_-$.

Proof. In order to prove the theorem we will apply Theorem 1.1 if (i) holds and Theorem 1.2 if (ii) holds. Observe that the cones Z_t are closed with respect to L^1-bounded a.s. convergence. The cone $\mathcal{A}$ is closed with respect to L^1-bounded a.s. convergence if (i) holds and it is closed in L^1 if (ii) holds.

Define random vectors $s_t^1, s_{t,u}^2 \in L_t^\infty$ by

$$s_t^{11} = 1, \quad s_{t,u}^{21} = \frac{1}{1 + \overline{R}_{t,u}}, \quad s_t^{1i} = \frac{\underline{S}_t^i + \overline{S}_t^i}{2} + \overline{D}_t^i, \quad s_{t,u}^{2i} = \frac{\underline{S}_t^i + \overline{S}_t^i}{2} \text{ for } i \geqslant 2.$$

Then for any $(x_t, y_t) \in \mathcal{Z}_t$ we have

$$0 \leqslant \left(x_t - \sum_{u \in \mathcal{T}_{t+}} \widetilde{y}_{t,u}\right)^+ \cdot \underline{S}_t + d_t(x_t) - \left(x_t - \sum_{u \in \mathcal{T}_{t+}} \widetilde{y}_{t,u}\right)^- \cdot \overline{S}_t$$

$$\leqslant x_t s_t^1 - \sum_{u \in \mathcal{T}_{t+}} y_{t,u} s_{t,u}^2.$$

From the margin requirement it follows that $y_{t,u} s_{t,u}^2 \geqslant 0$ for each $t \in \mathcal{T}_-$, $u \in \mathcal{T}_{t+}$, which implies that condition (a) in Theorem 1.1 and condition (a′) in Theorem 1.2 are fulfilled.

Suppose (i) holds. To verify conditions (b) and (c) in Theorem 3.1, take $0 \leqslant m < 1$ such that $m \geqslant M/R$ and $m\varepsilon/2 \geqslant (1-m)\max_{t,i}(\overline{S}_t^i + \overline{D}_t^i)$ a.s. It is straightforward to check that $my_{t,u}^+ s_{t,u}^2 \geqslant M\widetilde{y}_{t,u}^+ \underline{S}_t$ and $y_{t,u}^- s_{t,u}^2 \leqslant \widetilde{y}_{t,u}^- \overline{S}_t$ for any $(x_t, y_t) \in \mathcal{Z}_t$, and so $my_{t,u}^+ s_{t,u}^2 - y_{t,u}^- s_{t,u}^2 \geqslant 0$, as follows from the margin requirement. Thus condition (b) holds.

In order to check (c), put $\widehat{x}_t = (x_t^2, \dots, x_t^N)$ and observe that $x_t^+ s_t^1 - x_t^- s_t^1 \geqslant |\widehat{x}_t|\varepsilon/2$ for any $(x_t, y_t) \in \mathcal{Z}_t$ by virtue of the choice of s_t^1. Then

$$mx_t^+ s_t^1 - x_t^- s_t^1 = m(x_t^+ s_t^1 - x_t^- s_t^1) - (1-m)x_t^- s_t^1$$
$$\geqslant m|\widehat{x}_t|\varepsilon/2 - (1-m)x_t^- s_t^1 \geqslant |\widehat{x}_t|(m\varepsilon/2 - (1-m)\max_i(\overline{S}_t^i + \overline{D}_t^i)) \geqslant 0.$$

The second inequality is a consequence of the fact that if $x_t^1 < 0$ then $x_t^- s_t^1 \leq |\widehat{x}_t|\max_i(\overline{S}_t^i + \overline{D}_t^i)$, as it follows from the inequality $x_t^+ \underline{S}_t - x_t^- \overline{S}_t + d_t(x_t) \geq 0$, which in turn can be obtained by combining the self-financing condition and the margin requirement. Thus condition (c) in Theorem 3.1 holds.

Condition (d) is valid because if $\mathcal{A}_t \neq \{0\}$ then for any $a_t \in \mathcal{A}_t$ we have $a_t s_t^1 \geqslant a_t^+ \underline{S}_t - a_t^- \overline{S}_t + d_t(a_t)$, which yields $a_t s_t^1 \in A_t$ and $\mathbb{E}a_t s_t^1 \geq 0$. Thus condition (i) implies the hedging principle by virtue of Theorem 3.1.

If assumption (ii) is satisfied, then condition (b$'$) holds with $m = M/R$ (which can be proved in the same way as above), and so the hedging principle is valid in view of Theorem 3.3. $\qquad\square$

The next theorem provides a characterisation of consistent price systems in the model with a margin requirement. It extends the results of the papers [11, 53], which considered finite probability spaces and markets with linear time structure.

We denote by $Y_{t,u}^*(\omega)$ the positive dual cone of $Y_{t,u}(\omega)$, i.e. $Y_{t,u}^*(\omega) = \{q \in \mathbb{R}^N : qy \geqslant 0 \text{ for all } y \in Y_{t,u}(\omega)\}$.

Theorem 1.4. *In the stock market model with a margin requirement, a sequence $(p_t)_{t \in \mathcal{T}}$, $p_t \in L_t^\infty$, is a consistent price system if and only if each $p_t > 0$ and the following conditions hold.*

(i) For every $t \in \mathcal{T}$ such that $\mathcal{A}_t \neq \{0\}$ we have

$$\mathbb{E}vp_t^1 > 0 \text{ for any } v \in A_t \setminus \{0\}$$

and

$$\underline{S}^i_t + \underline{D}^i_t < \frac{p^i_t}{p^1_t} < \overline{S}^i_t + \overline{D}^i_t \quad a.s. \ \text{on the set } \{\omega : \underline{S}^i_t + \underline{D}^i_t < \overline{S}^i_t + \overline{D}^i_t\}$$

for all $i = 2, \ldots, N$.

(ii) There exist $\mathscr{F}_t$-measurable random variables $S^i_t \in [\underline{S}^i_t, \overline{S}^i_t]$, $D^i_t \in [\underline{D}^i_t, \overline{D}^i_t]$, $t \in \mathcal{T}$, $i = 2, \ldots, N$, such that

$$p^i_t = (S^i_t + D^i_t)p^1_t. \tag{1.10}$$

(iii) There exist $\mathscr{F}_t$-measurable random variables $B_{t,u} \in [1/(1+\overline{R}_{t,u}), 1/(1+\underline{R}_{t,u})]$ and random vectors $q_{t,u} \in Y^*_{t,u}$ a.s., $t \in \mathcal{T}_-$, $u \in \mathcal{T}_{t+}$, such that

$$\mathrm{E}(p^1_u \mid \mathscr{F}_t) = B_{t,u}p^1_t - q^1_{t,u}, \qquad \mathrm{E}(p^i_u \mid \mathscr{F}_t) = S^i_t p^1_t - q^i_{t,u}, \ i \geqslant 2. \tag{1.11}$$

Proof. Suppose the conditions listed above are satisfied for a sequence $(p_t)_{t \in \mathcal{T}}$, $0 < p_t \in L^\infty_t$. Put $S_t = (1, S^2_t, \ldots, S^N_t)$, $S'_{t,u} = (B_{t,u}, S^2_t, \ldots, S^N_t)$, $D_t = (0, D^2_t, \ldots, D^N_t)$. Then for any $(x_t, y_t) \in \mathcal{Z}_t$ we have

$$\mathrm{E}\left[x_t p_t - \sum_{u \in \mathcal{T}_{t+}} y_{t,u} p_u\right] = \mathrm{E}\left[x_t p_t - \sum_{u \in \mathcal{T}_{t+}} y_{t,u} \mathrm{E}(p_u \mid \mathscr{F}_t)\right]$$

$$= \mathrm{E}\left[\left(x_t(S_t + D_t) - \sum_{u \in \mathcal{T}_{t+}} y_{t,u} S'_{t,u}\right) p^1_t\right] + \sum_{u \in \mathcal{T}_{t+}} \mathrm{E} y_{t,u} q_{t,u}$$

$$\geqslant \mathrm{E}\left[\left(\left(x_t - \sum_{u \in \mathcal{T}_{t+}} \widetilde{y}_{t,u}\right) S_t + x_t D_t\right) p^1_t\right]$$

$$\geqslant \mathrm{E}\left[\left(\left(x_t - \sum_{u \in \mathcal{T}_{t+}} \widetilde{y}_{t,u}\right)^+ \underline{S}_t - \left(x_t - \sum_{u \in \mathcal{T}_{t+}} \widetilde{y}_{t,u}\right)^- \overline{S}_t + d_t(x_t)\right) p^1_t\right] \geqslant 0,$$

and so condition (1.2) involved in the definition of a consistent price system holds. Condition (1.3) follows from Proposition 1.3. Thus, $(p_t)_{t \in \mathcal{T}}$ is a consistent price system.

Let us prove the converse statement. Suppose $(p_t)_{t \in \mathcal{T}}$ is a consistent price system. It can be proved by induction over the sets $\mathcal{T}^K, \mathcal{T}^{K-1}, \ldots, \mathcal{T}^0$ (see their definition in the proof of Proposition 1.1) that each p_t is strictly positive.

Indeed, for any $t \notin \mathcal{T}_-$ any non-negative $0 \neq a_t \in L^1_t$ belongs to $\mathcal{A}^+_t$

(according to the assumption on p. 45), and so $\mathrm{E}a_t p_t > 0$ by virtue of (1.3), which implies $p_t > 0$ for $t \notin \mathcal{T}_-$. Suppose $p_u > 0$ for all $u \in \mathcal{T}^{\geqslant k}$ and consider some $t \in \mathcal{T}^{k-1}$. Take arbitrary non-negative $0 \neq x_t \in L_t^1$ and let $y_t = (y_{t,u})_{u \in \mathcal{T}_{t+}}$ with $y_{t,u} = x_t$ for some $u \in \mathcal{T}_{t+}$ and $y_{t,s} = 0$ for $s \neq u$. Since $(x_t, y_t) \in \mathcal{Z}_t$, from (1.2) we obtain $\mathrm{E}x_t p_t \geqslant \mathrm{E}x_t p_u > 0$, which proves the strict positivity of p_t.

For any $t \in \mathcal{T}$ where $\mathcal{A}_t \neq \{0\}$, the first part of condition (i) can be obtained by applying (1.3) to $(a_u)_{u \in T} \in \mathcal{A}^+$ with $a_u = 0$, $u \neq t$, and $a_t = (v, 0, \ldots, 0)$ for any $v \in A_t \setminus \{0\}$.

In order to check the second part, suppose $p_t^i/p_t^1 < \underline{S}_t^i + \underline{D}_t^i$ on a set $\Gamma \in \mathscr{F}_t$, $\mathrm{P}(\Gamma) > 0$, for some $i \geq 2$. Consider $a_t = (a_t^1, \ldots, a_t^N)$ with $a_t^1 = -(\underline{S}_t^i + \underline{D}_t^i)\mathbf{I}_\Gamma$, $a_t^i = \mathbf{I}_\Gamma$ and $a_t^j = 0$ for $j \neq 1, i$. We have $a_t \in \mathcal{A}_t$ and $\mathrm{E}a_t p_t < 0$, which contradicts (1.3). Thus $p_t^i/p_t^1 \geqslant \underline{S}_t^i + \underline{D}_t^i$, and similarly, $p_t^i/p_t^1 \leqslant \overline{S}_t^i + \overline{D}_t^i$.

If for the set $\Gamma' = \{\omega : \underline{S}_t^i + \underline{D}_t^i = p_t^i/p_t^1 < \overline{S}_t^i + \overline{D}_t^i\}$, we have $\mathrm{P}(\Gamma') > 0$, we can define a_t as above (with Γ' in place of Γ) and obtain that $a_t \in \mathcal{A}_t^+$ but $\mathrm{E}a_t p_t = 0$, which is a contradiction. Hence $p_t^i/p_t^1 > \underline{S}_t^i + \underline{D}_t^i$, and similarly, $p_t^i/p_t^1 < \overline{S}_t^i + \overline{D}_t^i$ a.s. on the set $\{\omega : \underline{S}_t^i + \underline{D}_t^i < \overline{S}_t^i + \overline{D}_t^i\}$, which proves the second part of (i).

Let us verify conditions (ii) and (iii). First we show that for almost each $\omega \in \Omega$ there exist $S_t^i(\omega)$, $D_t^i(\omega)$, $B_{t,u}(\omega)$, $q_{t,u}(\omega)$ satisfying (1.10)–(1.11), and then it will be shown that they can be chosen in the $\mathscr{F}_t$-measurable way.

For $t \in \mathcal{T}_-$, consider the cones defining the self-financing condition:

$$M_t(\omega) = \left\{ (x, y) \in \mathbb{R}^{N(1+|\mathcal{T}_{t+}|)} : \right.$$

$$\left(x_t - \sum_{u \in \mathcal{T}_{t+}} \widetilde{y}_{t,u} \right)^+ \cdot \underline{S}_t(\omega) + d_t(\omega)(x_t) \geqslant \left(x_t - \sum_{u \in \mathcal{T}_{t+}} \widetilde{y}_{t,u} \right)^- \cdot \overline{S}_t(\omega) \right\}.$$

Property (1.2) implies the following inclusion for each $t \in \mathcal{T}_-$:

$$\left(p_t, -\mathrm{E}[(p_u)_{u \in \mathcal{T}_{t+}} \mid \mathscr{F}_t] \right) \in \left(M_t \cap (\mathbb{R}^N \times Y_t) \right)^*$$

where we denote $Y_t = \{(y_{t,u})_{u \in \mathcal{T}_{t+}} : y_{t,u} \in Y_{t,u}\}$. It can be verified that for almost all ω, the cone $M_t(\omega)$, $t \in \mathcal{T}_-$, is positively dual to the set $M_t'(\omega)$

consisting of all pairs $(p, -q) \in \mathbb{R}^{N(1+|\mathcal{T}_{t+}|)}$ such that

$$p^1 = 1, \qquad p^i = S^i + D^i, \ i \geqslant 2,$$
$$q_u^1 = B_u, \qquad q_u^i = S^i, \ i \geqslant 2, \qquad \text{for } u \in \mathcal{T}_{t+},$$

with some $B_u \in [1/(1 + \overline{R}_{t,u}(\omega)), 1/(1 + \underline{R}_{t,u}(\omega))]$, $S^i \in [\underline{S}_t^i(\omega), \overline{S}_t^i(\omega)]$, $D^i \in [\underline{D}_t^i(\omega), \overline{D}_t^i(\omega)]$. Since $M_t'(\omega)$ is closed, we have $M_t^*(\omega) = \text{cone}(M_t'(\omega))$, which implies $M_t^*(\omega) \cap (\{0\} \times Y_t^*(\omega)) = \{0\}$, and so

$$\left(M_t \cap (\mathbb{R}^N \times Y_t)\right)^* = M_t^* + \{0\} \times Y_t^*$$

by virtue of Proposition 1.7. Thus, we obtain the existence of $B_{t,u}(\omega)$, $S_t^i(\omega)$, $D_t^i(\omega)$, $q_{t,u}(\omega)$ satisfying (1.10), (1.11) for $t \in \mathcal{T}_-$. In a similar way we obtain the existence of $S_t^i(\omega)$, $D_t^i(\omega)$ satisfying (1.10) for $t \notin \mathcal{T}_-$.

It remains to show that $B_{t,u}$, S_t^i, D_t^i, $q_{t,u}$ can be chosen $\mathscr{F}_t$-measurably. For $t \in \mathcal{T}_-$ consider the closed cones

$$V_t^{(1)}(\omega) = \big\{(B, S, D, q) : S^i \in [\underline{S}_t^i(\omega), \overline{S}_t^i(\omega)], \ D^i \in [\underline{D}_t^i(\omega), \overline{D}_t^i(\omega)], \ i \geqslant 2,$$
$$B_u \in [1/(1 + \overline{R}_{t,u}(\omega)), 1/(1 + \underline{R}_{t,u}(\omega))], \ u \in \mathcal{T}_{t+}\big\},$$
$$V_t^{(2)}(\omega) = \big\{(B, S, D, q) : p_t^i(\omega) = (S^i + D^i)p_t^1(\omega), \ i \geqslant 2,$$
$$\mathrm{E}(p_u^1 \mid \mathscr{F}_t)(\omega) = B_u p_t^1(\omega) - q_u^1, \ u \in \mathcal{T}_{t+},$$
$$\mathrm{E}(p_u^i \mid \mathscr{F}_t)(\omega) = S^i p_t^1(\omega) - q_u^i, \ i \geqslant 2, \ u \in \mathcal{T}_{t+}\big\}$$
$$V_t^{(3)}(\omega) = \big\{(B, S, D, q) : q_u \in Y_{t,u}^*(\omega), \ u \in \mathcal{T}_{t+}\big\}.$$

These cones consist of families (B, S, D, q) of vectors $B = (B_u)_{u \in \mathcal{T}_{t+}} \in \mathbb{R}^{|\mathcal{T}_{t+}|}$, $S = (S^i)_{i \geqslant 2} \in \mathbb{R}^{N-1}$, $D = (D^i)_{i \geqslant 2} \in \mathbb{R}^{N-1}$, $q = (q_u^i \mid u \in \mathcal{T}_{t+}, \ i \geqslant 1\} \in \mathbb{R}^{N|\mathcal{T}_{t+}|}$ satisfying the corresponding conditions. One can see that they are $\mathscr{F}_t$-measurable: $V_t^{(1)}$ is the volume bounded by $\mathscr{F}_t$-measurable hyperplanes, $V_t^{(2)}$ is the intersection of $\mathscr{F}_t$-measurable hyperplanes, and $V_t^{(3)}$ is positively dual to $\mathscr{F}_t$-measurable closed cone $\{0\} \times Y_{t,u}$ (see Remark 1.7). Consequently, their intersection is also $\mathscr{F}_t$-measurable, so there exists a measurable selection from it (Proposition 1.8), which provides the sought-for S_t^i, D_t^i, $B_{t,u}$, $q_{t,u}^i$.

