REASONING BY ANALOGY IN INDUCTIVE LOGIC

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By
Alexandra Hill
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Contents

Abstract

This thesis investigates ways of incorporating reasoning by analogy into Pure (Unary) Inductive Logic. We start with an analysis of similarity as distance, noting that this is the conception that has received most attention in the literature so far. Chapter 4 looks in some detail at the consequences of adopting Hamming Distance as our measure of similarity, which proves to be a strong requirement. Chapter 5 then examines various adaptations of Hamming Distance and proposes a subtle modification, further-away-ness, that generates a much larger class of solutions.

We then go on to look at a different notion of similarity and suggest that an isomorphic counterpart of a proposition in another language can be thought of as its analogue. Chapter 6 shows that the principle this idea motivates is related to Unary Language Invariance, and is widely satisfied.

Declaration

No portion of the work referred to in this thesis has been submitted in support of an application for another degree or qualification of this or any other university or other institute of learning.

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Chapter 1

Introduction

1.1 Historical and philosophical context

References to 'the' principle of induction are not hard to find in philosophical literature. Hume's famous treatise on the subject (thought it never actually uses the word 'induction') describes the principle as requiring "that instances of which we have had no experience, must resemble those of which we have had experience..." [15]. Following Carnap, we can call this reasoning from instances to instances "predictive inference" [1, p.207]. Karl Popper was more concerned with inferences to general laws: "First, it must be formulated in terms not only of 'instances' (as by Hume) but of universal regularities or laws" [28]. Russell, in [32], takes predictive inferences to be primary and gets very specific:

> The principle we are examining may be called the principle of induction, and its two parts may be stated as follows:
>
> (a) When a thing of a certain sort A has been found to be associated with a thing of a certain other sort B, and has never been found dissociated from a thing of the sort B, the greater the number of cases in which A and B have been associated, the greater is the probability that they will be associated in a fresh case in which one of them is known to be present;
>
> (b) Under the same circumstances, a sufficient number of cases of association will make the probability of a fresh association nearly a certainty, and will make it approach certainty without limit[32].

Such preliminary accounts of induction are too narrow, in more senses than one. Consider Russell's attempt to make 'the' principle of induction more precise. A moment's reflection on the kind of reasoning actually used in everyday as well as scientific discourse should reveal that his account leaves out arguments we would like to be included in a systematic investigation into the logic of induction. Consider, for example, determination of a radioactive material's half-life. Statistical analysis will reveal the time taken for various samples to decrease by half; from the data collected from samples, a general probabilistic statement is inferred. Such an argument does not fit into Russell's account. Furthermore, Russell goes too far in specifying that A must have "never been found dissociated from a thing of the sort B" [32] to license a predictive inference; the observance of one black sheep amongst hundreds of white ones will not invalidate our expectation that a sheep be white. Both parts of Russell's definition seem to be rather strong and not uncontentious principles, the absence of either not being incompatible with induction.

Carnap identifies the following five species of inductive inference (original italics):

> 1. The *direct inference*, that is, the inference from the population to a sample...
> 2. The *predictive inference*, that is, the inference from one sample to another sample not overlapping with the first... The special case where the second sample consists of only one individual is called the singular predictive inference...
> 3. The *inference by analogy*, the inference from one individual to another on the basis of their known similarity.
> 4. The *inverse inference*, the inference from a sample to the population...
> 5. The *universal inference*, the inference from a sample to a hypothesis of universal form [1, p.207].

Of course it is not news that induction is poorly understood, but what this does suggest is that it is best to think of inductive logic more generally as the logic of uncertain reasoning. Rather than state outright what we take to be 'the' principle of induction, we can investigate the logical properties of a wide range of principles (and maybe reassess some of them on the basis of our findings.) The question that motivates inductive logic is then quite general:

Q1 How should a rational agent assign beliefs?

The received wisdom is that any such assignment of belief should take the form of a probability function. Arguments for this doctrine, known as probabilism, take several forms. An early, influential pragmatic argument developed by Frank Ramsey [29], and Bruno de Finetti, [8] is known as the Dutch Book argument, which we will briefly sketch below (the full details can be found in [8].) Other notable defences of probabilism include James Joyce's argument from accuracy dominance [17] and arguments from calibration by Bas van Fraassen, [36], and Abner Shimony, [35].

Supposing that belief admits of degrees, one way of assessing how strongly an agent believes something is by asking whether she is willing to bet on it. The idea is that for any sentence θ, stake s and $0 \leq p \leq 1$, one of the following two bets will be acceptable to our agent.

(Bet1$_p$) Win $s(1-p)$ if θ is true, lose sp if θ is false.

(Bet2$_p$) Win sp if θ is false, lose $s(1-p)$ if θ is true.

Varying p will obviously affect our agent's choice, and by monitoring this we can judge their degree of belief in θ. Specifically, the proposal is that we take our agent's belief function w to assign to θ the supremum of the p such that Bet1$_p$ is acceptable to them. To be Dutch booked is to accept a series of bets which have the net effect of a guaranteed loss. It seems reasonable to require that a rational agent should not allow herself to be Dutch Booked. De Finetti showed that from this simple assumption we can derive the probability axioms for w.

Q1 then becomes:

Q2 How should a rational agent assign subjective probabilities?

Before proceeding with the substantive contribution of this thesis to this question, further mention should be made of Rudolf Carnap's approach, which has been highly influential on the development of this field. As suggested by the above quote, Carnap started from a very broad conception of induction and sought a systematic approach, as free as possible from pre-theoretic assumptions. Of particular importance is the requirement that our rational agent start from a point of zero knowledge. That is, all propositions that constitute their body of knowledge must be explicitly conditioned on when assigning probabilities, and no extra background knowledge be allowed to sneak in. This means that in particular, and as in deductive logic, we must work with an uninterpreted logical language; the validity of any assignments cannot depend on particular interpretations of the symbols. Since most philosophical debate over induction involves actual inferences in natural language, it is worth drawing a distinction as Carnap did between Pure and Applied Inductive Logic, our interest being with the former (henceforth, PIL.) For further explanation of the difference between these studies, see [[27], Chapter 1]. Note that this does not mean this thesis will never draw on interpretations to illustrate the formal principles; for example, we often casually allow that a colour property or an animal might be the interpretation of a predicate, but don't wish to commit ourselves to the idea that these things are primitive properties (or natural kinds, or have any other privileged philosophical status).

The philosophical problems of induction cannot be fully expunged from the subject, as it is philosophical ideas that motivate the mathematics. However, this is a primarily mathematical thesis; it will draw attention to the philosophical issues that arise but assume that their resolution is not neccessary for the development of PIL.

In particular, this thesis will look at the fourth kind of inference flagged up by Carnap above: inference by analogy. Chapter 2 will look at previous attempts to incorporate reasoning by analogy into inductive logic. Chapters 3, 4 and 5 follow Carnap's conception of similarity as arising out of distances between predicates, investigating the mathematical consequences of a number of principles similar in spirit to Carnap's Principle of Analogy (PA) [6]. Chapter 6 takes a rather different approach, looking at analogies between *structurally* similar propositions. This thesis develops and goes some way to answering the question of how inference by analogy can be incorporated into PIL. It does so by looking in a systematic way at the possible ways of formalising inference by analogy and the mathematical consequences of these.

1.2 Notation and basic principles

This mathematical setting for this thesis is the conventional (unary) context for PIL: first order predicate logic where the only non-logical symbols are finitely many unary predicate symbols $\{P_1, ..., P_q\}$ and countably many constant symbols $\{a_1, a_2, a_3, ...\}$. This language with q predicates we denote by L_q. The set of formulas of L_q we denote by FL_q, the set of sentences by SL_q, and the quantifier free sentences by $QFSL_q$.

We define the *atoms* of L_q as the 2^q mutually inconsistent and jointly exhaustive formulas of the form:

$$P_1^{\epsilon_1}(x) \wedge P_2^{\epsilon_2}(x) \wedge ... \wedge P_q^{\epsilon_q}(x)$$

where $\epsilon_i \in \{0, 1\}$ and $P_j^1 = P_j$, $P_j^0 = \neg P_j$.

We use $\alpha_1, ..., \alpha_{2^q}$ to denote the atoms of L_q. So, for example, the atoms of L_2 are

$\alpha_1(x) = P_1(x) \wedge P_2(x)$ $\quad$ $\alpha_2(x) = P_1(x) \wedge \neg P_2(x)$
$\alpha_3(x) = \neg P_1(x) \wedge P_2(x)$ $\quad$ $\alpha_4(x) = \neg P_1(x) \wedge \neg P_2(x)$

Another useful concept to introduce is that of a *state description.* A state description $\theta(a_{i_1}, ..., a_{i_n}) \in QFSL_q$ is a sentence of the form

$$\alpha_{h_1}(a_{i_1}) \wedge ... \wedge \alpha_{h_n}(a_{i_n})$$

where $h_i \in \{1, ..., 2^q\}$ for all $i \in \{1, ..., n\}$. By convention, when $\theta \in SL_q$ is written $\theta(a_{i_1}, ..., a_{i_n})$ then all constants appearing in θ are amongst the $a_{i_1}, ..., a_{i_n}$.

The definition of a *probability function on* L_q is a map $w : SL_q \to [0, 1]$ such that for all $\theta, \phi, \exists\psi(x) \in SL_q$:

1. If $\models \theta$ then $w(\theta) = 1$.
2. If $\models \neg(\theta \wedge \phi)$ then $w(\theta \vee \phi) = w(\theta) + w(\phi)$.
3. $w(\exists x\, \psi(x)) = \lim_{m\to\infty} w(\bigvee_{i=1}^m \psi(a_i))$.

Given a probability function w, the *conditional probability function* of $\theta \in SL$ given $\phi \in SL$ is given by:

$$w(\theta \,|\, \phi) = \frac{w(\theta \wedge \phi)}{w(\phi)}$$

We will adopt the convention that whenever there is a possibility of conditioning on zero, equations of the above form should be read as

$$w(\theta \mid \phi)w(\phi) = w(\theta \wedge \phi).$$

Our interest in inductive logic is to pick out probability functions on L_q which are arguably *logical* or *rational* in the sense that they could be the choice of a rational agent. Or to put it another way to discard probability functions which could be judged in some sense to be 'irrational'. We do this by imposing principles of rationality to narrow down our class of admissible functions. The principles to be introduced in this section are ones that seem to be particularly indispensable when formalising rational thinking and will be referred to frequently throughout. One very widely accepted principle is that the inherent symmetry between the constants should be respected by any rational probability function w on L_q. Precisely w should satisfy:

The Constant Exchangeability Principle (Ex)
For $\theta, \theta' \in QFSL_q$, if θ' is obtained from θ by replacing the distinct constant symbols $a_{i_1}, a_{i_2}, ...a_{i_m}$ in θ by distinct constant symbols $a_{k_1}, a_{k_2}, ...a_{k_m}$ respectively, then $w(\theta) = w(\theta')$.

Just as in deductive logic, particular interpretations of predicates and constants should not have any bearing on the validity of an inference. The only pertinent information about a constant is which sentences it appears in, not whether it happens to have been named a_1 or a_2. For example, just as the validity of a deductive argument

$P(a_1) \wedge P(a_2)$
$\therefore P(a_1)$

is unaffected by swapping a_1 and a_2 for any other two constants, the strength of an inductive assignment

$w(P(a_1) \mid \theta(a_1, ...a_n))$

should be similarly unaffected.
It being such a fundamental principle, this thesis will only consider probability functions that satisfy Ex. This allows us access to a powerful representation

theorem due to de Finetti, see [9].[1] In the statement of this theorem let

$$\mathbb{D}_q = \{\langle x_1, ...x_{2^q}\rangle \,|\, x_i \geq 0, \sum_{i=1}^{2^q} x_i = 1\}$$

For $\vec{b} = \langle b_1, b_2, ..., b_{2^q}\rangle \in \mathbb{D}_q$, let $w_{\vec{b}}$ be the probability function on L_q given by

$$w_{\vec{b}}\left(\bigwedge_{j=1}^{n} \alpha_{h_j}(a_{i_j})\right) = \prod_{i=1}^{2^q} b_i^{n_i}$$

where n_i is the number of occurrences of α_i amongst the α_{h_j}.

De Finetti's Representation Theorem
If the probability function w on L_q satisfies Ex then there is a (countably additive) measure μ on $\mathbb{D}_q$ such that for any state description $\theta \in L_q$.

$$w\,(\theta) = \int_{\mathbb{D}_q} w_{\vec{x}}(\theta)\, d\mu(\vec{x}) \tag{1.1}$$

Conversely if w is defined by (1.1) then w extends uniquely to a probability function on L_q satisfying Ex.

We refer to the measure μ here as the *de Finetti prior* of w.

As already suggested, we would also like the predicates to be exchangeable.

The Predicate Exchangeability Principle (Px)
For $\theta, \theta' \in QFSL_q$, if θ' is obtained from θ by replacing the distinct predicate symbols $P_{j_1}, P_{j_2}, ...P_{j_m}$ in θ by distinct predicate symbols $P_{s_1}, P_{s_2}, ...P_{s_m}$ respectively, then $w(\theta) = w(\theta')$.

A further principle suggested by the idea of treating the logical symbols symmetrically is that of strong negation:

The Strong Negation Principle (SN)
For $\theta, \theta' \in QFSL_q$, if θ' is obtained from θ by replacing each occurrence of $\pm P_i$ by $\mp P_i$ for some predicate P_i, then $w(\theta) = w(\theta')$.

[1] As given here the converse direction assumes a result due to Gaifman on restrictions of probability functions, see [10].

SN is motivated by similar considerations to those mentioned with respect to Ex and Px; in the absence of any knowledge about which properties the predicates designate, it seems irrational to treat a predicate and its negation differently. This is perhaps less obvious than the case for Px, and the status of SN will be discussed at greater length in the following chapter (section 2.2), but we mention it here as it is a widely adopted principle.

Almost all the probability functions that this thesis is concerned with will satisfy Px, and most SN, as well as Ex. However, it will always be made explicit in the statement of a theorem which principles the probability functions under consideration satisfy, so that there really are no hidden assumptions about rationality. What we are seeking are results that elucidate the consequences of adopting various basic principles while abstaining from making hard and fast judgments about which principles a rational agent should adopt. We frequently start with Px and SN (and always with Ex) for the reason that these are the principles a rational agent should be least willing to give up. If one were to have philosophical objections to Ex, Px or SN they would of course be at liberty to ignore the classes of functions they give rise to, but the theorems presented here would no doubt be of interest in any case.

Ex, Px and SN can all be classed as 'symmetry' principles. The next principle can be thought of as a 'relevance' principle, stating as it does that having seen an atom once is relevant to the probability of seeing it again.

The Principle of Instantial Relevance (PIR)
If w is a rational probability function, then

$$w(\alpha_i(a_n) \mid \alpha_i(a_{n+1}) \wedge \theta) \geq w(\alpha_i(a_j) \mid \theta)$$

for any $\theta \in SL_q$ not involving a_n or a_{n+1}.

PIR can be thought of as saying (roughly) that a rational person should find an event more likely in the case that they have witnessed an identical event before. So functions satisfying PIR will model a kind of singular predictive induction. For example, if in L_2 our predicates stand for being black and being a swan, PIR ensures that our expectation of seeing a black swan is raised (or at least not lowered) by having already seen one.

Pleasingly, it is easy to show via de Finetti's Representation Theorem that all probability functions satisfying Ex satisfy PIR. This result was first proved by Gaifman[10] but the proof given here follows that of Humburg [16].

Theorem 1. *If w is a probability function satisfying Ex, then w satisfies PIR.*

Proof. Write $\vec{a}$ for $a_1, a_2, ..., a_n$.
By de Finetti's Representation Theorem,

$$w(\theta(\vec{a})) = \int_{\mathbb{D}_q} w_{\vec{x}}(\theta) d\mu(\vec{x}) = A, \text{say},$$

$$w(\theta(\vec{a}) \wedge \alpha_i(a_{n+1})) = \int_{\mathbb{D}_q} x_i w_{\vec{x}}(\theta) d\mu(\vec{x}),$$

and

$$w(\theta(\vec{a}) \wedge \alpha_i(a_{n+1}) \wedge \alpha_i(a_{n+2})) = \int_{\mathbb{D}_q} x_i^2 w_{\vec{x}}(\theta) d\mu(\vec{x})$$

The inequality required by PIR is then

$$\int_{\mathbb{D}_q} x_i^2 w_{\vec{x}}(\theta) d\mu(\vec{x}) \int_{\mathbb{D}_q} w_{\vec{x}}(\theta) d\mu(\vec{x}) \geq \int_{\mathbb{D}_q} x_i w_{\vec{x}}(\theta) d\mu(\vec{x}) \int_{\mathbb{D}_q} x_i w_{\vec{x}}(\theta) d\mu(\vec{x})$$

which is equivalent to

$$A \int_{\mathbb{D}_q} x_i^2 w_{\vec{x}}(\theta) d\mu(\vec{x}) - \left(\int_{\mathbb{D}_q} x_i w_{\vec{x}}(\theta) d\mu(\vec{x}) \right)^2 \geq 0 \tag{1.2}$$

If $A = 0$, then all the integrals must be equal to zero and so this holds with equality. So suppose that $A \neq 0$. Then 1.2 is equivalent to

$$\int_{\mathbb{D}_q} \left(x_i A - \int_{\mathbb{D}_q} x_i w_{\vec{x}}(\theta) d\mu(\vec{x}) \right)^2 w_{\vec{x}}(\theta) d\mu(\vec{x}) \geq 0$$

which is obviously true, since the left hand side is the integral of a square so always non-negative.

□

This result perhaps provides another philosophical motivation for Ex; PIR seems to express one of the primary forms of inductive argument and accepting Ex guarantees PIR. The converse is not true; there are probability functions satisfying PIR that do not satisfy Ex (for an example of such a probability function see footnote 6 in [**?**].)

By accepting Ex, Px and SN we can thin out our class of probability functions, but we are still left with a very wide class. Carnap initially supposed that a single function could be defined that would fulfill all the requirements of inductive inference. It will become clear that, contrary to this original vision, the many plausible conditions on rationality that can be posited do not always result in overlapping classes of functions. Moreover, where classes of functions satisfy our requirements, there is no obvious way of picking a single function from the class. It becomes an increasingly unavoidable conclusion that there is no single, well-defined concept of what is rational. Russell's view then, when he said that "[n]ot to be absolutely certain is... one of the essential things in rationality" [33], seems to be correct in more than one respect; not only should rational belief admit of degrees, but our commitment to any particular belief function should be uncertain. In what follows, we shall see the effect that accepting various analogy principles has on this choice.

Chapter 2

Reasoning by analogy

2.1 Introduction

As mentioned in the introduction, Rudolf Carnap, on whose conception of inductive logic this thesis rests, considered inference by analogy to be one of the fundamental forms of inductive inference. His ideas for modelling analogical reasoning are discussed at some length in [6] (see especially p.32). Before presenting an original contribution to the mathematical problems of modeling analogy in PIL, we give an overview of this and more recent attempts to find functions that support analogical reasoning. First, some analysis of that which we seek to model.

Consider the person who knows that the vast majority of venomous snakes have slanted 'cat-like' eyes whereas the vast majority of non-venomous snakes have round eyes and pupils. If she comes across a snake unlike any she has seen before, if it has slanted eyes it would be supremely rational to stay well away from it. Even though she has never seen an identical creature before, it shares a sufficient number of properties with those in her experience for her to infer the likelihood of it sharing the further property of being venomous.

Another example: suppose you had only ever eaten green peppers and red peppers, and you disliked the taste of the former but liked the taste of the latter. Would you expect to like or dislike an orange pepper? Intuitively, we expect to like the orange pepper, it being similar to red. And for contrast, a non-example; suppose all the orange chairs in your experience have been very comfortable and you are presented with another orange chair by a manufacturer you have never come across before. You would reserve judgment as to whether or not this new chair will be comfortable, its similarity with the chairs in your experience not being a pertinent one.

Finally, we mention a different kind of analogy. In "Models and Analogies in Science" Mary Hesse discusses at some length the "conventional use of "analogy" in mathematical physics, as when Kelvin exhibited analogies between fluid flow, heat flow, electric induction, electric current, and magnetic field, by showing that all are describable by the same equations."[13] [1] Reasoning about entirely different concepts by relating them to some common structure may also be an important part of ancient Chinese logical thought as discussed by Jean-Paul Reding in [30]. To take a particular example,"'Study the sacrificial rites while affirming that there are no spirits, this is like studying the ceremonials of hospitality where there are no guests; this is like knotting nets while there are no fish.' (Mozi[])"[30]. As seen in this quote, this kind of analogy is closely related to metaphor.

The question of what makes a good analogy and a bad one is difficult to answer. For example, it seems as though similarity of colour is relevant when making predictions about taste, but not when making predictions about comfortableness; why does the analogy fail in the second case? The obvious answer is that we have extra background knowledge that tells us that colour is irrelevant to comfortableness. In a state of zero background knowledge, the observance of 100 comfortable orange chairs and 100 uncomfortable blue ones arguably would be rational grounds for expecting the next orange chair to be comfortable. For a rational belief function to reflect rationality it should, *in the absence of any other background knowledge*, respect analogy.

Unfortunately, finding natural examples of good and bad analogies can actually obfuscate the heart of the matter, for two reasons. Firstly, as already suggested, because it is very hard to specify the relevant background information present in any judgment. In Pure Inductive Logic we start from a position of zero background knowledge so any information relevant to the inductive inference must be made explicit, yet this is rarely the case in day to day discourse. Secondly, because the logical imperfections of natural language may well disguise the logical structure of the propositions we are considering. To investigate the rationality of inductive arguments phrased in plain English is not sensible. In his Logical Syntax of Language [2] Carnap points out that although the conclusions of physics

[1] For those familiar with the concept; any two theories with the same *Ramsey sentence* are analogues in this sense.

apply to natural physical objects, they are arrived at through investigation of idealized objects: 'a thin straight lever, ... a simple pendulum'[2], for example. In the same way, we can approach a study of rational thought by working in an idealized language. As mentioned in the introduction and following the convention in PIL, I will be working with a first-order language whose only extra logical symbols are a countable set of constant symbols $\{a_1, a_2, a_3, ...\}$ and a set of unary relation (predicate) symbols $\{P_1, ...P_q\}$. These symbols must be assumed to represent primitive, or 'simple' properties, whereas it is a fair assumption that many simple English words represent complex properties. Many philosophers motivate their proposed principles of analogy with natural language examples (for example, Jan Willem Romeijn with bachelors and maidens [31], Patrick Maher with swans [21]) and some appeal to particular interpretations of the predicate symbols is unavoidable. However the reader should remain wary of smuggling in background knowledge or being mislead by surface grammar when assessing any natural language examples in what follows.

Although similarity is at the heart of all good analogies, the above examples all rely on different types of similarity. We interpreted similarity of snakes to mean possession of a number of identical properties (being a snake, round eyed, venomous, etc.). To put it another way, we judge atomic sentences composed of these properties to be more or less similar depending on how many properties they have in common. Red and orange peppers, however, are similar not only because they possess some identical properties (size, shape, being peppers, etc.), but also because they possess colour properties which are themselves similar. As colours are likely to be the sort of thing we take as primitive properties, the closeness of two objects' colours cannot be explicated in terms of other simpler properties. We must have some kind of similarity relation defined on the primitive predicates themselves (by which, for example, red is more similar to orange than it is to yellow.) Of course not all properties will be comparable; it will only be possible to make judgements of similarity between properties drawn from the same domain (e.g. between colours, between shapes, between animals.) Finally, in the examples from mathematical physics, an analogy is made between propositions that appear to be entirely different in content while bearing some resemblance in structure. This kind of analogy could also be thought of as metaphor.

The latter kind of analogy seems to be an important type of logical reasoning. It is often found in the natural sciences; the corroborated physical theory of sound being used to motivate a physical theory of light, for example (see [13] for a fuller account of this comparison). It is also found in mathematics, where a result in one area often leads one to (rationally) expect a similar result in another; for example, the many analogues of the group isomorphism theorems in abstract algebra.

To recap, the three species of analogy identified here derive from

(i) Similarity between predicates within a single domain.

(ii) Similarity between atoms of a single language.

(iii) Similarity between sentences from disjoint languages.

To the best of our knowledge, only the former two types of analogy have previously been looked at in relation to PIL, and the next section provides a brief survey of notable work to this end. Chapter 3 will look in more detail at analogies of type (ii), taking a more systematic and mathematical approach than some of the previous work on this problem. Since the atoms of L_k can be thought of as a family of predicates of size 2^k, the methods and results of Chapter 3 can also be brought to bear on questions about analogies of type (i). The advantage of starting with atoms rather than families of primitive predicates is that there are natural ways in which atoms can be considered similar without recourse to particular interpretations of the predicates, whereas similarity between primitives must depend on their interpretation. Chapter 4 will go on to consider analogies of type (iii), which appear to be an entirely new consideration in the context of PIL.

2.2 Review of the literature on analogy

2.2.1 Similarity between Q-predicates

For the first kind of analogy we need some measure of similarity between predicates from within a particular domain, and before that, to make precise the notion of a 'domain'. Consider the unary predicate language L_q. If we suppose that the predicates can stand for any property (or even any 'natural' or 'projectible' property, supposing for a moment that this notion is well-defined) then we could take $q = 6$ and let them stand for colour properties (red, yellow, blue, green, orange, purple). The propositions expressed by the atomic sentences will then include the proposition that an object is red and yellow and not blue and not green and not orange and purple, for example. They will of course include the intelligible propositions to the effect that an object is one colour and no other, but these will only be 6 of the total 2^6. We are now in the awkward position that all these colour propositions have the same logical status, prior to any empirical information. Indeed, how could we ever determine the impossibility of satisfying the atomic formula 'x is red and blue, etc.'? At best, that our empirical observations will show up only sentences positing a single colour property of an object will allow us to reason inductively that the likelihood of seeing a simultaneously two or more coloured object is low, approaching zero as the number of observations increases. But when we claim to know that an object cannot be simultaneously both red and blue, it seems an unsatisfactory account of this knowledge to suppose that it is nothing more than a very confident inductive generalisation. A child will grasp this fact about coloured objects as soon as they grasp the very meaning of colour names, without needing a body of empirical evidence for it.

Carnap's answer to this was to consider *Attribute Spaces*[2] : geometrical spaces which can be partitioned into regions, each region corresponding to a predicate in the underlying language of PIL. The predicates arising from a single Attribute Space are thus mutually exclusive and jointly exhaustive, like the set of colours. Such a set $\{Q_1, ..., Q_k\}$ Carnap calls a *family*[6]. We will denote predicates within a Carnapian family by Q_i and refer to them as Q-predicates (following the notation introduced by Carnap), to distinguish them from our P_i which are not assumed to be mutually exclusive.

This added family structure could be thought of as a matter for applied rather than pure inductive logic. Suppose we want to reason about colours, and accordingly let $P_1, ..., P_6$ stand for the six main colour properties. Then we have the prior information that these properties are mutually exclusive and jointly exhaustive, which we can express as the conjunction of the sentences

$$\forall x \left(P_i(x) \leftrightarrow \left(\bigwedge_{\substack{j \neq i \\ j=1}}^{k} \neg P_j(x) \right) \right)$$

[2]More recently, Peter Gardenfors has fleshed out a similar theory of what he calls *Conceptual Spaces*, see [12]. One very interesting suggestion by Gardenfors is that a condition for a property to be projectible may be that the corresponding region of the Conceptual Space is convex.

for $i = 1, ..., 6$.
Denote this sentence by I and look at the functions w_I given by:

$$w_I(\theta) := w(\theta \,|\, I)$$

where w is a probability function on L_q satisfying Ex. Notice that the arguments for accepting Px and SN still apply to any function w, though not to w_I. In the case of SN, since w represents a probability function prior to any knowledge about 'red', it is entirely correct that we should have $w(x \text{ is red}) = w(x \text{ is not red})$. After conditioning on the proposition I that encodes some information about 'red', we get

$$\begin{aligned} w_I(x \text{ is red}) = w(x \text{ is red} \,|\, I) = 1/6 \\ < 5/6 = w(x \text{ is not red} \,|\, I) = w_I(x \text{ is not red}) \end{aligned}$$

hence w_I does not satisfy SN. In this example w_I will satisfy Px just as w does, but this will not always be the case. For example, extend the language to contain another two predicates P_7, P_8 which stand for being male and female, respectively, and let $I' = I \wedge (P_7 \leftrightarrow \neg P_8)$. Since w is prior to any knowledge about the predicates, we want to have $w(x \text{ is male}) = w(x \text{ is red})$. But as soon as we move to $w_{I'}$ we get

$$\begin{aligned} w_{I'}(x \text{ is male}) = w(x \text{ is male} \,|\, I') = 1/2 \\ > 1/6 = w(x \text{ is red} \,|\, I') = w_{I'}(x \text{ is red}) \end{aligned}$$

The advantage of the geometrical picture is that a notion of distance is immediate, and indeed Carnap suggests a measure of similarity between Q-predicates within a family based on a distance metric in the corresponding Attribute Space. However we will see that there are natural distance funtions between atoms that can be explored without recourse to any underlying Attribute Space. Given that we wish to avoid the philosophical commitments of Attribute Spaces, this will be our preferred approach. Note that any probability function defined on atoms could be thought of as defined on a family of Q-predicates instead, by just 'forgetting' the internal structure of the atoms. As such, the only possible limitation of looking at atoms is that the total number of atoms in a language must be a power of 2, whereas a family of Q-predicates could be of any size.

Other philosophers have not always used a distance function to motivate their measures of similarity (see [7], [19]) and Jan Willem Romeijn goes so far as to say that a "relevance function... need not comply to triangle inequalities"[31] though no further argument for this is given. Brian Skyrms [34], Roberto Festa [7], Theo Kuipers [19], Ilke Niiniluoto [24] and Maria de Maio [23] all consider a single family of primitive Q-predicates; their work on analogy is thus concerned with examples of type (i), although some can be applied to examples of type (ii). Jan Willem Romeijn [31] and Patrick Maher [21] both use what we have called atoms in their work on analogy, and so are concerned with analogy of the second type (ii) detailed in the introduction to this chapter.

In the search for probability functions that support reasoning by analogy, most previous work has proceeded by constructing candidate functions and demonstrating that they satisfy certain properties. A survey of these attempts will follow, but it is worth pointing out the difference between this approach and the one that we take this thesis. Firstly, there is not usually a sharp distinction between Pure and Applied Inductive Logic made. Moreover, the emphasis on individual probability functions in some philosophers' work is perhaps motivated by the desire to find 'the' rational probability function, allowing a precise quantification of the strength of an inductive argument as Carnap initially envisioned. However, in light of the many plausible and often mutually contradictory principles of rationality posed since the founding of PIL, this seems an unrealistic aim, and is not the purpose of this thesis. Rather than arguing the case for a particular probability function, we simply propose to investigate the logical consequences of adopting different principles and where possible, spell out the logical relationships between them.

When constructing probability functions that support arguments by analogy there has been a reluctance to stray too far from familiar functions. Central to the search for logical probability functions has been Carnap's proposed *Continuum of Inductive Methods* (see for example [5], [6]), and many of the attempts to construct a new function satisfying analogy involve modifying or mixing functions from this continuum. Members of the continuum are defined on a single family of Q-predicates, $\{Q_1, ...Q_k\}$ say[3], and are given as follows. For $0 < \lambda \leq \infty$, c_λ^k is

[3]Notice that the atoms of L_q have the same properties as a family of Q-predicates, so these functions could also be defined on atoms instead of L_q.

characterized by the special values

$$c_\lambda^k\left(Q_j(a_{n+1})\,|\,\bigwedge_{i=1}^n Q_{h_i}(a_i)\right)=\frac{n_j+\lambda k^{-1}}{n+\lambda}$$

where n_j is the number of occurrences of Q_j amongst the Q_{h_i}, whilst for $\lambda=0$ c_0^k is characterized by

$$c_0^k\left(\bigwedge_{i=1}^n Q_{h_i}(a_i)\right)=\begin{cases} k^{-1} \text{ if all the } h_i \text{ are equal,} \\ 0 \text{ otherwise.}\end{cases}$$

We will refer to functions of this kind as c_λ-functions. This approach can also be generalized to give what we shall refer to as $c_{\lambda\gamma}$-functions, where γ is any prior distribution $\langle\gamma_1,...\gamma_k\rangle$, $\sum_{i=1}^k \gamma_i=1$, and

$$c_{\lambda\gamma}^k\left(Q_j(a_{n+1})\,|\,\bigwedge_{i=1}^n Q_{h_i}(a_i)\right)=\frac{n_i+\lambda\gamma_j}{n+\lambda}$$

Functions from this continuum do not show the kind of analogy effects that Carnap found desirable. Precisely, they fail to satisfy his Principle of Analogy[6, p.46] given for a family of Q-predicates $\{Q_i\}$:

Carnap's Principle of Analogy (CA)
For predicates Q_i, Q_j, Q_k, if Q_i is more similar to Q_j than it is to Q_k, then

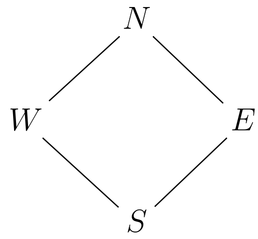
$$w(Q_i(a_{n+1})|\,Q_j(a_n)\wedge\theta)\geq w(Q_i(a_{n+1})|\,Q_k(a_n)\wedge\theta)$$

for any state description θ not containing a_n, a_{n+1}

Brian Skyrms [34] mixes four c_λ to give a probability function which exhibits some limited properties of analogical reasoning. He works in a language with four Q-predicates, and by way of illustration associates these with the four outcomes of a wheel of fortune: North, East, West and South[4]. The distance between these

[4]Similarly, the example of a roulette wheel is mentioned in [6, p.3].

outcomes can be pictured thus:



The idea here is that if the wheel frequently lands on North, one may be suspect that it is biased towards North, and thus expect another North. But also, if such a bias is revealed, it would be rational to consider the outcomes East and West more probable than South, given that they are closer to North. A single c_λ function is unable to make this second differentiation.

In our notation Skyrms proposes the probability function

$$w = 4^{-1}(c_{\lambda\gamma_N} + c_{\lambda\gamma_W} + c_{\lambda\gamma_E} + c_{\lambda\gamma_S})$$

where $\gamma_N = \langle \frac{1}{2}, \frac{1}{5}, \frac{1}{5}, \frac{1}{10} \rangle$, $\gamma_W = \langle \frac{1}{5}, \frac{1}{2}, \frac{1}{10}, \frac{1}{5} \rangle$, $\gamma_E = \langle \frac{1}{5}, \frac{1}{10}, \frac{1}{2}, \frac{1}{5} \rangle$, $\gamma_S = \langle \frac{1}{10}, \frac{1}{5}, \frac{1}{5}, \frac{1}{2} \rangle$
Skyrms explains the choice of w as corresponding to four equally probable 'metahypotheses' as to the probability distribution of outcomes:

H1 That $c_{\lambda\gamma_N}$ is the true probability function

H2 That $c_{\lambda\gamma_W}$ is the true probability function

H3 That $c_{\lambda\gamma_E}$ is the true probability function

H4 That $c_{\lambda\gamma_S}$ is the true probability function

Supposing then that we spin the wheel for the first time and it lands on North (N); then we can update the probability of each metahypothesis using Bayes' Theorem. For example,

$$w(H1 \mid N) = \frac{w(N \mid H1) w(H1)}{w(N)} = \frac{(1/2)(1/4)}{(1/4)} = 1/2$$

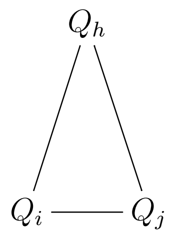
Similarly we get that $w(H2 \mid N) = 1/5$, $w(H3 \mid N) = 1/5$, $w(H4 \mid N) = 1/10$. Taking the sum of the probabilities given by each hypothesis weighted by these values, it can be shown that:

- $w(N|N) = 22/55$
- $w(W|N) = w(E|N) = 12/55$
- $w(S|N) = 9/55$,

where $w(W|N)$ is the probability of a West on the second spin of the wheel given that the first spin had already landed North, etc. So having seen a single north, the desired inequalities hold.

However, the analogy effects of Skyrms' proposed probability function break down very quickly when probabilities are conditioned on more than one previous outcome. For example, Roberto Festa, [7], shows that for $\theta = SSSSSWWWWW$, $w(N \mid E \wedge \theta) < w(N \mid S \wedge \theta)$, contradicting CA and the expectations of any rational agent.

