

Performance Analysis integrating Data Envelopment Analysis and Multiple Objective Linear Programming.

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Abbreviations

ADMU	Anti-ideal Decision Making Unit
AHP	Analytical Hierarchy Process
CCA	Component Correlation Analysis
CRS	Constant Return to Scale
DDEA	Discrimination Data Envelopment Analysis
DEA	Data Envelopment Analysis
DM	Decision Maker
DMU	Decision Making Unit
ER	Evidential Reasoning
GAA	Geometric Average Approach
GoDEA	Goal programming and Data Envelopment Analysis
GPM	Gradient Projection Method
IDEA	Imprecise Data Envelopment Analysis
IGP	Interactive Goal Programming
ISWT	Interactive Surrogate Worth Trade-off
IT	Information Technology
MCDM	Multiple Criterion Decision Making
MOLP	Multiple Objective Linear Program
MPS	Most Preferred Solution
SPOT	Sequential Proxy Optimization Trade-off
STEM	Step Method
STOM	Satisfactory Trade-off Method
TABRA	Target-based resource allocation
VIA	Visual Interactive Approach
VRS	Variable Return to Scale

Abstract

Firms or organisations implement performance assessment to improve productivity but evaluating the performance of firms or organisations may be complex and complicated due to the existence of conflicting objectives. Data Envelopment Analysis (DEA) is a non-parametric approach utilized to evaluate the relative efficiencies of decision making units (DMUs) within firms or organizations that perform similar tasks. Although DEA measures the relative efficiency of a set of DMUs the efficiency scores generated do not consider the decision maker's (DM's) or expert preferences.

DEA is used to measure efficiency and can be extended to include DM's and expert preferences by incorporating value judgements. Value judgements can be implemented by two techniques: weight restrictions or constructing an equivalence Multiple Objective Linear Programming (MOLP) model. Weight restrictions require prior knowledge to be provided by the DM and moreover the DM cannot interfere during the assessment analysis. On the other hand, the second approach enables the DM to interfere during performance assessment without prior knowledge whilst providing alternative objectives that allow the DM to reach the most preferred decision subject to available resources.

The main focus of this research was to establish interactive frameworks to allow the DM to set targets, according to his preferences, and to test alternatives that can realistically be measured through an interactive procedure. These frameworks are based on building an equivalence model between extended DEA and MOLP minimax formulation incorporating an interactive procedure. In this study two frameworks were established. The first is based on an equivalence model between DEA trade-off approach and MOLP minimax formulation which allows for incorporating DM's and expert preferences. The second is based on an equivalence model between DEA bounded model and MOLP minimax formulation. This allows for integrating DM's preferences through interactive steps to measure the whole efficiency score (i.e. best and worst efficiency) of individual DMU. In both approaches a gradient projection interactive approach is implemented to estimate, regionally, the most preferred solution along the efficient frontier. The second framework was further extended by including ranking based on the geometric average. All the frameworks developed and presented were tested through implementation on two real case studies.

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Dedication

This is for you, Mom. Thanks for always being there for me.....

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Chapter 1

Introduction to the thesis.

1.1 General overview

The aim of performance assessment is to improve the operations of business and organisation units with the intention of increasing productivity. Performance assessment is a complicated process due to the existence of multiple and conflicting objectives and attributes.

Data Envelopment Analysis (DEA), a non-parametric method, is a very powerful tool used for assessing performance, benchmarking and future planning. It is used to evaluate the performance of a set of peer entities called Decision Making Units (DMUs) relative to the best DMU's practice, which forms the efficient frontier. It was introduced by Charnes, Cooper, and Rhodes and is known as the CCR approach or model, (Charnes; 1978). Since then DEA has seen a great variety of applications in evaluating the performance of many kinds of entities and most extensions, i.e. incorporating value judgments, of the approach have resulted from these real life applications.

This research builds on these extensions of DEA and its integration with Multi Objective Linear Programming (MOLP) to provide three new interactive frameworks that are fully implemented and tested on two real life case studies.

1.2 Motivation for the research

The integration of assessing past performance and planning future targets involving DM and expert preferences is of increasing interest to support both management and planning control (Cooper, 2004). Although DEA has been developed for performance assessment many implementations and applications of DEA have not included the interaction of the decision maker(s) or expert(s) and therefore have not included their value judgements. The incorporation of value judgements into DEA can be satisfied by placing restrictions on weights linked to inputs or outputs or by developing equivalent Multiple Objective Linear Programming (MOLP) models which previously have been employed for target setting and future planning. The former approach of weight restrictions is difficult because of the need to identify the appropriate weights whilst the latter approach has the advantage of including the decision maker(s) or expert(s) judgements through the employment of interactive trade-off analysis. A third approach to incorporating DM preferences involves the construction of ordering preferences in which preference relationships are viewed in a fuzzy manner as “higher”, “lower”, “better” or worse” Liu (2006). Techniques for ordering preferences include: Pareto, K-cone and Lexicographic methods, which are all transitive (Liu, 1985 and Nemhauser, 1989).

Incorporating ranking in performance assessment and DEA is essential as it enables DMs to monitor the level of improvement of the targeted DMU among the others. Although there have been several attempts in the literature to incorporate ranking they have only provided a means of measuring the best efficiency (one-directional efficiency) but not a complete image of the performance of DMUs. In this research a new approach for ranking DMUs is presented which allows for measuring the performance of individual DMUs from two different perspectives (the best relative efficiency and the worst relative efficiency).

This work presents three interactive frameworks: production trade-off MOLP model, bounded MOLP model, and interactive ranking model. The origin of these models is to integrate DEA with a minimax MOLP model from which the Most Preferred Solution (MPS) is obtained interactively through gradient projection. The third framework employs the Geometric Average to incorporate the best and worst relative efficiencies for ranking the DMUs.

1.3 Research Questions

The aim of this research was to provide frameworks for better efficiency assessment and target setting leading to improved decision making. The research questions posed were:

- How effective and useful are DEA and its extensions in incorporating both DM and expert preferences when focussing on performance assessment, benchmarking and target setting?
- How can any equivalence of the extended DEA (i.e. trade-off approach and bounded model) and MOLP models be employed to establish an interactive framework for improving performance assessment?
- How can ranking be incorporated into performance assessment and what would be the effect on decision making?

1.4 Structure of the thesis

The thesis comprises six chapters including this introduction.

Chapter 2 is the main literature review and focuses on the essential background knowledge required to conduct the research of this work. The chapter begins by reviewing the concept of Data Envelopment Analysis (DEA) starting from the work of Farrell (1957) and follows the development of DEA and its extensions culminating with a discussion of its advantages and disadvantages. Then follows a critical review of Multi-Criteria Decision Making (MCDM) approaches including: basic Multiple Objective Linear Programming (MOLP), interactive MOLP, the integration of DEA and MOLP and the interactive Gradient Projection Approach (GPA) all of which form the basis of this research.

Chapter 3 begins with an introduction into the incorporation of Decision Maker (DM) and expert value judgements and reviews the production trade-off DEA/MOLP model before providing an illustrative example and presenting the steps required for an interactive framework. Then two case studies are presented to illustrate this interactive framework. The first case study considers decisions concerning several academic departments at the University of Bahrain and fully describes the framework process involving: selecting

variables, statistical analyses, identifying returns to scale and decisions for improvement of one department. The second case study considers decisions concerning several private hospitals in Bahrain and focuses on how hospital performance can be measured and the application of the framework to improve one specific hospital.

Chapter 4 begins with a critical review of relative efficiency and reviews the bounded DEA-MOLP approach before presenting an illustrative example and the steps required for the second interactive framework. The framework is then fully implemented and tested on The Bahrain International Hospital from the case study data presented in the previous chapter.

Chapter 5 begins with a critical review of ranking approaches and reviews the Geometric Average Approach (GAA) before presenting a third interactive framework and an illustrative example. The chapter concludes with a fully detailed implementation of the framework to the second case study of chapter 3 (The private hospitals of Bahrain).

Finally, chapter 6 presents an overview of the thesis and conclusions followed by a summary of the contributions that the research has made to knowledge before outlining possibilities and proposals for further research.

Chapter 2

Literature Review: Efficiency Analysis and Performance Assessment

2.1 Introduction

Improvement in an organization or firm can be achieved by taking appropriate decisions. Decision making is considered as the process of selecting a course or action from alternatives. Data Envelopment Analysis (DEA) and Multiple Criteria Decision Making (MCDM) have been developed to aid a Decision Maker (DM) to assess the performance of an organization and to choose the most efficient decision(s) among several alternatives.

Efficiency can be simply defined as the ratio of output to input. More output from less input results in relatively larger efficiency. If the greatest possible output is obtained then a state of absolute or optimum efficiency has been accomplished and it is not possible to become more efficient without making changes to assumed constants in system (e.g. production) processes.

DEA, which is also known as frontier analysis, is a very powerful tool and benchmarking technique which was formally developed by Charnes, Cooper and Rhodes in 1978 (Charnes, 1978) based on an original idea by Farrell (1975). It was first implemented to evaluate non-profit and public sector organizations in which the units being evaluated were known as DMUs (Decision Making Units). DEA has since been used to find ways to improve firms or organisation that have not been found by other techniques.

There is a wide variety of Multiple Criteria Decision Making (MCDM) tools and perhaps the most popular comprise those that come under the heading of Multiple Objective Linear Programming (MOLP), which can be used to solve problems with multiple conflicting linear objectives, criteria or attributes subject to linear constraints.

Relatively recently, Yang (2009) has shown that integrating DEA with MOLP techniques can produce a more comprehensive analysis tool to enable a decision maker interactively to achieve efficient decisions. This integration is reviewed and developed in Chapter 4.

2.1.1 Aims

The aim of this chapter is to present and critically examine the underlying literature upon which this thesis is based to build background knowledge about the DEA efficiency assessment approach. In summary the aims of this chapter are:

- A review of basic DEA and its applications
- Establishing solid background knowledge about DEA mathematical formulations and models.
- Investigating the methods used to incorporate value judgements in DEA.
- Studying the approaches that integrate DEA with MOLP
- Reviewing the methods used for ranking DMUs

2.1.2 Research questions

This chapter attempts to answer the following questions:

1. To what extent have DM preferences and expert preferences been included in DEA models?
2. How and to what extent has DMU efficiency been addressed in the literature?
3. To what extent has the ranking of DMUs been addressed in DEA models and how does this benefit the DM?

2.1.3 Methodology

This chapter critically examines and reviews the literature of DEA and its extensions and applications to identify any gaps that need to be addressed in order to achieve the research objectives of this thesis. The chapter begins with an introduction to DEA starting with the original concept proposed by Farrell (1957) and then follows its development and extensions to enable a discussion of its advantages and disadvantages to provide essential background knowledge about the origin of DEA and its major drawbacks. The chapter then examines Multi-Criteria Decision Making (MCDM) approaches, in particular MOLP, and how it can be made interactive for a Decision Maker (DM) and integrated with DEA, which is one of the desirable improvement in basic DEA. This will provide vital knowledge required for this study in which interactive DEA can be formulated.

2.1.4 Chapter structure

This chapter includes:

- Basic DEA models
- DEA mathematical formulations
- Extensions of DEA models
- Strengths and weaknesses of DEA models
- Multiple Criteria Decision Making (MCDM)
- Multiple Objective Linear Programming (MOLP) models
- Integration of DEA and MOLP

2.2 Data Envelopment Analysis

In this review, general developments in basic DEA and its applications in different sectors such as banking, health and education are considered.

DEA involves the use of linear programming methods to construct a non-parametric piece-wise surface (or frontier) over the data in the form of multiple inputs and multiple outputs. These data are collected for each DMU within an organization or firm. Efficiency measures are then calculated relative to this frontier. Several researchers have intensively studied the methodology of DEA such as Fare, Grosskopf and Lovell (1985, 1994), Seiford and Therall (1990), Lovell (1993), Ali and Seiford (1993), Lovell (1994), Charnes et al (1995), Seiford (1996), Cooper, Seiford and Tone (2000) and Thanassoulis (2001).

The piece-wise convex hull approach to frontier estimation, initially demonstrated by Farrell (1957), was considered by only few authors in the two decades following his work. Boles (1966), Shephard (1970) and Afriat (1972) proposed systematic mathematical programming methods that could achieve the task, but their work did not become wide spread until the paper of Charnes, Cooper and Rhodes (1978) in which *Data Envelopment Analysis* was first used. Since then a large numbers of papers have appeared which have extended and applied the DEA methodology.

Charnes, Cooper and Rhodes (1978) suggested an input oriented model that assumed constant returns to scale (CRS) known as the CCR model. Subsequent papers have considered alternative sets of assumptions, such as those by Fare, Grosskopf and Logan (1983) and Banker, Charnes and Cooper (1984) with variable returns to scale (VRS). The model proposed by this latter group is known as BCC model.

The idea behind DEA is to find a projection of inefficient DMUs on the efficient frontier by following a radial contraction or expansion, and hence, their efficiency score is modified relative to the highest efficiency score among DMUs. The efficient frontier can be defined as a surface on which the efficient DMUs lie (i.e. the surface that contains DMUs having the highest efficiency score among the others).

The underlying concept behind DEA was suggested by Farrell (1957) who proposed that the efficiency of a firm consisted of two components. Farrell (1957) defined these as: technical efficiency, which reflects the ability of a firm to obtain maximal output from a given set of input, and allocative efficiency, which reflects the ability of a firm to use the inputs in optimal proportions, given their respective prices and the production technology. Farrell's approach will be discussed in more detail in the next section.

The logical extension of Farrell's work was to provide a model that included multiple inputs and multiple outputs. This was achieved by dividing the sum of outputs over the sum of inputs with the assumption of different DMUs having a common set of weights for each set of inputs and outputs. However, Charnes et al. (1994) recognised that sharing common weights was a limitation on the model and they suggested individual weights for individual inputs and outputs.

Recall that DEA is based on comparing the efficiency of each DMU relative to an efficient frontier constructed from the highest outcome scores of DMUs within a firm or organization. The condition used for calculating optimality is well known as the Pareto optimality condition, which states that if one input is reduced then another input must be increased to maintain the same amount of output.

DEA has widespread applications for measuring performance in different sectors including banking, health and education.

In the banking sector DEA has been applied successfully to estimate banking performance (Brockett et al., 1997; Chen, 1997; Al-Shmmari and Salimi, 1998; Lorenzo-Vivas et al., 2000 Ayadi et al., 1998) and the performance of bank branches see for example Avkiran (1999).

In the health sector DEA has been applied to estimate the performance of hospitals, nursing homes, and physicians see for example Hollingsworth et al. (1999) who counted 91 DEA studies in health care. His finding led several researchers to address essential questions to help formulate hypotheses in subsequent studies. Most of the hospital studies have merely illustrated DEA as a methodology and demonstrated its potential, however,

some studies tried to produce useful results for a DM. For example, a study of 22 hospitals in the National Health Service in the United Kingdom used a four output, five input model (Kerr et al., 1999).

In the education sector, Hanushek (1986) surveyed 147 studies to characterize the achievements of individual students resulting from consuming inputs related to school characteristics e.g. teachers' qualifications and curricula. A subsequent similar study, by Wenger (2000) added 'completion of schooling' to the output and showed that 'school completion' is essential in a production theoretic perspective. There are many studies in this sector, see for example Ray and Mukherjee (1998), Mincer (1974) and Beharman and Birdsall (1983). Some studies concentrate on higher education, see for example Flegg et al (2004) and Johns (2005 and 2006).

DEA is a non-parametric approach has been found to be more applicable in contrast with parametric methods such as ratio analysis or regression. DEA is an extreme point approach, which compares each DMU with those that have best practice unlike statistical approaches based on central tendency methods which compare DMUs relative to the average DMU. Ratio analysis suffers from a major drawback in that a different ratio may indicate the performance of a unit ambiguously in different directions; whilst regression analysis yields average or central tendency behaviour of observations (Ganley and Cubbin, 1992; Seiford and Tharall, 1990). Furthermore, DEA does not require a production function, pre-described probability distributions or any assumptions about the distribution of standard errors or the efficiency residual in the production function and therefore is easier to implement (Banker, 1992 and Cubbin and Tzanidakis, 1998).

2.2.1 Farrell Efficiency

The concept of the piece-wise-linear convex hull approach was initially presented by Farrell (1957). In his study, the following assumptions were applied:

1. Efficiency measures were based on radial uniform contractions or expansions from inefficient elements to the frontier.
2. The production frontier was specified as the most pessimistic piecewise linear envelopment of the data (Koopmans, 1951).
3. The frontier was calculated through solving systems of linear equations, obeying the two conditions on the unit isoquant:
 - i. That its slope is not positive.
 - ii. That no observed point lies between it and the origin.

The most important feature of Farrell's approach is that overall efficiency could be decomposed into 'technical' and 'allocated' (price) efficiency as defined below and explained through Figure 2.1 in which inputs x_1 and x_2 produce output y .

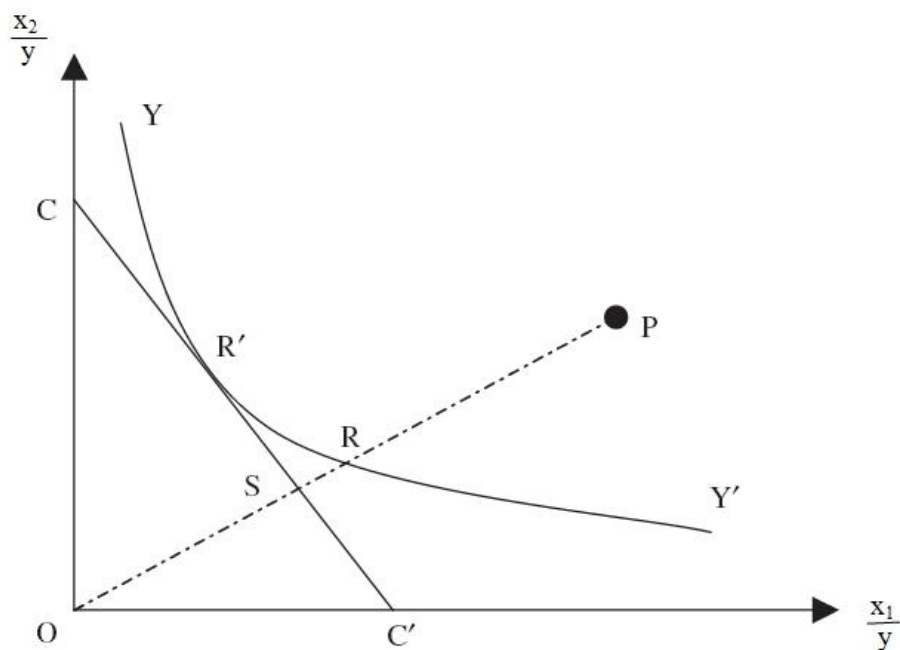


Figure 2.1 Farrell's efficiency Analysis

Assuming constant returns to scale (CRS) the unit isoquant YY' shows the minimum combination of inputs per unit of output needed to produce a unit of output. Every package of inputs along the unit isoquant is considered as technically efficient while any point above and to the right of it, such as point P, defines a technically inefficient producer because the input package that is being used is more than enough to produce a unit of output. The distance RP along the ray OP measures the technical inefficiency of a producer located at point P and represents the amount by which all inputs can be divided without decreasing the amount of output. The technical inefficiency level associated to package P can be expressed by the ratio RP/OP and therefore the technical efficiency of the producer under analysis (1-RP/OP) is given by:

$$\text{Technical efficiency} = \frac{OR}{OP}$$

If information on market prices is known and cost minimization is assumed in such a way that the input price ratio is reflected by the slope of the isocost-line CC' then allocative inefficiency can also be derived from the unit isoquant plotted in Figure 2.1. The relevant distance is given by the line segment SR, which in relative terms is the ratio SR/OR. With respect to the least cost combination of inputs given by point R', the ratio indicates the cost reduction that a producer would be able to reach if it moved from a technically but not allocatively efficient input (R) to a both technically and allocatively efficient one (R'). Therefore, the allocative efficiency of the producer at point P is given by

$$\text{Allocative efficiency} = \frac{OS}{OR}$$

Together with the concepts of technical efficiency and allocative efficiency, Farrell (1957) also described a measure of what he termed overall efficiency as:

$$\frac{OR}{OP} \times \frac{OS}{OR} = \frac{OS}{OP}$$

where the distance involved in its definition (SP) can also be analyzed in terms of cost reduction.

The natural extension of the Farrell's approach was to include multiple inputs and multiple outputs to the model. This was achieved by Charnes, Cooper and Rhodes, (1978) which is known as CCR model. The technical efficiency of a DMU in this case can be expressed as:

$$\text{Technical efficiency} = \frac{\sum \text{weighted output}}{\sum \text{weighted input}} \quad 2.0$$

The weights are varied to approach the optimum solution (i.e. max or min) of the DEA problem.

2.2.2 Basic DEA models

DEA models have been developed over the last decade and have been demonstrated as a collection of models and extension to the original work of Charnes, Cooper and Rhodes (1978). For instance, these models vary in their orientation (i.e. input oriented or output oriented) and constant return to scale (CRS) or variable return to scale (VRS). The original DEA model was a non-linear model in which the efficiency (the ratio of outputs to inputs) is maximised. This involves finding values for the weights (see equation 2.1) such that the efficiency of the i^{th} firm is maximised. The optimum solution is measured for each observed DMU and this optimisation is repeated for each of the DMUs within a firm. The optimal weights are obtained by solving the mathematical programming problem:

$$\begin{aligned} \text{Max } h &= \frac{\sum_{r=1}^s u_r y_{rj_o}}{\sum_{i=1}^m v_i x_{ij_o}} \\ \text{Subject to} & \\ & \frac{\sum_{r=1}^s u_r y_{rj}}{\sum_{i=1}^m v_i x_{ij}} \leq 1 \quad j = 1, \dots, n \\ & u_r, v_i \geq 0 \\ & \text{for all } r = 1, \dots, s \text{ and } i = 1, \dots, m \end{aligned} \quad (2.1)$$

where:

j_0 is the observed DMU

h is the efficiency score of the DMU₀ under analysis;

n is number of DMUs under analysis;

s is the number of outputs;

m is the number of inputs;

$y_{rj} = \{y_{1j}, y_{2j}, \dots, y_{rj}, \dots, y_{sj}\}$ is the vector of outputs for DMU _{j} with y_{rj} being the value of output r for DMU _{j} .

$x_{ij} = \{x_{1j}, x_{2j}, \dots, x_{ij}, \dots, x_{mj}\}$ is the vector of inputs for DMU _{k} with x_{jk} being the value of input j for DMU _{k} ;

u and v are the vectors of multipliers, representing the weights for inputs (x_i) and outputs (y_r) respectively.

For a given set of n DMUs within a firm, the model allows for the calculation of the optimal set of the input weights $[u_w]_i = 1$ and output weights $[v_i]_j = 1$ that maximize its efficiency score h for an individual DMU₀,

Charnes, Cooper, and Rhodes (1981) defined efficiency by reference to the orientation chosen:

- i. Output oriented model, in which a DMU is inefficient if it is possible to improve any output without increasing any input or decreasing any other output.
- ii. Input oriented model, in which a DMU is inefficient if it is possible to decrease any input without increasing any other input and decreasing any output.

Hence, a DMU is defined to be inefficient whenever one of the above conditions are not satisfied, i.e. (i) and (ii). An efficiency score less than one indicates that the linear combination of other units forming the sample could produce the vector of outputs using smaller vectors of inputs. Mathematically, a DMU is efficient when the ratio of the output to the input is equal to one (i.e. $h = 1$).

Charnes (1981) introduced small non-Archimedean infinitesimal ε ($\sim 10^{-5}$ or 10^{-6}) to avoid zero production of the DMUs. So, the model can be expressed as:

$$\begin{aligned}
 \text{Max } h &= \frac{\sum_{r=1}^s u_r y_{rj_0}}{\sum_{i=1}^m v_i x_{ij_0}} \\
 \text{Subject to} \\
 \frac{\sum_{r=1}^s u_r y_{rj}}{\sum_{i=1}^m v_i x_{ij}} &\leq 1 \quad j = 1, \dots, n \\
 u_r, v_i &\geq \varepsilon \\
 &\text{for all } r = 1, \dots, s \text{ and } i = 1, \dots, m
 \end{aligned} \tag{2.2}$$

The above non-linear model can be converted to a linear model as demonstrated by Charnes and Cooper (1978) by letting $\sum_i v_i x_{ij_0} = 1$. By using this transformation the CCR input oriented model is established and can be written as:

$$\begin{aligned}
 \text{Max } h_0 &= \sum_{r=1}^s u_r y_{rj_0} \\
 \text{Subject to} \\
 \sum_{i=1}^m v_i x_{ij_0} &= 1 \\
 \sum_{r=1}^s u_r y_{rj} - \sum_{i=1}^m v_i x_{ij} &\leq 0, \quad j = 1, \dots, n \\
 u_r, v_i &\geq 0 \\
 &\text{for all } r = 1, \dots, s \text{ and } i = 1, \dots, m
 \end{aligned} \tag{2.3a}$$

This form of DEA model is known as the *multiplier* form. On the other hand, CCR output oriented model is produced by letting $\sum_r u_r y_{rj_0} = 1$. In this case, the efficiency score is equal to $1/h_0$ and the mathematical formulation of the problem can be written as:

$$\text{Min } h_0 = \sum_{r=1}^s v_r x_{ij_0}$$

Subject to

$$\sum_{i=1}^m u_r y_{rj_0} = 1 \quad (2.4a)$$

$$\sum_{i=1}^m v_i x_{ij} - \sum_{r=1}^s u_r y_{rj} \geq 0 \quad j = 1, \dots, n$$

$$u_r, v_i \geq 0$$

for all $r = 1, \dots, s$ and $i = 1, \dots, m$

The CCR input model is solved by using any optimisation method to find the vectors \mathbf{u} and \mathbf{v} which maximise the targeted firm (the efficiency of the firm equals h_0) whereas minimizing the targeted firm using the CCR output oriented formulation yields reciprocal efficiency (i.e. efficiency = $1/h_0$)

If the previous formulation (i.e. input oriented and output oriented) is defined to be the primal then the dual form can be written as:

Dual CCR input oriented

$$\text{Min } h = \theta_{j_0}$$

Subject to

$$\theta_{j_0} x_{ij_0} - \sum_{i=1}^m \lambda_j x_{ij} \geq 0, \quad j = 1, \dots, n$$

$$y_{rj_0} - \sum_{j=1}^n \lambda_j y_{rj} \leq 0 \quad (2.3b)$$

$$\lambda_j \geq 0$$

for all $i = 1, \dots, m$ and $r = 1, \dots, s$

Dual CCR output oriented

$$\text{Max } h = \theta_{j_0}$$

Subject to

$$\theta_{j_0} y_{rj_0} - \sum_{r=1}^s \lambda_j y_{rj} \leq 0, \quad j = 1, \dots, n \quad (2.4b)$$

$$x_{ij_0} - \sum_{i=1}^m \lambda_j x_{ij} \geq 0$$

$$\lambda_j \geq 0$$

for all $i = 1, \dots, m$ and $r = 1, \dots, s$

The proposed CCR models were developed to deal with CRS where the input consumed is proportional to the output produced. However, imperfect competition, government regulations, constraints on finance etc, may not result in constant returns to scale (CRS). This led various researchers, such as Afriat (1972), Fare Grosskopf and Logan (1983) and Banker, Charnes and Cooper (1984) to suggest adjusting the CRS DEA model to account for variable returns to scale (VRS). This type of model is known as BCC model and it changes the CCR model by adding the convexity constraint (i.e. $\sum_j \lambda_j = 1$).

Figure 2.2 illustrates a comparison of efficiencies for five DMUs (A, B, C, D and E) calculated using CCR input oriented (CRS, solid line) and BCC input oriented (VRS, dashed line).

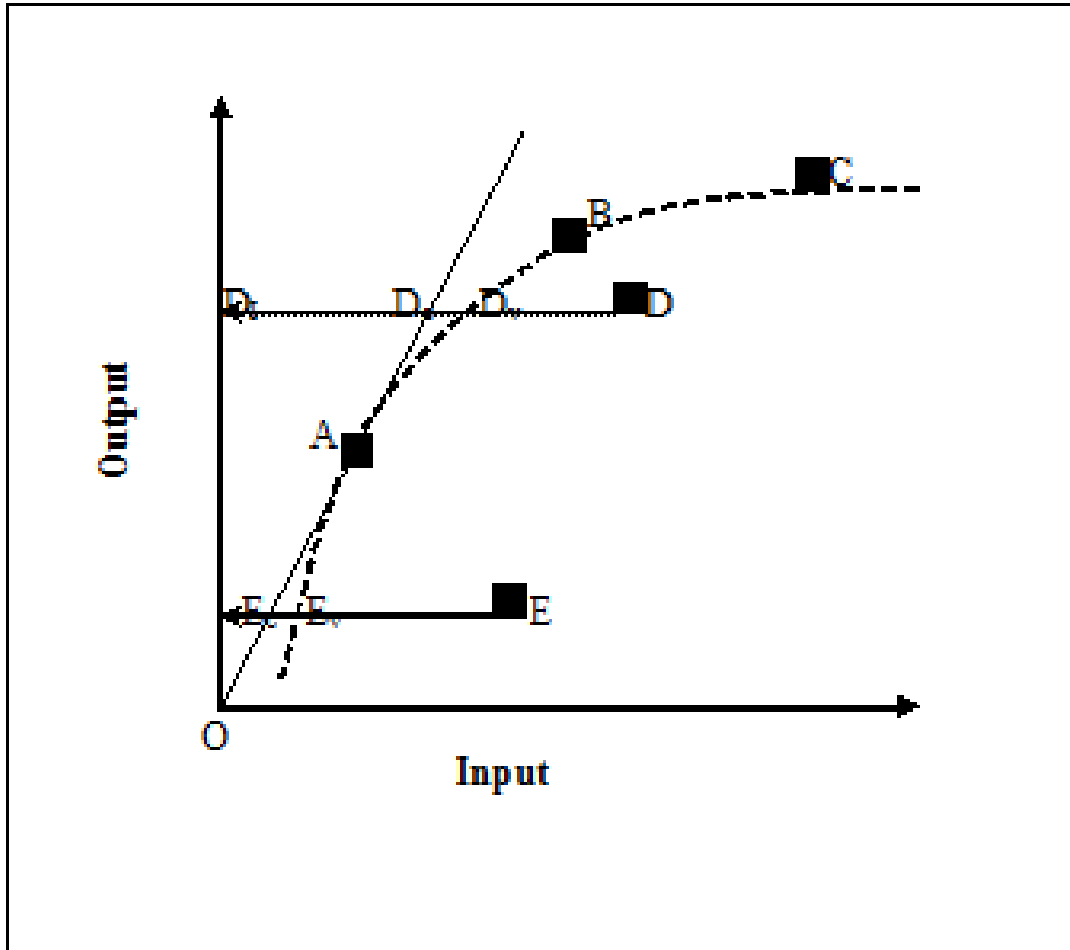


Figure 2.2 Computing the Efficiency using VRS (dashed line) and CRS (solid line).

Using the CCR model the only efficient DMU is A. However, using the BCC model DMUs A, B and C are efficient whilst D and E are inefficient.

The BCC mathematical formulation is written as:

Output oriented Primal form

$$\begin{aligned}
 \text{Min } h_0 &= \sum_{r=1}^s v_r x_{rj_0} \\
 \text{Subject to} \\
 \sum_{r=1}^s u_r y_{rj_0} &= 1 \\
 \sum_{i=1}^m v_i x_{ij} - \sum_{r=1}^s u_r y_{rj} - u_0 &\geq 0 \quad j = 1, \dots, n \\
 u_r, v_i &\geq \varepsilon
 \end{aligned} \tag{2.5a}$$

for all $r = 1, \dots, s$ and $i = 1, \dots, m$

Output oriented Dual form

$$\begin{aligned}
 \text{Max } h &= \theta_{j_0} \\
 \text{Subject to} \\
 \theta_{j_0} y_{rj_0} - \sum_{r=1}^s \lambda_j y_{rj} &\leq 0, \quad j = 1, \dots, n \\
 x_{ij_0} - \sum_{i=1}^m \lambda_j x_{ij} &\geq 0 \\
 \sum_{j=1}^n \lambda_j &= 1 \\
 \lambda_j &\geq 0
 \end{aligned} \tag{2.5b}$$

for all $i = 1, \dots, m$ and $r = 1, \dots, s$

The transformation from the CCR model (2.4b) to the BCC model (2.5b) is achieved by adding convexity constraints $\sum_j \lambda_j = 1$. This ensures that the targeted DMU will be scaled according to Ali et al. (1995). It is worth noting that, in 2.5a, u_0 plays the key role of expressing the returns to scale. If $u_0 = 0$ (or $\sum_j \lambda_j = 0$) then there are constant returns to scale (CRS), whilst $u_0 > 0$ has local decreasing returns to scale and $u_0 < 0$ has local increasing returns to scale

2.2.3 Extensions to the basic DEA model

In this section developments of the basic DEA model are considered. Initially the focus is on general developments and integrating DEA with value judgements followed by an intensive review of the trade-off approach, the bounded model and the geometric average approach.

Since the formulation of the original DEA model by Charnes, Cooper, and Rhodes (1978) many developments have been demonstrated. For example, additional control of weights may be needed especially if dealing with a large number of input and output variables and this may cause DEA to be less discriminating. Moreover, the decision maker may put restrictions on the weights based on his preference structure about the relative importance of different inputs and outputs.

Inputs and outputs are assumed to be discretionary (i.e. they are controllable) and can be set for decision-maker. However, in reality some inputs and outputs are considered to be exogenous and non-discretionary, (e.g. a bank's branches would not have absolute control over the numbers of deposits processed). Consequently, Charnes (1994) suggested categorising the input and the output variables into two subsets: discretionary and non-discretionary.

In practice DEA can be described as the projection of scores, generated from the ratio of multi-input to multi-output on the efficient frontier (envelopment) surface. Any inefficient DMU can be written as a linear combination of an efficient DMU. For example, in Figure 2.2 the inefficient DMU D can be written as $D = \lambda_1 A + \lambda_2 B$ by using BCC model. Thus, for any DMU₀ defined by input and output vectors (x_{ij_0}, y_{rj_0}) , the following transformation can be applied:

$$(x_{ij_0}, y_{rj_0}) \xrightarrow{\text{DEA dual transformation}} \left(x_{ij_0} = \sum_j \lambda_j x_{ij}, y_{rj_0} = \sum_j \lambda_j y_{rj} \right) \quad (2.6)$$

2.2.3.1 DEA and value judgements

The term value judgement was defined by Allen et al. (1997) as “logical structures, incorporated within an efficiency assessment study, reflecting the Decision Makers preferences in the process of assessing efficiency”. Hence, value judgements incorporated into DEA reflect prior views or information in efficiency assessments and have different implications for DEA outcomes. Moreover, it allows for identifying a relationship between certain inputs and outputs adhering to marginal rates of substitution, which can be used to support the resource allocation decision.

To ensure that results are consistent with the purpose of the decision making process value judgements may be needed to constrain weights. Weight restrictions permit DMs to determine the relative importance of inputs and outputs which helps to increase the power of DEA and give better efficiency estimates by incorporating advice from managers and experts. The incorporation of value judgements in DEA models has been extensively studied by several authors, such as Allen et al. (1997), Thanassoulis, Portela and Allen (2004). Weight restrictions have been introduced by imposing both upper and lower bounds on individual weights (Dyson and Thanassoulis ,1988); identifying an assurance region (Thompson et al, 1990); restricting composite inputs and outputs, weight ratios and proportions (Wong and Basely,1990); and through the cone ratio concept (Harness et al., 1990, 1994). Weight restrictions may provide a discriminatory framework, coupled with value judgements, by defining ranges of acceptable weights or assurance regions for selecting the preferred efficient DMU, and also ensure the efficiencies scores reflect the

performance of DMUs of all the input and output variables. Another issue to consider is the instability of the efficiency scores in weight restrictions when data are slightly perturbed; Scheel and Scholtes (1998) suggested identifying perturbations a priori to avoid instability in efficiency scores.

The incorporation of value judgement into DEA models can be divided into three categories: (a) restrictions implemented to control the weights of the input and output, i.e. *weight restrictions*, (b) transformation of the input and output to seek satisfaction of the value judgements, i.e. *cone ratio approach*, and (c) restrictions on virtual inputs and outputs.

In category (a) the types of weight restriction can be described by the generalised linear programme:

$$\begin{aligned}
& \text{Max } h_0 = \sum_{r=1}^s u_r y_{rj_0} \\
& \text{subject to} \\
& \sum_i^m v_i x_{ij_0} = C \\
& \sum_{r=1}^s u_r y_{ri} - \sum_{i=1}^m v_i x_{ij_0} \leq 0, \quad j = 1, \dots, N \\
& \kappa_i v_i + \kappa_{i+1} v_{i+1} \leq v_{i+2} \quad r1 \\
& \alpha_i \leq \frac{v_i}{v_{i+1}} \leq \beta_i \quad r2 \quad (2.7) \\
& \gamma_i v_i \geq u_r \quad r3 \\
& \delta_i \leq v_i \leq \tau_i \quad r4 \\
& \rho_r \leq u_r \leq \eta_r \quad r5 \\
& -v_i \leq -\varepsilon \quad i = 1, \dots, m \\
& -\mu_r \leq -\varepsilon \quad r = 1, \dots, s
\end{aligned}$$

Where u_r and v_i are the weights for inputs and outputs respectively; the Greek letters ($\kappa_i, \alpha_i, \beta_i, \gamma_i, \delta_i, \pi_i, \rho_r, \eta_r$) are constants and used to describe the relative importance of the input and output factors; the normalised constant, C (normally either 1 or 100), defines the upper limit on efficiency ratings and prevents the input and output weights being inflated or totally ignored; and the constraints labelled r_1 to r_5 relate to constraints introduced in the literature to restrict the input and output weights and cannot be used simultaneously as they depend on the prior knowledge of the DM.

Weight restrictions were first considered by Dyson and Thanassoulis (1988) by introducing constraint r_1 in formulation (2.7). They implemented regression analysis to determine lower bounds on the DEA weights in a single-input multiple-output DEA model. In 1990, Wong and Beasley proposed the constraint r_2 , to calculate the relative ordering of inputs and outputs as well as to reflect the marginal rate of substitution. The upper (β_i) and the lower (α_i) limits in constraint r_2 were set according to Pedraja-Chapparo et al., (1997).

In 1990, Thompson et al. developed the assurance region model characterised by constraints r_3 in formulation (2.7) where $\gamma_i v_i \geq u_r$ represents the relationship between weights of inputs and outputs (i.e. the weights of inputs and outputs were combined rather than separated as in the DEA basic model). This approach allows flexibility in changing the assurance region until it reaches a level of efficiency as viewed by the DM. Furthermore, Allen et al. (1997) noticed that the bound values of the assurance regions depend on the scaling of inputs and outputs and consequently constraints r_4 and r_5 were suggested. Whilst the assurance region concept adds flexibility and more precise controls in the efficiency measurements it can also be applied to relax tight conditions imposed by the allocative efficiency DEA model. The drawback of this approach is that it requires great deal of information to impose proper conditions on all inputs and outputs.

The cone ratio approach, category (b), was first demonstrated by Charnes et al. (1990) to incorporate preferences of DMs. The original data reflect a range of complex issues involving personal judgement and prior knowledge and are difficult to consider. Brockett et al., 1997 restricted the possible range for the virtual weights by which the original data can be transformed. The efficiency of DMUs is measured by comparison to a previously chosen DMU considered to be more efficient. It is worth noting that the cone ratio approach did not add constraints as in the assurance region approach

In 1992, Thanassoulis and Dyson reported a general structure of weight-based techniques. Their model set an ideal target from among the reference DMUs that dominate the DMU under observation. Moreover, the model allows for estimating the amount needed for the DMU to become efficient and permits the DM to integrate value judgements to satisfy his preferences in target setting by increasing output and decreasing inputs to achieve efficiency. The idea behind the weight-based general preference structure is to divide the DMUs into two groups such that one group contains those DMUs that need to be improved and the others are those that need to be kept without changing (i.e. maintained at their current level). Hence, an inefficient DMU can have a target set to improve their outcome by lowering inputs level and raising outputs level.

The model proposed by Zhu (1996), which is known as the weight-based preference structure DEA model, is able to reflect the relative degree of desirability by controlling inputs and outputs through the introduction of preference weights into the objective function of the non-radial CCR model. Zhu's model adds more flexibility in changing the inputs and outputs compared to the model of Thanassoulis and Dyson (1992) which only allowed for changing inputs for a fixed output. The most important feature of the Zhu model is its ability to integrate preference structures to achieve efficiency targets and to produce an efficiency score of individual DMU by changing both inputs and outputs without restrictions. By using this model, uncontrollable inputs and/or outputs can be regulated, the breakdown of efficiency in terms of allocation and technical efficiency can be measured, and the number of relatively efficient DMUs can be reduced depending on the choice of preference weights.

Athanassopoulos (1995) combined goal programming and DEA (GoDEA) to overcome the shortcomings of the Thanassoulis and Dyson (1992) model because their model deals with DMUs individually. The GoDEA model considers global targets and global resources of organisations to provide solutions to identify which level of inputs and outputs can be improved. The GoDEA model sets a global target such that DMU improves their outcome. Hence, this model applies the concept of equity, efficiency and effectiveness that reflects the satisfaction of the main goal of global achievements or effectiveness of the system by increasing the contribution of each DMU to the main goal. Another essential feature of GoDEA is that it does not need prior information about each DMU because it assumes some information about the ideal input and output target predefined by DM. GoDEA

minimises the objective function deviational variables used in constraint sets individually and globally. Unlike basic DEA, GoDEA considers the incorporation of global targets for controllable inputs and outputs. Furthermore, it is able to solve both target and resource allocation based problems by calculating the interactions between DMUs and incorporating the DM's preferences as well as selecting the efficient technologies for each DMU in order to achieve the global target levels.

GoDEA model was developed to measure global targets and the target-based resource allocation (TABRA) approach was developed to adopt a principle-agent paradigm such that the principal is the central managements and the agent is the DMU management. TABRA focuses on the trade-offs between the objectives of individual DMUs in term of marginal rates of substituting between inputs and outputs by setting global performance targets (the effectiveness) and the allocation of resources (the equity) which depend on prerequisite of each DMU. The method seeks for the most preferred combination of inputs and outputs that maximise the accomplishment of all goals whereas the objective function is to minimise the goal deviational variables assigned by the explicit representation of the trade-offs between effectiveness, efficiency and quality.

Further improvement in DEA was reported by Thanassoulis and Simpson (2000). In their model, unobserved DMUs, which are derived from Pareto-efficient observed DMUs, are used to integrate value judgements. It has been achieved by combining a set of unobserved DMUs with a set of observed DMUs through which efficiencies of the observed DMUs can be replicated by carefully chosen unobserved DMUs. In fact, the basic DEA model is used to identify the efficient and non-enveloped DMUs. From any non-enveloped DMUs, anchor DMUs are determined by the concept of super-efficiency in which outputs of individual DMU are adjusted to build the unobserved DMUs, together with DM's value judgement. The unobserved are then constructed by setting the lowest permissible level for each output of each anchor DMU, which in turn produces a positive slack value at its optimal solution. In essence, unobserved DMUs use modifications of observed DMUs as additional DMUs to augment the comparative set in a manner that reflects value judgements. Nevertheless it is not conclusive to say that unobserved DMU approach is a better approach, it provides an alternative method to that of weight restrictions. The unobserved DMUs might not be user-friendly, but unobserved DMU offer more flexibility to identify trade-offs between inputs and outputs, and target setting under the unobserved

DMU is deemed more significant since the observed input-output mix of Pareto-inefficient DMU concerned is mainly preserved.

Although the above models can provide better discrimination between efficient and inefficient DMUs, they cannot be interpreted as a realistic improvement (Allen et al, 1997, Tanassoulis and Allen, 1998). For example, most of the mentioned models design the weight restrictions according to the marginal importance of inputs and outputs, whilst some other models include monetary considerations which reflect input costs and output prices. In these models the composite sum of inefficient DMUs is not generally technologically feasible (producible). To overcome this drawback Podinoviski (2004) developed a model called the trade-off approach, which incorporates production trade-offs into the dual or primal DEA model. Two ways were proposed to incorporate the production trade-offs: first, by adding terms to the dual to modify the composite DMUs and second, by using weight restrictions in the primal. Although the proposed approach may expand the production possibility set the radial targets of inefficient DMUs are always producible. Furthermore, the efficiency estimates retain their traditional meaning as the realistic ultimate radial improvement factors. However, the trade-off approach fails to clearly address how the DMs preferences are involved; it does not provide alternative objectives to guide the DM to the most preferred solution; *a priori* knowledge is required and solving the models is complicated (Podinoviski, 2007). To overcome these drawbacks an interactive framework has been developed to integrate the trade-off approach with MOLP. The concepts of the trade-off approach and MOLP are discussed in more detail below to lay the groundwork for the interactive framework presented in subsequent chapters.

2.2.3.2 The production trade-off DEA model

In the trade-off approach value judgements are incorporated explicitly by externally adding P and Q vectors, for input and output respectively, into the constraints of the basic DEA dual programme. These two vectors are specified to handle the trade-off contributions, i.e. value judgements, and are defined by the DM *a priori* based on expertise or survey. By this modification, the basic dual formulation DEA programme presented in section 2.2.2 can be expressed as:

Output oriented dual formulation

$$\text{Max } h = \theta_{j_0}$$

Subject to

$$\begin{aligned} \sum_{j=1}^n \lambda_j y_{rj} + \sum_{t=1}^K \pi_t Q_t &\geq \theta_{j_0} y_{rj_0}, & r = 1, \dots, s; t = 1, \dots, K \\ \sum_{j=1}^n \lambda_j x_{ij} + \sum_{t=1}^K \pi_t P_t &\leq x_{ij_0}, & i = 1, \dots, m; \lambda_j \geq 0 \text{ for all } j \\ \theta_{j_0} &\geq 1 \\ \lambda_j, \pi_t &\geq 0 \end{aligned} \tag{2.8a}$$

Input oriented dual formulation

$$\text{Min } h = \theta_{j_0}$$

Subject to

$$\begin{aligned} \sum_{j=1}^n \lambda_j y_{rj} + \sum_{t=1}^K \pi_t Q_t &\geq y_{rj_0}, & r = 1, \dots, s; t = 1, \dots, K \\ \sum_{j=1}^n \lambda_j x_{ij} + \sum_{t=1}^K \pi_t P_t &\leq \theta_{j_0} x_{ij_0}, & i = 1, \dots, m; \lambda_j \geq 0 \text{ for all } j \\ \theta_{j_0} &\geq 1 \\ \lambda_j, \pi_t &\geq 0 \end{aligned} \tag{2.9a}$$

Where K is the number of production trade-off judgements made by the DM with the index t denoting each specific judgement. $P_t \in R^m$ and $Q_t \in R^s$ are the two trade-off vectors of

inputs and outputs respectively, which translate the perceived importance between inputs and output variables into relations between the corresponding weights. For example, the higher weights join more expensive inputs and outputs with higher prices. To understand the meaning of P and Q trade-off vectors, consider the primal formulations for both the input and output oriented models.