Condition (ii) for $t = T$ can be proved in a similar way. $\qquad \square$

1.6 Appendix 1: auxiliary results from functional analysis

In this appendix we provide several results from functional analysis used in this and the next chapters.

Let $(B, \mathcal{B}, \eta)$ be a measurable space with a finite measure η. Let L^1 and L^∞ be the spaces of all (equivalence classes of) integrable and essentially bounded functions with the norms $\|\cdot\|_1$ and $\|\cdot\|_\infty$, respectively. We put $\mathrm{E}x := \int x \, d\eta$ for any $x \in L^1$. For a cone $\mathcal{A}$, we denote $\mathcal{A}^+ = \mathcal{A} \setminus (-\mathcal{A})$.

Proposition 1.4 (a version of the Kreps-Yan theorem). *Let $\mathcal{H}$ and $\mathcal{A}$ be closed cones in L^1 such that $\mathcal{A}^+ \neq \emptyset$, $-\mathcal{A} \subset \mathcal{H}$, $\mathcal{A}^+ \cap \mathcal{H} = \emptyset$ and there exists $q \in L^\infty$ satisfying $\mathrm{E}qx > 0$ for all $x \in \mathcal{A}^+$. Then there exists $p \in L^\infty$ such that $\mathrm{E}px > 0$ for all $x \in \mathcal{A}^+$ and $\mathrm{E}px \leqslant 0$ for all $x \in \mathcal{H}$.*

Proof. If $\mathcal{A} \cap (-\mathcal{A}) = \{0\}$ (i.e. $\mathcal{A}^+ = \mathcal{A} \setminus \{0\}$), the proposition follows from a result by Rokhlin: see Theorem 1.1 in [52]. In the general case, consider the closed cones

$$\mathcal{A}^n = \{x \in \mathcal{A} : \mathrm{E}qx \geqslant n^{-1}\|x\|\}.$$

Then $\mathcal{A}^n \cap (-\mathcal{A}^n) = \{0\}$, and by applying Rokhlin's theorem to $\mathcal{H}$ and $\mathcal{A}^n$, we find $p^n \in L^\infty$ such that $\mathrm{E}p^n x > 0$ for any $x \in \mathcal{A}^n \setminus \{0\}$ and $\mathrm{E}p^n x \leqslant 0$ for all $x \in \mathcal{H}$. Without loss of generality it may be assumed that $\|p^n\|_\infty = 1$. Then the proposition holds with $p = \sum\limits_{n=1}^{\infty} 2^{-n} p^n$. $\qquad \square$

Remark 1.6. Note that if the space L^1 is *separable*, there always exists $q \in L^\infty$ such that $\mathrm{E}qx > 0$ for all $x \in \mathcal{A}^+$ ($\neq \emptyset$), and Proposition 1.4 follows from Theorem 5 in [8]. In a non-separable L^1, there might exist pointed cones $\mathcal{A}$ without functionals $q \in L^\infty$, see Section 1.7.

Proposition 1.5. *Let $\mathcal{H}$ and $\mathcal{A}$ be closed cones in L^1 such that $\mathcal{A}^+ \neq \emptyset$, $-\mathcal{A} \subset \mathcal{H}$, $\mathcal{A}^+ \cap \mathcal{H} = \emptyset$ and there exists $q \in L^\infty$ such that $\mathrm{E}qx > 0$ for all $x \in \mathcal{A}^+$. Then for any $y \in L^1$ the following conditions are equivalent:*
  (a) $y \in \mathcal{H}$;
  (b) $\mathrm{E}py \leqslant 0$ for any $p \in L^\infty$ such that $\mathrm{E}px > 0$ for all $x \in \mathcal{A}^+$ and $\mathrm{E}px \leqslant 0$
    for all $x \in \mathcal{H}$.

Proof. Clearly (a) implies (b). To prove the converse implication suppose $y \notin \mathcal{H}$. By separating the point y from the closed convex set $\mathcal{H}$, we construct $p_1 \in L^\infty$ such that $\mathrm{E}p_1 y > 0$ and $\mathrm{E}p_1 x \leqslant 0$ for all $x \in \mathcal{H}$. Since $\mathcal{H} \supset -\mathcal{A}$, we have $\mathrm{E}p_1 x \geqslant 0$ for all $x \in \mathcal{A}$.

According to Proposition 1.4, there exists $p_2 \in L^\infty$ such that $\mathrm{E}p_2 x \leqslant 0$ for any $x \in \mathcal{H}$ and $\mathrm{E}p_2 x > 0$ for any $x \in \mathcal{A}^+$. By choosing $\lambda > 0$ large enough, we get $\mathrm{E}py > 0$ for $p = \lambda p_1 + p_2$. On the other hand, $\mathrm{E}px > 0$ for all $x \in \mathcal{A}^+$ and $\mathrm{E}px \leqslant 0$ for all $x \in \mathcal{H}$. A contradiction. $\qquad\square$

Proposition 1.6 (Komlós theorem, [38]). *Let $\{x^\lambda\}_{\lambda \in \Lambda}$ be a family of functions in L^1 such that $\sup_{\lambda \in \Lambda} \|x^\lambda\|_1 < \infty$. Then there exists $x \in L^1$ and a sequence x^{λ_n} such that any its subsequence $x^{\lambda_{n_k}}$ Cesàro converges to x a.s., i.e. $k^{-1}(x^{\lambda_{n_1}} + \ldots + x^{\lambda_{n_k}}) \to x$ a.s.*

For the reader's convenience, we provide an auxiliary (known) result used in Theorem 1.4. For a cone $A \subset \mathbb{R}^n$ we denote by A^* its *positive dual* cone, i.e. $A^* = \{p \in \mathbb{R}^n : pa \geqslant 0 \text{ for all } a \in A\}$. It is well known that $A^{**} = A$ for any closed cone $A \subset \mathbb{R}^n$.

Proposition 1.7. *Let A and B be closed cones in $\mathbb{R}^n$ such that $A^* \cap (-B^*) = \{0\}$. Then $(A \cap B)^* = A^* + B^*$.*

Proof. It is straightforward to prove that $(A \cap B)^* \supset A^* + B^*$. In order to prove $(A \cap B)^* \subset A^* + B^*$, observe that $A \cap B \supset (A^* + B^*)^*$. Indeed, if $x \in (A^* + B^*)^*$ then $px \geqslant 0$ for any $p \in A^*$, so $x \in A^{**} = A$, and, similarly, $x \in B$. Consequently, $(A \cap B)^* \subset (A^* + B^*)^{**} = A^* + B^*$, where we use that $A^* + B^*$ is closed because A^* and B^* are closed and $A^* \cap (-B^*) = \{0\}$, see [50, Corollary 9.1.3]. $\qquad\square$

In the rest of the appendix we provide results on random measurable sets which were used in Section 1.5.

Let $(\Omega, \mathscr{F}, \mathrm{P})$ be a complete probability space and X a complete separable metric space endowed with the Borel σ-algebra $\mathscr{B}(X)$. By definition, a *(closed) random set* Z on $(\Omega, \mathscr{F}, \mathrm{P})$ is a set-valued mapping $Z\colon \Omega \to 2^X$ from Ω to the family of all subsets of X such that $Z(\omega)$ is closed a.s.

A random set Z is called *measurable* if for any open set $G \subset X$

$$\{\omega : Z(\omega) \cap G \neq \emptyset\} \in \mathscr{F}.$$

A measurable function $\xi \colon \Omega \to X$ is called a *measurable selection* of a random set Z if $\xi(\omega) \in Z(\omega)$ a.s.

The following propositions assemble properties of measurable random sets. Their proofs can be found in [6, Ch. III] and [43, Ch. 2].

Proposition 1.8. *For a set-valued mapping $Z \colon \Omega \to 2^X$ such that $Z(\omega)$ is closed and non-empty a.s. the following conditions are equivalents:*

(a) Z is measurable;

(b) for any $x \in X$ the distance $d(x, Z(\omega))$ is a measurable function from $(\Omega, \mathscr{F})$ to $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$;

(c) there exists a sequence $\{\xi_n\}_{n \geqslant 0}$ of measurable selections of Z such that $Z(\omega) = \mathrm{cl}\{\xi_n(\omega), \ n \geqslant 0\}$ a.s.;

(d) $\{\omega : Z(\omega) \cap B \neq 0\} \in \mathscr{F}$ for any $B \in \mathscr{B}(X)$;

(e) $\mathrm{Graph}(X) = \{(\omega, x) \in \Omega \times X : x \in Z(\omega)\} \in \mathscr{F} \otimes \mathscr{B}(X)$.

Proposition 1.9. *Any countable intersection $Z(\omega) = \bigcap\limits_{n \geqslant 0} Z_n(\omega)$ of measurable random sets Z_n, $n \geqslant 0$, is also measurable.*

Remark 1.7. 1) If $Z(\omega) = \{x \in \mathbb{R}^n : x \cdot p(\omega) = a(\omega)\}$ is a random hyperplane in $\mathbb{R}^n$ (possibly, $Z(\omega) = \mathbb{R}^n$ or $Z(\omega) = \emptyset$ for some ω), where $p \colon \Omega \to \mathbb{R}^n$ is a measurable vector, and $a \colon \Omega \to \mathbb{R}$ is random variable, then Z is measurable. This follows from that the distance between any point $x \in \mathbb{R}^n$ and this hyperplane is a measurable function of $p(\omega)$ and $a(\omega)$.

2) For a random measurable set Z in $\mathbb{R}^n$, which is non-empty a.s., its positively dual cone $Z^*(\omega) = \{p \in \mathbb{R}^n : p \cdot x \geqslant 0 \text{ for any } x \in Z(\omega)\}$ is also measurable. Indeed, let $\{\xi_n\}_{n \geqslant 0}$ be a sequence of measurable selections of Z such that $Z = \mathrm{cl}\{\xi_n, n \geqslant 0\}$ a.s. Then $Z_n(\omega) = \{p \in \mathbb{R}^n : p \cdot \xi_n(\omega) \geqslant 0\}$ are measurable random sets, and hence $Z^* = \bigcap\limits_{n \geqslant 0} Z_n$ is also measurable.

1.7 Appendix 2: linear functionals in non-separable L^1 spaces

The aim of this appendix is to provide an example[3] of a non-separable space L^1 that contains a pointed non-empty cone $\mathcal{A}$ such that there is no linear functional $p \in L^\infty$ strictly positive on $\mathcal{A} \setminus \{0\}$. This example shows that assumption (A) imposed in Section 1.1 is essential.

Let $(\Omega, \mathscr{F}, \mathrm{P})$ be a probability space, where Ω is the space of all functions $\omega \colon [0,1] \to \mathbb{R}$, $\mathscr{F}$ is the product of a continuum of copies of the Borel σ-algebra $\mathscr{B}$ on $\mathbb{R}$, and P is the probability measure on $\mathscr{F}$ such that the (canonical) random variables $\xi_t(\omega) := \omega(t)$, $t \in [0,1]$, have standard Gaussian distribution ($\xi_t \sim \mathscr{N}(0,1)$) and are independent.

Let $\mathcal{A}$ be the closed convex cone in $L^1 = L^1(\Omega, \mathscr{F}, \mathrm{P})$ spanned on the system of random variables ξ_t.

First we prove that the cone $\mathcal{A}$ is pointed, i.e. $\mathcal{A} \cap (-\mathcal{A}) = \{0\}$. Indeed, suppose the contrary: there exists a non-zero random variable $Z \in L^1$, a sequence $\xi_{t_1}, \xi_{t_2}, \ldots$ of distinct random variables ξ_{t_i}, and two sequences

$$A_k = \sum_{i=1}^{k} a_i^k \xi_{t_i}, \qquad B_k = \sum_{i=1}^{k} b_i^k \xi_{t_i}$$

of non-negative linear combinations of ξ_{t_i} such that

$$A_k \to Z, \ B_k \to -Z \ (\text{in } L^1).$$

This implies

$$C_k := \sum_{i=1}^{k} c_i^k \xi_{t_i} \to 0 \ (\text{in } L^1) \quad \text{for } c_i^k := a_i^k + b_i^k. \tag{1.12}$$

Since ξ_{t_i} are independent and standard Gaussian, C_k is a Gaussian random variable with zero expectation and variance $v^k = \sum_{i=1}^{k}(c_i^k)^2$. Thus $\mathrm{E}|C_k| = \sqrt{v^k} \cdot \mathrm{E}|X|$, where $X \sim \mathscr{N}(0,1)$. According to (1.12), $\mathrm{E}|C_k| \to 0$, and

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[3]The idea of this example was suggested by Prof. W. Schachermayer, [54].

consequently,

$$0 = \lim_{k \to \infty} v^k = \lim_{k \to \infty} \sum_{i=1}^{k} (a_i^k + b_i^k)^2 \geqslant \lim_{k \to \infty} \sum_{i=1}^{k} (a_i^k)^2,$$

where the last inequality holds because $a_i^k \geq 0$ and $b_i^k \geq 0$. Therefore

$$\mathrm{E}A_k^2 = \sum_{i=1}^{k} (a_i^k)^2 \to 0,$$

which implies that $\mathrm{E}|A_k| \to 0$, and hence $Z = 0$. This is a contradiction meaning that $\mathcal{A}$ is a pointed cone.

Now we prove that there are no continuous linear functionals on L^1 which are strictly positive on $\mathcal{A} \setminus \{0\}$. Suppose the contrary: such a functional π exists. Let p be a function in L^∞ such that $\langle \pi, y \rangle = \mathrm{E}py$, $y \in L^1$.

It is well-known that any $\mathscr{F}$-measurable function p on Ω can be represented in the form

$$p = f(\xi_{t_1}, \xi_{t_1}, \ldots),$$

where $\{t_1, t_2, \ldots\}$ is a countable subset of $[0, 1]$ and $f(\cdot, \cdot, \ldots)$ is a measurable function on the product of a countable number of copies of $(\mathbb{R}, \mathscr{B})$. Consider any $t \notin \{t_1, t_2, \ldots\}$. The random variable ξ_t is independent of $\xi_{t_1}, \xi_{t_1}, \ldots$ and hence independent of p. Consequently,

$$\mathrm{E}p\xi_t = \mathrm{E}p \cdot \mathrm{E}\xi_t = 0,$$

which is a contradiction.

Chapter 2

Utility maximisation in multimarket trading

In this chapter we continue studying the general model of interconnected markets described in the previous chapter and consider the question of choosing optimal trading strategies in this model. We propose a problem where a trader needs to manage a portfolio of assets choosing the level of consumption and allocating her wealth between the assets in the way that maximises the utility from trading. The main result of the chapter provides conditions for the existence of *supporting prices*, which allow to decompose the *multi-period constrained* utility maximisation problem and to reduce it to a family of *single-period unconstrained* problems.

Problems of utility maximisation in multi-period trading have been widely studied in the literature (typically for the "linear" time structure; see e.g. [1, 10, 13, 25, 42]). The most closely related model to our research is the framework of *von Neumann – Gale random dynamical systems* introduced by von Neumann [68] and Gale [25, 27] for the deterministic case and later extended to the stochastic case by Dynkin, Radner and others (see e.g. [1, 12, 13, 48, 50]). Originally, this framework addressed utility maximisation problems in models of a growing economy, where a planner needs to choose between consumption of commodities and using them for further production. Our model extends results of the von Neumann – Gale framework to financial markets. The main conceptual difference between financial and economic models consists in the possibility of *short sales*, when a trader may borrow assets from the broker. Mathematically this is expressed in that coordinates of portfolio vectors may be negative, while in economic models commodity vectors are typically non-negative.

Similarly to Chapter 1, in order to obtain the main results, we systematically use the idea of *margin requirements* on admissible portfolios, which limit allowed leverage. As it has already been mentioned, requirements of this type present in one form or another in all real financial markets and thus are fully justified from the applied point of view. We prove the existence of supporting prices in a market with a margin requirement under a condition on the size of the margin.

The chapter is organised as follows. In Section 2.1 we describe the general problem of utility maximisation in the multimarket model and establish the existence of optimal trading strategies which deliver maximum values of utility functionals. In Section 2.2 we prove the main result on the existence of supporting prices in the model. Section 2.3 contains an application of the general result to a specific model of an asset market with transaction costs and portfolio constraints. In Section 2.4 we construct two examples of natural market models, where supporting prices do not exist. Section 2.5 provides an auxiliary result from convex analysis, the Duality Theorem, which plays the central role in the proof of the existence of supporting prices.

2.1 Optimal trading strategies in the model of interconnected asset markets

In this chapter we use the general market model of Chapter 1, slightly modifying it to allow consideration of utility maximisation problems.

It is assumed that a system of interconnected markets is described by a probability space $(\Omega, \mathscr{F}, \mathrm{P})$, a directed acyclic graph G with a finite set of nodes $\mathcal{T}$, which are interpreted as different trading sessions, a family of σ-algebras $\mathscr{F}_t$, $t \in \mathcal{T}$, representing random factors affecting each trading session, and a measurable space of assets $(\Theta, \mathscr{J}, \mu)$ with a finite measure μ. The meaning of these objects is the same as in Chapter 1.

For a subset $T \subset \mathcal{T}$ we denote by $L_T^1 = L_T^1(\Omega \times \Theta)$ the space of all integrable functions $y \colon T \times \Omega \times \Theta \to R$, and let $L_{t+}^1 = L_{\mathcal{T}_{t+}}^1$ for each $t \in \mathcal{T}_-$.

To keep further notation concise, we define $L_{t+}^1 := \{0\}$ for each sink node $t \notin \mathcal{T}_-$ (which can be interpreted as that no distribution of assets is done in terminal trading sessions).

Trading constraints in the model are defined by some given (convex) cones $\mathcal{Z}_t \subset L_t^1 \otimes L_{t+}^1, t \in \mathcal{T}$. In this chapter, from the beginning, we assume that $\mathcal{Z}_t$ are closed with respect to L^1-bounded a.s.-convergence, and, hence, are also closed in L^1 (which follows from the fact that from any sequence converging in L^1 it is possible to extract a subsequence converging with probability one).