Having demonstrated the limitations of Skyrms' proposed function, Festa generalizes Skyrms' method in an attempt to improve the scope of the analogy effects displayed there. He does so successfully for a hypothetical situation in which we know that for some ordered triple $\langle Q_i, Q_j, Q_h \rangle$, Q_j is more similar to Q_i than Q_h is, while Q_i and Q_j are equally similar to Q_h. In other words, the distance between the three Q-predicates can be pictured thus



Starting from the requirement that a mixture w of $c_{\lambda\gamma}$ functions satisfies

$$w(Q_i(a_{n+1}) \mid Q_j(a_n) \wedge \theta) > w(Q_i(a_{n+1}) \mid Q_h(a_n) \wedge \theta)$$

for any state description θ not involving a_n or a_{n+1}, Festa derives some constraints on the γ vectors and concludes that w can be composed of as many different $c_{\lambda\gamma}$ functions as one likes, as long as they satisfy the following conditions:
For all pairs of vectors $\gamma_x = \langle \gamma_{x_1}, \gamma_{x_2}, ..., \gamma_{x_k} \rangle$, $\gamma_y = \langle \gamma_{y_1}, \gamma_{y_2}, ..., \gamma_{y_k} \rangle$, either

$$\gamma_{x_i} \geq \gamma_{y_i}, \gamma_{x_j} \geq \gamma_{y_j} \text{ and } \gamma_{x_h} \leq \gamma_{y_h}$$

or the same statement with x and y swapped. And for at least one pair of vectors, γ_u, γ_v,

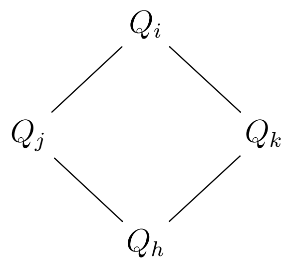
$$\gamma_{u_i} > \gamma_{v_i} \text{ and } \gamma_{u_j} > \gamma_{v_j}$$

or the same statement with u and v swapped. And for at least one pair of vectors, γ_w, γ_z,

$$\gamma_{w_i} > \gamma_{z_i} \text{ and } \gamma_{w_h} < \gamma_{z_h}$$

or the same statement with w and z swapped.

A mixture satisfying the above conditions will have the desired effects regarding Q_i, Q_j, Q_h. However, as Festa points out, it is actually impossible to extend this kind of system to deal with a wheel of fortune type situation, that is when we have four predicates with distances that can be pictured



So once again, this 'solution' is rather limited.

Maria Concetta di Maio [23] also follows Carnap's ideas very closely, but rather than investigating CA she looks at another condition suggested by Carnap as a 'tentative principle' [6],p.45.

Principle of Linearity
For any state description θ where n_i is the number of occurrences of Q_i and n_j is the number of occurrences of Q_j, let $\theta_p^{i,j}$ be the conjunction of predicates that resembles θ except for having p occurrences of Q_i and $n_i + n_j - p$ occurrences of Q_j. Then for a_n not already appearing in θ,

$$w(Q_i(a_n) \,|\, \theta_p^{i,j}) \text{ is a linear function of } p$$

She looks at this in the context of Ex and two other conditions:

1. $w(Q_i(a_n)) = w(Q_j(a_n))$, $\forall i, j$.

2. In a family of just two Q-predicates, $w(Q_1(a_n) \,|\, \theta_p^{1,2}) = w(Q_2(a_n) \,|\, \theta_p^{2,1})$

If we were working with atoms instead of a family of Q-predicates, both conditions would follow from Px + SN.

Di Maio shows that this principle defines a class of functions containing the λ-continuum but also containing functions sensitive to analogy. Unfortunately, the kind of analogy principle she shows to be satisfied by functions in this class seems to involve conditions on the past evidence as well as on several parameters. Moreover, the linearity condition seems somewhat arbitrary and Carnap himself did not regard it to be the most plausible candidate for an axiom of rationality (see [6, p.45]).

Finally, Theo Kuipers suggests a principle that he calls 'virtual analogy' (VA) [19]:

Virtual Analogy Principle (VA)
Let $\{Q_i\}$ be a family of Q-predicates and d a distance function between Q-predicates. Then for any i, j, k such that $d(Q_i, Q_j) < d(Q_i, Q_k)$,

$$\begin{aligned} & w(Q_j(a_{n+1}) \,|\, Q_j(a_n) \wedge \theta) - w(Q_j(a_{n+1}) \,|\, Q_i(a_n) \wedge \theta) \\ & \qquad < w(Q_k(a_{n+1}) \,|\, Q_k(a_n) \wedge \theta) - w(Q_k(a_{n+1}) \,|\, Q_i(a_n) \wedge \theta) \end{aligned}$$

for any state description θ not involving a_n, a_{n+1}.

A comparison between CA and VA will be made in the next section when considering distance functions on atoms rather than predicates. Both Kuipers, [19] and Niiniluoto, [24] proceed by weighting $c_{\lambda\gamma}$ functions by a further parameter. We mention these only briefly as while this approach ensures some of the desired analogical properties, the resulting functions violate Ex, which we regard as a fatal failing.

2.2.2 Similarity between atoms

If, instead of looking at a single family of Q-predicates, we look at a language L_q, there is another way of combining $c_{\lambda\gamma}$-functions: we may define $c_{\lambda\gamma}$-functions on the primitive predicates $\{P_1, P_2, ..., P_q\}$ and then look at the functions on atoms of L_q that this will give rise to. This idea is utilised by both Maher, [21], and Romeijn, [31]. To begin with, both take L_2 with the four atoms:

$$\alpha_1(x) = P_1(x) \wedge P_2(x) \qquad \alpha_2(x) = P_1(x) \wedge \neg P_2(x)$$
$$\alpha_3(x) = P_1(x) \wedge \neg P_2(x) \quad \alpha_4(x) = \neg P_1(x) \wedge \neg P_2(x)$$

If we let $c_{\lambda\gamma_i}$ be a function defined for $\{P_i, \neg P_i\}$, this immediately gives rise to a function w on atoms by taking

$$w(\alpha_1(a_n) \,|\, \theta) = c_{\lambda\gamma_1}(P_1(a_n) \,|\, \theta_1) c_{\lambda\gamma_2}(P_2(a_n) \,|\, \theta_2)$$
$$w(\alpha_2(a_n) \,|\, \theta) = c_{\lambda\gamma_1}(P_1(a_n) \,|\, \theta_1) c_{\lambda\gamma_2}(\neg P_2(a_n) \,|\, \theta_2)$$
$$w(\alpha_3(a_n) \,|\, \theta) = c_{\lambda\gamma_1}(\neg P_1(a_n) \,|\, \theta_1) c_{\lambda\gamma_2}(P_2(a_n) \,|\, \theta_2)$$
$$w(\alpha_4(a_n) \,|\, \theta) = c_{\lambda\gamma_1}(\neg P_1(a_n) \,|\, \theta_1) c_{\lambda\gamma_2}(\neg P_2(a_n) \,|\, \theta_2)$$

for any state description θ not involving a_n, where θ_i is the conjunction of the $\pm P_i$ conjuncts in θ.

Maher does not make explicit what general principle of analogy this will satisfy, but does mention, for example, that

$$w(\alpha_1(a_{n+1}) \,|\, \alpha_2(a_n) \wedge \theta) > w(\alpha_1(a_{n+1}) \,|\, \alpha_4(a_n) \wedge \theta)$$

for any state description θ not involving a_n, a_{n+1}. Intuitively, this is because the function looks at each conjunct in α_1 and then looks for similar conjuncts in the past evidence. Since α_2 has one similar conjunct (P_1) and α_4 none, the former will offer more inductive support than the latter.

Taking (as Maher does) $d(\alpha_i, \alpha_j)$ to be the number of predicates on which α_i and α_j differ, w does satisfy CA given for the atoms of L_2 rather than on a family of primitive predicates. In other words, for atoms $\alpha_i, \alpha_j, \alpha_k$, if $d(\,\alpha_i, \alpha_j) < d(\,\alpha_i, \alpha_k)$ then

$$w(\alpha_i(a_{n+1})|\, \alpha_j(a_n) \wedge \theta) \geq w(\alpha_i(a_{n+1})|\, \alpha_k(a_n) \wedge \theta)$$

for any state description θ not containing a_n, a_{n+1}

Notice that VA can be given in a similar fashion for the atoms of L_2, but for the same distance function, w defined as above will not in general satisfy VA. As a counter-example, take $\gamma = 2^{-1}$, λ to be 1 and θ to be 3 copies of α_1. Then

$w(\alpha_2 \,|\alpha_2 \wedge \alpha_1^3) - w(\alpha_2 \,|\, \alpha_1 \wedge \alpha_1^3)$

$$\begin{aligned}
&= c_1(P_1 \,|\, P_1^4)c_1(\neg P_2 \,|\, \neg P_2P_2^3) - c_1(P_1 \,|\, P_1^4)c_1(\neg P_2 \,|\, P_2^4)\\
&= \frac{(3+1+1/2)(1+1/2) - (3+1+1/2)(1/2)}{(3+1+1)^2}\\
&= 9/50
\end{aligned}$$

Whereas

$w(\alpha_4 \,|\alpha_4 \wedge \alpha_1^3) - w(\alpha_4 \,|\, \alpha_1 \wedge \alpha_1^3)$

$$\begin{aligned}
&= c_1(\neg P_1 \,|\, \neg P_1P_1^3)c_1(\neg P_2 \,|\, \neg P_2P_2^3) - c_1(\neg P_1 \,|\, P_1^4)c_1(\neg P_2 \,|\, P_2^4)\\
&= \frac{(1+1/2)(1+1/2) - (1/2)(1/2)}{(3+1+1)^2}\\
&= 4/50
\end{aligned}$$

Above and wherever no ambiguity can arise, the instantiating constants are left implicit (it should be assumed that each atom is instantiated by a distinct constant.)

Of course there are many other probability functions that will satisfy CA; Chapter 4 will explore this further. But as Maher points out, the problem with taking the predicates as independent in this way is that it makes no difference to w which object the predicates are instantiated by. For example, as far as w is concerned, $\alpha_1(a_1) \wedge \alpha_2(a_2) \wedge \alpha_3(a_3)$ is indistinguishable from $\alpha_1(a_1) \wedge \alpha_1(a_2) \wedge \alpha_4(a_3)$ since in both cases there are two instances of P_1, two instances of P_2 and one instance of each of $\neg P_1, \neg P_2$. Intuitively, one's expectation of seeing a black swan should be increased more by seeing lots of black swans and lots of white cats than by seeing lots of white swans and black cats. But w would give

$$w(\text{ a black swan} \,|\, n \text{ black swans and } m \text{ white cats})$$

$$= w(\text{ a black swan} \,|\, n \text{ black cats and } m \text{ white swans}).$$

This observation suggests the question: does every function that respects analogy have this unwelcome property? If analogical reasoning is described according to a strict form of CA in which the conditioning evidence θ can be any quantifier free sentence, then the answer is yes, as will be shown in Chapter 4.

For this reason Maher follows Carnap in proposing a weighted average of the above w and a single c_λ-function defined on the atoms. Since for any c_λ-function we have

$$c_\lambda(\text{ a black swan} \,|\, n \text{ black swans and } m \text{ white cats})$$

$$> c_\lambda(\text{ a black swan} \,|\, n \text{ black cats and } m \text{ white swans}),$$

the weighted average will give a similar inequality. The weighting is done by a parameter η, where the final function is:

$$u := \eta w + (1 - \eta) c_\lambda$$

Such a function u will not in general satisfy CA, as Maher notes. Maher's proposal differs only slightly from Carnap's, but does so in two ways. Firstly, his η is defined as the probability that the predicates are statistically independent, $u(I)$. Maher's idea is that if they are independent, w is an appropriate function, whereas if they are dependent, c_λ is more appropriate, and in [21] these two assumptions are enshrined as axioms. But we are none the wiser as to what principle u actually satisfies, nor how many other probability functions might satisfy a similar principle.

The second slight divergence from Carnap is that Maher does not suppose that the predicates and their negations should have equal initial probabilities. So he would allow the use of $c_{\lambda\gamma}$-functions for various γ in the place of the c_λ-functions Carnap thought appropriate. This is defended in [21] by the following argument:

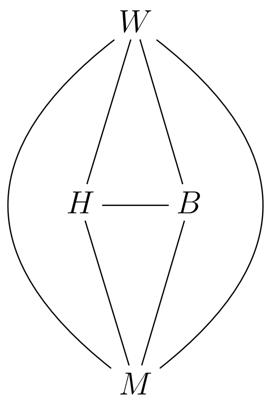
> For example, since being a raven is just one of many comparable things that an individual could be, I would say rationality requires that in the absence of any evidence one should think of an unobserved individual is more probably a non-raven than a raven.[21]

This of course depends on how you choose your language, but if we know that a raven is just one of many *comparable* things an individual can be then all of these things should be represented in the language and we should be conditioning on the fact that the property of being not a raven is logically equivalent to the disjunction of these other properties [5]. As suggested above (page 23,) in applying PIL to

[5] If Maher has in mind that a suitable partition of the space of animals could give rise to a family of predicates: {Ravens, Non-Ravens}, one possible objection is that this may violate the requirement that properties be convex.

inductive arguments about ravens, we should move from a probability function w satisfying Px and SN to one w_I according to which it *will* be more likely to see a non-raven than a raven. L_2 will just not be appropriate for modelling inductive arguments about ravens.

Like Maher, Romeijn proceeds by defining functions on the primitive predicates rather than the atoms. He works in L_2 and to illustrate his method interprets P_1 as the property of being male and P_2 as the property of being married, so that the four atoms correspond to the properties of being a husband, a bachelor, a wife and a maiden. Interestingly, Romeijn suggests that similarity can be quantified using any (symmetric) 'relevance relation' between atoms which need not correspond to a distance function; as an example, he defines a relevance function ρ such that $\rho(\text{husbands}, \text{bachelors}) > \rho(\text{husbands}, \text{wives})$. ρ can be represented graphically, where a shorter line means a higher value, by



Since ρ "need not comply with triangular inequalities"[31], only a direct line between atoms should be read as representing their relevance to each other.

He then defines a function which resembles a $c_{\lambda\gamma}$ function but uses three different sets of parameters, one for the special values of the form $w(\pm P_1(a_n) \mid \theta)$, one for the special values $w(\pm P_2(a_n) \mid \theta \wedge P_1(a_n))$ and a third for the special values $w(\pm P_2(a_n) \mid \theta \wedge \neg P_1(a_n))$, where θ is a state description on L_2 not involving a_n.

These parameters are defined to depend on ρ, and the resulting function on atoms satisfies a principle equivalent to a generalisation of CA using ρ rather than a distance function in the obvious way. As Romeijn notes, one major limitation of this model is the asymmetry with which it treats predicates. In particular, his function does not satisfy Px. Moreover, it remains to give a classification of all the probability functions consistent with this principle.

Chapter 3

Distance and similarity

Before looking at specific ways in which atoms can be similar to one another, we give a schema for a class of analogy principles based on similarity of atoms. As mentioned in Chapter 2, in the context of pure, uninterpreted inductive logic we have no non-arbitrary measure of similarity between the primitive predicates of L_q. For our purposes we are not too concerned with a quantitative notion of similarity between atoms; a qualitative one will be enough. What we need then is a binary relation between unordered pairs as follows:

Definition 2. *Let A_q be the set of atoms of L_q, i.e. the set $\{\alpha_1, ..., \alpha_{2^q}\}$. Then ${}^q\!>_S$ is a similarity relation just if it is a relation on $\{A_q \times A_q\} \times \{A_q \times A_q\}$ defined only between unordered pairs that have one element in common, and satisfying $\{\alpha_i, \alpha_i\} \;{}^q\!>_S\; \{\alpha_i, \alpha_j\}$ whenever $i \neq j$.*

We can now introduce the following class of analogy principles.

Analogy Principle for $>_S$ (AP$_S$)

For any atoms $\alpha_i, \alpha_j, \alpha_k$ of L_q such that $\{\alpha_i, \alpha_j\} \;{}^q\!>_S\; \{\alpha_i, \alpha_k\}$,

$$w(\alpha_i(a_n) \,|\, \alpha_j(a_{n+1}) \wedge \theta) \geq w(\alpha_i(a_n) \,|\, \alpha_k(a_{n+1}) \wedge \theta)$$

for any $\theta \in QFSL_q$ not containing a_n, a_{n+1}.

We also introduce the following class of stronger principles, in which the inequalities are required to be strict.

Strong Analogy Principle for $>_S$ (SAP$_S$)

For any atoms $\alpha_i, \alpha_j, \alpha_k$ of L_q such that $\{\alpha_i, \alpha_j\} \;{}^q\!>_S\; \{\alpha_i, \alpha_k\}$,

$$w(\alpha_i(a_n) \,|\, \alpha_j(a_{n+1}) \wedge \theta) > w(\alpha_i(a_n) \,|\, \alpha_k(a_{n+1}) \wedge \theta)$$

for any $\theta \in QFSL_1$ not containing a_n, a_{n+1}.

And finally the class of weaker principles, in which the inequalities are only required to hold for state descriptions.

State Description Analogy Principle for $>_S$ (SDAP$_S$)
For any atoms $\alpha_i, \alpha_j, \alpha_k$ of L_q such that $\{\alpha_i, \alpha_j\} \,{}^q\!>_S \{\alpha_i, \alpha_k\}$,

$$w(\alpha_i(a_n) \,|\, \alpha_j(a_{n+1}) \wedge \theta) \geq w(\alpha_i(a_n) \,|\, \alpha_k(a_{n+1}) \wedge \theta)$$

for any state desciption $\theta \in QFSL_q$ not containing a_n, a_{n+1}.

The next section will explore the idea that any measure of dissimilarity should take the form of a distance function, and that our binary similarity relation be derived from this.

3.1 Distance

Let us grant firstly that any measure of dissimilarity between the atoms of L_q should take the form of a distance function.

Definition 3. *For a set X, a function $d : X \times X \to \mathbb{R}$ is a distance function iff:*

(i) $d(x_i, x_j) = d(x_j, x_i)$, $\forall i, j$

(ii) $0 = d(x_i, x_i) < d(x_i, x_j)$, $\forall i, j$ *such that* $i \neq j$

(iii) $d(x_i, x_j) \leq d(x_i, x_k) + d(x_k, x_j)$, $\forall i, j, k$

Let $\bigwedge_{i=1}^{q} P_i^{\epsilon_i}(x)$ be an atom of L_q, where $\epsilon_i \in \{0, 1\}$ and $P_i^1 = P_i$, $P_i^0 = \neg P_i$. Taking X to be the set $\{\langle \epsilon_1, ...\epsilon_q \rangle \,|\, \epsilon_i \in \{0, 1\}, \forall i\}$, we then have several choices for distance functions between the vectors in X and hence the atoms of L_q. Some different distance functions will result in the same comparative statements about distances between atoms; for example, recall the following definitions.

Definition 4. *The Manhattan distance between two vectors $\vec{x}, \vec{y}$, which we will denote $||\vec{x} - \vec{y}||_M$ is given by*

$$||\vec{x} - \vec{y}||_M = \sum_{i=1}^{q} |x_i - y_i|$$

Definition 5. *The Euclidean distance between two vectors $\vec{x}, \vec{y}$, which we will denote $||\vec{x} - \vec{y}||_E$ is given by*

$$||\vec{x} - \vec{y}||_E = \sqrt{\sum_{i=1}^{q} (x_i - y_i)^2}$$

Notice that both the Manhattan and the Euclidean distances generate the same sets of inequalities between distances. For example, in L_2 these are:

1. $d(\alpha_i, \alpha_i) < d(\alpha_i, \alpha_j)$ whenever $i \neq j$
2. $d(\alpha_1, \alpha_2), d(\alpha_1, \alpha_3), d(\alpha_4, \alpha_2), d(\alpha_4, \alpha_3) < d(\alpha_1, \alpha_4), d(\alpha_2, \alpha_3)$

And for contrast, recall another notion of distance, the Chebyshev distance:

Definition 6. *The Chebyshev distance between two vectors $\vec{x}, \vec{y}$, which we will denote $||\vec{x} - \vec{y}||_C$ is given by*

$$||\vec{x} - \vec{y}||_C = \max |x_i - y_i|$$

If we take the Chebyshev distance, then the only inequalities generated for L_q are the ones of the form $d(\alpha_i, \alpha_i) < d(\alpha_i, \alpha_j)$, where $i \neq j$.

Any distance function we take will give rise to a similarity relation $>_S$[1] by specifying that $\{\alpha_i, \alpha_j\} >_S \{\alpha_i, \alpha_k\}$ if and only if $d(\alpha_i, \alpha_j) < d(\alpha_i, \alpha_k)$.

If we take the Chebyshev distance and corresponding similarity relation $>_C$, say, then SDAP$_C$ is known as the Strong Principle of Instantial Relevance (SPIR) [27].

If we take the Manhattan distance, this is equivalent to the Hamming Distance between atoms,

Definition 7. *The Hamming Distance between two atoms, α_i and α_j, is the number of predicates P such that $\alpha_i(a_n) \vDash P(a_n)$ if and only if $\alpha_j(a_n) \vDash \neg P(a_n)$. We denote the Hamming Distance by $|\, \alpha_i - \alpha_j \,|$.*

and we will use AP$_H$ to distinguish the resulting analogy principle. As already noted, the Euclidean distance will give rise to exactly the same similarity relation, so can also motivate AP$_H$. Hamming Distance seems a particularly natural

[1] Where no ambiguity will arise, we will omit the index q and just write $>_S$.

distance function for PIL, given that it has a certain symmetry in the way it treats predicates and their negations and ascribes the same weight to a difference in any predicate. In fact, the permutations of atoms that preserve Hamming Distances are exactly those permutations licensed by SN and Px, as the following demonstrates.

Given a permutation σ of $\{1, 2, 3, \ldots, 2^q\}$ we shall also denote by σ the permutation of atoms given by $\alpha_i(x) \mapsto \alpha_{\sigma(i)}(x)$ and the permutation of $\mathbb{D}^{2^q}$ given by

$$\langle x_1, x_2, x_3, \ldots, x_{2^q}\rangle \mapsto \langle x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}, \ldots, x_{\sigma(2^q)}\rangle.$$

Let $\mathcal{P}_q$ be the set of permutations of $\{1, 2, 3, \ldots, 2^q\}$ such that the corresponding permutation of atoms can be effected by permuting predicates and transposing $\pm P_i(x)$.

Theorem 8. *Let σ be a permutation of $\{1, 2, 3, \ldots, 2^q\}$. Then, as a permutation of atoms, σ preserves Hamming distance if and only if $\sigma \in \mathcal{P}_q$.*

Proof. The right to left implication is clear. In the other direction suppose that σ is a permutation of atoms such that for all i, j

$$|\sigma(\alpha_i) - \sigma(\alpha_j)| = |\alpha_i - \alpha_j| \,. \tag{3.1}$$

Let $\alpha_1 = P_1 \wedge P_2 \wedge \ldots \wedge P_k$ and define $f : \{\pm P_i \,|\, 1 \leq i \leq 2^q\,\} \to \{\,\pm P_i \,|\, 1 \leq i \leq 2^q\,\}$ as follows:

$$f(\pm P_i) = \begin{cases} \pm P_i & \text{if } P_i \text{ is a conjunct in } \sigma(\alpha_1), \\ \mp P_i & \text{if } \neg P_i \text{ is a conjunct in } \sigma(\alpha_1). \end{cases}$$

Let τ be the permutation of $\{\,1, 2, \ldots, 2^q\}$ such that on atoms

$$\tau(\pm P_1 \wedge \pm P_2 \wedge \ldots \wedge \pm P_q) = f(\pm P_1) \wedge f(\pm P_2) \wedge \ldots f(\pm P_q).$$

Notice that $\tau \in \mathcal{P}_q$ so to show that $\sigma \in \mathcal{P}_q$ it is enough to show that $\tau\sigma \in \mathcal{P}_q$.

Clearly $\tau\sigma$ also satisfies (3.1) and $\tau\sigma(\alpha_1) = \alpha_1$. Hence by (3.1) if k propositional variables are negated in α_i then the same must be true of $\tau\sigma(\alpha_i)$. Given $1 \leq k \leq q$ let α_{i_k} be the atom with just P_k negated. So $\tau\sigma(\alpha_{i_k})$ is an atom with just one negated propositional variable, say it is $P_{\nu(k)}$. Clearly ν must be a permutation of $\{1, 2, \ldots, q\}$ since if $k \neq j$ then

$$|\alpha_{i_k} - \alpha_{i_j}| = 2$$

so

$$|\tau\sigma(\alpha_{i_k}) - \tau\sigma(\alpha_{i_j})| = 2.$$

Let η be the permutation of atoms given by permuting the predicates according to ν, i.e.

$$\eta(\pm P_1 \wedge \pm P_2 \wedge \ldots \wedge \pm P_q) \equiv \pm P_{\nu(1)} \wedge \pm P_{\nu(2)} \wedge \ldots \wedge \pm P_{\nu(q)}.$$

Then $\eta \in \mathcal{P}_q$ and as a permutation of atoms $\eta^{-1}\tau\sigma$ preserves Hamming distance and is the identity on all atoms with at most one negated predicate. But then $\eta^{-1}\tau\sigma$ must be the identity on *all* atoms since, for example,

$$\neg P_1 \wedge \neg P_2 \wedge \ldots \wedge \neg P_k \wedge P_{k+1} \wedge P_{k+2} \wedge \ldots \wedge P_q$$

is the unique atom which is distance k from α_1 and distance $k+1$ from

$$P_1 \wedge P_2 \wedge \ldots \wedge P_k \wedge \neg P_{k+1} \wedge P_{k+2} \wedge P_{k+2} \wedge \ldots \wedge P_{q-1} \wedge P_q$$

$$P_1 \wedge P_2 \wedge \ldots \wedge P_k \wedge P_{k+1} \wedge \neg P_{k+2} \wedge P_{k+2} \wedge \ldots \wedge P_{q-1} \wedge P_q$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$P_1 \wedge P_2 \wedge \ldots \wedge P_k \wedge P_{k+1} \wedge P_{k+2} \wedge P_{k+2} \wedge \ldots \wedge P_{q-1} \wedge \neg P_q$$

and these distances must be preserved after applying $\eta^{-1}\tau\sigma$. Hence $\eta^{-1}\tau\sigma =$ Identity $\in \mathcal{P}_q$ so $\sigma = \tau^{-1}\eta \in \mathcal{P}_q$ as required. □

For this reason, AP_H will be of particular interest to us, and is the focus of the next chapter.

Before considering non-distance based notions of similarity, we make one more observation about this way of treating analogy. If we accept distance as the fundamental measure of dissimilarity between predicates, then from the first distance function axiom we have that $d(P_i, \neg P_i) = d(\neg P_i, P_i)$ for any predicate P_i. This alone suggests a symmetry in the way we treat predicates and their negations. For any distance function on the atoms of L_2 that is a function of the distances between the primitive predicates, this fact guarantees that $d(\alpha_1, \alpha_2) = d(\alpha_3, \alpha_4)$, for example. Of course since AP_S only depends on the strict inequalities generated by d, this does not commit us to $w(\alpha_1 \,|\, \alpha_2) = w(\alpha_3 \,|\, \alpha_4)$ or any other such equalities but it certainly suggests SN as a background assumption. Px is not implied in the same way, since we could take our distance function to be a weighted sum,

$$d(\langle \epsilon_1, ...\epsilon_q \rangle, \langle \tau_1, ...\tau_q \rangle) = \lambda_1 d(\epsilon_1, \tau_1) + \lambda_2 d(\epsilon_2, \tau_2) + ... + \lambda_q d(\epsilon_q, \tau_q),$$

with different λ_i. For example, define d on the atoms of L_2 by

$$d(\langle \epsilon_1, \epsilon_2 \rangle, \langle \tau_1, \tau_2 \rangle) = d(\epsilon_1, \tau_1) + 2d(\epsilon_2, \tau_2),$$

and let $>_S$ be the similarity function that arises. Then we get that $d(\alpha_1, \alpha_2) < d(\alpha_1, \alpha_3)$, hence SAP_S implies $w(\alpha_1 \,|\, \alpha_2) > w(\alpha_1 \,|\, \alpha_3)$, in violation of Px.

3.2 Similarity without distance

As mentioned, the idea that similarity derives from distance between predicates and their negations in some sense implies the acceptance of SN. If we want to remain agnostic about SN, perhaps we should consider alternatives.

Another way of looking at Hamming Distance is that it measures the number of SN licensed permutations required to transform one atom to another; the fewer permutations required, the closer two atoms are to one another. The permutations licensed by SN are $(\alpha_1\, \alpha_2)(\alpha_3\, \alpha_4)$ and $(\alpha_1\, \alpha_3)(\alpha_2\, \alpha_4)$. It only requires the application of a single one of these to transform α_1 into α_2 or α_3, but it requires the application of both to transform α_1 to α_4.

Consider the following Weak Negation principle:

Weak Negation (WN)
For any sentence θ, if θ' is the result of simultaneously swapping every instance of $\pm P_i$ in θ for $\mp P_i$, for all i, then $w(\theta') = w(\theta)$.

On L_2, WN licenses only the permutation $(\alpha_1\, \alpha_4)(\alpha_2\, \alpha_3)$. It takes one application of this to transform α_1 to α_4 and it is impossible to transform α_1 to α_2 or α_3. Denote by $\alpha*$ the atom that is obtained from α by application of the WN licensed permutation. Then we can define the similarity relation $>_W$ by:

Definition 9. $\{\alpha_i, \alpha_j\} >_W \{\alpha_i, \alpha_k\}$ *iff* $k \neq j, k \neq i$ *and* $i = j$ *or* $\alpha_j = \alpha_i*$

The principle AP_W will be explored further in Chapter 3.

Another objection to the distance based notion of similarity might be the idea that while distance does matter, the path taken to get from one point to another is also relevant. While $d(\alpha_i, \alpha_j)$ may be strictly less than $d(\alpha_i, \alpha_k)$, if we've 'travelled' by completely different routes in each case, these are not directly comparable. For example, consider L_3. Any distance relation will have something to say about the values of

$$d(P_1 \wedge P_2 \wedge P_3, P_1 \wedge \neg P_2 \wedge \neg P_3)$$

and

$$d(P_1 \wedge P_2 \wedge P_3, \neg P_1 \wedge P_2 \wedge P_3)$$

If our distance function is consistent with Px (by which we mean the resulting analogy principle is) then the latter must be less than or equal to the former; on the other hand, if we weight a difference in P_1 very highly, then perhaps the latter could be larger. But we might still take that any such comparison between distances does not allow us to say which pair is more similar to one another, those similarities being in entirely different domains. In that case, a betweenness relation might be more appropriate than a distance relation.

Definition 10. *For atoms*

$$\alpha_i = \bigwedge_{n=1}^{q} P_n^{\epsilon_n}, \alpha_j = \bigwedge_{n=1}^{q} P_n^{\nu_n}, \alpha_k = \bigwedge_{n=1}^{q} P_n^{\tau_n}$$

say that α_k *is further away from* α_i *than* α_j *is if they are different atoms and for* $1 \leq n \leq q$,

$$\epsilon_n = \tau_n \Rightarrow \epsilon_n = \nu_n.$$

Definition 11. *Let* $>_F$ *be the similarity relation such that* $\{\alpha_i, \alpha_j\} >_F \{\alpha_i, \alpha_k\}$ *iff* α_k *is further away from* α_i *than* α_j *is.*

Notice that ${}^2\!>_F$ and ${}^2\!>_H$ are the same relation, hence AP_H holds for w on L_2 if and only if AP_F does[2]. AP_F will thus also receive some attention in Chapter 4.

[2]Whereas ${}^q\!>_F \neq {}^q\!>_H$ for $q > 2$.

3.3 Some useful results about (S)AP$_s$

Our first observation is that SAP$_S$ and the well known Atom Exchangeability Principle (Ax) (see below) are mutually inconsistent for many similarity relations $>_S$.

The Atom Exchangeability Principle (Ax)
If σ is a permutation of $1, 2, \ldots, 2^q$, then

$$w\left(\bigwedge_{j=1}^{n} \alpha_{i_j}(a_j)\right) = w\left(\bigwedge_{j=1}^{n} \alpha_{\sigma(i_j)}(a_j)\right)$$

If there are distinct i, j, k such that

$$\{\alpha_i, \alpha_j\} >_S \{\alpha_i, \alpha_k\}$$

then the stated inconsistency is immediate since for distinct atoms $\alpha_i(x), \alpha_j(x), \alpha_k(x)$ Ax gives that

$$w(\alpha_i(a_2) \mid \alpha_j(a_1)) = w(\alpha_i(a_2) \mid \alpha_k(a_1)) \tag{3.2}$$

wheras SAP$_S$ prescribes strict inequality in (3.2). If there is no such triple, then AP$_S$ is equivalent to SPIR. SPIR fails for some w satisfying Ax but is also satisfied by some (see [26].)

Of course this was entirely to be expected since whereas the principles Ex, Px, SN seem to simply capture an evident symmetry in the language, to demand symmetry between atoms makes a more substantial claim. To assume that atoms can be exchanged without restriction is to suppose that the distinguishing features of the atoms have no relevance for a rational probability function. In particular, this means that such a function cannot be affected by varying degrees of similarity between atoms; in other words, it precludes analogy by similarity of atoms. This result is of some further relevance here because Ax is a widely accepted rational principle in Inductive Logic, for example it holds for Carnap's c_λ^q Continuum and the continuum arising in [25]. Hence most SAP$_S$ principles will fail widely amongst the familiar rational probability functions considered in Inductive Logic.

Our next proposition shows that AP$_S$ and SAP$_S$ are often preserved under marginalization, a result which will play an important role later when we come to consider various (S)AP$_S$ for L_q with $q > 2$. To consider what it means for (S)AP$_S$

to hold for languages of different sizes we first need to say something about when we take a similarity relation ${}^{q+1}\!>_S$ to be the extension of a similarity relation ${}^{q}\!>_S$.

Suppose that ${}^{q}\!>_S$ is a similarity relation on L_q. For any atom $\alpha(x)$ of L_q write $\alpha^+(x)$ for $\alpha(x) \wedge P_{q+1}(x)$ and $\alpha^-(x)$ for $\alpha(x) \wedge \neg P_{q+1}(x)$. Then ${}^{q+1}\!>_S$ is an extension of ${}^{q}\!>_S$ if for every α, β, γ such that $\{\alpha, \beta\}\ {}^{q}\!>_S \{\alpha, \gamma\}$,

$$\begin{array}{ccc} \{\alpha^+, \beta^+\} & {}^{q+1}\!>_S & \{\alpha^+, \gamma^+\} \\ \{\alpha^+, \beta^-) & {}^{q+1}\!>_S & \{\alpha^+, \gamma^-\} \\ \{\alpha^-, \beta^+\} & {}^{q+1}\!>_S & \{\alpha^-, \gamma^+\} \\ \{\alpha^-, \beta^-\} & {}^{q+1}\!>_S & \{\alpha^-, \gamma^-\} \end{array} \tag{3.3}$$

We can now prove the following proposition.

Proposition 12. *Suppose that ${}^{q}\!>_S$ is a similarity relation on L_q and ${}^{q+1}\!>_S$ an extension to L_{q+1}. If the probability function w on L_{q+1} satisfies Ex + SN + AP_S, then the restriction of w to SL_q also satisfies AP_S.*

Proof. Let w be a probability function satisfying AP_S on L_{q+1} and let w' be its restriction to L_q. Let $\alpha(x), \beta(x), \gamma(x)$ be atoms of L' such that

$$\{\alpha, \beta\}\ {}^{q}\!>_S \{\alpha, \gamma\}.$$

Let α^+ and α^- be defined as above (and similarly for $\beta^\pm, \gamma^\pm$.) By assumption, we have

$$\begin{array}{ccc} \{\alpha^+, \beta^+\} & {}^{q+1}\!>_S & \{\alpha^+, \gamma^+\} \\ \{\alpha^+, \beta^-) & {}^{q+1}\!>_S & \{\alpha^+, \gamma^-\} \end{array} \tag{3.4}$$

Let $\theta \in SL'$, so θ does not contain any occurrences of P_q. Then since w satisfies SN,

$$w(\alpha^- \wedge \theta) = w(\alpha^+ \wedge \theta), w(\alpha^+ \wedge \theta) = w(\alpha^- \wedge \theta),$$

and similarly for β, γ. [Here, following our earlier stated abbreviation, we are leaving the instantiating constants implicit].