Output oriented primal formulation

$$\begin{aligned}
 \text{Min } h_0 &= \sum_{i=1}^m v_i x_{ij_0} \\
 \text{Subject to} \\
 &\sum_r^s u_r y_{rj_0} = 1 \\
 &\sum_{r=1}^s u_r y_{rj} - \sum_{i=1}^m v_i x_{ij} \leq 0 \\
 &\sum_{t=1}^K u_t Q_t - \sum_{t=1}^K v_t P_t \leq 0 \\
 &u_r, v_i \geq 0
 \end{aligned} \tag{2.8b}$$

Input oriented primal formulation

$$\begin{aligned}
 \text{Max } h_0 &= \sum_{r=1}^s u_r y_{rj_0} \\
 \text{Subject to} \\
 &\sum_r^s v_i x_{ij_0} = 1 \\
 &\sum_{r=1}^s u_r y_{rj} - \sum_{i=1}^m v_i x_{ij} \leq 0 \\
 &\sum_{t=1}^K u_t Q_t - \sum_{t=1}^K v_t P_t \leq 0 \\
 &u_r, v_i \geq 0
 \end{aligned} \tag{2.9b}$$

Note that both primal programmes (i.e. input-oriented and output-oriented) share the same additional homogeneous weight restriction constraint. This indicates that the additional weight restriction constraint can be used in the both input-output oriented without any modifications, unlike the non-homogeneous weight restriction constraints such as weight bounds. There is an important feature that needs to be considered in using this model. P and Q vectors should be changed within the range by which the production possibility set maximum can be expanded (i.e. maximum enlargement of the area under the origin efficient frontier for P and Q equals zero). The new generated efficient frontier (the expanded one) should reflect realistic situations in which the radial targets of inefficient DMUs are always producible and the efficiency measure retains its traditional meaning as the realistic ultimate radial improvement factor. Although expansion under the area of efficient frontier has taken place as a consequence of adding P and Q vectors, the inefficient DMUs can be projected on the new (realistic) efficient frontier by following a radial path. P and Q vectors should be chosen with caution because the transformation of (\bar{x}, \bar{y}) into $(\bar{x}\lambda + \sum \pi P, \bar{y}\lambda + \sum \pi Q)$ must belong to R^+ (i.e. satisfy the *Closedness* axiom see Podinovski, 2004). For instance, if vectors of input and output are $\bar{x} = (2,0,0)$ and $\bar{y} = (1)$ respectively then P and Q vectors cannot be chosen such that the transformation of $(\bar{x}\lambda + \sum \pi P, \bar{y}\lambda + \sum \pi Q)$ becomes negative, e.g. ($P = (-1,-1,-1)$). This causes violation of the *Closedness* property because the transformation to (\bar{x}, \bar{y}) becomes negative (i.e. equals $(1,-1,-1)$). Hence, this transformation should be applied to \bar{x} vector of the form $(2, \varepsilon_1, \varepsilon_2)$ where ε_1 and ε_2 must be > 0 . Another important feature of this model is that it adds more flexibility in controlling inputs and outputs as one can vary input and/or output data to achieve the DMs preferences. This approach will be extended in the next chapter by producing an equivalent MOLP model to establish an interactive framework based on the gradient projection approach.

To illustrate the above discussion, consider the following example (Podinovski, 2004) of three similar hypothetical university departments (Table 2.1).

Department	No. of staff	No. of undergraduate students	No. of Masters students
1	100	700	700
2	100	1200	600
3	100	1300	250

Table 2.1 Hypothetical university data

Let the following judgements be agreed on by the three departments:

(J₁) No extra resources can be claimed if the number of master students is reduced by one and the undergraduate students are increased by one, i.e. $P_1 = \mathbf{(0)}, Q_1 = (1, -1)^T$

(J₂) No extra resources can be claimed if the number of master students is increased by one and the undergraduate students are reduced by two, i.e. $P_2 = \mathbf{(0)}, Q_2 = (-2, 1)^T$

(J₃) In order to increase the undergraduate students by five, it is sufficient to increase the number of staff by one, i.e. $P_3 = \mathbf{(1)}, Q_3 = (5, 0)^T$

For the output-orientation CRS model, the above judgements can be incorporated in both the dual and primal formulation as shown below:

Dual formulation

$$\text{Max } h = \theta_{j_0}$$

subject to

$$700\lambda_1 + 1200\lambda_2 + 1300\lambda_3 + 1\pi_1 - 2\pi_2 + 5\pi_3 \geq 700\theta_{j_0} \tag{2.9a}$$

$$700\lambda_1 + 600\lambda_2 + 250\lambda_3 - 1\pi_1 + 1\pi_2 + 0\pi_3 \geq 700\theta_{j_0}$$

$$100\lambda_1 + 100\lambda_2 + 100\lambda_3 + 0\pi_1 + 0\pi_2 + 1\pi_3 \geq 100$$

$$\lambda_1, \lambda_2, \lambda_3, \pi_1, \pi_2, \pi_3 \geq 0, \theta_{j_0} \geq 1$$

In the above formulation, the left-hand side of the constraints contain the decision variables $(\lambda_1, \lambda_2, \lambda_3)$ which represent a composite DMU feasible in CRS whilst (π_1, π_2, π_3) modifies this composite unit according to the DM preferences (i.e. judgements) J₁, J₂ and J₃. It is worth recalling that π belongs to the same feasible decision space as λ and its action is to compensate for the extension on the space due to the addition of the P and Q vectors

Primal formulation

$$\begin{aligned} \text{Min } h_0 &= 100v_1 \\ \text{subject to} \\ 700u_1 + 700u_2 &= 1 \\ 100v_1 - 700u_1 - 700u_2 &\geq 0 \\ 100v_1 - 1200u_1 - 600u_2 &\geq 0 \\ 100v_1 - 1300u_1 - 250u_2 &\geq 0 \\ u_1 - u_2 &\leq 0 \\ -2u_1 + u_2 &\leq 0 \\ 5u_1 - v_1 &\leq 0 \\ u_1, u_2, u_3, v_1 &\geq 0 \end{aligned} \tag{2.9b}$$

In this Primal formulation of model 2.9a the first four constraints are basic DEA requirements whilst the next three inequalities are as a consequence of adding the DM preferences (judgements), J_1 , J_2 , and J_3 .

2.2.3.3 The bounded DEA model

DEA models have taken the optimistic point of view in which the relative efficiencies reflect the best relative efficiencies or optimistic efficiencies. The optimum solution will be between 1 and 0, that is the efficient DMUs have score of 1 and the inefficient DMUs have score of less than 1 but greater than 0. In this case the DMUs are classified into two groups: optimistic non-efficient and optimistic efficient. Efficiency assessment can also be viewed from a pessimistic prospective. Thus the efficiency assessment will reflect worst relative efficiency and will have worst DMUs and non-worst DMUs. The efficiency scores from this point of view will be greater than 1. Contrary to the best efficient optimistic DMUs that form a convex efficient frontier, the worst relative efficiency pessimistic DMUs form a concave inefficient frontier.

A few studies have considered dealing with assessment of DMU efficiencies from both an optimistic and pessimistic point of view. Entani et al (2002) propose upper and lower limits to overcome the difficulties of combining the best and worst relative efficiencies because they are incomparable. Doyle et al (1995) proposed three pairs of upper and lower

bound evaluation models. These mathematical formulations are not considered any further in this review.

More recently in 2007, Wang Y-M and Yang J-B developed a model that is able to integrate the views of both optimists and pessimists in one model, which they called the *bounded DEA model*. The problems arising from incomparability between optimistic and pessimistic views were overcome by introducing a virtual anti-ideal DMU (ADMU). Conceptually, the ADMU is defined to be a DMU that consumes most inputs (x^{\max}) only to produce least outputs (y^{\min}). Although ADMU is a virtual DMU, it may reflect real practical situations because a waste of resources is allowed for in the theory of production. The efficiency can be found by solving the following LP.

$$\begin{aligned}
 \text{Max } \theta_{ADMU} &= \sum_{r=1}^s u_r y_r^{\min} \\
 \text{subject to} \\
 \sum_{r=1}^s u_r y_{rj} - \sum_{i=1}^m v_i x_{ij} &\leq 0 \quad j = 1, \dots, n \\
 \sum_{i=1}^m v_i x_i^{\max} &= 1 \\
 u_r, v_i &\geq \varepsilon \quad r = 1, \dots, s; \quad i = 1, \dots, m
 \end{aligned} \tag{2.10}$$

where ε is the non-Archimedean infinitesimal.

By using the anti-ideal DMU, which allows for estimating the efficiencies of DMUs in the interval $[\theta_{ADMU}, 1]$, the following pair of fractional, optimistic (Max) and pessimistic (Min), LP DEA models can be written:

$$\text{Max(Min)} \quad h_0 = \frac{\sum_{r=1}^s u_r y_{r0}}{\sum_{i=1}^m v_i x_{i0}}$$

subject to

$$\theta_{ADMU} \leq \frac{\sum_{r=1}^s u_r y_{rj}}{\sum_{i=1}^m v_i x_{ij}} \leq 1 \quad j = 1, \dots, n \quad (2.11a)$$

$$u_r, v_i \geq \varepsilon \quad r = 1, \dots, s; \quad i = 1, \dots, m$$

which are equivalent to:

$$\text{Max(Min)} \quad h_0 = \sum_{r=1}^s u_r y_{rj_0}$$

subject to

$$\sum_r u_r y_{rj} - \sum_{i=1}^m v_i x_{ij} \leq 0, \quad j = 1, \dots, n \quad (2.11b)$$

$$\sum_{r=1}^s u_r y_{rj} - \sum_{i=1}^m v_i (\theta_{ADMU} x_{ij}) \geq 0, \quad j = 1, \dots, n$$

$$\sum_{i=1}^m v_i x_{ij_0} = 1$$

$$u_r, v_i \geq \varepsilon \quad r = 1, \dots, s; \quad i = 1, \dots, m$$

One approach to finding pessimistic and optimistic DMUs is to identify the worst DMU among the others, allowing DM to take most suitable decision so that he can see the action of each DEA from different point of view. An equivalent MOLP model based on minimax formulation will be presented later. To measure the worst and most efficient DMU, the projection iterative technique is introduced below.

2.2.3.4 The Geometric Average Approach

Measuring DMU activity from different points of view (i.e. optimistic and pessimistic) offers a complete view of the DMU that enables DMs to take appropriate decisions. It is essential for the DM to identify the overall activities of individual DMUs from both extreme views. The Geometric Average Approach (GAA) calculates the overall performance of each DMU much easier than the previous models of Doyle et al (1995) and Entani et al (2002). In practise, optimistic and pessimistic efficiencies can be used to form an interval with lower bound (pessimistic) and upper bound (optimistic). This can be achieved by using an adjustment coefficient as reported by Wang Y.M. et al. (2007). The adjustment coefficient is restricted to be between zero and one and has the ability to adjust the pessimistic efficiency which in this case can be written as: $\tilde{\psi}_j^* = \alpha \psi_j^*$ where: α is the adjustment coefficient, ψ_j^* is the pessimistic DMU and $\tilde{\psi}_j^*$ is the adjusted pessimistic DMU. To form the interval the adjusted pessimistic, DMU efficiency should fulfil the following equality $\tilde{\psi}_j^* = \alpha \psi_j^* < \theta_j^*$ where θ_j^* is the optimistic efficiency and it runs over j^{th} DMUs such that $\alpha < \min_{j \in \{1, \dots, n\}} \left\{ \frac{\theta_j^*}{\psi_j^*} \right\}$. The interval can be written as $\left[\alpha \psi_j^*, \theta_j^* \right]$ ($j = 1, \dots, n$). To simplify the comparison of the interval numbers Wang et al (2007) proposed the comparison of their midpoints calculated by using arithmetic average $\frac{(\alpha \psi_j^* + \theta_j^*)}{2}$. However, finding α is problematic but this parameter plays an important role in rankings of DMUs efficiency scores. To overcome this problem, a geometric average is suggested in which α can be dismissed because the geometric average can be expressed as $\sqrt{\alpha \psi_j^* \theta_j^*} = \sqrt{\alpha} \sqrt{\psi_j^* \theta_j^*}$ so eliminating $\sqrt{\alpha}$ from the geometric average will not affect the rankings among the DMUs.

The DEA model to measure the overall efficiency score based on the geometric average can be written as (Wang et al, 2007):

$$\begin{aligned} & \text{Max(Min)} \left\{ \frac{\sum_r^s u_r y_{rj_0}}{\sum_i^m v_i x_{ij_0}} \right\} \\ & \text{subject to} \end{aligned} \tag{2.12}$$

$$\alpha \leq \frac{\sum_r^s u_r y_{rj_0}}{\sum_i^m v_i x_{ij_0}} \leq 1, \quad j = 1, \dots, n$$

$$u_r, v_i \geq \varepsilon, \quad r = 1, \dots, s; \quad i = 1, \dots, m$$

The above programme can be converted to output oriented and can be written as

$$\begin{aligned} & \text{Min(Max)} \sum_{i=1}^m v_i x_{ij_0} \\ & \text{subject to} \end{aligned}$$

$$\begin{aligned} & \sum_{r=1}^s u_r y_{rj_0} = 1 \\ & \sum_{r=1}^s u_r y_{rj} - \alpha \sum_{i=1}^m v_i x_{ij} \geq 0, \quad j = 1, \dots, n \\ & -\sum_{r=1}^s u_r y_{rj} + \sum_{i=1}^m v_i x_{ij} \geq 0 \\ & u_r, v_i \geq \varepsilon \end{aligned} \tag{2.13}$$

The purpose of converting the geometric average model to the above formulation is to derive an equivalent MOLP model (see the next section) that allows for estimating the most preferred overall efficiency score for each DMU.

2.2.3.5 Ranking DMUs in DEA

Lack of discrimination in DEA applications has led researchers to seek methods to rank DMUs. Furthermore, DMs are interested in ranking because such information can be used to identify best practice DMUs against which others can be compared. The integration of ranking DMUs in DEA has been studied by several researchers. A method of ranking was initially proposed by Charnes et al., (1985a) in which the DMU benchmark was identified by counting the number of times that an efficient DMU becomes the benchmarking unit for others. However, finding the reference set is not an easy task. Another approach was suggested by Charnes et al., (1962) by changing the rate of outputs and evaluating the change of efficiency score but this was never clarified. In 1986, Sexton et al proposed the cross efficiency method which uses the weights obtained from solving n-linear problems with the DMUs' evaluated efficiency being stored in a matrix. Each row of this matrix contains the cross efficiency score of DMUs and the average of the matrix rows is computed as a ranking measure. The outcome of this method is that DEA models may sometimes have more than one solution. Another technique was proposed by Thompson et al (1992) based on assurance regions by which the number of efficient DMUs may be decreased but finding suitable weight is not easy. In 2001, Adler et al suggested a ranking approach based on decreasing the number of inputs and outputs by utilizing component analysis but this method failed to provide complete ranking whilst in 2007 Wang et al utilized the geometric average for ranking DMUs.

2.2.4 Advantages and disadvantages of DEA

Advantages

DEA is a framework analysis which accommodates a comprehensive view of organisational performance. It is an appropriate tool for assessing performances of an organisation or firm. This is partially due to the fact that a multitude of subjective factors affect the quality and productivity of a service that needs to be well managed. Although there is no clear relationship between inputs (consumed) and outputs (produced) in DEA Charnes et al (1994) identified three extremely useful features:

- Each DMU can be characterized individually.
- Inefficient DMUs are improved by projecting them on the efficient frontier (envelopment).
- It facilitates making inferences for each DMU among the inferences on the DMUs' general profile.

Charnes et al (1994) also reported a complementary list of other advantages of DEA:

- Multiple-output and multiple-input can be handled in various DMUs measurements.
- A focus on a best-practice frontier, instead of on central-tendencies, i.e. every DMU is compared to an efficient unit or a combination of efficient units. The comparison, therefore, leads to sources of inefficiency of DMUs that do not belong to the frontier.
- No restrictions are imposed on the functional form relating inputs to outputs.

Disadvantages

Although traditional DEA models are considered a powerful tool for efficiency assessment, many limitations have been identified:

- As DEA is an extreme point technique, it is very sensitive to noise (even for symmetrical noise with a zero mean) that may cause significant errors in efficiency measurements.
- Statistical hypothesis tests are difficult because DEA is a non-parametric.
- The standard formulation of DEA is based on separate linear programmes for each DMU, which is computationally demanding.
- Basic DEA suffer from other limitations as reported by Athanassoulis (1993):
 - * Difficulties in aggregating different aspects of efficiency especially whenever DMUs perform multiple activities.
 - * Insensitivity to intangible and categorical components (e.g. service quality in a bank branches).
- There are crucial problems related to mixing multiple dimensions in the analyses. For instance, consider a DMU performing two different activities; the DMU could be found efficient in the first activity but inefficient in the second. For example, a bank's branches are a single platform that management uses to sell financial services to customers as well as providing more traditional banking services such as processing deposits or loan. Furthermore, it is difficult to simultaneously assess the sales efficiency and the service efficiency of the branch. Because the relevant inputs and outputs for individual activities are not directly comparable, the analyst would have to run two DEA models one for sales and the other for services. DEA is intended for estimating the relative efficiency of a DMU but not specifically addressing absolute efficiency. In other words, it measures how well the DMU is

behaving compared to its peers (the set of efficient units) but not compared to a theoretical maximum. Therefore, two major problems arise:

- * It is impossible to rank efficient units absolutely because all DMUs located on the frontier surface have 100% efficient score.
 - * From managerial point of view, it may be more useful to compare DMUs to a frontier of absolute best performance. So the analyst would be able to better detect, for example, a DMUs network's true inefficiency. In fact, one might argue that efficient units may not be efficient enough, so the created frontier does not reflect the real potential of the DMU network.
-
- There is no specifically robust methodology for evaluating or testing the appropriateness of a set of factors in an efficiency study. A DEA model can indicate how efficient a specific DMU is out of a given set of factors, and what its efficiency score is. It does not indicate, however, whether the chosen factors can provide the right efficiency.

2.3 Multiple Criteria Decision Making approaches

In the real world, many objectives need to be addressed to control a certain system and to make a firm or organisation follow an optimum direction. However, sometimes these objectives may be conflicting. This can lead to trade-offs between objectives and has resulted in multiple objective optimisation techniques being used as a practical tool to control such systems and direct firms and organisations in an optimum direction

Different techniques of Multi-Criteria Decision Methods (MCDM) have been established. These techniques often integrate common methodologies, for instance, a utility function includes a set of judgement criteria, a set of decision variables, and a means of selecting alternatives or solutions.

Analytical Hierarchy Process (AHP) and multiple attribute utility theory were demonstrated by Saaty (1980). Methods used for solving MCDM problems were reviewed by Hwang and Yoon (1981). However, Stewart (1992) argued that these methods were unsatisfactory because they failed to provide some degree of theoretical and/or empirical justification. Stewart (1992) and separately Dyer et al (1992) developed new methods that could produce consistent and rational results, capable of dealing with uncertainties and providing transparency to the analysis process. More recently, Yang and Singh (1994) and Yang and Sen (1997) reported a method, called Evidential Reasoning (ER), which considers both qualitative and quantitative data, can deal with missing values, random numbers and subjective judgements with uncertainty.

Multiple Objective Linear Programming (MOLP) deals with well defined objective functions under certain constraints. MOLP problems are usually solved by maximising or minimising objective functions within a feasible set such as finding the optimum solution of the utility function under certain constraints with feeding information from the Decision Maker (DM). While MOLP does not allow for choosing the alternatives it assigns a target alternative as the essential alternative to be satisfied. This leads to interactive procedures facilitating trade-off analysis.

2.3.1 Multiple Objective Linear Programming (MOLP)

Generally a multiple objective optimisation problem symbolising various plans and ambitions of the DM can be written as:

$$\begin{aligned} & \text{Max}[f_1(\lambda), f_2(\lambda), \dots, f_i(\lambda), \dots, f_s(\lambda)] \\ & \text{Subject to} \\ & \quad \lambda \in \Omega \\ & \text{where} \\ & \quad \Omega = \{\lambda / g_l(\lambda) \leq 0, h_m(\lambda) = 0; l = 1, \dots, L, m = 1, \dots, M\} \end{aligned} \tag{2.13}$$

Where:

λ is an n -dimensional column vector with elements being decision variables,

$f(\lambda)$ are continuous and differentiable objective functions,

$g(\lambda)$ and $h(\lambda)$ are continuously differentiable equality and inequality constraints respectively.

The $f(\lambda)$ are conflicting objective functions and there are many solutions that could optimise $f(\lambda)$ simultaneously. Conceptually, a feasible solution λ^* is said to be efficient if there does not exist any $\lambda \in \Omega$ ($\lambda \neq \lambda^*$), such that $f(\lambda) \leq f(\lambda^*)$ and $f(\lambda) \neq f(\lambda^*)$, where f_i ($i = 1, 2, \dots, s$) are assumed to be minimised. Furthermore, a feasible solution ξ^* is said to be weakly efficient if there does not exist any $\lambda \in \Omega$ ($\lambda \neq \lambda^*$), such that $f(\lambda) < f(\lambda^*)$, where f_i ($i = 1, 2, \dots, s$) are assumed to be minimised. Conventionally, an efficient solution refers to a non-dominated solution or non-inferior or Pareto optimal solution.

One of the essential issues that needed to be incorporated with MCDM problems was to design a framework to allow DMs to interfere directly in deciding their course of action and provide a set of available alternatives. Keeney and Raiffa (1993) suggested that the interactions of subjective values and trade-offs could lead a suitable solution for complex decision making problems with uncertainties as described below.

2.3.2 Interactive MOLP Methods

This section was mainly adapted from the reference (Multiobjective optimisation and control, G.P. Liu, J.B. Yang, and J.F. Whidborne, 2003)

Generally, interactive methods are implemented whenever *a priori* knowledge about decision problem is not available. The DM is allowed to search for different solutions along the efficient frontier composed of all efficient solutions, in order to obtain the most preferred solution (MPS) that maximises the DM's explicit or implicit utility function. Decision makers provide their vision about the weighted objectives and the alternatives. At each interaction, the current solution is adapted to the structure of preferences of the DM. This interactive process is designed to drive the DM towards the MPS, or at least to a good solution, which is acceptable by the DM. For this reason, interactive methods are powerful tools for solving multiple objective optimisation problems.

The interactive methods used for solving MCDM problems can be classified, according to the methods used in providing preference information by the DM, into three categories: local trade-off methods, efficient generation methods, and Goal or reference point methods. Trade-off methods include: the GDF algorithm (Geoffrion Dyer and Feinberg; 1972), Interactive Goal Programming (IGP) (Dyer; 1972), Sequential Proxy Optimisation approach (SPOT) (Sakawa; 1982), Interactive Surrogate Worth Trade-off (ISWT) (Chankong and Haimes; 1982), and Gradient Projection (Yang; 1999). The efficient generation methods include: Zionts-Walleenius's method (Zionts and Walleenius; 1967) and the Tchebyshev method (Steuer and Choo; 1983). In these methods the DM needs to assess the most suitable solution among alternatives. Finally, in goal or reference point methods the DM provides a target value or reference level to be achieved as in the Step Method (STEM) (Banayoun et al; 1971); Wierzbicki's (Wierzbicki; 1980); Satisficing Trade-off Method (STOM) (Nakayama and Sawaragi; 1984); Visual Interactive Approach (VIA) (Korhonen and Laakso; 1986); LBS method (Jaszkiewicz and Slowinski; 1994), and Guess method (Buchanan; 1997).

In the GDF algorithm developed by Geoffrion et al. (1972) the utility function (which is differentiable, continuous, concave, and increasing) comprises objective functions that are

concave and a set of convex decision spaces. The underlying algorithm of this approach is based on that of Frank and Wolf (1956) in which the optimum solutions of a utility function are reached by calculating the gradient of this function that increases its direction. The optimising process moves sequentially from one objective to another under the condition of ascending step length. Hence, the first round of this process will produce a solution that is used for the second round and so on. Furthermore, at each round the DM provides local trade-off information on objectives or local weights. Mathematically, if each round of the iterative process is h , the solution of the previous weights of individual object x^{h-1} are determined as $w^h = (w_1^h, w_2^h, w_3^h, \dots, w_k^h)$ in interactive method with the DM. The optimal solution y^h should follow an ascending direction of the relevant utility function which can be derived from:

$$\begin{aligned} \text{Max} \quad & \left(\sum_{i=1}^k w_i^h \nabla_x f_i(x^{h-1}) \right) \cdot y \\ \text{Subject to} \quad & \\ & y \in \Omega \end{aligned} \tag{2.14}$$

Although the GDF method and Zionts-Wallenius's method shared the same mathematical formulation in Zionts-Wallenius's method the multicriteria simplex technique is used to reach the optimum solution. At each iteration all non-basic efficient variables for the current solution are provided and in each step the DM must provide his opinion and judgements about various calculated solutions.

A method that depends on implicit trade-offs was proposed by Steuer and Choo (1983). This method mainly benefited from Tchebycheff approach in which DM must select one possible solution from different solutions produced at each round of iterative process. For instance, the solution for each iterative process is used to project the ideal vector to the current criterion vector such that a new weights interval is generated and the DM chooses one solution from various solutions. In case the DM disagrees with any of obtained solutions another set of solution can be offered by the next iteration but at each step of the iterative process the weight's interval become smaller. The idea of this method is to minimize the weight's interval in each step of iterative process whereby:

$$\begin{aligned}
& \text{Min} \quad \left(\alpha + \rho \sum (f_i^* - f_i(x)) \right) \\
& \text{Subject to} \\
& w_i (f_i^* - f_i(x)) \leq \alpha \quad i = 1, \dots, k \\
& x \in \Omega \quad \alpha \geq 0
\end{aligned} \tag{2.15}$$

Another method that can be categorised as an implicit trade-off method is the STEM method by Benayoun et al. (1971). This method relies on minimising the Tchebycheff distance towards an ideal solution in the criterion space. The parameters of the distance formula and the feasible space can be changed by a normalised weighting method depending on the DM preferences in the early solution. This method allows the DM to monitor the direction of the optimum. Mathematically the STEM method can be expressed as:

$$\begin{aligned}
& \text{Min}_{x, \alpha} \quad \alpha \\
& \text{Subject to} \\
& w_i (f_i^* - f_i(x)) \leq \alpha \quad i \in \{1, \dots, k\} - J^h \\
& f_j(x) \geq f_j(x^{h-1}) - \Delta f_j^h \quad j \in J^h \\
& f_i(x) \geq f_i(x^{h-1}) \quad i \in \{1, \dots, k\} - J^h \\
& x \in \Omega \quad \alpha \geq 0
\end{aligned} \tag{2.16}$$

Where $f_i^* = \max_{x \in \Omega} f_i(x)$ $i = 1, \dots, k$ are considered to be the ideal value generated from maximising the objective function individually.

There are group of methods based on a pre-defined reference point in which the optimum solution is to reach this reference point such as that proposed by Wierzbicki (1977) Solution are divided into two types: efficient and weakly efficient and Wierzbicki (1980) reported that an achievement function provides technical support to the reference point method for producing an efficient solution, while the trade-off method allows the variation of the achievement function to assess the reference point. Basically, this method is based

on an iterative process in which an achievements function is minimised through the reference point defined by the DM. A set of reference points can be found such that each point differs from the original reference points by some component. This technique allows the DM to allocate one of the solutions on the efficient frontier obtained by minimising the achievement function. Although the function's weights are fixed during this process the reference point may be varied according to the views of the DM. The weights associated with the reference points are either supplied by the DM or can be set arbitrarily using a normalising factor.

Mathematically, the achievement function can be expressed as:

$$s(\bar{q}, f(x), w) = \max_{i=1, \dots, k} \{w_i (\bar{q}_i - f_i(x))\} + \rho \sum_{i=1}^k (\bar{q}_i - f_i(x)) \quad (2.17)$$

where:

$w = (w_1, \dots, w_k)$ is the weights vector,

$\bar{q} = (\bar{q}_1, \dots, \bar{q}_k)$ is the reference point

ρ is a small positive value.

Along with an addressed solution (x^{h-1}, f^{h-1}) , the DM must provide another reference point q^{-h} . Using this provided point the criterion vector can then be written as $f^{h,0}$ at which the achievement function is minimised.

The formulation can be written as:

$$\begin{aligned} \text{Min}_{x \in \Omega} s(q^{-h}, f(x), w) \text{ which is similar to } & \left\{ \alpha + \rho \sum_{i=1}^k (q_i^{-h} - f_i(x)) \right\} \\ \text{Subject to} & \\ w_i (q_i^{-h} - f_i(x)) \leq \alpha, \quad & i = 1, \dots, k; x \in \Omega \end{aligned} \quad (2.18)$$

The application of these two vectors (q^{-h} and $f^{h,0}$) is to generate k reference points:

$$q^{-h,j} = q^{-h} - (0, \dots, 0, \delta^{j,h}, 0, \dots, 0) \quad , \delta^{j,h} = \|q^{-h} - f^{h,0}\|_2 \quad j = 1 \dots k \quad (2.19)$$

associated with criterion vectors which can be expressed as:

$$\text{Min}_{x \in \Omega} s(q^{-h,j}, f(x), w) \Rightarrow f^{h,j} \quad j = 1 \dots k \quad (2.20)$$

From these $k+1$ criterion vectors, $\{f^{h,0}, f^{h,1}, \dots, f^k\}$, the DM must select one of the generated solution as the MPS written as $(x^h, f(x^h))$.

Another method used in MOLP methods, called VIA, was proposed by Korhonen and Laakso (1986). This method is also based on an achievement function in which a direction with a certain step size is resolved in the DM's utility function. This method is constrained by maximising the utility function, i.e. trying to find the direction at each iterative step that yields an increase in the utility function. The starting reference direction associated with the vision of the DM is projected onto the efficient set, as well as several points relative to the segment that join this reference point, and the criterion vector of the previous iteration. An improvement, which is indicated by the increasing direction of the utility function, will lead the DM to interactively choose a specific direction rather than choose arbitrarily. This is usually achieved by using the AHP method to determine the direction in which the utility function is being increased.

The STOM method developed by Nakayama and Sawaragi (1984) is similar to Wierzbicki's (1977) method. In STOM the achievements function is iteratively improved starting with a reference point which does not change through the whole process. At each iterative step the DM must provide the reference levels for the objectives that need to be improved, maintained and relaxed. Eventually, it is not essential for the DM to provide the reference values for the objective function that are to be relaxed. The reference point is determined by using the trade-off process by which Lagrange multipliers are applied to minimise the objective function. Mathematically, with the existence of a solution (x^{h-1}, f^{h-1}) , the DM must provide the objectives function that needs to be improved $\{f_i, i \in I_n^h\}$ with the amount of improvement $\{\Delta f_i, i \in I_n^h\}$, the objectives function that is preferred to be maintained $\{f_i, i \in I_m^h\}$ and the objective function selected to be relaxed $\{f_i, i \in I_o^h\}$ with the amounts of relaxation as $\{\Delta f_i, i \in I_o^h\}$.

The change of the new reference point $q^h = (q_1^h, \dots, q_k^h)$ can be expressed as:

$$\begin{aligned}
 q_i^h &= f_i^{h-1} + \Delta f_i^h & \forall_i \in I_n^h \\
 q_i^h &= f_i^{h-1} & \forall_i \in I_m^h \\
 q_i^h &= f_i^{h-1} - \Delta f_i^h & \forall_i \in I_o^h
 \end{aligned} \tag{2.21}$$

The structure of the problem in this case can be written as:

$\text{Min}_{x \in \Omega} s \left(f^*, f(x), \frac{1}{f^* - q^h} \right)$ and can be formulated as:

$$\text{Min}_{x, \alpha} \alpha$$

Subject to

$$w_i^h (f_i^* - f_i(x)) \leq \alpha \quad i = 1, \dots, k$$

$$x \in \Omega \quad \alpha \geq 0$$

(2.22)

where

$$w_i^h = \frac{1}{f_i^* - q_i^h} \quad \forall i = 1, \dots, k$$

The ISWT (Chankong and Haimes, 1978) and SPOT (Sakawa, 1982) approaches depend on the trade-off methods that use constraints to generate immediate MOLP solutions. The difference between these two approaches occurs in the methodology of the trade-off process being applied.

All of the above methods are, conceptually, trade-off methods in which the objective function is varied due to the interaction with the DM.

The GUESS (Buchanan, 1997) and LBS (Slowinski, 1994) methods are based on the trade-off method initialized with providing a reference point by the DM. The methods employ a projection of the gradient of optimal solutions on the efficient frontier proposed by the DM. The applications of the reference points are to normalise the achievement function between 0 and 1.

Another method that depends on a different methodology used in trade-off processing is the IGP method developed by Dyer (1972). In this approach the DM must accommodate uncertain goals for an individual objective function at the beginning of the iterative process.

MOLP methods obtain optimum solutions that coincide with the DM preferences. These solutions may result in a convergence of the utility function due to interactive methods which guide the DM to MPS. The MOLP method is an efficient tool that can be implemented interactively.

Yang (1999) reviewed frequently used interactive methods and identified that they had a number of problems including unclear termination criterion. For example, Geoffrion's method is based on the strict assumption that the preferences supplied by the DM must not only be consistent but also monotonic with respect to an implicit utility function that is difficult to satisfy. Yang also argues that the DM may find it easier to provide the required information in the early iterations than in later iterations and without proven convergence, the solution process of an interactive method may be stopped prematurely and the obtained compromised solution may not be the MPS. The STEM method fails to identify the best compromise solution because the DM is allowed to search the efficient frontier in a restricted and irreversible manner and there is no guideline provided to clarify the benefit of continuing the trade-off between sacrificed objectives. The ISTM method may search for compromise solutions but fails to produce clear termination criterion. Unlike the above mentioned methods, the gradient projection method has clear termination criterion and this is presented in section 2.3.4 because it will be implemented in the work of this thesis.

2.3.3 Integration of DEA and MOLP models

As a decision maker of an organisation setting targets that reflect the required progression of DMUs is vital. Integrating MOLP with DEA provides powerful tool that enables the DM to allocate his targets in terms of input and output. Furthermore, it distinguishes between efficient and inefficient DMUs and it recommends the best optimal direction of allocating resources in order to turn the inefficient DMUs into efficient DMUs.

Combining DEA with MOLP has been studied by several researchers (Tavares, 2002). Initially, Golany (1988) demonstrated a method of selecting the optimum input from DEA analysis and set it as a target for the inefficient units. A method to set targets based on the weights preference format was developed by Zhu (1996) and Thanassoulis and Dyson (1992). Another method was proposed by Athanassopoulos (1995 and 1998) which examined the integration of planning and resource allocation and incorporated the notion of global target achievements. In 1998, Joro et al proposed a method that uses MOLP methodology to choose the targets by combining the CCR output and input models. Analogous to the Joro model, Korhonen et al (1998 and 2002) and Joro et al (2003) illustrated the possibility of finding the best inputs and outputs for the decision maker. Recently, Yang et al (2009) incorporated DEA within MOLP and proved the equivalence between the CCR output oriented and the minimax formulation.

2.3.4 Interactive Gradient Projection Method

The shortcomings of most interactive techniques (i.e. lack of rigorous termination criterion), are addressed by the Gradient Projection Method (GPM). This method is based on marginal rate of substitution (utility gradient) to represent the DMs' preferences, which is similar to the GDF approach of Geoffrion et al (1972). The interactive process obtained using GPM calculates the projection of the gradient onto the tangent plane of the efficient frontier to identify the trade-off direction (Li and Yang, 1996; Yang, 1999; Li et al., 1999). The local minimum of the utility function is satisfied whenever the projection of the trade-off direction to the tangent line around a certain point on the efficient plane is zero.

Minimax is a method that can be implemented to solve most MOLP formulations (see equation 2.23) (Steuer and Choo, 1993; Yang, 1999 and Li, 2002). Suppose λ is the optimum solution of equation (2.23) and f_r^* the maximum feasible value of the r^{th} objective f_r then the efficient solution can be established by using the following formulation, where s is the number of objectives:

$$\begin{aligned} & \text{Minmax}_{\lambda} \{w_r (f_r^* - f_r(\lambda))\} \quad r = 1, \dots, s \\ & \text{Subject to} \\ & \quad \lambda \in \Omega \end{aligned} \tag{2.23}$$

A special formulation of the weighted minimax formulation is called the ideal point formulation such that $f^* = [f_1^*, \dots, f_r^*, \dots, f_s^*]$ sets the ideal point as the reference point. In the minimax formulation, for a given weight vector the DM is assumed to be satisfied with an efficient solution $\lambda \in \Omega$ at which $f(\lambda)$ has the shortest distance from f^* measured in the objective space. Another formulation of the weighted minimax MOLP can be derived by introducing an auxiliary variable θ that is similar to formulation (2.19) and (2.20) (for more information see Lightner and Director, 1981; Yang and Li, 2002) which can be expressed as:

$$\begin{aligned} & \text{Min}_{\lambda \in \Omega} \theta \\ & \text{subject to} \\ & \quad w_r (f_r^* - f_r(\lambda)) \leq \theta; \quad r = 1, \dots, s \\ & \quad \lambda \in \Omega \end{aligned} \tag{2.24}$$

The above formulation (2.24) is employed to design interactive steps supporting the DM to search for the MPS on the efficient frontier by incrementally changing the weighting parameters w_r for objectives $r=1, \dots, s$.

The steps to reach the optimal solution using this method are as follows. For the targeted *DMU*, at the first step of iteration t an initial positive weight vector $w^t = \{w_1^t \cdots w_r^t \cdots w_s^t\}$ is set. The optimal solution of the minimax MOLP formulation (2.22) is then given by $\lambda^t = \{\lambda_1^t \cdots \lambda_r^t \cdots \lambda_s^t\}$, which must be an efficient solution and the optimal value of the dual variables (also called shadow prices) of the r^{th} objective constraint $w_r (f_r^* - f_r(\lambda)) \leq \theta$ is

given by β_r^t . Letting $f(\lambda^t) = [f_1(\lambda^t) \cdots f_r(\lambda^t) \cdots f_s(\lambda^t)]^T$ represent the corresponding efficient solution in the objective space, it is proved (Yang and Li, 2002) that the normal vector N^t at $f(\lambda^t)$ on the efficient frontier is given by:

$$N^t = [w_1^t \beta_1^t, w_2^t \beta_2^t, \dots, w_r^t \beta_r^t, \dots, w_s^t \beta_s^t]^T \quad (2.25)$$

In linear programming, β_r^t can be generated using most existing software packages at no extra cost and for the first iteration all DMUs have an equal chance, hence w_r is assumed to be a unit vector ($[1, \dots, 1]^T$). Suppose the implicit utility function of the DM is denoted by $u(f(\lambda))$ then the gradient of $u(f(\lambda))$ at the solution $f(\lambda^t)$ is given by:

$$\nabla u^t = \frac{\partial u}{\partial f(\lambda^t)} = \left[\frac{\partial u}{\partial f_1^t} \cdots \frac{\partial u}{\partial f_r^t} \cdots \frac{\partial u}{\partial f_s^t} \right]^T \quad (2.26)$$

In practice $u(f(\lambda))$ is unknown but the utility gradient ∇u^t can be calculated using the local preference information of the DM, similar to marginal rates of substitution.

If the solution of $f(\lambda^t)$ is MPS then it is necessary as well as sufficient in a convex case (e.g. linear case) that the normal vector N^t is proportional to the utility gradient ∇u^t at $f(\lambda^t)$ hence:

$$N^t = [w_1^t \beta_1^t \cdots w_r^t \beta_r^t \cdots w_s^t \beta_s^t]^T \alpha \left[\frac{\partial u}{\partial f_1^t} \cdots \frac{\partial u}{\partial f_r^t} \cdots \frac{\partial u}{\partial f_s^t} \right]^T \quad (2.27)$$

If the optimal conditions are satisfied the gradient, $(\vec{G} = \nabla u^t)$ as shown in Figure 2.3, would be proportional to the normal vector $(\vec{N} = [w\beta]^T)$ and the dashed line would overlap the dotted line or point E_2 would overlap E_3 . If the optimal condition is not satisfied, as in Figure 2.3, then the gradient $(\vec{G} = \nabla u^t)$ can be projected onto the tangent plane as shown by the dotted line of the efficient frontier at $f(\lambda^t)$. This indicates a direction by which the

DM's utility can be improved. The projection, denoted by Δu^t as shown in Figure 2.3, is given by:

$$\Delta u^t = [\Delta f_1^t \cdots \Delta f_r^t \cdots \Delta f_s^t]^T = -\nabla u^t + \frac{[(\nabla u^t)^T N^t]}{[(N^t)^T N^t]} N^t \quad (2.28)$$

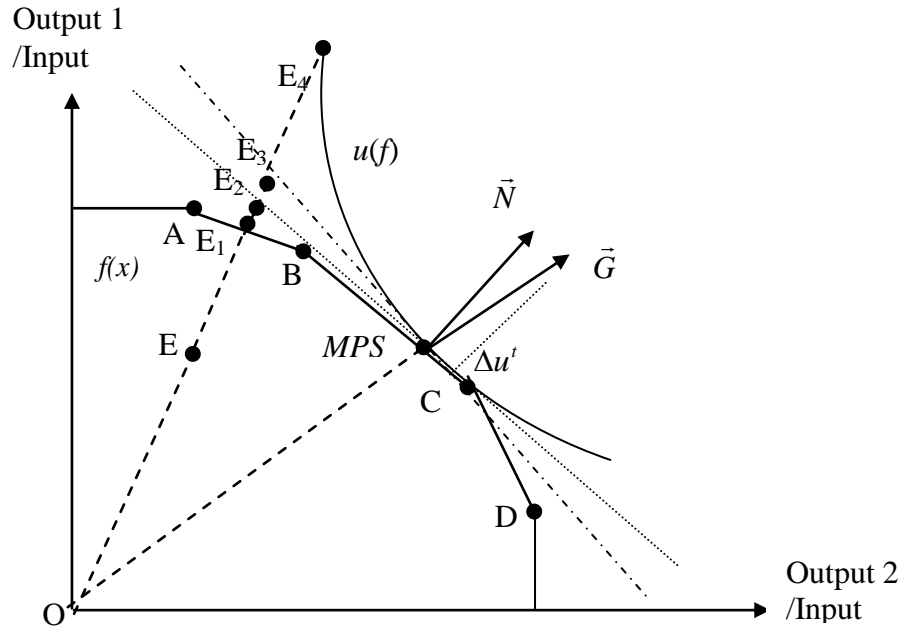


Figure 2.3: Projection of utility gradient onto tangent plane of efficient frontier (Yang, 2009)

As the utility function is not known explicitly a utility gradient is estimated by indifference trade-offs based on marginal rate of substitution, M , which may be provided by the DM. The first objective f_1 is set as the reference objective and the indifference trade-off M_{1r}^t between the first and the r^{th} objectives and the marginal rate of substitution M^t at $f(\lambda^t)$ are given by:

$$M_{1r}^t = -\frac{df_1^t}{df_r^t} \quad \text{and} \quad M^t = [1, M_{12}^t \cdots M_{1r}^t \cdots M_{1s}^t]^T \quad (2.29)$$

where df_1^t is the change in $f_1(\lambda)$ that can be exactly offset by the change df_r^t in $f_r(\lambda)$ with the overall utility kept constant, given that all other objectives remain unchanged.

Yang (2009) shows that the gradient of the utility function ∇u^t given in (2.24) is proportional to the marginal rate of substitution M^t at $f(\lambda^t)$ as follows:

$$M^t = \frac{\nabla u^t}{\partial u / \partial f_1^t} \quad (2.30)$$

where the MPS is reached, the following optimal indifference trade-off between $f_1(\lambda)$ and the r^{th} objective can be estimated by applying the following equation

$$df_r^t = -df_1^t \frac{w_1^t \beta_1^t}{w_r^t \beta_r^t} \quad (2.31)$$

The optimal indifference trade-off can also be used to check whether the MPS has been achieved. If the MPS has not been achieved then the projection Δu^t can be calculated using M^t and denoted by $\Delta \bar{u}^t$, which should provide a new trade-off direction to improve the DM's utility function as follows:

$$\Delta \bar{u}^t = [\Delta \bar{f}_1^t \cdots \Delta \bar{f}_r^t \cdots \Delta \bar{f}_s^t]^T = -M^t + \frac{(M^t)^T N^t}{(N^t)^T N^t} N^t \quad (2.32)$$

Next, the trade-off step size is determined. Suppose $\bar{\alpha}$ is a trade-off step which can be separated into two parts where:

$$\bar{\alpha} = \alpha_{\max} \alpha \quad (2.33)$$

α_{\max} is the largest permissible step-size and α is a regulating factor with $0 \leq \alpha \leq 1$.

Given f_r^+ ($= \bar{f}_{rj_0}$) is the maximum value generated from the pay-off table and f_r^t is the current value of the r^{th} objective and Δf_r^t as the projection of the r^{th} objective that needs to be sacrificed. The maximum step size is then determined as follows:

$$\alpha_{\max}^t = \frac{f_r^+ - f_r^t}{|\Delta \bar{f}_r^t|} \quad (2.34)$$

An explicit trade-off table, as illustrated in Table 2.2 below, can then be used to help find the trade-off step size $\bar{\alpha}$ using the following general formula:

$$f_r(\bar{\alpha}) = f_r(\lambda^t) + \bar{\alpha} \Delta \bar{f}_r^t \quad (2.35)$$

α	$f_1(\bar{\alpha})$	$f_2(\bar{\alpha})$...	$f_s(\bar{\alpha})$
0.1	$f_1(\lambda^t) + 0.1\alpha_{\max} \Delta \bar{f}_1^t$	$f_2(\lambda^t) + 0.1\alpha_{\max} \Delta \bar{f}_2^t$...	$f_s(\lambda^t) + 0.1\alpha_{\max} \Delta \bar{f}_s^t$
0.2	$f_1(\lambda^t) + 0.2\alpha_{\max} \Delta \bar{f}_1^t$	$f_2(\lambda^t) + 0.2\alpha_{\max} \Delta \bar{f}_2^t$...	$f_s(\lambda^t) + 0.2\alpha_{\max} \Delta \bar{f}_s^t$
.
.
.
1.0	$f_1(\lambda^t) + 1.0\alpha_{\max} \Delta \bar{f}_1^t$	$f_2(\lambda^t) + 1.0\alpha_{\max} \Delta \bar{f}_2^t$...	$f_s(\lambda^t) + 1.0\alpha_{\max} \Delta \bar{f}_s^t$

Table 2.2 Trade-off step size table

In Table 2.2 the DM may determine the step size by analysing the trade-offs among the outputs along the tangent plane of the efficient frontier. Once the step size is decided the weighting parameters w_r can be updated as follows.

$$w_r^{t+1} = \frac{f_1(\lambda^*) - [f_1(\lambda^t) + \bar{\alpha} \Delta \bar{f}_1^t]}{f_r(\lambda^*) - [f_r(\lambda^t) + \bar{\alpha} \Delta \bar{f}_r^t]} \quad r = 1, \dots, s \quad (2.36)$$

Replacing w_r by w_r^{t+1} in formulation (2.22) and solving it leads to a new solution λ^{t+1} which should have a higher utility than λ^t for a sufficient small $\bar{\alpha}$ or $u(\lambda^{t+1}) > u(\lambda^t)$.

The interactive and trade-off analysis process continues until the DM agrees with the optimal trade-offs between the objectives and when the MPS that maximises the implicit utility function of the DM is found the interactive process is terminated.

2.4 Summary

Most methodological extensions of the basic DEA followed an application driven path as a result of the application of the method to real life problems. In this research the desire is to incorporate value judgement in DEA. Having studied the literature, there appear to be two streams for incorporating value judgements. The first stream imposes restrictions on weights of inputs or outputs, which requires prior information provided by DM (Charnes et al; 1978, Charnes et al; 1990, Thanassoulis; 1995). The second stream employs an equivalence MOLP model to utilize the MOLP solution methods, which enables value judgements to be incorporated interactively.

Although there are well established equivalence MOLP models none of them includes measuring the whole efficiency (i.e. best and worst efficiency) of DMUs or utilises both an interactive and ranking procedure. Furthermore, they do not integrate the DM's and expert preferences in a well defined model, which would improve performance assessment and benchmarking, notably target setting and resource allocation.