A *trading strategy* ζ is a family of functions $\zeta = (x_t, y_t)_{t \in \mathcal{T}}$ such that

$$(x_t, y_t) \in \mathcal{Z}_t \text{ for each } t \in \mathcal{T}.$$

For each $t \in \mathcal{T}_-$, the function x_t represents the portfolio held before one buys and sells assets in the session t and the function $y_t = (y_{t,u})_{u \in \mathcal{T}_{t+}}$ specifies the distribution of assets to the sessions $u \in \mathcal{T}_{t+}$. If $t \notin \mathcal{T}_-$ (i.e. t is a sink node), $x_t \in L_t^1$ specifies the *terminal portfolio* obtained in this session, and $y_t := 0$ by the definition of L_{t+}^1. Note that in Chapter 1 trading strategies are indexed by $t \in \mathcal{T}_-$, while in this chapter we also take into account terminal pairs $(x_t, 0)$ for sink nodes $t \notin \mathcal{T}_-$.

The definition of a *contract* in the market is the same as in Chapter 1: it is a family of portfolios

$$\gamma = (c_t)_{t \in \mathcal{T}}, \qquad c_t \in L_t^1,$$

where c_t stands for the portfolio which has to be delivered – according to this contract – in the trading session t.

In this chapter we will consider only *exact hedging (replication)* of contracts, and will say that a trading strategy $\zeta = (x_t, y_t)_{t \in \mathcal{T}}$ *hedges* a contract $\gamma = (c_t)_{t \in \mathcal{T}}$ if

$$x_t = y_{t-} - c_t \text{ for each } t \in \mathcal{T},$$

where $y_{t-} := \sum_{s \in \mathcal{T}_{t-}} y_{s,t}$ for $t \in \mathcal{T}_+$ is the amount of assets delivered to the trading session t, and $y_{t-} := 0$ for a source node $t \notin \mathcal{T}_+$. In terms of the previous chapter, the set of families of risk-acceptable portfolios is $\mathcal{A} = \{0\}$.

Suppose that with each $t \in \mathcal{T}$, a utility function $u_t \colon L_t^1 \times L_{t+}^1 \to \mathbb{R}$ is associated. The *utility* of a trading strategy $\zeta = (x_t, y_t)_{t \in \mathcal{T}}$ is defined by

$$u(\zeta) = \sum_{t \in \mathcal{T}} u_t(x_t, y_t).$$

Further, it is assumed that each function u_t is upper semi-continuous (with respect to the topology induced by the norm in L^1) and concave, and, hence, u is also upper semi-continuous and concave.

If t is not a sink node (i.e. $t \in \mathcal{T}_-$), the utility function u_t measures the gain obtained from rearranging a portfolio x_t into a family of portfolios y_t. A simple example of such a function is

$$u_t^{(1)}(x_t, y_t) = \mathrm{E}\, f\!\left(\int_\Theta \left(x_t - \sum_{u \in \mathcal{T}_{t+}} y_{t,u} \right) S_t d\mu \right) \qquad (2.1)$$

for a function $S_t = S_t(\omega, \theta)$ of *asset prices* and a deterministic utility function $f \colon \mathbb{R} \to \mathbb{R}$. Here, the utility of a pair (x_t, y_t) is equal to the expected f-utility of the monetary gain from selling assets and buying new ones.

A more sophisticated model is

$$u_t^{(2)}(x_t, y_t) = \mathrm{E}f\!\left(\int_\Theta \left[\left(x_t - \sum_{u \in \mathcal{T}_{t+}} y_{t,u} \right)^+ \underline{S}_t - \left(x_t - \sum_{u \in \mathcal{T}_{t+}} y_{t,u} \right)^- \overline{S}_t \right] d\mu \right), \quad (2.2)$$

where $\underline{S}_t \leqslant \overline{S}_t$ are *bid* and *ask price* functions.

If $t \notin \mathcal{T}_-$, the value of $u_t(x_t, y_t)$ with $y_t \equiv 0$, represents the utility of a terminal portfolio x_t and can be also defined, for example, by formulae (2.1) or (2.2), putting $y_{t,u} := 0$.

Remark 2.1. It is worth mentioning that the model proposed is able to include the case when one receives a gain from just *holding* assets, not necessarily consuming them. This circumstance was especially important in von Neumann – Gale models of a growing economy, where additional utility can be obtained from availability of goods whose consumption does not reduce their quantity (e.g. housing, works of art, etc.). The approach can also be used to model irrationalities in people's actions, e.g. when the valuation of additional gain or loss depends on the current wealth (such as

the *endowment effect*, see [34]).

In the financial setting, a typical example is a gain obtained from dividend payments. The corresponding utility function can be of the form

$$u_t^{(3)}(x_t, y_t) = \mathrm{E}\, g\left(\int_\Theta x_t D_t d\mu\right) + \mathrm{E}\, f\left(\int_\Theta \Big(x_t - \sum_{u \in \mathcal{T}_{t+}} y_{t,u}\Big) S_t d\mu\right),$$

where $D_t = D_t(\omega, \theta)$ is a function of *dividend rates* and g is some deterministic utility function.

If γ is a hedgeable contract (there exists at least one trading strategy hedging it), by $U(\gamma)$ we denote the supremum of the utilities of all trading strategies hedging γ, i. e.

$$U(\gamma) := \sup\{u(\zeta) \mid \zeta \text{ hedges } \gamma\}.$$

A trading strategy $\widehat{\zeta}$ is called an *optimal trading strategy* for γ if $u(\widehat{\zeta}) = U(\gamma)$, i. e. ζ maximises the utility from trading with the contract γ.

Remark 2.2. Note that U is a concave function. Indeed, if $\gamma = \alpha\gamma_1 + (1-\alpha)\gamma_2$ for hedgeable contracts γ_1, γ_2 and $\alpha \in [0,1]$, then for any $\varepsilon > 0$ there exist trading strategies ζ_1, ζ_2 hedging γ_1, γ_2 such that $U(\gamma_i) \leqslant u(\zeta_i) + \varepsilon$. Consequently, $\zeta = \alpha\zeta_1 + (1-\alpha)\zeta_2$ hedges γ and

$$U(\gamma) \geqslant u(\zeta) \geqslant \alpha u(\zeta_1) + (1-\alpha)u(\zeta_2) \geqslant \alpha U(\gamma_1) + (1-\alpha)U(\gamma_2) - \varepsilon.$$

Passing to the limit $\varepsilon \to 0$, we obtain $U(\gamma) \geqslant \alpha U(\gamma_1) + (1-\alpha)U(\gamma_2)$.

The next theorem states that conditions (a), (b) of Theorem 1.1 guarantee the existence of optimal trading strategies for hedgeable contracts.

Theorem 2.1. *Suppose there exist functions $s_t^1, s_{t,u}^2 \in L_t^\infty$, $t \in \mathcal{T}_-$, $u \in \mathcal{T}_{t+}$, with values in $[\underline{s}, \overline{s}]$, where $\underline{s} > 0$, $\overline{s} \geqslant 1$, and a constant $0 \leqslant m < 1$ such that for all $t \in \mathcal{T}$, $u \in \mathcal{T}_{t+}$, $(x_t, y_t) \in \mathcal{Z}_t$ the following conditions are satisfied:*

(a) $\mathrm{E}x_t s_t^1 \geqslant \mathrm{E}y_{t,u} s_{t,u}^2$;

(b) $m\mathrm{E}y_{t,u}^+ s_{t,u}^2 \geqslant \mathrm{E}y_{t,u}^- s_{t,u}^2$.

Then an optimal trading strategy exists for any hedgeable contract.

The functions s_t^1, $s_{t,u}^2$ can be interpreted as some systems of asset prices in the same way as in Theorem 1.1.

Proof. Let γ be a hedgeable contract, we need to show that there exists a trading strategy $\widehat{\zeta}$ hedging γ such that $u(\widehat{\zeta}) = U(\gamma)$.

As follows from the proof of Theorem 1.1, there exists a constant C (namely, $C = (1 + m)/(1 - m) \cdot \overline{s}/\underline{s}$) such that $\|y_{t,u}\| \leq C\|x_t\|$ for any $(x_t, y_t) \in \mathcal{Z}_t$, $t \in \mathcal{T}_-$, $u \in \mathcal{T}_{t+}$, which implies that the set of all trading strategies hedging γ is bounded in L^1.

Take a sequence of strategies ζ^i hedging γ such that $u(\zeta^i) \to U(\gamma)$. By using the Komlós theorem (see Proposition 1.6), we find a subsequence ζ^{i_j} Cesàro-convergent a.s. to some $\widehat{\zeta}$ (i.e. $\widehat{\zeta^j} := j^{-1}(\zeta^{i_1} + \ldots + \zeta^{i_j}) \to \widehat{\zeta}$). Since the cones $\mathcal{Z}_t$ are closed with respect to L^1-bounded a.s.-convergence, $\widehat{\zeta}$ is a trading strategy and it hedges γ. Then we find

$$u(\widehat{\zeta}) \geqslant \limsup_{j \to \infty} u(\widehat{\zeta^j}) \geqslant \limsup_{j \to \infty} \frac{1}{j} \sum_{k=1}^{j} u(\zeta^{i_k}) = U(\gamma),$$

where in the first inequality we use that u is upper semicontinuous, and in the second inequality we use that u is concave. Thus $\widehat{\zeta}$ is the sought-for trading strategy. $\qquad\square$

Remark 2.3. Under the conditions of the theorem, the set of all hedgeable contracts is a closed cone in $L_{\mathcal{T}}^1$, as follows from the proof of Theorem 1.1.

2.2 Supporting prices: the definition and conditions of existence

In this section we introduce the central notion of the chapter, a *system of supporting prices* for a hedgeable contract, and provide conditions which guarantee its existence.

Further we always assume that cones $\mathcal{Z}_t$ and utility functions u_t satisfy

the following natural conditions of *monotonicity*:

if $(x_t, y_t) \in \mathcal{Z}_t$, then $(x'_t, y_t) \in \mathcal{Z}_t$ for any $x'_t \in L^1_t$ such that $x'_t \geqslant x_t$, (2.3)

$u(x'_t, y_t) \geqslant u(x_t, y_t)$ for any $x_t, x'_t \in L^1_t$, $y_t \in L^1_{t+}$ such that $x'_t \geqslant x_t$, (2.4)

where here and below all inequalities for random variables are understood to hold with probability one.

These two properties imply that if $\gamma = (c_t)_{t \in \mathcal{T}}$, $\gamma' = (c'_t)_{t \in \mathcal{T}}$ are two hedgeable contracts and $\gamma' \leqslant \gamma$ ($c'_t \leqslant c_t$ for each $t \in \mathcal{T}$), then $U(\gamma') \geqslant U(\gamma)$, i.e. a contract with a smaller amount of payments provides greater utility. Indeed, if γ can be hedged by a trading strategy $\zeta = (x_t, y_t)_{t \in \mathcal{T}}$, then γ' can be hedged by $\zeta' = (x'_t, y_t)_{t \in \mathcal{T}}$ with $x'_t = x_t + c_t - c'_t$, and $u(\zeta') \geqslant u(\zeta)$.

Moreover, (2.3) implies that any non-positive contract $\gamma = (c_t)_{t \in \mathcal{T}}$ (i.e. a contract according to which one does not make any payment, but only receives assets) is hedgeable, e.g. by the strategy $\zeta = (x_t, y_t)_{t \in \mathcal{T}}$ with $x_t = c_t$, $y_t = 0$.

A family of functions $(p_t)_{t \in \mathcal{T}}$, $p_t \in L^\infty_t$, is called a *system of supporting prices* for a hedgeable contract $\gamma = (c_t)_{t \in \mathcal{T}}$ if for any family of pairs $(x_t, y_t)_{t \in \mathcal{T}}$, $(x_t, y_t) \in \mathcal{Z}_t$, it holds that

$$U(\gamma) \geqslant \sum_{t \in \mathcal{T}} \left[u_t(x_t, y_t) + \mathrm{E}\Big(\sum_{u \in \mathcal{T}_{t+}} y_{t,u} p_u - x_t p_t - c_t p_t \Big) \right], \qquad (2.5)$$

where $y_{t,u} p_u := 0$ for a sink node t.

The financial interpretation of supporting prices is that $\mathrm{E}[\sum_u y_{t,u} p_u - x_t p_t]$ is equal to the expected profit from choosing the pair (x_t, y_t), where $\mathrm{E} x_t p_t$ is interpreted as the expected cost of the portfolio x_t in the session t and $\mathrm{E} \sum_u y_{t,u} p_u$ is the expected cost of the portfolios $y_{t,u}$ in the "future" sessions $u \in \mathcal{T}_{t+}$. The quantity $\mathrm{E} c_t p_t$ is the expected cost of the portfolio c_t delivered according to the contract.

Observe that each p_t is non-negative – otherwise for $x_t = r p_t^-$, $y_t = 0$ we have $(x_t, y_t) \in \mathcal{Z}_t$, but the right-hand side of (2.5) exceeds the left-hand side for a large real $r > 0$. Moreover, each p_t is strictly positive ($p_t(\omega, \theta) > 0$ a.s.) if $u_t(r\mathbf{I}_\Gamma, 0) \to +\infty$ whenever $r \to +\infty$ for any set $\Gamma \in \mathscr{F} \otimes \mathscr{J}$ of strictly

positive measure (otherwise take $x_t = r\mathbf{I}(p_t = 0)$, $y_t = 0$ to obtain the same contradiction).

The following theorem provides sufficient conditions for the existence of supporting prices for non-positive contracts in the general model. A particular asset market model with transaction costs and portfolio constraints will be studied in the next section. Note that even in natural market models supporting prices may not exist for contracts whose portfolio positions can be positive. An example will be provided in Section 1.5.

Theorem 2.2. *Supporting prices exist for any contract $\gamma \leqslant 0$ if conditions (a), (b) of Theorem 2.1 and the following three conditions hold:*

(c) *if $(x_t, y_t) \in \mathcal{Z}_t$ for some $t \in \mathcal{T}_-$, then $(x_t, 0) \in \mathcal{Z}_t$ and $(y_{t,u}, 0) \in \mathcal{Z}_u$ for each $u \in \mathcal{T}_{t+}$;*

(d) *there exists a constant A such that if $(x_t, y_t), (x'_t, 0) \in \mathcal{Z}_t$ for some $t \in \mathcal{T}_-$, then there exists y'_t such that $(x'_t, y'_t) \in \mathcal{Z}_t$ and $\|y_t - y'_t\| \leqslant A\|x_t - x'_t\|$;*

(e) *there exists a constant B such that if $(x_t, y_t) \in \mathcal{Z}_t$ and $(x'_t, y'_t) \in \mathcal{Z}_t$ for some $t \in \mathcal{T}$, then $|u_t(x_t, y_t) - u_t(x'_t, y'_t)| \leqslant B(\|x_t - x'_t\| + \|y_t - y'_t\|)$.*

Condition (c) can be interpreted as a "safety" requirement meaning that it should be possible to liquidate a portfolio at any time. Condition (d) is a continuity property of the cones $\mathcal{Z}_t$ and condition (e) is a Lipschitz-continuity property of the utility functions.

In order to prove the theorem, we first establish an auxiliary result showing that the *steepness* of the function U is bounded from above at any contract $\gamma \leqslant 0$ (see Section 2.5 for the definitions of the *steepness* and a *support* of a function). This will allow us to apply the Duality theorem to obtain a *support* for U, which will provide a system of supporting prices.

We will use the notation $\mathcal{H}$ for the closed cone of all hedgeable contracts.

Lemma 2.1. *If conditions (a)–(e) are satisfied, the steepness of the function $U \colon \mathcal{H} \to \mathbb{R}$ is bounded from above at any contract $\gamma \leqslant 0$.*

Proof. Let K denote the maximal length of a directed path in the graph. Assuming, without loss of generality, that $A \geqslant 1$, we will show that for any

non-positive contract $\gamma = (c_t)_{t \in \mathcal{T}}$, and any hedgeable contract $\gamma' = (c'_t)_{t \in \mathcal{T}}$ we have $U(\gamma') - U(\gamma) \leqslant B(2A)^{K+1} \|\gamma' - \gamma\|$.

Observe that it is sufficient to consider only the case $\gamma' \leqslant \gamma$, because for $\gamma'' = \min(\gamma, \gamma')$ we have $U(\gamma'') \geqslant U(\gamma')$ according to (2.3)–(2.4) and $\|\gamma'' - \gamma\| \leqslant \|\gamma' - \gamma\|$.

Let $\varepsilon = \gamma - \gamma'$ (i.e. $\varepsilon = (\varepsilon_t)_{t \in \mathcal{T}}$, where $\varepsilon_t = c_t - c'_t$) and $\zeta' = (x'_t, y'_t)_{t \in \mathcal{T}}$ be a trading strategy at which $U(\gamma')$ is attained. Using assumptions (c)–(d) we will construct a trading strategy $\zeta = (x_t, y_t)_{t \in \mathcal{T}}$ hedging γ such that

$$\sum_{t \in \mathcal{T}} \|x'_t - x_t\| \leqslant (2A)^K \|\varepsilon\|, \qquad \sum_{t \in \mathcal{T}} \|y'_t - y_t\| \leqslant A(2A)^K \|\varepsilon\|.$$

Let $\kappa(t)$ denote the maximal length of a directed path emanating from a node $t \in \mathcal{T}$. Define the sets $\mathcal{T}^k := \{t \in \mathcal{T} : \kappa(t) = K - k\}$, $k = 0, \ldots, K$, which form a partition of $\mathcal{T}$ such that if there is a path from $t \in \mathcal{T}^k$ to $s \in \mathcal{T}^n$, then $k < n$. Note that $\mathcal{T}^0$ contains only source nodes, and $\mathcal{T}^K$ is the set of all the sink nodes. Put also $\mathcal{T}^{\leqslant k} = \bigcup_{n \leqslant k} \mathcal{T}^n$, $\mathcal{T}^{\geqslant k} = \bigcup_{n \geqslant k} \mathcal{T}^n$.