Hence

$$w'(\alpha \mid \beta \wedge \theta) = \frac{w((\alpha^+ \vee \alpha^-) \wedge (\beta^+ \vee \beta^-) \wedge \theta)}{w((\beta^+ \vee \beta^-) \wedge \theta)}$$

$$\begin{aligned}
&= \frac{w(\alpha^+ \wedge \beta^+ \wedge \theta) + w(\alpha^+ \wedge \beta^- \wedge \theta) + w(\alpha^- \wedge \beta^+ \wedge \theta) + w(\alpha^- \wedge \beta^- \wedge \theta)}{w(\beta^+ \wedge \theta) + w(\beta^- \wedge \theta)} \\
&= \frac{w(\alpha^+ \wedge \beta^+ \wedge \theta) + w(\alpha^+ \wedge \beta^- \wedge \theta) + w(\alpha^+ \wedge \beta^- \wedge \theta) + w(\alpha^+ \wedge \beta^+ \wedge \theta)}{w(\beta^+ \wedge \theta) + w(\beta^+ \wedge \theta)} \\
&= \frac{2w(\alpha^+ \wedge \beta^+ \wedge \theta) + 2w(\alpha^+ \wedge \beta^- \wedge \theta)}{2w(\beta^+ \wedge \theta)} \\
&= w(\alpha^+ \mid \beta^+ \wedge \theta) + w(\alpha^+ \mid \beta^- \wedge \theta) \\
&\geq w(\alpha^+ \mid \gamma^+ \wedge \theta) + w(\alpha^+ \mid \gamma^- \wedge \theta) \qquad \text{by (3.4), AP}_S \text{ on } L_{q+1} \\
&= w'(\alpha \mid \gamma \wedge \phi)
\end{aligned}$$

so w' satisfies AP_S on L_q, as required. □

Note that the same result holds for SAP_S, mutatis mutandis.

Clearly, the similarity relation derived from Hamming Distance on L_{q+1} is an extension of the similarity relation derived from Hamming Distance on L_q. Moreover, since Hamming Distance (as well as all other distances mentioned in the previous section) suggests SN as a background assumption, it is natural to look at (S)AP_H in the presence of SN, in which case Proposition 12 shows that it will be preserved under marginalisation. This means that in seeking to classify the probability functions satisfying these conditions we can start by looking at small languages.

Our next proposition concerns the smallest language, L_1, where there is only a single predicate. For this language Carnap's probability function c_∞^1 has the discrete de Finetti prior which puts all the measure on the single point $\langle \frac{1}{2}, \frac{1}{2} \rangle$ and c_0^1 has discrete de Finetti prior which splits the measure equally between the points $\langle 1, 0 \rangle, \langle 0, 1 \rangle$.

Proposition 13. *Suppose that the probability function w on L_1 satisfies Ex. Then w satisfies AP_S for any $>_S$.*

Proof. By the definition of a similarity relation, any $>_S$ satisfies

$$\{P_1, P_1\}, \{\neg P_1, \neg P_1\} >_S \{P_1, \neg P_1\}$$

and these are the only possible inequalities relevant to L_1. Hence to show that w satisfies AP_S it is enough to show that

$$w(P(a_{n+1}) \,|\, P(a_n) \wedge \phi) \geq w(P(a_{n+1}) \,|\, \neg P(a_n) \wedge \phi) \tag{3.5}$$

where as usual $\phi \in QFSL_1$ does not contain a_n or a_{n+1}. This is a well known example of SPIR, see [27], but for completeness we will sketch the proof.

Using the Disjunctive Normal Form Theorem let

$$\phi(a_1, a_2, \ldots, a_n) \equiv \bigvee_{j=1}^{r} \bigwedge_{i=1}^{n} P^{\epsilon_{ij}}(a_i)$$

where the $\epsilon_{ij} \in \{0, 1\}$ and $P^1 = P, P^0 = \neg P$. By de Finetti's Representation Theorem (and our convention on page 13), to show (3.5) it is sufficient to show that when the denominators are non-zero,

$$\frac{\int x^2 f(x)\, d\mu(x)}{\int x f(x)\, d\mu(x)} \geq \frac{\int x(1-x) f(x)\, d\mu(x)}{\int (1-x) f(x)\, d\mu(x)}$$

where $f(x) = \sum_{j=1}^{r} x^{\sum_i \epsilon_{ij}} (1-x)^{n - \sum_i \epsilon_{ij}}$ and all the integrals are over $\mathbb{D}_1$. Simplifying and subtracting the right hand side gives

$$\int x^2 f(x) d\mu(x) \int f(x) d\mu(x) - \left(\int x f(x) d\mu(x) \right)^2 \geq 0,$$

equivalently

$$\int f(x) \left(x - \frac{\int y f(y)\, d\mu(y)}{\int f(y)\, d\mu(y)} \right)^2 d\mu(x) \geq 0. \tag{3.6}$$

The result follows. □

Note that the only way equality can hold in (3.6) for all $f(x)$ is if μ concentrates all the measure on the three points $\langle 1, 0 \rangle, \langle 0, 1 \rangle, \langle a, 1-a \rangle \in \mathbb{D}_2$ for some $0 < a < 1$. This means that if we make the further assumption that w satisfies SN we would have to have that $\mu(\{\langle 0,1\rangle\}) = \mu(\{\langle 1,0\rangle\})$ and $a = 1/2$ so

$$w = \mu(\{\langle \frac{1}{2}, \frac{1}{2}\rangle\})c_\infty^1 + (1 - \mu(\{\langle \frac{1}{2}, \frac{1}{2}\rangle\}))c_0^1.$$

Hence, given the extra constraint of SN, the corresponding result holds for SAP$_S$ just when w is not of the above form.

Finally we note that for any $>_S$, SAP$_S$ implies the Principle of Regularity (REG):

The Principle of Regularity (REG)
For any consistent $\theta \in QFSL$, $w(\theta) \neq 0$.

For if w were to fail REG we would have $w(\theta(a_1, ..., a_n)) = 0$ for some $\theta(a_1, ..., a_n) \in QFSL$. But then we would have

$$w(\alpha_i(a_{n+2}) \,|\, \alpha_j(a_{n+1}) \wedge \theta(a_1, ..., a_n)) = w(\alpha_i(a_{n+2}) \,|\, \alpha_k(a_{n+1}) \wedge \theta(a_1, ..., a_n))$$

for any atoms $\alpha_i, \alpha_j, \alpha_k$, whereas SAP$_S$ requires strict inequality in some instances.

Having introduced the key ideas, in the next Chapter we turn to look at (S)AP$_H$ in more detail.

Chapter 4

Hamming Distance and analogy

This Chapter will look more closely at Hamming Distance and the resulting AP_H, SAP_H and SDAP_H. The close relationship between Hamming Distance and the symmetry requirements of Px + SN makes this a particularly natural way to formulate analogy between atoms in PIL.

In consequence of Proposition 12 it is instructive to first consider probability functions satisfying AP_H on languages with few predicates. By Proposition 13 AP_H will hold on L_1 for any w satisfying Ex, so we begin with the case of AP_H for L_2.

4.1 L_2 and the Wheel of Fortune

Fix the ordering of the atoms of L_2 as usual by $\alpha_1(x) = P_1(x) \wedge P_2(x)$, $\alpha_2(x) = P_1(x) \wedge \neg P_2(x)$, $\alpha_3(x) = \neg P_1(x) \wedge P_2(x)$, $\alpha_4(x) = \neg P_1(x) \wedge \neg P_2(x)$. The Hamming Distance between the atoms can then be pictured thus:

$$\begin{tikzcd}
& \alpha_1 \arrow[dl, dash] \arrow[dr, dash] & \\
\alpha_3 \arrow[dr, dash] & & \alpha_2 \arrow[dl, dash] \\
& \alpha_4 &
\end{tikzcd}$$

Notie that this situation is actually a model of Skyrm's 'Wheel of Fortune', with the atoms representing respectively the compass points North, East, West, South.

For $\vec{b} \in \mathbb{D}_q$ define $w_{\vec{b}}$ to be the probability function on L_q given by

$$w_{\vec{b}}\left(\bigwedge_{i=1}^{m} \alpha_{h_i}(a_{r_i})\right) = \prod_{i=1}^{m} b_{h_i} = \prod_{j=1}^{2^q} b_j^{n_j}$$

where n_j is the number of times that α_j appears amongst the α_{h_i}.

For $\langle a,b,c,d\rangle \in \mathbb{D}_2$ let $y_{\langle a,b,c,d\rangle}$ be the probability function given by

$$8^{-1}(w_{\langle a,b,c,d\rangle}+w_{\langle a,c,b,d\rangle}+w_{\langle b,a,d,c\rangle}+w_{\langle c,a,d,b\rangle}+w_{\langle b,d,a,c\rangle}+w_{\langle c,d,a,b\rangle}+w_{\langle d,b,c,a\rangle}+w_{\langle d,c,b,a\rangle}).$$

Note that the eight summands correspond to the permutations of $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ that preserve Hamming Distance, where the atoms are associated with a, b, c and d respectively. Also note that in the case $a = b = c = d = 1/4$ all these summands are equal and

$$y_{\langle \frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4}\rangle} = w_{\langle \frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4}\rangle} = c_\infty^2.$$

Proposition 14. *The following probability functions on L_2 satisfy Ex, Px, SN and AP_H:*

(i) $y_{\langle a,b,c,d\rangle}$ *when* $\langle a,b,c,d\rangle \in \mathbb{D}_2$, $a \geq b \geq c \geq d$, $ad = bc$ *and* $a = b$ *or* $b = c$.

(ii) $\lambda y_{\langle a,a,b,b\rangle} + (1-\lambda)c_\infty^2$ *when* $\langle a,a,b,b\rangle \in \mathbb{D}_2$, $a > b \geq 0$ *and* $0 \leq \lambda \leq 1$.

(iii) $\lambda y_{\langle 1,0,0,0\rangle} + (1-\lambda)y_{\langle \frac{1}{2},\frac{1}{2},0,0\rangle}$ *when* $0 \leq \lambda \leq 1$.

Proof. (i) Clearly $y_{\langle a,b,c,d\rangle}$ satisfies Ex since it is a convex combination of probability functions satisfying Ex. Also, by Theorem 8, it satisfies Px and SN as it is invariant under permutations of the atoms that preserve Hamming Distance. It remains only to show that $y_{\langle a,b,c,d\rangle}$ satisfies AP$_H$; in other words, we need to show that the following inequalities hold for any $\phi \in QFSL_2$:

$$y_{\langle a,b,c,d\rangle}(\alpha_1 \,|\, \alpha_2 \wedge \phi) \geq y_{\langle a,b,c,d\rangle}(\alpha_1 \,|\, \alpha_4 \wedge \phi) \tag{4.1}$$

$$y_{\langle a,b,c,d\rangle}(\alpha_1 \,|\, \alpha_1 \wedge \phi) \geq y_{\langle a,b,c,d\rangle}(\alpha_1 \,|\, \alpha_2 \wedge \phi) \tag{4.2}$$

This is a straightforward calculation which we include for the sake of completeness (and later reference) though the trusting reader may at this point wish to skip it.

Let $N_1 = w_{\langle a,b,c,d\rangle}(\phi), N_2 = w_{\langle a,c,b,d\rangle}(\phi), E_1 = w_{\langle b,a,d,c\rangle}(\phi), E_2 = w_{\langle c,a,d,b\rangle}(\phi), W_1 = w_{\langle b,d,a,c\rangle}(\phi), W_2 = w_{\langle c,d,a,b\rangle}(\phi), S_1 = w_{\langle d,b,c,a\rangle}(\phi), S_2 = w_{\langle d,c,b,a\rangle}(\phi)$.

Writing out (4.1) using the definition of $y_{\langle a,b,c,d\rangle}$ and canceling the 8^{-1} factors, we obtain

$$\frac{abN_1 + acN_2 + baE_1 + caE_2 + bdW_1 + cdW_2 + dbS_1 + dcS_2}{bN_1 + cN_2 + aE_1 + aE_2 + dW_1 + dW_2 + bS_1 + cS_2} \geq \frac{adN_1 + adN_2 + bcE_1 + cbE_2 + bcW_1 + cbW_2 + daS_1 + daS_2}{dN_1 + dN_2 + cE_1 + bE_2 + cW_1 + bW_2 + aS_1 + aS_2}$$

Multiplying out gives that the following sum should be non-negative.

<u>Sum 1.</u>

$$\begin{aligned}
&N_1E_1(bc-ad)(a-b) + N_1E_2(b^2-ad)(a-c) + N_1W_1(bc-d^2)(a-b)+\\
&N_1W_2(b^2-d^2)(a-c) + N_1S_1b(a-d)^2 + N_1S_2(ab-cd)(a-d)+\\
&N_2E_1(c^2-ad)(a-b) + N_2E_2\colon\ (ad-bc)(c-a) + N_2W_1(c^2-d^2)(a-b)+\\
&N_2W_2(bc-d^2)(a-c) + N_2S_1(ac-db)(a-d) + N_2S_2c(a-d)^2+\\
&E_1W_1a(b-c)^2 + E_1W_2(ab-cd)(b-c) + E_1S_1(a^2-bc)(b-d)+\\
&E_1S_2(a^2-c^2)(b-d) + E_2W_1(ac-bd)(c-b) + E_2S_1(a^2-b^2)(c-d)+\\
&E_2S_2(a^2-bc)(c-d) + W_1W_2d(b-c)^2 + W_1S_1(ad-bc)(b-d)+\\
&W_1S_2(ad-c^2)(b-d) + W_2S_1(ad-b^2)(c-d) + W_2S_2(ad-bc)(c-d)
\end{aligned}$$

If $ad = bc$ and $a \geq b \geq c \geq d$ then all these terms, except for the terms in N_2E_1 and W_2S_1, are greater than or equal to zero (and not all zero unless $a = b = c = d$.) If in addition $a = b$ then also $c = d$ (since $ad = bc$) and the terms in N_2E_1 and W_2S_1 are both equal to zero. Similarly if $b = c$, then $ad = b^2 = c^2$ so again the terms in N_2E_1 and W_2S_1 are both zero. Hence the required inequality holds.

Writing out (4.2) using the definition of $y_{\langle a,b,c,d\rangle}$ and canceling the 8^{-1} factors, we obtain

$$\frac{a^2N_1 + a^2N_2 + b^2E_1 + c^2E_2 + b^2W_1 + c^2W_2 + d^2S_1 + d^2S_2}{aN_1 + aN_2 + bE_1 + cE_2 + bW_1 + cW_2 + dS_1 + dS_2} \geq \frac{abN_1 + acN_2 + baE_1 + caE_2 + bdW_1 + cdW_2 + dbS_1 + dcS_2}{bN_1 + cN_2 + aE_1 + aE_2 + dW_1 + dW_2 + bS_1 + cS_2}$$

Multiplying out gives that the following sum should be non-negative.

<u>Sum 2</u>

$$\begin{aligned}
&N_1E_1(a^2-b^2)(a-b)+N_1E_2(a^2-bc)(a-c)+N_1W_1(ad-b^2)(a-b)+\\
&N_1W_2(ad-bc)(a-c)+N_1S_1b(a-d)^2+N_1S_2(ac-bd)(a-d)+\\
&N_2E_1(a^2-bc)(a-b)+N_2E_2(a^2-c^2)(a-c)+N_2W_1(ad-bc)(a-b)+\\
&N_2W_2(ad-c^2)(a-c)+N_2S_1(ab-cd)(a-d)+N_2S_2c(a-d)^2+\\
&E_1E_2a(b-c)^2+E_1W_2(bd-ac)(b-c)+E_1S_1(b^2-ad)(b-d)+\\
&E_1S_2(bc-ad)(b-d)+E_2W_1(ab-cd)(b-c)+E_2S_1(bc-ad)(c-d)+\\
&E_2S_2(c^2-ad)(c-d)+W_1W_2d(b-c)^2+W_1S_1(b^2-d^2)(b-d)+\\
&W_1S_2(bc-d^2)(b-d)+W_2S_1(bc-d^2)(c-d)+W_2S_2(c^2-d^2)(c-d)
\end{aligned}$$

If $ad=bc$ and $a\geq b\geq c\geq d$ then all these terms except those in N_1W_1, E_1W_2, E_2S_2 are greater than or equal to zero. If $a=b$ then also $c=d$ (since $ad=bc$) and the terms in N_1W_1, E_1W_2, E_2S_2 are zero whilst if $b=c$ then $ad=b^2=c^2$ and again the terms in N_1W_1, E_1W_2, E_2S_2 are zero. Hence the required inequality holds in this case too.

(ii) This part is proved similarly. Taking $N=w_{\langle a,a,b,b\rangle}(\phi), E=w_{\langle b,a,b,a\rangle}(\phi), W=w_{\langle a,b,a,b\rangle}(\phi), S=w_{\langle b,b,a,a\rangle}(\phi), K=c^2_\infty(\phi)$, inequality (4.1) becomes

$$\frac{4^{-1}\lambda(a^2N+baE+abW+b^2S)+4^{-2}(1-\lambda)K}{4^{-1}\lambda(aN+aE+bW+bS)+4^{-1}(1-\lambda)K} \geq \frac{4^{-1}\lambda(abN+abE+abW+abS)+4^{-2}(1-\lambda)K}{4^{-1}\lambda(bN+aE+bW+aS)+4^{-1}(1-\lambda)K}.$$

Canceling factors of 4^{-1} and multiplying out gives that the following sum should be non-negative.

$$\begin{aligned}
&\lambda^2(NEa(a-b)^2+NS(a^2-b^2)(a-b)+WSb(a-b)^2)+\\
&\lambda(1-\lambda)(NK(a-b)(a-4^{-1})+SK(a-b)(4^{-1}-b))
\end{aligned}$$

Clearly this is non-negative since $a\geq b$ and $2a+2b=1$ (so $a\geq 4^{-1}\geq b$).

With these same values for N, K etc. (4.2) becomes

$$\frac{4^{-1}\lambda(a^2N+b^2E+a^2W+b^2S)+4^{-2}(1-\lambda)K}{4^{-1}\lambda(aN+bE+aW+bS)+4^{-1}(1-\lambda)K} \geq \frac{4^{-1}\lambda(a^2N+abE+abW+b^2S)+4^{-2}(1-\lambda)K}{4^{-1}\lambda(aN+aE+bW+bS)+4^{-1}(1-\lambda)K}.$$

Canceling factors of 4^{-1} and multiplying out gives that the following sum should be non-negative,

$$\lambda^2(NEa(a-b)^2 + EW(a^2-b^2)(a-b) + WSb(a-b)^2)+$$
$$\lambda(1-\lambda)(EK(a-b)(4^{-1}-b) + WK(a-b)(a-4^{-1})),$$

which again it clearly is.

For part (iii), let $N = w_{\langle 1,0,0,0\rangle}(\phi), E = w_{\langle 0,1,0,0\rangle}(\phi), W = w_{\langle 0,0,1,0\rangle}(\phi), S = w_{\langle 0,0,0,1\rangle}(\phi), n = w_{\langle \frac{1}{2},\frac{1}{2},0,0\rangle}(\phi), e = w_{\langle 0,\frac{1}{2},0,\frac{1}{2}\rangle}(\phi), w = w_{\langle \frac{1}{2},0,\frac{1}{2},0\rangle}(\phi), s = w_{\langle 0,0,\frac{1}{2},\frac{1}{2}\rangle}(\phi)$
Ignoring the (trivial by our convention as given on page 13) cases when a denominator is zero note that

$$(\lambda y_{\langle 1,0,0,0\rangle} + (1-\lambda) y_{\langle \frac{1}{2},\frac{1}{2},0,0\rangle})(\alpha_1 \mid \alpha_4 \wedge \phi) = 0$$

for all ϕ, while

$$(\lambda y_{\langle 1,0,0,0\rangle} + (1-\lambda) y_{\langle \frac{1}{2},\frac{1}{2},0,0\rangle})(\alpha_1 \mid \alpha_2 \wedge \phi) = \frac{(1-\lambda)16^{-1}n}{(\lambda 4^{-1}E + (1-\lambda)8^{-1}(n+e)}$$

and

$$\lambda y_{\langle 1,0,0,0\rangle} + (1-\lambda) y_{\langle \frac{1}{2},\frac{1}{2},0,0\rangle}(\alpha_1 \mid \alpha_1 \wedge \phi) = \frac{\lambda 4^{-1}N + (1-\lambda)16^{-1}(n+w)}{\lambda 4^{-1}N + (1-\lambda)8^{-1}(n+w)}$$

Clearly then, (4.1) holds. For (4.2) to hold, canceling factors of 16^{-1} and cross multiplying we see that the following sum must be non-negative:

$$16\lambda^2 NE + 2(1-\lambda)^2(ne+ew)+$$
$$4\lambda(1-\lambda)(Nn + 2Ne + En + Ew),$$

which it clearly is. □

Noticing that the terms $N_1, N_2, E_1, \ldots, S_2, N, E, W, S, K$ will always be strictly positive for consistent ϕ when $d > 0$ in (i) and $b, \lambda > 0$ in (ii) it can be seen that we will have SAP$_H$ in case (i) if in addition $a > d > 0$ and in case (ii) if in addition $a > b > 0$ and $\lambda > 0$. We will never have SAP$_H$ in case (iii) since there are consistent ϕ for which as N, E, W, S, n, e, w and s are all zero.

It is interesting to note that despite the cases (iii) and (ii) with $b = 0$ of this proposition the probability functions

$$w = \lambda y_{\langle 1,0,0,0\rangle} + (1-\lambda)c_\infty^2$$

do not satisfy AP_H when $0 < \lambda < 1$, since in this case, for example,

$$w(\alpha_1 \mid \alpha_2 \wedge \alpha_2) = \frac{w(\alpha_1 \wedge \alpha_2 \wedge \alpha_2)}{w(\alpha_2 \wedge \alpha_2)} = \frac{(1-\lambda)1/4^3}{\lambda + (1-\lambda)1/4^2}$$

while

$$w(\alpha_1 \mid \alpha_4 \wedge \alpha_2) = \frac{w(\alpha_1 \wedge \alpha_4 \wedge \alpha_2)}{w(\alpha_4 \wedge \alpha_2)} = \frac{(1-\lambda)1/4^3}{(1-\lambda)1/4^2}$$

The main part of this chapter will now be devoted to showing the converse to Proposition 14: that any probability function on L_2 satisfying Ex, Px, SN and AP_H must be one of the functions above. We first need some lemmata.

Lemma 15. *If the probability function w on L_2 satisfies Ex, Px, SN then there is a countably additive measure μ on $\mathbb{D}_2$ such that for S a Borel subset of $\mathbb{D}_2$ and $\sigma \in \mathcal{P}_2$, $\mu(\sigma(S)) = \mu(S)$ and*

$$w = \int_{\mathbb{D}_2} y_{\langle x_1, x_2, x_3, x_4 \rangle} \, d\mu(\vec{x}).$$

Proof. By the version of de Finetti's Representation Theorem given earlier and the definition of $w_{\langle x_1, x_2, x_3, x_4 \rangle}$

$$w = \int_{\mathbb{D}_2} w_{\langle x_1, x_2, x_3, x_4 \rangle} \, d\mu(\vec{x})$$

for some countably additive measure μ on $\mathbb{D}_2$. Since for any $\sigma \in \mathcal{P}_2$,

$$w\left(\bigwedge_{j=1}^{k} \alpha_{h_j}(a_j)\right) = w\left(\bigwedge_{j=1}^{k} \alpha_{\sigma(h_j)}(a_j)\right),$$

$$\int_{\mathbb{D}_2} w_{\langle x_1, x_2, x_3, x_4 \rangle} \, d\mu(\vec{x}) = \int_{\mathbb{D}_2} w_{\langle x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}, x_{\sigma(4)} \rangle} \, d\mu(\vec{x}) \tag{4.3}$$

$$= \int_{\mathbb{D}_2} w_{\langle x_1, x_2, x_3, x_4 \rangle} \, d\mu(\sigma^{-1}(\vec{x})). \tag{4.4}$$

Hence from (4.4), since $|\mathcal{P}_2| = 8$,

$$w = \int_{\mathbb{D}_2} w_{\langle x_1, x_2, x_3, x_4 \rangle} 8^{-1} \sum_{\sigma \in \mathcal{P}_2} \mu(\sigma^{-1}(\vec{x})).$$

The measure

$$8^{-1} \sum_{\sigma \in \mathcal{P}_2} \mu\sigma^{-1}$$

has the required invariance property and taking μ' to be this measure and using (4.3) for μ' gives

$$w = \int_{\mathbb{D}_2} 8^{-1} \left(\sum_{\sigma \in \mathcal{P}_2} w_{\langle x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}, x_{\sigma(4)} \rangle} \right) d\mu'(\vec{x}),$$

which is the required conclusion since

$$y_{\langle x_1, x_2, x_3, x_4 \rangle} = 8^{-1} \left(\sum_{\sigma \in \mathcal{P}_2} w_{\langle x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}, x_{\sigma(4)} \rangle} \right).$$

□

In the next two lemmas assume that w is a probability function on L_2 satisfying AP_H and Ex, Px, SN with de Finetti representation

$$w = \int_{\mathbb{D}_2} y_{\vec{x}} \, d\mu(\vec{x})$$

and, by Lemma 15, μ invariant under permutations σ of $\mathbb{D}_2$ for $\sigma \in \mathcal{P}_2$.

Recall that $\vec{b} \in \mathbb{D}_2$ is said to be *in the support of* μ if for every open subset S of $\mathbb{R}^4$ containing $\vec{b}$, $\mu(S \cap \mathbb{D}_2) > 0$. Notice that if μ is as in Lemma 15 and $\langle b_1, b_2, b_3, b_4 \rangle$ is in the support of μ then so is $\langle b_{\sigma(1)}, b_{\sigma(2)}, b_{\sigma(3)}, b_{\sigma(4)} \rangle$ for $\sigma \in \mathcal{P}_2$.

We shall be needing the following result.

Lemma 16. *Let $\langle b_1, b_2, \ldots, b_{2^q} \rangle \in \mathbb{D}_q$ be in the support of μ and $k_1, k_2, \ldots, k_{2^q} \in \mathbb{N}$. Then*

$$\lim_{m \to \infty} \frac{\int_{\mathbb{D}_q} \prod_{i=1}^{2^q} x_i^{[mb_i]+k_i} \, d\mu(\vec{x})}{\int_{\mathbb{D}_q} \prod_{i=1}^{2^q} x_i^{[mb_i]} \, d\mu(\vec{x})} = \prod_{i=1}^{2^q} b_i^{k_i},$$

where as usual $[mb_i]$ is the integer part of mb_i.

Proof. We begin by showing that for any natural numbers $r_1, r_2, \ldots, r_{2^q}$, possibly equal to 0, and for any $\nu > 0$

$$\left| \frac{\int_{\mathbb{D}_q} \prod_{i=1}^{2^q} x_i^{mb_i+n_i} \, d\mu(\vec{x})}{\int_{\mathbb{D}_q} \prod_{i=1}^{2^q} x_i^{mb_i} \, d\mu(\vec{x})} - \prod_{i=1}^{2^q} b_i^{n_i} \right| < \nu. \tag{4.5}$$

for all m eventually, where $n_i = [(m+s)b_i] - (m+s)b_i + sb_i + r_i$ for some fixed $s \geq 2/b_i$ if $b_i \neq 0$ and $n_i = r_i$ if $b_i = 0$, $i = 1, 2, \ldots, 2^q$. Let $h \geq 1$ be an upper bound on the n_i (for all m) and let

$$A_i = \begin{cases} [b_i, h] & \text{if } b_i \neq 0, \\ \{b_i, r_i\} & \text{if } b_i = 0. \end{cases}$$

Notice that the A_i do not depend on m and $n_i \in A_i$ for all m, $i = 1, 2, \ldots, 2^q$

Since the function

$$f : \langle x_1, x_2, \ldots, x_{2^{q+1}} \rangle \mapsto \prod_{i=1}^{2^q} x_i^{x_{2^q+i}}$$

is uniformly continuous on $\mathbb{D}_q \times \prod_{i=1}^{2^q} A_i$ we can pick $0 < \epsilon < \nu$ such that for $\vec{z}, \vec{t} \in \mathbb{D}_q \times \prod_{i=1}^{2^q} A_i$,

$$\left| \prod_{i=1}^{2^q} z_i^{z_{2^q+i}} - \prod_{i=1}^{2^q} t_i^{t_{2^q+i}} \right| < \nu/2 \text{ whenever } |\vec{z} - \vec{t}| < \epsilon. \tag{4.6}$$

Also since the function $\prod_{i=1}^{2^q} x_i^{b_i}$ takes its maximum value on $\mathbb{D}_q$ at $\vec{x} = \vec{b}$ there is a $\delta > 0$ such that

$$\prod_{i=1}^{2^q} b_i^{b_i} > \prod_{i=1}^{2^q} y_i^{b_i} + 2\delta \text{ whenever } |\vec{y} - \vec{b}| \geq \epsilon, \ \vec{y} \in \mathbb{D}_q.$$

Again by the uniform continuity of the function f we can choose $\epsilon' < \epsilon$ such that for $\vec{z}, \vec{t} \in \mathbb{D}_q \times \prod_{i=1}^{2^q} A_i$,

$$\left| \prod_{i=1}^{2^q} z_i^{z_{2^q+i}} - \prod_{i=1}^{2^q} t_i^{t_{2^q+i}} \right| < \delta \text{ whenever } |\vec{z} - \vec{t}| < \epsilon'. \tag{4.7}$$

Hence for any $\vec{x}, \vec{y} \in \mathbb{D}_q$ with $|\vec{x} - \vec{b}| < \epsilon'$, $|\vec{y} - \vec{b}| \geq \epsilon$,

$$\begin{aligned} \left| \prod_{i=1}^{2^q} x_i^{b_i} - \prod_{i=1}^{4} y_i^{b_i} \right| &\geq \left| \left| \prod_{i=1}^{2^q} b_i^{b_i} - \prod_{i=1}^{2^q} y_i^{b_i} \right| - \left| \prod_{i=1}^{2^q} b_i^{b_i} - \prod_{i=1}^{2^q} x_i^{b_i} \right| \right| \\ &> |2\delta - \delta| = \delta. \end{aligned}$$

For any such $\vec{x}, \vec{y}$ then,

$$\prod_{i=1}^{2^q} y_i^{b_i} < \prod_{i=1}^{2^q} x_i^{b_i} - \delta \leq \prod_{i=1}^{2^q} x_i^{b_i}(1-\delta)$$

so

$$\prod_{i=1}^{2^q} y_i^{mb_i} < \prod_{i=1}^{2^q} x_i^{mb_i}(1-\delta)^m.$$

Let I_m denote the integral

$$\frac{\int_{\mathbb{D}_q} \prod_{i=1}^{2^q} x_i^{mb_i+n_i} d\mu(\vec{x})}{\int_{\mathbb{D}_q} \prod_{i=1}^{2^q} x_i^{mb_i} d\mu(\vec{x})}.$$

Then

$$I_m = \frac{\int_{N_\epsilon(\vec{b})} \prod_{i=1}^{2^q} x_i^{mb_i+n_i} d\mu(\vec{x}) + \int_{\neg N_\epsilon(\vec{b})} \prod_{i=1}^{2^q} x_i^{mb_i+n_i} d\mu(\vec{x})}{\int_{N_\epsilon(\vec{b})} \prod_{i=1}^{2^q} x_i^{mb_i} d\mu(\vec{x}) + \int_{\neg N_\epsilon(\vec{b})} \prod_{i=1}^{2^q} x_i^{mb_i} d\mu(\vec{x})} \tag{4.8}$$

where as usual $N_\epsilon(\vec{b}) = \{\vec{x} \in \mathbb{D}_q \mid |\vec{x} - \vec{b}| < \epsilon\}$.
We have that,

$$\begin{aligned}
\int_{\neg N_\epsilon(\vec{b})} \prod_{i=1}^{2^q} x_i^{mb_i+n_i} d\mu(\vec{x}) &\leq \int_{\neg N_\epsilon(\vec{b})} \prod_{i=1}^{2^q} x_i^{mb_i} d\mu(\vec{x}) \\
&\leq \int_{\neg N_\epsilon(\vec{b})} \inf_{N_{\epsilon'}(\vec{b})} (\prod_{i=1}^{2^q} x_i^{mb_i}) d\mu(\vec{x})(1-\delta)^m \\
&= \inf_{N_{\epsilon'}(\vec{b})} (\prod_{i=1}^{2^q} x_i^{mb_i}) \mu(\neg N_\epsilon(\vec{b}))(1-\delta)^m.
\end{aligned}$$

Also,

$$\begin{aligned}
\int_{N_{\epsilon'}(\vec{b})} \prod_{i=1}^{2^q} x_i^{mb_i} d\mu(\vec{x}) &\geq \int_{N_{\epsilon'}(\vec{b})} \inf_{N_{\epsilon'}(\vec{b})} (\prod_{i=1}^{2^q} x_i^{mb_i}) d\mu(\vec{x}) \\
&= \inf_{N_{\epsilon'}(\vec{b})} (\prod_{i=1}^{2^q} x_i^{mb_i}) \mu(N_{\epsilon'}(\vec{b})),
\end{aligned}$$

so

$$\begin{aligned}\int_{\neg N_\epsilon(\vec{b})} \prod_{i=1}^{2^q} x_i^{mb_i+n_i} d\mu(\vec{x}) &\leq \int_{N_{\epsilon'}(\vec{b})} \prod_{i=1}^{2^q} x_i^{mb_i} d\mu(\vec{x}) \frac{\mu(\neg N_\epsilon(\vec{b}))}{\mu(N_{\epsilon'}(\vec{b}))}(1-\delta)^m \\ &\leq \int_{N_\epsilon(\vec{b})} \prod_{i=1}^{2^q} x_i^{mb_i} d\mu(\vec{x}) \frac{\mu(\neg N_\epsilon(\vec{b}))}{\mu(N_{\epsilon'}(\vec{b}))}(1-\delta)^m. \quad (4.9)\end{aligned}$$

Let $\underline{d_m}$ and $\overline{d_m}$ respectively be the minimum and maximum values of $\prod_{x=1}^{2^q} x_i^{n_i}$ for $\vec{x}$ from the closure of the set $N_\epsilon(\vec{b})$. Then

$$\underline{d_m} \leq \frac{\int_{N_\epsilon(\vec{b})} \prod_{i=1}^{2^q} x_i^{mb_i+n_i} d\mu(\vec{x})}{\int_{N_\epsilon(\vec{b})} \prod_{i=1}^{2^q} x_i^{mb_i} d\mu(\vec{x})} \leq \overline{d_m}$$

so there is some constant, d_m say, such that

$$d_m = \frac{\int_{N_\epsilon(\vec{b})} \prod_{i=1}^{2^q} x_i^{mb_i+n_i} d\mu(\vec{x})}{\int_{N_\epsilon(\vec{b})} \prod_{i=1}^{2^q} x_i^{mb_i} d\mu(\vec{x})} = \prod_{i=1}^{2^q} a_i^{n_i}$$

for some $\vec{a} \in N_\epsilon(\vec{b})$. By (4.7), $|\, d_m - \prod_{i=1}^{2^q} b_i^{n_i}| < \nu/2$.