Chapter 3

The production trade-off MOLP equivalent model

3.1 Introduction

In DEA models prior preference knowledge needs to be identified and this is normally provided by a DM through value judgements, which are subjective and difficult to be determined. Moreover, it is impossible to assume a well-defined explicit utility function that expresses the DM's preferences because of the conflict of interests or incompleteness of available information. Developing an interactive framework based on trade-off analysis and incorporating DM preferences to locate the MPS on the efficient frontier could be achieved by using the gradient projection and local search method (Yang, 1999). By using these methods the DM can examine a spectrum of technically achievable targets that make him/her more confident in making the best decisions. The gradient projection approach uses the marginal rate of substitution (utility gradient) together with a normal vector on the efficient frontier determined by employing a weighted minimax formulation. The normal vector is used to create an optimal condition to terminate the interactive procedure as it provides the optimal indifference trade-off that the MPS must satisfy. Otherwise, the normal vector provides a direction along which a better efficient solution could be achieved.

Recent developments in the integration of MOLP with DEA have heightened the need for incorporating value judgements. Decision makers need to improve the discrimination of variables and the performance of DMUs in an interactive manner. To date there has been little work done in this area. The framework developed in this research as reported in this chapter unveils a new technique in this field by employing the production trade-off approach through which value judgements are considered by a board within an organisation. The production trade-off approach incorporates two additional vectors P (for input variables) and Q (for output variables) in the DEA (input or output orientation) dual formulation that appear as weight restrictions in the primal formulation. The inclusion of

these two vectors improves the discrimination of variables and because they are chosen by experts or survey they can achieve a more realistic outcome through the board choosing the P and Q vectors. Consequently, the DM's preferences can be incorporated in the efficiency analysis through an interactive process.

3.1.1 Aims

The purpose of this chapter is to establish a framework in which target setting and resource allocation decisions can be supported through an interactive process. This is achieved by providing a new equivalent MOLP-DEA model based on the production trade-off approach (Podinovski, 2004, 2005). The proposed framework provides a powerful tool as it allows the DM to improve the targeted DMU of real situations by incorporating P and Q vectors. In this chapter, we aim to:

- Integrate MOLP with DEA through the production trade-off approach.
- Establish a framework in which the DM can have more involvement in managing and improving the targeted DMU.

3.1.2 Research Questions

This chapter is designed to answer the following questions:

1. How can the MOLP-DEA equivalence model provide a tool that enables the DM to set targets according to his/her preferences?
2. How can the proposed framework be used to improve a targeted DMU?
3. What is the range of targets that the DM can set?
4. What are the resources that should be changed (increased or decreased) to satisfy the DM targets?

3.1.3 Methodology

To address the above questions this chapter considers the required theoretical research and two case studies. The theoretical research establishes an MOLP-DEA equivalence model employing the production trade-off approach through which a framework is built for conducting an interactive analysis. The two case studies illustrate the application of the model to improve the performances of departments in the University of Bahrain and private hospitals in Bahrain.

3.1.4 Chapter Overview

The chapter presents:

- An overview of incorporating value judgements
- Selecting input and output variables
- The theoretical model
- An example of the production trade-off DEA-MOLP model.
- Case study#1: The University of Bahrain
- Case study#2: Private hospitals in Bahrain

3.2 An overview of incorporating value judgements

Value judgement models have been developed to improve the discrimination of efficiency analysis. They are formulated by imposing additional restrictions that take into account the marginal importance of individual inputs and outputs, which incorporate prior information about the assessment efficiency of DMUs. This prior information can be included by employing various techniques which have different implications on the assessed relative efficiency of DMUs (Allen et al, 1997 and Thanossoulis, 2001).

There are two ways to incorporate value judgements in DEA assessment either in primal DEA formulation as weight restrictions or in the dual DEA formulation as an additional term modifying composite unit. The later formulation falls under the umbrella name of the trade-off approach (Podinovski, 2004; Podinovski, 2005). Although weight bounds have often been employed (Sarrico et al, 1997 and Sueyoshi, 1999) there is no clear evidence of agreement about the meaning of these restrictions and the outcomes of DEA are significantly affected by the application of these bounds, which has resulted in ambiguities and confusion. It has been proposed that the trade-off approach overcomes the drawback of value judgements structured by weight bounds and that this type of approach can produce realistic improvements (Podinovski, 2004). For instance, by using the trade-off approach inefficient DMUs will reflect real situations located in the feasible region which can be produced.

In 2007, Podinovski proposed an algorithm to integrate target setting and resource allocation using the trade-off approach. To do so, he defined three stages as follows:

- Solve the model 2.8a or 2.9a (inputs or outputs respectively) by considering the type of returns to scale (i.e. CRS or VRS) and if VRS, the condition $\sum \lambda = 1$ needs to be added.
- Look for possible non-radial improvements to the radial target. This can be achieved maximizing the sum of residual slacks subject to the explicit condition in which the efficient target has only positive inputs. The proposed model is written as

$$\begin{aligned}
& \text{Max } \sum_{r=1}^s e_r + \sum_{i=1}^m d_i \\
& \text{subject to} \\
& \quad \bar{y}\lambda + \sum_{t=1}^K \pi_t Q_t - e = y^* \\
& \quad x\lambda + \sum_{t=1}^K \pi_t P_t + w + d = x^* \\
& \quad \bar{x}\lambda + \sum_{t=1}^K \pi_t P_t + w \geq 0 \\
& \quad \lambda, \pi, e, w, d \geq 0
\end{aligned} \tag{3.1}$$

where the vectors w , m , and d are of same dimension. In case of VRS, equality $\sum_{j=1}^n \lambda_j = 1$ is added to the constraints in model (3.1).

- Identify a reference set of efficient peer observable DMUs, which is achieved by maximising the sum of w_i ; ensuring the efficiency of the observable DMU. This is achieved by solving the following model

$$\begin{aligned}
& \text{Max } \sum_{i=1}^m w_i \\
& \text{subject to} \\
& \quad \bar{y}\lambda + \sum_{t=1}^K \pi_t Q_t = y^* + e' \\
& \quad \bar{x}\lambda + \sum_{t=1}^K \pi_t P_t + w = x^* - d' \\
& \quad \lambda, \pi, w \geq 0
\end{aligned} \tag{3.2}$$

where e' and d' are fixed and determined by optimising model 3.1.

In this research a flexible framework incorporating target setting and resource allocation was developed. The flexibility of the developed model can be emphasized by active participation of the DM in renewing his preferences during efficiency analysis resulting in the most preferred solution based on his views. An MOLP-trade-off model is implemented by incorporating a DEA output-oriented approach with MOLP through the Minimax

method. The MOLP-trade-off model can be solved iteratively by using the gradient projection method.

The next section contains a theoretical derivation of MOLP-trade-off model followed by an illustrative example. Real applications of the model are then presented through two empirical case studies, one at the University of Bahrain (UOB) and the other at several private hospitals in Bahrain.

3.3 The production trade-off DEA-MOLP model

The aim of this section is to present the MOLP- production trade-off approach that employs an interactive framework based on the gradient projection technique to determine the MPS on the efficient frontier for target setting and resource allocation.

3.3.1 The theoretical derivation

The general form of the MOLP problem formulation can be written as a maximization problem of multiple objectives within a utility function and can be expressed as:

$$\begin{aligned} & \text{Max } [f_1(\lambda), f_2(\lambda), \dots, f_r(\lambda), \dots, f_s(\lambda)] \\ & \text{Subject to} \\ & \quad \lambda \in \Omega \end{aligned} \tag{3.3}$$

where

$$\Omega = \{ \lambda / g_l(\lambda) \leq 0, h_m(\lambda) = 0; l = 1, \dots, L, m = 1, \dots, M \}$$

The minimax formulation, which represents a special form of equation (3.3), can be expressed as:

$$\begin{aligned}
& \text{Min}_{\lambda \in \Omega} \theta \\
& \text{Subject to} \\
& w_r [f_r^* - f_r(\lambda)] \leq \theta \\
& \lambda \in \Omega
\end{aligned} \tag{3.4}$$

The equivalent MOLP (for more information see Yang, 2009) of the output oriented model can be written as:

$$\begin{aligned}
& \text{Min } \theta \\
& \text{Subject to} \\
& w_r \left(f_r^* - \sum_{j=1}^n \lambda_j y_{rj} \right) \leq \theta, \quad r = 1, \dots, s \\
& \lambda \in \Omega = \left\{ \begin{array}{l} \lambda \left| \sum_{j=1}^n \lambda_j x_{ij} \leq x_{ij_0}, \quad i = 1, \dots, m, j = 1, \dots, n, t = 1, \dots, K \right. \\ \text{for all } \lambda \geq 0 \end{array} \right\}
\end{aligned} \tag{3.5}$$

According to the production trade-off approach (see section 2.2.3.2) the above formulation can be modified by adding the two vectors \mathbf{P} (for input data) and \mathbf{Q} (for output data). Hence, the problem can be written as:

$$\begin{aligned}
& \text{Max } \theta_{j_0} \\
& \text{Subject to} \\
& \theta_{j_0} y_{rj_0} - \left(\sum_{j=1}^n \lambda_j y_{rj} + \sum_{t=1}^K \pi_t Q_t \right) \leq 0, \quad r = 1, \dots, s \\
& \sum_{j=1}^n \lambda_j x_{ij} + \sum_{t=1}^K \pi_t P_t \leq x_{ij_0}, \quad i = 1, \dots, m; \lambda_j \geq 0 \text{ for all } j \\
& \lambda, \pi \geq 0, \theta_{j_0} \text{ sign free}
\end{aligned} \tag{3.6}$$

In order to derive MOLP equivalent model, the following definitions are used:

$$w_r = \frac{1}{y_{rj_0}} \quad (3.7)$$

$$f_r^* = \frac{F^{\max}}{w_r} = y_{rj_0} F^{\max} \quad (3.8)$$

$$F^{\max} = \max_{1 \leq r \leq s} \{w_r \bar{f}_{rj_0}\} = \max_{1 \leq r \leq s} \left\{ \frac{\bar{f}_{rj_0}}{y_{rj_0}} \right\} \quad (3.9)$$

$$f_r(\lambda, \pi) = \sum_{j=1}^n \lambda_j y_{rj} + \sum_{t=1}^k \pi_t Q_{rt} \quad r = 1, \dots, s; \quad \lambda = [\lambda_1, \dots, \lambda_n]^T \text{ and } \pi = [\pi_1, \dots, \pi_k]^T \quad (3.10)$$

\bar{f}_{rj_0} is the maximum feasible value of the r th composite output for the observed DMU₀.

Note that $\bar{f}_{rj_0} = [\bar{f}_{1j_0}, \dots, \bar{f}_{sj_0}]^T$ is the ideal point in the objective space spanned by

$$f_r(\lambda) = \sum_{j=1}^n \lambda_j y_{rj} \quad (r = 1, \dots, s) \text{ and } \lambda = (\lambda_1, \dots, \lambda_n)^T \text{ on } \Omega_{j_0}.$$

By using the above definitions, formulation (3.6) can be written as

Max θ_{j_0}

Subject to

$$\frac{1}{w_r} \theta_{j_0} - f_r(\lambda, \pi) \leq 0, \quad r = 1, \dots, s \quad (3.11)$$

$$\lambda, \pi \in \Omega_{j_0} = \left\{ \lambda, \pi \left| \sum_{r=1}^s \lambda_j x_{ij} + \sum_{t=1}^k \pi_t P_t \leq x_{ij_0}, i = 1, \dots, m; t = 1, \dots, K, \text{ and } \lambda_j, \pi_t \geq 0 \right. \right\}$$

Model (3.11) can be modified to be similar to model (3.4). This can be achieved by the following manipulations:

$$\begin{aligned}
& \frac{1}{w_r} \theta_{j_0} - f_r(\lambda, \pi) \leq 0 \\
& \Leftrightarrow f_r^* - f_r(\lambda, \pi) \leq f_r^* - \frac{1}{w_r} \theta_{j_0} \quad \text{"}\Leftrightarrow\text{" means equals} \\
& \Leftrightarrow w_r (f_r^* - f_r(\lambda, \pi)) \leq w_r f_r^* - \theta_{j_0} \\
& \Leftrightarrow w_r (f_r^* - f_r(\lambda, \pi)) \leq F^{\max} - \theta_{j_0} \\
& \Leftrightarrow w_r (f_r^* - f_r(\lambda, \pi)) \leq \theta, \quad \text{where } \theta = F^{\max} - \theta_{j_0}
\end{aligned} \tag{3.12}$$

Furthermore, the modification on the objective function can be written as:

$$\begin{aligned}
\text{Max } \theta_{j_0} &= \text{Min } (-\theta_{j_0}) \\
&= \text{Min } (F^{\max} - \theta_{j_0}) \\
&= \text{Min } \theta
\end{aligned} \tag{3.13}$$

Thus, the equivalent MOLP Minimax model to the production trade-off approach output oriented model in dual form can be written as:

$$\begin{aligned}
& \text{Min } \theta \\
& \text{Subject to} \\
& \quad w_r [f_r^* - f_r(\lambda, \pi)] \leq \theta \\
& \quad \lambda, \pi \in \Omega
\end{aligned} \tag{3.14}$$

If the data of DMUs obey Variable Returns to Scale (VRS), formulation (3.6) can be modified by imposing the convexity constraint $\sum_{j=1}^n \lambda_j = 1$ and so the equivalent MOLP formulation can be expressed as:

$$\begin{aligned}
& \text{Min } \theta \\
& \text{Subject to} \\
& w_r \left[f_r^* - \left(\sum_{j=1}^n \lambda_j y_{rj} + \sum_{t=1}^K \pi_t Q_t \right) \right] \leq \theta, \quad r = 1, \dots, s \\
& \lambda, \pi \in \Omega_{j_0} = \left\{ \lambda, \pi \left| \sum_{j=1}^n \lambda_j x_{ij} + \sum_{t=1}^K \pi_t P_t \leq x_{ij_0}, \quad i = 1, \dots, m; \quad t = 1, \dots, K, \text{ and } \lambda_j, \pi_t \geq 0 \right. \right\}
\end{aligned} \tag{3.15}$$

It is worth noting that π belongs to the same feasible decision space as λ and its action is to compensate for the extension on the space due to the addition of the P and Q vectors as defined and explained in section 2.2.3.2.

3.3.2 An illustrative example

This example illustrates the equivalence of the derived MOLP model (3.15) with DEA CCR output oriented dual model (3.6). The data were collected from the University of Bahrain (UOB). The data covered seven DMUs (colleges) as follows: Art, Business, Education, Engineering, Science, IT and Law. Each DMU had two inputs: number of academic staff (x_1) and number of students (x_2) and two outputs: the number of graduates (y_1) and the number of publications (y_2).

DMU	Inputs		Outputs	
	x_1	x_2	y_1	y_2
1	176	3895	473	29
2	62	5337	693	19
3	80	2552	716	26
4	99	1605	392	89
5	76	954	149	55
6	47	2247	300	32
7	30	1182	55	12

Table 3.1 Inputs and Outputs

Relative efficiency results were obtained by applying the output oriented CCR dual DEA model (3.6) as shown in Table 3.2 where $P = (0, 0)^T$ and $Q = (-1, 1)^T$ are the trade-off vectors for inputs and outputs respectively and the λ s are the decision variables.

DMU	DEA Score ($1/\theta$)	Intensity (θ)	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7
1	0.441	2.266	0.000	0.000	1.301	0.358	0.000	0.000	0.000
2	1.000	1.000	0.000	1.000	0.000	0.000	0.000	0.000	0.000
3	1.000	1.000	0.000	0.000	1.000	0.000	0.000	0.000	0.000
4	1.000	1.000	0.000	0.000	0.000	1.000	0.000	0.000	0.000
5	1.000	1.000	0.000	0.000	0.000	0.000	1.000	0.000	0.000
6	0.984	1.016	0.000	0.265	0.000	0.309	0.000	0.000	0.000
7	0.450	2.223	0.000	0.008	0.000	0.298	0.000	0.000	0.000

Table 3.2 The efficiency scores obtained from applying formulation (3.6)

Table 3.2 shows the relative efficiency scores and the composite inputs and outputs for all DMUs. The bold scores are the inefficient DMUs (1, 6 and 7). The score of these DMUs are: 44.1%, 98.4% and 45.0% respectively whilst the DMUs that have efficiency scores of unity lie on the efficient frontier.

Recall that the importance of incorporating MOLP into DEA is to allow the interaction of DMs in the efficiency evaluation procedure. Table 3.3 presents the equivalence of $\theta = F^{\max} - \theta_{j_o}$ for individual DMUs. F^{\max} was generated by feeding in the trade-off vectors P and Q and solving the following programme for each DMU to build the pay-off table:

$$\begin{aligned} \text{Max } f_r(\lambda, \pi) &= \sum_{j=1}^n \lambda_j y_{rj} + \sum_{t=1}^K \pi_t Q_t, & r=1, \dots, s \text{ and } \lambda &= [\lambda_1, \dots, \lambda_n]^T \\ \text{Subject to} & & & \\ \lambda_j, \pi_t \in \Omega_{j_o} &= \left\{ \lambda_j, \pi_j \left| \sum_{j=1}^n \lambda_j x_{ij} + \sum_{t=1}^K \pi_t P_t \leq x_{ij_0}, i=1, \dots, m; t=1, \dots, K \text{ and } \lambda_j, \pi_t \geq 0 \right. \right\} \end{aligned} \quad (3.16)$$

along with using definition (3.9).

Max f_1		Max f_2		Max(f_1)/ y_1	Max(f_2)/ y_2	F^{\max}
f_1	f_2	f_1	f_2			
1092.80	39.68	696.89	158.22	2.31	5.46	5.46
693.00	19.00	245.49	55.74	1.00	2.93	2.93
716.00	26.00	316.77	71.92	1.00	2.77	2.77
450.31	16.35	392.00	89.00	1.15	1.00	1.15
267.66	9.72	149.00	55.00	1.80	1.00	1.80
451.39	15.02	186.10	42.25	1.50	1.32	1.50
277.75	9.67	118.79	26.97	5.05	2.25	5.05

Table 3.3 The pay-off table showing the results of solving formulation (3.16).

The equivalence of MOLP to the DEA dual model is satisfied by solving (3.15) and (3.6) respectively. If $\theta = F^{\max} - \theta_{j_o}$ is fulfilled then equivalence is achieved.

<u>DEA model</u>			<u>Minimax model</u>						
DMU	$1/\theta_{jo}$	θ_{jo}	F^{max}	<u>input and output variables</u>				θ	$F^{max}-\theta_{jo}$
				x_1	x_2	y_1	y_2		
1	0.441	2.266	5.46	176	3895	473	29	3.19	3.19
2	1.000	1.000	2.93	62	5337	693	19	1.93	1.93
3	1.000	1.000	2.77	80	2552	716	26	1.77	1.77
4	1.000	1.000	1.15	99	1605	392	89	0.15	0.15
5	1.000	1.000	1.80	76	954	149	55	0.80	0.80
6	0.984	1.016	1.50	47	2247	300	32	0.49	0.49
7	0.450	2.223	5.05	30	1182	55	12	2.83	2.83

Table 3.4 The equivalence of the DEA outputs dual and minimax formulation.

Table 3.4 shows the result obtained from solving (3.15). It also shows the values of F^{max} measured using (3.16). The last two columns demonstrate the intensity of the equivalent MOLP-Trade-off approach formulation (3.15), θ measures and the values of $F^{max}-\theta_{jo}$. F^{max} indicates the maximum allowable range that outputs can reach and θ_{jo} is the optimum intensity resulting from (3.6).

Before proceeding to the case study it is worth to briefly mention the interactive steps that were developed in chapter 2

3.3.3 The interactive steps

The interactive steps of the framework are as follows:

- Identifying the trade-off vectors P and Q by expertise or questionnaire.
- Solving model 3.4 to find the composite output of the targeted DMU. For instance, finding the projection of inefficient DMU point on the efficient frontier. The initial parameters are determined in this step.
- Generating the pay-off table to identify the range through which the targets can be selected.
- Calculating the weighting vectors by using a perturbation technique to start the interactive procedure.

-
- Solving model 3.13 to find the decision variables.
 - Measuring the normal vector and checking the optimality condition
 - Finding the trade-off direction, and then the step size table to update the weighting vector for the next iterative step.

3.4 Case study#1: The University Of Bahrain (UOB)

3.4.1 Selecting the variables

The DEA approach is significantly affected by the number of inputs and outputs, the higher the number of inputs and outputs the less discerning the analysis. The guideline for choosing the number of inputs and outputs is less than one third of the number of DMUs $(m + s) < \frac{n}{3}$ (Friedman and Sinuany-Stern, 1998).

Correlation is used to improve discrimination among DMUs. For instance, if some of the input or output variables are highly correlated, one or more of these input or output variables might be eliminated as they have least impact on DEA outcomes (Jenkins, 2003). On the other hand, input variables that have high correlation coefficient with output variables are recommended to be involved in the variables set.

In this work multivariate statistical analysis was implemented to specify the number of variables to be used in the analysis. This was based on partial covariance analysis developed by Jenkins and Anderson (2003). This approach compares the sum of covariance of certain variable(s) relative to the sum of the covariance of the total variables so that most information appears in the retained variables.

Both correlation analysis and multivariate analysis, including partial covariance and cluster approach, were implemented. They were applied to reduce the inputs and outputs for performing DEA analysis on the data collected from the University of Bahrain shown in Table 3.5 below

The data covered seven DMUs (colleges) as follows: Art, Business, Education, Engineering, Science, IT and Law. Each DMU has four inputs: number of full time academic staff (I_1), number of teaching assistants (I_2), number of registered students (I_3), and expenses in Bahraini Dinars (I_4); and two outputs: number of graduates (O_1) and number of research publications (O_2).

DMU	College	Inputs				Outputs	
		No. Staff Full time	TA	Registered Students	Expenses (BD)	Graduate Students	Research Published
1	Art	150	28	3517	50496	496	99
2	Business	53	3	5033	51801	632	20
3	Education	77	8	2002	108595	489	16
4	Engineering	102	6	1401	103846	359	75
5	Science	76	1	902	44261	143	81
6	IT	47	12	2140	21458	234	23
7	Law	27	1	1145	32458	147	15

Table 3.5 Data collected from The University of Bahrain (UOB) (where: TA denotes Teaching Assistants and BD denotes Bahraini Dinars).

3.3.2 Correlation analysis

Correlation is used to improve discrimination among DMUs. For instance, if some of the input or output variables are highly correlated, one or more of these input or output variables might be eliminated as they have least impact on DEA outcomes. On the other hand, input variables that have high correlation coefficient with output variables are recommended to be involved in the variables set. Correlation analysis was applied to examine the relationship between the variables in Table 3.5.

DMU	I_1	I_2	I_3	I_4	O_1	O_2
I_1	1					
I_2	0.763	1				
I_3	0.179	0.352	1			
I_4	0.379	-0.0439	-0.0922	1		
O_1	0.388	0.357	0.852	0.433	1	
O_2	0.837	0.487	-0.103	0.0971	-0.0442	1

Table 3.6 Results of correlation analysis.

Table 3.6 presents the correlation analysis outcome. The yellow shaded area in Table 3.6 shows the relationship (correlation) between input and output variables and the bold data are those that need to be omitted.

From the correlation matrix (Table 3.6), input variables I₁, I₃, or I₄ could be omitted, whilst of the output variables the analysis suggests that O₂ should be removed. Clearly, using this analysis is not sufficient to decide which variable(s) can most appropriately be omitted. For instance, which variable(s), if omitted, have the least effect on DEA outcome?

3.3.3 Multivariate Analysis

Multivariate analysis based on partial covariance was implemented to investigate the variable(s) that if they were removed would result in least information loss. The effect on the efficiency if one of the variables is removed is considered first and the results are presented in Table 3.7 below, where I and O represent input and output variables respectively, whilst the numbers before them indicate which variables have been omitted. For example, the column headed 3I, 2O,1I shows the results from only having 3 inputs (3I) and 2 outputs (2O) when input 1 is omitted (1I).

DMU	4I,2O,0	3I,2O,1I	3I,2O,2I	3I,2O,3I	3I,2O,4I	4I,1O,1O	4I,1O,2O
1	1.000	1.000	1.000	1.000	0.787	1.000	1.000
2	1.000	1.000	1.000	1.000	1.000	0.354	1.000
3	1.000	1.000	1.000	0.715	1.000	0.195	1.000
4	1.000	1.000	1.000	0.822	1.000	0.690	1.000
5	1.000	1.000	1.000	1.000	1.000	1.000	1.000
6	0.969	0.969	0.969	0.969	0.728	0.570	0.894
7	0.822	0.762	0.818	0.768	0.822	0.521	0.900
Relative Variance	1.16E+09	1.16E+09	1.16E+09	1.15E+09	2.20E+06	1.32E+03	3.62E+04
Relative Variance percentage	100.00	100.00	100.00	99.81	0.19	3.521	96.479
No. Of Efficient DMUs	5	5	5	3	4	2	5

Table 3.7 Efficiency changes due to omitting one variable.

Table 3.7 shows the changes to efficiency resulting from applying basic DEA (i.e. dual CCR output oriented model) with different inputs and outputs variables omitted. The three bottom rows show the variance, variance percentage and number of efficient DMU changed as an effect of omitting a variable.

Based on the results in Table 3.7 input 1 (Full time staff), input 2 (Teaching Assistants) or output 2 (Research Publications) can be omitted without losing information because the number of efficient DMUs does not change. On the other hand, much information could be missed if input 3 (Registered Students) or output 1 (Graduated Students) are removed. It is worth noting that omitting output 1 (Graduated Students) omits most information, i.e. number of the efficient DMUs is reduced to 2.

For more discernment in the DEA, further reduction in the number of variables was attempted. Partial covariance was again employed but this time when more than one input variable was removed. Note that the column headings are defined as in Table 3.7, for example, column 1 shows the results from the omission of inputs 1 and 2 (i.e. 12I) whilst column 2 shows the results from omitting inputs 1 and 3 (i.e. 13I).

From the results in Table 3.8, one remarkable point can be seen, that all the information is retained (i.e. no loss) when both inputs 1 and 2 are omitted simultaneously.

DMU	2I,2O,12I	2I,2O,13I	2I,2O,14I	2I,2O,23O	2I,2O,24I	2I,2O,34I	1I,2O,234I	1I,2O,134I	1I,2O,124I	1I,2O,123I
1	1.000	1.000	0.550	1.000	0.821	0.742	0.742	0.097	0.550	1.000
2	1.000	1.000	0.729	1.000	1.000	1.000	1.000	1.000	0.490	1.000
3	1.000	0.370	0.978	0.538	1.000	0.538	0.538	0.290	0.953	0.370
4	1.000	0.525	1.000	0.817	1.000	0.817	0.817	0.329	1.000	0.368
5	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.933
6	0.980	0.980	0.427	0.980	0.757	0.695	0.695	0.098	0.427	0.980
7	0.724	0.566	0.719	0.777	0.861	0.777	0.777	0.741	0.501	0.409
Relative Variance	1.16E+09	2195185	1154349650	1.15E+09	1.15E+09	1744.286	1654	90.28571	2193530.6	1.15E+09
Relative Variance percentage	100.000%	0.190%	99.810%	99.810%	99.810%	0.000%	0.000%	0.000%	0.190%	99.810%
No. Of Efficient DMUs	5	3	2	3	4	2	2	2	2	2

Table 3.8 Efficiency changes due to omitting more than one variable.

3.4.4 Cluster Analysis

Cluster analysis groups data objects based on information found in the data that describes the objects and their relationship. The goal is that the objects within a group be similar (or related) to one another and different from (or unrelated to) the objects in the other groups. The greater the similarity (or homogeneity) within a group and the greater the difference between groups the better or more distinct the cluster.

There are several techniques for conducting cluster analysis with binary data, all of which involve calculating distances between groups of data based upon the observed variables and then applying one of the standard cluster analysis algorithms to these distances. A popular group of these measures designed for binary data is known collectively as matching coefficients (Dillon and Goldstein, 1984). There are many techniques of matching coefficient, all of which take as their main goal the measurement of response set similarity between any two groups. The logic underlying these methods is that two individuals should be viewed as similar if they share a common pattern of attribute among the binary variables (Snijders et al, 1990).

In this study cluster analysis was implemented to measure the similarity and dissimilarity within the input variables. If input variables share a common pattern of attribute one of them may be omitted. The omitted inputs are those that contain least information.

A hierarchical clustering (dendrogram) is used to present both the cluster-subcluster relationships and the order through which the clusters were merged or split (Figure 3.1).

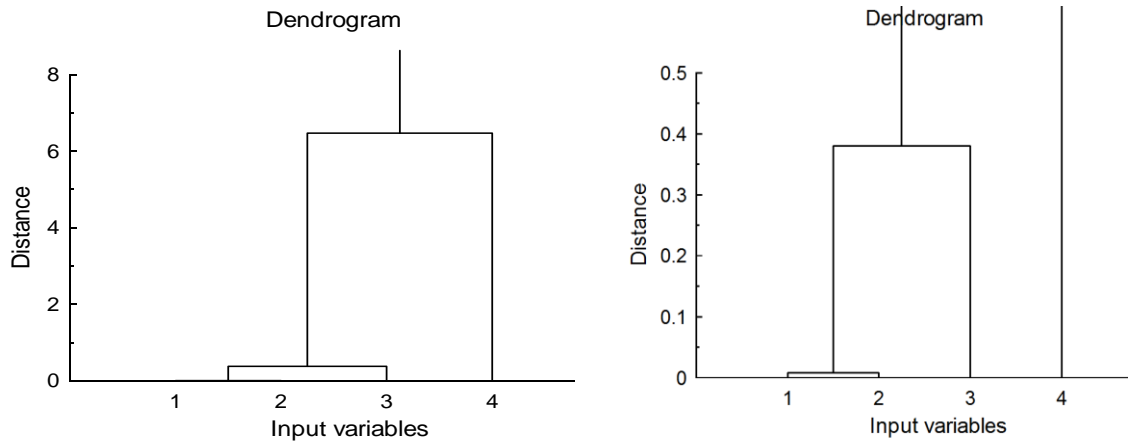


Figure 3.1 Dendrograms for the input variables (The right hand dendrogram clarifies the similarity between inputs 1 and 2 by magnifying the left hand dendrogram).

Cluster analysis suggests that input 1 (Full-time Staff) or input 2 (Teaching Assistants) can be eliminated resulting in 3 inputs and 2 outputs. Table 3.9 shows the data after removing input 2 (Teaching Assistants).

DMU	Colleges	Inputs			Outputs	
		Full time staff	Registered Students	Expenses	Graduate Students	Published papers
1	Art	150	3517	50496	496	99
2	Business	53	5033	51801	632	20
3	Education	77	2002	108595	489	16
4	Engineering	102	1401	103846	359	75
5	Science	76	902	44261	143	81
6	IT	47	2140	21458	234	23
7	Law	27	1145	32458	147	15

Table 3.9 The UOB data after cluster analysis

3.4.5 Identifying the type of returns to scale

Identifying of the type of returns to scale, i.e. CCR with Constant Returns to Scale (CRS) or BCC with Variable Returns to Scale (VRS) is essential in DEA analysis. Failing to do so might result in inconsistent inefficiency scores. For example, if VRS is wrongly implemented the resulting efficiency scores will be greater than the true efficiency scores because of the restrictive property of the VRS which assumes that the efficient frontier always produces a closer of the envelopment of the data. Hence, it is essential to examine the returns to scale properties, which can be satisfied by using an hypothesis test. The hypothesis test allows identification of the type of data whether it is CRS or VRS so it should be adopted for a particular case study.

Several authors have implemented a two-sample t-test (a hypothesis test used for small samples) to identify the type of returns to scale (Camanho and Dyson, 2005; Banker et al, 1996; Banker et al, 1993). In practice the hypothesis test compares the mean of two samples to identify the probability that the two samples are likely to come from the same population.

Say β_i and β_j are the population means for the distributions of DEA outcome scores from applying the CCR and BCC models. The null hypothesis assumes that there is no difference between the mean of the two samples whilst the alternative hypothesis opposes it, i.e.

$$\begin{aligned} H_0 : \beta_i - \beta_j &= 0 \quad \left(\text{null hypothesis} \right) \\ H_1 : \beta_i - \beta_j &\neq 0 \quad \left(\text{alternative hypothesis} \right) \end{aligned} \tag{3.18}$$

For small samples of n_i and n_j , where $n < 30$, the t-test can be written as:

$$t = \frac{(\bar{x}_i - \bar{x}_j) - (\beta_i - \beta_j)}{\sqrt{s^2 \left(\frac{1}{n_i} + \frac{1}{n_j} \right)}} \tag{3.19}$$

Where \bar{x}_i is the mean $\left(= \frac{\sum x_i}{n_i} \right)$ and s_i^2 is the variance $\left(= \frac{\sum (x_i - \bar{x}_i)^2}{n_i - 1} \right)$.

In the case that the null hypothesis does not pertain i.e. $\beta_i - \beta_j \neq 0$ within the 5% significance interval, it is rejected and the p -value is investigated. The p -value indicates the observed probability of obtaining the sample results when the null hypothesis is assumed to be true (Kinnear and Gray, 2004). If so, the β_i and β_j are not equal, which implies that the CCR and BCC models have different outcome scores. Hence, it infers that data are more likely exhibiting VRS, in which case the BCC model should be used.

DMU	DEA Efficiency Scores	
	CRS (CCR)	VRS (BCC)
Art	1.000	1.000
Business	1.000	1.000
Education	1.000	1.000
Engineering	1.000	1.000
Science	1.000	1.000
IT	0.980	0.980
Law	0.933	0.933
Mean	0.988	0.988

Table 3.10 The outcomes of CRS (CCR) and VRS (BCC) and their means.

Table 3.10 shows that the DEA efficiency scores for CRS (i.e. CCR model) and VRS (i.e. BCC model) have similar means. This suggests that the data are more likely to exhibit constant returns to scale (CRS).

3.4.6 Improving the Law Department

In this case study the steps of the interactive framework were applied to the real data collected from the University of Bahrain (see Table 3.9). This data was entered into Excel Solver and the Lingo software with Excel Solver used to identify the efficiency scores and Lingo used to determine the normal vector. The results are shown in Table 3.11 which was obtained by running the output-oriented CCR dual DEA model (2.8a)

DMU	Dep.	DEA Score(1/θ)	Intensity (θ)	λ ₁	λ ₂	λ ₃	λ ₄	λ ₅	λ ₆	λ ₇
1	Art	1.000	1.000	1.000	0.000	0.000	0.000	0.000	0.000	0.000
2	Bus.	1.000	1.000	0.000	1.000	0.000	0.000	0.000	0.000	0.000
3	Edu.	1.000	1.000	0.000	0.000	1.000	0.000	0.000	0.000	0.000
4	Eng.	1.000	1.000	0.000	0.000	0.000	1.000	0.000	0.000	0.000
5	Sci.	1.000	1.000	0.000	0.000	0.000	0.000	1.000	0.000	0.000
6	IT	0.980	1.021	0.191	0.228	0.000	0.000	0.000	0.000	0.000
7	Law	0.861	1.161	0.000	0.181	0.000	0.141	0.040	0.000	0.000

Table 3.11 The DEA CCR model outcome score.

Table 3.11 shows two inefficient departments: IT and Law (in bold). The Law department was considered because it had the lowest efficiency score of 86% and its virtual or composite components on the efficient frontier are a convex combination of: 0.181 of the Business Department, 0.141 of Engineering Department and 0.040 of the Science Department. It is worth noting that the sums of the convex combinations for these inefficient departments is not normal (i.e. $\sum_{j=1}^7 \lambda_j \neq 1$).

In the first step the P and Q vectors need to be determined and this was achieved by interviewing the department staff. Recall that P and Q may cause an increase on the feasible region and if this occurs then it could imply that the collected raw data could be considered realistic. The staff agreed that no extra academic staff would be needed to increase publications by one but this would be at the expense of graduates, which needed

to be decreased by one. Therefore, the P and Q vectors could be written as $P=(0,0)$ and $Q=(-1,1)$. This seems reasonable because to publish more research, the staff need to have more time to undertake the research and write the publications and this might be achieved by decreasing their teaching load through reducing the number of students.

By including these decisions and running model (2.8a) it was found that DMU5, DMU6 and DMU7 became inefficient as shown by the italic entries in Table 3.12 below.

DMU	$1/\theta$	θ	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7
1	1.000	1.000	1.000	0.000	0.000	0.000	0.000	0.000	0.000
2	1.000	1.000	0.000	1.000	0.000	0.000	0.000	0.000	0.000
3	1.000	1.000	0.000	0.000	1.000	0.000	0.000	0.000	0.000
4	1.000	1.000	0.000	0.000	0.000	1.000	0.000	0.000	0.000
5	<i>0.990</i>	<i>1.010</i>	<i>0.108</i>	<i>0.000</i>	<i>0.000</i>	<i>0.374</i>	<i>0.000</i>	<i>0.000</i>	<i>0.000</i>
6	<i>0.952</i>	<i>1.051</i>	<i>0.000</i>	<i>0.414</i>	<i>0.000</i>	<i>0.000</i>	<i>0.000</i>	<i>0.000</i>	<i>0.000</i>
7	<i>0.780</i>	<i>1.283</i>	<i>0.017</i>	<i>0.123</i>	<i>0.232</i>	<i>0.000</i>	<i>0.000</i>	<i>0.000</i>	<i>0.000</i>

Table 3.12 Changes in efficiency after adding the P and Q vectors.

The range by which the objectives can be varied is determined by generating the pay-off table using model 3.16. Table 3.13 shows that to improve each DMU other DMUs need to be sacrificed, i.e. DMU1 (Art Department) needs to be excluded. Furthermore, it shows that for the inefficient DMUs (5, 6 and 7) both outputs can be further improved. For example, the Law Department f_1 can be increased from 50.80 up to 199.88 and f_2 from 7.89 up to 28.78. Hence, for DMU7 (Law Department) the maximum composite output for \bar{f}_1 is 277.75 whereas for \bar{f}_2 is 26.97. Hence, the maximum values of the both outputs can be expressed as a maximum output vector as $\bar{f} = [199.88, 28.78]$ which was used in the next step of the analysis.

		Max f_1		Max f_2		Maximum Values	
DMU	Dep	f_1	f_2	f_1	f_2	f_1	f_2
1	Art	496.00	99.00	496.00	99.00	496.00	99.00
2	Bus.	632.00	20.00	99.72	56.49	632.00	56.49
3	Edu.	489.00	16.00	144.88	82.07	489.00	82.07
4	Eng.	359.00	75.00	191.92	108.71	359.00	108.71
5.	Sci	208.24	9.57	143.00	81.00	208.24	81.00
6	IT	261.80	8.28	122.73	40.33	261.80	40.33
7	Law	199.88	7.89	50.80	28.78	199.88	28.78

Table 3.13 The new pay-off table.

The interactive procedure was started by solving model (3.14) after changing the weight values of both objectives. The initial starting optimal solution came from applying the basic dual DEA model (2.8a) for the Law Department. The composite output will be located on the efficient frontier (the DM can accept or reject this point) and was: $\lambda^0 = (0.0174, 0.123, 0.232, 0.000, 0.000, 0.000, 0.000)^T$ with $f_1^t(\lambda^0) = 199.88$ and $f_2^t(\lambda^0) = 28.78$.

Note that the maximum value of output 1 obtained from the pay-off table is similar to that obtained from the DEA model (2.8a), i.e. $\bar{f}_1 = f_1^t = f_1(\lambda^0) = 199.88$. This reflects that the composite unit is an extreme point on the efficient frontier and f_1 is dominant. At this point the weighted vector can be written as $w_r = (1, 0)^T$ and the normal vector as $N^0 = (1, 0)^T$.

If the DM did not agree with these initial trade-offs then this implies that this solution is not the MPS. We can search for the MPS around \bar{f}_1 or \bar{f}_2 by making a small variation in both of them (i.e. sacrificing of one of the DMU in favour of another), which could not be done in basic DEA models. By considering f_1 as a reference objective that can be sacrificed a slight decrease in f_1^t can be made. For instance, f_1^t can be chosen as 199.87 ($< \bar{f}_1 = 199.88$) and hence the targeted objective values denoted by $f_r(\lambda^0) = (199.87, 7.90)$ measured by solving model (3.16) with the new weights calculated as

$$w_1 = \frac{1}{199.88 - 199.87} = 100 \text{ and } w_2 = \frac{1}{28.78 - 7.90} = 0.048. \text{ The latter is related to a}$$

perturbation technique often employed in interactive MOLP and is used to start the interactive process. The composite output unit for this change was $\lambda^0 = (0.0174, 0.123, 0.232, 0.000139, 0.000, 0.000, 0.000)^T$ generated by solving formulation (3.16) and the normal vector was estimated by equation (2.27) to be $N^0 = (0.028, 0.048)^T$. At this point the department is a linear convex combination of DMU1, DMU2, DMU3 and DMU4 (Arts Department 0.0174, Business Department 0.123, Education Department 0.232 and Engineering Department 0.000139) at which $f_1(\lambda^0) = 199.87$ and $f_2(\lambda^0) = 7.90$. It is worth noting that the latter values are closer to that obtained from the composite output DEA model (2.8a), so the efficiency score point will be closer to the efficient frontier. Note that f_j does not exceed the maximum value obtained from the pay-off table.

<u>Objectives</u>		<u>Decision variables</u>							<u>Normal vectors</u>	
$f_1(\lambda^0)$	$f_2(\lambda^0)$	λ_1^0	λ_2^0	λ_3^0	λ_4^0	λ_5^0	λ_6^0	λ_7^0	N_1^0	N_2^0
199.87	7.90	0.0173	0.123	0.232	0.000139	0.00	0.00	0.00	0.028	0.048

Table 3.14 The objectives, decision variables and normal vectors for the initial values.

The optimum indifference trade-offs can be measured by equation (2.24) as $df^0 = (1.00, 0.583)^T$. If the DM does not agree with this indifference trade-off, which reflects that the initial target values are not the most preferred outcome, then he can suggest new set of indifference trade-offs. The staff in the Law Department suggested the marginal rate of substitution as $M^0 = (1.00, 0.500)^T$. This will increase f_2 by two ($f_2^{new} = f_2^{old} + 2$), for a unit reduction in f_1 ($f_1^{new} = f_1^{old} - 1$). Table 3.15 below compares the old and new indifference trade-offs.

Indifference trade-offs	Values
Old trade-offs	$(199.87, 7.90) \leftrightarrow (199.87-1.00, 7.90+1.72)$
New trade-offs	$(199.87, 7.90) \leftrightarrow (199.87-1.00, 7.90+2.00)$

Table 3.15 Comparison between old and new trade-off values.

The new trade-off proposed by the DM was to improve research output (increase publications) (i.e. f_2) at the expense of reducing the number of students. The new trade-off direction is determined by applying the gradient projection method using equation (2.32) resulting in $\Delta \bar{f}^0 = (-0.528, 0.308)^T$. To calculate the maximum allowable trade-off step size, equation (2.34) was implemented resulting in $(\alpha_{\max}^0 \sim 68)$ which was then used to build the step sizes in Table 3.16. Table 3.16 shows decreasing f_1 and increasing f_2 for each incremental step size between the current estimates of 9.68 of f_2 and its maximum feasible value of 28.78. The LHS of Table 3.16 shows increments of 0.1 (C10) for α between 0 and 1 whilst the RHS shows increments of 0.01 (C100) for α between 0 and 0.1. The purple shaded area shows the MPS.

C10			C100		
α	$f_1(\bar{\alpha})$	$f_2(\bar{\alpha})$	α	$f_1(\bar{\alpha})$	$f_2(\bar{\alpha})$
0.0	199.87	7.90	0.00	199.87	7.90
0.1	196.29	9.99	0.01	199.51	8.11
0.2	192.71	12.08	0.02	199.15	8.32
0.3	189.13	14.16	0.03	198.80	8.53
0.4	185.54	16.25	0.04	198.44	8.74
0.5	181.96	18.34	0.05	198.08	8.94
0.6	178.38	20.43	0.06	197.72	9.15
0.7	174.80	22.52	0.07	197.36	9.36
0.8	171.22	24.60	0.08	197.00	9.57
0.9	167.64	26.69	0.09	196.65	9.78
1.0	164.06	28.78	0.10	196.29	9.99

Table 3.16 Trade-off steps.

At the next iteration, the weighting vector was determined by using equation (2.36) where $w^1 = (18.78, 1.00)^T$. Feeding the updated weights in formulation (3.16) to estimate the convex combination $\lambda^1 = (0.0124, 0.128, 0.216, 0.171, 0.000, 0.000)^T$ of the observed DMU the new efficient solution was measured as $f(\lambda^1) = (198.8, 8.53)^T$ and the Normal vector calculated as $N^1 = (0.57, 0.97)^T$. Note that the updated Normal vector is a multiple of the previous one (i.e. $N^1 \sim 20N^0$). This result was expected because the normal vector should locate on the line segment between the efficient point obtained from DEA models and the MPS. The optimal indifference in this case being $df^1 = (1.00, 0.583)^T$.

Staff in the Law Department were looking for more publications requiring an optimal indifference trade-off of (198.80-1.00, 8.53+1.78), which is unsatisfactory because they proposed alternative indifference trade-off of (198.80-1.00, 8.53+2.22). Subsequently, the marginal rate of substitution can be approximated as $M^1 = (1.00, 0.450)^T$, the gradient projection as $\Delta \bar{f}^1 = (-0.537, 0.313)^T$ and the maximum step size as $\alpha_{\max}^1 \sim 63$.

Now suppose the DM sets the target level of f_2 to 10.11. As before step size tables of C10 and C100 were constructed with $w^2 = (4.83, 1.00)^T$ and feeding w^2 in formulation (3.16), the new convex combination of the observed DMU became: $\lambda^2 = (0.000, 0.142, 0.174, 0.0601, 0.000, 0.000, 0.000)^T$ and the new efficient solution was $f(\lambda^2) = (196.01, 10.11)^T$. The new normal vector was $N^2 = (0.52, 0.90)^T$ being a multiple of initial normal vector (i.e. $N^2 \sim 18N^0$). The indifference trade-off was estimated as $df^2 = (1.00, 0.578)^T$. The staff accepted this optimal indifference trade-off (198.86-7.55, 20.00+1,) and the process was terminated with $f(\lambda^2) = (199.86, 20.00)^T$ which became the MPS. The calculations are shown in Table 3.17 below.

t	λ^t	f^t	θ	M^t	N^t	$\Delta \bar{f}^t$	α_{\max}	α	$w^{t+!}$
0	$\begin{pmatrix} 0.0174 \\ 0.1230 \\ 0.2320 \\ 0.0002 \\ 0.0000 \\ 0.0000 \\ 0.0000 \end{pmatrix}$	$\begin{pmatrix} 199.87 \\ 7.90 \end{pmatrix}$	1.0	$\begin{pmatrix} 1.00 \\ 0.50 \end{pmatrix}$	$\begin{pmatrix} 0.028 \\ 0.048 \end{pmatrix}$	$\begin{pmatrix} -0.53 \\ 0.31 \end{pmatrix}$	68	0.03	$\begin{pmatrix} 18.78 \\ 1.00 \end{pmatrix}$
1	$\begin{pmatrix} 0.0124 \\ 0.1280 \\ 0.2160 \\ 0.1710 \\ 0.0000 \\ 0.0000 \\ 0.0000 \end{pmatrix}$	$\begin{pmatrix} 198.80 \\ 8.53 \end{pmatrix}$	20	$\begin{pmatrix} 1.00 \\ 0.45 \end{pmatrix}$	$\begin{pmatrix} 0.57 \\ 0.97 \end{pmatrix}$	$\begin{pmatrix} -0.537 \\ 0.313 \end{pmatrix}$	63	0.08	$\begin{pmatrix} 4.83 \\ 1.00 \end{pmatrix}$
2	$\begin{pmatrix} 0.0000 \\ 0.1418 \\ 0.1736 \\ 0.0600 \\ 0.0000 \\ 0.0000 \\ 0.0000 \end{pmatrix}$	$\begin{pmatrix} 196.01 \\ 10.11 \end{pmatrix}$	1.0	$\begin{pmatrix} 1.00 \\ 1.73 \end{pmatrix}$	$\begin{pmatrix} 0.52 \\ 0.90 \end{pmatrix}$	----	----	----	----

Table 3.17 The steps of the interactive procedure up to the second iteration.