Let us show by induction that for each $k \leqslant K$ it is possible to find a family of pairs $(x_t, y_t) \in \mathcal{Z}_t$, $t \in \mathcal{T}^{\leqslant k}$, such that for each $t \in \mathcal{T}^{\leqslant k}$

$$x_t = y_{t-} - c_t \tag{2.6}$$

and

$$\sum_{t \in \mathcal{T}^{\leqslant k}} \|x'_t - x_t\| \leqslant (2A)^k \sum_{t \in \mathcal{T}^{\leqslant k}} \|\varepsilon_t\|, \qquad \|y'_t - y_t\| \leqslant A \|x'_t - x_t\|. \tag{2.7}$$

For $k = 0$ we can find such a family of pairs for each $t \in \mathcal{T}^0$ by defining $x_t = -c_t$, and using assumption (d) choosing y_t such that $(x_t, y_t) \in \mathcal{Z}_t$ and $\|y'_t - y_t\| \leqslant A \|x'_t - x_t\|$ (observe that $(x_t, 0) \in \mathcal{Z}_t$ because $c_t \leqslant 0$ and monotonicity assumption (2.3) holds).

If a family $(x_t, y_t) \in \mathcal{Z}_t$, $t \in \mathcal{T}^{\leqslant k}$, satisfying (2.6)–(2.7), is found for some $k < K$, define $x_t = y_{t-} - c_t$ for each $t \in \mathcal{T}^{k+1}$, where $y_{t-} = 0$ if t is a source

node, and $y_{t-} = \sum\limits_{s \in \mathcal{T}_{t-}} y_{s,t}$ otherwise. Then

$$
\begin{aligned}
\sum_{t \in \mathcal{T}^{\leqslant k+1}} \|x'_t - x_t\| &\leqslant \sum_{t \in \mathcal{T}^{\leqslant k+1}} \left(\|y'_{t-} - y_{t-}\| + \|\varepsilon_t\| \right) \\
&\leqslant \sum_{t \in \mathcal{T}^{\leqslant k}} \|y'_t - y_t\| + \sum_{t \in \mathcal{T}^{\leqslant k+1}} \|\varepsilon_t\| \\
&\leqslant A(2A)^k \sum_{t \in \mathcal{T}^{\leqslant k}} \|\varepsilon_t\| + \sum_{t \in \mathcal{T}^{\leqslant k+1}} \|\varepsilon_t\| \leqslant (2A)^{k+1} \sum_{t \in \mathcal{T}^{\leqslant k+1}} \|\varepsilon_t\|.
\end{aligned}
$$

Observe that $(x_t, 0) \in \mathcal{Z}_t$ for $t \in \mathcal{T}^{k+1}$ according to assumption (2.3) because $(y_{t-}, 0) \in \mathcal{Z}_t$ as follows from (c), and $c_t \leqslant 0$. Consequently, using (d) it is possible to find y_t, $t \in \mathcal{T}^{k+1}$, such that $(x_t, y_t) \in \mathcal{Z}_t$, $\|y'_t - y_t\| \leqslant A\|x'_t - x_t\|$.

Proceeding by induction, we find a family $\zeta = (x_t, y_t)_{t \in \mathcal{T}}$ satisfying (2.6)–(2.7) for $k = K$. Then from (2.6) it follows that ζ is a trading strategy hedging the contract γ. From (2.7) and property (e) if follows that $u(\zeta') - u(\zeta) \leqslant B(2A)^{K+1}\|\varepsilon\|$, which proves the lemma. $\qquad\square$

Proof of Theorem 2.2. Consider any contract $\gamma \leqslant 0$. According to Proposition 2.1 (see Section 2.5), there exists a *support* of the function $U \colon \mathcal{H} \to \mathbb{R}$ at γ, and so there exists $p = (p_t)_{t \in \mathcal{T}}$, $p_t \in L_t^\infty$, such that

$$
U(\gamma') - U(\gamma) \leqslant \mathrm{E}[(\gamma - \gamma')p] \text{ for any } \gamma' \in \mathcal{H}
$$

(p can be taken as the negative of the support).

Take any family $\zeta = (x_t, y_t)_{t \in \mathcal{T}}$ such that $(x_t, y_t) \in \mathcal{Z}_t$ for each $t \in \mathcal{T}$. This family is a trading strategy hedging the contract $\gamma' = (c'_t)_{t \in \mathcal{T}}$ with $c'_t = y_{t-} - x_t$. Consequently

$$
\begin{aligned}
U(\gamma) \geqslant U(\gamma') + \mathrm{E}[(\gamma' - \gamma)p] &\geqslant u(\zeta) + \mathrm{E}[(\gamma' - \gamma)p] \\
&= \sum_{t \in \mathcal{T}} u_t(x_t, y_t) + \sum_{t \in \mathcal{T}} \mathrm{E}[(c'_t - c_t)p_t] \\
&= \sum_{t \in \mathcal{T}} u_t(x_t, y_t) + \sum_{t \in \mathcal{T}} \mathrm{E}[(y_{t-} - x_t - c_t)p_t].
\end{aligned}
$$

Rearranging the terms in the second sum,

$$\sum_{t \in \mathcal{T}} \mathrm{E}[(y_{t-} - x_t - c_t)p_t] = \mathrm{E}\left[\sum_{t \in \mathcal{T}} \sum_{s \in \mathcal{T}_{t-}} y_{s,t} p_t\right] - \mathrm{E}\left[\sum_{t \in \mathcal{T}} [(x_t + c_t)p_t]\right]$$

$$= \mathrm{E}\left[\sum_{t \in \mathcal{T}} \sum_{u \in \mathcal{T}_{t+}} y_{t,u} p_u\right] - \mathrm{E}\left[\sum_{t \in \mathcal{T}} [(x_t + c_t)p_t]\right],$$

we obtain (2.5), which shows that $(p_t)_{t \in \mathcal{T}}$ is a system of supporting prices. $\quad\square$

2.3 Supporting prices in an asset market with transaction costs and portfolio constraints

In this section we prove the existence of supporting prices in a model of an asset market proposed in Section 1.1 (and modified in a way to fit the general model of Section 2.1).

Recall that the model considers a market of N assets represented by a probability space $(\Omega, \mathscr{F}, \mathrm{P})$, a directed acyclic graph G with a finite set of nodes $\mathcal{T}$, σ-algebras $\mathscr{F}_t$ associated with the nodes of the graph, and the finite space of assets $(\Theta, \mathscr{J}, \mu)$, where $\Theta = \{1, \ldots, N\}$, $\mathscr{J} = 2^\Theta$, $\mu\{i\} = 1$ for each $i = 1, \ldots, N$. It is assumed that $\mathscr{F}_t \subset \mathscr{F}_u$ for any $t \in \mathcal{T}_-$, $u \in \mathcal{T}_{t+}$, and each σ-algebra $\mathscr{F}_t$ is completed by all $\mathscr{F}$-measurable sets of measure 0.

Asset $i = 1$ is interpreted as cash (deposited with a bank account) and assets $i = 2, \ldots, N$ as stock. For each $t \in \mathcal{T}$, the bid and the ask stock prices are denoted by $\underline{S}_t^i \leqslant \overline{S}_t^i$ and the dividend rates for long and short stock positions are denoted by $\underline{D}_t^i \leqslant \overline{D}_t^i$. Random variables $\underline{R}_{t,u} \leqslant \overline{R}_{t,u}$ stand for the interest rates for lending and borrowing cash between the sessions $t \in \mathcal{T}_-$ and $u \in \mathcal{T}_{t+}$.

For each $t \in \mathcal{T}_-$ random $\mathscr{F}_t$-measurable closed cones $Y_{t,u}(\omega)$, $u \in \mathcal{T}_{t+}$, are given and describe portfolio constraints. We define the cones $\mathcal{Z}_t$ by formula (1.7) for $t \in \mathcal{T}_-$, and for sink nodes $t \notin \mathcal{T}_-$ we let

$$\mathcal{Z}_t = \left\{ (x_t, 0) \in L_t^1 \otimes \{0\} : x_t^+ \underline{S}_t + d_t(x_t) \geqslant x_t^- \overline{S}_t \text{ a.s.} \right\},$$

which consists of pairs $(x_t, 0)$, where the portfolio x_t has non-negative liquidation value with probability one.

It is assumed that the market satisfies margin requirement (1.9),

$$M\widetilde{y}^+_{t,u}\underline{S}_t(\omega) \geq \widetilde{y}^-_{t,u}\overline{S}_t(\omega) \text{ for any } y_t \in Y_{t,u}(\omega), \ t \in \mathcal{T}_-, \ u \in \mathcal{T}_{t+},$$

where M is a constant such that $0 \leqslant M < R$ with $R \leq (1 + \underline{R}_{t,u})/(1 + \overline{R}_{t,u})$.

Theorem 2.3. *1) In the model of a stock market with a margin requirement, an optimal trading strategy exists for any hedgeable contract γ.*

2) If the constant M satisfies the inequality

$$M \leqslant \frac{\min\left\{\operatorname*{ess\,inf}_{\omega,i} \frac{\underline{S}^i_u + \underline{D}^i_u}{\underline{S}^i_t}, \ \operatorname*{ess\,inf}_{\omega}(1 + \underline{R}_{t,u})\right\}}{\max\left\{\operatorname*{ess\,sup}_{\omega,i} \frac{\overline{S}^i_u + \overline{D}^i_u}{\overline{S}^i_t}, \ \operatorname*{ess\,sup}_{\omega}(1 + \overline{R}_{t,u})\right\}}, \quad t \in \mathcal{T}_-, \ u \in \mathcal{T}_{t+}, \qquad (2.8)$$

and the utility functions u_t, $t \in \mathcal{T}$, satisfy assumption (e) of Theorem 2.2, then a system of supporting prices exists for any contract $\gamma \leqslant 0$.

Inequality (2.8) requires that the margin should be large enough (equivalently, the constant M should be small enough), which protects a trader if the price change in an unfavourable way: it will be seen in the proof that this inequality implies the validity of condition (c) in Theorem 2.2.

Proof. In the proof of Theorem 1.3 conditions (a), (b) of Theorem 1.1, and, hence, of Theorem 2.1, were verified for the random vectors $s^1_t, s^2_{t,u} \in L^\infty_t$ with

$$s^{11}_t = 1, \quad s^{21}_{t,u} = \frac{1}{1 + \overline{R}_{t,u}}, \quad s^{1i}_t = \frac{\underline{S}^i_t + \overline{S}^i_t}{2} + \overline{D}^i_t, \quad s^{2i}_{t,u} = \frac{\underline{S}^i_t + \overline{S}^i_t}{2} \text{ for } i \geqslant 2.$$

Thus, the first claim follows from Theorem 2.1.

Now we prove the second claim by verifying conditions (c)–(d) of Theorem 2.2. In order to verify (c), observe that if $(x_t, y_t) \in \mathcal{Z}_t$ for some $t \in \mathcal{T}_-$ then

$$\sum_{u \in \mathcal{T}_{t+}} (\widetilde{y}^+_{t,u}\underline{S}_t - \widetilde{y}^-_{t,u}\overline{S}_t) \geqslant 0$$

as follows from the margin requirement. Summing this inequality with the self-financing condition and using the superlinearity (concavity and positive

homogeneity) of the function $x \mapsto x^+\underline{S}_t - x^-\overline{S}_t$, we obtain $x_t^+\underline{S}_t + d_t(x_t) \geqslant x_t^-\overline{S}_t$, i.e. $(x_t, 0) \in \mathcal{Z}_t$.

Using the notation $\underline{S}_t = (1, \underline{S}_t^2, \ldots, \underline{S}_t^N)$, $\overline{S}_t = (1, \overline{S}_t^2, \ldots, \overline{S}_t^N)$ for the price vectors, and $\underline{D}_t = (0, \underline{D}_t^2, \ldots, \underline{D}_t^N)$, $\overline{D}_t = (0, \overline{D}_t^2, \ldots, \overline{D}_t^N)$ for the dividend rates vectors, we obtain the following chain of inequalities for all $t \in \mathcal{T}_-$, $u \in \mathcal{T}_{t+}$:

$$y_{t,u}^+(\underline{S}_u + \underline{D}_u) \geqslant \widetilde{y}_{t,u}^+\underline{S}_t \cdot \min\left\{ \operatorname*{ess\,inf}_{\omega,i} \frac{\underline{S}_u^i + \underline{D}_u^i}{\underline{S}_t^i}, \; \operatorname*{ess\,inf}_{\omega}(1 + \underline{R}_{t,u}) \right\}$$

$$\geqslant M\widetilde{y}_{t,u}^+\underline{S}_t \cdot \max\left\{ \operatorname*{ess\,sup}_{\omega,i} \frac{\overline{S}_u^i + \overline{D}_u^i}{\overline{S}_t^i}, \; \operatorname*{ess\,sup}_{\omega}(1 + \overline{R}_{t,u}) \right\}$$

$$\geqslant \widetilde{y}_{t,u}^-\overline{S}_t \cdot \max\left\{ \operatorname*{ess\,sup}_{\omega,i} \frac{\overline{S}_u^i + \overline{D}_u^i}{\overline{S}_t^i}, \; \operatorname*{ess\,sup}_{\omega}(1 + \overline{R}_{t,u}) \right\}$$

$$\geqslant y_{t,u}^-(\overline{S}_u + \overline{D}_u),$$

where in the second inequality we used (2.8) and in the third inequality we used the margin requirement. This implies $(y_{t,u}, 0) \in \mathcal{Z}_u$ and proves the validity of condition (c).

Let us prove condition (d) for the constant A defined by

$$A = \widetilde{A}\operatorname*{ess\,sup}_{\omega,t,u}(1 + \overline{R}_{t,u}), \quad \text{where } \widetilde{A} = \left(\frac{\overline{S} + \overline{D}}{\underline{S}}\right)^2 \cdot \frac{1 + M}{1 - M}.$$

If $|\widetilde{y}_t(\omega)| \leqslant \widetilde{A}|x_t(\omega) - x_t'(\omega)|$ for some $\omega \in \Omega$, define $y_t'(\omega) = 0$, otherwise let $y_t'(\omega) = \lambda(\omega)y_t(\omega)$, where $0 < \lambda(\omega) \leqslant 1$ is such that $|\widetilde{y}_t(\omega) - \widetilde{y}_t'(\omega)| = \widetilde{A}|x_t(\omega) - x_t'(\omega)|$. Observe that in the latter case we have

$$(1 - \lambda(\omega))|\widetilde{y}_t^+(\omega)| \geqslant |x_t(\omega) - x_t'(\omega)| \cdot \widetilde{A} \frac{\underline{S}}{(1 + M)\overline{S}} \tag{2.9}$$

as is possible to obtain from the margin requirement: indeed, $|\widetilde{y}_{t,u}^-| \leqslant |\widetilde{y}_t^+| \cdot M\overline{S}/\underline{S}$, so $\widetilde{A}|x_t - x_t'| = |\widetilde{y}_t - \widetilde{y}_t'| = (1 - \lambda)|\widetilde{y}_t| \leqslant (1 - \lambda)|\widetilde{y}_t^+|(1 + M\overline{S}/\underline{S})$ a.s. on the set $\{\omega : |\widetilde{y}_t(\omega)| > \widetilde{A}|x_t(\omega) - x_t'(\omega)|\}$, which implies (2.9).

Then on the set $\{\omega : |\widetilde{y}_t(\omega)| \leqslant A|x_t(\omega) - x_t'(\omega)|\}$ we have

$$(x_t' - \widetilde{y}_t)^+\underline{S}_t - (x_t' - \widetilde{y}_t)^-\overline{S}_t + d_t(x_t') = x_t'^+\underline{S}_t - x_t'^-\overline{S}_t + d_t(x_t') \geqslant 0.$$

On its complement we find

$$(x'_t - \widetilde{y}'_t)^+ \underline{S}_t - (x'_t - \widetilde{y}'_t)^- \overline{S}_t + d_t(x'_t)$$

$$\geqslant (x_t - \widetilde{y}_t)^+ \underline{S}_t - (x_t - \widetilde{y}_t)^- \overline{S}_t + d_t(x_t) + (y_t - \widetilde{y}'_t)^+ \underline{S}_t - (y_t - \widetilde{y}'_t)^- \overline{S}_t$$

$$+ (x_t - x'_t)^+ \underline{S}_t - (x_t - x'_t)^- \overline{S}_t + d_t(x'_t - x_t)$$

$$\geqslant (1 - \lambda(\omega))(\widetilde{y}^+_t \underline{S}_t - \widetilde{y}^-_t \overline{S}_t) - |x_t - x'_t|(\overline{S} + \overline{D}) \geqslant 0.$$

The first inequality follows from that $x'_t - \widetilde{y}'_t = x_t - \widetilde{y}_t + \widetilde{y}_t - \widetilde{y}'_t + x'_t - x_t$ and the functions $x \mapsto x^+ \underline{S}_t + x^- \overline{S}_t$ and $x \mapsto d_t(x)$ are superlinear. The second inequality is valid because $(x_t - \widetilde{y}_t)^+ \underline{S}_t - (x_t - \widetilde{y}_t)^- \overline{S}_t + d_t(x_t) \geqslant 0$ for $(x_t, y_t) \in \mathcal{Z}_t$, $y'_t(\omega) = \lambda(\omega)y_t(\omega)$, and $(x_t - x'_t)^+ \underline{S}_t - (x_t - x'_t)^- \overline{S}_t \geqslant -|x_t - x'_t|\overline{S}$. In the last inequality we use that $\widetilde{y}^+_t \underline{S}_t - \widetilde{y}^-_t \overline{S}_t \geqslant (1 - M)\widetilde{y}^+_t \underline{S}_t$ according to the margin requirement, and apply (2.9).