Using this, (4.8) and (4.9) we see that for sufficiently large m,

$$I_m - \prod_{i=1}^{2^q} b_i^{n_i} \leq d_m + \frac{\mu(\neg N_\epsilon(\vec{b}))}{\mu(N_{\epsilon'}(\vec{b}))}(1-\delta)^m - \prod_{i=1}^{2^q} b_i^{n_i} < \nu.$$

Also for large m,

$$\begin{aligned}I_m - \prod_{i=1}^{2^q} b_i^{n_i} &\geq \frac{d_m}{1+\frac{\mu(N_\epsilon(\vec{b}))}{\mu(N_{\epsilon'}(\vec{b}))}(1-\delta)^m} - \prod_{i=1}^{2^q} b_i^{n_i} \\ &> \frac{\prod_{i=1}^{2^q} b_i^{n_i} - \nu/2}{1+\nu/2} - \prod_{i=1}^{2^q} b_i^{n_i} \\ &\geq -\nu.\end{aligned}$$

This completes the proof of (4.5). By taking the limit of the ratio of expressions as in (4.5) when the $r_i = k_i$ and when the $r_i = 0$ we now obtain as required that

$$\lim_{m\to\infty} \frac{\int_{\mathbb{D}_q} \prod_{i=1}^{2^q} x_i^{[(m+s)b_i]+k_i}\, d\mu(\vec{x})}{\int_{\mathbb{D}_2} \prod_{i=1}^{2^q} x_i^{[(m+s)b_i]}\, d\mu(\vec{x})} = \prod_{i=1}^{2^q} b_i^{k_i}.$$

□

Lemma 17. *Let the probability function w have de Finetti prior μ and suppose that $\langle b_1, \ldots, b_{2^q}\rangle, \langle c_1, \ldots, c_{2^q}\rangle$ are in the support of μ and $\prod_{i=1}^{2^q} b_i^{b_i}, \prod_{i=1}^{2^q} c_i^{c_i} < 1$. Then there are increasing sequences k_n, j_n and $0 < \lambda (< \infty)$ such that*

$$\lim_{n\to\infty} \frac{\int_{\mathbb{D}_{2^q}} \prod_{i=1}^{2^q} x_i^{[k_n b_i]}\, d\mu(\vec{x})}{\int_{\mathbb{D}_{2^q}} \prod_{i=1}^{2^q} x_i^{[j_n c_i]}\, d\mu(\vec{x})} = \lambda.$$

Proof. Pick small $\epsilon > 0$. From the proof of the previous lemma there is a $\nu > 0$ such that sufficiently large m

$$(1+\nu)\int_{N_\epsilon(\vec{b})} \prod_{i=1}^{2^q} x_i^{[mb_i]}\, d\mu(\vec{x}) \geq \int_{\mathbb{D}_{2^q}} \prod_{i=1}^{2^q} x_i^{[mb_i]}\, d\mu(\vec{x}) \geq \int_{N_\epsilon(\vec{b})} \prod_{i=1}^{2^q} x_i^{[mb_i]}\, d\mu(\vec{x}). \tag{4.10}$$

The sequence (in m)

$$\int_{N_\epsilon(\vec{b})} \prod_{i=1}^{2^q} x_i^{[mb_i]}\, d\mu(\vec{x})$$

is decreasing to 0 (strictly for infinitely many m) since $\prod_{i=1}^{2^q} b_i^{b_i} < 1$. Indeed

$$\int_{N_\epsilon(\vec{b})} \prod_{i=1}^{2^q} x_i^{[mb_i]}\, d\mu(\vec{x}) \geq \int_{N_\epsilon(\vec{b})} \prod_{i=1}^{2^q} x_i^{[(m+1)b_i]}\, d\mu(\vec{x}) \geq \gamma \int_{N_\epsilon(\vec{b})} \prod_{i=1}^{2^q} x_i^{[mb_i]}\, d\mu(\vec{x}) \tag{4.11}$$

where $\gamma > 0$ is at most the minimum of the function $\prod_{b_i \neq 0} x_i$ on the closure of $N_\epsilon(\vec{b})$. Similarly for $\vec{c}$, so we may assume that this same γ works there too.

Using (4.11) we can now produce increasing (infinitely often strictly) sequences $j_n, k_n \in \mathbb{N}$ such that

$$\int_{N_\epsilon(\vec{b})} \prod_{i=1}^{2^q} x_i^{[j_n b_i]}\, d\mu(\vec{x}) \geq \int_{N_\epsilon(\vec{c})} \prod_{i=1}^{2^q} x_i^{[(k_n c_i]}\, d\mu(\vec{x})] \geq \gamma \int_{N_\epsilon(\vec{b})} \prod_{i=1}^{2^q} x_i^{[j_{(n+1)} b_i]}\, d\mu(\vec{x})$$

$$\int_{N_\epsilon(\vec{c})} \prod_{i=1}^{2^q} x_i^{[k_{(n+1)} c_i]}\, d\mu(\vec{x}) \geq \int_{N_\epsilon(\vec{c})} \prod_{i=1}^{2^q} x_i^{[(j_{(n+1)} c_i]}\, d\mu(\vec{x})] \geq \gamma \int_{N_\epsilon(\vec{c})} \prod_{i=1}^{2^q} x_i^{[k_{(n+1)} c_i]}\, d\mu(\vec{x}).$$

From these inequalities we obtain that for all $n > 0$,

$$\gamma \;\leq\; \frac{\int_{N_\epsilon(\vec{b})} \prod_{i=1}^{2^q} x_i^{[j_n b_i]}\, d\mu(\vec{x})}{\int_{N_\epsilon(\vec{c})} \prod_{i=1}^{2^q} x_i^{[k_n c_i]}\, d\mu} \;\leq\; \gamma^{-1}$$

and with (4.10)

$$\gamma(1+\nu)^{-1} \;\leq\; \frac{\int_{\mathbb{D}_{2^q}} \prod_{i=1}^{2^q} x_i^{[j_n b_i]}\, d\mu(\vec{x})}{\int_{\mathbb{D}_{2^q}} \prod_{i=1}^{2^q} x_i^{[k_n c_i]}\, d\mu(\vec{x})} \;\leq\; \gamma^{-1}(1+\nu). \tag{4.12}$$

The sequence in (4.12) has a convergent subsequence, to λ say, and the lemma follows.

□

Corollary 18. *Let w be a probability function with de Finetti prior μ and let $\vec{b}, \vec{c}$ be distinct support points of μ such that $\prod_{i=1}^{2^q} b_i^{b_i}, \prod_{i=1}^{2^q} c_i^{c_i} < 1$. Then there exists $\lambda > 0$ and state descriptions ϕ_n, ψ_n such that for any $r_1, \ldots, r_{2^q} \in \mathbb{N}$,*

$$\lim_{n\to\infty} w\left(\bigwedge_{i=1}^{2^q} \alpha_i^{r_i} \mid \phi_n(a_1,\ldots,a_{s_n}) \vee \psi_n(a_1,\ldots,a_{t_n})\right)$$
$$= (1+\lambda)^{-1} \prod_{i=1}^{2^q} b_i^{r_i} + \lambda(1+\lambda)^{-1} \prod_{i=1}^{2^q} c_i^{r_i}.$$

Proof. Let j_n, k_n, λ be as in Lemma 17 and $\phi_n(a_1,\ldots,a_{s_n})$ be the conjunction of $[j_n b_i]$ copies of $\alpha_i(x)$ for $i = 1,\ldots,2^q$ instantiated by $a_1,\ldots,a_{s_n}$, so $s_n = \sum_{i=1}^{2^q}[j_n b_i]$. Similarly let $\psi_n(a_1,\ldots,a_{t_n})$ be the conjunction of $[k_n c_i]$ copies of $\alpha_i(x)$ for $i = 1,\ldots,2^q$ instantiated by $a_1,\ldots,a_{t_n}$, so $t_n = \sum_{i=1}^{2^q}[k_n c_i]$. Let δ_n be such that

$$\int_{\mathbb{D}_{2^q}} \prod_{i=1}^{2^q} x_i^{[j_n b_i]}\, d\mu(\vec{x}) = (1+\delta_n)\lambda \int_{\mathbb{D}_{2^q}} \prod_{i=1}^{2^q} x_i^{[k_n c_i]}\, d\mu(\vec{x}),$$

so $\delta_n \to 0$ as $n \to \infty$.

Then

$$w\left(\bigwedge_{i=1}^{2^q} \alpha_i^{r_i} \mid \phi_n(a_1,\ldots,a_{s_n}) \vee \psi_n(a_1,\ldots,a_{t_n})\right)$$
$$= \frac{\int_{\mathbb{D}_{2^q}} \prod_{i=1}^{2^q} x_i^{[j_n b_i]+r_i}\, d\mu(\vec{x}) + \int_{\mathbb{D}_{2^q}} \prod_{i=1}^{2^q} x_i^{[k_n c_i]+r_i}\, d\mu(\vec{x})}{\int_{\mathbb{D}_{2^q}} \prod_{i=1}^{2^q} x_i^{[j_n b_i]}\, d\mu(\vec{x}) + \int_{\mathbb{D}_{2^q}} \prod_{i=1}^{2^q} x_i^{[k_n c_i]}\, d\mu(\vec{x})}$$

$$= \frac{\int_{\mathbb{D}_{2^q}} \prod_{i=1}^{2^q} x_i^{[j_n b_i]+r_i} \, d\mu(\vec{x})}{(1+\lambda(1+\delta_n)) \int_{\mathbb{D}_{2^q}} \prod_{i=1}^{2^q} x_i^{[j_n b_i]} \, d\mu(\vec{x})}$$

$$+ \quad \frac{\int_{\mathbb{D}_{2^q}} \prod_{i=1}^{2^q} x_i^{[k_n c_i]+r_i} \, d\mu(\vec{x})}{(1+\lambda^{-1}(1+\delta_n)^{-1}) \int_{\mathbb{D}_{2^q}} \prod_{i=1}^{2^q} x_i^{[k_n c_i]} \, d\mu(\vec{x})}.$$

Taking the limit as $n \to \infty$ now gives, by Lemma 17,

$$(1+\lambda)^{-1} \prod_{i=1}^{2^q} b_i^{r_i} + \lambda(1+\lambda)^{-1} \prod_{i=1}^{2^q} c_i^{r_i},$$

as required.

□

We are now in a position to prove the converse to Proposition 14. In what follows, let w be a probability function satisfying AP_H with de Finetti prior μ.

Theorem 19. *If the probability function w on L_2 satisfies Ex, Px, SN and AP_H, then for some countably additive measure μ on*

$$\begin{aligned} \mathbb{A}_2 \quad = \quad & \{\langle x_1, x_2, x_3, x_4\rangle \in \mathbb{D}_2 \,|\, y_{\langle x_1,x_2,x_3,x_4\rangle} = y_{\langle a,b,c,d\rangle} \textit{ for some } \langle a,b,c,d\rangle \in \mathbb{D}_2 \\ & \qquad \textit{with } ad = bc \textit{ and } a = b \textit{ or } b = c)\}, \end{aligned}$$

$$w = \int_{\mathbb{A}_2} y_{\langle x_1,x_2,x_3,x_4\rangle} \, d\mu(\vec{x}).$$

Furthermore we may take μ to be invariant under the $\sigma \in \mathcal{P}_2$.

Proof. By Lemma 15 we know that

$$w = \int_{\mathbb{D}_2} y_{\langle x_1,x_2,x_3,x_4\rangle} \, d\mu(\vec{x})$$

for some such measure μ so all that remains is to show that μ gives all the measure to points from $\mathbb{A}_2$.

So suppose that $\vec{b} = \langle b_1, b_2, b_3, b_4\rangle$ is in the support of μ. We can assume, without loss of generality since we have invariance under the permutations in $\mathcal{P}_2$, that $b_1 \geq b_2, b_3, b_4$ and $b_2 \geq b_3$. Clearly if all the b_i are equal then $\vec{b}$ is in $\mathbb{A}_2$. So now suppose that not all the b_i are equal and to start with that at most one of them is zero.

Suppose firstly that $b_1 \neq b_2$. Note that if $\langle b_1, b_2, b_3, b_4 \rangle$ is in the support of μ then so is $\langle b_2, b_1, b_4, b_3 \rangle$. So by Corollary 18, we can find state descriptions ϕ_n, ψ_n such that for some $\lambda > 0$,

$$\lim_{n\to\infty} w(\alpha_1^{j_1}\alpha_2^{j_2}\alpha_3^{j_3}\alpha_4^{j_4} \,|\, \phi_n \vee \psi_n) = (1+\lambda)^{-1}(b_1^{j_1}b_2^{j_2}b_3^{j_3}b_4^{j_4} + \lambda b_2^{j_1}b_1^{j_2}b_4^{j_3}b_3^{j_4})$$

Taking $j_3 = j_4 = 0$ and the cases $j_1 = j_2 = 1$ and $j_2 = 1, j_1 = 0$ we get that

$$\lim_{n\to\infty} w(\alpha_1 \,|\, \alpha_2 \wedge (\phi_n \vee \psi_n)) = \frac{b_1b_2 + \lambda b_2b_1}{b_2 + \lambda b_1}$$

and taking $j_2 = j_3 = 0$ and the cases $j_1 = j_4 = 1$ and $j_4 = 1, j_1 = 0$ we get that

$$\lim_{n\to\infty} w(\alpha_1 \,|\, \alpha_4 \wedge (\phi_n \vee \psi_n)) = \frac{b_1b_4 + \lambda b_2b_3}{b_4 + \lambda b_3}$$

AP$_H$ requires that $w(\alpha_1 \,|\, \alpha_2 \wedge \theta) > w(\alpha_1 \,|\, \alpha_4 \wedge \theta)$ for any $\theta \in QFSL$; so, in particular, $w(\alpha_1 \,|\, \alpha_2 \wedge (\phi_n \vee \psi_n)) > w(\alpha_1 \,|\, \alpha_4 \wedge (\phi_n \vee \psi_n))$ for all n. This means that we cannot have

$$\lim_{n\to\infty} w(\alpha_1 \,|\, \alpha_2 \wedge (\phi_n \vee \psi_n)) < \lim_{n\to\infty} w(\alpha_1 \,|\, \alpha_4 \wedge (\phi_n \vee \psi_n))$$

and so we must have

$$\frac{b_1b_2 + \lambda b_2b_1}{b_2 + \lambda b_1} \geq \frac{b_1b_4 + \lambda b_2b_3}{b_4 + \lambda b_3}$$

which simplifies to

$$\lambda(b_2b_3 - b_1b_4)(b_1 - b_2) \geq 0.$$

Hence

$$(b_2b_3 - b_1b_4)(b_1 - b_2) \geq 0$$

and we can conclude that

$$b_2b_3 \geq b_1b_4$$

Similarly, using the fact that $\langle b_1, b_3, b_2, b_4 \rangle$ and $\langle b_2, b_4, b_1, b_3 \rangle$ must be in the support of μ and considering

$$\lim_{n\to\infty} w(\alpha_1 \,|\, \alpha_1 \wedge (\phi_n \vee \psi_n))$$

and

$$\lim_{n\to\infty} w(\alpha_1 \,|\, \alpha_2 \wedge (\phi_n \vee \psi_n))$$

AP_H gives that

$$\frac{b_1^2 + \lambda b_2^2}{b_1 + \lambda b_2} \geq \frac{b_1 b_3 + \lambda b_2 b_4}{b_3 + \lambda b_4}$$

for some $\lambda > 0$, which simplifies to

$$\lambda(b_1 b_4 - b_2 b_3)(b_1 - b_2) \geq 0$$

Hence

$$(b_1 b_4 - b_2 b_3)(b_1 - b_2) \geq 0$$

and can conclude that

$$b_1 b_4 = b_2 b_3$$

Notice that what Corollary 18 is allowing us to do here is to isolate any of the coefficients in Sum 1, Sum 2 etc. in the proof of Proposition 14. For example, directly above we have isolated the coefficients of the $N_1 E_1$ and $N_2 W_1$ terms in the first and second sums respectively.

It is clear that we can do this for any coefficient, so we will simply say that in the above we used $N_1 E_1$ and $N_2 W_1$. Similarly, using $E_2 S_1$ in Sum 1 we get that

$$(b_1^2 - b_2^2)(b_3 - b_4) \geq 0$$

and so $b_3 \geq b_4$. Then using $E_2 W_1$ in the same sum gives that

$$(b_1 b_3 - b_2 b_4)(b_3 - b_2) \geq 0$$

which with our assumption that $b_2 \geq b_3$ forces $b_2 = b_3$. So $\vec{b}$ is in $\mathbb{A}_2$ as required.

Still assuming that at most one b_i is zero, suppose now that $b_1 = b_2$. Notice firstly that using $W_2 S_2$ in the Sum 1 gives that $b_1 \neq b_3$, since otherwise we would have $b_1(b_4 - b_1)(b_1 - b_4) \geq 0$ and so $b_1 = b_2 = b_3 = b_4$. Now using $E_2 W_1$ from Sum 1 we get that

$$0 \leq (b_1 b_3 - b_2 b_4)(b_3 - b_2) = b_1(b_3 - b_4)(b_3 - b_1)$$

and so $b_3 \leq b_4$. Also, by E_2S_2,

$$0 \leq (b_1^2 - b_2b_3)(b_3 - b_4) = b_1(b_1 - b_3)(b_3 - b_4)$$

and so we must have $b_3 = b_4$ (and so $b_1b_4 = b_2b_3$). Hence once again we have that $\vec{b}$ is in $\mathbb{A}_2$, as required.

We finally consider that case when more than one of b_1, b_2, b_3, b_4 is zero. Clearly $b_1 > 0$. If $b_2 = b_3 = b_4 = 0$ then $\langle b_1, b_2, b_3, b_4\rangle \in \mathbb{A}_2$. The remaining cases are when just b_3, b_4 are zero and when b_2, b_3 are zero. If $b_3 = b_4 = 0$ then using N_1W_1 from Sum 2 we obtain

$$(b_1b_4 - b_2^2)(b_1 - b_2) \geq 0$$

which with the other assumptions forces $b_1 = b_2$ so $\langle b_1, b_2, b_3, b_4\rangle \in \mathbb{A}_2$. Finally if $b_2 = b_3 = 0$ then using W_2S_2 from Sum 1 gives the contradiction

$$(b_1b_4 - b_2b_3)(b_3 - b_4) \geq 0$$

thus concluding the proof.

□

Put another way Theorem 19 tells us that if $\langle b_1, b_2, b_3, b_4\rangle$ is in the support of μ then one of the following hold:

(A) $y_{\langle b_1,b_2,b_3,b_4\rangle} = y_{\langle 1,0,0,0\rangle}$,

(B) $y_{\langle b_1,b_2,b_3,b_4\rangle} = y_{\langle \frac{1}{2},\frac{1}{2},0,0\rangle}$,

(C) $y_{\langle b_1,b_2,b_3,b_4\rangle} = y_{\langle \frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4}\rangle}$,

(D) $y_{\langle b_1,b_2,b_3,b_4\rangle} = y_{\langle a,a,b,b\rangle}$ for some $a > b > 0$ (so $b = 1/2 - a$),

(E) $y_{\langle b_1,b_2,b_3,b_4\rangle} = y_{\langle a,b,b,c\rangle}$ for some $a > b > 0$, $c = b^2/a$.

In fact this result can be strengthened further to give a complete classification of the probability functions satisfying AP_H (+Ex+Px+SN) on L_2.

Theorem 20. *Let w be a probability function on L_2 satisfying Ex, Px, SN. Then w satisfies AP_H just if one of the following hold:*

(1) $w = y_{\langle a,b,b,c\rangle}$ *for some* $\langle a, b, b, c\rangle \in \mathbb{D}_2$ *with* $a > b \geq 0$, $c = b^2/a$.

(2) $w = \lambda y_{\langle a,a,b,b\rangle} + (1-\lambda)c_\infty^2$ *for some* $\langle a,a,b,b\rangle \in \mathbb{D}_2$ *with* $a \geq b \geq 0$ *and* $0 \leq \lambda \leq 1$.

(3) $w = \lambda y_{\langle 1,0,0,0\rangle} + (1-\lambda)y_{\langle \frac{1}{2},\frac{1}{2},0,0\rangle}$ *with* $0 \leq \lambda \leq 1$.

Proof. The reverse direction follows from Proposition 51. In the other direction let μ be as in Theorem 19 and suppose that $\vec{b} = \langle b_1, b_2, b_2, b_4\rangle$ and $\vec{c} = \langle c_1, c_2, c_3, c_4\rangle$ are distinct points in the support of μ and

$$y_{\langle b_1,b_2,b_3,b_4\rangle} \neq y_{\langle c_1,c_2,c_3,c_4\rangle}.$$

We first show that none of the following are possible.

(i) $y_{\langle b_1,b_2,b_3,b_4\rangle} = y_{\langle a,b,b,c\rangle}$ for some $a > b, c = b^2/a$, and
$y_{\langle c_1,c_2,c_3,c_4\rangle} = y_{\langle \alpha,\beta,\beta,\gamma\rangle}$ where $\alpha \geq \beta$ and$\gamma = \beta^2/\alpha$.

(ii) $y_{\langle b_1,b_2,b_3,b_4\rangle} = y_{\langle a,b,b,c\rangle}$ for some $a > b, c = b^2/a$, and
$y_{\langle c_1,c_2,c_3,c_4\rangle} = y_{\langle \alpha,\alpha,\beta,\beta\rangle}$ where $\alpha \geq \beta$ and either $b > 0$ or $\beta > 0$.

(iii) $y_{\langle b_1,b_2,b_3,b_4\rangle} = y_{\langle a,a,b,b\rangle}$ where $a > b$, and
$y_{\langle c_1,c_2,c_3,c_4\rangle} = y_{\langle \alpha,\alpha,\beta,\beta\rangle}$ where $\alpha > \beta$.

Concerning (i), notice that

$$\langle a,b,b,b^2/a\rangle = \langle (1+z)^{-2}, z(1+z)^{-2}, z(1+z)^{-2}, z^2(1+z)^{-2}\rangle$$

and

$$\langle \alpha,\beta,\beta,\beta^2/\alpha\rangle = \langle (1+w)^{-2}, w(1+w)^{-2}, w(1+w)^{-2}, w^2(1+w)^{-2}\rangle$$

for some $0 \leq z, w \leq 1$.

If $a = \alpha$ we would have $z = w$, contradicting the assumption that $y_{\langle b_1,b_2,b_3,b_4\rangle} \neq y_{\langle c_1,c_2,c_3,c_4\rangle}$. So we can assume that $a \neq \alpha$.

Since $a > 1/4$ and $\alpha \geq 1/4$, we can assume without loss of generality that $a > \alpha$. Notice that this forces $z < w$ and so $b = z(1+z)^{-2} < w(1+w)^{-2} = \beta$, and $c = z^2(1+z)^{-2} < w^2(1+w)^{-2} = \gamma$.

Since $\langle b,c,a,b\rangle$, $\langle \beta,\gamma,\alpha,\beta\rangle$ must be in the support of μ, by Corollary 18 we can find sentences ϕ_n,ψ_n such that for some λ,

$$\lim_{n\to\infty} w(\alpha_1^{j_1}\alpha_2^{j_2}\alpha_3^{j_3}\alpha_4^{j_4}\,|\,\phi_n\vee\psi_n) = (1+\lambda)^{-1}(b^{j_1}c^{j_2}a^{j_3}b^{j_4}+\lambda\beta^{j_1}\gamma^{j_2}\alpha^{j_3}\beta^{j_4}). \quad (4.13)$$

By taking $j_2=j_4=0$ and the cases $j_3=1$, $j_1=0$ and $j_3=j_1=1$ we get that

$$\lim_{n\to\infty} w(\alpha_1\,|\,\alpha_3\wedge(\phi_n\vee\psi_n)) = \frac{(ba+\lambda\beta\alpha)}{(a+\lambda\alpha)}$$

and taking $j_2=j_3=0$ and the cases $j_4=1$, $j_1=0$ and $j_4=j_1=1$ we get that

$$\lim_{n\to\infty} w(\alpha_1\,|\,\alpha_4\wedge(\phi_n\vee\psi_n)) = \frac{(b^2+\lambda\beta^2)}{(b+\lambda\beta)}$$

Multiplying out the inequality we get from AP_H, this gives that

$$\lambda(\alpha b-\beta a)(\beta-b)\geq 0$$

and since $\lambda>0$,

$$(\alpha b-\beta a)(\beta-b)\geq 0$$

which is impossible.

Now suppose that (ii) holds. Note that $2\alpha\geq\alpha+\beta=1/2$, hence $\alpha\geq 1/4$. By our previous observation, if $b\geq 1/4$ then $z(1+z)^{-2}\geq 1/4$, giving

$$0\geq(1-z)^2$$

which is only satisfiable if $z=1$, meaning $a=1/4=b$, in contradiction of our assumption that $a>b$. So we have that $\alpha\geq 1/4>b$.

Using the corresponding version of (4.13) (with the same j_i) but now for the points $\langle b,c,a,b\rangle$, $\langle\alpha,\alpha,\beta,\beta\rangle$, we obtain the inequality

$$\beta(b-a)(\alpha-b)\geq 0$$

which, since $a>b$, is impossible unless $\beta=0$.

Similarly, taking the points $\langle b,c,a,b\rangle$, $\langle\alpha,\alpha,\beta,\beta\rangle$ and the cases $j_1=2,j_2=j_3=j_4=0$ and $j_1=1,j_2=j_3=j_4=0$, and the cases $j_1=j_2=1,j_3=j_4=0$ and

$j_2 = 1, j_1 = j_3 = j_4 = 0$, and again using the corresponding versions of (4.13) we derive from the requirement from AP_H that

$$\frac{b^2 + \lambda\alpha^2}{b + \lambda\alpha} \geq \frac{bc + \lambda\alpha^2}{c + \lambda\alpha}.$$

On multiplying out this yields

$$\alpha(b - c)(b - \alpha) \geq 0,$$

implying that $b = c = 0$.

Finally suppose that (iii) is the case. We can assume that $a > \alpha$. Using the corresponding version of (4.13) again but now with the points $\langle a, a, b, b\rangle$, $\langle \alpha, \beta, \alpha, \beta\rangle$, the cases $j_1 = 2, j_2 = j_3 = j_4 = 0$ and $j_1 = 1, j_2 = j_3 = j_4 = 0$, and the cases $j_1 = j_2 = 1, j_3 = j_4 = 0$ and $j_2 = 1, j_1 = j_3 = j_4 = 0$, we derive from the requirement of AP_H that

$$\frac{(a^2 + \lambda\alpha^2)}{(a + \lambda\alpha)} \geq \frac{(a^2 + \lambda\alpha\beta)}{(a + \lambda\beta)}.$$

Multiplying out this inequality gives that

$$\lambda a(\beta - \alpha)(a - \alpha) \geq 0$$

which again is impossible.

Combining the above with the observation following Theorem 19 we see that the support of μ can only be one of:

(a) $\{\sigma\langle a, b, b, c\rangle \mid \sigma \in \mathcal{P}_2\}$ for some $\langle a, b, b, c\rangle \in \mathbb{D}_2$ with $a > b \geq 0$, $c = b^2/a$;

(b) $\{\sigma\langle a, a, b, b\rangle \mid \sigma \in \mathcal{P}_2\}$ for some $\langle a, a, b, b\rangle \in \mathbb{D}_2$ with $a > b \geq 0$;

(c) The image under the $\sigma \in \mathcal{P}$ of $\{\langle a, a, b, b\rangle, \langle \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\rangle, \}$ where $a > b \geq 0$.

The result now follows using Proposition 14. □

Directly from this theorem and the remark following Proposition 14 we obtain that for w a probability function on L_2 satisfying Ex, Px, SN, w satisfies SAP_H just if one of the following hold:

(1) $w = y_{\langle a,b,b,c\rangle}$ for some $\langle a, b, b, c\rangle \in \mathbb{D}_2$ with $a > b > 0$, $c = b^2/a$.

(2) $w = \lambda y_{\langle a,a,b,b\rangle} + (1-\lambda)c_\infty^2$ for some $\langle a,a,b,b\rangle \in \mathbb{D}_2$ with $a > b > 0$ and $0 < \lambda \leq 1$.

4.2 AP$_H$ for languages with more than 2 predicates

We begin by considering L_3. Fix an ordering of the atoms of L_3 by

$$\begin{array}{ll}
\alpha_1(x) = P_1(x) \wedge P_2(x) \wedge P_3(x) & \alpha_5(x) = \neg P_1(x) \wedge P_2(x) \wedge P_3(x) \\
\alpha_2(x) = P_1(x) \wedge P_2(x) \wedge \neg P_3(x) & \alpha_6(x) = \neg P_1(x) \wedge P_2(x) \wedge \neg P_3(x) \\
\alpha_3(x) = P_1(x) \wedge \neg P_2(x) \wedge P_3(x) & \alpha_7(x) = \neg P_1(x) \wedge \neg P_2(x) \wedge P_3(x) \\
\alpha_4(x) = P_1(x) \wedge \neg P_2(x) \wedge \neg P_3(x) & \alpha_8(x) = \neg P_1(x) \wedge \neg P_2(x) \wedge \neg P_3(x).
\end{array}$$

As in the case of L_2 we define $y_{\vec{c}}$ for $\vec{c} \in \mathbb{D}_3$ by

$$y_{\vec{c}} = |\mathcal{P}_3|^{-1} \sum_{\sigma \in \mathcal{P}_3} w_{\sigma(\vec{c})}$$

where $\mathcal{P}_3$ is the set of Hamming distance preserving permutations of the atoms of L_3. From Theorem 8 on page 37 it follows that $|\mathcal{P}_3| = 48$.

For the next proposition it will be useful to observe that because of repeated terms $y_{\langle \frac{1}{2},\frac{1}{2},0,0,0,0,0,0\rangle}$ simplifies to

$$\begin{aligned}
12^{-1}(&w_{\langle \frac{1}{2},\frac{1}{2},0,0,0,0,0,0\rangle} + w_{\langle \frac{1}{2},0,\frac{1}{2},0,0,0,0,0\rangle} + w_{\langle \frac{1}{2},0,0,0,\frac{1}{2},0,0,0\rangle} \\
&+ w_{\langle 0,\frac{1}{2},0,\frac{1}{2},0,0,0,0\rangle} + w_{\langle 0,\frac{1}{2},0,0,0,\frac{1}{2},0,0\rangle} + w_{\langle 0,0,\frac{1}{2},\frac{1}{2},0,0,0,0\rangle} \\
&+ w_{\langle 0,0,\frac{1}{2},0,0,0,\frac{1}{2},0\rangle} + w_{\langle 0,0,0,\frac{1}{2},0,0,0,\frac{1}{2}\rangle} + w_{\langle 0,0,0,0,\frac{1}{2},\frac{1}{2},0,0\rangle} \\
&+ w_{\langle 0,0,0,0,\frac{1}{2},0,\frac{1}{2},0\rangle} + w_{\langle 0,0,0,0,0,\frac{1}{2},0,\frac{1}{2}\rangle} + w_{\langle 0,0,0,0,0,0,\frac{1}{2},\frac{1}{2}\rangle}).
\end{aligned} \tag{4.14}$$

Proposition 21. *For $0 \leq \lambda \leq 1$ the probability function*

$$w = \lambda y_{\langle \frac{1}{2},\frac{1}{2},0,\dots,0\rangle} + (1-\lambda)c_0^3$$

on L_3 satisfies Ex, Px, SN and AP$_H$.

Proof. The only non-trivial part here is to show that w satisfies AP_H. Let $t_1, \dots, t_{12}$ denote the values given by each of the summands of $y_{\langle \frac{1}{2}, \frac{1}{2}, 0, \dots, 0\rangle}$ to $\phi \in QFSL_3$ in the order in (4.14).

Note that

$$w(\alpha_1 \mid \alpha_1 \wedge \phi) = \frac{\lambda 48^{-1}(t_1+t_2+t_3) + (1-\lambda)8^{-1} w_{\langle 1,0,0,\dots,0\rangle}(\phi)}{\lambda 24^{-1}(t_1+t_2+t_3) + (1-\lambda)8^{-1} w_{\langle 1,0,0,\dots,0\rangle}(\phi)}$$

$$w(\alpha_1 \mid \alpha_2 \wedge \phi) = \frac{\lambda 48^{-1} t_1}{\lambda 24^{-1}(t_1+t_4+t_5) + (1-\lambda)8^{-1} w_{\langle 0,1,0,\dots,0\rangle}(\phi)}$$

and

$$w(\alpha_1 \wedge \alpha_i \wedge \phi) = 0$$

for all α_i such that $|\alpha_1 - \alpha_i| > 1$.

The only thing we need check then is that

$$w(\alpha_1 \mid \alpha_1 \wedge \phi) \geq w(\alpha_1 \mid \alpha_2 \wedge \phi)$$

(since $w(\alpha_1 \mid \alpha_1 \wedge \phi) \geq w(\alpha_1 \mid \alpha_3 \wedge \phi)$ and $w(\alpha_1 \mid \alpha_1 \wedge \phi) \geq w(\alpha_1 \mid \alpha_5 \wedge \phi)$ will follow similarly.) But this reduces to the following sum being non-negative

$$\begin{aligned} &18^{-1}\lambda^2(t_1+t_2+t_3)(t_4+t_5) + (1-\lambda)^2 w_{\langle 1,0,0,\dots,0\rangle}(\phi) w_{\langle 0,1,0,\dots,0\rangle}(\phi) \\ &+6^{-1}\lambda(1-\lambda)(w_{\langle 0,1,0,\dots,0\rangle}(\phi)(t_1+t_2+t_3+2t_4+2t_5) + t_1 w_{\langle 1,0,0,\dots,0\rangle}(\phi)), \end{aligned}$$

which it clearly is. □

Theorem 22. c^3_∞ *and the* $\lambda y_{\langle \frac{1}{2}, \frac{1}{2}, 0, \dots, 0\rangle} + (1-\lambda)c^3_0$ *for* $0 \leq \lambda \leq 1$ *are the only probability functions on* L_3 *satisfying Ex, Px, SN and* AP_H*.*

Proof. Suppose that w is a probability function on L_3 not of the above types that satisfies Ex, Px, SN and AP_H, and let μ is its de Finetti prior. Let w_2 be the restriction of w to L_2.Then w_2 is given by the measure μ_2 such that for any $A \subseteq \mathbb{D}_2$

$$\mu_2(A) = \mu\{\langle x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8\rangle \in \mathbb{D}_3 \mid \langle x_1+x_2, x_3+x_4, x_5+x_6, x_7+x_8\rangle \in A\}$$

By Theorem 19 there are three possibilities for w_2. In the first case

$$w_2 = y_{\langle a,b,b,a^{-1}b^2\rangle}$$

for some $a > b \geq 0$. Pick $\langle x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8 \rangle$ in the support of μ such that

$$\langle x_1 + x_2, x_3 + x_4, x_5 + x_6, x_7 + x_8 \rangle = \langle a, b, b, a^{-1}b^2 \rangle$$

By the same reasoning as that used in Theorem 19 (for L_3 instead of L_2) we can assume that μ is invariant under permutations from $\mathcal{P}_3$, so in fact we have a series of possible equations:

$$\left.\begin{array}{l} \langle x_1 + x_3, x_2 + x_4, x_5 + x_7, x_6 + x_8 \rangle, \\ \langle x_1 + x_5, x_3 + x_7, x_2 + x_6, x_4 + x_8 \rangle \end{array}\right\} = \left\{\begin{array}{l} \langle a, b, b, a^{-1}b^2 \rangle, \\ \text{or } \langle b, a, a^{-1}b^2, b \rangle, \\ \text{or } \langle b, a^{-1}b^2, a, b \rangle, \\ \text{or } \langle a^{-1}b^2, b, b, a \rangle, \end{array}\right.$$

By considering cases[1] we find that the only possible solutions to such a system of equations are:

$$\langle x_1, a - x_1, a - x_1, b - a + x_1, a - x_1, b - a + x_1, b - a + x_1, a^{-1}b^2 + a - b - x_1 \rangle,$$

$$\langle x_1, a - x_1, b - x_1, x_1, b - x_1, x_1, a^{-1}b^2 - b + x_1, b - x_1 \rangle.$$

The second of these is actually the same as the first after applying the permutation in $\mathcal{P}_3$ which transposes x_1 with x_2, x_3 with x_4, x_5 with x_6 and x_7 with x_8 so it is enough to consider just the first of these.

For readability, let this point be denoted

$$\langle \alpha, \beta, \beta, \gamma, \beta, \gamma, \gamma, \delta \rangle.$$

Using Corollary 18 for L_3, the points

$$\langle \beta, \gamma, \alpha, \beta, \gamma, \delta, \beta, \gamma \rangle, \quad \langle \gamma, \beta, \beta, \alpha, \delta, \gamma, \gamma, \beta \rangle$$

and appropriate past evidence $(\phi_n \vee \psi_n)$, AP_H requires that

$$\lim_{n \to \infty} w(\alpha_1 \,|\, \alpha_2 \wedge (\phi_n \vee \psi_n)) \geq \lim_{n \to \infty} w(\alpha_1 \,|\, \alpha_7 \wedge (\phi_n \vee \psi_n)),$$

[1] See Appendix A for details.

which gives the inequality

$$\frac{(\beta\gamma+\lambda\gamma\beta)}{\gamma+\lambda\beta} \geq \frac{\beta^2+\lambda\gamma^2}{\beta+\lambda\gamma}$$

for some $\lambda > 0$. This simplifies to

$$\lambda(\gamma^2-\beta^2)(\beta-\gamma) \geq 0$$

and so $\gamma = \beta$.