The DEA model suggests that to improve the efficiency of the Law Department, graduated students should be increased by 36% and published papers increased by 47% as can be seen from Table 3.18. These results were obtained with composite outputs resulting from projecting the Law Department on the efficient frontier. These results (efficiency scores) were obtained by comparing the targeted DMU score, which is inefficient, relative to efficient DMUs. It is worth noting that by using the basic CCR model these results could not be improved or changed and hence the DM has no choice but to accept them.

Unlike basic DEA, which compares the efficiency of the targeted DMU relative to the others, the developed DEA-MOLP integrated model seeks the most preferred solution (MPS) according to DM preferences. For instance, in the Law Department the DM managed to increase publications up to 28%; from the value proposed by CCR model but to do so the number of graduated students should be reduced by almost 2% and expenses should be raised by 0.10%.

	Inputs			Outputs	
	Academic Staff	Registered Students.	Expenses	Graduated Students.	Published Papers
Raw value	27	1145	32458	147	15
DEA results	27	1145	32458	200	8
Improvements %	0.00%	0.00%	0.00%	35.97%	47.37%
Raw value	27	1145	32458	147	15
DEA_MOLP results	27	1145	32425	196	10
Improvements %	0.00%	0.00%	0.10%	33.34%	32.58%
DEA results	27	1145	32458	200	8
DEA_MOLP results	27	1145	32425	196	10
Improvements %	0.00%	0.00%	0.10%	1.94%	28.11%

Table 3.18 Summary of target setting and resource allocation for the Law Department.

In comparison the DEA_MOLP integrated model allows alternative outcomes according to DM preferences which, in this case adds more flexibility to the basic DEA solution. Nevertheless the basic DEA method did find the composite outcome of the inefficient DMU projection on the efficient frontier.

3.5 Case study#2: Private Hospitals in Bahrain

The data were collected from the Ministry of Health in Bahrain (MHB) and comprised ten DMUs (hospitals). Similar procedures to those used in section 3.3.1, were implemented to select the variables, three inputs (Physicians, Nurses and Dentists) and two outputs (In-patients and Out-patients) as shown in Table 3.19 (see the Appendices for a detailed analysis).

DMU	Hospital	Physicians	Nurses	Dentists	Out-patients treated	In-patients admitted
1	American Mission	68	109	15	186294	2528
2	Awali	18	43	2	37671	942
3	Bahrain International	58	62	6	85939	3924
4	Ibn Al-Nafees	39	48	1	96987	3483
5	Bahrain Specialist	43	50	2	83406	4449
6	Gulf Dental Speciality	1	16	10	14228	27
7	Al Amal	13	16	2	13617	12
8	Al Hillal	18	22	1	41013	431
9	Salmaniya Medical	625	1677	6	518287	46738
10	Bahrain Defence	229	758	12	460496	24616

Table 3.19 The raw data after a statistical analysis was used to identify the variables.

3.5.1 Measuring hospital performance

The CCR DEA model was applied to measure the performance of each DMU relative to the others within the group and the results are shown in Table 3.20 below. The bold figures in Table 3.20 indicate the inefficient DMUs.

DMU	1/θ	θ	λ ₁	λ ₂	λ ₃	λ ₄	λ ₅	λ ₆	λ ₇	λ ₈	λ ₉	λ ₁₀
1	1.00	1.00	1.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
2	0.81	1.24	0.12	0.00	0.00	0.26	0.00	0.00	0.00	0.00	0.00	0.00
3	0.77	1.30	0.00	0.00	0.00	0.51	0.75	0.00	0.00	0.00	0.00	0.00
4	1.00	1.00	0.00	0.00	0.00	1.00	0.00	0.00	0.00	0.00	0.00	0.00
5	1.00	1.00	0.00	0.00	0.00	0.00	1.00	0.00	0.00	0.00	0.00	0.00
6	1.00	1.00	0.00	0.00	0.00	0.00	0.00	1.00	0.00	0.00	0.00	0.00
7	0.42	2.37	0.00	0.00	0.00	0.33	0.00	0.00	0.00	0.00	0.00	0.00
8	0.92	1.08	0.00	0.00	0.00	0.46	0.00	0.00	0.00	0.00	0.00	0.00
9	1.00	1.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	1.00	0.00
10	1.00	1.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	1.00

Table 3.20 Inefficient DMUs scores obtained from using the CCR DEA analysis.

The DM within the Ministry of Health in Bahrain desired that In-patients in the Bahrain Defence hospital should be increased by 3 with no extra resources. By setting the Q vector to $(0, 3)^T$ and the P vector to $(0, 0)^T$ and running model (3.16) the outcome scores shown in Table 3.21 were obtained.

DMU	1/θ	θ	λ ₁	λ ₂	λ ₃	λ ₄	λ ₅	λ ₆	λ ₇	λ ₈	λ ₉	λ ₁₀
1	1.00	1.00	1.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
2	0.81	1.24	0.12	0.00	0.00	0.26	0.00	0.00	0.00	0.00	0.00	0.00
3	0.69	1.46	0.00	0.00	0.00	1.29	0.00	0.00	0.00	0.00	0.00	0.00
4	1.00	1.00	0.00	0.00	0.00	1.00	0.00	0.00	0.00	0.00	0.00	0.00
5	0.83	1.21	0.00	0.00	0.00	1.04	0.00	0.00	0.00	0.00	0.00	0.00
6	1.00	1.00	0.00	0.00	0.00	0.00	0.00	1.00	0.00	0.00	0.00	0.00
7	<i>0.42</i>	<i>2.37</i>	<i>0.00</i>	<i>0.00</i>	<i>0.00</i>	<i>0.33</i>	<i>0.00</i>	<i>0.00</i>	<i>0.00</i>	<i>0.00</i>	<i>0.00</i>	<i>0.00</i>
8	0.92	1.08	0.00	0.00	0.00	0.46	0.00	0.00	0.00	0.00	0.00	0.00
9	0.89	1.12	0.00	0.00	0.00	6.00	0.00	0.00	0.00	0.00	0.00	0.00
10	0.80	1.25	0.46	0.00	0.00	5.07	0.00	0.00	0.00	0.00	0.00	0.00

Table 3.21 The outcomes after adding the P and Q vectors.

In Table 3.21 the inefficient DMUs are shown in bold and the observed DMU in italics and the addition of the DM preferences has caused several DMUs to have scores less than one, for example, DMU5, DMU9, and DMU10.

3.5.2 Improving the Al Amal private hospital

The DM also wanted to improve the performance of DMU7 (Al Amal Hospital), (shown by italics in Table 3.21) and therefore the production trade-off MOLP model was implemented. Initially the trade-off DEA model (2.8a) was used to measure the performance of the observable DMU among the others. The DM had the choice to accept this outcome, which provided the combination of decision variables of the observable DMU (i.e. a point on the efficient frontier) or to start the interactive procedure by implementing the perturbation technique. In this case, the MPS is satisfied and the maximum composite from the pay-off table is similar to the trade-off DEA model (2.8a), i.e. $f_1^0(\text{payoff}) = f_1^0(\text{trade offs DEA}) = 32329$, hence, it is suitable to have f_1 as a reference.

DMU	Hospital	Max f_1		Max f_2		f_1	f_2
		f_1	f_2	f_1	f_2		
1	American Mission	186294	2528	132890	7092	186294	7092
2	Awali	46758	1196	35646	1904	46758	1904
3	Bahrain International	125275	4498	103423	5517	125275	5517
4	Ibn Al-Nafees	96987	3483	96987	3483	96987	3483
5.	Bahrain Specialist	101028	3628	83406	4449	101028	4449
6	Gulf Dental Speciality	14228	27	2011	108	14228	108
7	Al Amal	32329	1161	25245	1346	32329	1347
8	Al Hillal	44452	1596	34950	1864	44452	1864
9	Salmaniya Medical	581922	20898	518287	46738	581922	46738
10	Bahrain Defence	577434	18813	460496	24616	577434	24616

Table 3.22 The maximum values of the output variables from the pay-off table.

The iterative procedure was started by making small changes to f_1 and f_2 of $f_1^0 = 32328.99$ (< 32329) and $f_2^0 = 1161.01$ (> 1161). The initial weighted vector was $w_1^0 = \frac{1}{32329 - 32328.99} = 0.01$ and $w_2^0 = \frac{1}{1346.7 - 1161.01} = 0.00538$ which was fed into model (3.14) to find the composite output combination of $\lambda^0 = (0.000, 0.000, 0.000, 0.333, 0.000, 0.000, 0.000, 0.000, 0.000, 0.000)$ and the normal vector, calculated by using equation (2.27), as $N^0 = (0.000151, 0.00538)^T$. The optimal indifference trade-off vector for $f(\lambda^0)$ was estimated by using equation (2.24) with $df^0 = (1.00, 0.028)^T$.

<u>Objectives</u>		<u>Decision variables</u>							<u>Normal vectors</u>	
$f_1(\lambda^0)$	$f_2(\lambda^0)$	λ_1^0	λ_2^0	λ_3^0	λ_4^0	λ_5^0	λ_6^0	λ_7^0	N_1^0	N_2^0
577433.6	18812.7	0.462	0.000	0.000	5.066	0.000	0.000	0.000	0.000377	0.000172

Table 3.23 The parameters used to start the interactive procedure

The new marginal rate of substitution was $M = (1.00, 30.0)$. Table 3.24 below compares the old and new indifference trade-offs.

<u>Indifference trade-offs</u>	<u>Values</u>
Old trade-offs	$(32329, 1161) \leftrightarrow (32329-1.00, 1161+35.6)$
New trade-offs	$(32329, 1161) \leftrightarrow (32329-1.00, 1161+30.0)$

Table 3.24 Comparing the old and new indifference trade-offs.

The gradient projection direction of this change was calculated using equation (2.32) giving $\Delta \bar{f}^1 = (-0.158, 0.004)^T$. The maximum step size was measured as $\alpha_{\max}^0 \sim 41909$ using equation (2.34) from which the step size table was established and f_1 and f_2 were generated as shown in Table 3.25. As in the previous section, Table 3.25 shows increments of 0.1 (C10) for α between 0 and 1 on the LHS whilst the RHS shows increments of 0.01 (C100) for α between 0 and 0.1. The purple shaded area shows the MPS.

C10		
α	$f_1(\bar{\alpha})$	$f_2(\bar{\alpha})$
0.0	32328.99	1161.01
0.1	31667.38	1179.58
0.2	31005.77	1198.15
0.3	30344.15	1216.72
0.4	29682.54	1235.29
0.5	29020.93	1253.86
0.6	28359.32	1272.43
0.7	27697.71	1291.00
0.8	27036.10	1309.57
0.9	26374.48	1328.13
1.0	25712.87	1346.70

C100		
α	$f_1(\bar{\alpha})$	$f_2(\bar{\alpha})$
0.20	31005.77	1198.15
0.21	30939.61	1200.01
0.22	30873.44	1201.86
0.23	30807.28	1203.72
0.24	30741.12	1205.58
0.25	30674.96	1207.43
0.26	30608.80	1209.29
0.27	30542.64	1211.15
0.28	30476.48	1213.00
0.29	30410.32	1214.86
0.30	30344.15	1216.72

Table 3.25 .Trade-off steps

It is worth noting that f_1 decreases whereas f_2 increases without exceeding the maximum value of f_2 obtained from the pay-off table.

According to the DM in the Al Amal hospital, in-patients should be increased by 3.4% in order to increase their revenue and be able to hire more staff but this will be at the expense of out-patients. The decrease in out-patients can be found from Table 3.25. New targets were set as $f^1 = (30940, 1200)$ with the convex combination $\lambda^1 = (0.000, 0.000, 0.000, 0.219, 0.103, 0.0204, 0.000, 0.000, 0.000, 0.000)^T$ which was generated by supplying the new weighted vector $w^1 = (0.11, 1.00)$ in model (3.16). As before the weighted vector was determined by using equation (2.36) with the normal vector $N^1 = (0.0222, 0.790)^T$ estimated by equation (2.27). It is worth noting that this target setting results in a convex combination of Bahrain International hospital (DMU3), Ibn Al Nafees hospital (DMU3), and Bahrain Specialist hospital (DMU5).

t	λ^t	f^t	θ	M^t	N^t	$\Delta \bar{f}^t$	α_{\max}	α	w^{t+1}
0	$\begin{pmatrix} 0.000 \\ 0.000 \\ 0.000 \\ 0.333 \\ 0.000 \\ 0.000 \\ 0.000 \\ 0.000 \\ 0.000 \\ 0.000 \\ 0.000 \end{pmatrix}$	$\begin{pmatrix} 32328.99 \\ 1161.01 \end{pmatrix}$	1.00	$\begin{pmatrix} 1.00 \\ 30.0 \end{pmatrix}$	$\begin{pmatrix} 0.000151 \\ 0.00539 \end{pmatrix}$	$\begin{pmatrix} -0.158 \\ 0.004 \end{pmatrix}$	41909	0.21	$\begin{pmatrix} 0.11 \\ 1.00 \end{pmatrix}$
1	$\begin{pmatrix} 0.000 \\ 0.000 \\ 0.000 \\ 0.261 \\ 0.066 \\ 0.013 \\ 0.000 \\ 0.000 \\ 0.000 \\ 0.000 \end{pmatrix}$	$\begin{pmatrix} 30940 \\ 1200 \end{pmatrix}$	147	$\begin{pmatrix} 1.00 \\ 20.0 \end{pmatrix}$	$\begin{pmatrix} 0.0222 \\ 0.790 \end{pmatrix}$	---	---	----	----

Table 3.26 Summary of the steps of the interactive procedure up to the first iteration.

The DEA outcome indicates that the Al Amal hospital could be improved by increasing the both Out-patients and In-patients up to 32329 and 1161 respectively. It also shows that no extra Physicians or Nurses are needed but that Dentists should be decreased by up to 83%.

	Inputs			Outputs	
	Physicians	Nurses	Dentists	Out-patients treated	In-patient admitted
Raw value	13	16	2	13617	12
DEA results	13	16	0.33	32329	1161
Improvements %	0.00%	0.00%	83.33%	137.42%	9575%
Raw value	13	16	2	13617	12
MOLP-DEA results	13	16	0.52	30940.16	1200.06
Improvements %	0.00%	0.00%	73.93%	127.22%	9901%
DEA results	13	16	0.33	32329	1161
MOLP-DEA results	13	16	0.52	30940.16	1200.06
Improvements %	0.00%	0.00%	56.42%	4.30%	3%

Table 3.27 Summary of targets and resource allocations for Al Amal hospital

However, the MOLP-DEA approach gave target composite outputs of (30940, 1200) for Out-patients and In-patients, which suggests that these new targets could be achieved by decreasing Dentists by 74% with no changes to Physicians or Nurses.

In comparing the DEA and MOLP-DEA outputs, Table 3.27 shows that only Dentists should be decreased by almost 56% to obtain an increase in In-patients of 3% at the expense of a reduction in Out-patients of 4%.

3.6 Summary

DEA is a technique used to aid the DM to improve DMUs performance whilst MOLP is a technique that can be used to explore various possible alternative solutions. The core of the interactive framework developed in this chapter is the integration of MOLP and DEA employing the production trade-off approach. This allows the DMs to examine various realistic targets and explore which of them could be achieved technically.

The developed framework has two phases: (a) the method of choosing P and Q and (b) the iterative procedure. The iterative MOLP procedure is achieved by using the interactive gradient projection method, which involves determining normal vectors on the efficient frontier. The normal vector provides a vigorous measure to test whether the most preferred decision is achieved which maximise the implicit utility function of the DM, or whether the DM should be satisfied with the optimal indifference trade-offs. The projection of the utility gradient onto the tangent plane of the efficient frontier using the normal vector can guide the DM to manage a direction by which the targeted DMU could be improved.

The equivalent MOLP-DEA model was tested by using real data collected from the University of Bahrain in which the number of input and output variables was selected by a statistical analysis.

The findings of this study have important implications for future practice. First, the P and Q vectors are incorporated in MOLP-DEA model for improving the discrimination of the DEA model. Second, constructing a well-defined framework realistically allows the DMs to manage improving the targeted DMU in a technical manner.

Chapter 4

Integrating bounded DEA and minimax MOLP

4.1 Introduction

DEA was developed to analyse the relative efficiency of DMUs. For best performance of a DMU relative efficiency is less than or equal to one. On the other hand, for worst performance relative efficiency measure is within the range of greater than or equal to one. Contrary to the best relative efficiencies, which define the efficient frontier, the worst relative efficiencies define an inefficient frontier. Investigating the best and worst efficiency allows DMs to estimate and assess overall efficiency of DMUs from different points of view. Recently, Wang and Yang (2007) developed a model, called the bounded model, which allows the DMs to measure the best and worst performance using a unified model (i.e. upper and lower bounds of the DMUs). The values of the best and worst efficiency are determined by introducing a virtual anti-ideal DMU, which is defined as the absolutely worst performance among all DMUs. Although the bounded model enables the best and worst efficiency to be measured, the DM has no choice but to accept the results without any alternatives. Furthermore, the bounded model is not designed to include the DMs preference or value judgements in assessing the DMUs and setting targets and therefore DMs are restricted in their ability to change resource (i.e. input) values.

4.1.1 Aims

The main goal of this study is to establish an interactive framework based on integrating MOLP and the bounded DEA model through which the DM can exam various achievable targets and hence choose the most suitable one according to his preferences. Moreover, by using the developed framework the DM is allowed to examine several alternatives of costs, outputs, inputs and savings by making each inefficient DMU efficient.

4.1.2 Research Questions

- How can the bounded DEA model be integrated with the minimax MOLP approach?
- How can the proposed framework aid the DMs to have a proper decision?
- What are the achievable targets that enable the DMs to choose from?
- How can we incorporate the DM preferences?
- How can we guide the DMs to satisfy their goals?

4.1.3 Research methodology

This study consists of two parts: a theoretical derivation and a case study. The theoretical part presents an equivalent MOLP-DEA model by integrating the MOLP minimax method and the bounded DEA model proposed by Wang and Yang. The case study was conducted by collecting data from a private hospital in Bahrain. A framework is presented and implemented to improve the performance of the International Hospital of Bahrain.

4.1.4 Chapter structure

This chapter presents the following:

- An overview of relative efficiency
- The bounded DEA-MOLP model
 - Theoretical derivation
 - An illustrative example
- An interactive framework
- A case study: The Bahrain International Hospital
- Chapter summary

4.2 Overview of relative efficiency

Conventional DEA assumes that all the data comprise exact and precise numerical values. However, in some cases imprecise data such as ordinal data, as reported Cook et al. (1993 and 1996) and interval data as reported by Cooper et al. (1999) may be collected. A typical example is faculty positions, e.g. full professor, associate professor, assistant professor, etc., and academic degrees such as doctorate, master and bachelor. Interval data were reported by Cooper et al. (1999, 2001), which is associated with the imprecise DEA model (IDEA).

Intuitively efficiency scores of imprecise data are imprecise as well. For instance, the efficiency scores appear in ranges, rather than being exact. Using imprecise data induces a non-linear analysis that adds more complexity in finding the optimum solution. Cooper et al. (1999) proposed a transformation method to tackle this problem and the scale transformation was simplified by Zhu (2003) to reduce the computational burden. The efficiency scores of this approach could be used only to measure the upper bound, Kao (2006).

Overall efficiency of a DMU within a firm or organisation can be measured by finding the worst relative efficiency and the best relative efficiency. The best relative efficiencies are those scores that have values between zero and one, whereas the worst relative efficiency

are those that have relative efficiency scores greater than or equal to one (Wang and Yang, 2007). Hence, DMU efficiency scores measured in different ranges have different meanings and measuring the efficiency scores in some of these ranges causes one-sided efficiency. In their proposed model DMU efficiencies are estimated within the range of the interval defined by the upper and lower bound determined through introducing virtual anti-ideal DMU (the lowest DMU performance). Previous models proposed to measure the worst and best relative efficiency have been presented by Doyle et al (1995) and Entani (2002), both sharing the same structure, Wang and Yang (2007).

In this chapter, the integrated model of MOLP and bounded DEA model presented by Wang and Yang (2007) are further developed to build a framework which incorporates the DM's preferences interactively through the DEA assessment procedure.

4.3 The bounded DEA-MOLP model

The aim of this section is to present the theoretical derivation of the integrated MOLP-DEA model, which allows for the interaction of a DM during DEA-oriented preference assessment and planning. The projection gradient interactive method is implemented to explore the most preferred performance target according to the DM preferences.

In 2000, Parkan and Wang reported that the worst and the best relative efficiency can be calculated by using the two DEA models. The DEA model for the best efficiency has efficiency scores between zero and one, whereas the DEA model for worst efficiency has an efficiency scores greater than one (Parkan and Wang, 2000).

The dual output oriented DEA model used to measure the best efficiency can be written as:

$$\begin{aligned}
 & \text{Max } h = \theta_{j_0} + \varepsilon(s_r^+ + s_i^-) \\
 & \text{Subject to} \\
 & \theta_{j_0} y_{rj_0} - \sum_{j=1}^n \lambda_j y_{rj} + s_r^+ \leq 0, \quad r = 1, \dots, s \\
 & \sum_{j=1}^n \lambda_j x_{ij} + s_i^- \leq x_{ij_0}, \quad i = 1, \dots, m \\
 & \lambda_j, s_r^+, s_i^- \geq 0
 \end{aligned} \tag{4.1}$$

The dual output oriented DEA model used to measure the worst efficiency can be written as:

$$\begin{aligned}
 & \text{Max } h = \theta_{j_0} - \varepsilon(s_r^+ + s_i^-) \\
 & \text{Subject to} \\
 & \theta_{j_0} y_{rj_0} - \sum_{j=1}^n \lambda_j y_{rj} - s_r^+ \geq 0, \quad r = 1, \dots, s \\
 & \sum_{j=1}^n \lambda_j x_{ij} - s_i^- \geq x_{ij_0}, \quad i = 1, \dots, m \\
 & \lambda_j, s_r^+, s_i^- \geq 0
 \end{aligned} \tag{4.2}$$

4.3.1 The equivalent MOLP model theoretical derivation

Formulation (2.10) for the bounded DEA model can be divided into two models: optimistic and pessimistic. The Optimistic Output Oriented model in primal form can be written as:

$$\begin{aligned}
 \text{Min } h_0 &= \sum_{r=1}^s v_r x_{ij_0} \\
 \text{subject to} & \\
 & \sum_{r=1}^s u_r y_{rj} - \sum_{i=1}^m v_i x_{ij} \leq 0, \quad j = 1, \dots, n \\
 & \sum_{r=1}^s u_r y_{rj} - \sum_{i=1}^m v_i (\theta_{ADMU} x_{ij}) \geq 0, \quad j = 1, \dots, n \\
 & \sum_{r=1}^s u_r y_{rj_0} = 1 \\
 & u_r, v_i \geq \varepsilon
 \end{aligned} \tag{4.3}$$

The dual form of the above model (4.3) is:

$$\begin{aligned}
 \text{Max } h &= \theta_{j_0} + \varepsilon (s_r^+ + s_i^-) \\
 \text{Subject to} & \\
 & \theta_{j_0} y_{rj_0} - \left[\sum_{j=1}^n (\lambda_j - \beta_j) y_{rj} - s_r^+ \right] \leq 0, \quad r = 1, \dots, s \\
 & \sum_{j=1}^n (\lambda_j - \theta_{ADMU} \beta_j) x_{ij} - x_{ij_0} + s_i^- \leq 0, \quad i = 1, \dots, m \\
 & \lambda_j, \beta_j, s_r^+, s_i^- \geq 0
 \end{aligned} \tag{4.4}$$

By using the following manipulation:

$$\text{Let } f_r(\lambda, \beta) = \sum_{j=1}^n (\lambda_j - \beta_j) y_{rj} - s_r^+$$

Then

$$\frac{1}{w_r} \theta_{j_0} - f_r(\lambda, \beta) \leq 0$$

$$\Leftrightarrow f_r^{\max} - f_r(\lambda, \beta) \leq f_r^{\max} - \frac{1}{w_r} \theta_{j_0} \quad \text{"}\Leftrightarrow\text{" means equals} \quad (4.5)$$

$$\Leftrightarrow w_r (f_r^{\max} - f_r(\lambda, \beta)) \leq w_r f_r^{\max} - \theta_{j_0}$$

$$\Leftrightarrow w_r (f_r^{\max} - f_r(\lambda, \beta)) \leq F^{\max} - \theta_{j_0}$$

$$\Leftrightarrow w_r (f_r^{\max} - f_r(\lambda, \beta)) \leq \theta, \quad \text{where } \theta = F^{\max} - \theta_{j_0}$$

and the objective function transformation as:

$$\begin{aligned} \text{Max } \theta_{j_0} &= \text{Min}(-\theta_{j_0}) = \text{Min}(F^{\max} - \theta_{j_0}) \\ &= \text{Min}(\theta) \end{aligned}$$

Where (4.6)

$$F^{\max} = \max_{1 \leq r \leq s} \{w_r \bar{f}_{rj_0}\} = \max_{1 \leq r \leq s} \left\{ \frac{\bar{f}_{rj_0}}{y_{rj_0}} \right\}$$

The equivalent MOLP model based on the minimax approach for the upper bound can be expressed as:

$$\text{Min } h = \theta - \varepsilon (s_r^+ + s_i^-)$$

Subject to

$$w_r \left[f_r^{\max} - \left(\sum_{j=1}^n (\lambda_j - \beta_j) y_{rj} - s_r^+ \right) \right] \leq \theta \quad (4.7)$$

$$\lambda_j, \beta_j \in \Omega = \left\{ \lambda_j, \beta_j \left| \sum_{j=1}^n (\lambda_j - \theta_{ADMU} \beta_j) x_{ij} + s_i^- \leq x_{i_0}, i = 1, \dots, m \text{ and } \lambda_j, \beta_j \geq 0 \right. \right\}$$

Note that

$$\theta = F^{\max} - \theta_{j_0} \geq w_r \bar{f}_{rj_0} - \theta_{j_0} \geq w_r f_r(\lambda) - \theta_{j_0} \geq 0; \quad r = 1, \dots, s$$

and

$$f^{\max} = \frac{F^{\max}}{w_r} \geq \frac{w_r \bar{f}_{rj_0}}{w_r} = \bar{f}_{rj_0} \max_{\lambda \in \Omega_{j_0}} f_r(\lambda) \quad (4.8)$$

On the other hand, the Pessimistic Output Oriented model in primal form can be written as:

$$\begin{aligned} \text{Max } h_0 &= \sum_{i=1}^m v_i x_{ij_0} \\ \text{subject to} \\ \sum_{r=1}^s u_r y_{rj} - \sum_{i=1}^m v_i x_{ij} &\leq 0, \quad j = 1, \dots, n \\ \sum_{r=1}^s u_r y_{rj} - \sum_{i=1}^m v_i (\theta_{ADMU} x_{ij}) &\geq 0, \quad j = 1, \dots, n \\ \sum_{r=1}^s u_r y_{rj_0} &= 1 \\ u_r, v_i &\geq \varepsilon \end{aligned} \quad (4.9)$$

The dual form for the above model (4.9) will be:

$$\begin{aligned} \text{Min } h &= \theta_{j_0} - \varepsilon (s_r^+ + s_i^-) \\ \text{Subject to} \\ \theta_{j_0} y_{rj_0} + \left[\sum_{j=1}^n (\lambda_j - \beta_j) y_{rj} - s_r^+ \right] &\geq 0 \\ \sum_{j=1}^n (\theta_{ADMU} \beta_j - \lambda_j) x_{ij} - x_{ij_0} - s_i^- &\geq 0 \\ \lambda_j, \beta_j, s_r^+, s_i^- &\geq 0 \end{aligned} \quad (4.10)$$

By using the following transformation:

$$\text{Let } f_r(\lambda, \beta) = \sum_{j=1}^n (\lambda_j - \beta_j) y_{rj} - s_r^+$$

Then

$$\begin{aligned} \frac{1}{w_r} \theta_{j_0} + f_r(\lambda, \beta) &\geq 0 \\ \Leftrightarrow \frac{1}{w_r} \theta_{j_0} &\geq -f_r(\lambda, \beta), & \text{"}\Leftrightarrow\text{" means equals} & \quad (4.11) \\ \Leftrightarrow \theta_{j_0} &\geq -w_r f_r(\lambda, \beta) \\ \Leftrightarrow F^{\min} + \theta_{j_0} &\geq F^{\min} - w_r f_r(\lambda, \beta) \\ \Leftrightarrow w_r (f_r^{\min} - f_r(\lambda, \beta)) &\leq F^{\min} + \theta_{j_0} \\ \Leftrightarrow w_r (f_r^{\min} - f_r(\lambda, \beta)) &\leq \theta, & \text{where } \theta = F^{\min} + \theta_{j_0} \end{aligned}$$

and the objective function transformation as:

$$\begin{aligned} \text{Min } \theta_{j_0} &= \text{Min} (F^{\min} + \theta_{j_0}) \\ &= \text{Min}(\theta) \end{aligned} \quad (4.12)$$

Where

$$F^{\min} = \min_{1 \leq r \leq s} \{w_r \bar{f}_r\} = \min_{1 \leq r \leq s} \left\{ \frac{\bar{f}_r}{y_r} \right\}$$

The equivalent MOLP pessimistic model (lower bound) is:

$$\begin{aligned} \text{Min } h &= \theta - \varepsilon (s_r^+ + s_i^-) \\ \text{Subject to} & \\ w_r \left[f_r^{\min} - \left(\sum_{j=1}^n (\beta_j - \lambda_j) y_{rj} + s_r^+ \right) \right] &\leq \theta \\ \beta_j, \lambda_j \in \Omega &= \left\{ \beta_j, \lambda_j \left| \sum_{j=1}^n (\theta_{ADMU} \beta_j - \lambda_j) x_{ij} - s_i^- \geq x_{ij_0}, i = 1, \dots, m \text{ and } \beta_j, \lambda_j \geq 0 \right. \right\} \end{aligned} \quad (4.13)$$

Note that:

$$\theta = F^{\min} - \theta_{j_0} \leq w_r \bar{f}_r - \theta_{j_0} \leq w_r f_r(\lambda) - \theta_{j_0} \geq 0; \quad r = 1, \dots, s$$

(4.14)

and

$$f_r^{\min} = \frac{F^{\min}}{w_r} \leq \frac{w_r \bar{f}_r}{w_r} = \bar{f}_r = \min_{\lambda \in \Omega_{j_0}} f_r(\lambda);$$

4.3.2 An illustrative example

The equivalence of the derived MOLP model with the DEA CCR output oriented dual model is demonstrated through the hospital data in case study #2 presented in section 3.5. For convenience the raw data is reproduced in Table 4.1 below.

DMU	Inputs			Outputs	
	Physicians	Nurses	Dentists	Out-Patients	In-Patients
1	68	109	15	186294	2528
2	18	43	2	37671	942
3	58	62	6	85939	3924
4	39	48	1	96987	3483
5	43	50	2	83406	4449
6	1	16	10	14228	27
7	13	16	2	13617	12
8	18	22	1	41013	431
9	625	1677	6	518287	46738
10	229	758	12	460496	24616
Most inputs and least outputs	625	1677	15	13617	12

Table 4.1 Input and output data collected from MHB.

The Anti-ideal DMU (θ_{ADMU}) is defined as the most inputs to produce the least outputs. Hence, its performance is the worst among all other DMUs and it should have the smallest efficiency and can be estimated by using the following model

$$\text{Max } \theta_{ADMU} = \sum_{r=1}^s u_r y_r^{\min}$$

Subject to

$$\sum_{r=1}^s u_r y_{rj} - \sum_{i=1}^m v_i x_{ij} \leq 0, \quad j = 1, \dots, n \quad (4.15)$$

$$\sum_{i=1}^m v_i x_i^{\max} = 1$$

$$u_r, v_i \geq \varepsilon$$

Feeding the data presented in Table 4.1 in the above model (4.15), the anti-ideal DMU can be measured and is equal to 0.00936. To calculate the lower and upper limits the DEA in dual form was applied and Table 4.2 below presents the results.

DMU	The best efficiency	The worst efficiency	Bounded DEA model
1	1.0000	2.4202	[0.0227, 1.000]
2	0.8057	1.5171	[0.0142, 0.806]
3	0.7671	1.5349	[0.0144, 0.767]
4	1.0000	2.6192	[0.0245, 1.000]
5	1.0000	2.1380	[0.0200, 1.000]
6	1.0000	1.0000	[0.0094, 1.000]
7	0.4212	1.0000	[0.0049, 0.421]
8	0.9226	2.2370	[0.0209, 0.923]
9	1.0000	1.0000	[0.0094, 1.000]
10	1.0000	1.4509	[0.0136, 1.000]
Anti-ideal DMU (θ_{ADMU})	0.00936		

Table 4.2 Best and worst efficiencies from applying DEA models 4.1 and 4.2.

The anti-ideal DMU was measured using model (4.15). The last column of Table 4.2 shows the results obtained from using the bounded model (4.4) for the upper bound and the bounded model (4.10) for the lower bound.

It is worth noting that for the upper limits, the best efficiency in Table 4.2 does not change from implementing the DEA or bounded models. However, the lower limits do change

when applying the bounded model which comes from the multiplication of the lower limits by the Anti-ideal DMU, (i.e. the worst efficiency multiplied by the Anti-ideal DMU).

In order to apply the MOLP bounded model, F^{max} and F^{min} should be measured. F^{max} expresses the maximum value of the upper bound, whereas F^{min} highlights the minimum value of the lower bound. To calculate F^{max} for the upper bound data, a pay-off table shown in Table 4.3 below was generated by using the following programme:

$$\begin{aligned}
 & \text{Max } \sum_{j=1}^n (\lambda_j - \beta_j) y_{rj} \\
 & \text{Subject to} \\
 & \sum_{j=1}^n (\lambda_j - \theta_{ADMU} \beta_j) x_{ij} \leq x_{ij_0} \\
 & \lambda_j, \beta_j \geq 0, r = 1, \dots, s \text{ and } i = 1, \dots, m
 \end{aligned} \tag{4.16}$$

Max f_1		Max f_2		f_1/y	f_2/y	F^{max}
f_1	f_2	f_1	f_2			
186294	2528	132891	7092	1	3	3
46758	1196	35646	1904	1	2	2
125275	4499	103423	5517	1	1	2
96987	3483	96987	3483	1	1	1
101028	3628	83406	4449	1	1	1
14228	27	2011	108	1	4	4
32329	1161	25245	1347	2	112	112
44452	1596	34950	1864	1	4	4
581922	20898	518287	46738	1	1	1
577434	18813	460496	24616	1	1	1

Table 4.3 Results obtained from applying model 4.16.

On the other hand, the F^{min} estimates presented in Table 4.4 below were calculated by using the following linear programme:

$$\text{Min } \sum_{j=1}^n (\lambda_j - \beta_j) y_{rj}$$

Subject to

$$\sum_{j=1}^n (\lambda_j - \theta_{ADMU} \beta_j) x_{ij} \geq x_{ij_0}$$

$$\lambda_j, \beta_j \geq 0, r = 1, \dots, s \text{ and } i = 1, \dots, m$$

(4.17)

Min f_1		Min f_2		f_1/y	f_2/y	F^{min}
f_1	f_2	f_1	f_2			
131753	4502	131753	4502	1	2	1
46953	1153	46953	1153	1	1	1
85939	3924	85939	3924	1	1	1
63174	2621	63174	2621	1	1	1
67158	2898	67158	2898	1	1	1
14228	27	14228	27	1	1	1
21058	874	21058	874	2	73	2
29026	1210	29026	1210	1	3	1
1787339	39537	1787339	39537	4	1	1
777424	14096	777424	14096	2	1	1

Table 4.4 Results obtained from applying model 4.17.

The equivalence of the bounded DEA model and the bounded MOLP model for the lower limits (worst efficiency) and upper limits (best efficiency) was tested. For the upper limits this was established by showing the equivalence of the equation $\theta = F^{\max} - \theta_{j_0}$ (see Table 4.5 below).

DEA model			Minimax model							
DMU	$1/\theta_{j_0}$	θ_{j_0}	F^{max}	Composite inputs and outputs					θ	$F^{max}-\theta_{j_0}$
				x_1	x_2	x_3	y_1	y_2		
1	1.000	1.000	2.81	68	109	15	186294	2528	1.81	1.81
2	0.806	1.241	2.02	18	43	2	37671	942	0.78	0.78
3	0.767	1.304	1.46	58	62	6	85939	3924	0.15	0.16
4	1.000	1.000	1.00	39	48	1	96987	3483	0.00	0.00
5	1.000	1.000	1.21	43	50	2	83406	4449	0.21	0.21
6	1.000	1.000	3.98	1	16	10	14228	27	2.98	2.98
7	0.421	2.374	112.23	13	16	2	13617	12	109.85	109.86
8	0.923	1.084	4.33	18	22	1	41013	431	3.24	3.25
9	1.000	1.000	1.12	625	1677	6	518287	46738	0.12	0.12
10	1.000	1.000	1.25	229	758	12	460496	24616	0.25	0.25

Table 4.5 Equivalence of the bounded DEA and MOLP models for upper limits.

The equivalence of the lower limits (worst efficiency) for the bounded DEA model and bounded MOLP model was achieved by establishing $\theta = F^{min} + \theta_{j_0}$ (see Table 4.6).

DEA model			Minimax model							
DMU	$1/\theta_{j_0}$	θ_{j_0}	F^{min}	Composite inputs and outputs					θ	$F^{min}+\theta_{j_0}$
				x_1	x_2	x_3	y_1	y_2		
1	0.0227	44.15	0.033	68	109	15	186294	2528	44.85	44.18
2	0.0142	70.43	0.034	18	43	2	37671	942	71.65	70.46
3	0.0144	69.61	0.014	58	62	6	85939	3924	70.61	69.62
4	0.0245	40.79	0.010	39	48	1	96987	3483	41.44	40.80
5	0.0200	49.97	0.009	43	50	2	83406	4449	50.62	49.98
6	0.0094	106.84	1.000	1	16	10	14228	27	107.84	107.84
7	0.0094	106.84	0.976	13	16	2	13617	12	108.39	107.82
8	0.0209	47.76	0.039	18	22	1	41013	431	48.47	47.80
9	0.0094	106.84	0.027	625	1677	6	518287	46738	107.69	106.87
10	0.0136	73.64	0.023	229	758	12	460496	24616	74.21	73.66

Table 4.6 Equivalence of the bounded DEA and MOLP models for lower limits

4.3.3 An interactive Framework

The interactive procedure was implemented to improve the Bahrain International Hospital and consists of the following steps (see chapter 2):

- Generate the pay-off table
 - Lower bound
 - Upper bound
- Generate the initial efficient solution
- Calculate the normal vector and checking of the optimality condition
- Determine the trade-off direction
- Determine the trade-off step size
- Update the weighting vector

4.4 Case study: The Bahrain International Hospital

The interactive approach was started by employing the DEA model to calculate the linear combination of the observed DMU3 (Bahrain International Hospital), which represents its projection on the efficient frontier. The DM had the choice to accept these results, in which case the obtained results became the most preferred solution (MPS) and the process would be terminated. If DM rejected the results then the interactive procedure would be initiated by perturbing the composite output (i.e. changing their values slightly).

If one of the composite outputs, obtained from DEA model, matches the one from those obtained from the pay-off table, the composite output unit is considered to be an extreme point on the efficient frontier. As a result the weight and the normal vectors of any of the composite output will be either zero or one and a weighted value of unity for any of composite output represents its dominance.

In this case f_2 was chosen as a reference for perturbation purposes. The composite outputs could be written as: $f_1(\lambda^0, \beta^0) = (103423.65, 0.00)$ and $f_2(\lambda^0, \beta^0) = (5516.75, 0.00)$ with the decision variables $\lambda^0 = (0, 0, 0, 1.24, 0, 0, 0, 0, 0, 0)^T$ and $\beta^0 = (0, 0, 0, 0, 0, 0, 0, 0, 0)^T$ calculated using equation (4.3) after feeding it the weighted vector $w^0 = (100, 4.58e-5)$. This showed that DMU3 was a multiple of DMU4. The limited interval that allowed varying both f_1 and f_2 were calculated from the pay-off table (Table 4.3) generated for the individual DMU using equation (4.8). The normal vector, was found by applying equation (2.27) giving $N^0 = (4.6e-5, 98.2e-5)^T$. Table 4.7 below summarises the initial parameters that would be used in the next step. Note that the β decision variable is omitted because its value was zero.

<u>Objectives</u>		<u>Decision variables</u>						<u>Normal vectors</u>		
$f_1(\lambda^0)$	$f_2(\lambda^0)$	λ_1^0	λ_2^0	λ_3^0	λ_4^0	λ_5^0	...	λ_{10}^0	N_1^0	N_2^0
10.34e4	4.97e3	0.00	0.00	0.00	1.24	0.00	0.00	0.00	46e-5	98.2e-5

Table 4.7 The initial starting parameters used in the next iterative step.

Indifference trade-off was estimated by equation (2.24) as $df^0 = (21.35, 1.00)^T$ and the DM had the choice to accept it and if not then other trade-off values could have been chosen. The latter case indicates that the initial target parameters were not the most preferred ones. Unlike the basic DEA model, this framework allows the DM to choose other indifference trade-offs in which the marginal rate of substitution should be $M^0 = (0.02, 1.00)^T$. Note that marginal rate must be chosen so that f_i must not exceed the maximum feasible value of obtained from the pay-off table (i.e. the value of 125274.88) see the Table 4.8 below.

Indifference trade-offs	Values
Old trade-offs	$(103423.65, 5516.75) \leftrightarrow (103423.65 + 21.35, 5516.75 - 1.00)$
New trade-offs	$(103423.65, 5516.75) \leftrightarrow (103423.65 + 50.00, 5516.75 - 1.00)$

Table 4.8 Old and new trade-off values.

The DM in the Bahrain International Hospital agreed that to improve the hospital services, output1 (Out-patients) should be reduced by 5% (i.e. output1 = 109323.33) whereby output2 (In-patients) should be increased by a certain amount that could be found by steps taken below.

As in sections 3.3 and 3.4, Table 4.9 shows increments of 0.1 (C10) for α between 0 and 1 on the LHS whilst the RHS shows increments of 0.01 (C100) for α between 0 and 0.1. The purple shaded area shows the MPS.

C10			C100		
α	$f_1(\bar{\alpha})$	$f_2(\bar{\alpha})$	α	$f_1(\bar{\alpha})$	$f_2(\bar{\alpha})$
0.0	103423.45	5516.7500	0.20	107793.74	5312.03
0.1	105608.59	5414.3910	0.21	108012.25	5301.80
0.2	107793.74	5312.0320	0.22	108230.76	5291.56
0.3	109978.88	5209.6729	0.23	108449.28	5281.32
0.4	112164.02	5107.3139	0.24	108667.79	5271.09
0.5	114349.16	5004.9549	0.25	108886.31	5260.85
0.6	116534.31	4902.5959	0.26	109104.82	5250.62
0.7	118719.45	4800.2369	0.27	109323.33	5240.38
0.8	120904.59	4697.8779	0.28	109541.85	5230.14
0.9	123089.73	4595.5188	0.29	109760.36	5219.91
1.0	125274.88	4493.1598	0.30	109978.88	5209.67

Table 4.9 Trade-off steps

The value that reduces the input2 (In-patients) by 5% was initially found between the step sizes interval $\alpha \in (0.2, 0.3)$, which was found at the step size $\alpha = 0.27$ and as a result $f_1 = 109323.33$.

If the DM disagreed with the above improvement of f_1 and sought alternative options then the next iterative step ($t=1$), would be carried out by calculating the weighted vector using equation (2.36) with $w^1 = (1.00, 57.72)^T$. In this case, entering this weighted vector in formulation (3.14) the updated decision variables would become $\lambda^1 = (0, 0, 0, 0.35, 0.90, 0, 0, 0, 0)^T$ and $\beta^1 = (0, 0, 0, 0, 0, 0, 0, 0, 0)^T$. Subsequently, the normal vector could be estimated as $N^1 = (0.73, 15.65)^T$ and the optimal indifference as $df^1 = (21.44, 1.00)^T$. The DM was unsatisfied with this indifference trade-off $(109323.33 + 21.44, 5240.38 - 1.00)$ and was looking to reduce the In-patients by 10% from the original value (i.e. reduced by another 5% from the previous target). The new marginal rate of substitution was chosen to be $M^1 = (0.015, 1.00)^T$ which lead to determining a new direction of improvement by using the gradient projection method giving $\Delta \bar{f}^1 = (3.16e-4, -1.47e-3)^T$. The maximum step size value was $\alpha_{\max}^1 = 505168.65$ and the nearest target value was found at step size 0.37.

Further improvement could be achieved by implementing more iterative steps depending on the DM preferences. As mentioned above the DM was looking to improve the Out-patients variable at the expense of the In-patients variable. The parameters generated from each iterative step are summarised in Table 4.10.

t	λ^t	f^t	θ	M^t	N^t	$\Delta \bar{f}^t$	α_{\max}	α	w^{t+1}
0	$\begin{pmatrix} 0.00 \\ 0.00 \\ 0.00 \\ 0.00 \\ 1.24 \\ 0.00 \\ 0.00 \\ 0.00 \\ 0.00 \\ 0.00 \end{pmatrix}$	$\begin{pmatrix} 103424 \\ 5517 \end{pmatrix}$	1.0	$\begin{pmatrix} 0.02 \\ 1.00 \end{pmatrix}$	$\begin{pmatrix} 4.6e-5 \\ 98.2e-5 \end{pmatrix}$	$\begin{pmatrix} 0.0268 \\ -0.00125 \end{pmatrix}$	815826	0.27	$\begin{pmatrix} 1.00 \\ 57.72 \end{pmatrix}$
1	$\begin{pmatrix} 0.00 \\ 0.00 \\ 0.00 \\ 0.35 \\ 0.90 \\ 0.00 \\ 0.00 \\ 0.00 \\ 0.00 \\ 0.00 \end{pmatrix}$	$\begin{pmatrix} 109323 \\ 5240 \end{pmatrix}$	15927	$\begin{pmatrix} 0.015 \\ 1.00 \end{pmatrix}$	$\begin{pmatrix} 0.73 \\ 15.65 \end{pmatrix}$	$\begin{pmatrix} 0.0316 \\ -0.00147 \end{pmatrix}$	505169	0.37	$\begin{pmatrix} 1.00 \\ 18.22 \end{pmatrix}$
2	$\begin{pmatrix} 0.00 \\ 0.00 \\ 0.00 \\ 0.70 \\ 0.57 \\ 0.00 \\ 0.00 \\ 0.00 \\ 0.00 \\ 0.00 \end{pmatrix}$	$\begin{pmatrix} 115244 \\ 4966 \end{pmatrix}$	100030	$\begin{pmatrix} 0.0155 \\ 1.00 \end{pmatrix}$	$\begin{pmatrix} 0.46 \\ 9.85 \end{pmatrix}$	----	----	----	----

Table 4.10 Summary of the parameters generated in each interactive step.

Unlike in DEA, improving the Bahrain International Hospital by projecting it on the efficient frontier, the proposed framework allowed for setting different targets and allocating resources so that the DM could choose the most preferred solution according to his preferences. The DEA composite outputs for Bahrain International Hospital were given by (103423, 5516) for In-patients and Out-patients. Further improvement required by the

DM preferences could be achieved using the developed framework (Table 4.10). DEA proposes that for the Bahrain International Hospital to be efficient both outputs (Out-patients and In-patients) should be increased by 30% alongside with decreasing the Physicians and Dentists by 10% and 66% respectively. Up to this stage no more analysis could be done and the DM had no further alternative choices.