Consequently, using that $y'_{t,u} \in Y_{t,u}$ a.s. as follows from the construction of y'_t, we get $(x'_t, y'_t) \in \mathcal{Z}_t$ and $|\widetilde{y}'_t - \widetilde{y}_t| \leqslant \widetilde{A}|x'_t - x_t|$, so $|y'_t - y_t| \leqslant A|x'_t - x_t|$, which proves the validity of (d), and completes the proof of the theorem. $\square$

2.4 Examples when supporting prices do not exist

We provide two examples, when supporting prices do not exist for certain hedgeable contracts. In the first example we consider a market with the margin requirement, where M is not small enough, and in the second example we consider a contract with positive coordinates. These examples show that the assumptions of Theorem 2.3 cannot be relaxed.

In the both examples we let G be the linear graph with the nodes $t = 0, 1$ (i.e. the graph $0 \to 1$) and the probability space $(\Omega, \mathscr{F}, P)$ consist of $\Omega = [0, 1]$, the Borel σ-algebra $\mathscr{F}$ on Ω, and the Lebesgue measure P. We assume $\mathscr{F}_0$ is the trivial σ-algebra (i.e. it consists of all sets of probability 0 and 1) and put $\mathscr{F}_1 = \mathscr{F}$. In the model, two assets are traded in the market

with the prices $S_t = (S_t^1, S_t^2)$, $t = 0, 1$, given by

$$S_0^1 = 1/2, \quad S_1^1(\omega) = \begin{cases} 3, & 0 \leqslant \omega < 1/2, \\ 1, & 1/2 \leqslant \omega \leqslant 1, \end{cases}$$

$$S_0^2 = 1/2, \quad S_1^2(\omega) = 1 + \omega.$$

The cones $\mathcal{Z}_t$ are defined by

$$\mathcal{Z}_0 = \{(x, y) : (x - y)S_0 \geqslant 0, \ My^+ S_0 \geqslant y^- S_0\},$$
$$\mathcal{Z}_1 = \{(x, y) : xS_1 \geqslant 0, \ y = 0\},$$

where $M > 0$. The utility functions are

$$u_0(x, y) = (x - y)S_0, \quad u_1(x, y) = \mathrm{E}[(x - y)S_1].$$

Example 1. Suppose $M > 1/2$. Consider the contract $\gamma = (c_0, c_1)$ with $c_0 = (-1, 0)$, $c_1 = (0, 0)$. Let us show that $U(\gamma) = 5/2$. Indeed, for any trading strategy $\zeta = (x_t, y_t)_{t=0,1}$ hedging γ we have

$$x_0^1 - y_0^1 + x_0^2 - y_0^2 \geqslant 0, \qquad y_0^1 + 2y_0^2 \geqslant 0,$$

where the first inequality is the self-financing condition of the cone $\mathcal{Z}_0$ and the second inequality follows from that if $(y_0, 0) \equiv (x_1, 0) \in \mathcal{Z}_1$ for a constant vector y_0 then $y_0^1 S_1^1 + y_0^2 S_1^2 \geqslant 0$, which is possible only if $y_0^1 + 2y_0^2 \geqslant 0$ since $S_1^1(\omega) = 1$, $S_1^2(\omega) = 2$ for $\omega = 1$.

If ζ hedges γ, we have $x_0^1 = 1$, $x_0^2 = 0$, so

$$y_0^1 + y_0^2 \leqslant 1, \qquad y_0^1 + 2y_0^2 \geqslant 0.$$

Multiplying the first inequality by 2 and subtracting the second one multiplied by $1/2$ we obtain $3y_0^1/2 + y_0^2 \leqslant 2$. On the other hand, for any constant vectors x, y we have

$$u_0(x, y) = (x^1 + x^2 - y^1 - y^2)/2$$
$$u_1(y, 0) = y^1 \mathrm{E}S_1^1 + y^2 \mathrm{E}S_1^2 = 2y^1 + 3y^2/2,$$

so $u(\zeta) = u_0(x_0, y_0) + u_1(y_0, 0) = 1/2 + 3y_0^1/2 + y_0^2 \leqslant 5/2$. Thus $U(\gamma) \leqslant 5/2$.

The value $5/2$ is attained at the strategy with $x_0 = (1, 0)$, $y_0 = x_1 = (2, -1)$, $y_1 = (0, 0)$, which hedges γ if $M \geqslant 1/2$ as one can verify.

Suppose there exist supporting prices p_0, p_1 for the contract γ. Then for any $(x_t, y_t) \in \mathcal{Z}_t$, $t = 0, 1$, we have

$$5/2 = U(\gamma) \geqslant u_0(x_0, y_0) + u_1(x_1, y_1) - (c_0 + x_0)p_0 + \mathrm{E}[(y_0 - x_1)p_1]. \quad (2.10)$$

Take the following (x_t, y_t):

$$x_0 = (1, 0), \quad y_0 = (2 + \varepsilon, -1 - \varepsilon),$$
$$x_1 = (2 + \varepsilon + f_\varepsilon, -1 - \varepsilon), \quad y_1 = (0, 0),$$

where $\varepsilon \in (0, \ 2M - 1]$ is an arbitrary number and the function $f_\varepsilon = f_\varepsilon(\omega)$ is given by

$$f_\varepsilon(\omega) = \varepsilon \cdot \mathbf{I}\{\omega \geqslant 1/(1 + \varepsilon)\}$$

It is straightforward to check that $(x_0, y_0) \in \mathcal{Z}_0$ and $(x_1, y_1) \in \mathcal{Z}_1$.

Observe that $u_0(x_0, y_0) = 0$, $u_1(x_1, y_1) = \mathrm{E}[(2 + \varepsilon + f_\varepsilon)S_1^1 - (1 + \varepsilon)S_1^2] \geqslant (2 + \varepsilon)\mathrm{E}S_1^1 - (1 + \varepsilon)\mathrm{E}S_2^1 = (2 + \varepsilon) \cdot 2 - (1 + \varepsilon) \cdot (3/2) = 5/2 + \varepsilon/2$, and $\mathrm{E}(y_0 - x_1)p_1 = -\mathrm{E}f_\varepsilon p_1^1$, so from (2.10) we have

$$5/2 \geqslant 5/2 + \varepsilon/2 - \mathrm{E}f_\varepsilon p_1^1.$$

However, since $p_1 \in L_1^\infty$, there is a constant p such that $|p_1^1| \leqslant p$ a.s., so $-\mathrm{E}f_\varepsilon p_1 \geqslant -p\mathrm{E}f_\varepsilon = -p\varepsilon^2/(1 + \varepsilon)$, i.e.

$$5/2 \geqslant 5/2 + \varepsilon/2 - p \cdot \varepsilon^2/(1 + \varepsilon),$$

which is impossible for $\varepsilon > 0$ small enough. The contradiction means that supporting prices do not exist.

Example 2. Let now $M \geqslant 0$ be arbitrary (in particular, when $M = 0$, the markets forbids *short sales*). Consider the contracts $\gamma^\varepsilon = (c_0, c_1^\varepsilon)$ with $c_0 = (-2, 0)$, $c_1^\varepsilon = (0, 1 + f_\varepsilon)$, where $\varepsilon \in [0, 1/2]$ is an arbitrary number and f_ε is the function from Example 1.

Let us show that $U(\gamma^\varepsilon) = 5/2 - \varepsilon - \mathrm{E}f_\varepsilon S_1^2$. Indeed, for any trading

strategy $\zeta = (x_t, y_t)_{t=0,1}$ hedging γ^ε we have

$$y_0^1 + y_0^2 \leqslant 2, \qquad y_0^1 + 2(y_0^2 - 1 - \varepsilon) \geqslant 0,$$

where the first inequality follows from the self-financing condition of the cone $\mathcal{Z}_0$, and the validity of the second inequality follows from that $S_1^1(\omega) = 1$, $S_1^2(\omega) = 2$ for $\omega = 1$.

Multiplying the first inequality by 2 and subtracting the second one multiplied by $1/2$ we obtain $3y_0^1/2 + y_0^2 \leqslant 3 - \varepsilon$. Consequently,

$$
\begin{aligned}
u(\zeta) &= u_0(x_0, y_0) + u_1(x_1, y_1) = u_0(x_0, y_0) + u_1(y_0 + c_1^\varepsilon, 0) \\
&= 1 - (y_0^1 + y_0^2)/2 + \mathrm{E}y_0^1 S_1^1 + \mathrm{E}(y_0^2 - 1 - f_\varepsilon)S_1^2 \\
&= 3y_0^1/2 + y_0^2 - 1/2 - \mathrm{E}f_\varepsilon S_1^2 \\
&\leqslant 5/2 - \varepsilon - \mathrm{E}f_\varepsilon S_1^2,
\end{aligned}
$$

so $U(\gamma) \leqslant 5/2 - \varepsilon - O(\varepsilon^2)$, where $O(\varepsilon^2) = \mathrm{E}f_\varepsilon S_1^2 = (2\varepsilon^2 + 3\varepsilon^3/2)/(1+\varepsilon)^2$. Observe that the value $5/2 - \varepsilon - \mathrm{E}f_\varepsilon S_1^2$ is attained at the trading strategy with $x_0 = (2, 0)$, $y_0 = (2 - 2\varepsilon, 2\varepsilon)$, $x_1 = (2 - 2\varepsilon, 2\varepsilon - 1 - f_\varepsilon)$, $y_1 = (0, 0)$.

If supporting prices existed, we would have

$$U(\gamma^\varepsilon) \geqslant u_0(x_0, y_0) + u_1(x_1, y_1) + (c_0 - x_0)p_0 + \mathrm{E}(y_0 - x_1 + c_1^\varepsilon)p_1$$

for any $(x_t, y_t) \in \mathcal{Z}_t$, $t = 0, 1$.

Take $x_0 = y_0 = (2, 0)$, $x_1 = (2, -1)$, $y_1 = (0, 0)$, which satisfy $(x_t, y_t) \in \mathcal{Z}_t$, and give

$$5/2 - \varepsilon - O(\varepsilon^2) \geqslant 5/2 - \mathrm{E}f_\varepsilon p_1.$$

As it was shown in Example 1, we have $\mathrm{E}f_\varepsilon p_1 = O(\varepsilon^2)$, so the above inequality is impossible for $\varepsilon > 0$ small enough, implying that supporting prices do not exist in the model.

2.5 Appendix: geometric duality

Let L be a normed linear space and L^* its dual. Let f be a real-valued concave function defined on a convex set $X \subset L$.

The *steepness* of f from a point $x \in X$ to a point $x' \in X$, $x' \neq x$, is defined by

$$s_x(x') = \frac{f(x') - f(x)}{\|x' - x\|}.$$

An element $p \in L^*$ is called a *support* of the function f at a point $x \in X$ if

$$f(x') - f(x) \leqslant p \cdot (x' - x) \text{ for any } x' \in X.$$

The set of all supports of f at x is denoted by Σ_x (which may be empty).

For a given $x \in X$, consider the two problems:

(1) the *primal problem*:

$$\text{find} \sup_{x' \in X \setminus \{x\}} s_x(x');$$

(2) the *dual problem*:

$$\text{find} \inf_{p \in \Sigma_x} \|p\|.$$

The following theorem connects the primal and the dual problems and plays an important role in many questions of convex analysis.

Proposition 2.1 (The Duality Theorem, [26])**.** *The set Σ_x is not empty if and only if $s_x(x')$ is bounded from above. In this case, if x is not a point where f attains its global maximum, the values of the two problems are equal.*

Chapter 3

Detection of changepoints in asset prices

In this chapter applications of sequential statistical methods to detecting trend changes in asset prices are considered. We propose a model where one holds an asset and wants to sell it before a fixed time T. The price of the asset initially increases but may start decreasing at an unknown moment of time. The problem consists in detecting this change in the trend in order to sell the asset for a high price.

The solution will be based on methods of *changepoint* (or *disorder*) detection theory. The optimal moment of selling the asset will be represented as the first moment of time when some statistic (the *Shiryaev-Roberts statistic* or, equivalently, the *posterior probability process*), constructed from the price sequence, exceeds a certain time-dependent threshold.

The mathematical theory of changepoint detection started developing in 1920-1950s in connection with problems of production quality control and radiolocation (see a survey in [65]). Later it was also applied in the financial context (see e. g. [62]). Changepoint detection methods can be divided into the two groups: Bayesian and minimax methods. In the Bayesian setting, a changepoint (a moment of disorder) is an unobservable random variable with a known prior distribution, while in the minimax setting it is an unknown parameter. In this chapter we follow the Bayesian approach.

The problem we consider was previously studied in [2, 14, 56] in the case of continuous time when the evolution of an asset price is described by a geometric Brownian motion, which changes its value of the drift coefficient at an unknown moment of time (from a "favourable" value to an "unfavourable" one). The result we obtain extends the results available in the literature to the case of discrete time. Moreover, unlike the majority of previous results, which assume the changepoint is exponentially distributed, we do not impose

any assumption on its prior distribution.

The first section of the chapter describes the model of asset prices with changepoints. In Section 3.2 we introduce the basic statistics and formulate the main theorem, which is proved in Section 3.3. Section 3.4 contains results of numerical simulations.

3.1 The problem of selling an asset with a changepoint

We propose a model describing the price S of an asset at moments of time $t = 0, 1, \ldots, T$ driven by a geometric Gaussian random walk with logarithmic mean and variance (μ_1, σ_1) up to an unknown moment of time θ and (μ_2, σ_2) after θ. The moment θ will be interpreted as the point when the trend of the asset changes from a favourable value to an unfavourable one, and is called a *changepoint* (or a *moment of disorder*[1]).

Let $\xi = (\xi_t)_{t=1}^T$ be a sequence of i.i.d. (independent and identically distributed) standard normal random variables defined on a probability space $(\Omega, \mathscr{F}, \mathrm{P})$ and let the sequence $S = (S_t)_{t=0}^T$ be defined by its logarithmic increments as follows:

$$S_0 = 1, \qquad \log \frac{S_t}{S_{t-1}} = \begin{cases} \mu_1 + \sigma_1 \xi_t, & t < \theta, \\ \mu_2 + \sigma_2 \xi_t, & t \geqslant \theta, \end{cases} \quad \text{for } t = 1, 2, \ldots, T,$$

where $\mu_1, \mu_2 \in \mathbb{R}$, $\sigma_1, \sigma_2 > 0$ are known numbers, $\theta \in \{1, 2, \ldots, T+1\}$ is an unknown parameter. The equality $S_0 = 1$ means that the prices are measured relative to the price at time $t = 0$ and does not reduce the generality of the model.

We follow the Bayesian approach and assume that θ is a random variable defined on $(\Omega, \mathscr{F}, \mathrm{P})$, independent of the sequence ξ_t and taking values in the set $\{1, 2, \ldots, T+1\}$ with known prior probabilities $p_t = \mathrm{P}(\theta = t)$. The value p_1 is the probability that the changepoint appears from the beginning

---

[1]The term "disorder" comes from applications of the theory in production quality control problems, where θ can represent the moment of equipment breakage (a disorder) and increase of defective products.

of the sequence S_t , and p_{T+1} is the probability that the changepoint does not appear within the time horizon $[0, T]$. The prior distribution function of θ is denoted by $G(t) = \sum_{u \leqslant t} p_u$.

Let $U_\alpha \colon \mathbb{R}_+ \to \mathbb{R}$, $\alpha \leqslant 1$, be the family of functions defined as follows[2]:

$$U_\alpha(x) = \alpha x^\alpha \text{ for } \alpha \neq 0, \qquad U_0(x) = \log x.$$

The problem considered in this chapter consists in finding the moment of time τ which maximises the utility from selling the asset provided one holds it at time $t = 0$ and needs to sell it by $t = T$.

Let $\mathbb{F} = (\mathscr{F}_t)_{t=0}^T$, $\mathscr{F}_t = \sigma(S_u;\ u \leqslant t)$, be the filtration generated by the price sequence S. By definition, a moment τ when one sells the asset should be a *stopping time* of the filtration $\mathbb{F}$, i.e. τ should be a random variable taking values in the set $\{0, 1, \dots, T\}$ such that $\{\omega : \tau(\omega) \leqslant t\} \in \mathscr{F}_t$ for any $t = 0, \dots, T$. The class of all stopping times $\tau \leqslant T$ of $\mathbb{F}$ is denoted by $\mathfrak{M}$. The notion of a stopping time reflects the concept that a decision to sell the asset at time t should be based only on the information obtained from the "historical" prices $S_0, S_1, \dots, S_t$ and should not rely on the "future" prices S_{t+1}, S_{t+2}, which are unknown at time t.

Mathematically, the problem of optimal selling of the asset with respect to the utility function U_α is formulated as the *optimal stopping problem*

$$V_\alpha = \sup_{\tau \in \mathfrak{M}} \mathrm{E} U_\alpha(S_\tau). \tag{3.1}$$

Its solution consists in finding the value V_α and the *optimal stopping time* τ_α^*, at which the supremum is attained (if such a stopping time exists).

It will be assumed that

$$\mu_1 > -\frac{\alpha \sigma_1^2}{2}, \qquad \mu_2 < -\frac{\alpha \sigma_2^2}{2}. \tag{3.2}$$

In this case the sequence $(u_t^\alpha)_{t=0}^T$, $u_t^\alpha = U_\alpha(S_t)$, is a submartingale[3] when

---

[2]These functions are from capital growth theory and are known to maximize long run asymptotic growth of wealth; see [5, 37, 40]. $U_0(x)$ corresponds to the full Kelly strategy and $U_\alpha(x)$, $\alpha < 0$, correspond to fractional Kelly strategies blending cash with the optimal full Kelly portfolio.