Replacing γ by β throughout, by the same reasoning, with the same points and $(\phi_n \vee \psi_n)$, AP_H requires that

$$\lim_{n\to\infty} w(\alpha_3 \,|\, \alpha_4 \wedge (\phi_n \vee \psi_n)) \geq \lim_{n\to\infty} w(\alpha_3 \,|\, \alpha_2 \wedge (\phi_n \vee \psi_n))$$

which gives the inequality

$$\frac{(\alpha\beta+\lambda\beta\alpha)}{\beta+\lambda\alpha} \geq \frac{\alpha\beta+\lambda\beta^2}{\beta+\lambda\beta}$$

which simplifies to

$$-\lambda\beta(\alpha-\beta)^2 \geq 0$$

This is impossible unless $\alpha = \beta$, contradicting $a > b$, or $\beta = 0$, contradicting the assumption that w is not c_0^3.

In the second case,

$$w_2 = \lambda y_{\langle a,a,b,b\rangle} + (1-\lambda)c_\infty^2$$

for some $a > b \geq 0$. We again get a series of equations, which we may take to be

$$\langle x_1+x_2, x_3+x_4, x_5+x_6, x_7+x_8\rangle = \langle a,a,b,b\rangle$$

together with

$$\left.\begin{array}{l}\langle x_1+x_3, x_2+x_4, x_5+x_7, x_6+x_8\rangle, \\ \langle x_1+x_5, x_3+x_7, x_2+x_6, x_4+x_8\rangle\end{array}\right\} = \left\{\begin{array}{l}\langle a,a,b,b\rangle, \\ \text{or } \langle a,b,a,b\rangle, \\ \text{or } \langle b,a,b,a\rangle, \\ \text{or } \langle b,b,a,a\rangle, \\ \text{or } \langle \frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4}\rangle,\end{array}\right.$$

since taking the vector $\langle \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4} \rangle$ in all three cases would yield the solution

$$\langle x, \tfrac{1}{4} - x, \tfrac{1}{4} - x, x, \tfrac{1}{4} - x, x, x, \tfrac{1}{4} - x \rangle$$

and the usual argument for this point and

$$\langle \tfrac{1}{4} - x, x, x, \tfrac{1}{4} - x, x, \tfrac{1}{4} - x, \tfrac{1}{4} - x, x \rangle$$

with $w(\alpha_1 \,|\, \alpha_2 \wedge (\phi_n \vee \psi_n))$ and $w(\alpha_1 \,|\, \alpha_4 \wedge (\phi_n \vee \psi_n))$ forces $x = 1/8$, and hence forces $w = c^3_\infty$.

Again one can check[2] that the only possible solution (up to a permutation from $\mathcal{P}_3$) to such a system of equations is:

$$\langle x_1, a - x_1, a - x_1, x_1, \tfrac{1}{4} - x_1, \tfrac{1}{4} - a + x_1, \tfrac{1}{4} - a + x_1, \tfrac{1}{4} - x_1 \rangle.$$

Let this be denoted

$$\langle \alpha, \beta, \beta, \alpha, \gamma, \delta, \delta, \gamma \rangle$$

Notice that α, γ cannot both be zero.

Using Corollary 18 for L_3, the points

$$\langle \gamma, \alpha, \delta, \beta, \delta, \beta, \gamma, \alpha \rangle, \langle \alpha, \gamma, \beta, \delta, \beta, \delta, \alpha, \gamma \rangle$$

and appropriate past evidence $(\phi_n \vee \psi_n)$, AP_H requires that

$$\lim_{n\to\infty} w(\alpha_1 \,|\, \alpha_2 \wedge (\phi_n \vee \psi_n)) \geq \lim_{n\to\infty} w(\alpha_1 \,|\, \alpha_7 \wedge (\phi_n \vee \psi_n))$$

which gives the inequality

$$\frac{\gamma\alpha + \lambda\alpha\gamma}{\alpha + \lambda\gamma} \geq \frac{\gamma^2 + \lambda\alpha^2}{\gamma + \lambda\alpha}$$

for some $\lambda > 0$, which simplifies to

$$-\lambda(\alpha - \gamma)(\alpha^2 - \gamma^2) \geq 0$$

so $\alpha = \gamma$ which gives $x_1 = \frac{1}{4} - x_1$ and so $\alpha = x_1 = \frac{1}{8}$. Note that this means that $\beta > \alpha > \delta$, since $a > 1/4$.

[2]Details may be found in Appendix A.

By the same reasoning, using the same points and replacing γ by α throughout, AP_H requires that

$$\lim_{n\to\infty} w(\alpha_3 \,|\, \alpha_4 \wedge (\phi_n \vee \psi_n)) \geq \lim_{n\to\infty} w(\alpha_3 \,|\, \alpha_2 \wedge (\phi_n \vee \psi_n))$$

which gives the inequality

$$\frac{\delta\beta + \lambda\beta\delta}{\beta + \lambda\delta} \geq \frac{\alpha\delta + \lambda\beta\alpha}{\alpha + \lambda\alpha}$$

for some $\lambda > 0$, which simplifies to

$$-\lambda\alpha(\beta - \delta)^2 \geq 0$$

which is impossible by our previous observation that $\beta > \alpha > \delta$.

In the third and final case,

$$w = \lambda y_{\langle 1,0,0,0\rangle} + (1-\lambda) y_{\langle \frac{1}{2},\frac{1}{2},0,0\rangle}$$

It is easy to see that the only functions whose restriction to L_2 result in such a function are those of the form

$$\lambda y_{\langle \frac{1}{2},\frac{1}{2},0,0,0,0,0,0\rangle} + (1-\lambda)c_0^3, \quad 0 \leq \lambda \leq 1,$$

details of this may be found in Appendix A. □

Having determined the probability functions on L_3 which satisfy Ex, Px, SN and AP_H we are now in a position to do the same for L_k when $k \geq 3$. To this end let $y^k_{\langle \frac{1}{2},\frac{1}{2},0,\ldots,0\rangle}$, where there are $2^k - 2$ zeros, be the obvious analog of $y_{\langle \frac{1}{2},\frac{1}{2},0,0,0,0,0,,0\rangle}$ on L_3. Then by the direct generalization of the methods in proof of Theorem 22 we obtain:

Corollary 23. *For $k \geq 3$ the only probability functions on L_k satisfying Ex, Px, SN and AP_H are c^k_∞ and those of the form $\lambda y^k_{\langle \frac{1}{2},\frac{1}{2},0,\ldots,0\rangle} + (1-\lambda)c^k_0$ for some $0 \leq \lambda \leq 1$. Furthermore, none of these satisfies SAP_H.*

4.3 Weakening the background conditions

Despite the rather natural link between SN and Hamming Distance, one way to weaken the apparently very strong condition Ex+SN+Px+SAP$_H$ would be to replace SN by WN. Of course WN follows straight-forwardly from SN, so the $y_{\vec{x}}$ probability functions will satisfy WN, but we would hope to have more functions satisfying SAP$_H$ if we replace SN with WN. In fact this is not such an unnatural picture. We initially thought of the atoms of L_2 as the vertices of a wheel of fortune. In that case, not only did the lengths of the sides between vertices correspond to Hamming Distance, but the symmetries of the wheel of fortune corresponded exactly to the permutations of atoms given by Px + SN. In the following diagram, lengths of sides still represent Hamming Distance, but the symmetries correspond to the permutations of atoms given by Px + WN.

$$\begin{tikzcd}
& \alpha_1 \arrow[dl, dash] \arrow[dr, dash] & \\
\alpha_2 \arrow[dr, dash] & & \alpha_3 \arrow[dl, dash] \\
& \alpha_4 &
\end{tikzcd}$$

The four symmetries of the diamond correspond to the four permutations of atoms given by (14)(23), (14)(2)(3), (1)(4)(23), and (1)(2)(3)(4), which are exactly those licensed by Px and WN. Define the function $z_{\langle a,b,c,d\rangle}$ to be

$$4^{-1}\left(w_{\langle a,b,c,d\rangle} + w_{\langle d,b,c,a\rangle} + w_{\langle a,c,b,d\rangle} + w_{\langle d,c,b,a\rangle}\right)$$

We then have the following:

Lemma 24. *If w satisfies Px, WN and AP$_H$ on L_2, then for any point $\langle a,b,c,d\rangle$ in the support of its de Finetti prior μ with $a \geq d$ and $b \geq c$, at least one of the following holds:*

1. $a = d$
2. $b = c$
3. $ac = bd$

Proof. If $\langle a,b,c,d\rangle$ is in the support of μ then so is $\langle d,c,b,a\rangle$, so by Corollary 18 we can find state descriptions ϕ_n, ψ_n such that for some $\lambda > 0$,

$$\lim_{n\to\infty} w\left(\alpha_1^{j_1}\alpha_2^{j_2}\alpha_3^{j_3}\alpha_4^{j_4} \mid \phi_n \vee \psi_n\right) = (1+\lambda)^{-1}\left(a^{j_1}b^{j_2}c^{j_3}d^{j_4} + \lambda d^{j_1}c^{j_2}b^{j_3}a^{j_4}\right)$$

Taking $j_2 = j_3 = j_4 = 0$ and the cases $j_1 = 2$ and $j_1 = 1$ we get that

$$\lim_{n\to\infty} w(\alpha_1 \mid \alpha_1 \wedge (\phi_n \vee \psi_n)) = \frac{a^2 + \lambda d^2}{a + \lambda d}$$

and taking $j_3 = j_4 = 0$ and the cases $j_1 = j_2 = 1$ and $j_1 = 0, j_2 = 1$ we get that

$$\lim_{n\to\infty} w(\alpha_1 \mid \alpha_2 \wedge (\phi_n \vee \psi_n)) = \frac{ab + \lambda dc}{b + \lambda c}$$

AP_H requires that $w(\alpha_1 \mid \alpha_1 \wedge \theta) \geq w(\alpha_1 \mid \alpha_2 \wedge \theta)$ for any $\theta \in SL$; so, in particular, $w(\alpha_1 \mid \alpha_1 \wedge (\phi_n \vee \psi_n)) \geq w(\alpha_1 \mid \alpha_2 \wedge (\phi_n \vee \psi_n))$ for all n. This means that we cannot have

$$\lim_{n\to\infty} w(\alpha_1 \mid \alpha_1 \wedge (\phi_n \vee \psi_n)) < \lim_{n\to\infty} w(\alpha_1 \mid \alpha_2 \wedge (\phi_n \vee \psi_n))$$

and so we must have

$$\frac{a^2 + \lambda d^2}{a + \lambda d} \geq \frac{ab + \lambda dc}{b + \lambda c}$$

which simplifies to

$$\lambda(ac - bd)(a - d) \geq 0.$$

Hence

$$(ac - bd)(a - d) \geq 0. \tag{4.15}$$

Using the same points and taking $j_1 = j_3 = j_4 = 0$ with cases $j_2 = 2$ and $j_2 = 1$ we get

$$\lim_{n\to\infty} w(\alpha_2 \mid \alpha_2 \wedge (\phi_n \vee \psi_n)) = \frac{b^2 + \lambda c^2}{b + \lambda c}$$

and taking $j_3 = j_4 = 0$ with cases $j_2 = j_1 = 1$ and $j_2 = 0, j_1 = 1$ we get that

$$\lim_{n\to\infty} w(\alpha_2 \mid \alpha_1 \wedge (\phi_n \vee \psi_n)) = \frac{ba + \lambda cd}{a + \lambda d}.$$

AP_H then requires

$$\frac{b^2 + \lambda c^2}{b + \lambda c} \geq \frac{ba + \lambda cd}{a + \lambda d}$$

which simplifies to

$$\lambda(bd - ac)(b - c) \geq 0$$

and so we must have

$$(bd - ac)(b - c) \geq 0. \tag{4.16}$$

Either 1. holds, in which case both 4.15 and 4.16 hold, or $a > d$ in which case by 4.1 we must have $ac \geq bd$. Then for 4.16 to hold, either $ac = bd$ or $b = c$ and so either 3. or 2. holds, respectively.

□

Theorem 25. *The only functions satisfying Px + WN + SAP$_H$ on L_2 are the following:*

1. $\lambda z_{\langle a,b,b,c\rangle} + (1-\lambda)z_{\langle \alpha,\beta,\gamma,\alpha\rangle}$, *with* $\beta > \gamma$, $a > c$, $a\gamma = b\alpha = \beta c$ *and* $0 < \lambda < 1$.

2. $\lambda z_{\langle a,b,c,d\rangle} + (1-\lambda)z_{\langle \alpha,\beta,\beta,\alpha\rangle}$ *with* $a > d, b > c, ac = bd, \alpha b = a\beta, \beta d = \alpha c$, *and* $0 < \lambda \leq 1$.

Proof. Any probability function satisfying Px + WN must be of the form

$$\int_{\mathbb{D}_{2^q}} z_{\vec{x}} d\mu(\vec{x}).$$

Moreover, there must be at least one point $\langle a, b, c, d\rangle$ in the support of μ with a, b, c, d non-zero and $a \neq d$. For otherwise we would have

$$w(\alpha_1 \,|\, \alpha_1 \wedge \alpha_1 \wedge \alpha_2 \wedge \alpha_3 \wedge \alpha_4) = w(\alpha_1 \,|\, \alpha_4 \wedge \alpha_1 \wedge \alpha_2 \wedge \alpha_3 \wedge \alpha_4),$$

contradicting SAP$_H$. By Px and WN, $\langle a, c, b, d\rangle$, $\langle d, b, c, a\rangle$ and $\langle d, c, b, a\rangle$ must also be in the support of μ, so we can assume without loss of generality that $a > d$ and $b \geq c$. Similarly, there must be at least one point $\langle \alpha, \beta, \gamma, \delta\rangle$ in the support of μ with $\alpha, \beta, \gamma, \delta$ non-zero and $\beta \neq \gamma$, for otherwise we would have

$$w(\alpha_2 \,|\, \alpha_2 \wedge \alpha_1 \wedge \alpha_2 \wedge \alpha_3 \wedge \alpha_4) = w(\alpha_2 \,|\, \alpha_3 \wedge \alpha_1 \wedge \alpha_2 \wedge \alpha_3 \wedge \alpha_4),$$

contradicting SAP$_H$. By Px and WN, we can assume without loss of generality that $\beta > \gamma$ and $\alpha \geq \delta$. Then by Lemma 24, we know that we have the following four options:

	$b=c$	$ac=bd$
$\alpha=\delta$	(i)	(ii)
$\alpha\gamma=\beta\delta$	(iii)	(iv)

Moreover, using Corollary 18, all pairs of points with the first taken from

$$\{\langle a,b,c,d\rangle,\langle a,c,b,d\rangle,\langle d,b,c,a\rangle,\langle d,c,b,a\rangle\}$$

and the second from

$$\{\langle \alpha,\beta,\gamma,\delta\rangle,\langle \alpha,\gamma,\beta,\delta\rangle,\langle \delta,\beta,\gamma,\alpha\rangle\ \langle \delta,\gamma,\beta,\alpha\}$$

and all appropriate choices of j_i, SAP_H requires that all of the following expressions must be non-negative.

$(a-\alpha)(a\beta-\alpha c)$	$(a-\alpha)(a\gamma-\alpha b)$	$(a-\alpha)(b\delta-\gamma d)$	$(a-\alpha)(c\delta-\beta d)$
$(a-\delta)(a\beta-b\delta)$	$(a-\delta)(a\beta-c\delta)$	$(a-\delta)(a\gamma-b\delta)$	$(a-\delta)(a\gamma-c\delta)$
$(a-\delta)(\alpha b-\beta d)$	$(a-\delta)(\alpha b-\gamma d)$	$(a-\delta)(\alpha c-\beta d)$	$(a-\delta)(\alpha c-\gamma d)$
$(\alpha-d)(a\beta-b\delta)$	$(\alpha-d)(a\beta-c\delta)$	$(\alpha-d)(a\gamma-b\delta)$	$(\alpha-d)(a\gamma-c\delta)$
$(\alpha-d)(\alpha b-\beta d)$	$(\alpha-d)(\alpha b-\gamma d)$	$(\alpha-d)(\alpha c-\beta d)$	$(\alpha-d)(\alpha c-\gamma d)$
$(\delta-d)(a\beta-\alpha c)$	$(\delta-d)(a\gamma-\alpha b)$	$(\delta-d)(b\delta-\gamma d)$	$(\delta-d)(c\delta-\beta d)$
$(b-\beta)(\gamma d-\alpha c)$	$(b-\beta)(\alpha b-\beta d)$	$(b-\beta)(a\gamma-c\delta)$	$(b-\beta)(b\delta-a\beta)$
$(b-\gamma)(\beta d-c\delta)$	$(b-\gamma)(\beta d-\alpha c)$	$(b-\gamma)(a\beta-c\delta)$	$(b-\gamma)(a\beta-\alpha c)$
$(b-\gamma)(b\delta-\gamma d)$	$(b-\gamma)(b\delta-a\gamma)$	$(b-\gamma)(\alpha b-\gamma d)$	$(b-\gamma)(\alpha b-a\gamma)$
$(\beta-c)(\beta d-c\delta)$	$(\beta-c)(\beta d-\alpha c)$	$(\beta-c)(a\beta-c\delta)$	$(\beta-c)(a\beta-\alpha c)$
$(\beta-c)(b\delta-a\gamma)$	$(\beta-c)(b\delta-a\gamma)$	$(\beta-c)(\alpha b-\gamma d)$	$(\beta-c)(\alpha b-a\gamma)$
$(\gamma-c)(\gamma d-\alpha c)$	$(\gamma-c)(\alpha b-\beta d)$	$(\gamma-c)(a\gamma-c\delta)$	$(\gamma-c)(b\delta-a\beta)$

Given these constraints, option (i) forces $a>\alpha=\delta>d$, $\beta>b=c>\gamma$ and $a\gamma=b\alpha=\beta d$. Derivation of this can be found in Appendix B, and that all constraints are satisfied under these conditions can be seen by inspection of the above list. Options (ii) and (iii) are unsatisfiable, and option (iv) forces $a=\alpha$, $b=\beta$, $c=\gamma$ and $d=\delta$[3] .We now have two options to consider. Either

(a) $\langle a,b,b,c\rangle$, $\langle \alpha,\beta,\gamma,\alpha\rangle$ belong to the support of μ, with the condition that $a>c$, $\beta>\gamma$ and $a\gamma=b\alpha=\beta c$. Then any further points in the support of μ must satisfy either $x_1=x_4$ or $x_2=x_3$ or both.

[3]Details may be found in Appendix B

(b) $\langle a, b, c, d\rangle$ with $a > d$, $b > c$ and $ac = bd$ belongs to the support of μ, and any further points in the support of μ are of the form $\langle x, y, y, x\rangle$.

In case (a), that the probability function must be of the form $\lambda z_{\langle a,b,b,c\rangle} + (1 - \lambda)z_{\langle \alpha,\beta,\gamma,\alpha\rangle}$ can be seen as follows. Suppose that there was another point $\langle x, y, y, z\rangle$ in the support of μ. If x, y and z are all non-zero, then as well as the conditions $a\gamma = b\alpha = \beta c$, we would have the conditions $x\gamma = y\alpha = \beta z$. But then $\gamma(a-x) = \alpha(b - y) = \beta(c - z)$. By assumption, α, β and γ are all non-zero, and so this forces $a = x, b = y, c = z$.

On the other hand, if $x = 0$, then $(b-y)(xb-yd) \geq 0$ entails $y \geq b$ or $y = 0$, and $(b - y)(yd - xc) \geq 0$ entails $b \geq y$ or $y = 0$. If $b = y$, then $(y - c)(xb - yd) \geq 0$ becomes $-yd(b - c) \geq 0$, which is a contradiction. So the only consistent option is that $y = 0$. In this case, $(y - c)(bz - ay) \geq 0$ becomes $-cbz \geq 0$, implying $z = 0$. But then $x + 2y + z = 0$, which is a contradiction. If we start from the assumption that $z = 0$, then $(b-y)(bz-ay) \geq 0$ and $(b-y)(ay-cz) \geq 0$ jointly entail either $b = y$ or $y = 0$, and the same contradiction results. If we start from the assumption that $y = 0$, then as already seen, $(y - c)(bz - ay) \geq 0$ becomes $-cbz \geq 0$, implying $z = 0$; also $(y-c)(xb-yd) \geq 0$ becomes $-bxc \geq 0$, implying $x = 0$ and again we have a contradiction.

The assumption of another point $\langle x, y, z, x\rangle$ in the support of μ distinct from $\langle \alpha, \beta, \gamma, \alpha\rangle$ results in a similar contradiction.

In case (b), either the function is simply $z_{\langle a,b,c,d\rangle}$ or the support of μ includes another point of the form $\langle \alpha, \beta, \beta, \alpha\rangle$. Suppose firstly that $\alpha > a$. Then from the requirement that $(a - \alpha)(a\beta - \alpha c) \geq 0$, we must have $\alpha c \geq a\beta \geq \beta d$ and from the requirement that $(a - \alpha)(c\delta - \beta d) \geq 0$ we have that $\beta d \geq c\delta = c\alpha$, hence $\alpha c = a\beta = \beta d$. Since $a > d$, this forces $\beta = 0$ and $\alpha = 1/2$, and hence $c = 0$. But $c \neq 0$ by assumption, so this case is ruled out.

So now suppose that $\alpha = a$. Then $\alpha c = ac = bd$, so $(b-\beta)(\gamma d-\alpha c) \geq 0$ becomes $(b - \beta)(\beta d - bd) \geq 0$ and hence either $d = 0$ or $b = \beta$. In the latter case, then $\alpha = a > d$ and $\beta = b > c$, contradicting the fact that $2\alpha+2\beta = 1 = a+b+c+d$. In the former case, $d = 0$, we have $\alpha c = bd = 0$, so either $\alpha = a = 0$ or $c = 0$, in contradiction of our inital assumptions.

So the only remaining possibility is that $a > \alpha$. In this case, $bd = ac > \alpha c$, and from $(a - \alpha)(\alpha c - \beta d) \geq 0$ we know that $\alpha c \geq \beta d$, hence $bd > \beta d$ and so $b > \beta$.

Then $(b-\beta)(b\delta - a\beta) \geq 0$ and $(a-\alpha)(a\gamma - \alpha b) \geq 0$ together force $\alpha b = a\beta$, and $(a-\alpha)(c\delta - \beta d) \geq 0$ and $(b-\beta)(\gamma d - \alpha c) \geq 0$ together force $\beta d = \alpha c$. Observation of the above list of constraints shows that under these conditions all are satisfied.

Finally, suppose that there were another point $\langle x, y, y, x\rangle$ in the support of μ, with $x \neq \alpha$; without loss of generality let $x > \alpha$, so $y < \beta$. Then by the same reasoning as above, we would have to have $ay = xb$, and thus $b(\alpha - x) = a(\beta - y)$, but since a and b are both positive, this is a contradiction. Hence case (b) amounts to item 2. in the statement of the theorem.

□

Notice that in the event that the probability function is of the first form, $\lambda z_{\langle a,b,b,c\rangle} + (1-\lambda) z_{\langle \alpha,\beta,\gamma,\alpha\rangle}$, if $\lambda = 2^{-1}$, $\alpha = b, \beta = a$ and $\gamma = c$, then it satisfies SN and is as in Theorem 19. In the event that the probability function is of the second form, $\lambda z_{\langle a,b,c,d\rangle} + (1-\lambda) z_{\langle \alpha,\beta,\beta,\alpha\rangle}$ and $\alpha = \beta$, $a = c$, $b = d$, then it satisfies SN and again is as in Theorem 19.
We note here that we do not have an analogue of Theorem 22 for these weaker background conditions, because the proof of that theorem depends on Proposition 12 which depends on SN. One suggestion for future research is to determine whether Proposition 12 can be extended to probability functions not satisfying SN.

Given that any picture of Hamming Distance on L_2 must have symmetries corresponding to permutations licensed by Px and WN, it does not make sense to weaken the background conditions any further. Instead we move on in the next section to look at the weaker principle SDAP_H, which is much more widely satisfied.

4.4 State Description Analogy for Hamming Distance

Having looked at AP_H and SAP_H, it might be objected that allowing the evidence conditioned on, θ, to be any sentence of $QFSL$ is too permissive. The proof that SAP_H (+Ex+Px+SN) is unsatisfiable in L_3 and larger languages depends essentially on our being able to condition on disjunctions of state descriptions, so perhaps there is something pathological about these cases. This is not obvious, since we can come up with plausible stories in which our past evidence would be a disjunction of this kind; for example, two experts give inconsistent accounts and we don't know which to choose from. However, Carnap's CA and the various other principles discussed in Chapter 2 only consider state descriptions as admissible past evidence, so we turn now to SDAP_H (as defined on page 35).

On L_1 we have only two atoms: P_1 and $\neg P_1$, hence as an immediate Corollary of Proposition 13, for any $>_S$, SDAP_S holds for any exchangeable function. It is easy to demonstrate examples of functions satisfying SDAP_H on L_2. Notice that any state description $\theta \in QFSL_2$ can be written as the conjunction of two state descriptions θ_1 and θ_2 on the languages $\{P_1\}$ and $\{P_2\}$, respectively. Then if w_1, w_2 are any two exchangeable functions on L_1, for any state description $\theta \in QFSL_2$ define a probability function $w_1 \times w_2$ by

$$(w_1 \times w_2)(\theta) := w_1(\theta_1)w_2(\theta_2).$$

In other words, if μ_1, μ_2 are the de Finetti priors of w_1, w_2 respectively, we have

$$\begin{aligned}
&(w_1 \times w_2)(\alpha_1^{n_1} \wedge \alpha_2^{n_2} \wedge \alpha_3^{n_3} \wedge \alpha_4^{n_4}) \\
&\quad = w_1(P_1^{n_1+n_2} \wedge \neg P_1^{n_3+n_4}) \times w_1(P_2^{n_1+n_3} \wedge \neg P_2^{n_2+n_4}) \\
&\quad = \int_{\mathbb{D}_1} x^{n_1+n_2}(1-x)^{n_3+n_4} d\mu_1(\vec{x}) \times \int_{\mathbb{D}_1} x^{n_1+n_3}(1-x)^{n_2+n_4} d\mu_2(\vec{x})
\end{aligned}$$

where α_i^m denotes m conjuncts of the form $\alpha_i(a_j)$.
Maher's $c_\lambda \times c_\lambda$ ([21]) was of this form, but clearly the construction works just as well for any two exchangeable functions on L_1, and that SDAP_H holds can be seen as follows:

For any state description $\theta = \alpha_1^{n_1}\alpha_2^{n_2}\alpha_3^{n_3}\alpha_4^{n_4}$, let $f_\theta(x)$ denote the function $x^{n_1+n_2}(1-x)^{n_3+n_4}$, $g_\theta(x)$ the function $x^{n_1+n_3}(1-x)^{n_2+n_4}$.

We have

$$\begin{aligned}
(w_1 \times w_2)(\alpha_1 \,|\, \alpha_2 \wedge \theta) &= \frac{\int_{\mathbb{D}_1} x^2 f_\theta(x) d\mu_1(\vec{x})}{\int_{\mathbb{D}_1} x f_\theta(x) d\mu_1(\vec{x})} \times \frac{\int_{\mathbb{D}_1} x(1-x) g_\theta(x) d\mu_1(\vec{x})}{\int_{\mathbb{D}_1} (1-x) g_\theta(x) d\mu_1(\vec{x})} \\
&\geq \frac{\int_{\mathbb{D}_1} x(1-x) f_\theta(x) d\mu_1(\vec{x})}{\int_{\mathbb{D}_1} (1-x) f_\theta(x) d\mu_1(\vec{x})} \times \frac{\int_{\mathbb{D}_1} x(1-x) g_\theta(x) d\mu_1(\vec{x})}{\int_{\mathbb{D}_1} (1-x) g_\theta(x) d\mu_1(\vec{x})} \\
&= (w_1 \times w_2)(\alpha_1 \,|\, \alpha_4 \wedge \theta)
\end{aligned}$$

And similarly for the other inequalities required by SDAP$_H$. Notice that $w_1 \times w_2$ will satisfy SN just if both w_1 and w_2 do, and will satisfy Px just if w_1 and w_2 are the same function. Clearly this kind of function construction can be extended to languages of any size, but unfortunately does not always generate functions satisfying SDAP$_H$ on larger languages. In fact, SDAP$_H$ will frequently fail even on L_3:

Proposition 26. *Suppose c_λ is a member of Carnap's Continuum on L_1. Then $w = c_\lambda \times c_\lambda \times c_\lambda$ does not satisfy SDAP$_H$.*

Proof. Suppose w is as above and let $\theta = \alpha_2^n$. Then

$$\begin{aligned} w(\alpha_1 \,|\, \alpha_2 \wedge \theta) &= c_\lambda(P_1 \,|\, P_1^{n+1}) \times c_\lambda(P_2 \,|\, P_2^{n+1}) \times c_\lambda(P_3 \,|\, \neg P_3^{n+1}) \\ &= \frac{(n+1+\lambda/2)(n+1+\lambda/2)(\lambda/2)}{(n+1+\lambda)^3} \end{aligned}$$

And

$$\begin{aligned} w(\alpha_1 \,|\, \alpha_7 \wedge \theta) &= c_\lambda(P_1 \,|\, \neg P_1 \wedge P_1^n) \times c_\lambda(P_2 \,|\, \neg P_2 \wedge P_2^n) \times c_\lambda(P_3 \,|\, P_3 \wedge \neg P_3^n) \\ &= \frac{(n+\lambda/2)(n+\lambda/2)(1+\lambda/2)}{(n+1+\lambda)^3} \end{aligned}$$

Multiplying out and simplifying we get that

$$w(\alpha_1 \,|\, \alpha_2 \wedge \theta) < w(\alpha_1 \,|\, \alpha_7 \wedge \theta)$$

is equivalent to

$$\lambda(2+\lambda) < 4n^2$$

which is clearly satisfiable for any λ by taking a large enough n. □

In order to completely classify the functions that satisfy SDAP$_H$, we begin by considering functions on L_2.

Our first conjecture was that any w on L_2 satisfying SDAP$_H$ would be a product of two functions on L_1 (indeed the square of a single function if Px is to hold). However this is clearly not the case, for we already know that $y_{\langle a,a,b,b\rangle}$ satisfies AP$_H$ and hence SDAP$_H$ for any $0 \leq a, b \leq 1$, $a+b = 1/2$, but that this cannot be a product is seen as follows: The de Finetti measure of $y_{\langle a,a,b,b\rangle}$, μ, gives positive measure to the point

$$\langle a, a, b, b\rangle = \left\langle \frac{x}{2(x+y)}, \frac{x}{2(x+y)}, \frac{y}{2(x+y)}, \frac{y}{2(x+y)} \right\rangle.$$

Hence if there were a function on L_1 that squared to give $y_{\langle a,a,b,b,\rangle}$, its de Finetti measure would have in its support the points

$$\langle \frac{x}{x+y}, \frac{y}{x+y} \rangle$$

and

$$\langle 1/2, 1/2\rangle.$$

But then μ would also have in its support

$$\langle 1/2, 1/2\rangle \times \langle 1/2, 1/2\rangle = \langle 1/4, 1/4, 1/4, 1/4\rangle$$

and

$$\left\langle \frac{x}{x+y}, \frac{y}{x+y} \right\rangle \times \left\langle \frac{x}{x+y}, \frac{y}{x+y} \right\rangle = \left\langle \frac{x^2}{(x+y)^2}, \frac{xy}{(x+y)^2}, \frac{xy}{(x+y)^2}, \frac{y^2}{(x+y)^2} \right\rangle,$$

which is not the case.

All the functions satisfying AP_H have the property that for any point $\langle a, b, c, d\rangle$ in the support of their de Finetti measures, $ad = bc$. This is also a property of any product function. A weaker conjecture would therefore be that that all functions satisfying SDAP_H must have this property. Notice that this could only be a necessary, but not a sufficient condition for satisfying SDAP_H, as the following example demonstrates.

Let

$$\vec{z_1} := \left\langle \frac{36}{49}, \frac{6}{49}, \frac{6}{49}, \frac{1}{49} \right\rangle$$

and

$$\vec{z_2} := \left\langle \frac{1600}{1682}, \frac{40}{1681}, \frac{40}{1681}, \frac{1}{1681} \right\rangle$$

and consider the probability function $2^{-1}(y_{\vec{z_1}} + y_{\vec{z_2}})$

Then every point $\langle a, b, c, d\rangle$ in the support of the corresponding de Finetti measure has the property that $ad = bc$ but, for example,

$$2^{-1}(y_{\vec{z_1}} + y_{\vec{z_2}})(\alpha_1 \mid \alpha_2 \wedge \bigwedge_{i=1}^{5} \alpha_2) \approx 0.04$$

while

$$2^{-1}(y_{\vec{z_1}} + y_{\vec{z_2}})(\alpha_1 \mid \alpha_4 \wedge \bigwedge_{i=1}^{5} \alpha_2) \approx 0.08.^4 \tag{4.17}$$

We do have the limited result that if $ad = bc$ holds for all points in the support of μ, the corresponding probability function satisfies SDAP_H as long as the state description conditioned on is sufficiently small.

Proposition 27. *If w is a probability function on L_2 satisfying Ex, Px and SN and every point $\langle a, b, c, d\rangle$ in the support of its de Finetti measure μ is such that $ad = bc$, then w satisfies:*

(i) $w(\alpha_1 \mid \alpha_1) > w(\alpha_1 \mid \alpha_2)$

(ii) $w(\alpha_1 \mid \alpha_2) > w(\alpha_1 \mid \alpha_4)$

Proof. By Px and SN,

$$w = \int_{\mathbb{D}_2} y_{\vec{x}} \, d\mu(\vec{x})$$

and $w(\alpha_1) = w(\alpha_2) = w(\alpha_3) = w(\alpha_4)$. In fact, (i) rests only on Px and SN and does not require the property that $ad = bc$. Since $w(\alpha_1) = w(\alpha_2)$, it is sufficient to show that

$$w(\alpha_1 \wedge \alpha_1) > w(\alpha_1 \wedge \alpha_2)$$

which is equivalent to

$$\int_{\mathbb{D}_2} \left(2a^2 + 2b^2 + 2c^2 + 2d^2\right) d\mu(\vec{x}) - \int_{\mathbb{D}_2} (2ab + 2ac + 2bd + 2cd) \, d\mu(\vec{x}) > 0$$

Simplifying the left hand side gives

$$\int_{\mathbb{D}_2} \left[(a-b)^2 + (a-c)^2 + (b-d)^2 + (c-d)^2\right] d\mu(\vec{x}) > 0$$

which is clearly non-negative, and indeed positive unless $w = c_\infty$.

[4] See Appendix A for calculations.

For inequality (ii), we need to show that

$$w(\alpha_1 \wedge \alpha_2) > w(\alpha_1 \wedge \alpha_4)$$

Equivalently,

$$\int_{\mathbb{D}_2} (2ab + 2ac + 2bd + 2cd)\ d\mu(\vec{x}) - \int_{\mathbb{D}_2} (4ad + 4bc)\ d\mu(\vec{x}) > 0$$

Simplifying, this gives

$$\int_{\mathbb{D}_2} 2(a-c)(b-d) + 2(a-b)(c-d)\, d\mu(\vec{x}) > 0$$

We can assume without loss of generality that $a \geq b, c, d$ and, unless $w = c_\infty$, $a > b$ for at least one point $\langle a, b, c, d\rangle$. Since we are assuming that $ad = bc$, we then also have $c > d$ for this point. The inequality then holds as required.

□

Proposition 27 thus demonstrates a class of probability functions displaying the limited properties of analogy that Skyrms shows to be satisfiable in [34].

If we restrict our attention to single $y_{\vec{x}}$ functions, we can prove that $ad = bc$ is a necessary and sufficient condition for SDAP_H to hold in the presence of Ex + Px + SN.

We will need the following Lemma.