	<u>Inputs</u>			<u>Outputs</u>	
	Physicians	Nurses	Dentists	Out-patients treated	In-patients admitted
Raw value	58.00	62.00	6.00	85939.00	3924.00
DEA results	52.16	62.00	2.01	112035.70	5115.58
Improvements %	10.07%	0.00%	66.47%	30.37%	30.37%
Raw value	58.00	62.00	6.00	85939.00	3924.00
MOLP-t=1	52.35	61.80	2.15	109010.85	5223.15
Improvements %	9.74%	0.32%	64.17%	26.85%	33.11%
Raw value	58.00	62.00	6.00	85939.00	3924.00
MOLP-t=2	51.81	62.10	1.84	115432.32	4974.03
Improvements %	10.67%	0.16%	69.33%	34.32%	26.76%
DEA results	52.16	62.00	2.01	112035.70	5115.58
MOLP-t=1	52.35	61.80	2.15	109010.85	5223.15
MOLP-t=2	51.81	62.10	1.84	115432.32	4974.03
Improvements % (MOLP-t=1/DEA)	0.37%	0.32%	6.88%	2.70%	2.10%
Improvements % (MOLP-t=2/DEA)	0.67%	0.16%	8.53%	3.03%	2.77%
Improvements % (MOLP-t=2/MOLP-t=1)	1.03%	0.49%	14.42%	5.89%	4.77%

Table 4.11 Comparison of the DEA and MOLP developed models.

Based on the data (recorded in Table 4.11), DEA suggested that In-patients and Out-patients could be improved by up to 30.4%, which could be achieved by reducing Physicians by 10.1% and Dentists by 66.5%. It is worth noting that, using the DEA model, the DM has no choice but to accept these results and the developed model strengthens DEA by allowing alternatives according to the DM's preferences.

Initially the DM preferences were to raise In-patients by 33.1%. The developed model showed that this would be at the expense of reducing Out-patients by 26.8% and that this would have required a 9.7% reduction in Physicians, a 0.32% reduction of Nurses and 64.2% reduction of Dentists.

The last row in Table 4.11 compares the changes that occurred according to DM preferences. In comparing the DEA outcomes with first iterative step (MOLP-t=1), the output variables (Out-patients and In-patients) were increased by 2.7% and 2.1% respectively and the resources should be changed as follows: Physicians should be increased by 0.37%, Nurses should be decreased by 0.32%, and Dentists should be increased by 6.88%.

Similar to the above by comparing DEA outcomes with the second iterative step (MOLP-t=2) Out-patients were increased by 3.03% but In-patients were decreased by 2.77% and this required a reduction of Physicians and Dentists by 0.67% and 8.53% respectively and an increase in Nurses of 0.16%.

Finally, by comparing the first iterative (MOLP-t=1) with the second iterative (MOLP-t=2), Out-patients increased by 5.89%, In-patients decreased by 4.77%; and this required Physicians and Nurses to be increased by 1.03% and 0.49 respectively whilst Dentists should be decreased by 14.4%

As can be seen from the interactive trade-off steps (Table 4.11), the DM placed more emphasis on decreasing the Physicians, aiming to increase the revenue, i.e. reducing the amount paid out in Physician salaries. In the first interactive step the DM increased Out-patients and In-patient by 26.8% and 33.1% respectively with Physicians being reduced by 9.74% and in the second interactive step, In-patients were reduced by 7% and Out-patients increased by up to 8% (compared to the first interactive step) with Physicians decreased by 10.7%.

In brief, we have shown the power of the equivalent MOLP model in establishing a framework that allows target setting, based on DM preferences and resource allocation that needs to be adjusted accordingly.

4.5 Summary

In this chapter an interactive framework was presented and illustrated. The origin of this framework is the integration between MOLP and DEA underlying the bounded model proposed by Wang and Yang (2007). An equivalent relationship was built between the output-oriented dual bounded model and the minimax formulation. The bounded model is constructed by unifying the best efficiency, which has values between zero and one, and the worst efficiency, which has values greater than or equal to one, of DEA models. It was achieved by using a virtual anti-ideal DMU, which has the worst efficiency among all other DMUs. For instance, the anti-ideal DMU utilizes the most inputs to produce the least outputs. The proposed framework can be used to support a comprehensive efficiency analysis by measuring the best and worst efficiency of the DMUs and can incorporate the DMs preferences and guide them to make better decisions on setting realistic targets values and allocating resources.

The interactive gradient projection method was implemented to explore target setting and resource allocation. The advantage of this method is the determination of a normal vector on the efficient frontier, which provides a realistic estimate to examine whether the most preferred solution is satisfied i.e. that which maximises the DM's implicit utility function. Moreover, the projection of a utility gradient onto a tangent plane of the efficient frontier yields directions through which the DMs utility can be further improved.

The findings of this study enhance the application of DEA by incorporating the DM's preferences and providing an overview of the DEA problem from different sides of the targeted DMU in an interactive manner. An important practical implication is that it can help deepen the understanding of appropriate ways to make inefficient DMU efficient by looking at the problem from optimistic and pessimistic viewpoints.

Chapter 5

Interactive MOLP and ranking frameworks

5.1 Introduction

It is possible to take two extreme points of view (optimistic and pessimistic) in the bounded DEA model and these lead to two different efficiency estimates of individual DMUs: the best relative efficiency and the worst relative efficiency. The best relative efficiency scores have values between zero and one, whilst the worst relative efficiency scores are greater than or equal to one. Combining these best and worst relative efficiencies could provide a better understanding of the practice of each DMU and provide the decision-maker (DM) with a panoramic view of the performance of individual DMUs, resulting in an improved decision making process. Ranking is essential for the DM as it allows him/her to test the impact of choices on the practice of other DMUs. Several ranking approaches have appeared in the literature and the geometric average approach (GAA) has been proposed for providing an overall ranking of DMUs' efficiency (Wang, Chin, and Yang, 2007). Although the geometric average approach could provide an overall ranking, the original suggestion based on the bounded DEA model allowed no interference/interactivity by the DM in the analysis and consequently he/she would be given no alternative but just to accept the resulting optimum solutions from the analysis.

5.1.1 Aims

The purpose of this chapter is to establish a framework that integrates the interactive (bounded) MOLP model developed in the previous chapter with a ranking approach based on the geometric average approach. By using the proposed framework, the DM is able to set targets and to choose the most suitable one that satisfies his/her preferences. Further, the ranking procedure enables the DM to examine any improvements that may take place after the iterative steps are performed.

5.1.2 Research questions

This chapter investigates the following research questions:

- How can the ranking of DMUs be integrated with an MOLP interactive procedure?
- How can the DM be supported in making the most suitable decisions?

5.1.3 Methodology

The new proposed framework is based on a thorough investigation of the relevant models and methods that have already been published. In this chapter, the feasibility of the framework is initially demonstrated through an illustrative example using secondary data and then the applicability of the framework is demonstrated through a case study of private hospitals in Bahrain in order to improve the performance of one specific hospital.

5.1.4 Chapter structure

This chapter consists of:

- An overview of ranking approaches
- A development of the Geometric Average Approach (GAA)
- An illustrative example of GAA
- A case study: The hospitals of Bahrain

5.2 An overview of ranking approaches

In the literature, there are various ranking approaches using DEA and these are critically examined in this section.

Ranking DMUs in DEA was introduced by Sexton et al. (1986). The cross-evaluation matrix analysis was recommended by Doyle and Green (1994) who argued that DMs do not always have a reason to use assurance regions. Generally, the cross-evaluation matrix approach was based on measuring efficiency scores of DMUs by using the optimal weights obtained from solving DEA models. DEA cross-efficiency scores can be represented by a cross-efficiency matrix generated from the following equation:

$$h_{kj} = \frac{\sum_{r=1}^s u_r y_{rj}}{\sum_{i=1}^m v_i x_{ij}}, \quad k = 1, \dots, n, j = 1, \dots, n \quad (5.1)$$

where h_{kj} represents the score for j in the DEA run of unit k . For instance, unit j is evaluated by the weights of unit k . It is worth noting that the values of the elements in the cross-efficiency matrix are between zero and one, $0 \leq h_{kj} \leq 1$, where the diagonal elements constitute the standard DEA efficiency outcome. In other words, $h_{kk} = 1$ means that DMU k is efficient and $h_{kk} < 1$ means that DMU k is inefficient. If the weights obtained from DEA models do not have unique values, Goal Programming methods can be implemented. Based on Sexton (1986), for example, the Goal Programming could be either aggressive or benevolent. In the aggressive context, the selected DMU should minimize self-efficiency with the secondary objective of minimizing the other DMUs cross-efficiency rankings. On the other hand, the benevolent secondary objective would be to maximize all DMUs cross-efficiency rankings. Similar implementations of Goal Programming can be found in the literature (Oral, 1991).

Application of the cross-efficiency ranking method in the DEA context utilizes the results of the efficiency matrix in order to rank the units. If h_{kj} is an element in the matrix then \bar{h}_k ,

which is the average of cross efficiency scores of k elements, can be defined as $n^{-1} \sum_{j=1}^n h_{kj}$. It

is worth noting that the median or the variance can also be used instead of the average (Green et al. 1996). Moreover, it has been argued that \bar{h}_k is more representative than the diagonal elements h_{kk} , which represents the basic DEA efficiency score. This is because \bar{h}_k considers all elements in the cross-efficiency matrix including the diagonal one. The maximum value of \bar{h}_k is 1, which occurs if DMU k is efficient in all the runs, i.e. all DMUs evaluate that DMU k is efficient. In order to rank DMUs, a DMU with the highest score is ranked as one and those with lower scores are ranked as n . Although the \bar{h}_k score is comparable as it utilizes the weights of all DMUs equally but a drawback occurs in this technique when the multiplier weights lose their connection with subsequent evaluation.

The shortcomings of this method were discussed by Doyle and Green (1994). They proposed the idea of a “maverick index”, which estimates the dispersion of each element from the basic DEA efficiency scores, i.e. h_{kk} is achieved by using the following relationship as:

$$M_k = \frac{h_{kk} - e_k}{e_k}, \quad \text{where} \quad e_k = \frac{1}{(1-n)} \sum_{j \neq k} h_{jk} \quad (5.2)$$

The higher value of M_k , the more a DMU could be considered a maverick. Note that other authors, for example Doyle and Green (1994), later argued that this scaling is similar to a benchmarking process through which the efficiency of DMUs results from self-assessment, where the elements reaching higher M_k scores fail to appear in the reference set of the inefficient DMUs. On the other hand, those elements that achieve low M_k scores distribute all around.

A super-efficiency ranking approach was proposed by Anderson and Peterson (1993) which enables extreme efficient units to achieve efficiency scores of more than one as described by the following model.

$$\begin{aligned}
 h_k &= \text{Max} \sum_{r=1}^s u_r y_{rk} \\
 &\text{subject to} \\
 &\sum_{i=1}^m v_i x_{ij} - \sum_{r=1}^s u_r y_{rj} \geq 0 \text{ for } j = 1, \dots, n, j \neq k \\
 &\sum v_i x_{ik} = 1 \\
 &u_r \geq \varepsilon \text{ for } r = 1, \dots, s \\
 &v_i \geq \varepsilon \text{ for } i = 1, \dots, s
 \end{aligned} \tag{5.3}$$

The dual formulation of the above model (5.3), which determines the distance between the Pareto frontier with and without unit k , i.e. for $= \{1, \dots, n, j \neq k\}$, is given by:

$$\begin{aligned}
 &\text{Min } \theta_k \\
 &\text{subject to} \\
 &\sum_{j \in J} \lambda_{kj} x_{ij} \leq \theta_k x_{ik} \text{ for } i = 1, \dots, m \\
 &\sum_{j \in J} \lambda_{kj} y_{rj} \geq y_{rk} \text{ for } r = 1, \dots, s \\
 &\lambda_{kj} \geq 0 \text{ for } j = 1, \dots, n
 \end{aligned} \tag{5.4}$$

There are several shortcomings of this model. First, the DEA objective function is referred to as a ranking value, in spite of the fact that individual unit is assessed based on different weights. In reality, the ranking value obtained by applying this formulation can be considered as the ratio between the weights of each unit k and a virtual unit that is closest to it on the efficient frontier.

Second, the problem caused by the high ranking value of the “specialized” DMU was addressed by Sueyoshi (1999) by introducing special bounds such as:

$$\begin{aligned}
 v_i &\geq \frac{1}{(m+s)\max_j(x_{ij})} \\
 u_r &\geq \frac{1}{(m+s)\max_j(y_{rj})}
 \end{aligned}
 \tag{5.5}$$

Moreover, to limit the super-efficiency score, Sueyoshi proposed an Adjusted Index Number (AIN) relationship as shown below:

$$\text{AIN}_k = 1 + \frac{\left(\delta_k^* - \min_{j \in E} \delta_j^* \right)}{\left(\max_{j \in E} \delta_j^* - \min_{j \in E} \delta_j^* \right)}
 \tag{5.6}$$

where E is the set of efficient DMUs.

Third infeasibility might occur and, if so, the ranking of all DMUs could not be achieved. Furthermore, the extreme efficiency of DMUs, which is described by Thrall (1996), indicates that the super-efficiency CCR model may be infeasible. Several authors such as Zhu (1996a), Dula and Hickman (1997), and Seiford and Zhu (1999) provide important information about the conditions by which the super-efficiency model becomes infeasible. One suggestion to deal with the infeasibility problem in the super-efficiency approach, proposed by Mehrabian et al. (1999), is to modify the dual formulation as shown by the model:

$$\begin{aligned}
 &\text{Min } \theta_k \\
 &\text{subject to} \\
 &\quad - \sum_{j \in J} \lambda_{kj} x_{ij} + x_{ik} - \theta_k \geq 0 \text{ for } i = 1, \dots, m \\
 &\quad \sum_{j \in J} \lambda_{kj} y_{rj} \geq y_{rk} \text{ for } r = 1, \dots, s \\
 &\quad \lambda_{kj} \geq 0 \text{ for } j = 1, \dots, n
 \end{aligned}
 \tag{5.7}$$

The simplicity of the super-efficiency model attracted interest by several researchers. For example, Hashimoto (1997) incorporated this model with assurance regions in order to rank all DMUs but to provide a complete ranking of the DMUs further preferences need to be added for each DMU, i.e.

$$\begin{aligned}
 h_k &= \text{Max} \sum_{r=1}^s u_r y_{rk} \\
 &\text{subject to} \\
 &\sum_{r=1}^s u_r y_{rj} \leq 1 \text{ for } j = 1, \dots, n, j \neq k \\
 &u_r - u_{r+1} \geq \varepsilon \text{ for } r = 1, \dots, s-1 \\
 &u_s \geq \varepsilon \text{ for } r = 1, \dots, s \\
 &u_r - u_{r+1} + u_{r+2} \geq 0 \text{ for } r = 1, \dots, s-2
 \end{aligned} \tag{5.8}$$

where u_r is the sequence of weights assigned to the r^{th} place. This is a preference voting system, i.e. a ranked voting system in which each DM selects, ranks and votes for the top s DMUs among the n DMUs. The use of assurance regions is to prevent the specialization pitfall caused by using the standard super-efficiency model.

The benchmark ranking approach developed by Togersen et al. (1996) enabled the complete ranking of DMUs. The method was based on estimating the importance of the observable DMU as a benchmark for inefficient DMUs. This approach works in two stages. First, an additive model is applied to measure the slacks through which efficient DMUs are identified as those that have zero slacks. Second, to order all DMUs, the following model is used:

$$\begin{aligned}
 \frac{1}{E_k} &= \text{Max } \theta_k \\
 &\text{subject to} \\
 &-\sum_{j \in V} \lambda_{kj} x_{ij} - s_{ik}^- + x_k \geq 0 \text{ for } i = 1, \dots, m \\
 &\sum_{j \in V} \lambda_{kj} y_{rj} - \theta_k y_{rk} - s_{rk}^+ = 0 \text{ for } r = 1, \dots, s \\
 &\sum_{j \in V} \lambda_{kj} = 1
 \end{aligned} \tag{5.9}$$

In order to rank the efficient DMUs and evaluate which are of particular importance, the benchmarking measure aggregates the individual reference weights as shown in the equation (5.10) below:

$$\rho_k^r \equiv \frac{\sum_{j=1}^n \lambda_{jk} (y_{rj}^p - y_{rj})}{(y_r^p - y_r)}$$

$$\forall k = 1, \dots, V, r = 1, \dots, s \quad (5.10)$$

where

$$y_{rj}^p = \frac{y_{rj}}{E_j} + \sigma_{rj}$$

For an efficient DMU k , the benchmark ρ_k^r , estimates the ratio of the total combined potential increase in output over the k^{th} DMU that acts as a reference. Note that the scores of efficient DMUs determine the entire potential within each variable. The mean value of ρ_k can then be found, which is then used for ranking all efficient DMUs completely.

Ranking methods reported in the literature utilize statistical techniques together with DEA to provide a complete ranking and three approaches have been reported in the literature: Canonical Correlation analysis (CCA), linear discriminant analysis (DDEA) and discriminant analysis of ratios (DR/DEA).

Canonical Correlation Analysis (CCA) is an extension of simple regression analysis and deals with multiple inputs and multiple outputs (Johnson and Wichern, 2002). CCA determines the weighting vector for inputs and outputs that is common to all DMUs. In particular, CCA finds vectors Z_j and W_j , which are the linear sum of input variables and output variables respectively, which can be written as:

$$Z_j = \sum_{i=1}^n V_i x_{ij} \quad \text{for } i = 1, \dots, m$$

$$W_j = \sum_{r=1}^n U_r y_{rj} \quad \text{for } r = 1, \dots, s \quad (5.11)$$

The aim of CCA is to determine the two vectors V' and U' in order to maximize r_{ZW} the correlation coefficient of the composite inputs, Z , to the composite outputs, W , by using the following programme (Hardoon, 2004):

$$\begin{aligned} \text{Max } r_{ZW} &= \frac{V'S_{xy}U}{\sqrt{(V'S_{xx}V)(U'S_{yy}U)}} \\ &\text{subject to} \\ &V'S_{xx}V = 1 \\ &U'S_{yy}U = 1 \end{aligned} \tag{5.12}$$

where S_{xx} , S_{yy} and S_{xy} are the matrices of the sums of squares and sums of the products of the variables respectively. Note that the ratio of V' and U' is determined up to a proportional constant and feasibility is achieved for positive weight values.

Friedman and Sinuany-Stern (1997) proposed using the ratio between Z_j and W_j as a scaling score, T_j . Then the largest eigenvalue obtained from CCA is used to find the common weights for the linear combinations and can be written as:

$$T_j = \frac{W_j}{Z_j} = \frac{\sum_{r=1}^s U_r y_{rj}}{\sum_{i=1}^m V_i x_{ij}}, \quad j = 1, \dots, n \tag{5.13}$$

Note that T_j is unbounded as the DEA score is limited to values between 0 and 1 and ranking is based on the scaling ratios instead of DMUs absolute measures.

Linear discriminant analysis (DDEA) is a ranking method developed by Sinuany-Stern et al. in 1994 based on finding a score function for ranking DMUs. The score function is determined by dividing the DMUs into two groups: efficient and inefficient. By integrating conventional discriminant analysis, the following model can be built:

$$D_j = \sum_{r=1}^s u_r y_{rj} + \sum_{i=1}^m v_i (-x_{ij}), \quad j = 1, \dots, n \quad (5.14)$$

where the critical D_c , which divides DMUs into two groups (efficient and inefficient) is the midpoint of the discriminant function. Subsequently, the efficient DMUs are those that have values greater than D_c i.e. $D_j > D_c$ (Morrison, 1976). Note that u_r and v_i have closed form solutions and a DDEA score is feasible only if all the weights are non-negative.

Discriminant analysis of ratios (DR/DEA) is another ranking method originally proposed by Sinuany-Stern and Friedman (1998). It integrates discriminant ratio analysis with DEA aiming to avoid the problem caused by infeasibility found in the previous approach. To do so, a ratio function is established by using the linear combination vectors of inputs and outputs defined in the previous approach. While DEA allows estimating different efficiency scores for individual DMUs, DR/DEA provides common weights for all DMUs. The idea behind this approach is to determine the weights through which the ratio score function optimally discriminates between two groups of DMUs on a one-dimensional scale, i.e. efficient and inefficient DMUs. The ratio and the arithmetic mean can be expressed as:

$$T_j = \frac{\sum_{r=1}^s u_r y_{rj}}{\sum_{i=1}^m v_i x_{ij}}, \quad j = 1, \dots, n \quad (5.15)$$

$$\bar{T}_1 = \frac{1}{n_1} \sum_{j=1}^{n_1} T_j \quad \text{and} \quad \bar{T}_2 = \frac{1}{n_2} \sum_{j=1}^{n_2} T_j$$

where n_1 and n_2 are the number of efficient and inefficient DMUs, obtained from DEA model, respectively. The weighted mean of efficient, n_1 , and inefficient, n_2 , DMUs can be written as $\bar{T} = (n_1 \bar{T}_1 + n_2 \bar{T}_2) n^{-1}$. The ratio for common weights that needs to be measured between the group variance of T , ($SS_B(T)$) and the within- group variance of T , ($SS_W(T)$) should be maximized as described by the following model:

$$\max_{u_r, v_i} \lambda = \max_{u_r, v_i} \frac{SS_B(T)}{SS_W(T)}$$

where

$$SS_B(T) = n_1(\bar{T}_1 - \bar{T})^2 + n_2(\bar{T}_2 - \bar{T})^2 \quad (5.17)$$

$$SS_W(T) = \sum_{j=1}^m (T_j - \bar{T}_1)^2 + \sum_{j=n_1+1}^n (T_j - \bar{T}_2)^2$$

Hence, the ratio between the composite inputs and outputs, T_j , represents the efficiency scores for DMU j ; thereby DMUs with higher ratio scores will be at the top and those with lower ratio scores will be at the bottom. It is worth noting that for negative weight values, non-negative constraints have to be added to the optimization model and a non-linear optimization method needs to be implemented to solve the problem.

So far, none of the methods mentioned above provide a ranking of inefficient DMUs. For instance, the benchmarking and super-efficiency methods only attempt to rank efficient DMUs assigned in the standard DEA models. However, (Cooper and Tone, 1997) claim that the cross-efficiency matrix method and the associated statistical techniques do address the problem by stating that the original efficiency value “will generally be determined from different facets” An attempt to rank inefficient DMUs was derived by Bardhan et al. in 1996 through a Measure of Inefficient Dominance (MID). This technique was based on slack-adjusted DEA models whereby overall estimates of inefficiency can be measured by equation (5.18):

$$0 \leq 1 - \frac{\sum_{i=1}^m \left(\frac{s_i^*}{x_{ik}} \right) + \sum_{r=1}^s \left(\frac{\sigma_r^*}{y_{rk}} \right)}{m + s} \leq 1 \quad (5.18)$$

The MID index ranks the inefficient DMUs according to their average proportional inefficiency in all inputs and outputs. While the benchmarking approach ranks the efficient DMUs, the MID index ranks the inefficient DMUs.

Golany (1988) introduced Multiple Criterion Decision Making (MCDM) into DEA. In his approach, DEA and multiple-objective linear programming were combined interactively. Although MCDM does not allow complete ranking a discussion was provided about using the preliminary details of input and output variables in order to refine the preferential

power of the DEA model. Using MCDM as an interactive framework for ranking, a procedure was developed in which the importance of inputs and outputs are chosen by the DM. Even though the latter is considered as a drawback of the approach, several researchers such as Golany (1988), Kornbluth (1991), Thanassoulis and Dyson (1992), Golany and Roll (1994), Zhu (1996b) and Halme et al. (1999) incorporated discriminatory details into DEA models. A group of researchers (Thompson, 1986; Dyson and Thanassoulis, 1988; Charnes et al., 1989 and 1990; Cook and Kress, 1990a and 1990b; Thompson et al., 1990; Wong and Beasley, 1990; Cook and Johnston, 1992 and Green and Doyle, 1995) applied an assurance region or cone-ratio model to specify the importance of inputs or outputs through which DMUs can be scaled.

A number of authors suggested that a discrimination factor should be constructed by imposing ratio constraints on the multipliers and a lower bound (Cook and Kress, 1990a, 1991 and 1994; Cook et al., 1993 and 1996). This modification can almost ensure a single efficient DMU. For example, when considering aggregation of votes whereby y_{rj} is the number of r^{th} placed votes received by candidate j , an intensity function $d(r, \varepsilon)$ can be defined in the DEA model as:

Max ε

Subject to

$$\begin{aligned} \sum_{r=1}^s u_r y_{rj} &\leq 1, \quad j = 1, \dots, n \\ u_r - u_{r+1} - d(r, \varepsilon) &\geq 0 \\ u_r - d(r, \varepsilon) &\geq 0 \\ u_r, \varepsilon &\geq 0 \end{aligned} \tag{5.18}$$

The outcome of the above model depends on the discriminatory function, $d(r,\varepsilon)$. It is linear whenever the difference between the ranks is linear; however, this is not always the case as reported by Green et al. (1996), because the form of $d(r,\varepsilon)$ limits the ranking results. Moreover, mathematically, the shape of the function used in this analysis is not clear.

A two-stage methodology was developed for fully ranking DMUs by utilizing fuzzy logic (Karsak, 1998; Hogaard, 1999). In the first stage, efficiency scores using DEA were identified whilst the second stage incorporated fuzzy logic to rank the efficient units using expert information. Several authors argued that DEA should be considered as another methodology within MCDM (e.g. Troutt, 1995; Li and Reeves, 1999; and Sinuany-Stern et al, 2000). In order to distinguish between efficient DMUs, Troutt (1995) proposed a maximin efficiency ratio model as:

$$\text{Max}_{u_r, v_i} \left(\text{Min}_k \frac{\sum_{r=1}^s u_r y_{rk}}{\sum_{i=1}^m v_i x_{ik}} \right)$$

Subject to

$$\frac{\sum_{r=1}^s u_r y_{rk}}{\sum_{i=1}^m v_i x_{ik}} \leq 1 \quad \forall k \tag{5.19}$$

$$\sum_{r=1}^s u_r = 1$$

$$u_r, v_i \geq 0 \quad \forall r, i$$

To increase the discrimination between DMUs, Li and Reeves (1999) proposed incorporating multiple objectives to the basic DEA as in the following model:

$$\begin{aligned}
& \text{Min } d_k \\
& \text{Min } M \\
& \text{Min } \sum_{j=1}^n d_j \\
& \text{Subject to} \\
& -\sum_{i=1}^m v_i x_{ij} + \sum_{r=1}^s u_r y_{rj} + d_i = 0 \quad \text{for } j = 1, \dots, n \quad (5.20) \\
& \sum_{i=1}^m v_i x_{ik} = 1 \\
& M - d_i \geq 0 \\
& u_r, v_i, d_j \leq 0 \quad \forall r, i, j
\end{aligned}$$

In the above model (5.20) three objectives are used. The first objective is the same as the standard CCR model. The second is to minimize the maximum deviation (slack variable) and the last is to minimize the total deviations. The first objective increases discrimination between DMUs whilst the second and third objectives do not provide the full ranking of DMUs.

Some other researchers, such as Sinuany-Stern et al. (2000) suggested using the Analytical Hierarchical Process (AHP/DEA) for ranking DMUs in two stages. In the first stage, the DEA model is applied in pairs to the units whilst the second stage depends on the first stage outcome to generate a pair wise comparison matrix.

5.3 The Geometric Average Approach (GAA)

The standard DEA approach provides a means of measuring the best efficiency which does not provide a complete image of the performance of DMUs. In this section, we propose a new approach for ranking DMUs based on the bounded model discussed in the previous chapter, which allows measuring the performance of individual DMU from two different perspectives (the best relative efficiency and the worst relative efficiency). The geometric average of these two efficiencies could provide a more useful measure of a DMU's efficiency (i.e. overall efficiency).

5.3.1 The theoretical background of GAA

To apply GAA the overall efficiency for each DMU is determined by finding the optimistic efficiency (the best relative efficiency, or upper bound) and the pessimistic efficiency (the worst relative efficiency, or lower bound) (Wang et al, 2007).

The optimistic efficiency DEA output linear programming model (for efficiency scores between 0 and 1) can be written as:

$$\begin{aligned} \text{Min } h_0 &= \sum_{i=1}^m v_i x_{ij_0} \\ \text{Subject to} \\ \sum_{r=1}^s u_r y_{rj} - \sum_{i=1}^m v_i x_{ij} &\leq 0 \quad j = 1, \dots, n \\ \sum_{r=1}^s u_r y_{rj_0} &= 1 \\ u_r, v_i &\geq \varepsilon \end{aligned} \tag{5.21}$$

and the dual formulation is:

$$\begin{aligned}
\text{Max } h &= \theta_{j_0} + \varepsilon(s_r^+ + s_i^-) \\
\text{Subject to} \\
\theta_{j_0} y_{rj_0} - \sum_{j=1}^n \lambda_j y_{rj} + s_r^+ &\leq 0, & r = 1, \dots, s \\
\sum_{j=1}^n \lambda_j x_{ij} + s_i^- &\leq x_{ij_0}, & i = 1, \dots, m \\
\lambda_j, s_r^+, s_i^- &\geq 0
\end{aligned} \tag{5.22}$$

The pessimistic efficiency DEA output linear programme (for efficiency scores greater than 1) can be written as:

$$\begin{aligned}
\text{Max } h_0 &= \sum_{i=1}^m v_i x_{ij_0} \\
\text{Subject to} \\
\sum_{r=1}^s u_r y_{rj} - \sum_{i=1}^m v_i x_{ij} &\geq 0, & j = 1, \dots, n \\
\sum_{r=1}^s u_r y_{rj_0} &= 1 \\
u_r, v_i &\geq \varepsilon
\end{aligned} \tag{5.23a}$$

Which, in the bounded model can be written:

$$\begin{aligned}
\text{Max } h_0 &= \sum_{i=1}^m v_i x_{ij_0} \\
\text{subject to} \\
\sum_{r=1}^s u_r y_{rj} - \sum_{i=1}^m v_i x_{ij} &\leq 0, & j = 1, \dots, n \\
\sum_{r=1}^s u_r y_{rj} - \sum_{i=1}^m v_i (\alpha x_{ij}) &\geq 0, & j = 1, \dots, n \\
\sum_{r=1}^s u_r y_{rj_0} &= 1 \\
u_r, v_i &\geq \varepsilon
\end{aligned} \tag{5.23b}$$

whilst the dual formulation can be expressed as:

$$\text{Min } h = \theta_{j_0} - \varepsilon(s_r^+ + s_i^-)$$

Subject to

$$\begin{aligned} \theta_{j_0} y_{rj_0} - \sum_{j=1}^n \lambda_j y_{rj} - s_r^+ &\geq 0, & r = 1, \dots, s \\ \sum_{j=1}^n \lambda_j x_{ij} - s_i^- &\geq x_{ij_0}, & i = 1, \dots, m \\ \lambda_j, s_r^+, s_i^- &\geq 0 \end{aligned} \quad (5.24a)$$

and for the bounded model:

$$\text{Min } h = \theta_{j_0} - \varepsilon(s_r^+ + s_i^-)$$

Subject to

$$\begin{aligned} \theta_{j_0} y_{rj_0} + \left[\sum_{j=1}^n (\lambda_j - \beta_j) y_{rj} - s_r^+ \right] &\geq 0 \\ \sum_{j=1}^n (\alpha \beta_j - \lambda_j) x_{ij} - x_{ij_0} - s_i^- &\geq 0 \\ \lambda_j, \beta_j, s_r^+, s_i^- &\geq 0 \end{aligned} \quad (5.24b)$$

The geometric average is expressed as $\sqrt{\alpha \psi_j^* \theta_j^*} = \sqrt{\alpha} \sqrt{\psi_j^* \theta_j^*}$, where the terms $\alpha \psi_j^*$ and θ_j^* in the geometric average are the lower bound (i.e. worst efficiency) and the upper bound (i.e. the best efficiency). The difficulty of estimating the constant (α) in the bounded model is a major drawback of this model. Implementation is investigated through an illustrative example and case study later in this chapter.

5.3.2 The stages of the framework

The proposed framework consists of two main stages:

- **Interactive stage**
 - » Generate the pay-off table Lower bound and Upper bound
 - » Generate the initial efficient solution
 - » Calculate the normal vector and checking of the optimality condition
 - » Determine the trade-off direction

- » Determine the trade-off step size
- » Update the weighting vector
- **Ranking stage**
 - » The overall ranking using geometric average approach

5.3.3 An illustrative example

The application of the proposed framework for improving a targeted DMU in an interactive manner and ranking it within the other DMUs is illustrated through a simple data set shown in Table 5.1.

DMU	Input (x)	Output 1 (y_1)	Output 2 (y_2)
1	1	2	5
2	2	2	4
3	3	6	6
4	1	3	2
5	2	6	2

Table 5.1 Simple data set consisting of one input and two outputs variables.

To assess the DMU performance the output-oriented CCR dual model was implemented. Table 5.2 shows that DMU 2 and DMU 3 are inefficient and their efficiency scores are 45% and 73% respectively. The composite output for DMU 2 on the efficient frontier is a linear combination of 1.60 DMU 1 and 0.4 DMU 4.

DMU	Efficiency score	Decision variables				
1	1.00	1.00	0.00	0.00	0.00	0.00
2	0.45	1.60	0.00	0.00	0.40	0.00
3	0.73	0.75	0.00	0.00	2.25	0.00
4	1.00	0.00	0.00	0.00	1.00	0.00
5	1.00	0.00	0.00	0.00	0.00	1.00
θ_{ADMU}		0.2424				

Table 5.2 The output-oriented CCR model (2.4b) applied to the data in Table 5.1

DMU 2 will be the targeted for improvement and its ranking among other DMUs will be tested. Table 5.3 presents the initial ranking of the DMUs obtained from applying the Geometric Average Approach. The scores of DMUs were produced from the lower and upper bound models (4.4 and 4.10) and the geometric average using $\sqrt{\psi_j^* \theta_j^*}$ developed in the previous chapter. The Anti-ideal DMU was measured by using model 4.16 giving $\theta_{ADMU} = 0.2424$. The targeted DMU (i.e. DMU 2) is initially ranked fifth.

DMU	Lower bound (ψ_j^*)	Upper bound (θ_j^*)	Geometric Average	Ranking
1	0.4848	1.0000	0.6963	1
2	0.2424	0.4545	0.3320	5
3	0.2909	0.7273	0.4600	4
4	0.3394	1.0000	0.5826	2
5	0.2424	1.0000	0.4924	3

Table 5.3 Lower/upper bounds and geometric average (target DMU is shown bold).

Recall that the developed framework consists of two stages: an interactive procedure and a ranking stage. The interactive procedure was started by constructing a pay-off table to find the allowable range in which objectives were to be chosen by using model (4.16). Table 5.4 shows the trade-off feasible ranges of the DMUs.

DMU	Max f_1		Max f_2		Max f_1/y_1	Max f_2/y_2	F^{max}
	f_1	f_2	f_1	f_2			
1	3	2	2	5	1.50	1.00	1.50
2	6	4	4	10	3.00	2.50	3.00
3	9	6	6	15	1.50	2.50	2.50
4	3	2	2	5	1.00	2.50	2.50
5	6	4	4	10	1.00	5.00	5.00

Table 5.4 The pay-off table. (The targeted DMU is shown bold)

Table 5.4 shows the allowable ranges when each composite output of individual DMU is maximized. For the targeted DMU (DMU2), the possible ranges of the objectives are as follows: f_1 can be varied between 4 and 6 and f_2 can be varied between 4 and 10. The

maximum allowable feasible value for each objective is: $f = [f_1, f_2]^T$ or $f = [6, 10]^T$, which is the ideal point of the composite outputs.

5.3.3.1 Stage #1: The interactive MOLP

The interaction was started by solving model (4.4) to generate the composite output of the targeted DMU on the efficient frontier. The DM has the choice of accepting them otherwise trade-off between the objectives f_1 and f_2 takes place. If f_2 is considered to be a reference objective, it could be perturbed or reduced by a small amount (e.g. reduced by 0.01). In this case the $f(\lambda^0) = [4.01, 99.9]^T$ was used to generate the composite output. Note that the resultant solution is almost similar, but not exact, to those obtained from the *DEA* model for $f(\lambda^0) = [4.00, 10.0]^T$. Hence, the initial decision variables are given by $\lambda^0 = [1.996, 0.000, 0.000, 0.003, 0.000]^T$ and the normal vectors $N^0 = [0.5, 0.17]^T$ are estimated using equation (2.27). Hence, the initial solution of DMU2 is a linear combination of 1.996 of DMU1 and 0.003 of DMU4. The optimal indifference trade-off vector at the solution $f(\lambda^0)$ for a unit change of f_2 can be found by equation (2.31) as $df^0 = (0.34, 1.00)^T$ resulting in the initial indifference trade-offs $(4.01, 9.99) \Leftrightarrow (4.01 + 0.34, 9.99 - 1.00)$. Now suppose that the DM did not agree with obtained indifference trade-off and an alternative is proposed. If the DM prefers to increase f_1 at the expense of f_2 as: $(4.01, 9.99) \Leftrightarrow (4.01 + 0.44, 9.99 - 1.00)$ then the marginal rate of substitution can be generated by $M^0 = (2.3, 1.00)^T$.

The trade-off direction is calculated as $\Delta \bar{f}^0 = (0.26, -0.76)^T$ using the gradient projection method i.e. equation (2.28). The trade-off size can be found by calculating the maximum step size as $\alpha_{\max}^0 \approx 8$ using equation (2.34), which is used to build the following step size table (Table 5.5).

C10			C100		
α	y_1	y_2	α	y_1	y_2
0.00	4.01	9.990	0.00	4.01	9.990
0.10	4.21	9.989	0.01	4.03	9.932
0.20	4.41	9.988	0.02	4.05	9.873
0.30	4.61	9.987	0.03	4.07	9.814
0.40	4.81	9.986	0.04	4.09	9.756
0.50	5.01	9.985	0.05	4.11	9.697
0.60	5.20	9.984	0.06	4.13	9.639
0.70	5.40	9.983	0.07	4.15	9.580
0.80	5.60	9.982	0.08	4.17	9.523
0.90	5.80	9.981	0.09	4.20	9.463
1.00	6.00	9.980	0.10	4.21	9.405

Table 5.5 The step size table with $\bar{\alpha} \approx 0.05$ shown in bold.

Table 5.5 shows that while y_1 increases y_2 decreases. Note that y_1 should not exceed the maximum allowable value of 6. Suppose the DM sets the target level for f_1 at 4.11 which can be found between $\alpha = 0.00$ and $\alpha = 0.10$. This can be allocated more precisely in the C100 table above at $\alpha = 0.05$, which is $\bar{\alpha}$. Whenever $\bar{\alpha}$ is determined the weighting vector can be updated to find the decision variables and hence the resources can be allocated.

The weighting vector $w^1 = (1.00, 0.16)^T$ is calculated using equation (2.36) used to generate the decision variables: $\lambda^1 = [1.899, 0.000, 0.000, 0.101, 0.000]^T$ for the new targets $f(\lambda^1) = [4.11, 9.70]^T$.

5.3.3.2 Stage #2: Ranking DMUs by GAA

The ranking is essential for DMs to see the impact of their decisions. For instance, the ranking reveals whether the targeted DMU's position among the others is improved or reduced. The interactive outcomes, which are based on the DM's opinion, shows that in order to improve DMU2's efficiency, output1 and output2 should be increased by about 105% and 142% without changing the resources.

DMU	Input (x)	Output 1(y ₁)	Output 2(y ₂)
1	1	2	5
2	2▶2	2▶4.1	4▶9.7
3	3	6	6
4	1	3	2
5	2	6	2

Table 5.6 The improvement from the interactive stage (the target DMU is shown bold).

Table 5.6 indicates that for DMU2 to become efficient outputs 1 and 2 should be changed to 4.1 and 9.7 without requiring more resources. After applying the proposed changes, the Bounded Model and Geometric Average Approach (GAA $\sqrt{\psi_j^* \theta_j^*}$) are employed and the results are shown in Table 5.7.

DMU	Bounded Model		GAA Ranking	
	ψ_j^*	θ_j^*	After	before
1	0.2424	1.0000	3	1
2	0.2485	1.0000	2	5
3	0.2424	0.7273	5	4
4	0.3030	1.0000	1	2
5	0.2424	1.0000	4	3

Table 5.7 DMU ranking before and after changes. (Target DMU is shown bold)

Table 5.7 shows the results of applying the Bounded Model (i.e. 4.4), developed in the previous chapter, and the Geometric Average Approach ($GAA\sqrt{\psi_j\theta_j}$). DMU1 was moved from first to third place, DMU2 was raised from fifth to second place, DMU3 was moved from fourth to fifth place, DMU4 was raised from second to first place, and DMU5 was moved from third to fourth place. The change in ranking of each DMU was caused by the fact that the pessimistic efficiency of individual DMUs is assessed relative to all the other DMUs. The pessimistic efficiency was computed using all the input and output information rather than only one input and one output.

The DM may agree with the improvement for the targeted DMU and if so then the process is terminated. Otherwise, new targets may be proposed requiring a re-run of the first stage of the interactive procedure and the second stage of ranking procedure in the developed framework.

5.4 Case Study: Hospitals of Bahrain

In this section, the proposed framework is applied to the private hospitals of Bahrain with the aim to improve the Awali hospital. The data of the case study is shown in Table 5.8.

DMU	Hospital	Physicians	Nurses	Dentists	Out-patients treated	In patients admitted
1	American Mission	68	109	15	186294	2528
2	Awali	18	43	2	37671	942
3	Bahrain International	58	62	6	85939	3924
4	Ibn Al-Nafees	39	48	1	96987	3483
5	Bahrain Specialist	43	50	2	83406	4449
6	Gulf Dental Speciality	1	16	10	14228	27
7	Al Amal	13	16	2	13617	12
8	Al Hillal	18	22	1	41013	431
9	Salmaniya Medical	625	1677	6	518287	46738
10	Bahrain Defence	229	758	12	460496	24616

Table 5.8 The data collected from the Ministry of Health in Bahrain.

By applying the above, models (5.22 for optimistic model and 5.24 for pessimistic model) and the dual bounded models shown in the previous chapter (model 4.4 for the upper bound and model 4.10 for the lower bound) the following results are obtained:

DMU	The best efficiency	The worst efficiency	Bounded DEA model
1	1.0000	2.4202	[0.0227, 1.000]
2	0.8057	1.5171	[0.0142, 0.806]
3	0.7671	1.5349	[0.0144,0.767]
4	1.0000	2.6192	[0.0245,1.000]
5	1.0000	2.1380	[0.0200,1.000]
6	1.0000	1.0000	[0.0094, 1.000]
7	0.4212	1.0000	[0.0049,0.421]
8	0.9226	2.2370	[0.0209,0.923]
9	1.0000	1.0000	[0.0094,1.000]
10	1.0000	1.4509	[0.0136,1.000]
Anti-ideal DMU	0.00936		

Table 5.9 Pessimistic and optimistic efficiencies from DEA and bounded models

5.4.1 Stage #1: Interactive MOLP

The output-oriented CCR dual model was run for the data in table (5.8) to determine the relative efficiency scores. As shown in Table 5.10, DMUs 2, 3, 7, and 8 (hospitals: Awali, Bahrain International, Al Amal, and Al Hillal respectively) were found to be inefficient related to all ten hospitals.

DMU	1/θ	θ	λ ₁	λ ₂	λ ₃	λ ₄	λ ₅	λ ₆	λ ₇	λ ₈	λ ₉	λ ₁₀
1	1.00	1.00	1.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
2	0.81	1.24	0.12	0.00	0.00	0.26	0.00	0.00	0.00	0.00	0.00	0.00
3	0.77	1.30	0.00	0.00	0.00	0.51	0.75	0.00	0.00	0.00	0.00	0.00
4	1.00	1.00	0.00	0.00	0.00	1.00	0.00	0.00	0.00	0.00	0.00	0.00
5	1.00	1.00	0.00	0.00	0.00	0.00	1.00	0.00	0.00	0.00	0.00	0.00
6	1.00	1.00	0.00	0.00	0.00	0.00	0.00	1.00	0.00	0.00	0.00	0.00
7	0.42	2.37	0.00	0.00	0.00	0.33	0.00	0.00	0.00	0.00	0.00	0.00
8	0.92	1.08	0.00	0.00	0.00	0.46	0.00	0.00	0.00	0.00	0.00	0.00
9	1.00	1.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	1.00	0.00
10	1.00	1.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	1.00

Table 5.10 DMU efficiency from CCR DEA analysis. (Inefficient DMUs bold).

For Awali, which is targeted for improvement, the efficiency score is 81% and its composite points on the efficient frontier are a linear combination of 0.12 of American Mission hospital and 0.26 of Ibn Al-Nafees hospital. The DM has the choice to accept or reject the generated composite outputs. If he accepts the results, the DEA composite input and output values will be the most preferred solution (MPS) for Awali hospital. If the obtained inputs and outputs are rejected then the interactive process is started by measuring the range of the output targets from which the DM can choose. This step is achieved by constructing the trade-off (Table 5.11) using formulation 3.14, which shows that the range of the composite output for f_1 is between 46758 and 35636 whereas for f_2 it is between 1196 and 1903. Hence the maximum feasible composite output vector is $\bar{f} = [46758, 1903]^T$, which represents the ideal point of the composite outputs.

Max f_1		Max f_2		f_1/y	f_2/y	F^{max}
f_1	f_2	f_1	f_2			
186294.00	2528.00	132890.66	7091.78	1.00	2.81	2.81
46758.02	1196.14	35646.20	1903.78	1.24	2.02	2.02
125274.87	4498.87	103423.44	5516.76	1.46	1.41	1.46
96987.00	3483.00	96987.00	3483.00	1.00	1.00	1.00
101028.12	3628.12	83406.00	4449.00	1.21	1.00	1.21
14228.00	27.00	2010.90	107.49	1.00	3.98	3.98
32329.00	1161.00	25245.08	1346.70	2.37	112.23	112.23
44452.37	1596.37	34949.62	1864.38	1.08	4.33	4.33
581922.00	20898.00	518287.00	46738.00	1.12	1.00	1.12
577433.69	18812.71	460496.00	24616.00	1.25	1.00	1.25

Table 5.11 The trade-off table where the target DMU is shown bold.

The resultant output of the DEA bounded model (4.4) for Awali hospital (DMU2) is $f^0 = [46758, 1196]^T$, where $\bar{f}_1 = f_1^0 = 46758$. This indicates that the DEA composite unit is an extreme point on the efficient frontier and hence the weighting vector can be written as $w_r = [1, 0]^T$ and is related to the normal vector $N^0 = [1, 0]^T$.

Establishing the equivalent MOLP model allows the user to vary f_1 , by a small amount such that $\bar{f}_1 > f_1^0$, e.g. $f_1^0 = 46758.01 (< \bar{f}_1 = 46758.02)$, where the maximum objective values of the starting interactive process of $f(\lambda^0, \beta^0) = [46758.01, 1196.15]^T$ can be determined by using model (4.7). The obtained values of the decision variables are $\lambda^0 = [0.12, 0, 0, 0, 0.26, 0, 0, 0, 0, 0]^T$ and $\beta^0 = [0, 0, 0, 0, 0, 0, 0, 0, 0]^T$ and the corresponding normal vector $N^0 = [0.0031, 0.0014]^T$ was obtained from equation (2.27). Similar to the DEA decision variables, Awali hospital is a linear combination of 0.12 American Mission hospital and 0.26 Ibn Al-Nafees hospital.

<u>Objectives</u>		<u>Decision variables</u>						<u>Normal vectors</u>		
$f_1(\lambda^0)$	$f_2(\lambda^0)$	λ_1^0	λ_2^0	λ_3^0	λ_4^0	λ_5^0	...	λ_{10}^0	N_1^0	N_2^0
4.7e4	1.2e3	0.12	0.00	0.00	0.26	0.00	0.00	0.00	31e-4	14e-4

Table 5.12 The initial starting parameters used in the next iterative step.