[3]A random sequence $(\zeta_t)_{t=0}^T$ is called a *submartingale* (resp., a *supermartingale* or a *martingale*) with respect to a filtration $(\mathscr{F}_t)_{t=0}^T$ if $\mathrm{E}(\zeta_t \mid \mathscr{F}_{t-1}) \geqslant \zeta_{t-1}$ (resp., $\mathrm{E}(\zeta_t \mid \mathscr{F}_{t-1}) \leqslant \zeta_{t-1}$ or $\mathrm{E}(\zeta_t \mid \mathscr{F}_{t-1}) = \zeta_{t-1}$) for each $t = 1, \dots, T$.

the logarithmic returns of S are i.i.d. $\mathcal{N}(\mu_1, \sigma_1^2)$ random variables, and a supermartingale when they are i.i.d. $\mathcal{N}(\mu_2, \sigma_2^2)$ random variables. In the former case, the value of u_t increases on average as t increases, so it is profitable to hold the asset, and in the latter case the average value of u_t decreases meaning that one needs to sell the asset as soon as possible. Consequently, the random variable θ represents the moment of time when holding the asset becomes unprofitable. Note that θ is not measurable with respect to $\mathscr{F}_t$ (except the trivial case when θ is a constant random variable), which can be interpreted as the *unobservability* of the changepoint.

It is also reasonable to assume that $p_1 > 0$ — otherwise it is clear that one should never stop at time $t = 0$ (i.e. the optimal stopping time $\tau_\alpha^* \geqslant 1$), and the problem can be reduced to the smaller time horizon $t = 1, 2, \ldots, T$. Nevertheless, the result of the main theorem in the next section will be still valid if $p_1 = 0$.

3.2 The structure of optimal selling times

In order to formulate the main result of the chapter, introduce auxiliary notation. Let $X = (X_t)_{t=1}^T$ denote the logarithmic returns of the prices:

$$X_t = \log \frac{S_t}{S_{t-1}}, \qquad t = 1, \ldots, T.$$

On the space $(\Omega, \mathscr{F}_T)$ introduce the family of probability measures P^u, $u = 1, \ldots, T+1$, generated by the sequence S with the value of the parameter $\theta \equiv u$. Following the standard notation of the changepoint detection theory, let[4] $\mathrm{P}^\infty \equiv \mathrm{P}^{T+1}$, and denote by $\mathrm{P}_t = \mathrm{P} \mid \mathscr{F}_t$, $\mathrm{P}_t^u = \mathrm{P}^u \mid \mathscr{F}_t$ the restrictions of the corresponding measures to the σ-algebra $\mathscr{F}_t$.

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[4]Typically, P^∞ denotes the measure when there is no change in the probability law of the observable sequence on the whole time horizon (i.e. the change "occurs" at time $t = \infty$). Since in the problem considered the time horizon is finite, this measure has the same meaning as P^{T+1}.

Introduce the *Shiryaev–Roberts statistic* $\psi = (\psi_t)_{t=0}^{T}$:

$$\psi_0 = 0, \qquad \psi_t = \sum_{u=1}^{t} \frac{d\mathrm{P}_t^u}{d\mathrm{P}_t^\infty} p_u \quad \text{for} \quad t = 1, \ldots, T.$$

Using that the density $d\mathrm{P}_t^u / d\mathrm{P}_t^\infty$ is given by the formula

$$\frac{d\mathrm{P}_t^u}{d\mathrm{P}_t^\infty} = \left(\frac{\sigma_1}{\sigma_2}\right)^{t-u+1} \cdot \exp\left(\sum_{i=u}^{t}\left[\frac{(X_i - \mu_1)^2}{2\sigma_1^2} - \frac{(X_i - \mu_2)^2}{2\sigma_2^2}\right]\right), \quad u \leqslant t,$$

$$\frac{d\mathrm{P}_t^u}{d\mathrm{P}_t^\infty} = 1, \qquad u > t,$$

it is straightforward to check that ψ_t satisfies the following recurrent formula:

$$\psi_t = (p_t + \psi_{t-1}) \cdot \frac{\sigma^1}{\sigma^2} \exp\left(\frac{(X_t - \mu_1)^2}{2\sigma_1^2} - \frac{(X_t - \mu_2)^2}{2\sigma_2^2}\right), \quad t = 1, \ldots, T. \quad (3.3)$$

Define recurrently the family of functions $V_\alpha(t, x)$ for $\alpha \leqslant 1$, $t = T, T - 1, \ldots, 0$, $x \geqslant 0$ as follows. For $\alpha = 0$ let

$$V_0(T, x) \equiv 0,$$
$$V_0(t, x) = \max\{0, \ \mu_2(x + p_{t+1}) + \mu_1(1 - G(t + 1)) + f_0(t, x)\},$$

where the function $f_0(t, x)$ is given by

$$f_0(t, x) = \int_{\mathbb{R}} V_0\left(t + 1, \ (p_{t+1} + x) \cdot \frac{\sigma_1}{\sigma_2} \exp\left(\frac{(z - \mu_1)^2}{2\sigma_1^2} - \frac{(z - \mu_2)^2}{2\sigma_2^2}\right)\right)$$
$$\times \frac{1}{\sigma_1\sqrt{2\pi}} \exp\left(-\frac{(z - \mu_1)^2}{2\sigma_1^2}\right) dz.$$

For $\alpha \neq 0$ define

$$V_\alpha(T, x) \equiv 0,$$
$$V_\alpha(t, x) = \max\{0, \ \alpha\beta^t\left[(\gamma - 1)(x + p_{t+1}) + (\beta - 1)(1 - G(t + 1))\right]$$
$$+ f_\alpha(t, x)\},$$

where

$$\beta = \exp\left(\alpha\mu_1 + \frac{\alpha^2\sigma_1^2}{2}\right), \quad \gamma = \exp\left(\alpha\mu_2 + \frac{\alpha^2\sigma_2^2}{2}\right),$$

and

$$f_\alpha(t,x) = \int_{\mathbb{R}} V_\alpha\left(t+1,\ (p_{t+1}+x)\cdot\frac{\sigma_1}{\sigma_2}\exp\left(\frac{(z-\mu_1)^2}{2\sigma_1^2}-\frac{(z-\mu_2)^2}{2\sigma_2^2}\right)\right)$$
$$\times\frac{1}{\sigma_1\sqrt{2\pi}}\exp\left(-\frac{(z-\mu_1-\alpha\sigma_1^2)^2}{2\sigma_1^2}\right)dz.$$

Below it will be shown that $V_\alpha(t,x)+\alpha$ represents the maximal expected gain one can obtain from selling the asset if the observation is started as time t with the value of the Shiryaev–Roberts statistic equal to x. In particular, the value V_α in problem (3.1) is equal to $V_\alpha(0,0)+\alpha$.

The main result of the chapter consists in the following theorem about the structure of optimal stopping times.

Theorem 3.1. *The following stopping time is optimal in problem (3.1):*

$$\tau_\alpha^* = \inf\{0 \leqslant t \leqslant T : \psi_t \geqslant b_\alpha^*(t)\}, \tag{3.4}$$

where the stopping boundary $b_\alpha^*(t)$, $t = 0,\ldots,T$, is given by

$$b_\alpha^*(t) = \inf\{x \geqslant 0 : V_\alpha(t,x) = 0\}.$$

The value V_α of problem (3.1) is equal to $V_\alpha(0,0)+\alpha$.

The theorem states that the optimal stopping time is the first moment of time when the Shiryaev–Roberts statistic exceeds the time-dependent threshold $b_\alpha^*(t)$. In order to find the function $b_\alpha^*(t)$ numerically, one first computes the functions V_α and then finds their minimal non-negative roots.

Note that it is also possible to express the optimal stopping time through the *posterior probability process* $\pi = (\pi_t)_{t=0}^T$ defined as the conditional probability

$$\pi_t = \mathrm{P}(\theta \leqslant t \mid \mathscr{F}_t).$$

Indeed, by the generalised Bayes formula (see e. g. [39, § 7.9]) we have

$$\pi_t = \frac{\displaystyle\sum_{u=1}^{t}\frac{d\mathrm{P}_t^u}{d\mathrm{P}_t}p_u}{\displaystyle\sum_{u=1}^{T+1}\frac{d\mathrm{P}_t^u}{d\mathrm{P}_t}p_u},\qquad 1-\pi_t = \frac{\displaystyle\sum_{u=t+1}^{T+1}\frac{d\mathrm{P}_t^u}{d\mathrm{P}_t}p_u}{\displaystyle\sum_{u=1}^{T+1}\frac{d\mathrm{P}_t^u}{d\mathrm{P}_t}p_u} = \frac{(1-G(t))\dfrac{d\mathrm{P}_t^\infty}{d\mathrm{P}_t}}{\displaystyle\sum_{u=1}^{T+1}\frac{d\mathrm{P}_t^u}{d\mathrm{P}_t}p_u}.$$

This implies

$$\psi_t = \frac{\pi_t}{1 - \pi_t}(1 - G(t)), \qquad \pi_t = \frac{\psi_t}{\psi_t + 1 - G(t)}.$$

Consequently, the optimal stopping time in problem (3.1) can be equivalently represented in the form

$$\tau_\alpha^* = \inf\{0 \leqslant t \leqslant T : \pi_t \geqslant \widetilde{b}_\alpha^*(t)\}, \text{ where } \widetilde{b}_\alpha^*(t) = \frac{b_\alpha^*(t)}{b_\alpha^*(t) + 1 - G(t)}.$$

This representation provides a clear interpretation of the optimal stopping time: one should sell the asset as soon as she becomes sufficiently confident that the disorder has already happened, which quantitatively means that the posterior probability π_t exceeds the threshold $\widetilde{b}_\alpha^*(t)$.

3.3 The proof of the main theorem

The proof of the theorem will be given in two steps. First, problem (3.1) will be reduced to an optimal stopping problem for the Shiryaev–Roberts statistic ψ with respect to a new probability measure on $(\Omega, \mathscr{F}_T)$. Then we show that ψ_t is a Markov sequence with respect to this measure and apply methods of Markovian optimal stopping theory to the problem.

Step 1. On the measure space $(\Omega, \mathscr{F}_T)$ introduce the family of probability measures Q^α, $\alpha \leqslant 1$, such that the logarithmic returns X_t, $t = 1, \ldots, T$, are i.i.d $\mathcal{N}(\mu_1 + \alpha\sigma_1^2, \sigma_1^2)$ random variables under Q^α. In particular, $Q^0 \equiv P^\infty$. For $t = 1, \ldots, T$, the explicit formula for the density dP_t/dQ_t^α, where $Q_t^\alpha = Q^\alpha \mid \mathscr{F}_t$, is given by the formula

$$\frac{dP_t}{dQ_t^\alpha} = \frac{dP_t}{dQ_t^0} \cdot \frac{dQ_t^0}{dQ_t^\alpha} = \left(\sum_{u=1}^{T+1} \frac{dP_t^u}{dP_t^\infty} p_u\right) \cdot \frac{dQ_t^0}{dQ_t^\alpha}$$

$$= \left(\sum_{u=1}^{t} \frac{dP_t^u}{dP_t^\infty} p_u + \sum_{u=t+1}^{T+1} p_u\right) \cdot \exp\left(\alpha\mu_1 t + \frac{\alpha^2\sigma_1^2}{2}t - \alpha\sum_{u=1}^{t} X_i\right)$$

$$= (\psi_t + 1 - G(t))\beta^t \cdot \exp\left(-\alpha\sum_{u=1}^{t} X_i\right).$$

We show that for any stopping time $\tau \in \mathfrak{M}$ it holds that

$$\mathrm{E}U_0(S_\tau) = \mathrm{E}^{\mathrm{Q}^0}\left[\sum_{t=1}^{\tau}\left[\mu_2\psi_t + \mu_1(1-G(t))\right]\right], \tag{3.5}$$

$$\mathrm{E}U_\alpha(S_\tau) = \alpha\mathrm{E}^{\mathrm{Q}^\alpha}\left[\sum_{t=1}^{\tau}\beta^{t-1}\left[(\beta-\beta/\gamma)\psi_t + (\beta-1)(1-G(t))\right]\right]$$
$$+ \alpha \quad \text{for } \alpha \neq 0, \tag{3.6}$$

where $\mathrm{E}^{\mathrm{Q}^\alpha}$ denotes the expectation with respect to Q^α. Here and below, we define $\sum_{t=1}^{\tau}[\ldots] = 0$ if $\tau = 0$.

In order to prove (3.5), observe that

$$\mathrm{E}U_0(S_\tau) = \mathrm{E}\sum_{t=1}^{\tau}X_t = \mathrm{E}^{\mathrm{Q}^0}\left[\frac{d\mathrm{P}_\tau}{d\mathrm{Q}_\tau^0}\sum_{t=1}^{\tau}X_t\right]$$
$$= \mathrm{E}^{\mathrm{Q}^0}\left[(\psi_\tau + 1 - G(\tau))\sum_{t=1}^{\tau}X_t\right].$$

Using the "discrete version" of the integration by parts formula,

$$a_t b_t = \sum_{s=1}^{t}a_s\Delta b_s + \sum_{s=1}^{t}b_{s-1}\Delta a_s + a_0 b_0, \tag{3.7}$$

valid for any sequences a_t and b_t with the notation $\Delta a_s = a_s - a_{s-1}$, $\Delta b_s = b_s - b_{s-1}$, we obtain

$$\mathrm{E}U_0(S_\tau) = \mathrm{E}^{\mathrm{Q}^0}\left[\sum_{t=1}^{\tau}(\psi_t + 1 - G(t))X_t + \sum_{t=1}^{\tau}\sum_{s=1}^{t-1}X_s(\psi_t - \psi_{t-1} - p_t)\right]. \tag{3.8}$$

The sequence $\psi_t + 1 - G(t)$ is a martingale with respect to the measure Q^0 since it is the sequence of the densities $d\mathrm{P}_t/d\mathrm{Q}_t^0$. This implies that the expectation of the second sum in the above formula is zero. Indeed, we have

$$\mathrm{E}^{\mathrm{Q}^0}\left[\sum_{t=1}^{\tau}\sum_{s=1}^{t-1}X_s(\psi_t - \psi_{t-1} - p_t)\right]$$
$$= \mathrm{E}^{\mathrm{Q}^0}\left[\sum_{t=1}^{T+1}\sum_{s=1}^{t-1}\mathrm{E}^{\mathrm{Q}^0}\left[X_s(\psi_t - \psi_{t-1} - p_t)\mathbf{I}(t \leqslant \tau)\mid \mathscr{F}_{t-1}\right]\right] = 0,$$

where we use that X_s and $\mathbf{I}(t \leqslant \tau)$ are $\mathscr{F}_{t-1}$-measurable random variables,

so they can be taken out of the conditional expectation, and

$$\mathrm{E}^{Q^0}(\psi_t - \psi_{t-1} - p_t \mid \mathscr{F}_{t-1}) = 0, \qquad t = 1, \ldots, T,$$

as follows from that $\psi_t + 1 - G(t)$ is a martingale.

For the first sum in (3.8) we have

$$\mathrm{E}^{Q^0}\left[\sum_{t=1}^{\tau}(\psi_t + 1 - G(t))X_t\right]$$

$$= \mathrm{E}^{Q^0}\left[\sum_{t=1}^{T+1}\mathrm{E}^{Q^0}\left((\psi_t + 1 - G(t))X_t\mathbf{I}(t \leqslant \tau) \mid \mathscr{F}_{t-1}\right)\right]$$

$$= \mathrm{E}^{Q^0}\left[\sum_{t=1}^{\tau}\left[\mathrm{E}^{Q^0}(p_t + \psi_{t-1})\mu_2 + \mu_1(1 - G(t))\right]\right]$$

$$= \mathrm{E}^{Q^0}\left[\sum_{t=1}^{\tau}[\mu_2\psi_t + \mu_1(1 - G(t))]\right].$$

In the second equality we use that $\mathbf{I}(t \leqslant \tau)$ is an $\mathscr{F}_{t-1}$-measurable random variable and it can be taken out of the conditional expectation, X_t is independent of $\mathscr{F}_{t-1}$, so $\mathrm{E}^{Q^0}[(1 - G(t))X_t \mid \mathscr{F}_{t-1}] = (1 - G(t))\mathrm{E}^{Q^0}X_t = (1 - G(t))\mu_1$, and, as follows from (3.3),

$$\mathrm{E}^{Q^0}(\psi_t X_t \mid \mathscr{F}_{t-1}) = (p_t + \psi_{t-1})\frac{\sigma_1}{\sigma_2}\mathrm{E}^{Q^0}\left[X_t \exp\left(\frac{(X_t - \mu_1)^2}{2\sigma_1^2} - \frac{(X_t - \mu_2)^2}{2\sigma_2^2}\right)\right]$$

$$= (p_t + \psi_{t-1})\mu_2.$$

In the third equality we use the representation

$$\mathrm{E}^{Q^0}\left[\sum_{t=1}^{\tau}(p_t + \psi_{t-1})\right] = \mathrm{E}^{Q^0}\left[\sum_{t=1}^{T+1}\mathrm{E}^{Q^0}(\psi_t \mid \mathscr{F}_{t-1})\mathbf{I}(t \leqslant \tau)\right]$$

$$= \mathrm{E}^{Q^0}\left[\sum_{t=1}^{T+1}\mathrm{E}^{Q^0}(\psi_t\mathbf{I}(t \leqslant \tau) \mid \mathscr{F}_{t-1})\right] = \mathrm{E}^{Q^0}\left[\sum_{t=1}^{\tau}\psi_t\right].$$

This proves formula (3.5).