Lemma 28. *Suppose* $1 > a_0 > a_1, a_2, ...a_k > 0$. *Then for some* $n \in \mathbb{N}$,

$$a_0^n > \sum_{i=1}^{k} a_i^n$$

Proof. First consider the case when $k = 2$. We can suppose without loss of generality that $a_1 > a_2$. Note that this means $\dfrac{a_0}{a_1} > 1 > \dfrac{a_2}{a_1}$.
So clearly if we take n to be sufficiently large, we get that

$$\left(\frac{a_0}{a_1}\right)^n > \left(\frac{a_2}{a_1}\right)^n + 1$$

But this is equivalent to $a_0^n > a_2^n + a_1^n$ and so the statement of the lemma holds.

Next suppose the statement holds for $k-1$, so $a_0^n > \sum_{i=1}^{k-1} a_i^n$ for some n.

Since $1 > a_0 > a_k$, we have that $a_k^n < a_k < a_0$. By similar reasoning to the first case, for sufficiently large m we get

$$(a_o^n)^m > \left(\sum_{i=1}^{k-1} a_i^n\right)^m + (a_k^n)^m > \sum_{i=1}^{k} a_i^{nm}$$

Hence the lemma holds for all k.

□

Proposition 29. *If $y_{\langle a,b,c,d\rangle}$ satisfies Ex, Px, SN and SDAP$_H$, then $ad = bc$.*

Proof. We prove this by supposing that $ad \neq bc$ and showing that this leads to a failure of SDAP$_H$. We can assume without loss of generality that $a \geq b, c, d$, with at least one inequality strict, and $b \geq c$, since such a point is obtainable from any $\langle x_1, x_2, x_3, x_4\rangle$ by a permutation from $\mathcal{P}_4$.. Assume firstly that $ad > bc$ (the case in which $ad < bc$ is similar).

Recall that SDAP$_H$ requires that for any state description θ,

$$y_{\langle a,b,c,d\rangle}(\alpha_1 \mid \alpha_2 \wedge \theta) > y_{\langle a,b,c,d\rangle}(\alpha_1 \mid \alpha_4 \wedge \theta)$$

As in Proposition 51, let $N_1 = w_{\langle a,b,c,d\rangle}(\phi)$, $N_2 = w_{\langle a,c,b,d\rangle}(\phi)$, $E_1 = w_{\langle b,a,d,c\rangle}(\phi)$, $E_2 = w_{\langle c,a,d,b\rangle}(\phi)$, $W_1 = w_{\langle b,d,a,c\rangle}(\phi)$, $W_2 = w_{\langle c,d,a,b\rangle}(\phi)$, $S_1 = w_{\langle d,b,c,a\rangle}(\phi)$, $S_2 = w_{\langle d,c,b,a\rangle}(\phi)$.

Writing out the inequality above using the definition of $y_{\langle a,b,c,d\rangle}$, we obtain

$$\frac{abN_1 + acN_2 + baE_1 + caE_2 + bdW_1 + cdW_2 + dbS_1 + dcS_2}{bN_1 + cN_2 + aE_1 + aE_2 + dW_1 + dW_2 + bS_1 + cS_2}$$
$$> \frac{adN_1 + adN_2 + bcE_1 + cbE_2 + bcW_1 + cbW_2 + daS_1 + daS_2}{dN_1 + dN_2 + bE_1 + cE_2 + cW_1 + bW_2 + aS_1 + aS_2}$$

which we can multiply out to get

$N_1E_1(abc + bad - a^2d - b^2c) + N_1E_2(ab^2 + cad - a^2d - cb^2) + N_1W_1(abc + bd^2 - ad^2 - b^2c) + N_1W_2(ab^2 + cd^2 - ad^2 - b^2c) + N_1S_1(a^2b + d^2b - adb - dab) + N_1S_2(a^2b + d^2c - adc - dab) + N_2E_1(ac^2 + bad - a^2d - bc^2) + N_2E_2(cad + acb - a^2d - c^2b) + N_2W_1(ac^2 + bd^2 - ad^2 - bc^2) + N_2W_2(acb + cd^2 - ad^2 - bc^2) + N_2S_1(a^2c + d^2b - adb - dac) + N_2S_2(a^2c + d^2c - 2adc) + E_1W_1(b^2a + c^2a - 2abc) + E_1W_2(b^2a + c^2d - bcd - bca) + E_1S_1(ba^2 + dbc - b^2c - da^2) + E_1S_2(ba^2 + dc^2 - bc^2 - da^2) + E_2W_1(c^2a +$

$b^2d - cbd - bca) + E_2S_1(ca^2 + db^2 - cb^2 - da^2) + E_2S_2(ca^2 + bcd - bc^2 - ad^2) + W_1W_2(c^2d + b^2d - 2bcd) + W_1S_1(bda + dbc - b^2c - d^2a) + W_1S_2(bda + dc^2 - bc^2 - d^2a) + W_2S_1(cda + b^2d - b^2c - d^2a) + W_2S_2(cda + dbc - bc^2 - d^2a) > 0$

Factorizing, the coefficients in the inequality become:

N_1E_1: $(bc - ad)(a - b)$	N_1E_2: $(b^2 - ad)(a - c)$
N_1W_1: $(bc - d^2)(a - b)$	N_1W_2: $(b^2 - d^2)(a - c)$
N_1S_1: $b(a - d)^2$	N_1S_2: $(ab - dc)(a - d)$
N_2E_1: $(c^2 - ad)(a - b)$	N_2E_2: $(ad - bc)(c - a)$
N_2W_1: $(c^2 - d^2)(a - b)$	N_2W_2: $(bc - d^2)(a - c)$
N_2S_1: $(ac - db)(a - d)$	N_2S_2: $c(a - d)^2$
E_1W_1: $a(b - c)^2$	E_1W_2: $(ab - cd)(b - c)$
E_1S_1: $(a^2 - bc)(b - d)$	E_1S_2: $(a^2 - c^2)(b - d)$
E_2W_1: $(ac - bd)(c - b)$	E_2W_2: $(a^2 - b^2)(c - d)$
E_2S_1: $(a^2 - bc)(c - d)$	E_2S_2: $d(b - c)^2$
W_1S_1: $(ad - bc)(b - d)$	W_1S_2: $(ad - c^2)(b - d)$
W_2S_1: $(ad - b^2)(c - d)$	W_2S_2: $(ad - bc)(c - d)$

We have, by assumption, that $a \geq b$ and $ad > bc$. If in fact $a > b$, the coefficient of N_1E_1 is negative. Let θ be n copies of α_1 and n copies of α_2 where n is some natural number.

Then we have $N_1 = a^nb^n, N_2 = a^nc^n, E_1 = b^na^n, E_2 = c^na^n, W_1 = b^nd^n, W_2 = c^nd^n, S_1 = d^nb^n, S_2 = d^nc^n$.

We first assume that $a > d$ and $b > c$, so $N_1E_1 = (ab)^{2n}$ is the biggest term in the expression.

Applying Lemma 28 we can see that if n is large enough, $N_1E_1 = (ab)^{2n}$ will be greater than the sum of the other terms. Since the coefficients are relatively small, if we take n to be sufficiently large we can ensure that

$(ad - bc)(a - b)N_1E_1 > N_1E_2(ab^2 + cad - a^2d - cb^2) + N_1W_1(abc + bd^2 - ad^2 - b^2c) + N_1W_2(ab^2 + cd^2 - ad^2 - b^2c) + N_1S_1(a^2b + d^2b - adb - dab) + N_1S_2(a^2b + d^2c - adc - dab) + N_2E_1(ac^2 + bad - a^2d - bc^2) + N_2E_2(cad + acb - a^2d - c^2b) + N_2W_1(ac^2 + bd^2 - ad^2 - bc^2) + N_2W_2(acb + cd^2 - ad^2 - bc^2) + N_2S_1(a^2c + d^2b - adb - dac) + N_2S_2(a^2c + d^2c - 2adc) + E_1W_1(b^2a + c^2a - 2abc) + E_1W_2(b^2a + c^2d - bcd - bca) + E_1S_1(ba^2 + dbc - b^2c - da^2) + E_1S_2(ba^2 + dc^2 - bc^2 - da^2) + E_2W_1(c^2a + b^2d - cbd - bca) + E_2S_1(ca^2 + db^2 - cb^2 - da^2) + E_2S_2(ca^2 + bcd - bc^2 - ad^2) +$

$W_1W_2(c^2d + b^2d - 2bcd) + W_1S_1(bda + dbc - b^2c - d^2a) + W_1S_2(bda + dc^2 - bc^2 - d^2a) + W_2S_1(cda + b^2d - b^2c - d^2a) + W_2S_2(cda + dbc - bc^2 - d^2a)$

And hence that

$0 > (bc - ad)(a - b)N_1E_1 + N_1E_2(ab^2 + cad - a^2d - cb^2) + N_1W_1(abc + bd^2 - ad^2 - b^2c) + N_1W_2(ab^2 + cd^2 - ad^2 - b^2c) + N_1S_1(a^2b + d^2b - adb - dab) + N_1S_2(a^2b + d^2c - adc - dab) + N_2E_1(ac^2 + bad - a^2d - bc^2) + N_2E_2(cad + acb - a^2d - c^2b) + N_2W_1(ac^2 + bd^2 - ad^2 - bc^2) + N_2W_2(acb + cd^2 - ad^2 - bc^2) + N_2S_1(a^2c + d^2b - adb - dac) + N_2S_2(a^2c + d^2c - 2adc) + E_1W_1(b^2a + c^2a - 2abc) + E_1W_2(b^2a + c^2d - bcd - bca) + E_1S_1(ba^2 + dbc - b^2c - da^2) + E_1S_2(ba^2 + dc^2 - bc^2 - da^2) + E_2W_1(c^2a + b^2d - cbd - bca) + E_2S_1(ca^2 + db^2 - cb^2 - da^2) + E_2S_2(ca^2 + bcd - bc^2 - ad^2) + W_1W_2(c^2d + b^2d - 2bcd) + W_1S_1(bda + dbc - b^2c - d^2a) + W_1S_2(bda + dc^2 - bc^2 - d^2a) + W_2S_1(cda + b^2d - b^2c - d^2a) + W_2S_2(cda + dbc - bc^2 - d^2a)$

which contradicts the requirement of SDAP_H.

Above we assumed that $a > d, b > c$. Notice that if $a = d$, then either $b = c$ and all coefficients are non-positive (so SDAP_H obviously fails), or $b > c$ and the only positive coefficients are those of E_2W_1 and E_1W_2, both of which equal $(abcd)^n$. As above, by Lemma 28, these are dominated by the larger terms, generating a contradiction.

If $a > d$ but $b = c$, then simply notice that the terms $N_1E_1, N_1E_2, N_2E_1, N_2E_2$ are equal and strictly larger than all others, and all have negative coefficients, so again the contradiction follows by Lemma 28.

If in fact $a = b$, then notice that either $c = d$, giving $ad = bc$, or the coefficient of W_2S_2 is negative. We can then generate a contradiction in a similar way to above by taking θ to be n copies of α_3 and n copies of α_4. If then $b > c$, W_2S_2 will be the biggest term in the expression and the contradiction follows. If $b = c$, then $W_1S_1, W_1S_2, W_2S_1, W_2S_2$ are equal and strictly larger than the other terms, and all have negative coefficients, so again the contradiction follows by Lemma 28.
The cases in which we assume $ad < bc$ proceed in a similar way.

□

To recap then, we know that any product of two probability functions satisfying Ex will satisfy SDAP_H, and we conjecture that if w satisfies SDAP_H the support of μ contains only points with $ad = bc$. To classify the functions satisfying SDAP_H

we must close the gap between these two conditions which intuitively seem very near to one another.

While $y_{\langle a,a,b,b\rangle}$ is not a product of two functions on L_1, it is an average of two products. Let w_1 have de Finetti measure μ_1 which gives equal measure to all points in the set

$$\{\langle 2a, 2b\rangle, \langle 2b, 2a\rangle\} \subset \mathbb{D}_1$$

and c_∞ on L_1 as usual have the de Finetti prior which puts all measure on the point $\langle 1/2, 1/2\rangle$. Then

$$y_{\langle,a,a,b,b\rangle} = 2^{-1}((w_1 \times c_\infty) + (c_\infty \times w_1)).$$

A natural conjecture then might be that all products and averages of products will satisfy SDAP$_H$. Unfortunately this too fails to capture the class correctly. Even the above construction can fail to satisfy SDAP$_H$ for different choices of w_1. For example, suppose μ_1 divides all measure equally between the points $\langle 0.8, 0.2\rangle, \langle 0.2, 0.8\rangle, \langle 0.9, 0.1\rangle$ and $\langle 0.1, 0.9\rangle$. Taking $\theta = \alpha_1^6\alpha_2^3$ we have

$$w(\alpha_1 \,|\, \alpha_1 \wedge \theta) < w(\alpha_1 \,|\, \alpha_2 \wedge \theta).^5 \tag{4.18}$$

At this point a full classification of the probability functions satisfying SDAP$_H$ proves elusive.

[5] See Appendix for calculations.

Chapter 5

Alternatives to Hamming Distance in L_2

5.1 Distance based alternatives

When looking at Hamming Distance we pictured the four atoms of L_2 as being the vertices of a tetrahedron with edges representing distances between them and symmetries of the tetrahedron representing the permutations of atoms licensed by a partcicular set of background conditions. When the symmetries of our tetrahedron corresponded to the permutations of atoms licensed by Px and SN, we were looking at a square, and when they corresponded to the permutations of atoms licensed by Px and WN we were looking at a diamond.

In exactly the same way, we can use any combination of symmetry principles to motivate distances. The permutations of atoms licensed by WN are (1)(2)(3)(4) and (14)(23). The only 2D geometrical object with the atoms as vertices and symmetries (1)(2)(3)(4) and (14)(23) is a parallelogram:

$$\begin{tikzcd} & \alpha_2 \arrow[rr, dash] & & \alpha_4 \\ \alpha_1 \arrow[ur, dash] \arrow[rr, dash] & & \alpha_3 \arrow[ur, dash] & \end{tikzcd} \tag{5.1}$$

or

$$\begin{tikzcd} & \alpha_3 \arrow[rr, dash] & & \alpha_4 \\ \alpha_1 \arrow[ur, dash] \arrow[rr, dash] & & \alpha_2 \arrow[ur, dash] & \end{tikzcd} \tag{5.2}$$

Let d_1 be a distance function that gives rise to parallelogram 5.1 and d_2 be a distance function that gives rise to parallelogram 5.2. One way of thinking of such distance functions is that for d_1, a difference in P_1 is more significant than a difference in P_2, and vice versa for d_2.

The distance function d_1 on L_2 generates a similarity relation $>_S$ with the following properties:

$$\{\alpha_1, \alpha_2\} >_S \{\alpha_1, \alpha_3\}$$

$$\{\alpha_1, \alpha_3\} >_S \{\alpha_1, \alpha_4\}$$

$$\{\alpha_2, \alpha_1\} >_S \{\alpha_2, \alpha_4\}$$

$$\{\alpha_2, \alpha_4\} >_S \{\alpha_2, \alpha_3\}$$

$$\{\alpha_3, \alpha_4\} >_S \{\alpha_3, \alpha_1\}$$

$$\{\alpha_3, \alpha_1\} >_S \{\alpha_3, \alpha_2\}$$

Lemma 30. *Suppose $>_S$ is as above, that w satisfies Ex, WN and AP$_S$ with de Finetti prior μ, and $\langle a, b, c, d\rangle$ is a point with $a \geq d$ in the support of μ. Then at least one of the following must hold:*

1. $a = d$

2. $b = c$

3. $ac = bd$

Proof. If $\langle a, b, c, d\rangle$ is in the support of μ then so is $\langle d, c, b, a\rangle$, so by Corollary 18 we can find state descriptions ϕ_n, ψ_n such that for some $\lambda > 0$,

$$\lim_{n\to\infty} w(\alpha_1^{j_1}\alpha_2^{j_2}\alpha_3^{j_3}\alpha_4^{j_4} \mid \phi_n \vee \psi_n) = (1+\lambda)^{-1}(a^{j_1}b^{j_2}c^{j_3}d^{j_4} + \lambda d^{j_1}c^{j_2}b^{j_3}a^{j_4})$$

Taking $j_2 = j_3 = j_4 = 0$ and the cases $j_1 = 2$ and $j_1 = 1$ we get that

$$\lim_{n\to\infty} w(\alpha_1 \mid \alpha_1 \wedge (\phi_n \vee \psi_n)) = \frac{a^2 + \lambda d^2}{a + \lambda d}$$

and taking $j_3 = j_4 = 0$ and the cases $j_1 = j_2 = 1$ and $j_1 = 0, j_2 = 1$ we get that

$$\lim_{n\to\infty} w(\alpha_1 \mid \alpha_2 \wedge (\phi_n \vee \psi_n)) = \frac{ab + \lambda dc}{b + \lambda c}$$

AP$_S$ requires that $w(\alpha_1 \mid \alpha_1 \wedge \theta) \geq w(\alpha_1 \mid \alpha_2 \wedge \theta)$ for any $\theta \in SL$; so, in particular, $w(\alpha_1 \mid \alpha_1 \wedge (\phi_n \vee \psi_n)) \geq w(\alpha_1 \mid \alpha_2 \wedge (\phi_n \vee \psi_n))$ for all n. This means that we cannot have

$$\lim_{n\to\infty} w(\alpha_1 \,|\, \alpha_1 \wedge (\phi_n \vee \psi_n)) < \lim_{n\to\infty} w(\alpha_1 \,|\, \alpha_2 \wedge (\phi_n \vee \psi_n))$$

and so we must have

$$\frac{a^2+\lambda d^2}{a+\lambda d} \geq \frac{ab+\lambda dc}{b+\lambda c}$$

which simplifies to

$$\lambda(ac-bd)(a-d) \geq 0.$$

Hence

$$(ac-bd)(a-d) \geq 0. \tag{5.3}$$

Using the same points and taking $j_1 = j_3 = j_4 = 0$ with cases $j_2 = 2$ and $j_2 = 1$ we get

$$\lim_{n\to\infty} w(\alpha_2 \,|\, \alpha_2 \wedge (\phi_n \vee \psi_n)) = \frac{b^2+\lambda c^2}{b+\lambda c}$$

and taking $j_3 = j_4 = 0$ with cases $j_2 = j_1 = 1$ and $j_2 = 0, j_1 = 1$ we get that

$$\lim_{n\to\infty} w(\alpha_2 \,|\, \alpha_1 \wedge (\phi_n \vee \psi_n)) = \frac{ba+\lambda cd}{a+\lambda d}.$$

AP_S then requires

$$\frac{b^2+\lambda c^2}{b+\lambda c} \geq \frac{ba+\lambda cd}{a+\lambda d}$$

which simplifies to

$$\lambda(bd-ac)(b-c) \geq 0$$

and so we must have

$$(bd-ac)(b-c) \geq 0. \tag{5.4}$$

Using the same points and taking $j_3 = j_4 = 0$ with cases $j_1 = j_2 = 1$ and $j_2 = 1, j_1 = 0$ we get

$$\lim_{n\to\infty} w(\alpha_1 \,|\, \alpha_2 \wedge (\phi_n \vee \psi_n)) = \frac{ab+\lambda dc}{b+\lambda c}$$

and taking $j_2 = j_4 = 0$ with cases $j_3 = j_1 = 1$ and $j_3 = 0, j_1 = 1$ we get that

$$\lim_{n\to\infty} w(\alpha_1 \,|\, \alpha_3 \wedge (\phi_n \vee \psi_n)) = \frac{ac+\lambda db}{c+\lambda b}.$$

AP_S then requires

$$\frac{ab+\lambda dc}{b+\lambda c} \geq \frac{ac+\lambda db}{c+\lambda b}$$

which simplifies to

$$\lambda(b^2-c^2)(a-d) \geq 0$$

and so we must have

$$(b^2-c^2)(a-d) \geq 0. \tag{5.5}$$

Either 1. holds, in which case 5.3, 5.4 and 5.5 all hold, or $a > d$. Then by 5.5 we must have $b \geq c$, and by 5.3 we must have $ac \geq bd$. But then for 5.4 to hold we must have either $b = c$ or $ac = bd$, hence 2. or 3. respectively.

□

Let w be a probability function with de Finetti prior μ satisfying Ex, Px, WN and AP_S. For any two points $\langle a,b,c,d\rangle$, $\langle \alpha,\beta,\gamma,\delta\rangle$ in the support of μ, the points $\langle d,c,b,a\rangle$ and $\langle \delta,\gamma,\beta,\alpha\rangle$ must also be in the support of μ. Using Corollary 18 with all pairs of points with the first taken from

$$\{\langle a,b,c,d\rangle, \langle d,c,b,a\rangle\}$$

and the second from

$$\{\langle \alpha,\beta,\gamma,\delta\rangle, \langle \delta,\gamma,\beta,\alpha\rangle\},$$

and all appropriate choices of j_i, AP_S requires that all the following expressions be non-negative:

$$\begin{array}{lll}
(a-\alpha)(a\beta-\alpha b) & (a-\alpha)(b\gamma-\beta c) & (a-\alpha)(c\delta-\gamma d) \\
(a-\delta)(a\gamma-b\delta) & (a-\delta)(b\beta-c\gamma) & (a-\delta)(\alpha c-\beta d) \\
(\alpha-d)(a\gamma-b\delta) & (\alpha-d)(b\beta-c\gamma) & (\alpha-d)(\alpha c-\beta d) \\
(\delta-d)(a\beta-\alpha b) & (\delta-d)(b\gamma-\beta c) & (\delta-d)(c\delta-\gamma d) \\
(b-\beta)(\alpha b-a\beta) & (b-\beta)(a\delta-\alpha d) & (b-\beta)(\gamma d-c\delta) \\
(b-\gamma)(b\delta-a\gamma) & (b-\gamma)(\beta d-\alpha c) & (b-\gamma)(a\alpha-\delta d) \\
(\beta-c)(b\delta-a\gamma) & (\beta-c)(\beta d-\alpha c) & (\beta-c)(a\alpha-\delta d) \\
(\gamma-c)(\alpha b-a\beta) & (\gamma-c)(a\delta-\alpha d) & (\gamma-c)(\gamma d-c\delta)
\end{array}$$

For brevity, we will restrict our attention to the stronger principle SAP_S.

Theorem 31. *There are no probability functions satisfying Ex, WN and SAP_S on L_2.*

Proof. We first show that any such functions would have to be of the form

$$\lambda 2^{-1}(w_{\langle a,b,c,d\rangle} + w_{\langle d,c,b,a\rangle}) + (1-\lambda)w_{\langle \alpha,\beta,\beta,\alpha\rangle}$$

where $a > d$ and $b > c, ac = bd$, $\alpha c = \beta d$, $a\beta = \alpha b$ and $0 < \lambda \leq 1$. But then it is easy to check that such a function gives equality in many of the inequalities required by SAP_S. For example, such a w gives

$$\begin{aligned} w(\alpha_1 \,|\, \alpha_1) - w(\alpha_1 \,|\, \alpha_2) &= \frac{a^2+d^2+\alpha^2}{a+d+\alpha} - \frac{ab+dc+\alpha\beta}{b+c+\beta} \\ &= \frac{(ac-bd)(a-d)+(a\beta-\alpha b)(a-\alpha)+(\beta d-\alpha c)(d-\alpha)}{(a+d+\alpha)b+c+\beta)} \\ &= 0. \end{aligned}$$

For w to satisfy SAP_S, at least one point with no zero entries in the support of its de Finetti prior μ must have the property that $x_1 > x_4$; similarly at least one point with no zero entries must have the property that $x_2 > x_3$. For otherwise we would have

$$w(\alpha_1 \,|\, \alpha_1 \wedge \alpha_1 \wedge \alpha_2 \wedge \alpha_3 \wedge \alpha_4) = w(\alpha_1 \,|\, \alpha_4 \wedge \alpha_1 \wedge \alpha_2 \wedge \alpha_3 \wedge \alpha_4)$$

or

$$w(\alpha_2 \,|\, \alpha_2 \wedge \alpha_1 \wedge \alpha_2 \wedge \alpha_3 \wedge \alpha_4) = w(\alpha_2 \,|\, \alpha_3 \wedge \alpha_1 \wedge \alpha_2 \wedge \alpha_3 \wedge \alpha_4).$$

So let $\langle a,b,c,d\rangle$ and $\langle \alpha,\beta,\gamma,\delta\rangle$ be these points, respectively. Then by Lemma 30, either $b = c$ or $ac = bd$ and either $\alpha = \delta$ or $\alpha\gamma = \beta\delta$, giving us the following four options to consider:

	$b = c$	$ac = bd$
$\alpha = \delta$	(i)	(ii)
$\alpha\gamma = \beta\delta$	(iii)	(iv)

We consider each in turn.
(i) Suppose firstly that $a > \alpha$. Then from $(a-\alpha)(b\gamma - \beta c) \geq 0$ we get that $b\gamma \geq \beta c = \beta b$. Also, since $a > \alpha = \delta$, $(a-\delta)(b\beta - c\gamma) \geq 0$ forces $b\beta \geq c\gamma = b\gamma$. So $\beta = \gamma$, contradicting $\beta \neq \gamma$.

So now suppose that $\alpha > a$. Then also $\delta > a$, and the same two inequalities as above force $\beta = \gamma$.

The only remaining option is that $a = \alpha$. In this case, $\delta = \alpha = a > d$, so we can generate exactly the same contradiction from $(\alpha - d)(b\beta - c\gamma) \geq 0$ and $(\delta - d)(b\gamma - \beta c) \geq 0$. So option (i) is unsatisfiable.

(ii) Suppose firstly that $\delta = \alpha > a > d$. Since by assumption $a > d > 0$ and $b, c > 0$, the condition $ac = bd$ gives $b > c$. Then $(a - \delta)(b\beta - c\gamma) \geq 0$ forces $c\gamma \geq b\beta$, which contradicts $\beta > \gamma$ and $b > c$.

So suppose next that $\delta = \alpha = a > d$. Then $(\delta - d)(a\beta - \alpha b) \geq 0$ forces $a\beta \geq \alpha b$, so $\beta \geq b$. But then to ensure that $a+b+c+d = 1 = \alpha+\beta+\gamma+\delta$ we must have $c > \gamma$. Then from $(\gamma - c)(a\delta - \alpha d) \geq 0$ we get $\alpha d \geq a\delta$, giving $d \geq \delta$ which is a contradiction.

The only remaining option is that $a > \alpha = \delta$. In this case, $bd = ac > \alpha c$, and from $(a-\delta)(\alpha c - \beta d) \geq 0$, $\alpha c \geq \beta d$. Hence $b > \beta$. Then from $(\alpha - d)(b\beta - c\gamma) \geq 0$, since $b > c$ and $\beta > \gamma$, we must have $\alpha \geq d$, that is $\delta \geq d$. From $(a - \alpha)(c\delta - \gamma d) \geq 0$ and $(b - \beta)(\gamma d - c\delta) \geq 0$ we get that $\gamma d = c\delta = c\alpha \geq \beta d$. But this contradicts $\beta > \gamma$.

(iii) is similarly unsatisfiable.

(iv) Firstly suppose that $a = \alpha$. Then from $(\delta - d)(a\beta - \alpha b) \geq 0$ and $(\gamma - c)(a\delta - \alpha d) \geq 0$ we get that $\delta > d$ implies $\beta \geq b$ and $\gamma \geq c$, contradicting $a + b + c + d = 1 = \alpha + \beta + \gamma + \delta$. So we must also have $\delta = d$. Now from $(b - \gamma)(a\alpha - \delta d) \geq 0$ and $(\beta - c)(a\alpha - \delta d) \geq 0$ we must have that $b \geq \gamma$ and $\beta \geq c$. Either $b = \gamma$, $\beta = c$, or $b > \gamma$, in which case $(b - \gamma)(b\delta - a\gamma) \geq 0$ and $(b - \gamma)(\beta d - \alpha c) \geq 0$ force $\beta = b$, and hence $\gamma = c$. So either way, $\langle \alpha, \beta, \gamma, \delta \rangle$ must be obtainable from $\langle a, b, c, d \rangle$ by a permutation licensed by Px.

If there is a further point $\langle x, y, z, w \rangle$ in the support of μ with $x \neq a$, then as we have seen in (ii) and (iii), such a point cannot have the properties $x > w, y = z$ or $x = w, y > z$. But we have also seen that such a point cannot have the properties $x > w, y > z, xz = yw$ (as this forces $x = y$.) By Lemma 30, the only remaining option is that $x = w, y = z$ (and so $xz = yw$.)

Next suppose that $a > \alpha$. Then also $a > \delta$ and so from $(a - \delta)(\alpha c - \beta d) \geq 0$ we get $\beta d \leq \alpha c < ac = bd$, hence $b > \beta$.

Now $(a - \alpha)(a\beta - \alpha b) \geq 0$ and $(b - \beta)(\alpha b - a\beta) \geq 0$ jointly entail $a\beta = \alpha b$, while $(a - \alpha)(c\delta - \gamma d) \geq 0$ and $(b - \beta)(\gamma d - c\delta) \geq 0$ jointly entail $c\delta = \gamma d$. Also note that $b > \beta \geq \gamma$, so $b > \gamma$. So $(a - \delta)(a\gamma - b\delta)$ and $(b - \gamma)(b\delta - a\gamma) \geq 0$ jointly entail $b\delta = a\gamma$, while $(a-\delta)(\alpha c - \beta d) \geq 0$ and $(b-\gamma)(\beta d - \alpha c) \geq 0$ jointly entail $\alpha c = \beta d$.

Note that in the above we have not used $\beta > \gamma$ or $a > d$, and so the same obtains if we start from the assumption that $\alpha > a$.

Now we get that

$$\alpha b - b\delta = a\beta - a\gamma$$

and

$$\beta d - \gamma d = \alpha c - c\delta$$

Hence

$$(b+d)(\alpha - \delta + \gamma - \beta) = (a+c)(\gamma - \beta + \alpha - \delta)$$

so either $\alpha - \delta + \gamma - \beta = 0$, hence $\alpha + \gamma = \beta + \delta = 1/2$ or $a + c = b + d = 1/2$. Suppose firstly that $\alpha + \gamma = \beta + \delta$. Then from $\alpha\gamma = \beta\delta$ we get

$$\alpha(1/2 - \alpha) = \beta(1/2 - \beta)$$

which rearranges to give

$$1/2(\alpha - \beta) = (\alpha - \beta)(\alpha + \beta)$$

so either $\alpha + \beta = 1/2$ or $\alpha = \beta$. In the former case, we now have $\delta = 1/2 - \beta = 1/2 - (1/2 - \alpha) = \alpha$, contradicting $\alpha > \delta$. So we must have $\alpha = \beta$ and hence $\gamma = \delta$. But now $b(\alpha - \delta) = a(\beta - \gamma)$ forces $a = b$ and similarly $d(\beta - \gamma) = c(\alpha - \delta)$ forces $d = c$. Notice that in the above we have not used the assumption that $a > \alpha$, so an exactly analogous argument shows that $a + c = b + d$ also forces $a = b, c = d, \alpha = \beta, \gamma = \delta$.

If we have points $\langle a, a, c, c\rangle$, $\langle c, c, a, a, \rangle$, $\langle \alpha, \alpha, \gamma, \gamma\rangle$, $\langle \gamma, \gamma, \alpha, \alpha, \rangle$ in the support of μ, in order for w to satisfy SAP_S we must have a further point $\langle x, y, z, w\rangle$ with x, y, z, w non-zero and $x \neq y$. Otherwise w would give the equality

$$w(\alpha_1 \,|\, \alpha_1 \wedge \alpha_1 \wedge \alpha_2 \wedge \alpha_3 \wedge \alpha_4) = w(\alpha_1 \,|\, \alpha_2 \wedge \alpha_1 \wedge \alpha_2 \wedge \alpha_3 \wedge \alpha_4).$$

As we have seen, in (ii) and (iii), such a point cannot have the properties $x > w, y = z$ or $x = w, y > z$. But we have also seen that such a point cannot have the properties $x > w, y > z, xz = yw$ (as this forces $x = y$.) By Lemma 30, the only remaining option is that $x = w, y = z$ (and so $xz = yw$.)

If $y = 0$, then $x = 1/2$, and $(y - c)(ax - xc) \geq 0$ gives a contradiction. Similarly, if $x = 0$, then $y = 1/2$, and $(x - c)(ay - yc) \geq 0$ gives a contradiction. But if $x, y \neq 0$, then $(a - x)(ya - yc) \geq 0$, $(a - y)(ax - xc) \geq 0$, $(a - x)(ay - xa) \geq 0$ and $(a - y)(xa - ay) \geq 0$ jointly force $x = y$, which is a contradiction.

We have thus shown that (i)-(iii) are unsatisfiable and (iv) entails that μ has in its support the points $\langle a, b, c, d\rangle$ and $\langle d, c, b, a\rangle$ with $ac = bd, a > d, b > c$, and any further point $\langle \alpha, \beta, \gamma, \delta\rangle$ in the support not obtainable from the first by WN licensed permutations must have $\alpha = \delta, \beta = \gamma$. We check finally whether any such $\langle \alpha, \beta, \gamma, \delta\rangle$ can appear in the support of μ.

If $\alpha = \delta = 0$, then $\beta = \gamma = 1/2$, and $(\alpha - d)(a\gamma - b\delta) \geq 0$ gives a contradiction. Similarly, if $\beta = \gamma = 0$, then $\alpha = \delta = 1/2$, and $(\beta - c)(b\delta - a\gamma) \geq 0$ gives a contradiction.

If $\alpha, \beta, \gamma, \delta \neq 0$ then $(b - \beta)(a\delta - \alpha d) \geq 0$ entails $(b - \beta)(a - d) \geq 0$ and $(\beta - c)(a\alpha - \delta d) \geq 0$ entails $(\beta - c)(a - d) \geq 0$, hence we have $b \geq \beta = \gamma \geq c$. Similarly, $(a - \delta)(b\beta - c\gamma) \geq 0$ entails $(a - \delta)(b - c) \geq 0$ and $(\alpha - d)(b\beta - c\gamma) \geq 0$ entails $(\alpha - d)(b - c) \geq 0$, hence we have $a \geq \alpha = \delta \geq d$. Furthermore, since $b > c$, at least one of $b > \beta = \gamma$ or $\beta = \gamma > c$ must hold. So since we have $(b - \beta)(\alpha b - a\beta) \geq 0$ and $(\gamma - c)(\alpha b - a\beta) \geq 0$, it must be the case that $\alpha b \geq a\beta$. And since $a > d$, at least one of $a > \alpha = \delta$ or $\alpha = \delta > d$ must hold, so from $(a - \alpha)(a\beta - \alpha b) \geq 0$ and $(\delta - d)(a\beta - \alpha b) \geq 0$ we can deduce $a\beta \geq \alpha b$. So in fact $\alpha b = a\beta$. Also from $(a - \alpha)(c\delta - \gamma d) \geq 0$, $(\delta - d)(c\delta - \gamma d) \geq 0$, $(b - \beta)(\gamma d - c\delta) \geq 0$ and $(\gamma - c)(\gamma d - c\delta) \geq 0$ we can deduce $\beta d = \gamma d = c\delta = c\alpha$.

It is easy to see that only one such extra point can appear in the support of μ. For suppose there was a further, distinct point $\langle x, y, y, x\rangle$ in the support of μ. Such a point would have to have the properties $a \geq x \geq d$, $b \geq y \geq c$, $ay = xb$, $yd = cx$. Then $\alpha b - xb = a\beta - ay$, giving

$$\alpha - x = \beta - y = (1/2 - \alpha) - (1/2 - x) = -\alpha + x$$

and so $\alpha = x$ (and $\beta = y$.)

We have thus proved that any probability function satisfying Ex + WN + SAP_S

must be of the form

$$\lambda 2^{-1}(w_{\langle a,b,c,d\rangle} + w_{\langle d,c,b,a\rangle}) + (1-\lambda)w_{\langle \alpha,\beta,\beta,\alpha\rangle}$$

where $a > d, b > c, ac = bd$ and $0 < \lambda \leq 1$.
Since any such function violates SAP_S, this completes the proof.