The first interactive step was started by allocating the reference objective f_i . The optimal indifference trade-off vector at $f(\lambda^0)$ for a unit change of f_i can be estimated by using equation (2.24) as $\Delta f^0 = [1.00, 0.45]^T$ resulting in the initial indifference trade-off of $(46758.01, 1196.15) \Leftrightarrow (46758.01 - 1.00, 1196.15 + 0.45)$. In case that the DM rejects this initial optimal indifference, which indicates that the initial target value is not the most preferred solution, a new set of indifference trade-offs may be provided by the DM, for example, $(46758.01, 1196.15) \Leftrightarrow (46758.01 - 1.00, 1196.15 + 0.40)$ with a marginal rate of substitution $M^0 = [1.00, 2.50]^T$. It is worth noting that the trade-off of $(1196.15 + 0.40)$ for f_2 does not exceed the maximum feasible value of 1903.78 obtained from the pay-off table.

<u>Indifference trade-offs</u>	<u>Values</u>
Old trade-offs	$(46758.01, 1196.15) \leftrightarrow (46758.01 - 1.00, 1196.15 + 0.45)$
New trade-offs	$(46758.01, 1196.15) \leftrightarrow (46758.01 - 1.00, 1196.15 + 0.40)$

Table 5.13 The old and new trade-off values.

The trade-off direction is determined as $\Delta \bar{f}^0 = [-0.019, 0.43]^T$. Hence, the DM prefers to improve f_2 at the expense of f_1 with maximum step of $\alpha_{\max}^0 \sim 16507$, which is implemented to build the step size table of 10 incremental steps. For more accurate targeted objective, 1000 incremental steps could also be established as shown in Table 5.14.

C10			C100			C1000		
α	f_1	f_2	α	f_1	f_2	α	$y1$	$y2$
0.0	46758.010	1196.150	0.40	46630.1801	1479.2020	0.420	46623.7886	1493.3546
0.1	46726.053	1266.913	0.41	46626.9843	1486.2783	0.421	46623.4690	1494.0622
0.2	46694.095	1337.676	0.42	46623.7886	1493.3546	0.422	46623.1494	1494.7699
0.3	46662.138	1408.439	0.43	46620.5928	1500.4309	0.423	46622.8298	1495.4775
0.4	46630.180	1479.202	0.44	46617.3971	1507.5072	0.424	46622.5103	1496.1851
0.5	46598.223	1549.965	0.45	46614.2013	1514.5835	0.425	46622.1907	1496.8928
0.6	46566.265	1620.728	0.46	46611.0056	1521.6598	0.426	46621.8711	1497.6004
0.7	46534.308	1691.491	0.47	46607.8098	1528.7361	0.427	46621.5515	1498.3080
0.8	46502.350	1762.254	0.48	46604.6141	1535.8124	0.428	46621.2320	1499.0156
0.9	46470.393	1833.017	0.49	46601.4183	1542.8887	0.429	46620.9124	1499.7233
1.0	46438.435	1903.780	0.50	46598.2226	1549.9650	0.430	46620.5928	1500.4309

Table 5.14 Trade-off step size table for increments of 10, 100, and 1000.

It is worth noting that while f_1 decreases, f_2 increases for each incremental step, and the maximum value of f_2 should not exceed the maximum feasible value obtained from the pay-off table, which is 1903.78. If the DM manages to increase f_2 up to 1500, i.e. increase it by 25%, this value can be allocated at $\alpha \sim 0.429$ as shown in Table 5.5, which corresponds to a decrease of f_1 to 46621, i.e. an approximate decrease of 0.3%.

By identifying the regulating parameters of the step size, the weighted vector can be updated using equation 3.14 as $w^1 = [1.00, 0.034]^T$. The obtained weight is used to determine the new decision variables corresponding to the new target set $f(\lambda^1, \beta^1) = [46621, 1500]^T$, and the related decision variables are found to be $\lambda^1 = [0.03, 0, 0, 0.41, 0, 0.12, 0, 0, 0, 0]^T$ and $\beta^0 = [0, 0, 0, 0, 0, 0, 0, 0, 0, 0]^T$, which corresponds to a new normal vector of $N^1 = [0.42, 0.19]^T$. If the DM does not agree with the obtained objectives, further iterative steps would be undertaken.

5.4.2 Stage #2: Ranking DMUs by GAA

For the purpose of understanding the improvement achieved by using MOLP-DEA optimistic bounded model, the initial data (i.e. 3 inputs and 2 outputs) for DMU2 were replaced by those data obtained from the developed model as shown in Table 5.15 below.

DMU	<u>Inputs</u>			<u>Outputs</u>	
	Physicians	Nurses	Dentists	Out-Patients	In-Patients
1	68	109	15	186294	2528
2	18	24.60	2	46620.15	1497.49
3	58	62	6	85939	3924
4	39	48	1	96987	3483
5	43	50	2	83406	4449
6	1	16	10	14228	27
7	13	16	2	13617	12
8	18	22	1	41013	431
9	625	1677	6	518287	46738
10	229	758	12	460496	24616

Table 5.15 The replaced data for DMU2

The optimistic and pessimistic bounded models were then applied to measure the best and the worst efficiency of DMUs. Table 5.16 below shows the effects of DEA on improving the performance of DMU2 before and after changing the data.

DMU	Bounded Model		GAA Ranking	
	$\theta_{\text{pessimistic}}$	$\theta_{\text{optimistic}}$	After	before
1	0.02	1.00	2	2
2	0.01	1.00	5	6
3	0.01	0.77	7	7
4	0.02	1.00	1	1
5	0.02	1.00	3	3
6	0.01	1.00	8	9
7	0.01	0.42	10	10
8	0.02	0.92	4	4
9	0.01	1.00	9	8
10	0.01	1.00	6	5

Table 5.16 The efficiency improvement achieved by using optimistic bounded MOLP_DEA resultant data for DMU2.

Table 5.16 shows that the ranking of the DMU2's efficiency is improved from the 6th 5th place. Note that some DMUs are ranked worse after the data for DMU2 are changed. For instance, there is a slight reduction in the ranking of DMU9 and DMU10.

It is worth noting that the analysis of improving the observable DMU is achieved by using what was referred to earlier as one-directional improvement. For instance, improvement is done towards the optimistic (upper bound frontier) direction. The pessimistic (lower bound frontier) direction of improvement will form the basis of a future study.

5.5 Summary

This chapter reviewed a wide range of ranking concepts which can be divided into four groups. The first group is based on a cross-efficient matrix in which DMUs are evaluated through both self and peer pressure. The second group is based on a super-efficiency approach where an efficient DMU receives a score greater than unity. The third group is based on benchmarking through which a DMU is highly ranked if it is chosen as a useful target against other DMUs. The fourth group incorporates multivariate statistical techniques into DEA including Canonical Correlation and Discriminant Analysis, which are used to calculate common weights.

The Geometric Average Approach (GAA) was proposed for ranking DMUs (Wang et al., 2007). The essence of this method is to find the optimistic frontier and the pessimistic frontier and then apply the geometric average formula. However, DEA/GAA does not allow the DM to interfere in the process. It could be strengthened by incorporating it with MOLP. Doing so would allow the DM to choose alternative data variables (resources or services), which could be done without changing the ranking of the observable DMUs.

Compared with the previous approaches in the literature, the developed framework consists of two stages: the interactive stage in which a DMU is studied deeply to gain insights from optimistic and pessimistic points of view, followed by a ranking stage based on taking the geometric average of these different points of view to provide a more realistic overall ranking.

Chapter 6

Conclusions, contributions to knowledge and future research.

6.1 Introduction

The integration of assessing past performance and planning future targets involving DM and expert preferences is of increasing interest to support both management and planning control (Cooper, 2004) and this tenet provided the motivation of this work. This research focuses on how DM and experts preferences could be incorporated without prior knowledge. The aim of the research was to incorporate value judgements which can be defined as “logical structures, incorporated within an efficiency assessment study, reflecting the DMs’ preferences in the process of assessing efficiency” (Allen et al, 1997). Extended DEA, which involves value judgement, was developed to measure individual DMU efficiency. MOLP was developed to solve problems having multiple conflicting objectives. This study set out to explore the integration of some extended DEA and MOLP models and identified their potential application through two real case studies in the medical and education sectors. Ranking is an essential tool that can be used to aid DMs to identify the level of improvement of each DMU based on their decisions and this thesis has also presented new frameworks of comprehensive performance assessment by integrating extended DEA and MOLP with ranking and illustrated their effectiveness through the two real case studies.

6.2 Conclusions

Incorporating value judgements can be achieved in two different ways, by weight restriction techniques or through an interactive procedure constructed by integrating DEA and MOLP models. In weight restriction techniques weights are supplied a priori by the DM, which are subjective and often difficult to identify. Integrating DEA models and MOLP models overcomes this problem and makes the analyses more practical. Furthermore, integrating DEA and MOLP models has several other advantages. (1) it enables the incorporation of DM's preferences through an interactive procedure without prior knowledge, (2) it aids the DM in reaching the most preferred solution (MPS) by incorporating the DM's influence in each iterative step, and (3) it provides alternative explicit and achievable targets which allows the DM to find the most suitable decision and resource allocation through a systematic procedure.

Integrating extended DEA and MOLP models enabled the construction of frameworks that can be employed as a comprehensive tool for performance assessment to provide valuable insights towards target setting, performance benchmarking and resource allocation. These frameworks are based on establishing the equivalent MOLP models to the extended DEA models employing the trade-off approach of Podinovski (2004) and the bounded model of Wang and Yang (2007). This paved the way to constructing an interactive trade-off analysis procedure using a local search approach and gradient projection method in which the DM's preferences are represented by the marginal rate of substitution to elicit the most preferred solutions on the efficient frontier. This approach enables DMs explicitly to set alternative targets and test them to see if they are achievable according to available resources before implementing the decisions. The most preferred solution (MPS) is iteratively identified on the efficient frontier in which the DM's utility function is maximised. The main advantage of the gradient projection method, among MOLP methods, is that the termination criterion is well defined and is satisfied whenever the normal vector coincides with the utility gradient of the targeted DMU.

The applicability of the bounded model was enhanced by incorporating a ranking method based on the geometric average approach in which DMU's practice is measured from two points of view (best and worst) to improve overall efficiency by identifying the level of

improvement. This yields the better understanding of DMU's practice and consequently is likely to result in better improvement.

As set out in Chapter 1, the main purpose of the study was to construct frameworks to allow the incorporation of decision maker and expert preferences in some extended DEA models, and to study the impact of the developed frameworks on improving performance assessment using real data. More specifically, the objectives of the study were to:

- To investigate the effectiveness and usefulness of DEA and its extensions in incorporating both DM and expert preferences when focussing on performance assessment, benchmarking and target setting.
- To investigate how any equivalence between the extended DEA (i.e. trade-off approach and bounded model) and the MOLP models can be employed to establish an interactive framework for improving performance assessment.
- To investigate how ranking can be incorporated into performance assessment and what the affect on decision making.

These objectives were pursued through the research presented in Chapters 2 to 5.

Chapter 2 is divided into three main reviews: basic DEA and its applications to real life problems, extended DEA, and interactive MOLP methods. Basic DEA was reviewed to provide essential background knowledge to initiate this study and understand the foundations and mathematical formulation of Farrell (1957) and the linear optimization of the multiple inputs/outputs problem of Charnes et al (1978) for constant/variable return to scale. Extended DEA was reviewed particularly for applications to real life problems. The inclusion of value judgements followed the two streams already mentioned: weight restrictions or deriving an equivalent MOLP model to enable incorporating the DMs' preferences through an interactive procedure. The review of weight restrictions included: direct restrictions (e.g. Thomas et al, 1986; Dyson et al, 1998; Banker et al, 1989, and Thanassoulis et al, 1995); adjusting the observed input-output levels to capture value judgements (e.g. Charnes et al., 1990; Kornbluth, 1991; and Cook et al, 1992) and restricting the virtual inputs and outputs (e.g. Wong and Beasley, 1989; Beasley, 1990). The review of equivalent MOLP-DEA models began with considering the similarity between DEA and MCDM, specifically MOLP (e.g. Joro et al, 1998; Agrell and Tind,

2001; Stewart, 1994 and 1996). Further reviews included the interactive model through which the DM could allocate a set of inputs levels as resources and choose the most preferred set of outputs levels from a set of alternative points located on the efficient frontier (Golany, 1988), interactive multiple goal programming where preference information is provided interactively by the DM (Post and Sponk, 1999) achieved by adjusting the upper and lower feasible boundaries of both inputs and outputs levels, and more recently, the super-ideal point model showing that the dual formulation of the DEA output-oriented and MOLP are identical (Yang et al, 2009).

The review also investigated the application of ranking in DEA. Ranking is vital in performance assessment where the DM is often interested in ranking to refine the evaluation of DMU's practice. Furthermore, ranking helps the DM to identify the level of improvement on a targeted DMU after the analysis has taken place. Although there are a number of ranking methods presented in the literature (Adler et al, 2002), in this research a ranking procedure that employing a geometric average (Wang and Yang, 2007) was considered most appropriate because it includes both optimistic and pessimistic views.

Finally, MOLP interactive methods were reviewed to identify a suitable method that could be realistically implemented. Among MOLP interactive methods, gradient projection method was preferred because the criterion for termination of the procedure is well defined (i.e. when the projection of the utility gradient becomes zero). The method employs the projection of the gradient onto the tangent plane of the efficient frontier to identify the trade-off direction (Li and Yang, 1996; Yang, 1999; and Li et al., 1999).

Chapter 3 was mainly about establishing a framework to allow the incorporation of DM and expert preferences without prior knowledge. This was achieved by providing a theoretical derivation of the equivalent MOLP/extended DEA model based on the trade-off approach proposed by Podinovski (2004) and illustrated through a real case study in the educational sector (the University of Bahrain). Several statistical tests were applied to select the number of DEA variables including: correlation, multivariate and cluster analyses with constant and variable returns to scale being considered. The interactive method used to calculate the most preferred solution was gradient projection method identified in Chapter 2.

In any organisation, it is essential to consider sub-manager (e.g. bank branch managers, heads of departments of colleges or hospitals, etc) preferences. This will provide more realistic performance assessment leading to a better way to improve each DMU practice and hence better productivity. The proposed framework provides rich insights into performance assessment with realistic MPS targets incorporating both executive and sub-manager preferences. Sub-manager preferences are integrated within the body of the extended DEA programme whilst the executive preferences are incorporated through the interactive procedure.

Each nation's future wealth and competitive position in a globalized world depends increasingly on its ability to create quality knowledge. Higher education is an essential resource for the creation and application of knowledge. Hence, decision makers are increasingly concerned about assessing the performance of higher education to improve its productivity. The developed framework provides a comprehensive analysis for quality control and future planning in higher education as demonstrated through the first of the two case studies in the University of Bahrain (UOB).

Similarly, the health of a nation also contributes to the productivity of its people and consequently health services need be efficient. To be efficient, clinical and non-clinical managers must perform an intensive performance analysis by incorporating both clinical managers, who manage physicians' decision making (i.e. patient management), and non-clinical managers, who make the best use of all hospital assets by managing operations (i.e. practice management). Therefore, patient and practice require extraordinary amount of coordination and commitment for performance improvement. The proposed framework enables managers to better control and manage expenditures through a realistic and a well established explicit interactive procedure by allowing them to test their targets and identify whether they are achievable according to their available resources as demonstrated in the second case study in the Hospitals of Bahrain.

These two case studies were extensively used in Chapters 3, 4 and 5 to test and demonstrate the applicability of the methods and procedures employed in this thesis.

In Chapter 4 the bounded DEA model of Wang and Yang (2007) was used as the basis of an interactive framework to measure the targeted DMUs' practice from two points of view:

best efficiency (between 0 and 1) and worst efficiency (greater than 1). Wang and Yang (2007) presented the anti-ideal DMU (ADMU) defined as the DMU that consumes the most inputs to produce the least output. The framework developed in this chapter enhanced this bounded model by considering the DMs' preferences that can provide more realistic performance assessment results. The interactive steps were tested by applying the gradient projection method described above and in greater detail in Chapter 2. The methodology of the interactive MOLP method is to elicit DM's preferences to search for the MPS along the efficient frontier for the targeted DMU. The MPS can provide better measures for efficiency score than incorporating DM's preferences and support target setting and resource allocation. The proposed framework was applied to the real case studies previously described.

The process of integrating ranking within efficiency analysis was presented in chapter 5. Although there are several well established method of ranking (Adler, 2002), the proposed ranking technique was considered to be more systematic and realistic. For instance, the developed framework not only incorporates the DM's preferences but also allows the DM to identify the level of improvement that may take place according to his judgements. As such the developed framework helps the DM to choose the best decision that could be achieved based on evaluable resources. The method (Wang and Yang, 2007) was then implemented to order the DMUs according to this geometric average score of the best and worst efficiency.

6.3 Summary of contributions to knowledge

In this thesis three new interactive frameworks were developed to comprehensively allow a DM to analyse DMUs practice. These frameworks are based on the methodology of integrating extended DEA models and MOLP models. The first framework was developed by integrating extended DEA model trade-off approach of Podinoviski (2004) and MOLP. The second framework was generated by integrating the extended DEA bounded model of Wang and Yang (2007) and the MOLP model. The third framework was established by incorporating a ranking procedure based on the geometric average of Wang and Yang (2007) from which an interactive bounded model framework was developed.

The main contributions of this study are therefore:

- The development of a framework that incorporates DM and expert preferences without prior knowledge about inputs or outputs in a systematic iterative way to produce more realistic performance assessment results, see Figure 6.1.
- An enhanced bounded model incorporating DM's preferences to measure the whole efficiency which yields more realistic picture about the targeted DMU practice. The developed model improves the targeted DMU from both an optimistic and/or a pessimistic view, see Figure 6.2a (optimistic), and Figure 6.2b (pessimistic).
- Employing the geometric average to rank DMUs based on measuring overall efficiency in an interactive manner to enable the DM to determine the level of improvement after the analysis has taken place in each iterative step, see Figure 6.3a (interactive stage), Figure 6.3b (ranking stage)

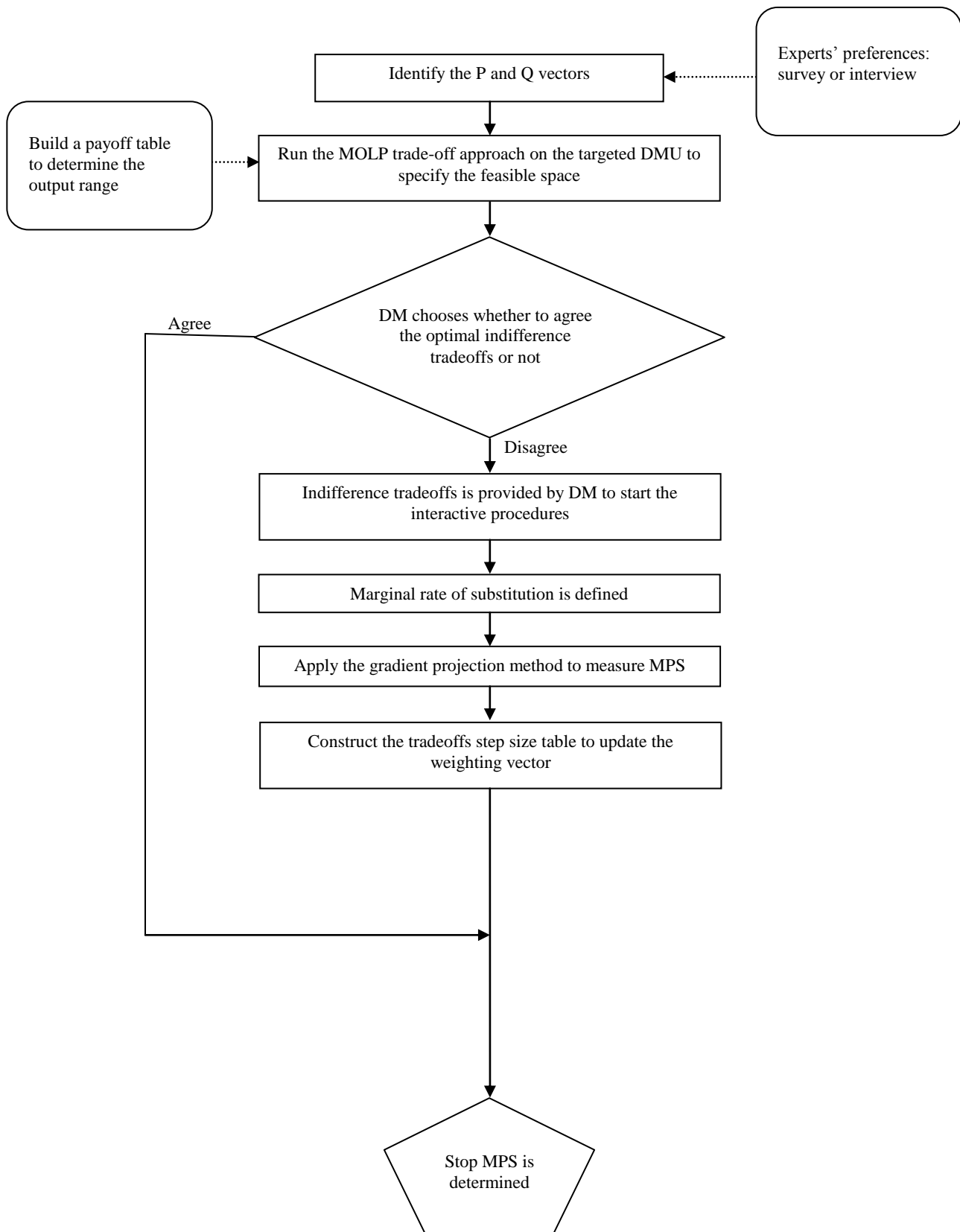


Figure 6.1: Flow chart of the MOLP trade-off approach

Optimistic interactive view:

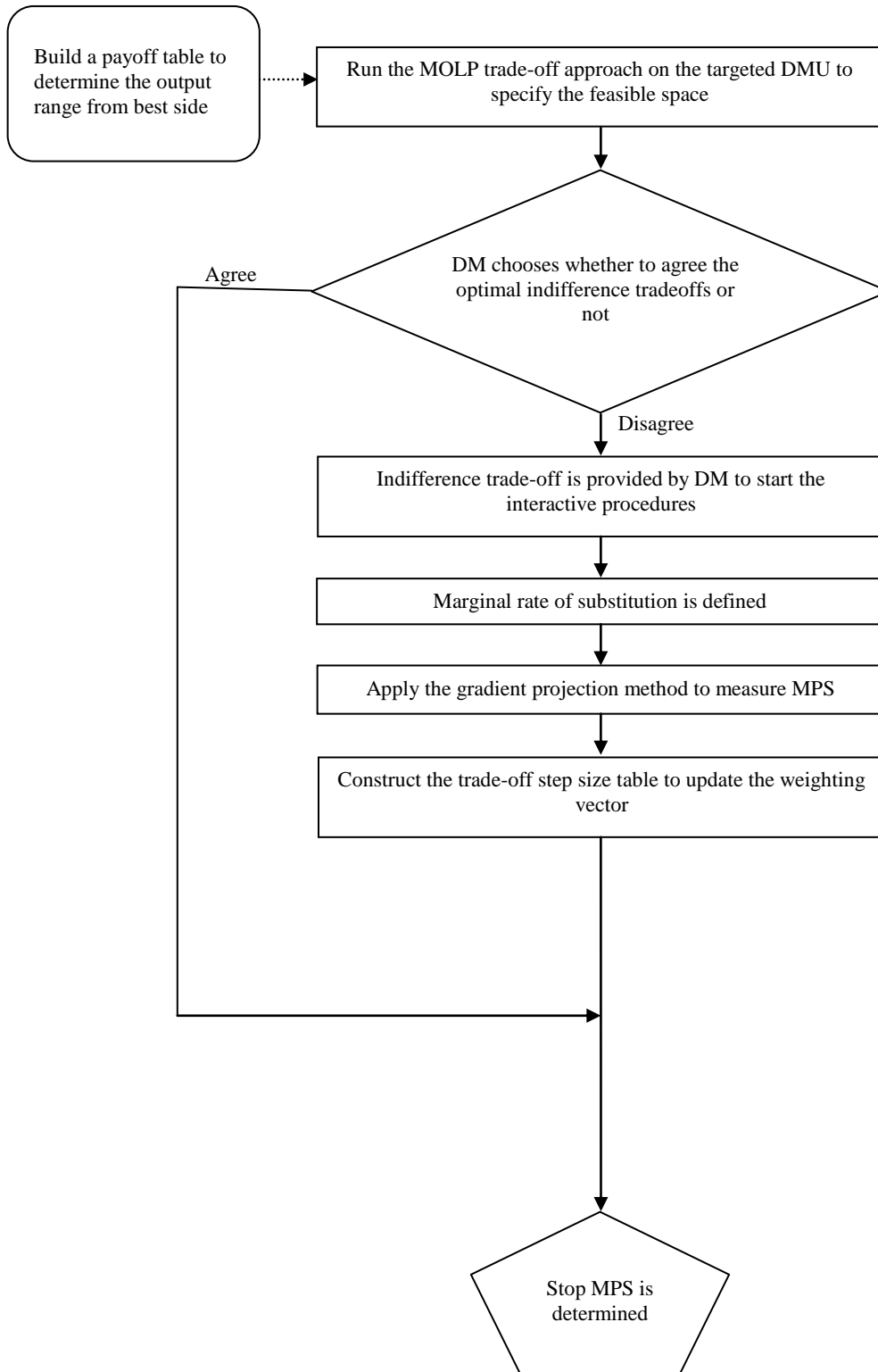


Figure 6.2a: Flow chart describing the optimistic interactive bounded model

Pessimistic interactive view:

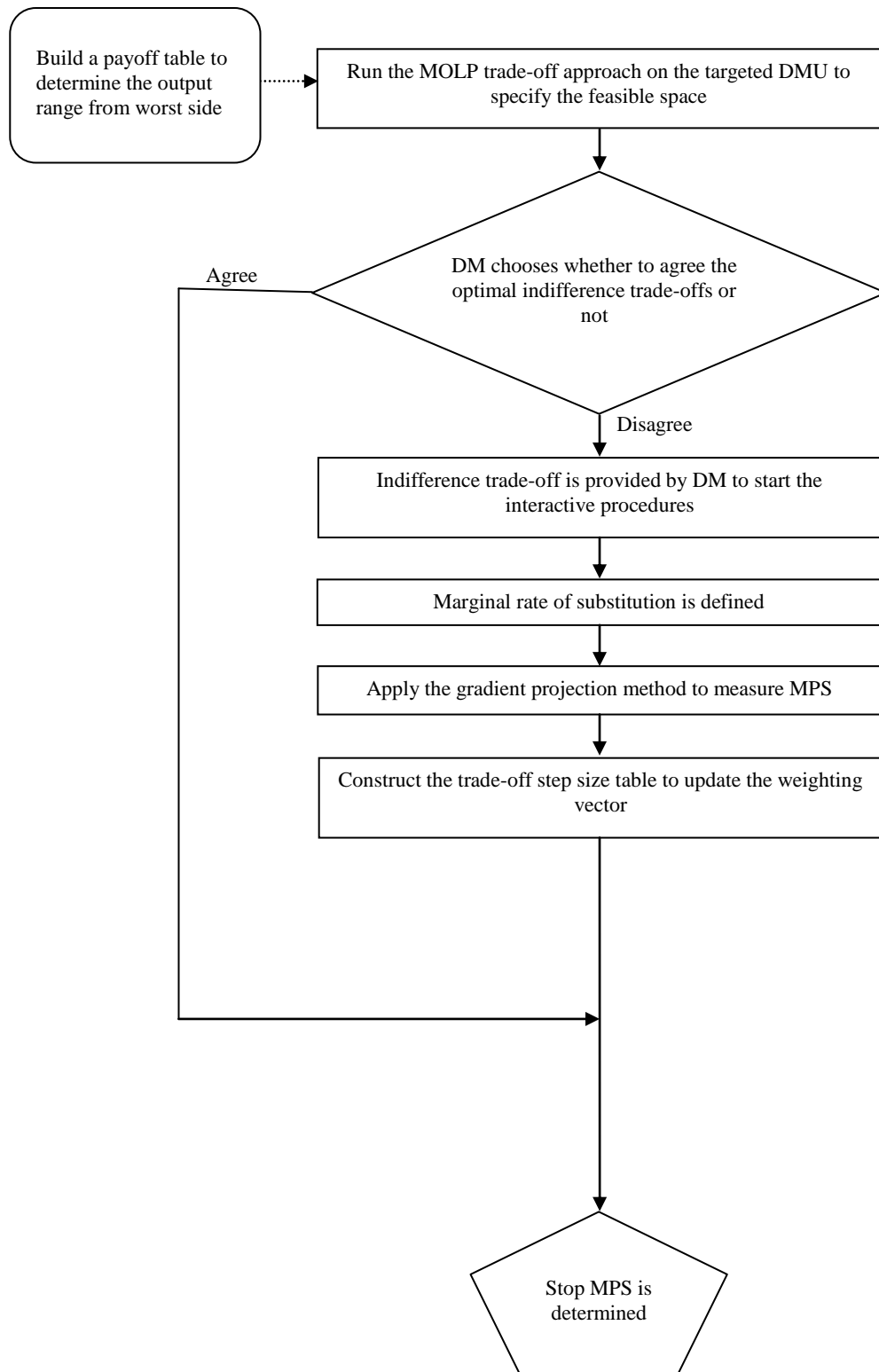


Figure 6.2b: Flow chart describing the pessimistic interactive bounded model

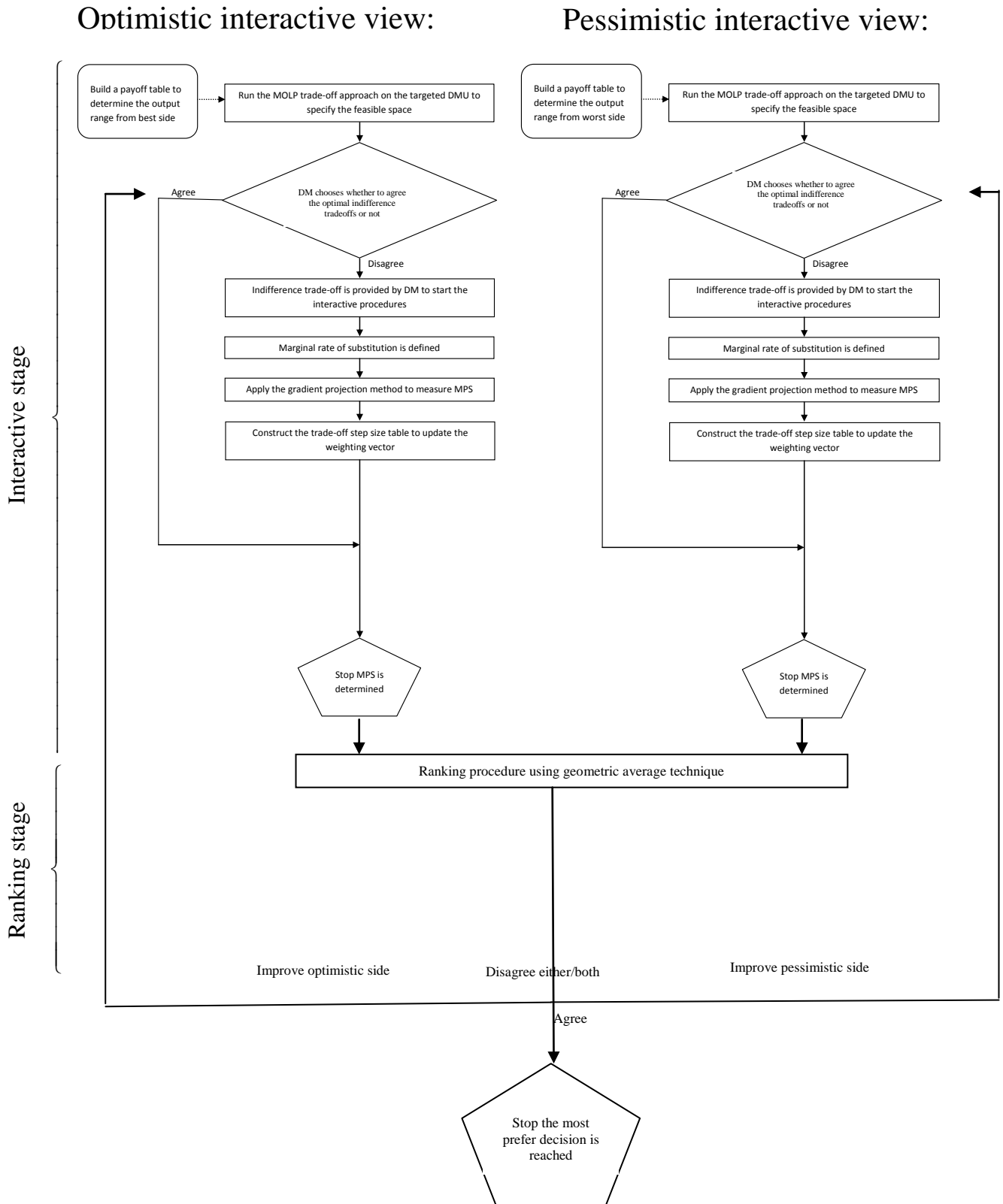


Figure 6.3: Flow chart of interactive bounded model with ranking

6.4 Future research

The DEA equivalent MOLP model explored in this work provides opportunities for considerable future research.

This research considered integrating MOLP methods and extended DEA with an output-oriented formulation based on a maximization approach. Further research could consider integrating extended DEA with an input-oriented formulation based on a minimization approach.

The bounded model could be further improved by considering preferences under uncertainty, where there are inconsistent judgements or imprecise data. Further, as the bounded model application is based on interval analysis, it could be enhanced by the use of fuzzy set theory. Fuzzy set theory deals with uncertain input-output environments caused by imprecise data such as missing judgement and forecasting data or ordinal preference information.

This thesis paves the way for investigating the incorporation of MOLP in other extended DEA models such as the Malmquist productivity index. The Malmquist productivity index was developed for forecasting based on previous knowledge and could be enhanced by aggregating DM preferences to improve the control of resources.

Finally, the developed frameworks were implemented on relatively small data sets (i.e. small number of DMUs) and it would be worth implementing the proposed frameworks using large data sets to test its stability and consistency.

References

- Adler, N. (2002). "International Dimensions of Organizational Behaviour" 4th edition, Cincinnati: South-Western College Publishing.
- Adler, N. and Golany, B. (2001). "Evaluation of deregulated airline networks using data envelopment analysis combined with principal components analysis with an application to Western Europe." European Journal of Operational Research **132**(2): 260-273.
- Adler, N. and Golany, B. (2001). "Including principal component weights to improve discrimination in DEA." Presented at: INFORMS annual DEA meeting Miami, United States of America, 11/4 to 11/7.
- Adler, N. and Golany, B. (2001). "Principal component analysis-Data envelopment analysis model with constraint." Presented at: 8th European Conference on Operational Research. Rotterdam Netherlands, 7/9 to 7/11.
- Alder, N., Friedman, L. and Sinuany-Stern, Z. (2002). "Review of ranking methods in the data envelopment analysis context." European Journal of Operational Research **140**: 249-265.
- Afriat, S.N. (1972). "Efficiency Estimation of Production Functions." International Economic Review **13**: 568-598.
- Agrell, P. and Tind, J. (2001). "A dual approach to non-convex frontier model." Journal of Productivity Analysis **16**: 129-147.
- Al-Shammari, M. and Salimi, A. (1998). "Modelling the operating efficiency of banks: a non-parametric methodology." Logistics information management **11**(1): 5-17.
- Ali, A. (1995). "Measuring inefficiency in DEA." Presented at INFORMS. International. Singapore, 625-628.
- Ali, A. and Seiford, L. (1995). "Components of efficiency evaluation in data envelopment analysis." European Journal of Operational Research **80**: 462-473.
- Ali, A. (1990). "Translation Invariance in Data Envelopment Analysis." Operations Research Letters **9**: 403-405.
- Ali, A., Seiford, L.M., (1993). "The mathematical programming approach to efficiency analysis" In: Fried, A.O., Lovell, A.K., and Schmidt, S.S. (Eds.) The Measurement of Productive Efficiency, Chapter 3, 120-159, Oxford University Press.
- Allen, R., Athanassopoulos A., Dyson, RG. and Thanassoulis, E. (1997). "Weights restrictions and value judgements in data envelopment analysis; Evolution, development and future directions." Annals of Operations Research **73**: 13-34.

-
- Andersen, P. and Petersen, N. (1993). "A procedure for ranking efficient units in data envelopment analysis." Management Science **39(10)**: 1261-1264.
- Anderson, D. R., Sweeney, D. J. and Williams, T. A. (1999). Statistics for Business and Economics. Seventh Edition.
- Athanassopoulos, A. (1995). "Goal programming and data envelopment analysis (GoDEA) for target-based multi-level planning: Allocating central grants to the Greek local authorities." European Journal of Operational Research **87**: 535-550.
- Athanassopoulos, A. (1998). "Decision support for target-based resource allocation of public services in multiunit and multilevel systems." Management Science **39(10)**: 1261-1264.
- Avkiran, N. (1999). "An application reference for data envelopment analysis in branch banking: helping the novice researcher." International Journal of Bank Marketing **17(5)**: 206-220.
- Ayadi, F., Adebayo, A. and Omolehinwa, E. (1998). "Bank performance measurement in a developing economy: An application of Data Envelopment Analysis", Managerial Finance **24(7)**: 5-16.
- Bai, J. and Kao, C. (2006). "On the Estimation and Inference of a Panel Co-integration Model with Cross-Sectional Dependence." In: Badi Baltagi (Ed.) Panel Data Econometrics: Theoretical Contributions and Empirical Applications.
- Banker, R. D. (1993). "Maximum likelihood, consistency and data envelopment analysis: a statistical foundation." Management Science **39**: 1265-1273.
- Banker, R. D., Charnes, A. and Cooper, W.W (1984). "Some models for estimating technical and scale inefficiencies in data envelopment analysis." Management Science **39**: 1078-1092.
- Banker, R. D. and Morey R.C. (1996). "Hypothesis tests using data envelopment analysis." Journal of Productivity Analysis **7**: 139-159.
- Banker, R.D. and Morey. R. C. (1989). "Incorporating value judgement in efficiency analysis." Research in Governmental and Nonprofit Accounting **5**: 245-267.
- Banker, R.D., Robert, M. and Thrall, M. (1989). "Estimation of returns to scale using data envelopment analysis. " Working Paper, 89-94, School of Urban and Public Affairs, Carnegie-Mellon University.
- Banker, R.D., Charnes, A., Cooper, W.W. and Clarke, R.L. (1989). "Constrained game formulations and interpretations for data envelopment analysis. " European Journal of Operational Research **40**: 299-308.

-
- Banker, R.D., Charnes, A., Cooper, W.W., Swarts, J. and Thomas, D.A. (1989). "An introduction to data envelopment analysis with some of the models and their uses." Research in Governmental and Nonprofit Accounting **5**: 125-163.
- Banker, R.D., Kauffman, R.J., Zweig, D., (1993). "Repository evaluation of software reuse." IEEE Transactions, Software Engineering. **19(4)**: 379-389.
- Banker, R.D., Chang, H.S. and Cooper, W.W., (1996). "Simulation studies of efficiency returns to scale and misspecification with non linear functions in DEA." Annals of Operations Research **66**: 233-253.
- Bardhan, Indranil R., William, Frank Bowlin., William Wager Cooper., Toshiyuki Sueyoshi., (1996). "Models and measures for efficiency in dominance in DEA Parts I&II" Journal of the Operations Research Society of Japan **39(3)**: 322-344.
- Beasley, J. E. (1990). "Comparing university departments." Omega, International Journal of Management Science **18(2)**: 171-183.
- Benayoun, R., Montgolfier, J., Tergny, J. and Laritchev, O. (1971). "Linear Programming with Multiple objective Functions. Step Method (STEM)", Mathematical Programming **1**: 366-375.
- Behrman, J.R. and Birdsall, N. (1983). "The quality of Schooling, quantity alone is misleading" American Economic Review **73(5)**: 928-946
- Boles, G.N. (1966). "Efficiency Squared-Efficient Computation of Efficiency Indexes." Proceedings of the 39th Annual Meeting of the Western Farm Economic Association 137-142.
- Brockett, P., Charnes C., Cooper, W., Huang, Z. and Sun, D. (1997). "Data transformations in DEA cone ratio envelopment approaches for monitoring bank performances." European Journal of Operational Research **98**: 2501-268.
- Buchanan, J. T. (1997). "A naive approach for solving MCDM problems: the GUESS method." Journal of the Operational Research Society **48**: 202-206.
- Camanho, A. S. and Dyson, R. (2005). "Cost efficiency, production and value-added models in the analysis of bank branch performance." Journal of the Operational Research Society **56**: 483-494.
- Chankong, V. and Haimes, Y. (1978). "The interactive Surrogate Worth Trade-off (ISWT) Method for Multiobjective Decision Making." In: Zionts, S. Multiple Criteria Problem Solving, Lecture Notes in Economics and Mathematical Systems, Springer-Verlag: 42-67.
- Chankong, V. and Haimes, Y.Y. (1982). Multiobjective Decision Making: Theory and Methodology, Elsevier North Holland, New York
-

Charnes, A., Cooper, W.W., (1962) " Programming with fractional function." Naval Research Logistic Quarter **9**: 181-185

Charnes, A. and Cooper, W. (1990). "Polyhedral cone-ratio DEA models with an illustrative application to large commercial banking." Journal of Econometrics 73-79.

Charnes, A., Cooper, W., Golany, B., Seiford, L. and Stutz J (1985). "Foundations of data envelopment analysis for Pareto-Koopman efficient empirical production functions." Journal of Econometrics **30**: 91-107.

Charnes, A., Cooper, W.W., Huang, Z.M. and Sun, D.B., (1990). "Polyhedral cone-ratio data envelopment analysis models with an illustrative application to large commercial banks." Journal of Econometrics **46**: 73-91.

Charnes, A., Cooper, W.W. and Li, S., (1989). "Using data envelopment analysis to evaluate the efficiency of economic performance by Chinese cities." Socio-Economic Planning Science **23**: 325-344.

Charnes, A., Cooper, W.W., and Rhodes, E. (1981). "Evaluating Program and managerial efficiency: An application of DEA to program follow through", Management Science, **27(6)**: 668-697.

Charnes, A., Cooper, W.W. and Rhodes, E. (1978). "Measuring the efficiency of decision making unit." European Journal of Operational Research **2**: 429-444.

Charnes A., Cooper, W.W., Lewin, A. Y. and Seiford L. M. (1994). Data Envelopment Analysis: Theory, Methodology and Applications. Kluwer.

Chen, T. (1997). "A measurement of the resource utilization efficiency of university library." International Journal of Production of Economics **53(1)**: 71-80.

Chen, T. (1998). "A study of efficiency evaluation in Taiwan's banks." International Journal of Service Industry Management **9(5)**: 402-415.

Cook, W.D., Kress, M.,(1990a)."A data envelopment model for aggregating preference rankings." Management Science **36(11)**: 1302-1310

Cook, W.D. and Kress,M., (1990b)." An m^{th} generation model for weak ranking of players in a tournament." Journal of the Operational Research Society **41(12)**: 1111-1119.

Cook, W. D. and Johnston, D.A., (1992). "Evaluating suppliers of complex systems: A multiple criteria approach." Journal of the Operational Research Society **43**: 1055-1061.

Cook, W.D., Johnston, D.A. and McCutcheon, D. (1992). "Implementations of robotics identifying efficient implementors." Omega, International Journal of Management Science **20(2)**: 227-239.

Cook, W. D. and Kress, M., (1994). "A multiple criteria composite index model for quantitative and qualitative data." European Journal of Operational Research **78**: 367-379.

Cook, W. D., Kress, M. and Seiford, L.M., (1993). "On the use of ordinal data in data envelopment analysis." Journal of the Operational Research Society **44**: 133-140.

Cook, W. D., Kress, M. and Seiford, L.M., (1996). "Data envelopment analysis in the presence of both quantitative and qualitative factors." Journal of the Operational Research Society **47**: 945-953.

Cooper, W.W., Park, K.S., Yu, G. (1999). "IDEA and AR IDEA: models for dealing with imprecise data in DEA," Management Science **45**: 597-607.

Cooper, W.W., Seiford, L.M. and Tone, K. (2000). "Data Envelopment Analysis: A comprehensive Text with Models, Applications, References and DEA-Solver Software, Kluwer Academic Publishers, Boston.

Cooper, W.W., Park, K.S., Yu, G.(2001). "An illustrative application of IDEA (imprecise data envelopment analysis) to a Korean mobile telecommunication company." Operations Research **49**: 807-820.

Cooper, W.W., Park, K.S. and Yu, G.(2001). "IDEA (imprecise data envelopment analysis) with CMDs (column maximum decision making units)." Journal of Operational Research Society **52**: 176-181

Cooper, W.W., Seiford, L.M. and Zhu, J. (2004). Handbook on Data Envelopment Analysis. Kluwer Academic Publishers, Boston.

Cooper, W.W., Kaoro, T., (1997). "Measures of inefficiency in data envelopment analysis and stochastic frontier estimation." European Journal of Operational Research **99(1)**: 72-88.

Cooper, W.W., Tone, K., (1997). "Measures of inefficiency in data envelopment analysis and stochastic frontier estimation," European Journal of Operational Research, **48**: 1101-1012.

Cubbin, J and Tzanidakis, G. (1998). "Regression versus data envelopment analysis for efficiency measurement: an application to the England and Wales regulated water industry" Utilities Policy **9(2)**: 75-85

Dillon, W.R., and Goldstein, M. (1984). "Multivariate Analysis; Methods and Applications. New York, John Wiley & Sons.

Doyle, J.R., Green, R.H. and Cook, W.D. (1995). " Upper and lower bound evaluation of multi-attribute objects: comparison model using linear programming, Organ. Behaviour Human Decision Process **64(3)**: 261-273

Doyle, J. and Green, R. (1994). "Efficiency and cross-efficiency in DEA: Derivations, meanings and uses." Journal of the Operational Research Society **39(6)**: 563-576.

Dula, J. H. and Hickman, B.L., (1997). "Effects of including the column being scored from the DEA envelopment LP technology matrix." Journal of the Operational Research Society **48**: 1001-1012.

Dyer, J. S. (1978). "Interactive Goal Programming." Management Science **19(1)**: 62-70.

Dyer, J.S., Fishburn, P.C., Steuer, R.E., Wallenius, J. and Zionts, S. (1992). "Multiple criteria decision making multi-attribute utility theory: the next ten years, Management Science **38(5)**: 645-654.

Dyer, J.S.,(1972)."Interactive goal programming."Management Science, **19(1)**: 62-70.

Dyson, R., Allen, R., Camanho, A., Podinovski, V., Sarrico, C. and Shale, E. (2001). "Pitfalls and Protocols in DEA." European Journal of Operational Research **132**: 245-259.

Dyson, R. and Thanasoulis E. (1988). "Reducing weight flexibility in data envelopment analysis." Journal of Operational Research Society **39(6)**: 563-576.

Dyson, R., Robert, G., Rachel, A., Camanho. A, and Shale, E. (1998). "Pitfalls and Protocols in DEA." Presented at (ESI XVI) in European Summer Institute XVI; Data Envelopment Analysis - 20 years on, Coventry, United Kingdom 8/16 to 8/26

Entani, T., Maeda, Y. and Tanaka, H. (2002) "Dual models of interval DEA and its extensions to interval data." European Journal of Operational Research **136**:32-45

Fare, R. and Grosskopf S. (1983). "Measuring output efficiency." European Journal of Operational Research **13**: 173-179.

Fare, R., Grosskopf, S. and Lovell, C.A.K. (1985). The measurement of Efficiency of Production, Kluwer Publishers. Boston,

Fare, R., Grosskopf, S. and Lovell, C.A.K. (1994). Production of Frontiers, Cambridge University Press.