Let us prove (3.6). We have

$$\mathrm{E}U_\alpha(S_\tau) = \alpha\mathrm{E}\exp\Big(\alpha\sum_{t=1}^{\tau}X_t\Big) = \alpha\mathrm{E}^{Q^\alpha}\Big[\frac{d\mathrm{P}_\tau}{d\mathrm{Q}_\tau^\alpha}\exp\Big(\alpha\sum_{t=1}^{\tau}X_t\Big)\Big]$$

$$= \alpha\mathrm{E}^{Q^\alpha}\big[(\psi_\tau + 1 - G(\tau))\beta^\tau\big]$$

$$= \alpha\mathrm{E}^{Q^\alpha}\Big[\sum_{t=1}^{\tau}\big[(\psi_t + 1 - G(t))(\beta^t - \beta^{t-1}) + \beta^{t-1}(\psi_t - \psi_{t-1} - p_t)\big]\Big] + \alpha,$$

where in the third equality we use the formula for the conditional density $d\mathrm{P}_t/d\mathrm{Q}_t^\alpha$, and in the last equality we use formula (3.7). Using representation (3.3), it is possible to verify the following expression for the conditional expectation $\mathrm{E}^{Q^\alpha}(\psi_t \mid \mathscr{F}_{t-1})$:

$$\mathrm{E}^{Q^\alpha}(\psi_t \mid \mathscr{F}_{t-1}) = \frac{\gamma}{\beta}(\psi_{t-1} + p_t) \text{ for } t = 1,\dots,T. \tag{3.9}$$

To check that the right-hand side (RH) of above expression for $\mathrm{E}U_\alpha(S_\tau)$ coincides with the right-hand side $(\overline{RH})$ of (3.6), we show that their difference is equal to zero:

$$(RH) - (\overline{RH}) = \alpha\mathrm{E}^{Q^\alpha}\Big[\sum_{t=1}^{\tau}\beta^{t-1}(\psi_t\beta/\gamma - \psi_{t-1} - p_t)\Big]$$

$$= \alpha\mathrm{E}^{Q^\alpha}\Big[\sum_{t=1}^{T+1}\beta^{t-1}(\psi_t\beta/\gamma - \psi_{t-1} - p_t)\mathbf{I}(t \leqslant \tau)\Big]$$

$$= \alpha\mathrm{E}^{Q^\alpha}\Big[\sum_{t=1}^{T+1}\mathrm{E}^{Q^\alpha}\big[\beta^{t-1}(\psi_t\beta/\gamma - \psi_{t-1} - p_t)\mathbf{I}(t \leqslant \tau) \mid \mathscr{F}_{t-1}\big]\Big]$$

$$= \alpha\mathrm{E}^{Q^\alpha}\Big[\sum_{t=1}^{T+1}\beta^{t-1}(\mathrm{E}^{Q^\alpha}(\psi_t \mid \mathscr{F}_{t-1})\beta/\gamma - \psi_{t-1} - p_t)\mathbf{I}(t \leqslant \tau)\Big] = 0,$$

where in the fourth equality we use that ψ_{t-1} and $\mathbf{I}(t \leqslant \tau)$ are $\mathscr{F}_{t-1}$-measurable random variables, so their conditional expectations coincides with themselves, and apply (3.9). This proves (3.6).

Step 2. For convenience of further notation, let $F_\alpha(t, \psi)$ denote the terms in the sums in (3.5)–(3.6):

$$F_0(t, x) = \mu_2 x + \mu_1(1 - G(t)),$$
$$F_\alpha(t, x) = \alpha\beta^{t-1}[(\beta - \beta/\gamma)x + (\beta - 1)(1 - G(t))]. \tag{3.10}$$

Representations (3.5)–(3.6) allow to reduce problem (3.1) to the optimal stopping problems for the Shiryaev–Roberts statistic ψ

$$V_\alpha = \sup_{\tau \in \mathfrak{M}} \mathrm{E}^{\mathrm{Q}^\alpha} \left[\sum_{u=1}^{\tau} F_\alpha(u, \psi_u) \right] + \alpha,$$

so that the optimal stopping times in these problems will be also optimal in problem (3.1).

Let $\mathfrak{M}_t$ denote the class of all stopping times τ of the filtration $\mathbb{F}$ such that $\tau \leqslant t$. In particular, $\mathfrak{M}_T = \mathfrak{M}$.

According to the results of [64, Ch. II, § 2.15], the Shiryaev–Roberts statistic is a Markov sequence with respect to the filtration $\mathbb{F}$ under each measure Q^α, $\alpha \leqslant 1$, since ψ_t is a function of ψ_{t-1} and X_t, while X_t form a sequence of independent random variables. Following the general theory of optimal stopping of Markov sequences (see e. g. [47, Ch. I]), introduce the family of the *value functions* $V_\alpha(t, x)$ for $t \in \{0, 1, \ldots, T\}$, $x \geqslant 0$:

$$V_\alpha(t, x) = \sup_{\tau \in \mathfrak{M}_{T-t}} \mathrm{E}^{\mathrm{Q}^\alpha} \left[\sum_{u=1}^{\tau} F_\alpha(t + u, \psi_u(t, x)) \right], \tag{3.11}$$

where $\psi(t, x) = (\psi_u(t, x))_{u=0}^{T-t}$ is a sequence of random variables defined by the recurrent formula

$$\psi_0(t, x) = x,$$

$$\psi_u(t, x) = (p_{t+u} + \psi_{u-1}(t, x)) \cdot \frac{\sigma_1}{\sigma_2} \exp\left(\frac{(X_u - \mu_1)^2}{2\sigma_1^2} - \frac{(X_u - \mu_2)^2}{2\sigma_2^2} \right)$$

with $X_1, X_2, \ldots$ being i.i.d. $\mathcal{N}(\mu_1 + \alpha\sigma_1^2, \sigma_1^2)$ random variables with respect to the measure Q^α. The sums in the definition of $V_\alpha(t, x)$ are equal to zero if $\tau = 0$, which, in particular, means that $V_\alpha(T, x) = 0$ for any $x \geqslant 0$.

The functions $V_\alpha(t, x)$ represent the maximal possible gain in the optimal stopping problem if the observation starts at time t with the value of the Shiryaev–Roberts statistic x. From formulae (3.5)–(3.6), it follows that original problem (3.1) corresponds to $t = 0$, $x = 0$, so the optimal stopping time for $V_\alpha(0, 0)$ will be the optimal stopping time in (3.1), and $V_\alpha = V_\alpha(0, 0) + a$.

The well-known result of the optimal stopping theory for Markov se-

quences (see [47, Theorem 1.8]) states that the value functions $V_\alpha(t, x)$ satisfy the following *Wald–Bellman equations* for $t = 0, \ldots, T-1$:

$$V_\alpha(t, x) = \max\{0, \; \mathrm{E}^{Q^\alpha}[F_\alpha(t+1, \psi_1(t, x)) + V_\alpha(t+1, \psi_1(t, x))]\}. \quad (3.12)$$

Here, 0 is the gain from *instantaneous stopping* in the problems at hand. Using that X_t are i.i.d $\mathcal{N}(\mu_1 + \alpha\sigma_1^2, \sigma_1^2)$ random variables with respect to Q^α and computing the expectations $\mathrm{E}^{Q^\alpha}[\ldots]$ in the above equation, we obtain that the functions $V_\alpha(t, x)$ satisfy the recurrent relations on p. 79.

From [47, Theorem 1.7] it follows that the optimal stopping time in problem (3.1) is the first moment of time when ψ_t enters the *stopping set* D_α:

$$D_\alpha = \{(t, x) : V_\alpha(t, x) = 0\}, \qquad \tau_\alpha^* = \inf\{t \geqslant 0 : (t, \psi_t) \in D_\alpha\}.$$

In order to prove representation (3.4), we show that for fixed t and α, the function $x \mapsto V_\alpha(t, x)$ is continuous and non-increasing, and there exists x such that $V_\alpha(t, x) = 0$. The non-increasing follows from that $\psi_u(t, x_1) \geqslant \psi_u(t, x_2)$ whenever $x_1 \geqslant x_2$, and the coefficients μ_2 and $\alpha\beta^t(1 - 1/\gamma)$ are negative in the formulae for $F_0(t, x)$ and $F_\alpha(t, x)$ respectively as follows from the assumption $\mu_2 < -\alpha\sigma_2^2/2$ (see (3.2)).

In order to prove the continuity of $x \mapsto V_\alpha(t, x)$, we show by induction over $t = T, T-1, \ldots, 0$ that for arbitrary $0 \leqslant x_1 \leqslant x_2$ it holds that

$$V_\alpha(t, x_1) - V_\alpha(t, x_2) \leqslant c_t^\alpha(x_2 - x_1) \quad (3.13)$$

with the constants

$$c_t^\alpha = \begin{cases} |\mu_2|(T-t), & \alpha = 0 \\ \alpha\beta^t(1 - \gamma^{T-t}), & \alpha \neq 0. \end{cases}$$

For $t = T$ the claim is valid, because $V_\alpha(T, x) = 0$ for all $x \geqslant 0$. Suppose it holds for some $t = s$ and consider $t = s - 1$. From the formulae on p. 79 it

follows that

$$V_\alpha(s-1, x_1) - V_\alpha(s-1, x_2)$$

$$\leqslant \begin{cases} |\mu_2|(x_2 - x_1) + f_0(s-1, x_1) - f_0(s-1, x_2), & \alpha = 0, \\ \alpha\beta^{s-1}(1-\gamma)(x_2 - x_1) + f_\alpha(s-1, x_1) - f_\alpha(s-1, x_2), & \alpha \neq 0. \end{cases}$$

Further, using the induction assumption for $t = s$, we find

$$f_\alpha(s-1, x_1) - f_\alpha(s-1, x_2) \leqslant \int_{\mathbb{R}} \frac{c_s^\alpha(x_2 - x_1)}{\sigma_2\sqrt{2\pi}} \exp\left(\frac{(z-\mu_1)^2}{2\sigma_1^2} - \frac{(z-\mu_2)^2}{2\sigma_2^2}\right)$$

$$\times \exp\left(\frac{(z-\mu_1 - \alpha\sigma_1^2)^2}{2\sigma_1^2}\right) dz = c_s^\alpha(x_2 - x_1)\frac{\gamma}{\beta}.$$

Combining it with the previous inequality we find for $\alpha = 0$

$$V_\alpha(s-1, x_1) - V_\alpha(s-1, x_2) \leqslant (x_2 - x_1)(|\mu_2| + c_s^\alpha) = c_{s-1}^\alpha(x_2 - x_1)$$

and for $\alpha \neq 0$

$$V_\alpha(s-1, x_1) - V_\alpha(s-1, x_2) \leqslant (x_2 - x_1)(\alpha\beta^{s-1}(1-\gamma) + c_s^\alpha\gamma/\beta)$$

$$= c_{s-1}^\alpha(x_2 - x_1),$$

which proves the claim. Since $0 \leqslant V_\alpha(t, x_1) - V_\alpha(t, x_2)$ because $x \mapsto V_\alpha(t, x)$ is a non-increasing function, we obtain that it is continuous.

Finally by induction over $t = T, T-1, \ldots, 0$ we prove that there exists a root $r_{\alpha,t}$ of the function $x \mapsto V_\alpha(t, x)$. For $t = T$ this is true since $V_\alpha(T, x) = 0$ for all $x \geqslant 0$. Suppose there exists a root for $t = s > 0$. Then for $t = s - 1$ and $x \to +\infty$ we have

$$E^{Q^\alpha} V_\alpha(s, \psi_1(s-1, x)) \leqslant \sup_{0 \leqslant y \leqslant r_{\alpha,s}} V_\alpha(s, y) \cdot Q^\alpha\{\psi_1(s-1, x) \leqslant r_{\alpha,s}\} \to 0$$

since $y \mapsto V_\alpha(s, y)$ is a continuous function and, hence, bounded on the segment $[0, r_{\alpha,s}]$, while $Q^\alpha\{\psi_1(s-1, x) \leqslant r_{\alpha,s}\} \to 0$ for $x \to +\infty$ as follows from the definition of $\psi_1(t, x)$. On the other hand, $F_\alpha(s-1, x) \to -\infty$ as $x \to +\infty$, which follows from that, according to the assumption $\mu_2 < -\alpha\sigma_2^2/2$ (see (3.2)), the coefficients μ_2 or $\alpha\beta^{t-1}(\beta - \beta/\gamma)$ in front of x in formula (3.10) are negative respectively in the case $\alpha = 0$ or $\alpha \neq 0$. Then

from (3.12) we obtain the existence of the root $r_{\alpha,s-1}$.

This completes the proof of the theorem.

Remark 3.1. Assumption $\mu_2 < -\alpha\sigma_2^2/2$ was used in the proof of the theorem to show that the functions $x \mapsto V_\alpha(t,x)$ have non-negative roots. Assumption $\mu_1 > -\alpha\sigma_1^2/2$ is not necessary for the proof, but if it does not hold, then problem (3.1) becomes trivial: from the recurrent formula for $V_\alpha(t,x)$ it is easy to see that $V_a(t,x) = 0$ for all $t = 0, 1, \ldots, T$, $x \geqslant 0$, so the optimal stopping time $\tau_\alpha^* = 0$.

Remark 3.2. Let us provide another proof of the continuity of the functions $x \mapsto V_\alpha(t,x)$, which directly uses their definition (3.11) without relying on the recurrent formulae.

Observe that for arbitrary $0 \leqslant x_1 \leqslant x_2$ it holds that

$$0 \leqslant V_\alpha(t,x_1) - V_\alpha(t,x_2) = V_\alpha(t,x_1) - \mathrm{E}^{Q^\alpha}\left[\sum_{u=1}^{\tau_\alpha^*(t,x_2)} F_\alpha(t+u, \psi_u(t,x_2))\right]$$

$$\leqslant \mathrm{E}^{Q^\alpha}\left[\sum_{u=1}^{\tau_\alpha^*(t,x_2)} \left[F_\alpha(t+u, \psi_u(t,x_1)) - F^\alpha(t+u, \psi_u(t,x_2))\right]\right]$$

$$\leqslant \begin{cases} \displaystyle\sum_{u=1}^{T-t} \mathrm{E}^{Q^0}\left[\mu_2(\psi_u(t,x_1) - \psi_u(t,x_2))\right], & \alpha = 0, \\ \displaystyle\sum_{u=1}^{T-t} \alpha\mathrm{E}^{Q^\alpha}\left[\beta^{t+u}(1 - 1/\gamma)(\psi_u(t,x_1) - \psi_u(t,x_2))\right], & \alpha \neq 0, \end{cases}$$

where $\tau_\alpha^*(t,x) = \inf\{u \geqslant 0 : (t+u, \psi_u(t,x)) \in D_\alpha\}$. From the definition of $\psi(t,x)$, using (3.9), we obtain

$$\mathrm{E}^{Q^\alpha}(\psi_u(t,x_1) - \psi_u(t,x_2)) = \mathrm{E}^{Q^\alpha}\left[\mathrm{E}^{Q^\alpha}(\psi_u(t,x_1) - \psi_u(t,x_2)) \mid \mathscr{F}_{u-1}\right]$$
$$= \frac{\gamma}{\beta}\mathrm{E}^{Q^\alpha}(\psi_{u-1}(t,x_1) - \psi_{u-1}(t,x_2)).$$

By induction we find $\mathrm{E}^{Q^\alpha}(\psi_u(t,x_1) - \psi_u(t,x_2)) = (\gamma/\beta)^u(x_1 - x_2)$, so

$$0 \leqslant V_\alpha(t,x_1) - V_\alpha(t,x_2) \leqslant \begin{cases} |\mu_2|(T-t)(x_2 - x_1), & \alpha = 0, \\ \alpha\beta^t(1 - \gamma^{T-t})(x_2 - x_1), & \alpha \neq 0, \end{cases}$$

which implies that the functions $x \mapsto V_\alpha(t,x)$ are continuous.

3.4 Numerical solutions

In this section we describe a method how the functions $V_\alpha(t,x)$ and the optimal stopping boundaries $b_\alpha^*(t)$ can be found numerically and consider several examples which illustrate the structure of random walks with change-points and the structure of the stopping boundaries.

In order to find the functions $V_\alpha(t,x)$ and $b_\alpha^*(t)$ numerically we take a partition of $\mathbb{R}_+$ by points $\{x_n\}_{n=0}^\infty$, $x_n = n\Delta$, where $\Delta > 0$ is a parameter, and compute the values $\overline{V}_\alpha(t,x_n)$ approximating $V_\alpha(t,x_n)$ and $\overline{b}_\alpha^*(t)$ approximating $b_\alpha^*(t)$ by backward induction over $t = T, T-1, \ldots, 0$.

For $t = T$, let $\overline{V}_\alpha(T,x_n) = 0$ for each $n \geqslant 0$. Suppose $\overline{V}_\alpha(s,x_n)$, $n \geqslant 0$, are found for some $s > 0$. In order to find $\overline{V}_\alpha(s-1,x_n)$, $n \geqslant 0$, define for any $x \geqslant 0$

$$\overline{V}_\alpha(s,x) = \sum_{n=0}^\infty \overline{V}_\alpha(s,x_n)\mathbf{I}\{x \in [x_n, x_{n+1})\}$$

and compute the values $\overline{V}_\alpha(s-1,x_n)$, $n \geqslant 0$, by formulae on p. 79 with $\overline{V}_\alpha(s,x)$ in place of $V_\alpha(s,x)$ in the formulae for $f_\alpha(s,x)$. This can be done in a finite number of steps, since after we find n such that $\overline{V}_\alpha(s-1,n) = 0$ then $\overline{V}_\alpha(s-1,n') = 0$ for all $n' \geqslant n$. Proceeding by induction over $t = T, T-1, \ldots, 0$ we find all the values $\overline{V}_\alpha(t,x_n)$, $n \geqslant 0$. Then for each $t = 0, \ldots, T$ define $\overline{b}_\alpha^*(t) = x_{n_0(\alpha,t)}$, where $n_0 = n_0(\alpha,t)$ is the smallest number n_0 such that $\overline{V}_\alpha(t,x_{n_0}) = 0$.

Let us show that the computational errors of the method (i. e. the differences $\overline{V}_\alpha(t,x) - V_\alpha(t,x)$ and $\overline{b}_\alpha^*(t) - b_\alpha^*(t)$) can be estimated by quantities proportional to Δ. It will be assumed that the integral in the formulae for $f_\alpha(t,x)$ and the elementary functions in the formulae for $\overline{V}_\alpha(t,x)$ can be computed exactly (or with negligible errors), so the computational errors appear only due to the approximation of $V_\alpha(t,x)$ by the functions $\overline{V}_\alpha(t,x)$.