□

There are two further combinations of symmetry principles that give rise to alternative distance functions. Firstly, SN in the absence of Px. Such distance functions would look like one of the following.

$$\begin{tikzcd} \alpha_1 \arrow[r, dash] \arrow[d, dash] & \alpha_3 \arrow[d, dash] \\ \alpha_2 \arrow[r, dash] & \alpha_4 \end{tikzcd} \tag{5.6}$$

$$\begin{tikzcd} \alpha_1 \arrow[r, dash] \arrow[d, dash] & \alpha_2 \arrow[d, dash] \\ \alpha_3 \arrow[r, dash] & \alpha_4 \end{tikzcd} \tag{5.7}$$

Notice that the similarity function $>_S$ that 5.6 gives rise to is actually the same one as in Theorem 31. An obvious corollary to that theorem then is:

Corollary 32. *There are no probability functions that satisfy Ex, SN + SAP_S.*

And finally, Px in the absence of any negation principle. Such distance functions could be as below.

$$\begin{tikzcd} & \alpha_1 \arrow[dl, dash] \arrow[dr, dash] & \\ \alpha_2 \arrow[dr, dash] & & \alpha_3 \arrow[dl, dash] \\ & \alpha_4 & \end{tikzcd} \tag{5.8}$$

$$\begin{tikzcd} & \alpha_4 \arrow[dl, dash] \arrow[dr, dash] & \\ \alpha_2 \arrow[dr, dash] & & \alpha_3 \arrow[dl, dash] \\ & \alpha_1 & \end{tikzcd} \tag{5.9}$$

If k is the distance function pictured by 5.8, let $>_K$ be the corresponding similarity relation on L_2. The following are properties of $>_K$.

$$\{\alpha_1, \alpha_2\} >_K \{\alpha_1, \alpha_4\}$$

$$\{\alpha_1, \alpha_3\} >_K \{\alpha_1, \alpha_4\}$$

$$\{\alpha_2, \alpha_4\} >_K \{\alpha_2, \alpha_1\}$$

$$\{\alpha_3, \alpha_4\} >_K \{\alpha_3, \alpha_1\}$$

$$\{\alpha_2, \alpha_4\} >_K \{\alpha_2, \alpha_3\}$$

$$\{\alpha_3, \alpha_4\} >_K \{\alpha_2, \alpha_3\}$$

$$\{\alpha_2, \alpha_3\} >_K \{\alpha_1, \alpha_4\}$$

Let w be a probability function with de Finetti prior μ satisfying Ex, Px and AP$_K$. For any two points $\langle a, b, c, d\rangle$, $\langle \alpha, \beta, \gamma, \delta\rangle$ in the support of μ, the points $\langle a, c, b, d\rangle$, $\langle \alpha, \gamma, \beta, \delta\rangle$ must also appear. Using Corollary 18 with all pairs of points with the first taken from

$$\{\langle a, b, c, d\rangle, \langle a, c, b, d\rangle\}$$

and the second from

$$\{\langle \alpha, \beta, \gamma, \delta\rangle, \langle \alpha, \gamma, \beta, \delta\rangle\},$$

and all appropriate choices of j_i, AP$_K$ requires that all the following expressions be non-negative:

$$\begin{array}{llll}
(a-\alpha)(a\beta-\alpha b) & (a-\alpha)(a\beta-\alpha c) & (a-\alpha)(a\gamma-\alpha b) & (a-\alpha)(a\gamma-\alpha c) \\
(a-\alpha)(b\delta-\beta d) & (a-\alpha)(b\delta-\gamma d) & (a-\alpha)(c\delta-\gamma d) & (a-\alpha)(c\delta-\beta d) \\
(\delta-d)(a\beta-\alpha b) & (\delta-d)(a\beta-\alpha c) & (\delta-d)(a\gamma-\alpha b) & (\delta-d)(a\gamma-\alpha c) \\
(\delta-d)(b\delta-\beta d) & (\delta-d)(b\delta-\gamma d) & (\delta-d)(c\delta-\gamma d) & (\delta-d)(c\delta-\beta d) \\
(b-\beta)(\alpha d-a\delta) & (b-\beta)(\gamma d-c\delta) & (b-\beta)(b\delta-\beta d) & \\
(c-\gamma)(\alpha d-a\delta) & (c-\gamma)(c\delta-\gamma d) & (c-\gamma)(\beta d-b\delta) & \\
(b-\gamma)(b\delta-\gamma d) & (b-\gamma)(\beta d-c\delta) & (b-\gamma)(\alpha d-a\delta) & \\
(\beta-c)(b\delta-\gamma d) & (\beta-c)(\beta d-c\delta) & (\beta-c)(a\delta-\alpha d) &
\end{array}$$

For w to satisfy SAP$_K$ there must be at least one point $\langle a, b, c, d\rangle$ in the support of μ with a, b, c, d non-zero and $b \neq c$, for otherwise w would give

$$w(\alpha_2 \,|\, \alpha_2 \wedge \alpha_1 \wedge \alpha_2 \wedge \alpha_3 \wedge \alpha_4) = w(\alpha_2 \,|\, \alpha_3 \wedge \alpha_1 \wedge \alpha_2 \wedge \alpha_3 \wedge \alpha_4).$$

Suppose without loss of generality that $b > c$. If w were simply the discrete probability function that splits all measure between $\langle a, b, c, d\rangle$ and $\langle a, c, b, d\rangle$ then it is easy to see that w would give equality in the weak inequalities required by AP$_K$.

For example,

$$w(\alpha_1 \mid \alpha_1) = \frac{2a^2}{2a} = a = \frac{a(b+c)}{b+c} = w(\alpha_1 \mid \alpha_2).$$

So for w to satisfy SAP$_K$, there must be another point $\langle \alpha, \beta, \gamma, \delta \rangle$ in the support of μ with $\alpha, \beta, \gamma, \delta$ non-zero and $a \neq \alpha$. Without loss of generality we can suppose that $\beta \geq \gamma$.

Suppose firstly that $a > \alpha$. Then from $(a-\alpha)(c\delta - \gamma d) \geq 0$ we get that $b\delta > c\delta \geq \beta d$, so from $(\delta - d)(b\delta - \beta d)$, $\delta \geq d$. Then $a\delta > \alpha d$, so from $(b-\beta)(\alpha d - a\delta) \geq 0$ and $(b-\beta)(b\delta - \beta d) \geq 0$ we get that $\beta = b$. Also from $(b-\gamma)(\alpha d - a\delta) \geq 0$ we get that $\beta \geq \gamma \geq b = \beta$ so $\gamma = \beta = b$. Finally, note that if $\delta = d$, then $(a-\alpha)(c\delta - \gamma d) \geq 0$ entails $(a-\alpha)(c-\gamma) \geq 0$ and so $c \geq \gamma = b$, contradicting $b > c$. So $\delta > d$.

That is, we have

$$\begin{array}{cccc} a & b & c & d \\ \vee & || & \wedge & \wedge \\ \alpha & \beta & \beta & \delta \end{array}$$

If $\alpha > a$, then from $(a-\alpha)(b\delta - \gamma d) \geq 0$ we get that $\gamma d \geq b\delta > c\delta$, so from $(\delta - d)(c\delta - \gamma d) \geq 0$ we get that $d \geq \delta$. Then $\alpha d > a\delta$, so from $(c-\gamma)(\alpha d - a\delta) \geq 0$ and $(c-\gamma)(c\delta - \gamma d) \geq 0$ we get that $\gamma = c$. Also, from $(\beta - c)(a\delta - \alpha d) \geq 0$ we get that $c \geq \beta$, so $c \geq \beta \geq \gamma = c$, hence in fact $\gamma = \beta = c$. Finally note that if $\delta = d$, then $(a-\alpha)(b\delta - \beta d) \geq 0$ entails $(a-\alpha)(b-\beta) \geq 0$ and so $c = \beta \geq b$, contradicting $b > c$.

That is, we have

$$\begin{array}{cccc} a & b & c & d \\ \wedge & \vee & || & \vee \\ \alpha & \beta & \beta & \delta \end{array}$$

Theorem 33. *If w is a probability function on L_2 satisfying Ex, Px and SAP$_K$, then it must have one of the following forms:*

1. $2^{-1}\lambda \left(w_{\langle a,b,c,d \rangle} + w_{\langle a,c,b,d \rangle} \right) + (1-\lambda) \int_{\mathbb{D}_2} w_{\langle x,b,b,1-x-2b \rangle} d\mu(x)$,
   for some $0 < \lambda < 1$ and some μ such that $\mu(x) = 0$ for any $x \geq a$.

2. $2^{-1}\lambda\left(w_{\langle a,b,c,d\rangle} + +w_{\langle a,c,b,d\rangle}\right) + (1-\lambda)\int_{\mathbb{D}_2} w_{\langle x,c,c,1-x-2c\rangle} d\mu(x)$,
for some $0 < \lambda < 1$ *and some* μ *such that* $\mu(x) = 0$ *for any* $x \leq a$.

Proof. We know already that w must have in the support of its de Finetti prior the points $\langle a,b,c,d\rangle, \langle a,c,b,d\rangle$ with $b > c$. We know that there must also be a point $\langle \alpha,\beta,\beta,\delta\rangle$ and one of

(i) $a > \alpha, d < \delta, \beta = b$,

(ii) $a < \alpha, d > \delta, \beta = c$.

Notice that in either case, if there was a further point $\langle x,y,z,w\rangle$ with $x = a$, then in fact these points would have to be the same. For then we would have $(w-d)(y-b) \geq 0$ and $(c-z)(d-w) \geq 0$ which, given $a+b+c+d = 1 = x+y+z+w$, are only jointly satisfiable if $w = d, z = c$ and $y = b$.

To complete the proof, we note that two points $\langle a_1,b_1,b_1,d_1\rangle$, $\langle a_2,b_2,b_2,d_2\rangle$ with $a_2 > a > a_1$ and $d_2 < d < d_1$ cannot both appear in the support of μ. For consider the requirement that $(b_1 - b_2)(a_1d_2 - a_2d_1) \geq 0$. Clearly $a_2d_1 > a_1d_2$, hence $b_2 \geq b_1$. But $b_2 = c < b = b_1$.

On the other hand, inspection of the list of constraints shows that two points $\langle a_1,b_1,b_1,d_1\rangle$, $\langle a_2,b_2,b_2,d_2\rangle$ with $a > a_1 > a_2$, $d < d_1 < d_2$, $b_1 = b_2 = b$ satisfy all the required conditions. Similarly, two points $\langle a_1,b_1,b_1,d_1\rangle$, $\langle a_2,b_2,b_2,d_2\rangle$ with $a_1 < a_2 < a$, $d_1 > d_2 > d$, $b_1 = b_2 = c$ will satisfy all conditions.

□

Notice that in 5.8, while α_4 must be closer to α_2 than α_3 is, α_1 could actually be further from α_2 than α_3 is. Let K' be a distance function with this further property. Then as well as the above conditions, $>_{K'}$ would have to satisfy:

$$\{\alpha_3,\alpha_2\} >_{K'} \{\alpha_3,\alpha_1\}$$

$$\{\alpha_2,\alpha_3\} >_{K'} \{\alpha_2,\alpha_1\}$$

Given these extra constraints, $\mathrm{AP}_{K'}$ would require that the following inequality holds for all $\theta \in QFSL$.

$$w(\alpha_2 \mid \alpha_3 \wedge \theta) \geq w(\alpha_2 \mid \alpha_1 \wedge \theta)$$

As seen above, at least one point in the support of μ must have the property that $b \neq c$ (otherwise we would have $w(\alpha_2 \,|\, \alpha_3) = w(\alpha_2 \,|\, \alpha_2)$, in contradiction of $\text{SAP}_{K'}$.) Using Corollary 18 with the points $\langle a, b, c, d\rangle$ and $\langle a, c, b, d\rangle$ and taking $j_1 = j_4 = 0$ with cases $j_2 = j_3 = 1$ and $j_3 = 1, j_2 = 0$, we get

$$\lim_{n\to\infty} w(\alpha_2 \,|\, \alpha_3 \wedge (\phi_n \vee \psi_n)) = \frac{bc + \lambda cb}{c + \lambda b}$$

Using the same points and taking $j_3 = j_4 = 0$ with cases $j_1 = j_2 = 1$ and $j_2 = 0, j_1 = 1$ we get

$$\lim_{n\to\infty} w(\alpha_2 \,|\, \alpha_1 \wedge (\phi_n \vee \psi_n)) = \frac{ba + \lambda ca}{a + \lambda a}$$

$\text{AP}_{K'}$ then requires

$$\frac{bc + \lambda cb}{c + \lambda b} \geq \frac{ba + \lambda ca}{a + \lambda a}$$

which gives

$$\lambda(c - b)(b - c) \geq 0$$

and since $\lambda > 0$, this is only possible if $b = c$, contradicting our assumption.
In other words, $\text{SAP}_{K'}$ is unsatisfiable, and $\text{AP}_{K'}$ only satisfiable if $b = c$ for all $\langle a, b, c, d\rangle$ in the support of μ.

5.2 Non-distance based alternatives

5.2.1 Structural similarity

As mentioned in the previous chapter, it might be objected that a principle like WN actually has another idea behind it; that, for example, an atom $P_1 \wedge P_2 \wedge P_3$ is in some way more similar to an atom $\neg P_1 \wedge \neg P_2 \wedge \neg P_3$ than it is to $P_1 \wedge P_2 \wedge \neg P_3$. WN seems to be motivated by the idea that the structure of an atom, in terms of how many predicates differ in sign, is a significant feature. Recall that we defined the similarity relation $>_W$ to be such that on L_2,

$$\{\alpha_1, \alpha_4\}, \{\alpha_2, \alpha_3\} >_W \{\alpha_1, \alpha_2\}, \{\alpha_1, \alpha_3\}, \{\alpha_4, \alpha_2\}, \{\alpha_4, \alpha_3\}$$

If we consider AP_W in the presence of Px and WN, we need the following inequalities to be satisfied;

1. $w(\alpha_1 \mid \alpha_1 \wedge \theta) \geq w(\alpha_1 \mid \alpha_4 \wedge \theta)$
2. $w(\alpha_1 \mid \alpha_4 \wedge \theta) \geq w(\alpha_1 \mid \alpha_2 \wedge \theta)$
3. $w(\alpha_2 \mid \alpha_2 \wedge \theta) \geq w(\alpha_2 \mid \alpha_3 \wedge \theta)$
4. $w(\alpha_2 \mid \alpha_3 \wedge \theta) \geq w(\alpha_2 \mid \alpha_1 \wedge \theta)$

Theorem 34. *If w is a probability function, with de Finetti prior μ, satisfying Ex, Px, WN and AP_W, then every point in the support of μ has one of the following forms: $\langle a,b,b,a\rangle$, $\langle a,0,0,b\rangle$, $\langle 0,a,b,0\rangle$.*

Proof. Suppose that the point $\langle a,b,c,d\rangle$ is in the support of μ. Then, by WN and Px, the point $\langle d,b,c,a\rangle$ is also in the support of μ. Using Corollary 18 with these points and the requirement from AP_W that

$$\lim_{n\to\infty} w(\alpha_1 \mid \alpha_4 \wedge (\phi_n \vee \psi_n)) \geq \lim_{n\to\infty} w(\alpha_1 \mid \alpha_2 \wedge (\phi_n \vee \psi_n)),$$

Corollary 18 gives that for some $\lambda > 0$,

$$\frac{ad+\lambda da}{d+\lambda a} \geq \frac{ab+\lambda db}{b+\lambda b}$$

which, cross-multiplying and expanding gives

$$-\lambda b(a-d)^2 \geq 0,$$

so either $b=0$ or $a=d$. Similarly, considering $\langle a,b,c,d\rangle$ and $\langle d,c,b,a\rangle$ gives that

$$-c(a-d)^2 \geq 0.$$

If $a \neq d$, we must have $b=c=0$.
Using Corollary 18, this time with the points $\langle a,b,c,d\rangle$ and $\langle a,c,b,d\rangle$ and the requirement from AP_W that

$$\lim_{n\to\infty} w(\alpha_2 \mid \alpha_3 \wedge (\phi_n \vee \psi_n)) \geq \lim_{n\to\infty} w(\alpha_2 \mid \alpha_1 \wedge (\phi_n \vee \psi_n)),$$

we get that

$$\frac{bc+\lambda cb}{c+\lambda b} \geq \frac{ba+\lambda ca}{a+\lambda a}$$

for some $\lambda > 0$, which is equivalent to

$$-a(b-c)^2 \geq 0$$

and the same inequality with $\langle d, b, c, a\rangle$ and $\langle d, c, b, a\rangle$ will give

$$-d(b-c)^2 \geq 0$$

So either $a = d = 0$ or $b = c$. This completes the proof. $\square$

Consider the sentence $\theta = \alpha_1 \wedge \alpha_2 \wedge \alpha_3 \wedge \alpha_4$ and the inequality required by AP_W:

$$w(\alpha_1 \mid \alpha_1 \wedge \theta) \geq w(\alpha_1 \mid \alpha_4 \wedge \theta)$$

Support points with zeros in will make no contribution to these probabilities, wheras all remaining support points have $x_1 = x_4$ and so force equality in this instance. So in fact we have

Corollary 35. *There are no probability functions that satisfy Ex, Px, WN and SAP_W.*

While this formulation of analogy proves unfruitful, the idea of similarity as deriving from structure will be explored further in the next chapter, with more success.

5.2.2 Further-away-ness

Returning for a moment to Hamming Distance, we noted in Chapter 3 that this is rather a crude measure of similarity as it assumes that all distances are comparable, even when they involve totally different predicates. Instead we suggested a subtler similarity relation $>_F$ (see page 40), arising from further-away-ness, and corresponding analogy principles. Not only is SAP_F more intuitively appealing than SAP_H, but the following section will show that (in the presence of Ex + Px + SN) it is satisfied by probability functions on languages of all sizes.

As already noted, $>_H$ and $>_F$ agree on L_2, so the class of probability functions satisfying AP_F on L_2 is exactly that class determined by AP_H.

Notice also that ${}^3\!>_F$ is an extension of ${}^2\!>_F$ as given by the definition on page 42, and so if we continue to take SN as a background assumption, by Proposition 12, a probability function satisfying AP_F on L_3 must marginalise to one satisfying AP_F (equivalently AP_H) on L_2. If we suppose that w satisfies Ex + Px + SN, its de Finetti prior μ must be invariant under permutations from $\mathcal{P}_3$; we know then, from Theorem 22, that the only candidate functions on L_3 are of the form $y_{\langle a,b,b,c,b,c,c,d\rangle}$ with $(a+b)(c+d) = (b+c)^2$ and $\nu y_{\langle a,b,b,a,c,d,d,c\rangle} + (1-\nu)c_0^3$ with $a+c = b+d = 4^{-1}$.

Proposition 36. *The probability function $\frac{3}{2}\lambda y_{\langle a,a,a,a,b,b,b,b\rangle} + (1-\frac{3}{2}\lambda)c_0^3$ satisfies Ex + Px + SN + AP_F on L_3 and is the only such probability function that marginalises to $\lambda y_{\langle 2a,2a,2b,2b\rangle} + (1-\lambda)c_0^2$ on L_2.*

Proof. We know that the only probability function on L_3 satisfying Ex + Px + SN and marginalising to something of the form $\lambda y_{\langle 2a,2a,2b,2b\rangle} + (1-\lambda)c_0^2$ on L_2 must contain in the support of its de Finetti prior μ, at least one $y_{\langle}\alpha,\beta,\beta,\alpha,\gamma,\delta,\delta,\gamma\rangle$ such that $\alpha+\beta = 2a$, $\gamma+\delta = 2b$ and $\alpha+\gamma = \beta+\delta$. So such a function must have in its support the points $\langle\alpha,\beta,\beta,\alpha,\gamma,\delta,\delta,\gamma\rangle$ and $\langle\beta,\alpha,\alpha,\beta,\delta,\gamma,\gamma,\delta\rangle$. Using these points and Corollary 18 consider the inequality required by AP_F:

$$\lim_{n\to\infty} w(\alpha_1 \,|\, \alpha_2 \wedge (\phi_n \vee \psi_n)) \geq \lim_{n\to\infty} w(\alpha_1 \,|\, \alpha_4 \wedge (\phi_n \vee \psi_n))$$

which gives

$$\frac{\alpha\beta + \lambda\beta\alpha}{\beta + \lambda\alpha} \geq \frac{\alpha^2 + \lambda\beta^2}{\alpha + \lambda\beta}$$

for some $\lambda > 0$. This simplifies to

$$\lambda(\beta^2 - \alpha^2)(\alpha - \beta) \geq 0$$

which is only satisfiable if $\alpha = \beta$ and hence $\alpha, \beta = a$. But since $\alpha+\gamma = \beta+\delta$ this forces $\gamma = \delta = b$. This means that our function must simply be

$$\nu y_{\langle a,a,a,a,b,b,b,b\rangle} + (1-\nu)c_0^3$$

We can now confirm that this function satisfies AP_F on L_3.
The inequalities we need to check for any $\theta \in$ QFSL are

(i) $w(\alpha_1 \,|\, \alpha_1 \wedge \theta) \geq w(\alpha_1 \,|\, \alpha_2 \wedge \theta)$

(ii) $w(\alpha_1 \,|\, \alpha_2 \wedge \theta) \geq w(\alpha_1 \,|\, \alpha_4 \wedge \theta)$

(iii) $w(\alpha_1 \mid \alpha_4 \wedge \theta) \geq w(\alpha_1 \mid \alpha_8 \wedge \theta)$

Let $t_1 = \frac{\nu}{6} w_{\langle a,a,a,a,b,b,b,b\rangle}(\theta)$, $t_2 = \frac{\nu}{6} w_{\langle a,a,b,b,a,a,b,b\rangle}(\theta)$, $t_3 = \frac{\nu}{6} w_{\langle a,b,a,b,a,b,a,b\rangle}(\theta)$, $t_4 = \frac{\nu}{6} w_{\langle b,b,b,b,a,a,a,a\rangle}(\theta)$, $t_5 = \frac{\nu}{6} w_{\langle b,b,a,a,b,b,a,a\rangle}(\theta)$, $t_6 = \frac{\nu}{6} w_{\langle b,a,b,a,b,a,b,a\rangle}(\theta)$ and $t_7 = (1-\nu)c_0^3(\theta)$.
Writing out (i) using the definition of w then, we have

$$\frac{a^2t_1 + a^2t_2 + a^2t_3 + b^2t_4 + b^2t_5 + b^2t_6 + 8^{-2}t_7}{at_1 + at_2 + at_3 + bt_4 + bt_5 + bt_6 + 8^{-1}t_7} \geq \frac{a^2t_1 + a^2t_2 + abt_3 + b^2t_4 + b^2t_5 + bat_6 + 8^{-2}t_7}{at_1 + at_2 + bt_3 + bt_4 + bt_5 + at_6 + 8^{-1}t_7}$$

Cross-multiplying and simplifying, this gives

$$\begin{aligned} t_1t_6a(a-b)(a-b) + t_2t_6a(a-b)(a-b) + t_3t_6a(a-b)(a-b) \\ +t_3t_7 8^{-1}(a-8^{-1})(a-b) + t_6t_7 8^{-1}(8^{-1}-b)(a-b) \geq 0 \end{aligned}$$

Similarly, (ii) is equivalent to

$$\frac{a^2t_1 + a^2t_2 + abt_3 + b^2t_4 + b^2t_5 + bat_6 + 8^{-2}t_7}{at_1 + at_2 + bt_3 + bt_4 + bt_5 + at_6 + 8^{-1}t_7} \geq \frac{a^2t_1 + abt_2 + abt_3 + b^2t_4 + bat_5 + bat_6 + 8^{-2}t_7}{at_1 + bt_2 + bt_3 + bt_4 + at_5 + at_6 + 8^{-1}t_7}$$

Which becomes

$$\begin{aligned} t_1t_5a(a-b)(a-b) + t_2t_4b(a-b)(a-b) + t_2t_6a(a-b)(a-b) + \\ t_3t_5b(a-b)(a-b) + t_2t_7 8^{-1}(a-8^{-1})(a-b) + t_5t_7 8^{-1}(8^{-1}-b)(a-b) \geq 0 \end{aligned}$$

and finally, (iii) is equivalent to

$$\frac{a^2t_1 + abt_2 + abt_3 + b^2t_4 + bat_5 + bat_6 + 8^{-2}t_7}{at_1 + bt_2 + bt_3 + bt_4 + at_5 + at_6 + 8^{-1}t_7} \geq \frac{abt_1 + abt_2 + abt_3 + bat_4 + bat_5 + bat_6 + 8^{-2}t_7}{bt_1 + bt_2 + bt_3 + at_4 + at_5 + at_6 + 8^{-1}t_7}$$

which simplifies to

$$\begin{aligned} t_1t_4(a^2-b^2)(a-b) + t_1t_5a(a-b)(a-b) + t_1t_6a(a-b)(a-b) + \\ t_2t_4b(a-b)(a-b) + t_3t_4b(a-b)(a-b) \\ +t_1t_7 8^{-1}(a-8^{-1})(a-b) + t_4t_7 8^{-1}(8^{-1}-b)(a-b) \geq 0. \end{aligned}$$

Clearly, all three inequalities hold and in fact are strict as long as $a \neq b$ and

$\nu > 0$. Finally, note that the measure given to $\langle a,a,a,a,b,b,b,b\rangle$ is $\nu/6$ but must be equal to the measure given in L_2 to $\langle 2a,2a,2b,2b\rangle$ which is $\lambda/4$, and so $\nu = 3/2\lambda$. □

Proposition 37. *c_∞ is the only probability function satisfying Ex + Px + SN + AP_F on L_3 that marginalises to a probability function of the form $y_{\langle ,a,b,b,b^2/ac\rangle}$ on L_2.*

Proof. As seen in Theorem 22, we know that such a function must have a point $\langle \alpha,\beta,\beta,\gamma,\beta,\gamma,\gamma,\delta\rangle$, with $\alpha+\beta = a$, $\beta+\gamma = b$, and $\gamma+\delta = b^2/a$ in the support of its de Finetti prior μ. By Px and SN, $\langle \gamma,\delta,\beta,\gamma,\beta,\gamma,\alpha,\beta\rangle$ must also be a support point. Using these two points, AP_F requires that

$$\lim_{n\to\infty} w(\alpha_1 \,|\, \alpha_1 \wedge (\phi_n \vee \psi_n)) \geq \lim_{n\to\infty} w(\alpha_1 \,|\, \alpha_2 \wedge (\phi_n \vee \psi_n))$$

which, by Corollary 18, gives

$$\frac{\alpha^2+\lambda\gamma^2}{\alpha+\lambda\gamma} \geq \frac{\alpha\beta+\lambda\gamma\delta}{\beta+\lambda\delta}$$

for some λ. Cross-multiplying and simplfying, this gives

$$(\alpha\delta-\beta\gamma)(\alpha-\gamma) \geq 0$$

We know from the properties of w on L_2, that $\alpha+\beta > \beta+\gamma$, and so $\alpha > \gamma$. So we can conclude that $\alpha\delta \geq \beta\gamma$. We also have that $\beta+\gamma > \gamma+\delta$ and so $\beta > \delta$. Using the same inequality but with the points $\langle \beta,\alpha,\gamma,\beta,\gamma,\beta,\delta,\gamma\rangle$ and $\langle \delta,\gamma,\gamma,\beta,\gamma,\beta,\beta,\alpha\rangle$, we get

$$(\beta\gamma-\alpha\delta)(\beta-\delta) \geq 0$$

and so $\beta\gamma \geq \alpha\delta$. So in fact we must have $\alpha\delta = \beta\gamma$.
Now consider the support points $\langle \alpha,\beta,\beta,\gamma,\beta,\gamma,\gamma,\delta\rangle$ and $\langle \beta,\gamma,\alpha,\beta,\gamma,\delta,\beta,\gamma\rangle$ and the requirement from AP_F that

$$\lim_{n\to\infty} w(\alpha_1 \,|\, \alpha_1 \wedge (\phi_n \vee \psi_n)) \geq \lim_{n\to\infty} w(\alpha_1 \,|\, \alpha_2 \wedge (\phi_n \vee \psi_n))$$

which, by Corollary 18, gives

$$\frac{\alpha^2+\lambda\beta^2}{\alpha+\lambda\beta} \geq \frac{\alpha\beta+\lambda\beta\gamma}{\beta+\lambda\gamma}$$

for some λ. Cross-multiplying and simplfying, this gives

$$(\alpha\gamma - \beta^2)(\alpha - \beta) \geq 0$$

Now using the points $\langle \alpha, \beta, \beta, \gamma, \beta, \gamma, \gamma, \delta \rangle$ and $\langle \beta, \alpha\gamma, \beta, \gamma, \beta, \delta, \gamma \rangle$ and the inequality

$$\lim_{n\to\infty} w(\alpha_1 \,|\, \alpha_2 \wedge (\phi_n \vee \psi_n)) \geq \lim_{n\to\infty} w(\alpha_1 \,|\, \alpha_6 \wedge (\phi_n \vee \psi_n))$$

we get, using Corollary 18, that

$$(\alpha\gamma - \beta^2)(\beta - \alpha) \geq 0.$$

Combining these two inequalities we can see that we must have either $\alpha = \beta$ or $\alpha\gamma = \beta^2$. If $\alpha = \beta$, then since $\alpha\delta = \beta\gamma$, we must also have $\gamma = \delta$. But then we know that $(\alpha + \alpha)(\gamma + \gamma) = (\alpha + \gamma)^2$, hence $0 = (\alpha - \gamma)^2$ and so $\alpha = \gamma$. Our function is then c_0^3 as required. On the other hand, if $\alpha\gamma = \beta^2$, we have $\beta^2 > \gamma^2$, hence $\beta > \gamma$, and $\alpha\gamma > \beta\delta$ since $\beta > \delta$. But using the points $\langle \beta, \alpha, \gamma, \beta, \gamma, \beta, \delta, \gamma \rangle$ and $\langle \gamma, \delta, \beta, \gamma, \beta, \gamma, \alpha, \beta \rangle$ and the inequality

$$\lim_{n\to\infty} w(\alpha_1 \,|\, \alpha_1 \wedge (\phi_n \vee \psi_n)) \geq \lim_{n\to\infty} w(\alpha_1 \,|\, \alpha_2 \wedge (\phi_n \vee \psi_n)),$$

Corollary 18 gives

$$(\beta\delta - \alpha\gamma)(\beta - \gamma) \geq 0$$

which contradicts the above observations.

□

This gives us the following result for SAP_F.

Corollary 38. *The only probability functions satisfying Ex + Px + SN + SAP_F on L_3 are of the form $\nu y_{\langle a,a,a,a,b,b,b,b \rangle} + (1-\nu)c_\infty^3$ with $a > b$ and $0 < \nu \leq 1$.*

This can be extended to bigger languages in the obvious way. Let $\langle a, ..., a, b, ..., b \rangle \in \mathbb{D}_q$ be the point with a as the first 2^{q-1} entries and b as the remaining 2^{q-1} entries, and $y_{\langle a,...,a,b,...,b \rangle}$ be the average of the $w_{\vec{x}}$ for all $\vec{x}$ obtainable from $\langle a, ..., a, b, ..., b \rangle$ by a permutation in $\mathcal{P}_q$. We first have the following Lemma.

Lemma 39. *Probability functions on L_q of the form $\nu y_{\langle a,...,a,b,...,b \rangle} + (1-\nu)c_\infty^q$ with $a > b$ and $0 < \nu \leq 1$ satisfy Ex + Px + SN + SAP_F.*

Proof. Let μ_q be the de Finetti prior of such a probability function. The satisfaction of Ex, Px and SN is immediate from the definition. To show that SAP$_F$ is satisfied we observe that there are a limited number of possible coefficients in any inequality obtained by cross-multiplying and simplifying those given by SAP$_F$.

Suppose that $\{\alpha_i, \alpha_j\} >_F \{\alpha_i, \alpha_k\}$. Suppose initially that $i = 1$ (without loss of generality, since any α_i can be transformed to α_1 by a permutation from $\mathcal{P}_q$) and notice that in the point $\vec{x} = \langle a, ..., a, b, ..., b\rangle$, if $x_j = b$, then we must have $x_k = b$, whereas in $\vec{y} = \langle b, ..., b, a, ..., a\rangle$, if $y_j = a$, then $y_k = a$. Since permutations of atoms according to Px and SN preserve Hamming Distance and hence further-away-ness, we in fact have that any point $\vec{y}$ obtained from $\vec{x}$ by such permutations has the property that $(y_j \neq y_i) \Rightarrow (y_k \neq y_i)$.

Now consider any two points in the support of μ_q, $\vec{p}, \vec{q}$, any $\theta \in QFSL$ and the corresponding terms $t_p = w_{\vec{p}}(\theta)$, $t_q = w_{\vec{q}}(\theta)$. Consider the inequality required by SAP$_F$,

$$\frac{p_i p_j t_p + q_i q_j t_q + ...}{p_j t_p + q_j t_q + ...} > \frac{p_i p_k t_p + q_i q_k t_q + ...}{p_k t_p + q_k t_q + ...}$$

Cross multiplying and simplifying, this gives

$$(p_i p_j q_k b + q_i q_j p_k - p_i p_k q_j - q_i q_k p_j) t_p t_q + ... > 0$$

We show that the coefficient of $t_p t_q$ cannot be negative and is sometimes strictly positive. Since this holds for arbitrary $\vec{p}, \vec{q}$, this ensures that the inequality holds as required. If $p_i = q_i$, then in fact

$$(p_i p_j q_k b + q_i q_j p_k - p_i p_k q_j - q_i q_k p_j) = p_i (p_j q_k + q_j p_k - p_k q_j - q_k p_j) = 0$$

So now consider the case when p_i differs from q_i, and that neither $\vec{p}$, nor $\vec{q}$ is the point $\langle 2^{-q}, ..., 2^{-q}\rangle$.

Without loss of generality we can assume $p_i = a$ and $q_i = b$. We know that $p_j = b \Rightarrow p_k = b$, and $q_j = a \Rightarrow q_k = a$. So we need to check the following 9 cases:

	$q_j = q_k = b$	$q_j = q_k = a$	$q_j = b, q_k = a$
$p_j = p_k = b$	(1)	(2)	(3)
$p_j = p_k = a$	(4)	(5)	(6)
$p_j = a, p_k = b$	(7)	(8)	(9)

Clearly cases (1),(2),(4) and (5) result in zero coefficients. The remaining cases are as follows:

(3) The inequality required by SAP_F is:

$$\frac{abt_p + b^2t_q + ...}{bt_p + bt_q + ...} > \frac{abt_p + bat_q + ...}{bt_p + at_q + ...}$$

After cross-multiplying and rearranging, this becomes:

$$(a^2b + b^3 - ab^2 - b^2a)t_pt_q + ... > 0$$

The coefficient of t_pt_q simplifies to $b(a-b)^2$ and so is positive.

(6) The inequality required by SAP_F is:

$$\frac{a^2t_p + b^2t_q + ...}{at_p + bt_q + ...} > \frac{a^2t_p + bat_q + ...}{at_p + at_q + ...}$$

After cross-multiplying and rearranging:

$$(a^3 + b^2a - a^2b - ba^2)t_pt_q + ... > 0$$

The coefficient of t_pt_q simplifies to $a(a-b)^2$ and so is positive.

(7) The inequality required by SAP_F is

$$\frac{a^2t_p + b^2t_q + ...}{at_p + bt_q + ...} > \frac{abt_p + b^2t_q + ...}{bt_p + bt_q + ...}$$

After cross-multiplying and rearranging:

$$(a^2b + b^3 - ab^2 - b^2a)t_pt_q + ... > 0$$

The coefficient of t_pt_q simplifies to $b(a-b)^2$ and so is positive.

(8) The inequality required by SAP_F is

$$\frac{a^2t_p + bat_q + ...}{at_p + at_q + ...} > \frac{abt_p + bat_q + ...}{bt_p + at_q + ...}$$

After cross-multiplying and rearranging:

$$(a^3 + b^2a - a^b - ba^2)t_pt_q + ... > 0$$

The coefficient of t_pt_qs simplifies to$a(a-b)^2$ which is positive.

(9) The inequality required by SAP_F is

$$\frac{a^2t_p + b^2t_q + ...}{at_p + bt_q + ...} > \frac{abt_p + bat_q + ...}{bt_p + at_q + ...}$$

After cross-multiplying and rearranging:

$$(a^3 + b^3 - ab^2 - ba^2)t_pt_q + ... > 0$$

The coefficient of t_pt_q simplifies to $(a^2 - b^2)(a - b)$ and so is positive.