Farrell, M. (1957). "The measurement of production efficiency." Journal of the Royal Statistical Society, Series A **120**: 253-278.

Flegg, A.T., Allen, D.O., Field, K. and Thurlow, T.W. (2004)."Measuring the efficiency of British universities:multi-period data envelopment analysis." Education Economics **12(3)**: 231-249.

Foreman, E. and Peniwati, K. (1998). "Aggregating individual judgements and priorities with the analytic hierarchy process." European Journal of Operational Research **108**: 165-169.

Forsund, F. and Sarafoglou, N. (2002). "On the Origins of Data Envelopment Analysis." Journal of Productivity Analysis **17**: 23-40.

Frank, M. and Wolfe, P. (1956). "An Algorithm for Quadratic Programming." Naval Research Logistics Quarterly **3,(1-2)**: 95-110.

Friedman, L. and Sinuany-Stern, Z. (1998). "Combining ranking scales and selecting variables in the DEA context." European Journal of Operational Research **25(9)**: 781-791.

Friedman, L. and Sinuany-Stern, Z. (1997). "Scaling units via the canonical correlation analysis in the DEA context." European Journal of Operational Research **100(3)**: 629-637

Ganley, J. and Cubbin, J. (1992). Public Sector Efficiency Measurement: Applications of Data Envelopment Analysis. Elsevier Publishing UK

Geoffrion, A., Dyer J. and Feinburg, A. (1972). "An interactive approach for multi-criterion optimization, with an application to the operation of an academic department." Management Science **19(4)**: 357-368.

Golany, B. (1995). "Evaluating efficiency-effectiveness-equality trade-offs: A data envelopment analysis approach." Management Science **41(7)**: 1172-1184.

Golany, B. and Roll, Y.A., (1994). "Incorporating standards via data envelopment analysis". In: Charnes, A, Cooper, W.W, Lewin, A and Seiford, L.M (Eds) Data Envelopment Analysis: Theory, Methodology and Applications. Springer

Green, R. H. and Doyle, J.R., (1995). "On maximizing discrimination in multiple criteria decision making." Journal of the Operational Research Society **46**: 192-204.

Halme, M. and Korhonen, P. (1999). "Restricting weight in value efficiency analysis". Presented at Sixth European Workshop on Efficiency and Productivity Analysis, Copenhagen Denmark 10/29 to 10/31.

Halme, M., Joro, T., Korhonen P., Salo, S. and Wallenius, J.,(1999)." Value efficiency approach to incorporating preference information in data envelopment analysis." Management Science, **45(1)**: 103-115.

Hanushek, E.A. (1986). "The Economics of Schooling; Production and Efficiency in Public Schools." Journal of Economic Literature **24(3)**: 1141-1177

Hashimoto, A. (1997). "A ranked voting system using a DEA/AR exclusion model: A note." European Journal of Operational Research **97**: 600-604.

Hollingsworth, B., Dawson, PJ. and Maniadakis, N. (1999)."Efficiency measurement of health care: a review of nonparametric methods and applications." Health Care Management Science **2**: 161-172

Hougaard, J. L. (1999). "Fuzzy scores of technical efficiency." European Journal of Operational Research **115**: 529-541.

Hwang C. Y. K. (1981). Multiple Attribute Decision Making Methods and Applications: A State-of-Art Survey. Springer-Verlag.

Jaszkiewicz, A. and Slowinski, R. (1994). "The Light Beam Search Over a Non-dominated Surface of a Multiple-objective Programming Problem." Multiple Criteria Decision Making - Proceedings of the Tenth International Conference: Expand and Enrich the Domains of Thinking and Application: 87-99.

Jenkins, L. and Anderson, M. (2003). "A multivariate statistical approach to reducing the number of variables in data envelopment analysis." European Journal of Operational Research **147**: 51-61.

Johnson, R.A., Wichern, D.W. (2002). Applied Multivariate Statistical Analysis, 5th edition, Prentice Hall, Englewood Cliffs, NJ.

Johnes, J. and Wei, Y. (2005). "Internet tools in the teaching of quantitative economics: Why gaps between potential and reality?." Journal of Further and Higher Education, **29(2)**: 125-141.

Johnes, J. (2006). "Data envelopment analysis and its application to the measurement of efficiency in higher education." Economics of Education Review, **25(3)**: 273-288.

Johnes, J. (2006), "Measuring the efficiency of universities: a comparison of multi-level modelling and data envelopment analysis." Bulletin of Economic Research **58(2)**: 75-104.

Joro, R., Korhonen, P. and Wallenius, J. (1998). "Structural Comparison of Data Envelopment Analysis and Multiple Objective Linear Programming." Management Science **44(7)**: 962-970.

Joro, R., Korhonen, P. and Zionts, S. (2003). "An interactive approach to improve estimates of value efficiency in data envelopment analysis." European Journal of Operational Research **149**: 688-699.

Karsak, E. E. (1998). "A two-phase robot selection procedure." Planning Review **9(7)**: 675-684.

Keeney, R. and Raiffa, H. (1993). Decision with Multiple Objectives: Preferences and Value Trade-offs. Cambridge University Press, Cambridge & New York

Kerr, C.A., Glass, J.C. and McCallion, J.M. (1990). "Best practice measures of resource utilization for hospitals: A useful complement in performance assessment." Public Administration **77(3)**: 639-650

Kinnear, P. R. and Gray, C. D. (2004). SPSS 12 Made Simple, Psychology Press.

Koopsman, T.C., (1951). "An analysis of production as an efficient combination of activities, In: Koopsman, T.C. (Ed.) Activity Analysis of Production and Allocation. John Wiley and Sons Inc.

Korhonen, P., Aapo, S. and Margareta, S. (1998). "Practical aspects of value efficiency analysis." Presented at: European Symposium on Data Envelopment Analysis;

Recent Developments and Applications, Wernigerode, Germany, 10/29 to 10/31.

Korhonen, P., Risto, T. and Syrki, W. (1998). "Value efficiency analysis of academic research." International Institute for Applied System Analysis. June. Report 9032.

Korhonen, P. and Laakso, J. (1986). "A Visual Interactive Method for Solving the Multiple Criteria Problem (VIA)." European Journal of Operational Research **24(2)**: 277-287.

Korhonen, P., Somismaa, M. and Siljamaki (2002). "On the use of Value Efficiency Analysis and some further developments." Journal of Productivity Analysis **17**: 49-65.

Kornbluth, J. S. H. (1991). "Analyzing policy effectiveness using cone restricted." Journal of the Operational Research Society **42**: 1097-1104.

Li, D. and Yang, J.B. (1996). "Interactive parametric minimax method for a class of composition problems." Journal of Mathematical Analysis and Applications. **198**: 64-83

Li, X. and Reeves, G.R., (1999). "A multiple criteria approach to data envelopment analysis." European Journal of Operational Research **115**: 507-5117.

Lightner, M. R. and Director, S.W. (1981). "Multiple criterion optimization for the design of electronics circuit." IEEE Transactions on Circuits and Systems **28**: 169-179.

Lorenzo-Vivas, A. and Pastor, J. (2002). "An efficiency comparison of European banking systems operating under different environmental conditions." Journal of Productivity Analysis **18**: 59-77.

Lovell, C.A.K. (1993). "Production frontiers and Productive efficiency". In: Fried, A.O., Lovell, A.K., and Schmidt, S.S., (Eds) The Measurement of Productive Efficiency, Chapter 1, 3-67, Oxford University Press.

Lovell, C.A.K. (1994), "Linear Programming Approaches to the Measurement and Analysis of Productive Efficiency." TOP **2**: 175-248.

Mehrabian, S., Alirezaee, M.R. and Jahanshahloo, G.R., (1999). "A complete efficiency ranking of decisions making units in data envelopment analysis." Computational Optimization and Application **14**: 261-266.

Morrison, D.F. (1976). Multivariate Statistical Methods. 2nd edition, McGraw-Hill

Nakayama, H. and Sawaragi, Y. (1984). "Satisficing Trade-off Method for Multiobjective Programming, An Interactive Decision Analysis." Lecture Notes in Economics and Mathematical Systems, Springer-Verlag **229**: 113-122.

Ngwenyama, O. K. and Bryson, N. (1999). "Eliciting and mapping qualitative preferences to numeric rankings in group decision making." European Journal of Operational Research **116**: 497-497.

-
- Oral, M., Ossama, K., Pascal, L.,(1991). A methodology for collective evaluation and selection of industrial R&D projects." Management Science, **37(7)**: 871-885, July.
- Parkan, C., Wang Y.M., (2000). "The worst possible relative efficiency analysis based on inefficient production frontier." Working Paper, Department of Management of Sciences, City University of Hongkong
- Pedraja-Chapparo, F., Salinas-Jimenez, J. and Smith, P.C (1997). "On the role of weight restrictions in data envelopment analysis." The Journal of Productivity Analysis **8(2)**: 215-230.
- Post, T. and Spronk, J. (1999). "Performance benchmarking using interactive data envelopment analysis." European Journal of Operational Research **115**: 472-487.
- Podinovski, V.V. (2005) "The explicit role of weight bounds in models of data envelopment analysis." Journal of the Operational Research Society **56**:1408-1418
- Podinovski, V.V. (2004) "Production trade-offs and weight restrictions in data envelopment analysis." Journal of the Operational Research Society **55**:1311-1322
- Ray, S.C., Mukherjee, K.,(Nov.1998)."A study of size efficiency in US banking: identifying banks that are too large." International Journal of Systematic Science **29(11)**: 1281-1294
- Ray, S.C., Mukherjee, K.,(Jul.1998)."Quantity, quality, and efficiency for a partially super-additive cost function, conneticut public school revisited." Journal of Productivity Analysis 10(1): 47-62.
- Saaty, T.L., (1980). The Analytic Hierarchy Process. Mc Graw Hill International.
- Sakawa, M. (1982). "Interactive Multiobjective Decision Making by the Sequential Proxy Optimization Technique: SPOT." European Journal of Operational Research **9(4)**: 386-396.
- Slowinski, R., Soniewicki, B.and Weglarz J. (1994). " DSS for multi-objective project scheduling." European Journal of Operation Research **79**: 220-229
- Sarrico, C., Dyson. R.G, (1997). "DEA in performance measurement in UK universities:The institutional perspective." Presented at: INFORMS conference (in DEA II) San Diego, United States of America.5/4 to 5/7.
- Scheel, H. and Scholtes, S. (1998). "Stability of DEA efficiency scores." Judge institute of Management Studies Working Paper 36, University of Cambridge.
- Seiford, L. M. and Zhu, J. (1999). "Infeasibility of super-efficiency data envelopment models, INFOR." **37(2)**: 174-187.

Seiford, L.M., Thrall, R.M. (1990). "Recent Developments in DEA: the mathematical programming approach to frontier analysis." Journal of Econometrics **46** :7-38

Seiford, L.M. (1996). "Data Envelopment Analysis: The Evolution of the State of the Art (1978-1995)." Journal of Productivity Analysis, **7**: 99-138.

Shepard, R.W. (1970). Theory of Cost and Production Function. Princeton University Press, New Jersey.

Sexton, T., Silkman R. and Hogan, A. (1986). "Data Envelopment Analysis: Critique and extensions." Measuring Efficiency: An Assessment of Data Envelopment Analysis **32**: 73–105

Sinuany-Stern, Z. and Friedman, L. (1998). "Data envelopment analysis and the discriminant analysis of ratios for ranking units." European Journal of Operational Research **111**: 470-478.

Sinuany-Stern, Z., Mehrez, A. and Barboy, A., (1994). "Academic departments' efficiency via data envelopment analysis." Computers and Operations Research **21(5)**: 543-556.

Sinuany-Stern, Z., Mehrez, A. and Hadad, Y. (2000). "An AHP/DEA methodology for ranking decision making units." International Transactions in Operational Research **7**: 109-124.

Steuer, R. E. and Choo, E. U. (1983). "An interactive weighted Tchebycheff procedure for multiple objective programming." Mathematical Programming **26**: 326-344.

Stewart, T, J. (1992). "A critical survey on the status of multiple criteria decision making theory and practice ." Omega International Journal of Management Science **38(5)**: 645-654.

Stewart, T. (1993). "Use of piecewise linear value functions in interactive decision support: a Monte Carlo study." Management Science **39**: 1369-1381.

Stewart, T. (1994). "Data envelopment analysis and multiple criteria decision making: a response." Omega, International Journal of Management Science **22(2)**: 205-206.

Stewart, T. (1996). "Relationships between data envelopment analysis and multi-criteria decision analysis." Journal of the Operational Research Society **47(5)**: 654-665.

Sueyoshi, T. (1999). "Data envelopment analysis non-parametric ranking test and index measurement: Slack-adjusted DEA and an application to Japanese agriculture cooperatives." Omega International Journal of Management Science **27**: 315-326.

Tavares, G. (2002). "Bibliography of Data Envelopment Analysis (1978-2001)." Rutcor Research Report Piscataway, New Jersey Rutgers University.

-
- Thannasoulis, E. (2001). Introduction to theory and application of data envelopment analysis: a foundation text with integrated software. Norwell, MA: Kluwer Academic Publishers.
- Thannasoulis, E. and Dyson, R. (1992). "Estimating Preferred Target Input-Output levels using Data Envelopment Analysis." European Journal of Operational Research **56**: 80-97.
- Thannasoulis, E., Portela, M., Allen, R. (2004). "Incorporating value judgments in DEA". Handbook on Data Envelopment Analysis. Kluwer Academic Publishers.
- Thannasoulis, E. and Allen, R. (1998). Simulating weights restrictions in data envelopment analysis by means of unobserved DMUs. Management Science **44**: 586-594.
- Thannasoulis, E., Aziz, B., Robert, G. and Dyson. (1995). "Exploring output quality targets in the provision of perinatal care in England using data envelopment analysis." European Journal of Operational Research **80(3)**: 588-607.
- Thannasoulis, E. (1995). "Assessing police forces in England and Wales Using DEA." European Journal of Operational Research **87(3)**: 641-657. December
- Thomas, D., Greffe, R., Grant, K.C.,(1986). "Application of data envelopment analysis to management audits of electric distribution utilities." In: Burnes, R (Ed.) Proceedings of the fifth NARUC biennial regulatory information conference 1783-1822.
- Thompson, R., Langemeier, L., Lee and C. Thrall, R. (1990). "The role of multiplier bounds in efficiency analysis with application to Kansas farming." Journal of Econometrics **46**: 93-108.
- Thompson, R.G., Singleton, F.D., Thrall, R.M. and Smith, B.A.,(1986). "Comparative site evaluations for locating a high energy physics lab in Texas." Interfaces **16**: 35-49
- Thompson, Russell, G., Lee, E. and Thrall R.M. (1992). "DEA/AR-efficiency of U.S. independent OIL/GAS producers over time" Computers and Operation Research **19(5)**: 377-391
- Thrall, R. M. (1996). "Duality, classification and slacks in data envelopment analysis." The Annals of Operations Research **66**: 109-138.
- Thrall, R. (1999). "What is the Economic Meaning of FDH?" Journal of Productivity Analysis **11**: 243-250.
- Torgersen, A. M., Forsund, F.R. and Kittelsen, S.A.C., (1996). "Slack-adjusted efficiency measures and ranking of efficient units." The Journal of Productivity Analysis **7**: 379-398.
- Wierzbicki, A. (1980). "The use of reference objectives and multi-objective optimization." In: G.Fandel and T. Gal (Eds.) Multiple Criteria Decision Making: Theory and Application Lecture Notes in Economics and Mathematical Systems, Springer, New York **177**: 468-486.

-
- Wierzbicki, A. P. (1977). "Basic properties of scalarizing functionals for multi-objective optimization." Optimization **8**: 55-60.
- Wang, Y.M., Chin, K.S., and Yang J.B. (2007). "Measuring the performances of decision-making unit using geometric average efficiency." Journal of the Operational Research Society **58**: 929-937.
- Wang, Y. M. and Yang, J. B. (2007) "Measuring the performances of decision making units using interval efficiencies." Journal of Computational and Applied Mathematics **198** : 253-267.
- Wong, Y. H. and Beasley, J. (1990). "Restricting weight flexibility in data envelopment analysis." Journal of the Operational Research Society **41(9)**: 829-835.
- Yang, J. B. (1999). "Gradient projection and local region search for multi-objective optimisation." European Journal of Operational Research **112**: 432-459.
- Yang, J. B. (2001). "Minimax reference point approach and its application for multi-objective optimisation." European Journal of Operational Research **131**: 31-61.
- Yang, J. B., Chen, C. and Zhang, Z. J. (1990). "The interactive step trade-off method (ISTM) for multi-objective optimisation." IEEE Transactions on Systems, Man, and Cybernetics **20**: 688-695.
- Yang, J. B., Wong, Y.H., Li, D. and Stewart T.J. (2009). "Integrating DEA-oriented performance assessment and target setting using interactive MOLP methods ." European Journal of Operational Research **195**: 205-222.
- Yang, J. B. and Li, D. (2002). "Normal vector identification and interactive trade-off analysis using minimax formulation in multi-objective optimisation." IEEE Transactions on Systems, Man, and Cybernetics - Part A: Systems and Humans **32(3)**: 305-319.
- Yang, J. B. and Sen, P. (1997). "Multiple Attribute Decision Evaluation of Complex Engineering Products using the Evidential Reasoning Approach." Journal of Engineering Design **8(3)**: 211-230.
- Yang, J. B. and Singh, M. (1994). "An evidential reasoning approach for multiple attribute decision making with uncertainty." IEEE Transactions on Systems, Man and Cybernetics, **24**: 1-18.
- Yang, J. B., Wong, Y. H., Xu, D. L. and Stewart, T. J. (2009). "Integrating *DEA*-oriented performance assessment and target setting using interactive *MOLP* methods." European Journal of Operational Research **195**: 205-222.

Zhu, J. (1996a). "Robustness of the efficient of the decision-making units in data envelopment analysis." European Journal of Operational Research **90**: 451-460.

Zhu, J. (1996b). "Data envelopment analysis with preference structure." Journal of the Operational Research Society **47**: 136-150.

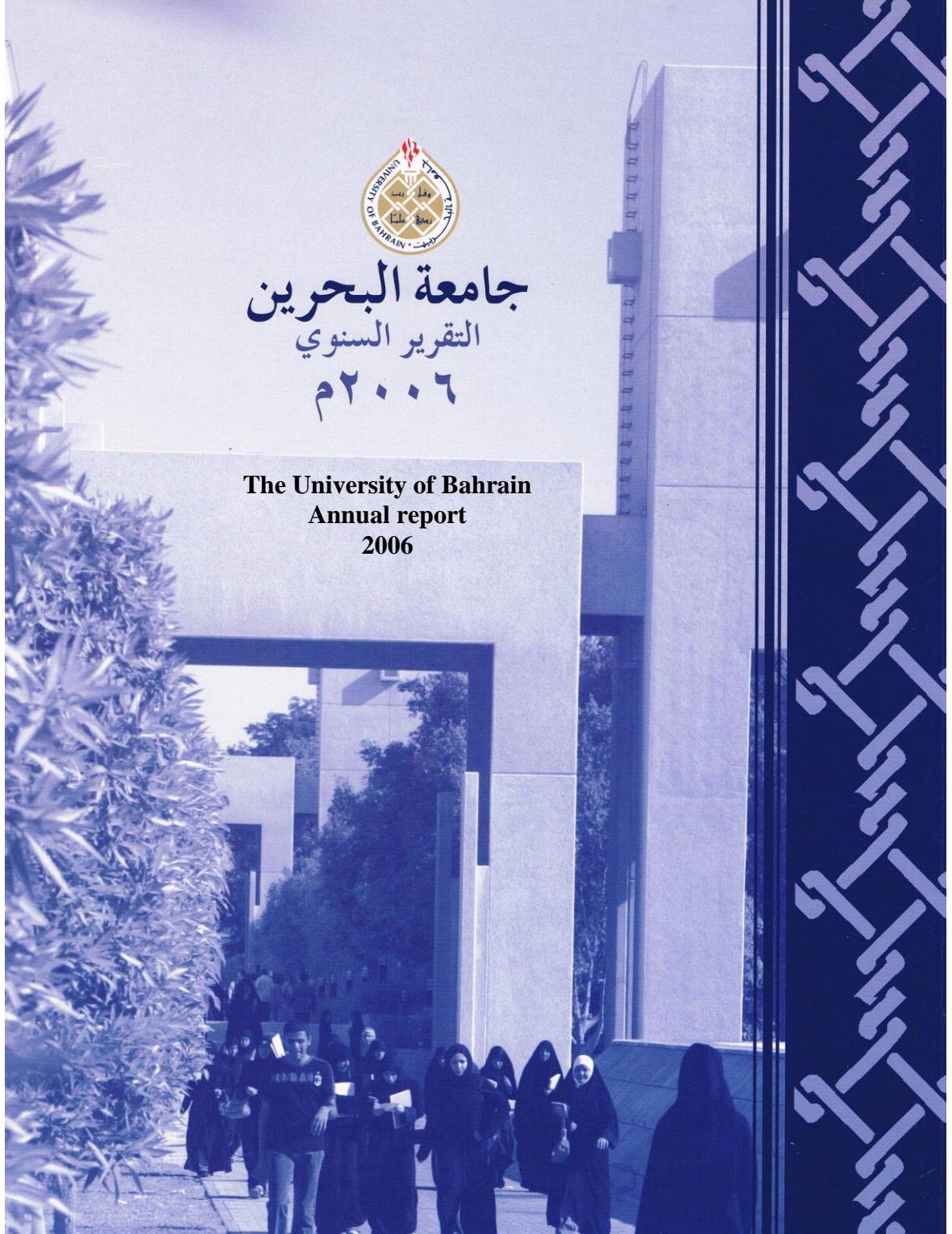
Zhu, J. (2003). "Imprecise data envelopment analysis (IDEA): a review and improvement with an application." European Journal of Operational Research **144**: 513-529.

Zionts, S. and Wallenius, J. (1976). "An Interactive Programming Method for Solving the Multiple Criteria Problem." Management Science **22(6)**: 652-663.

Zionts, S. and Wallenius, J. (1983). "An Interactive Multiple Objective Linear Programming Method for a Class of Underlying Utility Functions." Management Science **29(5)**: 519-529.

APPENDIX I:

University of Bahrain raw data.



القسم	البرنامج الأكاديمي	الفصل الأول			الفصل الثاني		
		طالب	طالبة	اجموع	طالب	طالبة	اجموع
الإعلام والسياحة والفنون	البكالوريوس في الإعلام	١٣١	٤٢٢	٥٥٣	١١٢	٤٢٨	٥٤٠
	البكالوريوس في السياحة	٥٣	١٠٨	١٦١	٥٤	١١١	١٦٥
	البكالوريوس في الإعلام - الفترة الثانية	٨٦	١٢٤	٢١٠	٧٥	١٠٥	١٨٠
العلوم الاجتماعية	البكالوريوس في الخدمة الاجتماعية	٨٦	٥١٠	٥٩٦	٧٤	٤٥٠	٥٢٤
	البكالوريوس في الجغرافيا التطبيقية ونظم المعلومات الجغرافية	١٩	١٤٣	١٦٢	١٨	١٠٨	١٢٦
	البكالوريوس في التاريخ	٢٤	١٣٢	١٥٦	٢٥	١٤٨	١٧٣
	البكالوريوس في علم الاجتماع	٣٧	٥٠٧	٥٤٤	٣٦	٤٦٧	٥٠٣
	البكالوريوس في الخدمة الاجتماعية - الفترة الثانية	٢٦	١٧٧	٢٠٣	٢١	١٦٤	١٨٥
	البكالوريوس في الجغرافيا التطبيقية ونظم المعلومات الجغرافية - الفترة الثانية	١٠	٩٨	١٠٨	٨	٩١	٩٩
	البكالوريوس في اللغة الإنجليزية	٥٠	٣٩٦	٤٤٦	٤٦	٣٨٦	٤٣٢
اللغة الإنجليزية وآدابها	البكالوريوس في اللغة الإنجليزية - الفترة الثانية	١٩	٧٤	٩٣	١٩	٦٦	٨٥
	الدبلوم المشارك في اللغة الفرنسية	١	-	١	-	-	-
اللغة العربية والدراسات الإسلامية	الماجستير في اللغة العربية	٥	٢	٧	٤	٤	٨
	البكالوريوس في اللغة العربية	٤٠	٣١	٧١	٤٦	٢١	٦٧
	البكالوريوس في الدراسات الإسلامية	٥٦	١٧٦	٢٣٢	٧٧	١٥٠	٢٢٧

القسم	البرنامج الأكاديمي	الفصل الأول			الفصل الثاني		
		طالب	طالبة	الاجموع	طالب	طالبة	الاجموع
الإدارة والتسويق	الماجستير في الإدارة التنفيذية	١٣	٥	١٨	٤	٢	٦
	البكالوريوس في إدارة الأعمال	٢١٧	٤٩٢	٧٠٩	١٩١	٤٩٠	٦٨١
	البكالوريوس في التسويق	٢٢٨	٣٢١	٥٤٩	٢١٦	٣٠١	٥١٧
	البكالوريوس في إدارة الأعمال - الفترة الثانية	٦١	٦٨	١٢٩	٥٠	٦٢	١١٢
	البكالوريوس في التسويق - الفترة الثانية	٥٠	٣٢	٨٢	٣٨	٢١	٥٩
	الدبلوم المشارك في إدارة الأعمال	١	٢	٣	-	٣	٣
	الدبلوم المشارك في التسويق	٢	٣	٥	-	١	١
	الدبلوم المشارك في إدارة الأعمال - الفترة الثانية	-	١	١	-	٢	٢
	الدبلوم المشارك في التسويق - الفترة الثانية	١	١	٢	١	١	٢
	دبلوم الدراسات التجارية	٣٥	٣٣	٦٨	٣٤	٣٣	٦٧
الإدارة المكتبية	دبلوم الإدارة المكتبية	٣	١٢٢	١٢٥	٣	٩٦	٩٩
	دبلوم الإدارة المكتبية - الفترة الثانية	١٠	٥١	٦١	٨	٤٠	٤٨
الاقتصاد والتمويل	البكالوريوس في الأعمال المصرفية والمالية	٣٢٥	٧٦٦	١٠٩١	٣١٢	٧٥٣	١٠٦٥
	البكالوريوس في الأعمال المصرفية والمالية - الفترة الثانية	٤٥	٨١	١٢٦	٣٣	٧١	١٠٤
	الدبلوم المشارك في الأعمال المصرفية والمالية	٣	٤	٧	-	-	-
	الدبلوم المشارك في الأعمال المصرفية والمالية - الفترة الثانية	-	١	١	-	١	١
الحاسبة	البكالوريوس في الحاسبة	٦٠٨	١١١٤	١٧٢٢	٥٣٠	١٠٤٧	١٥٧٧
	البكالوريوس في الحاسبة - الفترة الثانية	٩١	٩٨	١٨٩	٧٧	٧٥	١٥٢
	الدبلوم المشارك في الحاسبة	٤	٤	٨	٢	-	٢
	الدبلوم المشارك في الحاسبة - الفترة الثانية	٢	١	٣	-	٣	٣

القسم	البرنامج الأكاديمي	الفصل الأول			الفصل الثاني		
		طالب	طالبة	المجموع	طالب	طالبة	المجموع
التربية الرياضية	ماجستير التربية الرياضية في التدريب الرياضي	٢	-	٢	-	-	-
	البكالوريوس في التربية الرياضية	١٠٨	٢٥٥	٣٦٣	٩٥	٢٠٩	٣٠٤
تكنولوجيا التعليم	بكالوريوس التربية في تكنولوجيا التعليم والمعلومات	٤٥	٥٦٢	٦٠٧	٣٩	٥٠٧	٥٤٦
	الماجستير في القياس والتقويم التربوي	-	١	١	-	-	-
علم النفس	دبلوم الإرشاد النفسي والتربوي	٣	١٢	١٥	٣	١٢	١٥
	بكالوريوس التربية في علم النفس التربوي - فئات خاصة	١٤	٣٩١	٤٠٥	١٢	٣٦١	٣٧٣
	دبلوم الدراسات العليا في الإدارة المدرسية	١٨	١٠	٢٨	١٦	١٠	٢٦
الناهج وطرق التدريس	دبلوم التربية في التدريس	١٢٤	١٠٢	٢٢٦	١٢٥	٩٧	٢٢٢
	الدبلوم التخصصي لمعلمي الفصل	-	١	١	-	-	-
	البكالوريوس في التربية	٤٠	٧٠	١١٠	٣٥	٧٠	١٠٥
	بكالوريوس التربية في اللغة الإنجليزية	٥	١٠٨	١١٣	٣	٩٨	١٠١
	بكالوريوس التربية في الفنون التربوية	٢٨	٣٢٨	٣٥٦	٢٥	٢٨٤	٣٠٩

البرنامج الأكاديمي		الفصل الأول			الفصل الثاني		
	طالب	طالبة	المجموع	طالب	طالبة	المجموع	
الدبلوم المشارك في التعليم التطبيقي	١٦٢	٣٢١	٤٨٣	١	١٤	١٥	
الدبلوم المشارك في شبكات الحاسوب	-	-	-	١٨	٢٣	٤١	
الدبلوم المشارك في نظم المعلومات	-	-	-	٢١	٣٩	٦٠	
الدبلوم المشارك في تطبيقات الوسائط المتعددة	-	-	-	٩	٥٠	٥٩	
الدبلوم المشارك في الإدارة المكتبية	-	-	-	٨	٥٣	٦١	
الدبلوم المشارك في تجميع وصيانة الحاسوب	-	-	-	٢٣	١٨	٤١	
الدبلوم المشارك في تطوير البرمجيات وقواعد المعلومات	-	-	-	٢٤	٢٣	٤٧	
الدبلوم المشارك في إدارة وتصميم مواقع الإنترنت	-	-	-	٨	٤٤	٥٢	
الدبلوم المشارك في الدراسات التجارية	-	-	-	١٥	٤٧	٦٢	

القسم	البرنامج الأكاديمي	الفصل الأول			الفصل الثاني		
		طالب	طالبة	اجموع	طالب	طالبة	اجموع
علم الحاسوب	الماجستير في الحاسوبات	-	-	-	١	-	١
	البكالوريوس في علم الحاسوب	١٦٤	٣٣٩	٥٠٣	١٦٦	٢٩٤	٤٦٠
	البكالوريوس في علم الحاسوب - الفترة الثانية	٧٢	٢٥	٩٧	٥١	٢٠	٧١
	الدبلوم المشارك في علم الحاسوب	٥	٨	١٣	٧	١٣	٢٠
نظم المعلومات	الدبلوم المشارك في علم الحاسوب - الفترة الثانية	٢	٢	٤	٤	٣	٧
	البكالوريوس في نظم المعلومات	٢٦٦	٦١٥	٨٨١	٢٣٧	٥٤٤	٧٨١
	البكالوريوس في نظم المعلومات - الفترة الثانية	٧٦	٦٣	١٣٩	٦٧	٥١	١١٨
	الدبلوم المشارك في نظم المعلومات الإدارية	٤	٩	١٣	٥	١٤	١٩
هندسة الحاسوب	الدبلوم المشارك في نظم المعلومات الإدارية - الفترة الثانية	٦	٢	٨	٢	٢	٤
	البكالوريوس في هندسة الحاسوب	٢٣٢	٢٦١	٤٩٣	٢٠٩	٢٥٢	٤٦١
	الدبلوم المشارك في هندسة الحاسوب	٨	٥	١٣	١١	٦	١٧

القسم	البرنامج الأكاديمي	الفصل الأول			الفصل الثاني	
		طالب	طالبة	اجموع	طالب	طالبة
القانون العام والقانون الخاص	البكالوريوس في الحقوق	٢٣١	٧٠٤	٩٣٥	٢١٢	٦٨٠
	البكالوريوس في الحقوق - الفترة الثانية	٩١	٧٦	١٦٧	٧٧	٦٥
	الدبلوم المشارك في الحقوق	١	٢	٣	-	٣
	الدبلوم المشارك في الحقوق - الفترة الثانية	١	-	١	١	-

القسم	البرنامج الأكاديمي	الفصل الأول			الفصل الثاني		
		طالب	طالبة	المجموع	طالب	طالبة	المجموع
الرياضيات	الماجستير في الرياضيات	-	٢	٢	-	١	١
	البكالوريوس في الرياضيات	٨٩	٢٦١	٣٥٠	٩١	٢٦٦	٣٥٧
	البكالوريوس في الإحصاء وبحوث العمليات	١٦	١٧١	١٨٧	١٦	١٥٧	١٧٣
علوم الحياة	البكالوريوس في علوم الحياة	١٧	١٨٤	٢٠١	٢٤	١٩١	٢١٥
الفيزياء	البكالوريوس في الفيزياء	٧	٢٩	٣٦	٨	٢٦	٣٤
الكيمياء	البكالوريوس في الكيمياء	١٣	٨٥	٩٨	١٣	١٠٠	١١٣

القسم	البرنامج الأكاديمي	الفصل الأول			الفصل الثاني		
		طالب	طالبة	المجموع	طالب	طالبة	المجموع
الهندسة الكهربائية	الدكتوراه في الهندسة الكهربائية	٣	-	٣	٣	-	٣
	البكالوريوس في الهندسة الكهربائية	١٠٧	٤	١١١	١٢٤	٥	١٢٩
	البكالوريوس في الهندسة الإلكترونية	١٠١	٣٤	١٣٥	٩٩	٣٠	١٢٩
	الدبلوم المشارك في الهندسة الكهربائية	٧٤	١	٧٥	٧٤	٣	٧٧
	الدبلوم المشارك في الهندسة الإلكترونية	٣٨	٦	٤٤	٤٠	٦	٤٦
الهندسة الكيميائية	الماجستير في الهندسة الكيميائية	١	-	١	-	-	-
	البكالوريوس في الهندسة الكيميائية	٤٦	١٤	٦٠	٥٠	١١	٦١
	البكالوريوس في الأجهزة الدقيقة والتحكم	١١١	٣	١١٤	١٢٧	٦	١٣٣
	الدبلوم المشارك في الهندسة الكيميائية	١٢	٥	١٧	١٠	١	١١
	الدبلوم المشارك في الأجهزة الدقيقة والتحكم	٦٤	٢	٦٦	٥٤	١	٥٥
الهندسة المدنية والعمارة	البكالوريوس في الهندسة المدنية	٨٤	٥٩	١٤٣	٩٨	٦٥	١٦٣
	البكالوريوس في الهندسة المعمارية	٤٩	١٤٣	١٩٢	٣٩	١٢٥	١٦٤
	البكالوريوس في التصميم الداخلي	-	٧٨	٧٨	-	٧٨	٧٨
	الدبلوم المشارك في الهندسة المدنية	١٨	١٢	٣٠	١٩	٩	٢٨
	الدبلوم المشارك في الهندسة المعمارية	٢٠	١٢	٣٢	١٥	١٣	٢٨
الهندسة الميكانيكية	الدكتوراه في الهندسة الميكانيكية	٢	-	٢	٢	-	٢
	الماجستير في الإدارة الهندسية	١٢	٣	١٥	٤	-	٤
	البكالوريوس في الهندسة الميكانيكية	١٥٢	-	١٥٢	١٥٨	-	١٥٨
	البكالوريوس في علم المعادن	٢	٣	٥	٢	٣	٥
	الدبلوم المشارك في الهندسة الميكانيكية	٥٩	-	٥٩	٦١	-	٦١

الفصل الثاني		الفصل الأول		البرنامج الأكاديمي	
المجموع	طالبة	طالب	المجموع	طالبة	طالب
-	-	-	٤	١	٣

الماجستير في البيئة والتنمية المستدامة

الكلية	عدد الطلبة المسجلين				عدد الطلبة المقبولين			
	الفصل الثاني		الفصل الأول		الفصل الثاني		الفصل الأول	
	طالبة	طالب	طالبة	طالب	طالبة	طالب	طالبة	طالب
الآداب	٢٨٦٤	٦٦١	٣٠٣٥	٦٩٦	١٥٧	٤٢	٢٠٦	٦٠
إدارة الأعمال	٣٣٢٨	١٧١١	٣٤٦٣	١٨٥٠	٣٤٩	٢١٧	٦٧٧	٢٥٠
التربية	١٦٦٥	٣٦٦	١٨٥٧	٤٠١	-	-	٩٧	٩٧
التعليم التطبيقي	٣١١	١٢٧	٣٢١	١٦٢	-	-	٣٦١	٢٠٨
تقنية المعلومات	١٣٢٦	٨١٥	١٣٥٤	٨٤٨	١٧٩	٨٦	١٩٩	٧٠
الحقوق	٨٣٤	٣١١	٨٣٠	٣٣٨	٩٤	٢٣	٩٥	١٩
العلوم	٧٤٨	١٥٥	٨٧٧	١٦٧	١١	٥	٢٠٤	٤٥
الهندسة	٣٨٣	١٠٢٨	٤٥٩	١٢٨٦	-	-	١٠٤	٤٠٣
عمادة البحث العلمي	-	-	١	٣	-	-	-	-
الإجمالي	١٦٦٣٣		١٧٩٤٨		١١٦٣		٣٠٩٥	

عدد الطلبة الخريجين

الكلية	شهادة		دبلوم		دبلوم مشارك		بكالوريوس		دبلوم عال		ماجستير	
	طالب	طالبة	طالب	طالبة	طالب	طالبة	طالب	طالبة	طالب	طالبة	طالب	طالبة
الآداب	-	-	-	-	١	-	٨٧	٤٠٨	٢	٢	٢	٤
إدارة الأعمال	-	٣	٢٨	١٢٦	٦١	١٢٧	١٠٣	١٨٤	-	-	١٥	٤
التربية	-	-	١٣	١٧	-	-	٥٠	٤٠٩	٦٨	٥٨	٣١	٣٠
تقنية المعلومات	-	-	-	-	٦	٧	٩٢	١٢٩	-	-	٣	-
الحقوق	-	-	-	-	١٦	٣٤	٣٥	٦٢	-	-	-	-
العلوم	-	-	-	-	-	-	٢٦	١١٧	-	-	٢	٣
الهندسة	٣٨	١٢	٥٤	٣١	٧٥	١٦	٧٦	٥٧	-	-	١٥	٥
الإجمالي	٥٣	٥٣	٢٦٩	٣٤٣	١٨٣٥	١٣٠	١١٤					

APPENDIX II: Private hospitals raw data

6- Health and Health Services

6 - الصحة والخدمات الصحية

Health Services in Bahrain started in 1900's. Since then, the Government is working very hard to provide advance free medical care to all inhabitants of Bahrain. The Government has also adopted the World Health Organization Goal to achieve "Health For

ابتدأت الخدمات الصحية بمملكة البحرين مع بداية القرن الحالي . ومنذ ذلك الحين والمملكة تعمل جاهدة على تأمين العناية الصحية المتطورة المجانية لجميع المواطنين والمقيمين في البحرين ، ولقد تبنت المملكة شعار منظمة الصحة العالمية والداعي إلى توفير العناية الصحية .

This Chapter contains detailed data on patients using the health service facilities in both Government and private sectors. The in-patients and outpatients are classified according to some socio-economic characteristics. It has been Included Statistics.

يحتوي هذا الفصل على بيانات تفصيلية عن المستشفيات والمراكز الصحية بالبلاد والمترددين عليها سواء المرضى الداخليون أو الخارجيون وذلك طبقاً لبعض الخصائص الاقتصادية والاجتماعية وقد تم إضافة احصاءات لثلاثة مستشفيات جديدة تعمل في المملكة في هذا العدد من المجموعة .

Note : Ministry of Health Hospitals excludes data of Military Hospital and Recently Directorate of Health & Social Welfare (Previously Public Security Health Center).

ملاحظة : بيانات وزارة الصحة لا تشمل بيانات المستشفى العسكري وادارة الخدمات الصحية والاجتماعية حالياً (الأمن العام سابقاً) .

Sources of these chapter tables :

مصادر جداول هذا الفصل :

- Health Information Directorate / Ministry of Health.
- Military Hospital .
- Private Hospitals :
 - American Mission Hospital.
 - Bahrain International Hospital.
 - Awali Hospital.
 - Gulf Dental Speciality Hospital.
 - Ibn Al-Nafees Hospital.
 - Bahrain Specialist Hospital.
- Economic Planning Directorate / Ministry of Finance & National Economy.

- إدارة المعلومات الصحية / وزارة الصحة .
- المستشفى العسكري .
- المستشفيات الخاصة :
- مستشفى الإرسالية الأمريكية .
- مستشفى البحرين الدولي .
- مستشفى عوالي .
- مستشفى الخليج التخصصي للأسنان .
- مستشفى ابن النفيس .
- مستشفى البحرين التخصصي .
- إدارة التخطيط الطبيعي / وزارة المالية والاقتصاد الوطني .

المصدر : ادارة الميزانية العامة / وزارة المالية والاقتصاد الوطني

جدول 6.18 Table 6.18

احصاءات المستشفيات الخاصة (2005 - 2001)
Private Hospitals Statistics (2001 - 2005)

Item	السنة					البيان
	2005	2004	2003	2002	2001	
American Mission Hospital						مستشفى الأرسالية الأمريكية
Beds	40	40	40	40	40	الأسرة
Out-Patient Treated	154,698	153,235	153,206	135,826	139,074	المرضى الخارجيون
In-Patient Admitted	2,293	1,841	1,803	2,152	2,032	المرضى الداخليون الأيام التي قضاها المرضى في المستشفى
Patients Days in Hospital	5,169	4,887	4,862	5,614	5,292	إجمالي المواليد
Total Births	451	412	299	311	361	المواليد الأحياء
Live Births	449	410	298	306	360	المواليد الموتي
Still Births	2	2	1	5	1	الوفيات
Deaths	6	9	4	-	1	نسبة أشغال الأسرة (%)
Bed Occupancy (%)	35.4	33.4	33.3	38.5	36.2	متوسط عدد الأيام التي قضاها المريض بالمستشفى
Average Days						
Patients Admitted	2.3	2.7	2.9	2.6	2.6	
Awali Hospital						مستشفى عوالي
Beds	28	37	37	37	37	الأسرة
Out-Patient Treated	39,066	35,404	34,588	34,761	32,744	المرضى الخارجيون
In-Patient Admitted	1023	958	941	1,092	840	المرضى الداخليون الأيام التي قضاها المرضى في المستشفى
Patients Days in Hospital	3,197	2,977	3,142	3,512	3,105	إجمالي المواليد
Total Births	108	107	114	110	130	المواليد الأحياء
Live Births	108	107	114	109	130	المواليد الموتي
Still Births	-	-	-	1	-	الوفيات
Deaths	3	3	-	3	-	نسبة أشغال الأسرة (%)
Bed Occupancy (%)	31.3	28.1	30.7	37.4	36.4	متوسط عدد الأيام التي قضاها المريض بالمستشفى
Average Days						
Patients Admitted	2.8	2.8	3.0	3.1	3.7	
Bahrain International Hospital						مستشفى البحرين الدولي
Beds	57	57	65	65	65	الأسرة
Out-Patient Treated	70,746	92,362	64,182	46,904	33,142	المرضى الخارجيون
In-Patient Admitted	3,606	3,570	2,378	1,831	1,498	المرضى الداخليون الأيام التي قضاها المرضى في المستشفى
Patients Days in Hospital	8,870	7,050	5,916	4,018	3,652	إجمالي المواليد
Total Births	167	202	206	167	153	المواليد الأحياء
Live Births	166	200	206	165	153	المواليد الموتي
Still Births	1	2	-	2	-	الوفيات
Deaths	13	8	9	2	6	نسبة أشغال الأسرة (%)
Bed Occupancy (%)	42.6	41.5	33.0	23.1	20.2	متوسط عدد الأيام التي قضاها المريض بالمستشفى
Average Days						
Patients Admitted	2.5	2.5	2.9	2.6	2.8	

(...) Not Available

(...) غير متوفر

بعض بيانات عام 2004 لهذا الجدول لم يتم استلامها من المصدر وهما مستشفى الأرسالية الأمريكية والبحرين الدولي .

Part of the data for the year 2004 of this table still not received from the Source American Mission and Bahrain International Hospitals .

تابع جدول 6.18 Cont. Table 6.18

احصاءات المستشفيات الخاصة (2005 - 2001)
Private Hospitals Statistics (2001 - 2005)

Item	السنة					البيان
	Year 2005	Year 2004	Year 2003	Year 2002	Year 2001	
Gulf Dental Speciality Hospital (1)						مستشفى الخليج التخصصي للأسنان (1)
Beds	10	10	10	10	10	الأسرة
Total Out-Patient Visits	9,988	9,756	8,807	6,820	6,342	زيارات العيادات الخارجية
In-Patient Admitted	31	15	-	-	-	المرضى الداخليون
Laboratory Tests	31	15	-	5	14	فحوصات المختبر
Radiology Procedures	1,479	1,526	1,184	1,120	1,035	إجراءات الأشعة
Operations (Major/Minor)	31	15	3	11	45	الإجراءات الجراحية (الكبيرة/الصغيرة)
Bed Occupancy (%)	0.8	0.4	0.1	0.3	1.4	نسبة أشغال الأسرة (%)
Average Days						متوسط عدد الأيام
Patients Admitted	1.0	1.0	1.0	1.0	1.1	التي قضاها المريض بالمستشفى
Ibn Al-Nafees Hospital (2)						مستشفى ابن النفيس (2)
Beds	14	15	15	11	11	الأسرة
Out-Patient Treated	57,979	58,828	51,123	35,566	10,362	المرضى الخارجيون
In-Patient Admitted	2,500	2,087	1,741	1,525	401	المرضى الداخليون
Patients Days in Hospital	2,500	1,643	1,225	1,163	334	الأيام التي قضاها المرضى في المستشفى
Total Births	90	101	76	36	-	إجمالي المواليد
Live Births	90	101	75	36	-	المواليد الأحياء
Still Births	-	-	1	-	-	المواليد الموتي
Deaths	-	-	-	-	-	الوفيات
Bed Occupancy (%)	48.9	38.0	43.3	39.9	13.3	نسبة أشغال الأسرة (%)
Average Days						متوسط عدد الأيام
Patients Admitted	1.0	5.7	4.8	4.1	2.2	التي قضاها المريض بالمستشفى
Bahrain Specialist Hospital (3)						مستشفى البحرين التخصصي (3)
Beds	56	56	54	54	-	الأسرة
Out-Patient Treated	54,126	42,180	30,295	1,615	-	المرضى الخارجيون
In-Patient Admitted	3,592	2,392	1,521	175	-	المرضى الداخليون
Patients Days in Hospital	11,021	7,490	5,134	636	-	الأيام التي قضاها المرضى في المستشفى
Total Births	341	238	102	2	-	إجمالي المواليد
Live Births	339	237	101	2	-	المواليد الأحياء
Still Births	2	1	1	-	-	المواليد الموتي
Deaths	15	10	5	2	-	الوفيات
Bed Occupancy (%)	53.9	36.5	26.0	16.0	-	نسبة أشغال الأسرة (%)
Average Days						متوسط عدد الأيام
Patients Admitted	3.1	3.1	3.5	3.6	-	التي قضاها المريض بالمستشفى

(1) Opened on 14th December 1999

(2) Opened on April 2001

(3) Opened on 19th October 2002

(1) افتتح في 14 ديسمبر 1999

(2) افتتح في أبريل 2001

(3) افتتح في 19 أكتوبر 2002

جدول 6.19 Table 6.19

أطباء المستشفيات الخاصة حسب التخصص والجنسية والنوع - ديسمبر (2004 - 2005)
Private Hospitals Physicians By Profession, Nationality and Sex - December (2004 - 2005)

Profession	الجنسية/النوع Nationality/Sex									التخصص
	الجملة Total			غير بحريني Non-Bahraini			بحريني Bahraini			
	جملة Total	اناث Females	ذكور Males	جملة Total	اناث Females	ذكور Males	جملة Total	اناث Females	ذكور Males	
2004										
General Practitioners										ممارس عام وأمراض باطنية
Paediatrics										أمراض الأطفال
Pneumatology										الأمراض الصدرية
Dermatology & Venerology										الأمراض الجلدية والتناسلية
Anaesthetology										التخدير
General Syrgery										الجراحة العامة
Ophthalmology										أمراض العيون
E.N.T.										الأنف والأذن والحنجرة
Bacterology (Laboratory)										المختبرات
Radiology										الأشعة
Physiotherapy										العلاج الطبيعي
Orthopedics										معالجة تشوهات المفاصل والعظام
Gynaecology & Maternity										أمراض النساء والولادة
Others										أخرى
Total Physicians										جملة الاطباء
Dentists										اطباء الأسنان
2005⁽¹⁾										
General Practitioners	53	16	37	47	14	33	6	2	4	ممارس عام وأمراض باطنية
Paediatrics	24	11	13	20	8	12	4	3	1	أمراض الأطفال
Pneumatology	3	1	2	2	1	1	1	-	1	الأمراض الصدرية
Dermatology & Venerology	5	2	3	5	2	3	0	-	-	الأمراض الجلدية والتناسلية
Anaesthetology	8	3	5	8	3	5	0	-	-	التخدير
General Syrgery	16	1	15	15	1	14	1	-	1	الجراحة العامة
Ophthalmology	5	1	4	4	1	3	1	-	1	أمراض العيون
E.N.T.	8	1	7	7	-	7	1	1	0	الأنف والأذن والحنجرة
Bacterology (Laboratory)	5	1	4	5	1	4	0	-	-	المختبرات
Radiology	10	3	7	9	2	7	1	1	-	الأشعة
Physiotherapy	5	1	4	4	0	4	1	1	0	العلاج الطبيعي
Orthopedics	12	1	11	11	1	10	1	-	1	معالجة تشوهات المفاصل والعظام
Gynaecology & Maternity	22	18	4	21	17	4	1	1	-	أمراض النساء والولادة
Others	33	9	24	25	4	21	8	5	3	أخرى
Total Physicians	209	69	140	183	55	128	26	14	12	جملة الاطباء
Dentists	29	9	20	21	5	16	8	4	4	اطباء الأسنان

(1) Includes All Private Hospitals in Bahrain Kingdom, 3 hospitals opened during 2005 : Dr. Tariq Hospital, Noor Specialist Hospital, Urology & Plastic Surgery Hospital, in addition includes Joslin Diabetes Center.