Introduce the constants C_α^t, $t = 0, 1, \ldots, T$:

$$C_t^\alpha = \sum_{s=t}^T c_s^\alpha = \begin{cases} |\mu_2|\dfrac{(T-t)(T-t+1)}{2}, & \alpha = 0, \\[2ex] \alpha\left(\dfrac{\beta^{T+1} - \beta^t}{\beta - 1} - \dfrac{\beta^{T+1} - \gamma^{T+1}(\beta/\gamma)^t}{\gamma - \beta}\right), & \alpha \neq 0. \end{cases}$$

Theorem 3.2. *1) For all $t = 0, 1, \ldots, T$, $x \geqslant 0$, the estimate holds:*

$$0 \leqslant \overline{V}_\alpha(t, x) - V_\alpha(t, x) \leqslant C_t^\alpha \Delta.$$

2) For $t = 0, 1, \ldots, T$, the functions $b_\alpha^*(t)$, $\overline{b}_\alpha^*(t)$ satisfy the inequalities

$$\overline{b}_0^*(t) - \left(\frac{C_t^0}{|\mu_2|} + 2 \right) \Delta \leqslant b_0^*(t) \leqslant \overline{b}_0^*(t),$$

$$\overline{b}_\alpha^*(t) - \left(\frac{C_t^\alpha}{\alpha \beta^t (1 - \gamma)} + 2 \right) \Delta \leqslant b_\alpha^*(t) \leqslant \overline{b}_\alpha^*(t), \qquad \alpha \neq 0.$$

Proof. 1) The proof of the first statement is conducted by induction over $t = T, T - 1, \ldots, 0$. For $t = T$ the estimate is true because $V_\alpha(t, x) = \overline{V}_\alpha(t, x) = 0$. Suppose it holds for $t = s$ and let us prove it for $t = s - 1$.

According to the recurrent formulae on p. 79, $0 \leqslant \overline{V}_\alpha(s-1, x) - V_\alpha(s-1, x)$ because $0 \leqslant \overline{V}_\alpha(s, x) - V_\alpha(s, x)$ for all $x \geqslant 0$ as follows from the inductive assumption.

Let $n(x)$ denote the largest x_n not exceeding x. Then for $x \geqslant 0$ we have

$$\overline{V}_\alpha(s - 1, x) - V_\alpha(s - 1, x)$$
$$= \left[\overline{V}_\alpha(s - 1, x) - \overline{V}_\alpha(s - 1, x_{n(x)}) \right]$$
$$+ \left[\overline{V}_\alpha(s - 1, x_{n(x)}) - V_\alpha(s - 1, x_{n(x)}) \right]$$
$$+ \left[V_\alpha(s - 1, x_{n(x)}) - V_\alpha(s - 1, x) \right]$$
$$\leqslant 0 + C_s^\alpha \Delta + c_{s-1}^\alpha \Delta = C_{s-1}^\alpha \Delta,$$

where we use that the difference in the second line equals zero according to the definition of $\overline{V}_\alpha$, the difference in the fourth line is estimated from above by $c_{s-1}^\alpha (x - x_{n(x)}) \leqslant c_{s-1}^\alpha \Delta$ according to (3.13), and for the third line we use the inequality

$$\overline{V}_\alpha(s - 1, x_{n(x)}) - V_\alpha(s - 1, x_{n(x)}) \leqslant \overline{f}_\alpha(s - 1, x_n) - f_\alpha(s - 1, x_n)$$
$$\leqslant \int_{\mathbb{R}} \frac{C_s^\alpha \Delta}{\sigma_1 \sqrt{2\pi}} \exp\left(-\frac{(z - \mu_1 - \alpha \sigma_1^2)^2}{2\sigma_1^2} \right) dz = C_s^\alpha \Delta,$$

where the function $\overline{f}_\alpha$ is defined by the same formula as f_α, but with $\overline{V}_\alpha$ in place of V_α. The inequalities obtained prove the inductive step and, consequently, prove statement 1 of the theorem.

2) Fix $\alpha \leqslant 1$. Observe that $b_\alpha^*(t) \leqslant \bar{b}_\alpha^*(t)$ because $\overline{V}_\alpha(t, x) \geqslant V_\alpha(t, x)$ for all $t = 0, 1, \ldots, T$, $x \geqslant 0$.

Let us prove the lower inequalities. First, suppose for some $t \leqslant T$ there exists a non-negative integer number k such that

$$x_{n_0(t)-1} - x_k > \begin{cases} \dfrac{C_t^0}{|\mu_2|}\Delta, & \alpha = 0, \\[3mm] \dfrac{C_t^\alpha}{\alpha\beta^t(1-\gamma)}\Delta, & \alpha \neq 0 \end{cases} \tag{3.14}$$

(since α is fixed, it is omitted in the notation $n_0(\alpha, t)$). Let $k(t)$ denote the largest such integer number. Observe that for any $0 \leqslant k_1 \leqslant k_2 < n_0(t)$ it holds that

$$\overline{V}_\alpha(t, x_{k_1}) - V_\alpha(t, x_{k_2}) \geqslant \begin{cases} |\mu_2|(x_{k_2} - x_{k_1}), & \alpha = 0, \\[2mm] \alpha\beta^t(1-\gamma)(x_{k_2} - x_{k_1}), & \alpha \neq 0, \end{cases} \tag{3.15}$$

as follows from the definition of $\overline{V}_\alpha(t, x_k)$ by the formulae on p. 79. As a consequence,

$$V_\alpha(t, x_{k(t)}) \geqslant \overline{V}_\alpha(t, x_{k(t)}) - C_t^\alpha\Delta \geqslant \overline{V}_\alpha(t, x_{k(t)}) - \overline{V}_\alpha(t, x_{n_0(t)-1}) - C_t^\alpha\Delta > 0,$$

where in the first inequality we use statement 1, in the second inequality we use that $\overline{V}_\alpha \geqslant 0$, and in the last one we apply (3.14)–(3.15). This implies $b_\alpha^*(t) > x_{k(t)}$. Using that $k(t)$ is the largest integer number satisfying (3.14), we see that

$$x_{k(t)} \geqslant \begin{cases} x_{n_0(t)-1} - \left(\dfrac{C_t^0}{|\mu_2|} + 1\right)\Delta, & \alpha = 0, \\[4mm] x_{n_0(t)-1} - \left(\dfrac{C_t^\alpha}{\alpha\beta^t(1-\gamma)} + 1\right)\Delta, & \alpha \neq 0. \end{cases}$$

Consequently, if for some $t \leqslant T$ there exists a non-negative integer number satisfying (3.14), the lower inequality in statement 2 holds for this t because $x_{n_0(t)-1} = \bar{b}_\alpha^*(t) - \Delta$.

In the opposite case, we have $x_{n_0(t)-1} \leqslant C_t^0\Delta/|\mu_2|$ if $\alpha = 0$ and $x_{n_0(t)-1} \leqslant C_t^\alpha\Delta/(\alpha\beta^t(1-\gamma))$ if $\alpha \neq 0$, which also implies the validity of statement 2, since $b_\alpha^*(t) \geqslant 0$. $\qquad\square$

Now we provide examples of the solutions found numerically for certain values of the parameters. We let $T = 100$, the moment of disorder θ be uniformly distributed in the set $\{1, 2, \ldots, 101\}$, and consider the values of the parameters $\mu_1 = -\mu_2 = 2/T$, $\sigma_1 = \sigma_2 = 1/\sqrt{T}$. The choice of the parameters μ_1, μ_2 of order $1/T$ and the choice of σ_1, σ_2 of order $1/\sqrt{T}$ is due to that, as follows from the invariance principle (see e. g. [35, Theorem 12.9]), such a random walk with disorder converges weakly to a Brownian motion with disorder on $[0, 1]$ as $T \to \infty$, which presents interest in view of the results of the papers [2, 14, 56].

Figure 1 shows the optimal stopping boundaries for the Shiryaev–Roberts statistic in the problem with the above values of the parameters and $\alpha = 1, 0, -\frac{1}{2}, -1$. A larger value of α corresponds to a higher boundary.

Figure 2 presents the optimal stopping boundaries for the posterior probability process. Similarly, higher boundaries are obtained for larger values of $\alpha = 1, 0, -\frac{1}{2}, -1$.

Figures 3 and 4 show the value functions $V_0(t, x)$ for the Shiryaev–Roberts statistic and $\widetilde{V}_0(t, x) = V_0(t, x/(x + 1 - G(t)))$ for the posterior probability process, where $t = 10, 20, 30, \ldots, 90$. Higher lines correspond to smaller values of t.

On Figure 5 we provide three sample paths of the geometric Gaussian random walk with changepoints at $\theta = 25, 50, 75$. These paths are constructed from the same realisation of the sequence ξ_i of i.i.d $\mathcal{N}(0, 1)$ random variables, but with the three different values of θ.

The corresponding paths of the Shiryaev-Roberts statistic and the optimal stopping boundary $b_0^*(t)$ are shown on Figure 6. The optimal stopping times $\tau^* = 46, 57, 75$ (respectively, for $\theta = 25, 50, 75$) are found as the first moments of time the Shiryaev-Roberts statistic crosses the boundary.
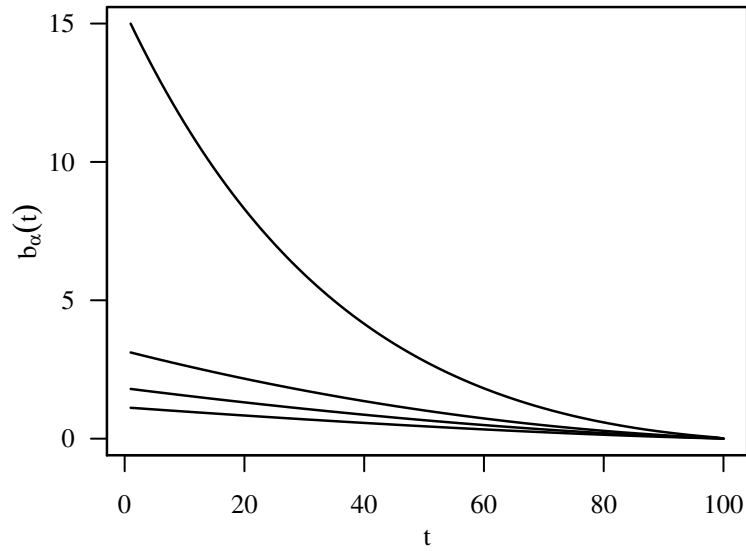
Figure 1. Stopping boundaries $b_\alpha^*(t)$ for the Shiryaev–Roberts statistic for $\alpha = 1, 0, -\frac{1}{2}, -1$ (higher lines correspond to larger values of α).
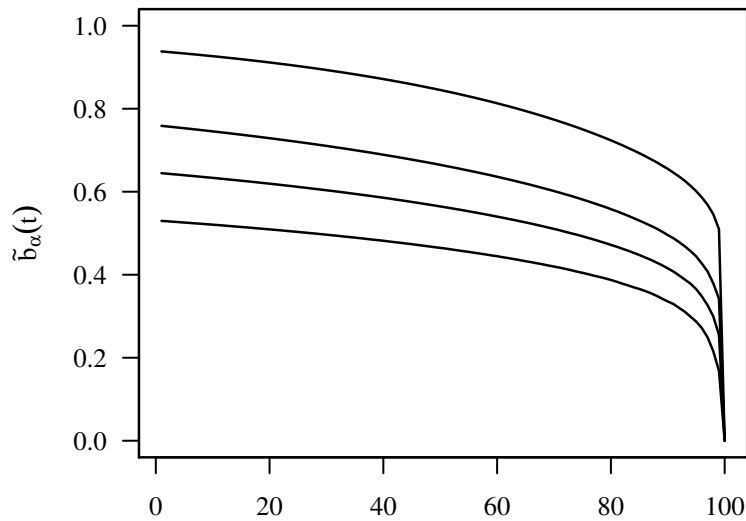

Figure 2. Stopping boundaries $\widetilde{b}_\alpha^*(t)$ for the posterior probability process for $\alpha = 1, 0, -\frac{1}{2}, -1$ (higher lines correspond to larger values of α).
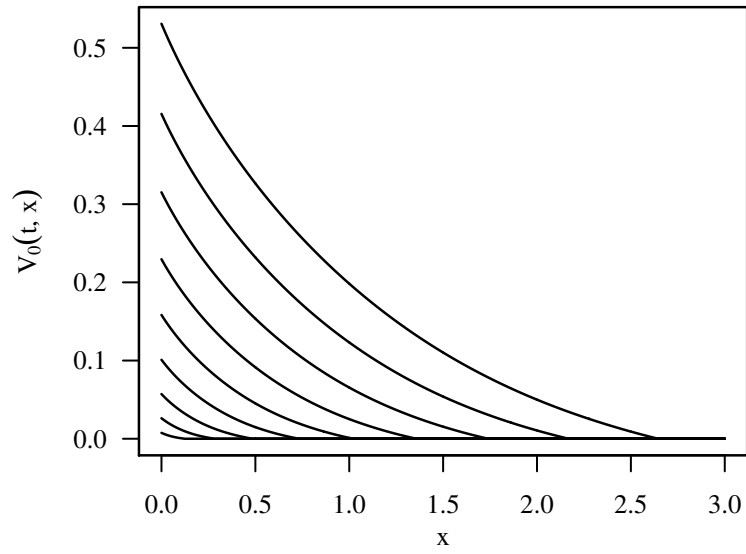
Figure 3. Value functions $V_0(t, x)$ for the Shiryaev–Roberts statistic for $t = 10, 20, \ldots, 90$ (higher lines correspond to smaller values of t).
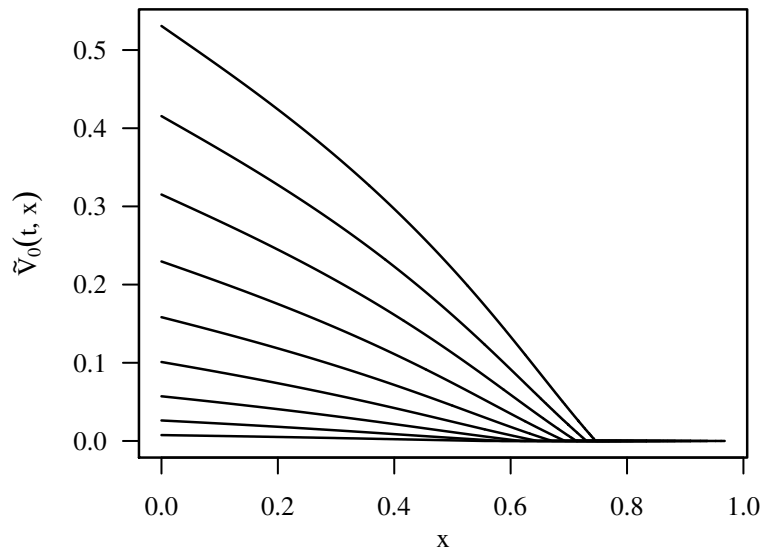
Figure 4. Value functions $\widetilde{V}_0(t, x)$ for the posterior probability process for $t = 10, 20, \ldots, 90$ (higher lines correspond to smaller values of t).
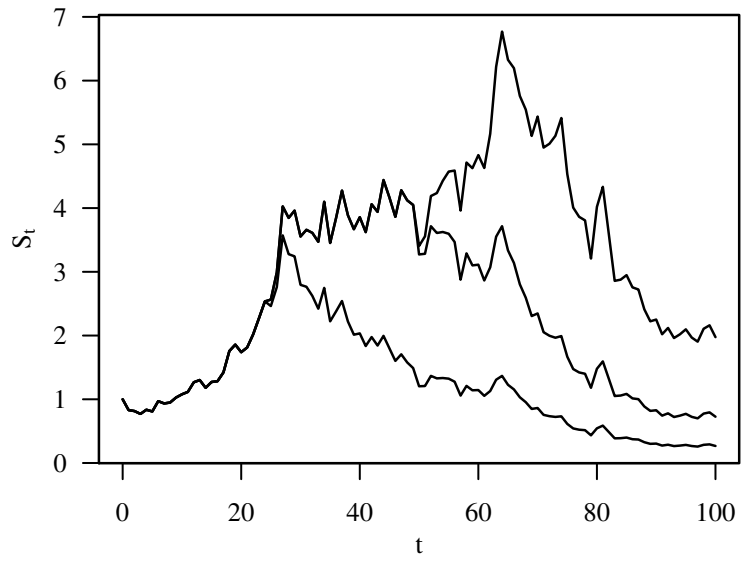
Figure 5. Sample paths of the random walk with changepoints $\theta = 25, 50, 75$.
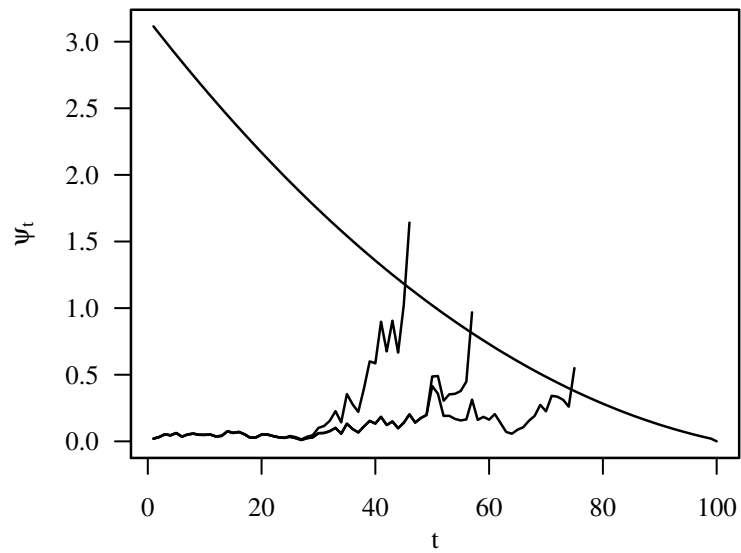

Figure 6. Paths of the Shiryaev–Roberts statistic with $\theta = 25, 50, 75$.

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