(1) يشمل احصاءات جميع المستشفيات الخاصة العاملة في البحرين ، منها 3 مستشفيات افتتحت عام 2005 وهي : مستشفى الدكتور طارق - مستشفى نور التخصصي - مستشفى جراحة المسالك البولية والتجميل ، كما يشمل كذلك على احصاءات مركز جوسلين للسكر الذي تم تشغيله في عام 2004 .

جدول 6.20 Table 6.20

المشتغلون بالمهن الطبية في المستشفيات الخاصة حسب الجنسية والنوع - ديسمبر (2004 - 2005)
Medical Staff in Private Hospitals By Nationality and Sex - December (2004 - 2005)

Profession	الجنسية/النوع									التخصص
	الجملة			غير بحريني			بحريني			
	جملة	اناث	ذكور	جملة	اناث	ذكور	جملة	اناث	ذكور	
Total	Females	Males	Total	Females	Males	Total	Females	Males		
2004										
Physicians										طبيب
Dentists										طبيب أسنان
Pharmacists										صيدلي
Asst. Pharmacists	Not Avilable from Source غير متوفر من المصدر									مساعد صيدلي
Nurses										ممرض
Asst. Nurse										مساعد ممرض
Dental Technicians										فني أسنان
Radiographers & Assts.										مصور ومساعد أشعة
Laboratory Tech.										محلل مختبر
Laboratory Assts.										مساعد محلل مختبر
Physiotherapist Asst.										فني علاج طبيعي
Aid Nurses										مضمد
Public Occupational Health Technician										فنيو الصحة المهنية
2005⁽¹⁾										
Physicians	209	63	146	190	52	138	19	11	8	طبيب
Dentists	29	9	20	21	5	16	8	4	4	طبيب أسنان
Pharmacists	38	25	13	32	19	13	6	6	-	صيدلي
Asst. Pharmacists	14	2	12	-	-	-	14	2	12	مساعد صيدلي
Nurses	365	345	20	341	281	59	24	13	11	ممرض
Asst. Nurse	47	41	6	31	31	0	16	10	6	مساعد ممرض
Dental Technicians	16	13	3	11	9	2	5	4	1	فني أسنان
Radiographers & Assts.	24	15	9	24	15	9	-	-	-	مصور ومساعد أشعة
Laboratory Tech.	38	22	16	35	20	15	3	2	1	محلل مختبر
Laboratory Assts.	-	-	-	-	-	-	-	-	-	مساعد محلل مختبر
Physiotherapist Asst.	15	10	5	14	10	4	1	-	1	فني علاج طبيعي
Aid Nurses	21	15	6	8	8	-	13	7	6	مضمد
Public Occupational Health Technician	7	1	6	3	-	3	4	1	3	فنيو الصحة المهنية

(1) Includes All Private Hospitals in Bahrain Kingdom,
3 hospitals opened during 2005 : Dr. Tariq Hospital,
Noor Specialist Hospital, Urology & Plastic Surgery
Hospital, in additional includes Joslin Diabetes Center.

(1) يشمل احصاءات جميع المستشفيات الخاصة العاملة في البحرين ، منها 3 مستشفيات
افتتحت عام 2005 وهي : مستشفى الدكتور طارق - مستشفى نور التخصصي - مستشفى
جراحة المسالك البولية والتجميل ، كما يشمل كذلك على احصاءات مركز جوسلين للسكر
الذي تم تشغيله في عام 2004 .

APPENDIX III: Selection of health variables

Years 2007		1	2	3	4	5	6
DMU	Hospitals	Inputs				Outputs	
		Beds	physicians	Nurses	Dentists	Out-patients treated	In patients admitted
1	American Mission Hospital	40	68	109	15	186294	2528
2	Awali hospital	28	18	43	2	37671	942
3	Bahrain International Hospital	57	58	62	6	85939	3924
4	Ibn Al-Nafees Hospital	18	39	48	1	96987	3483
5	Bahrain Specialist Hospital	56	43	50	2	83406	4449
6	Gulf Dental Speciality hospital	10	1	16	10	14228	27
7	Al Amal Hospital	20	13	16	2	13617	12
8	Al Hillal hospital	45	18	22	1	41013	431
9	Salmaniya Medical Hospital	874	625	1677	6	518287	46738
10	Bahrain Defence Hospital	339	229	758	12	460496	24616

	4i,2o,0	3i,2o,1i	3i,2o,2i	3i,2o,3i	3i,2o,4i	4i,1o,1o	4i,1o,2o
	1.000	1.000	0.864	1.000	1.000	0.406	1.000
	0.806	0.806	0.434	0.802	0.595	0.495	0.791
	0.767	0.767	0.767	0.686	0.767	0.748	0.686
	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	1.000	1.000	1.000	1.000	1.000	1.000	0.826
	1.000	1.000	0.440	1.000	1.000	1.000	1.000
	0.421	0.421	0.421	0.392	0.421	0.009	0.421
	0.923	0.923	0.923	0.901	0.923	0.231	0.923
	1.000	1.000	1.000	1.000	0.707	0.807	0.332
	1.000	1.000	0.545	1.000	1.000	1.000	0.785
R_Variance	3.29E+05	3.29E+05	2.92E+05	3.69E+04	3.29E+05	2.32E+08	3.40E+10
%R_Variance	100.000%	100.000%	88.788%	11.219%	99.992%	0.677%	99.323%
Efficient DMUs	6	6	3	6	5	4	3

	2i,2o,12i	2i,2o,13i	2i,2o,14i	2i,2o,23i	2i,2o,24i	2i,2o,34i	1i,2o,234i	1i,2o,134i	1i,2o,124i	1i,2o,123i
	0.846	1.000	0.985	0.864	0.864	1.000	0.864	0.457	0.846	0.128
	0.434	0.806	0.617	0.250	0.434	0.600	0.250	0.548	0.434	0.194
	0.767	0.666	0.767	0.356	0.767	0.686	0.356	0.641	0.767	0.151
	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.875	1.000	1.000
	1.000	1.000	1.000	0.582	1.000	0.992	0.411	0.963	1.000	0.447
	0.440	1.000	1.000	0.264	0.440	1.000	0.264	1.000	0.440	0.015
	0.421	0.395	0.421	0.126	0.421	0.269	0.126	0.074	0.421	0.070
	0.923	0.902	0.923	0.423	0.923	0.452	0.169	0.323	0.923	0.423
	1.000	1.000	0.703	1.000	0.368	0.705	0.276	0.696	0.313	1.000
	0.545	1.000	1.000	0.536	0.438	1.000	0.375	1.000	0.365	0.411
R_Variance	291770.8	291745.211	25.5666667	25.5666667	25.5666667	36843.07	0	36843.07	291745.2	25.5666667
%R_Variance	88.788%	88.781%	0.008%	0.008%	0.008%	11.212%	0.000%	11.212%	88.781%	0.008%
Efficient DMUs	3	6	4	1	2	4	1	2	2	2
Average =	0.738	0.877	0.842	0.540	0.665	0.770	0.409	0.658	0.651	0.384
ΔAverage =	0.154	0.015	0.050	0.352	0.226	0.121	0.482	0.234	0.241	0.508

t-Test: Two-Sample Assuming Equal Variances

	Variable		Efficiency Score:	
	1	2	CRS	VRS
Mean	0.891656	0.988609	1.000	1.000
Variance	0.035071	0.001298	0.806	1.000
Observations	10	10	0.767	0.886
Pooled Variance	0.018184		1.000	1.000
Hypothesized Mean Difference	0		1.000	1.000
df	18		1.000	1.000
t Stat	1.607682		0.421	1.000
P(T<=t) one-tail	0.062652		0.923	1.000
t Critical one-tail	1.734064		1.000	1.000
P(T<=t) two-tail	0.125304		1.000	1.000
t Critical two-tail	2.100922			

t-Test: Paired Two Sample for Means

	CRS	VRS
Mean	0.891656	0.988609
Variance	0.035071	0.001298
Observations	10	10
Pearson Correlation	0.233755	
Hypothesized Mean Difference	0	
df	9	
t Stat	-1.68228	
P(T<=t) one-tail	0.063404	
t Critical one-tail	1.833113	
P(T<=t) two-tail	0.126808	
t Critical two-tail	2.262157	

F-Test Two-Sample for Variances

	CRS	VRS
Mean	0.891656	0.988609
Variance	0.035071	0.001298
Observations	10	10
df	9	9
F	27.02643	
P(F<=f) one-tail	1.83E-05	
F Critical one-tail	3.178893	

t-Test: Two-Sample Assuming Unequal Variances

	Variable 1	Variable 2
Mean	0.891656	0.988609
Variance	0.035071	0.001298
Observations	10	10
Hypothesized Mean Difference	0	
df	10	
t Stat	-1.60768	
P(T<=t) one-tail	0.069492	
t Critical one-tail	1.812461	
P(T<=t) two-tail	0.138984	
t Critical two-tail	2.228139	

	Input_1	Input_2	Input_3	Input_4	Output_1	Output_2
Input_1	1					
Input_2	0.996	1				
Input_3	0.996	0.993	1			
Input_4	0.173	0.199	0.230	1		
Output_1	0.889	0.895	0.920	0.444	1	
Output_2	0.988	0.985	0.994	0.224	0.939	1

Omit 3

	Column 1	Column 2	Column 4	Column 5	Column 6
Column 1	1				
Column 2	0.996	1			
Column 4	0.173	0.199	1		
Column 5	0.889	0.895	0.444	1	
Column 6	0.988	0.985	0.224	0.939	1

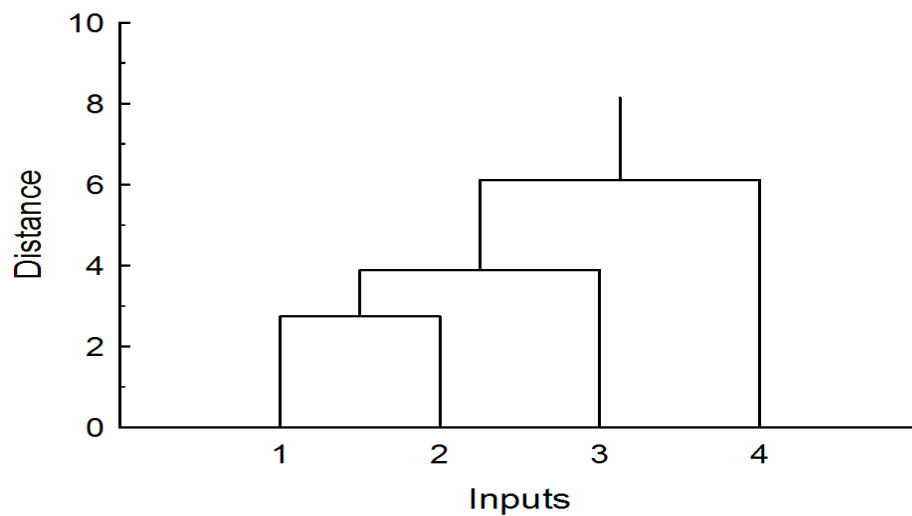
Omit 6

	Column 1	Column 2	Column 3	Column 4	Column 5
Column 1	1				
Column 2	0.993	1			
Column 3	0.199	0.230	1		
Column 4	0.895	0.920	0.444	1	
Column 5	0.985	0.994	0.224	0.939	1

Omit 1,3

	Column 2	Column 4	Column 5	Column 6
Column 1				
Column 2	1			
Column 4	0.199	1		
Column 5	0.895	0.444	1	
Column 6	0.985	0.224	0.939	1

Dendrogram



APPENDIX IV: MOLP/Trade-offs approach (Law department)

Primal formulation

$$\begin{aligned}
 & \text{Min } \sum_{i=1}^m v_i x_{io} \\
 & \text{subject to} \\
 & \sum_{r=1}^s u_r y_{ro} = 1 \\
 & \sum_{i=1}^m v_i x_{ij} + \sum_{r=1}^s u_r y_{ro} \geq 0 \\
 & \sum_{i=1}^m v_i p_i + \sum_{r=1}^s u_r q_r \leq 0 \\
 & v_i, u_r \geq \varepsilon
 \end{aligned}$$

$$\text{Min } 27v_1 + 1145v_2 + 3245v_3$$

Subject to

$$234u_1 + 23u_2 = 1$$

$$150v_1 + 3517v_2 + 50496v_3 - [496u_1 + 99u_2] \geq 0$$

$$53v_1 + 5033v_2 + 51801v_3 - [632u_1 + 20u_2] \geq 0$$

$$77v_1 + 2002v_2 + 108595v_3 - [489u_1 + 16u_2] \geq 0$$

$$102v_1 + 1401v_2 + 103846v_3 - [359u_1 + 75u_2] \geq 0$$

$$76v_1 + 902v_2 + 44261v_3 - [143u_1 + 81u_2] \geq 0$$

$$47v_1 + 2140v_2 + 2145v_3 - [234u_1 + 23u_2] \geq 0$$

$$27v_1 + 1145v_2 + 32458v_3 - [147u_1 + 15u_2] \geq 0$$

$$-u_1 + u_2 \leq 0$$

$$v_1, v_2, v_3, u_1, u_2 \geq 0$$

Dual formulation

$$\text{Max} \left\{ \theta + \varepsilon \left[\sum_{i=1}^m s_i^- + \sum_{r=1}^s s_r^+ \right] \right\}$$

Subject to

$$\sum_{j=1}^n \lambda_j x_{ij} - \sum_{t=1}^m \pi_t p_t + s_i^- \leq x_{io}$$
$$\sum_{j=1}^n \lambda_j y_{rj} + \sum_{t=1}^s \pi_t p_t - s_r^+ \geq \theta y_{ro}$$

$$\theta, \pi_t, \lambda_j, s^+, s^- \geq 0$$

$$\text{Max} \{ \theta + 10^{-10} [s^- + s^+] \}$$

subject to

$$150\lambda_1 + 53\lambda_2 + 77\lambda_3 + 102\lambda_4 + 76\lambda_5 + 29\lambda_6 + 27\lambda_7 + s^- \leq 27$$
$$3517\lambda_1 + 5033\lambda_2 + 2002\lambda_3 + 1401\lambda_4 + 902\lambda_5 + 1986\lambda_6 + 1145\lambda_7$$
$$+ s^- \leq 1145$$
$$50496\lambda_1 + 51801\lambda_2 + 108595\lambda_3 + 103846\lambda_4 + 44261\lambda_5$$
$$+ 21458\lambda_6 + 32458\lambda_7 + s^- \leq 32458$$

$$496\lambda_1 + 632\lambda_2 + 489\lambda_3 + 359\lambda_4 + 143\lambda_5 + 253\lambda_6 + 147\lambda_7 - \pi$$
$$+ s^+ \leq 147\theta$$
$$99\lambda_1 + 20\lambda_2 + 16\lambda_3 + 75\lambda_4 + 81\lambda_5 + 14\lambda_6 + 15\lambda_7 + \pi + s^+ \leq 15\theta$$
$$\theta, \pi, \lambda_j, s^+, s^- \geq 0$$

Trade-offs approach Excel sheet codes

DMU	X1	X2	X3	Y1	Y2
1	150	3517	50496	496	99
2	53	5033	51801	632	20
3	77	2002	108595	489	16
4	102	1401	103846	359	75
5	76	902	44261	143	81
6	47	2140	21458	234	23
7	27	1145	32458	147	15

DMU	1/θ	θ	λ1	λ2	λ3	λ4	λ5	λ6	λ7
1	1.000	1.000	1.000	0.000	0.000	0.000	0.000	0.000	0.000
2	1.000	1.000	0.000	1.000	0.000	0.000	0.000	0.000	0.000
3	1.000	1.000	0.000	0.000	1.000	0.000	0.000	0.000	0.000
4	1.000	1.000	0.000	0.000	0.000	1.000	0.000	0.000	0.000
5	0.990	1.010	0.108	0.000	0.000	0.374	0.000	0.000	0.000
6	0.952	1.051	0.000	0.414	0.000	0.000	0.000	0.000	0.000
7	0.780	1.283	0.017	0.123	0.232	0.000	0.000	0.000	0.000

max(f1)		max(f2)	
f1	f2	f1	f2
496.00	99.00	496.00	99.00
632.00	20.00	99.72	56.49
489.00	16.00	144.88	82.07
359.00	75.00	191.92	108.71
208.24	9.57	143.00	81.00
261.80	8.28	122.73	40.33
199.88	7.89	50.80	28.78

Sub DEA_VJ()

```
'  
' DEA_VJ Macro  
' Macro recorded 22/10/2008 by ASHOOR-PC  
'  
'  
  
Dim i, j, k, l, n As Integer  
l = 2  
Range("J2:J11").ClearContents  
Range("L31:AF43").ClearContents  
For i = 1 To Cells(25, 3)  
Range("J2: J11") = 1: Range("M7") = 1 ': Stop  
Range("R2:R3") = 1  
  
    'Inputs loop  
    For j = 0 To Cells(26, 3) - 1  
    Range("N" & 9 + j) = Cells(l, 2 + j)  
    Next  
  
    'Outputs loop  
    For k = 0 To Cells(27, 3) - 1  
    Range("Q" & 13 + k) = Cells(l, 6 + k)  
    Next  
    SolverSolve (True)  
    l = l + 1  
    For n = 1 To Cells(25, 3)  
    Cells(30 + i, 12 + n) = Range("J" & 1 + n)  
    'Cells(20 + i, 19 + n) = Range("R" & 1 + n)  
    Next  
    Cells(30 + i, 12) = Cells(7, 13)  
  
Next  
End Sub
```

Sub Payoff_VJ()

```
' Payoff_VJ Macro
' Macro recorded 07/08/2008 by Hala
'
'
Dim i, j, l, m, n, k As Integer
l = 2
m = 0
k = 0
Range("A27:AO40").ClearContents
'Range("P24:S43").ClearContents

' Output sumproduct
For m = 0 To Cells(14, 15) - 1
Range("L2:L8") = 1 ': Stop
Range("R3") = Range("N" & m + 2).Formula: ' Stop
For i = 0 To Cells(12, 15) - 1
Range("L2:L8") = 1 ': Stop
'For St. right handside, Input #
  For j = 0 To Cells(13, 15) - 1
    Range("S" & 5 + j) = Cells(l, 2 + j)
  Next
'Stop
  SolverSolve (True)
'
'Stop
'Writing results in tables
l = l + 1
If m = 0 Then
For n = 0 To Cells(14, 15) - 1
Cells(27 + k, 1 + n) = Cells(2 + n, 14)
Next
End If
If m = 1 Then
For n = 0 To Cells(14, 15) - 1
Cells(27 + k, 6 + n) = Cells(2 + n, 14)
Next
End If
If m = 2 Then
For n = 0 To Cells(14, 15) - 1
Cells(27 + k, 11 + n) = Cells(2 + n, 14)
Next
End If
If m = 3 Then
For n = 0 To Cells(14, 15) - 1
Cells(27 + k, 16 + n) = Cells(2 + n, 14)
Next
End If
```

```

If m = 4 Then
For n = 0 To Cells(14, 15) - 1
Cells(27 + k, 21 + n) = Cells(2 + n, 14)
Next
End If
k = k + 1
'Stop
Next
k = 0
l = 2
Next
End Sub

```

Sub DEA_MOLP_VJ()

```

Dim i, j, k, l, n As Integer
'Dim temp As Range
l = 2
Range("N29:X38").ClearContents
Range("L2: L8") = 1
Range("Q3") = 1
Range("T2:T3") = 1
For i = 1 To Cells(25, 3)
Range("L2: L8") = 1
'Range("Q3") = 1
Range("T2:T3") = 1

    'Inputs loop
    For j = 0 To Cells(26, 3) - 1
        Range("R" & 5 + j) = Cells(l, 2 + j)
    Next

    'Outputs loop
    For k = 0 To Cells(27, 3) - 1
        Range("Y" & 2 + k) = Cells(l, 6 + k)
        Range("W" & 2 + k) = Range("Y" & 2 + k) * Range("X" & 1 + i)
    Next

    'Stop
    'Start optimization
    SolverSolve (True)
    l = l + 1
    Range("N" & 28 + i) = Cells(3, 17)
    For n = 1 To Cells(25, 3)
        Cells(28 + i, 14 + n) = Range("L" & 1 + n)
    Next
    'Stop
    Next

End Sub

```

APPENDIX V: Bounded model formulation

Bahrain Private Hospitals (Bahrain International Hospital)

DMU	Physicians	Nurses	Dentists	Out-patients treated	In patients admitted	Max(X)	Min(Y)	V	U
1	68	109	15	186294	2528	625	13617	1E-09	6.87E-07
2	18	43	2	37671	942	1677	12	1E-09	1E-09
3	58	62	6	85939	3924	15	0	0.066667	1E-09
4	39	48	1	96987	3483	0		1E-09	
5	43	50	2	83406	4449				
6	1	16	10	14228	27				
7	13	16	2	13617	12				
8	18	22	1	41013	431				
9	625	1677	6	518287	46738				
10	229	758	12	460496	24616				

Primal formulation (Optimistic)

$$\text{Min } \theta_0 = \sum_{i=1}^m v_i x_{ij_0}$$

subject to

$$\sum_{r=1}^s u_r y_{rj} - \sum_{i=1}^m v_i x_{ij} \leq 0 \quad j=1, \dots, n$$

$$\sum_{r=1}^s u_r y_{rj} - \sum_{i=1}^m v_i (\theta_{ADMU}^* x_{ij}) \geq 0 \quad j=1, \dots, n$$

$$\sum_{r=1}^s u_r y_{rj_0} = 1$$

$$u_r \geq \varepsilon$$

$$v_i \geq \varepsilon$$

Primal formulation (Optimistic) : Bahrain International Hospital

$$\text{Min}\{13v_1 + 16v_2 + 2v_3\}$$

subject to

$$186294u_1 + 2528u_2 - [68v_1 + 109v_2 + 15v_3] \leq 0$$

$$186294u_1 + 2528u_2 - [68v_1 + 109v_2 + 15v_3] \leq 0$$

$$186294u_1 + 2528u_2 - [68v_1 + 109v_2 + 15v_3] \leq 0$$

$$186294u_1 + 2528u_2 - [68v_1 + 109v_2 + 15v_3] \leq 0$$

$$186294u_1 + 2528u_2 - [68v_1 + 109v_2 + 15v_3] \leq 0$$

$$186294u_1 + 2528u_2 - [68v_1 + 109v_2 + 15v_3] \leq 0$$

$$186294u_1 + 2528u_2 - [68v_1 + 109v_2 + 15v_3] \leq 0$$

$$186294u_1 + 2528u_2 - [68v_1 + 109v_2 + 15v_3] \leq 0$$

$$186294u_1 + 2528u_2 - [68v_1 + 109v_2 + 15v_3] \leq 0$$

$$186294u_1 + 2528u_2 - [68v_1 + 109v_2 + 15v_3] \leq 01$$

$$86294u_1 + 2528u_2 - 0.01[68v_1 + 109v_2 + 15v_3] \geq 0$$

$$186294u_1 + 2528u_2 - 0.01[68v_1 + 109v_2 + 15v_3] \geq 0$$

$$186294u_1 + 2528u_2 - 0.01[68v_1 + 109v_2 + 15v_3] \geq 0$$

$$186294u_1 + 2528u_2 - 0.01[68v_1 + 109v_2 + 15v_3] \geq 0$$

$$186294u_1 + 2528u_2 - 0.01[68v_1 + 109v_2 + 15v_3] \geq 0$$

$$186294u_1 + 2528u_2 - 0.01[68v_1 + 109v_2 + 15v_3] \geq 0$$

$$186294u_1 + 2528u_2 - 0.01[68v_1 + 109v_2 + 15v_3] \geq 0$$

$$186294u_1 + 2528u_2 - 0.01[68v_1 + 109v_2 + 15v_3] \geq 0$$

$$186294u_1 + 2528u_2 - 0.01[68v_1 + 109v_2 + 15v_3] \geq 0$$

$$186294u_1 + 2528u_2 - 0.01[68v_1 + 109v_2 + 15v_3] \geq 0$$

$$13617u_1 + 12u_2 = 1$$

$$u_r \geq \varepsilon$$

$$v_i \geq \varepsilon$$

Dual formulation (Optimistic)

$$\text{Max } h = \theta_0 + \varepsilon(s^+ + s^-)$$

Subject to

$$\left[\sum_{j=1}^n (\lambda_j - \beta_j) y_{rj} - s_r^+ \right] \leq \theta_0 y_{rj_0}, \quad r = 1, \dots, s$$

$$\sum_{j=1}^n (\lambda_j - \theta_{ADMU}^* \beta_j) x_{ij} + s_r^- \leq x_{ij_0}, \quad i = 1, \dots, m$$

$$\lambda, \beta, s^+, s^- \geq 0$$

Dual formulation (Optimistic): Bahrain International Hospital

$$\text{Max } \{ h = \theta_0 + \varepsilon(s^+ + s^-) \}$$

subject to

$$\begin{aligned} &186294(\lambda_1 - \beta_1) + 37671(\lambda_2 - \beta_2) + 58939(\lambda_3 - \beta_3) + 96987(\lambda_4 - \beta_4) \\ &\quad + 83406(\lambda_5 - \beta_5) + 14228(\lambda_6 - \beta_6) + 13617(\lambda_7 - \beta_7) \\ &\quad + 41013(\lambda_8 - \beta_8) + 518287(\lambda_9 - \beta_9) + 460496(\lambda_{10} - \beta_{10}) - s^+ \\ &\leq 13617 \theta_0 \end{aligned}$$

$$\begin{aligned} &2528(\lambda_1 - \beta_1) + 942(\lambda_2 - \beta_2) + 3924(\lambda_3 - \beta_3) + 3483(\lambda_4 - \beta_4) \\ &\quad + 4449(\lambda_5 - \beta_5) + 27(\lambda_6 - \beta_6) + 12(\lambda_7 - \beta_7) + 431(\lambda_8 - \beta_8) \\ &\quad + 46738(\lambda_9 - \beta_9) + 24616(\lambda_{10} - \beta_{10}) - s^+ \leq 2528 \theta_0 \end{aligned}$$

$$\begin{aligned} &68(\lambda_1 - 0.01\beta_1) + 18(\lambda_2 - 0.01\beta_2) + 58(\lambda_3 - 0.01\beta_3) + 39(\lambda_4 - 0.01\beta_4) \\ &\quad + 43(\lambda_5 - 0.01\beta_5) + 1(\lambda_6 - 0.01\beta_6) + 13(\lambda_7 - 0.01\beta_7) \\ &\quad + 18(\lambda_8 - 0.01\beta_8) + 625(\lambda_9 - 0.01\beta_9) + 229(\lambda_{10} - 0.01\beta_{10}) - s^+ \\ &\leq 13 \end{aligned}$$

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$$\begin{aligned} &(\lambda_1 - 0.01\beta_1) + 43(\lambda_2 - 0.01\beta_2) + 62(\lambda_3 - 0.01\beta_3) + 48(\lambda_4 - 0.01\beta_4) + \\ &50(\lambda_5 - 0.01\beta_5) + 16(\lambda_6 - 0.01\beta_6) + 16(\lambda_7 - 0.01\beta_7) + 22(\lambda_8 - \\ &0.01\beta_8) + 1677(\lambda_9 - 0.01\beta_9) + 758(\lambda_{10} - 0.01\beta_{10}) - s^+ \leq 16 \end{aligned}$$

$$\begin{aligned} &15(\lambda_1 - 0.01\beta_1) + 2(\lambda_2 - 0.01\beta_2) + 6(\lambda_3 - 0.01\beta_3) + 1(\lambda_4 - 0.01\beta_4) \\ &\quad + 2(\lambda_5 - 0.01\beta_5) + 10(\lambda_6 - 0.01\beta_6) + 2(\lambda_7 - 0.01\beta_7) \\ &\quad + 1(\lambda_8 - 0.01\beta_8) + 6(\lambda_9 - 0.01\beta_9) + 12(\lambda_{10} - 0.01\beta_{10}) \\ &\quad - s^+ \leq 2 \end{aligned}$$

$$\lambda, \beta, s^+, s^- \geq 0$$

Primal formulation (Pessimistic)

$$\text{Max } \theta_0 = \sum_r^s v_i x_{ij_0}$$

subject to

$$\sum_r^s u_r y_{rj} - \sum_i^m v_i x_{ij} \leq 0 \quad j = 1, \dots, n$$

$$\sum_r^s u_r y_{rj} - \sum_i^m v_i (\theta_{ADMU}^* x_{ij}) \geq 0 \quad j = 1, \dots, n$$

$$\sum_r^s u_r y_{rj_0} = 1$$

$$u_r \geq \varepsilon$$

$$v_i \geq \varepsilon$$

Primal formulation (Pessimistic): Bahrain International Hospital

$$\text{Max}\{13v_1 + 16v_2 + 2v_3\}$$

subject to

$$186294u_1 + 2528u_2 - [68v_1 + 109v_2 + 15v_3] \leq 0$$

$$186294u_1 + 2528u_2 - [68v_1 + 109v_2 + 15v_3] \leq 0$$

$$186294u_1 + 2528u_2 - [68v_1 + 109v_2 + 15v_3] \leq 0$$

$$186294u_1 + 2528u_2 - [68v_1 + 109v_2 + 15v_3] \leq 0$$

$$186294u_1 + 2528u_2 - [68v_1 + 109v_2 + 15v_3] \leq 0$$

$$186294u_1 + 2528u_2 - [68v_1 + 109v_2 + 15v_3] \leq 0$$

$$186294u_1 + 2528u_2 - [68v_1 + 109v_2 + 15v_3] \leq 0$$

$$186294u_1 + 2528u_2 - [68v_1 + 109v_2 + 15v_3] \leq 0$$

$$186294u_1 + 2528u_2 - [68v_1 + 109v_2 + 15v_3] \leq 0$$

$$186294u_1 + 2528u_2 - [68v_1 + 109v_2 + 15v_3] \leq 01$$

$$86294u_1 + 2528u_2 - 0.01[68v_1 + 109v_2 + 15v_3] \geq 0$$

$$186294u_1 + 2528u_2 - 0.01[68v_1 + 109v_2 + 15v_3] \geq 0$$

$$186294u_1 + 2528u_2 - 0.01[68v_1 + 109v_2 + 15v_3] \geq 0$$

$$186294u_1 + 2528u_2 - 0.01[68v_1 + 109v_2 + 15v_3] \geq 0$$

$$186294u_1 + 2528u_2 - 0.01[68v_1 + 109v_2 + 15v_3] \geq 0$$

$$186294u_1 + 2528u_2 - 0.01[68v_1 + 109v_2 + 15v_3] \geq 0$$

$$186294u_1 + 2528u_2 - 0.01[68v_1 + 109v_2 + 15v_3] \geq 0$$

$$186294u_1 + 2528u_2 - 0.01[68v_1 + 109v_2 + 15v_3] \geq 0$$

$$186294u_1 + 2528u_2 - 0.01[68v_1 + 109v_2 + 15v_3] \geq 0$$

$$186294u_1 + 2528u_2 - 0.01[68v_1 + 109v_2 + 15v_3] \geq 0$$

$$13617u_1 + 12u_2 = 1$$

$$u_r \geq \varepsilon$$

$$v_i \geq \varepsilon$$

Dual formulation (Pessimistic): Bahrain International Hospital

$$\text{Max} \{ h = \theta_0 + \varepsilon(s^+ + s^-) \}$$

subject to

$$\begin{aligned} 186294(\lambda_1 - \beta_1) + 37671(\lambda_2 - \beta_2) + 58939(\lambda_3 - \beta_3) + 96987(\lambda_4 - \beta_4) \\ + 83406(\lambda_5 - \beta_5) + 14228(\lambda_6 - \beta_6) + 13617(\lambda_7 - \beta_7) \\ + 41013(\lambda_8 - \beta_8) + 518287(\lambda_9 - \beta_9) + 460496(\lambda_{10} - \beta_{10}) - s^+ \\ \geq 13617 \theta_0 \end{aligned}$$

$$\begin{aligned} 2528(\lambda_1 - \beta_1) + 942(\lambda_2 - \beta_2) + 3924(\lambda_3 - \beta_3) + 3483(\lambda_4 - \beta_4) \\ + 4449(\lambda_5 - \beta_5) + 27(\lambda_6 - \beta_6) + 12(\lambda_7 - \beta_7) + 431(\lambda_8 - \beta_8) \\ + 46738(\lambda_9 - \beta_9) + 24616(\lambda_{10} - \beta_{10}) - s^+ \geq 2528 \theta_0 \end{aligned}$$

$$\begin{aligned} 68(\lambda_1 - 0.01\beta_1) + 18(\lambda_2 - 0.01\beta_2) + 58(\lambda_3 - 0.01\beta_3) + 39(\lambda_4 - 0.01\beta_4) \\ + 43(\lambda_5 - 0.01\beta_5) + 1(\lambda_6 - 0.01\beta_6) + 13(\lambda_7 - 0.01\beta_7) \\ + 18(\lambda_8 - 0.01\beta_8) + 625(\lambda_9 - 0.01\beta_9) + 229(\lambda_{10} - 0.01\beta_{10}) - s^+ \\ \geq 13 \end{aligned}$$

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$$\begin{aligned} (\lambda_1 - 0.01\beta_1) + 43(\lambda_2 - 0.01\beta_2) + 62(\lambda_3 - 0.01\beta_3) + 48(\lambda_4 - 0.01\beta_4) + \\ 50(\lambda_5 - 0.01\beta_5) + 16(\lambda_6 - 0.01\beta_6) + 16(\lambda_7 - 0.01\beta_7) + 22(\lambda_8 - \\ 0.01\beta_8) + 1677(\lambda_9 - 0.01\beta_9) + 758(\lambda_{10} - 0.01\beta_{10}) - s^+ \geq 16 \end{aligned}$$

$$\begin{aligned} 15(\lambda_1 - 0.01\beta_1) + 2(\lambda_2 - 0.01\beta_2) + 6(\lambda_3 - 0.01\beta_3) + 1(\lambda_4 - 0.01\beta_4) \\ + 2(\lambda_5 - 0.01\beta_5) + 10(\lambda_6 - 0.01\beta_6) + 2(\lambda_7 - 0.01\beta_7) \\ + 1(\lambda_8 - 0.01\beta_8) + 6(\lambda_9 - 0.01\beta_9) + 12(\lambda_{10} - 0.01\beta_{10}) \\ - s^+ \geq 2 \end{aligned}$$

$$\lambda, \beta, s^+, s^- \geq 0$$

DMU	Physicians	Nurses	Dentists	Out-patients treated	In patients admitted	Max(X)	Min(Y)	V	U
1	68	109	15	186294	2528	625	13617	1E-09	6.87E-07
2	18	43	2	37671	942	1677	12	1E-09	1E-09
3	58	62	6	85939	3924	15	0	0.066667	1E-09
4	39	48	1	96987	3483	0		1E-09	
5	43	50	2	83406	4449				
6	1	16	10	14228	27				
7	13	16	2	13617	12				
8	18	22	1	41013	431				
9	625	1677	6	518287	46738				
10	229	758	12	460496	24616				

Sub Anti_DMU()

```

'
' Anti_DMU Macro
' Macro recorded 28/02/2008 by ASHOOR
'
'
ActiveWindow.SmallScroll ToRight:=-3
SolverOk SetCell:="$N$8", MaxMinVal:=1, ValueOf:="0", ByChange:= _
"$K$2:$K$5,$L$2:$L$4"
SolverSolve (True)
End Sub

```

Sub Payoff_table_Pessimistic()

```
'  
Dim i, j, l, m, n, k As Integer  
'i = DMU  
'j = Input  
l = 2  
m = 0 'Output  
k = 0  
Range("A27:Y36").ClearContents  
'Range("P24:S43").ClearContents  
  
' Output sumproduct  
For m = 0 To Cells(19, 10) - 1  
Range("L2:L11") = 1: Range("M2:M11") = 1  
Range("N15") = _  
Range("R" & m + 2).Formula ' : Stop  
For i = 0 To Cells(17, 10) - 1  
Range("L2:L11") = 1: Range("M2:M11") = 1  
'For St. lift and right handside, Input #  
  For j = 0 To Cells(18, 10) - 1  
    'Range("M" & j + 17) = _  
    'Range("Q" & j + 2).Formula  
    Range("O" & 17 + j) = Cells(l, 2 + j)  
  Next  
'Stop  
  SolverSolve (True)  
'Stop  
l = l + 1  
If m = 0 Then  
For n = 0 To Cells(19, 10) - 1  
Cells(27 + k, 1 + n) = Cells(2 + n, 18)  
Next  
End If  
If m = 1 Then  
For n = 0 To Cells(19, 10) - 1  
Cells(27 + k, 6 + n) = Cells(2 + n, 18)  
Next  
End If  
If m = 2 Then  
For n = 0 To Cells(19, 10) - 1  
Cells(27 + k, 11 + n) = Cells(2 + n, 18)  
Next  
End If  
If m = 3 Then  
For n = 0 To Cells(19, 10) - 1  
Cells(27 + k, 16 + n) = Cells(2 + n, 18)  
Next  
End If  
If m = 4 Then  
For n = 0 To Cells(19, 10) - 1
```

```

Cells(27 + k, 21 + n) = Cells(2 + n, 18)
Next
End If
k = k + 1
'Stop
Next
k = 0
l = 2
Next
End Sub

Sub Optimestic_Bounded()
'
' Optimistic Macro
' Macro recorded 06/03/2008 by ASHOOR
'
'
Dim i, j, k, l, n, m As Integer
l = 2
Range("m24:m33").ClearContents
For i = 1 To Cells(24, 3)
Range("L2: L11") = 1: Range("N2:N11") = 1
Range("Z2") = 1
Range("AA2:AA4") = 1: Range("AB2:AB3") = 1

'Inputs loop
For j = 0 To Cells(25, 3) - 1
Range("W" & 9 + j) = 1 * Cells(l, j + 2)
Next

'Outputs loop
For k = 0 To Cells(26, 3) - 1
Range("W" & 13 + k) = 1 * Cells(l, k + 6)
Next

'Stop
SolverSolve (True)
'Stop
l = l + 1
'printing the results
Cells(23 + i, 13) = 1 / Cells(7, 19)
Next
l = 2

End Sub

```

Sub MOLP_Bounded()

```
'  
Dim i, j, k, l, n, r As Integer  
l = 2  
Range("N29:O33").ClearContents  
Range("Q14:Q23").ClearContents  
For i = 1 To Cells(11, 22)  
Range("L2: L11") = 1: Range("M2: M11") = 1  
Range("T2") = 1: Range("U2: U6") = 1  
Range("V2: V5") = 1  
  
'Inputs loop lifthand and righthand sides  
'Range("P" & j + 13) = Range("M" & j + 2).Formula  
For j = 0 To Cells(12, 22) - 1  
Range("Z" & j + 4) = Cells(l, 2 + j)  
  
Next  
  
'Outputs loop lifthand and righthand sides  
For k = 0 To Cells(13, 22) - 1  
Range("M" & 14 + k) = Cells(l, 6 + k)  
Range("N" & k + 14) = Range("L" & i + 13) * _  
Range("M" & k + 14)  
Next  
  
'Stop  
'Start optimization  
SolverSolve (True)  
l = l + 1  
Range("Q" & 13 + i) = Range("T2")  
Next  
l = 2  
End Sub
```

```

Sub Optimestic_Bounded()
'
' Optimistic Macro
' Macro recorded 06/03/2008 by ASHOOR
'
'
'
Dim i, j, k, l, n, m As Integer
l = 2
Range("m24:m33").ClearContents
For i = 1 To Cells(24, 3)
Range("L2: L11") = 1: Range("N2:N11") = 1
Range("Z2") = 1
Range("AA2:AA4") = 1: Range("AB2:AB3") = 1

'Inputs loop
For j = 0 To Cells(25, 3) - 1
Range("W" & 9 + j) = 1 * Cells(l, j + 2)
Next

'Outputs loop
For k = 0 To Cells(26, 3) - 1
Range("W" & 13 + k) = 1 * Cells(l, k + 6)
Next

'Stop
SolverSolve (True)
'Stop
l = l + 1
'printing the results
Cells(23 + i, 13) = 1 / Cells(7, 19)
Next
l = 2

End Sub

```

```

Payoff_table Macro
'Macro recorded 11/09/2008 by ASHOOR-PC
'
'
Dim i, j, l, m, n, k As Integer
'i = DMU
'j = Input
l = 2
m = 0 'Output
k = 0
Range("A27:Y36").ClearContents
'Range("P24:S43").ClearContents

' Output sumproduct
For m = 0 To Cells(19, 10) - 1
Range("L2:L11") = 1: Range("M2:M11") = 1
Range("N15") = _
Range("R" & m + 2).Formula: 'Stop
For i = 0 To Cells(17, 10) - 1
Range("L2:L11") = 1: Range("M2:M11") = 1
'For St. lift and right handside, Input #
  For j = 0 To Cells(18, 10) - 1
    'Range("M" & j + 17) = _
    'Range("Q" & j + 2).Formula
    Range("O" & 17 + j) = Cells(l, 2 + j)
  Next
'Stop
  SolverSolve (True)
'Stop
  l = l + 1
  If m = 0 Then
    For n = 0 To Cells(19, 10) - 1
      Cells(27 + k, 1 + n) = Cells(2 + n, 18)
    Next
  End If
  If m = 1 Then
    For n = 0 To Cells(19, 10) - 1
      Cells(27 + k, 6 + n) = Cells(2 + n, 18)
    Next
  End If
  If m = 2 Then
    For n = 0 To Cells(19, 10) - 1
      Cells(27 + k, 11 + n) = Cells(2 + n, 18)
    Next
  End If
  If m = 3 Then
    For n = 0 To Cells(19, 10) - 1
      Cells(27 + k, 16 + n) = Cells(2 + n, 18)
    Next
  End If
Next

```

```

End If
If m = 4 Then
For n = 0 To Cells(19, 10) - 1
Cells(27 + k, 21 + n) = Cells(2 + n, 18)
Next
End If
k = k + 1
'Stop
Next
k = 0
l = 2
Next
End Sub

```

Sub Pessimistic_Bounded()

```

'
' Pessimistic Macro
' Macro recorded 06/03/2008 by ASHOOR
'
'
Dim i, j, k, l, n, m As Integer
l = 2
Range("N24:N33").ClearContents
For i = 1 To Cells(24, 3)
Range("L2: L11") = 1: Range("N2:N11") = 1
Range("Z2") = 1
Range("AA2:AA4") = 1: Range("AB2:AB3") = 1

'Inputs loop
For j = 0 To Cells(25, 3) - 1
Range("W" & 9 + j) = 1 * Cells(l, j + 2)
Next

'Outputs loop
For k = 0 To Cells(26, 3) - 1
Range("W" & 13 + k) = 1 * Cells(l, k + 6)
Next

'Stop
'SolverOK MaxMinVal:=2
SolverSolve (True)
'Stop
l = l + 1
Cells(23 + i, 14) = 1 / Cells(7, 19)
Next
l = 2

```

End Sub

```
Sub Payoff_table()
'
' Payoff_table Macro
' Macro recorded 11/09/2008 by ASHOOR-PC
'
'
Dim i, j, l, m, n, k As Integer
'i = DMU
'j = Input
l = 2
m = 0 'Output
k = 0
Range("A27:Y36").ClearContents
'Range("P24:S43").ClearContents

' Output sumproduct
For m = 0 To Cells(19, 10) - 1
Range("L2:L11") = 1: Range("M2:M11") = 1
Range("N15") = _
Range("R" & m + 2).Formula: 'Stop
For i = 0 To Cells(17, 10) - 1
Range("L2:L11") = 1: Range("M2:M11") = 1
'For St. lift and right handside, Input #
  For j = 0 To Cells(18, 10) - 1
    'Range("M" & j + 17) = _
    'Range("Q" & j + 2).Formula
    Range("O" & 17 + j) = Cells(l, 2 + j)
  Next
'Stop
  SolverSolve (True)
'Stop
  l = l + 1
  If m = 0 Then
    For n = 0 To Cells(19, 10) - 1
      Cells(27 + k, 1 + n) = Cells(2 + n, 18)
    Next
  End If
  If m = 1 Then
    For n = 0 To Cells(19, 10) - 1
      Cells(27 + k, 6 + n) = Cells(2 + n, 18)
    Next
  End If
  If m = 2 Then
    For n = 0 To Cells(19, 10) - 1
      Cells(27 + k, 11 + n) = Cells(2 + n, 18)
    Next
  End If
End If
```

```

If m = 3 Then
For n = 0 To Cells(19, 10) - 1
Cells(27 + k, 16 + n) = Cells(2 + n, 18)
Next
End If
If m = 4 Then
For n = 0 To Cells(19, 10) - 1
Cells(27 + k, 21 + n) = Cells(2 + n, 18)
Next
End If
k = k + 1
'Stop
Next
k = 0
l = 2
Next
End Sub

Sub Optimistic()
'
' Optimistic Macro
' Macro recorded 15/03/2008 by ASHOOR
'
'
'   Dim i, j, k, l, n, m As Integer
l = 2
'Range("J2:J21").ClearContents
Range("l24:l33").ClearContents
'Range("K24:L30").ClearContents
For i = 1 To Cells(24, 3)
Range("J2: J11") = 1: Range("V2") = 1

'Inputs loop
For j = 0 To Cells(25, 3) - 1
Range("Q" & 9 + j) = 1 * Cells(l, j + 2)
Next

'Outputs loop
For k = 0 To Cells(26, 3) - 1
Range("Q" & 13 + k) = 1 * Cells(l, k + 6)
Next
'Stop
SolverSolve (True)
'Stop
l = l + 1
Cells(23 + i, 12) = Cells(7, 17)
Next
l = 2
End Sub

```