Finally we suppose that one of $\vec{p}$ or $\vec{q}$ is the point $\langle 2^{-q}, ..., 2^{-q}\rangle$. Suppose without loss of generality that $\vec{p}$ is this point. We have the following cases to consider.

$q_i = a$	$q_j = q_k = b$	$q_j = q_k = a$	$q_j = a, q_k = b$	$q_j = b, q_k = a$
	(1)	(2)	(3)	X
$q_i = b$	(4)	(5)	X	(6)

Clearly cases (1), (2), (3) and (5) result in zero coefficients.

(4) The inequality required by SAP_F is

$$\frac{(2^{-q})^2t_p + a^2t_q + ...}{2^{-q}t_p + at_q + ...} > \frac{(2^{-q})^2t_p + abt_q + ...}{2^{-q}t_p + bt_q + ...}$$

After cross-multiplying and rearranging:

$$(2^{-q})^2b + a^2 2^{-q} - (2^{-q})^2a - 2^{-q}ab)t_pt_q + ... > 0$$

The coefficient of t_pt_q simplifies to $2^{-q}(a - b)(a - 2^{-q})$ and so is positive.

(6) The inequality required by SAP_F is

$$\frac{(2^{-q})^2t_p + b^2t_q + ...}{2^{-q}t_p + bt_q + ...} > \frac{(2^{-q})^2t_p + bat_q + ...}{2^{-q}t_p + at_q + ...}$$

After cross-multiplying and rearranging:

$$(2^{-q})^2 a + b^2 2^{-q} - (2^{-q})^2 b - 2^{-q} ba) t_p t_q + ... > 0$$

The coefficient of $t_p t_q$ simplifies to $2^{-q}(a-b)(2^{-q}-b)$ and so is positive.

□

We can now prove the main theorem of this section.

Theorem 40. *Probability functions of the form $\nu y_{\langle a,...,a,b,...,b\rangle} + (1-\nu)c^q_\infty$ with $a > b$ and $0 < \nu \leq 1$ satisfy Ex + Px + SN + SAP$_F$ on L_q and are the only such functions for $q \geq 3$.*

Proof. Lemma 39 shows that such probability functions do indeed satisfy Ex + Px + SN + SAP$_F$. What remains is to show that they are the only such probability functions. The proof of this is by induction, with Corollary 38 as the base case.

For any q, where $\beta_1, ..., \beta_{2^{q-1}}$ are the atoms of L_{q-1}, we fix the ordering of the L_q atoms $\alpha_1, ..., \alpha_{2^q}$ by setting

$$\alpha_{2i-1} = \beta_i \wedge P_q, \quad \alpha_{2i} = \beta_i \wedge \neg P_q$$

Now suppose that the statement of the Theorem holds for L_{q-1} and consider L_q.

Firstly we show that a function of the form

$$\nu_2 y_{\langle a,...,a,b,...,b\rangle} + (1-\nu_2)c^q_\infty$$

for some $0 < \nu_2 \leq 1$ is the only probablity function on L_q satisfying Ex, Px, SN and SAP$_F$ and marginalising to

$$\nu y_{\langle 2a,...,2a,2b,...,2b\rangle} + (1-\nu)c^{q-1}_\infty$$

on L_{q-1}. Any function marginalising to the latter must have in the support of its de Finetti prior μ a point of the form

$$\langle x_1, 2a - x_1, x_2, 2a - x_2, ...\rangle.$$

By Px, we then have that a point

$$\langle x_1, x_2, 2a - x_1, 2a - x_2, ...\rangle$$

is also in the support of μ. If $x_1 + x_2 = 2^{-(q-1)}$ then also $(2a - x_1) + (2a - x_2) = 2^{-(q-1)}$, so $4a = 2\cdot 2^{-(q-1)}$, contradicting the inductive hypothesis. If $x_1 + x_2 = 2b$, then $2a - x_1 + 2a - x_2 = 2a$ gives that $4a - 2b = 2a$, while $2a - x_1 + 2a - x_2 = 2b$ gives that $4a - 2b = 2b$, either way contradicting the inductive hypothesis. So we must have that $x_1 + x_2 = 2a$.

Now note that SN requires that

$$\langle 2a - x_1, x_1, 2a - x_2, x_2, ...\rangle$$

is also in the support of μ, and SAP_F requires

$$\lim_{n\to\infty} w(\alpha_1 \,|\, \alpha_3 \wedge (\phi_n \vee \psi_n)) \geq \lim_{n\to\infty} w(\alpha_1 \,|\, \alpha_4 \wedge (\phi_n \vee \psi_n)).$$

So by Corollary 18 we have

$$\frac{x_1x_2 + \lambda(2a - x_1)(a - x_2)}{x_2 + \lambda(2a - x_2)} \geq \frac{x_1(2a - x_2) + \lambda(2a - x_1)x_2}{(2a - x_2) + \lambda x_2}$$

for some $\lambda > 0$. This simplifies to

$$(2x_1 - 2a)(2x_2 - 2a)) \geq 0$$

But since $x_1 + x_2 = 2a$, this can only hold if $2x_1 = 2x_2 = 2a$. So we have that $x_1 = x_2 = a$.

Exactly the same reasoning can be applied to all consecutive entries $x_i, 2a - x_i, x_j, 2a - x_j$ to deduce $x_i = x_j = a$ and to all consecutive entries $x_i, 2b - x_i, x_j, 2b - x_j$ to deduce $x_i = x_j = b$. Hence our original point must be

$$\langle a, a, a, a, ..., b, b, b, b\rangle.$$

Any point in the support of μ that contains consecutive entries $x_i, 2a - x_i, x_j, 2b - x_j$, by applying a permutation from $\mathcal{P}_q$ becomes one with consecutive entries $x_i, 2a - x_i, x_k, 2a - x_k$ and $x_l, 2b - x_l, x_j, 2b - x_j$. The proof that $x_i = a$ and $x_j = b$ then proceeds as above. So any such point is in fact obtained from the first by a permutation from $\mathcal{P}_q$.

Finally consider any point in the support of μ of the form

$$\langle x_1, 2^{-(q-1)} - x_1, x_2, 2^{-(q-1)} - x_2, ...\rangle.$$

By Px we know that a point

$$\langle x_1, x_2, 2^{-(q-1)} - x_1, 2^{-(q-1)} - x_2, ...\rangle$$

is also in the support of μ. If $x_1 + x_2 = 2a$, then

$$2^{-(q-1)} - x_1 + 2^{-(q-1)} - x_2 = 2^{-(q-2)} - 2a = 2b$$

and similarly, if $x_1 + x_2 = 2b$ then $2^{-(q-1)} - x_1 + 2^{-(q-1)} - x_2 = 2a$.

Now SN requires that $\langle 2^{-(q-1)} - x_1, x_1, 2^{-(q-1)} - x_2, x_2, ...\rangle$ is also in the support of μ, and SAP_F requires

$$\lim_{n\to\infty} w(\alpha_1 \,|\, \alpha_3 \wedge (\phi_n \vee \psi_n)) \geq \lim_{n\to\infty} w(\alpha_1 \,|\, \alpha_4 \wedge (\phi_n \vee \psi_n)).$$

So by Corollary 18 we have

$$\frac{x_1x_2 + \lambda(2^{-(q-1)} - x_1)(2^{-(q-1)} - x_2)}{x_2 + \lambda(2^{-(q-1)} - x_2)} \geq \frac{x_1(2^{-(q-1)} - x_2) + \lambda(2^{-(q-1)} - x_1)x_2}{(2^{-(q-1)} - x_2) + \lambda x_2}$$

for some $\lambda > 0$. This simplifies to

$$(2x_1 - 2^{-(q-1)})(2x_2 - 2^{-(q-1)})) \geq 0$$

So either $2^{-(q-1)} = 2x_1 = 2x_2$, or

$$2a = 2x_1 = 2x_2 > 2^{-(q-1)},$$

or

$$2b = 2x_1 = 2x_2 < 2^{-(q-1)}.$$

Furthermore, the rest of the entries in the vector are subject to the same reasoning. That is, any four consecutive entries

$$x_i, 2^{-(q-1)} - x_i, x_{i+1}, 2^{-(q-1)} - x_{i+1}$$

must be one of

$$2^{-q}, 2^{-q}, 2^{-q}, 2^{-q},$$

$$a, b, a, b,$$

or

$$b, a, b, a.$$

Note that we cannot have a point of the form

$$\langle 2^{-q}, 2^{-q}, 2^{-q}, 2^{-q}, ..., a, b, a, b, ...\rangle$$

in the support of μ. For if we did, Px would require that

$$\langle 2^{-q}, 2^{-q}, 2^{-q}, 2^{-q}, ..., a, a, b, b, ...\rangle$$

be in the support of μ_q. But then we would have

$$\langle 2^{-(q-1)}, 2^{-(q-1)}, ..., 2a, 2b, ...\rangle$$

in the support of μ_{q-1}, contradicting the inductive hypothesis. By exactly the same reasoning, we cannot have a point of the form

$$\langle 2^{-q}, 2^{-q}, 2^{-q}, 2^{-q}, ..., b, a, b, a, ...\rangle$$

in the support of μ. Finally note that we cannot have a point of the form

$$\langle a, b, a, b, ..., b, a, b, a, ...\rangle$$

In the support of μ. For if we did, Px would require that

$$\langle a, a, b, b, ..., b, b, a, a, ...\rangle$$

be in the support of μ_q, and this would require a point

$$\langle 2a, 2b, ..., 2b, 2a, ...\rangle$$

in the support of μ_{q-1}, contradicting the inductive hypothesis.

Hence we have confirmed that anything satisfying Ex, Px, SN and SAP$_F$ on L_q

and marginalising to $\nu_1 y_{\langle 2a,\dots,2a,2b,\dots,2b\rangle} + (1-\nu_1)c_\infty^{q-1}$ on L_{q-1} must be of the form $\nu_2 y_{\langle a,\dots,a,b,\dots,b\rangle} + (1-\nu_2)c_\infty^{q}$.

□

In contrast with Hamming Distance then, this subtler notion of further-away-ness gives rise to a strong analogy principle that is satisfiable on languages L_q for all $q \geq 1$.

Restricting our attention to state descriptions, again there many probability functions satisfying SDAP$_F$ for L_3. And again, in contrast with SDAP$_H$, we can identify a class of probability functions satisfying SDAP$_F$ on L_q for all $q \geq 1$.

Let w_1 be any exchangeable function on L_1, and define

$$w(\theta) := w_1(\theta_1) \times w_1(\theta_2) \times w_1(\theta_3)$$

for any state description θ, where θ_i is the conjunction of all the $\pm P_i$ occurring in θ. In other words, if μ_1 is the de Finetti prior of w_1, we have

$$\begin{aligned} w(\bigwedge_{i=1}^{8} \alpha_i^{n_i}) &= w_1(P_1^{m_1} \wedge \neg P_1^{N-m_1}) \times w_1(P_2^{m_2} \wedge \neg P_2^{N-m_2}) \times w_1(P_3^{m_3} \wedge \neg P_3^{N-m_3}) \\ &= \int_{\mathbb{D}} x^{m_1}(1-x)^{N-m_1} d\mu_1(\vec{x}) \times \int_{\mathbb{D}} x^{m_2}(1-x)^{N-m_2} d\mu_1(\vec{x}) \\ &\quad \times \int_{\mathbb{D}} x^{m_3}(1-x)^{N-m_3} d\mu_1(\vec{x}) \end{aligned}$$

where α_i^m denotes m conjuncts of the form $\alpha_i(a_j)$, $N = n_1 + \dots + n_8, m_1 = n_1 + n_2 + n_3 + n_4, m_2 = n_1 + n_2 + n_5 + n_6$, and $m_3 = n_1 + n_3 + n_5 + n_7$.

For any state description $\theta = \bigwedge_{i=1}^{8} \alpha_i^{n_i}$, let $f_\theta(x)$ denote the function $x^{m_1}(1-x)^{N-m_1}$, $g_\theta(x)$ the function $x^{m_2}(1-x)^{N-m_2}$ and $h_\theta(x)$ the function $x^{m_3}(1-x)^{N-m_3}$.

It should be intuitively clear that w will satisfy SDAP$_F$. For example, we have

$$\begin{aligned} w(\alpha_1 \,|\, \alpha_2 \wedge \theta) &= \frac{\int_{\mathbb{D}} x^2 f_\theta(x) d\mu_1(\vec{x})}{\int_{\mathbb{D}} x f_\theta(x) d\mu_1(\vec{x})} \times \frac{\int_{\mathbb{D}} x^2 g_\theta(x) d\mu_1(\vec{x})}{\int_{\mathbb{D}} x g_\theta(x) d\mu_1(\vec{x})} \times \frac{\int_{\mathbb{D}} x(x-1) h_\theta(x) d\mu_1(\vec{x})}{\int_{\mathbb{D}} (x-1) h_\theta(x) d\mu_1(\vec{x})} \\ &\geq \frac{\int_{\mathbb{D}} x^2 f_\theta(x) d\mu_1(\vec{x})}{\int_{\mathbb{D}} x f_\theta(x) d\mu_1(\vec{x})} \times \frac{\int_{\mathbb{D}} x(x-1) g_\theta(x) d\mu_1(\vec{x})}{\int_{\mathbb{D}} (x-1) g_\theta(x) d\mu_1(\vec{x})} \times \frac{\int_{\mathbb{D}} x(x-1) h_\theta(x) d\mu_1(\vec{x})}{\int_{\mathbb{D}} (x-1) h_\theta(x) d\mu_1(\vec{x})} \\ &= w(\alpha_1 \,|\, \alpha_4 \wedge \theta) \end{aligned}$$

And similarly for all other required inequalities. This way of forming probability functions can be extended to languages of size q, for all q, and will give a class of probability functions satisfying ULi with SDAP$_F$:

Proposition 41. *Let w be a probability function on L_1 satisfying Ex. Then for $q \geq 1$, the probability functions w^q given by*

$$w^q(\theta) = \prod_{i=1}^{q} w(\theta_i)$$

satisfy ULi with SDAP$_F$.

To relate the probability functions satisfying AP$_F$ to this class, note that the probability function $y_{\langle a,b,b,c,b,c,c,d\rangle}$ is the same as

$$y_{\langle x,1-x\rangle} \times y_{\langle x,1-x\rangle} \times y_{\langle x,1-x\rangle}$$

where $x^3 = a$, while the probability function $\nu y_{\langle a,a,a,a,b,b,b,b\rangle} + (1-\nu)c_0^3$ is the same as

$$\frac{\nu}{3}\left((y_{\langle x,1-x\rangle} \times c_0^2 \times c_0^2) + (c_0^2 \times y_{\langle x,1-x\rangle} \times c_0^2) + (c_0^2 \times c_0^2 \times y_{\langle x,1-x\rangle})\right) + (1-\nu)c_0^3 \tag{5.10}$$

where $x = 4a$.

Note that 5.10 is not a product if $\nu \neq 1$, so we know that there are probability functions satisfying SDAP$_F$ other than those described in Proposition 41. The situation then is similar to that for SDAP$_H$; although there are clearly many functions that satisfy SDAP$_F$, these remain to be classified.

Chapter 6

Similarity between sentences

The previous chapters consider analogies that derive from the sharing of similar or identical properties. For such examples of analogical reasoning, the various (S)AP$_S$ seem natural choices for formal principles, but as we have seen (S)AP$_H$ and others are too strong to be useful in languages larger than L_2. SDAP$_H$ is a weakening of AP$_H$ that is satisfied by a larger class of probability functions, though the classification of these proves elusive. But even SDAP$_H$ is sufficiently strong as to rule out many of the familiar probability functions used in inductive logic- for example, those from Carnap's Continuum [3] and the Nix-Paris Continuum [25]. However, as laid out in Chapter 2, an alternative kind of analogy involves similarity between whole sentences. This chapter will show that the principles inspired by this conception of similarity are consistent with Atom Exchangeability and in fact satisfied by all functions from both Carnap's Continuum and the Nix-Paris Continuum.

Recall that Atom Exchangeability was defined as follows:

The Atom Exchangeability Principle (Ax)
If σ is a permutation of $1, 2, ..., 2^q$, then

$$w\left(\bigwedge_{i=1}^{n} \alpha_{h_i}(a_i)\right) = w\left(\bigwedge_{i=1}^{n} \alpha_{\sigma(h_i)}(a_i)\right)$$

Note that Ax in the presence of Ex implies that the probability of a state description $\theta = \bigwedge_{i=1}^{n} \alpha_{h_i}(a_i)$ depends only on the multiset $\{n_1, n_2, ..., n_{2^q}\}$ where n_j is the number of times that the atom α_j appears amongst the α_{h_i}. We call this multiset the *spectrum* of θ.

Previous attempts to incorporate analogical reasoning into Inductive Logic, including Carnap's own, have considered analogy as deriving from the sharing of similar or identical properties. In other words, a principle has been sought that treats some atoms as more similar to one another than others. This approach means that Ax is violated. Although this may not be a failing of such principles, it is interesting to note that the conception of analogy presented in this section is consistent with Ax and is widely satisfied, including by functions of the afore-mentioned continuua.

6.1 The Counterpart Principle

As discussed in Chapter 2, propositions involving entirely different objects and properties can nevertheless possess some kind of similarity; this is what metaphor relies on. We therefore propose the following as a principle of analogical reasoning.

The Counterpart Principle (CP)
For any quantifier free sentence $\theta(a_1, ..., a_m)$, if $\theta'(a_{m+1}, ..., a_{2m})$ is obtained by replacing all predicate and constant symbols in θ by new ones,

$$w(\theta \,|\, \theta'\,) \geq w(\theta)$$

CP can be thought of as saying that for any sentence θ, having already seen the counterpart sentence θ' is at worst irrelevant, and at best offers inductive support for θ.

Our plan now is to show that CP is rather widely satisfied. We first need the following notion:

A probability function w on a language L is said to satisfy Unary Language Invariance, ULi, if there is a family of probability functions $w^{\mathcal{L}}$, one on each unary language $\mathcal{L}$, each satisfying Ex and Px, such that $w^L = w$ and whenever $\mathcal{L} \subset \mathcal{L}'$ then $w^{\mathcal{L}} = w^{\mathcal{L}'} \restriction S\mathcal{L}$.

w is said to satisfy ULi *with* Ax if in addition we can choose these $w^{\mathcal{L}}$ to satisfy Ax.

Note that ULi equivalently means that w can be extended to a probability function w_∞ on the infinite language $L_\infty = \{P_1, P_2, P_3, \ldots\}$ satisfying Px.

Theorem 42. *Let w satisfy Ex, Px and ULi. Then w satisfies the Counterpart Principle.*

Proof. Assume that w satisfies ULi and let w^+ be a probability function on the infinite (unary) language $L^+ = \{P_1, P_2, P_3, \ldots\}$ extending w and satisfying Ex + Px. Let θ, θ' be as in the statement of CP, without loss of generality assume that all the constant symbols appearing in θ are amongst $a_1, a_2, \ldots, a_k$, all the relation symbols appearing in θ are amongst $P_1, P_2, \ldots, P_j$ and for θ' they are correspondingly $a_{k+1}, a_{k+2}, \ldots, a_{2k}$, P_{j+1}, $P_{j+2}, \ldots, P_{2j}$. So with the obvious notation we can write

$$\theta = \theta(a_1, a_2, \ldots, a_k, P_1, P_2, \ldots, P_j),$$
$$\theta' = \theta(a_{k+1}, a_{k+2}, \ldots, a_{2k}, P_{j+1}, P_{j+2}, \ldots, P_{2j})$$

With this notation let

$$\theta_{i+1} = \theta(a_{ik+1}, a_{ik+2}, \ldots, a_{(i+1)k}, P_{ij+1}, P_{ij+2}, \ldots, P_{(i+1)j})$$

so $\theta_1 = \theta$, $\theta_2 = \theta'$. Let $\mathbb{L}$ be the unary language with a single unary relation symbol R and define $\tau : QFS\mathbb{L} \to QFSL^+$ by

$$\tau(R(a_i)) = \theta_i,$$
$$\tau(\neg\phi) = \neg\tau(\phi),$$
$$\tau(\phi \wedge \psi) = \tau(\phi) \wedge \tau(\psi), \qquad \text{etc.}$$

for $\phi, \psi \in QFS\mathbb{L}$.

Now set $v : QFS\mathbb{L} \to [0, 1]$ by

$$v(\phi) = w^+(\tau(\phi)).$$

Then since w^+ satisfies (P1-2) (on $QFSL$) so does v (on $QFS\mathbb{L}$). Also since w^+ satisfies Ex + Px, for $\phi \in QFS\mathbb{L}$, permuting the θ_i in $w(\tau(\phi))$ will leave this value unchanged so permuting the a_i in ϕ with leave $v(\phi)$ unchanged. Hence v satisfies Ex.

By Gaifman's Theorem v has an extension to a probability function on $S\mathbb{L}$ satisfying Ex and hence satisfying PIR. In particular then

$$v(R(a_1) \mid R(a_2)) \geq v(R(a_1)).$$

But since $\tau(R(a_1)) = \theta$, $\tau(R(a_2)) = \theta'$ this amounts to just the Counterpart Principle,

$$w(\theta \,|\, \theta') \geq w(\theta).$$

□

The condition ULi required for Theorem 42 holds for the c^L_λ of Carnap's Continuum and also for the w^δ_L of the Nix-Paris Continuum defined by

$$w^\delta_L = 2^{-q} \sum_{j=1}^{2^q} w_{\vec{e_j}}$$

where $\vec{e_j} = \langle \gamma, \gamma, \dots, \gamma, \gamma + \delta, \gamma, \dots, \gamma, \gamma \rangle$, the δ occurring in the jth coordinate, $\gamma = 2^{-q}(1 - \delta)$ and $0 \leq \delta \leq 1$. Indeed they both satisfy the stronger condition of ULi with Ax.

It is worth noting that we cannot do without ULi here; that is, Ex and Px alone do not guarantee that a probability function satisfies CP. As an example here let $q = 2$ and take w to be the probability function[1]

$$w = 4^{-1}(w_{\langle \frac{1}{2}, \frac{1}{2}, 0, 0 \rangle} + w_{\langle \frac{1}{2}, 0, \frac{1}{2}, 0 \rangle} + w_{\langle 0, \frac{1}{2}, 0, \frac{1}{2} \rangle} + w_{\langle 0, 0, \frac{1}{2}, \frac{1}{2} \rangle}).$$

Then w satisfies Ex and Px. However for $\theta = (P_1(a_1) \wedge \neg P_1(a_2))$, $\theta' = (P_2(a_3) \wedge \neg P_2(a_4))$, a straightforward calculation shows that

$$w(\theta \,|\, \theta') = 0 < w(\theta) = \tfrac{1}{8}.$$

Hence CP fails for this function.

A second argument for restricting attention here to probability functions satisfying ULi (equivalently to probability functions on L_∞) is that without it the lack of available predicates from which to form θ' from θ becomes a significant nuisance factor. Given this, and the fact that the main interest in Pure Inductive Logic is in probability functions satisfying Ax, we shall begin by limiting our attention to probability functions satisfying ULi with Ax.

Before moving on to consider ULi with Ax we introduce the following *irrelevance* principle.

[1]It is straightforward to see that convex combinations of probability functions also satisfy (P1-3) and hence are themselves probability functions.

Weak Irrelevance Principle (WIP)
If $\theta, \phi \in QFSL$ have no constant or predicate symbols in common then

$$w(\theta \mid \phi) = w(\theta).$$

Clearly WIP implies that CP holds with equality. By giving a function that satisfies WIP, hence CP, but not ULi, we can demonstrate that the converse to Theorem 42 does not hold.

Proposition 43. *There exist probability functions that satisfy Ex and CP but not ULi.*

Proof. Let $\vec{b} = \langle 2/3, 1/3 \rangle$. For any $k > 0$, any state description $\theta(\vec{a})$ on L_k can be written as a conjunction of two state description, $\theta_1(\vec{a})$ on L_1, and $\theta_2(\vec{a})$ on $L_k \setminus L_1$.

Define a probability function v_k on L_k by defining for any state description $\theta(\vec{a})$,

$$v_k(\theta(\vec{a})) = w_{\vec{b}}(\theta_1(\vec{a})) \cdot c_\infty(\theta_2(\vec{a}))$$

Any $\theta \in QFSL^+$ actually belongs to L_k for some k, so we can define v on L^+ by

$$v(\theta) = \sum_{\theta_i \models \theta} v_k(\theta_i)$$

where the θ_i are state descriptions on L_k.

Notice that v does not satisfy Px (and hence, does not satisfy ULi); $v(P_1(a_1)) = 2/3$ whereas $v(P_2(a_1)) = 1/2$. However we can show that v satisfies WIP.

To see this, let θ and ϕ be state descriptions from disjoint sublanguages of L^+, L and L' say. Suppose firstly that P_1 does not appear in either of them. Then

$$v(\theta \wedge \phi) = c_\infty(\theta \wedge \phi) = c_\infty(\theta) \cdot c_\infty(\phi) = v(\theta) \cdot v(\phi).$$

Now suppose that P_1 does appear in one of the two sentences; without loss of generality, suppose P_1 appears in θ. Then θ can be written as a conjunction of a state description θ_1 on L_1 and a further state description θ_2 on $L \setminus L_1$. Then

$$\begin{aligned} v(\theta \wedge \phi) &= v(\theta_1 \wedge \theta_2 \wedge \phi) \\ &= w_{\vec{b}}(\theta_1) \cdot c_\infty(\theta_2 \wedge \phi) \\ &= w_{\vec{b}}(\theta_1) \cdot c_\infty(\theta_2) \cdot c_\infty(\phi) \\ &= v(\theta) \cdot v(\phi) \end{aligned}$$

So WIP holds for state descriptions and from this WIP for all quantifier free sentences follows. For let θ, ϕ be any two quantifier free sentences from disjoint sublanguages of L^+, L and L'. Then

$$\begin{aligned} v(\theta \wedge \phi) &= v\left(\bigvee_{\theta_i \models \theta} \theta_i \wedge \bigvee_{\phi_i \models \phi} \phi_i \right) \\ &= \sum_{\theta_i \models \theta} \sum_{\phi_i \models \phi} v(\theta_i \wedge \phi_j) \\ &= \sum_{\theta_i \models \theta} \sum_{\phi_i \models \phi} v(\theta_i) \cdot v(\phi_j) \\ &= v(\theta) \cdot v(\phi) \end{aligned}$$

where the θ_i, ϕ_i are state descriptions on L and L' respectively.
Since v satisfies WIP, it will give

$$v(\theta \,|\, \theta') = v(\theta)$$

for all sentences θ and counterparts θ'. So v satisfies the Counterpart Principle (trivially) but does not satisfy Language Invariance. $\square$

6.2 CP and Ax

A particular class of well understood functions (see for example [25], [14], [27]) satisfying ULi with Ax and CP are those which satisfy the Weak Irrelevance Principle. Of course the motivations for WIP and CP are very different; in fact, they are clearly in tension with one another. WIP captures the intution that knowing a proposition about one set of properties and objects tells us nothing about the likelihood of a proposition involving a totally different set. CP suggests just the opposite – that facts about disjoint sets of properties and objects *can* be brought to bear on one another.

For probability functions satisfying Ex, Ax and WIP we have a precise characterization which we now explain because this notation will be required later.

Let $\mathbb{B}$ be the set of infinite sequences

$$\bar{p} = \langle p_0, p_1, p_2, p_3, ...\rangle$$

of reals such that $p_0 \geq 0, p_1 \geq p_2 \geq p_3 \geq ... \geq 0$ and

$$\sum_{i=0}^{\infty} p_i = 1.$$

For $\bar{p} \in \mathbb{B}$ and $f : \{1, 2, ..., n\} \to \{1, 2, ..., 2^q\}$ let $R_{\bar{p},n} = 1 - \sum_{j=1}^{n} p_j$ and designate

$$f(p) = \left\langle 2^{-q}R_{\bar{p},n} + \sum_{f(j)=1} p_j, 2^{-q}R_{\bar{p},n} + \sum_{f(j)=2} p_j, ..., 2^{-q}R_{\bar{p},n} + \sum_{f(j)=2^q} p_j \right\rangle \in \mathbb{D}_{2^q}.$$

Now let

$$u_n^{\bar{p},L} = 2^{-nq} \sum_f w_{f(\bar{p})}$$

where the f range over all functions $f : \{1, 2, ..., n\} \to \{1, 2, ..., 2^q\}$ and for $\theta \in QFSL$ define

$$u^{\bar{p},L}(\theta) = \lim_{n\to\infty} u_n^{\bar{p},L}(\theta).$$

This limit exists and $u^{\bar{p},L}$ extends to a probability function on L ((see [27]). The fact that the $u_n^{\bar{p},L}$ satisfy Ex and Ax carries over to $u^{\bar{p},L}$, indeed as we vary $\mathcal{L}$ the $u^{\bar{p},\mathcal{L}}$ form a language invariant family so $u^{\bar{p},L}$ satisfies ULi with Ax ([27]).

Notice that if $p_{n+1} = 0$ in $\bar{p}$ then $u^{\bar{p},L} = u_n^{\bar{p},L}$. In particular for $0 \leq \delta \leq 1$ and $\bar{p} = \langle 1 - \delta, \delta, 0, 0, \ldots\rangle$,

$$w_L^{\delta} = u^{\bar{p},L} = u_1^{\bar{p},L},$$

so these $u^{\bar{p},L}$ extend the Nix-Paris Continuum.

A generalization (to polyadic languages) of the following theorem is proved in [26]

Theorem 44. *The $u^{\bar{p},L}$ are exactly the probability functions on L satisfying ULi with Ax and WIP.*

This theorem then provides sufficient conditions under which we have equality in CP (in the presence of ULi with Ax) for *all* $\theta \in QFSL$. Apart from this cause for CP to not be strict there are also certain sentences θ, apart from the obvious $\top, \bot$, which guarantee equality. To describe these we first need to introduce some more notation.

For $\theta \in QFSL$, let $f_\theta(\bar{n})$ denote the number of state descriptions with spectrum $\tilde{n} = \{n_1, n_2, ..., n_{2^q}\}$ appearing in the Disjunctive Normal Form of θ[2]. Note that for any probability function w satisfying Ax and any sentence θ,

$$w(\theta) = \sum_{\tilde{n}} f_\theta(\tilde{n}) w(\bar{n})$$

where $w(\tilde{n})$ is the value of w on some/any state description with spectrum $\tilde{n}$.

The following lemma appears in [26] but for completeness we include a proof here.

Lemma 45. *Let $\theta \in QFSL$ be such that for any probability function w satisfying Ax,*

$$\sum_{\tilde{n}} f_\theta(\tilde{n}) w(\tilde{n}) = c \tag{6.1}$$

for some constant c. Then for each $\tilde{n}$, $f_\theta(\tilde{n}) = c f_\top(\tilde{n})$.

Proof. Given reals $s_1, s_2, ...s_{2^q} \geq 0$, and not all zero, let $v_{\vec{s}}$ be the probability function on L such that

$$v_{\vec{s}}(\tilde{n}) = (2^q!)^{-1} \sum_\sigma s_{\sigma(1)}^{n_1} s_{\sigma(2)}^{n_2} ... s_{\sigma(2^q)}^{n_{2^q}} (s_1 + s_2 + ... + s_{2^q})^{-m}$$

where σ ranges over all permutations of $1, 2, .., 2^q$ and $m = \sum_{i=1}^{2^q} n_i$. Then $v_{\vec{s}}$ satisfies Ax and (6.3) together with the fact that $v_{\vec{s}}(\top) = 1$ gives that

$$\begin{aligned} \sum_{\tilde{n}} f_\theta(\tilde{n}) (2^q!)^{-1} \sum_\sigma s_{\sigma(1)}^{n_1} s_{\sigma(2)}^{n_2} ... s_{\sigma(2^q)}^{n_{2^q}} &= c v_{\vec{s}}(\top)(s_1 + s_2 + ... + s_{2^q})^m \\ &= c \sum_{\tilde{n}} f_\top(\tilde{n}) (2^q!)^{-1} \sum_\sigma s_{\sigma(1)}^{n_1} s_{\sigma(2)}^{n_2} ... s_{\sigma(2^q)}^{n_{2^q}}. \end{aligned}$$

[2]For some fixed set constants which includes all the constants mentioned in θ though the particular fixed set is not important in what follows.

Since we can take each s_i to be algebraically independent this is only possible if the coefficients of $s_1^{n_1} s_2^{n_2} ... s_{2^q}^{n_{2^q}}$ on both sides agree, from which the result follows. $\square$

We shall refer to a $\theta \in QFSL$ such that $w(\theta) = c$ for all probability functions w on L satisfying Ax as being of *constant type*. Notice that in this case c must be rational with denominator (when in lowest form) which divides all the $f_\top(\tilde{n})$.

Since all functions satisfying WIP must trivially satisfy CP, we would like to restrict our attention now to those functions that do not satisfy WIP and determine which instances of CP give strict inequality. We are now in a position to show that there is a class of sentences for which no function (satisfying Ax and ULi) can return a strict inequality of the form given in CP. These are sentences of constant type defined above.

Theorem 46. *Let $\theta(a_1, ..., a_k)$ and $\phi(a_{k+1}, ..., a_{k+r})$ be quantifier free sentences with no predicate or constant symbols in common, and define L_θ and L_ϕ to be the sets of all predicates occurring in θ and ϕ respectively. Suppose also that $w(\theta)$ is constant for all probability functions w on L_θ satisfying Ex + Ax. Then*

$$w(\theta \mid \phi) = w(\theta)^3$$

for all w on $L_\theta \cup L_\phi$ satisfying Ex + Ax.

Proof. Suppose that $\theta(a_1, ..., a_k)$ and $\phi(a_{k+1}, ..., a_{k+r})$ are as in the statement of the theorem, and that

$$w(\theta) = m/n$$

for all probability functions w satisfying Ax.

By putting θ in Disjunctive Normal Form (DNF) we can express θ as a disjunction of state descriptions from L_θ. By Lemma 45,

$$f_\theta(\tilde{n}) = m/n f_\top(\tilde{n})$$

for all spectra $\tilde{n}$. For each spectrum $\tilde{n}$, take just $n^{-1} f_\top(\tilde{n})$ state descriptions from θ with that spectrum, to give some subset $\{\theta_1, \theta_2, ..., \theta_r\}$ of the state descriptions

[3] Note that $w(\theta \mid \phi) = w(\theta)$ is equivalent to $w(\phi \mid \theta) = w(\phi)$

from the DNF of θ. We then have that

$$w\left(\bigvee_{i=1}^{r}\theta_i\right) = \sum_{\tilde{n}} n^{-1} f_{\top}(\tilde{n}) w(\tilde{n}) = m^{-1} w(\theta).$$

For a given state description θ_i of spectrum $\tilde{n}$, it is possible to generate any of the other $f_{\top}(\tilde{n})$ state descriptions of the same spectrum by a combination of permuting atoms and permuting constants. So for each θ_i, $i = 1, ...r$, we can choose n permutations of state descriptions (given by permuting atoms and constants), $\sigma_1^i, ..., \sigma_n^i$ such that

$$\{\, \sigma_1^i(\theta_i), ..., \sigma_n^i(\theta_i), \ | \, i \in \{1, ..., r\}, spec(\theta_i) = \tilde{n} \,\}$$

is the set of all state descriptions with spectrum $\tilde{n}$.

For example, if $L_\theta = \{P, \neg P\}$ and $\theta = (P(a_1) \wedge P(a_2) \wedge P(a_3)) \vee (P(a_1) \wedge \neg P(a_2) \wedge P(a_3)) \vee (P(a_1) \wedge \neg P_1(a_2) \wedge \neg P(a_3)) \vee (\neg P(a_1) \wedge P(a_2) \wedge P(a_3))$ (so in this case $m = 1$ and $n = 2$), then we could choose σ_x^i such that σ_1^i is the identity, for all i, and

$$\begin{aligned}
\sigma_2^1(P(a_1) \wedge P(a_2) \wedge P(a_3)) &= \neg P(a_1) \wedge \neg P(a_2) \wedge \neg P(a_3)\\
\sigma_2^2(P(a_1) \wedge \neg P(a_2) \wedge P(a_3)) &= \neg P(a_1) \wedge P(a_2) \wedge \neg P(a_3)\\
\sigma_2^3(P(a_1) \wedge \neg P(a_2) \wedge \neg P(a_3)) &= \neg P(a_1) \wedge \neg P(a_2) \wedge P(a_3)\\
\sigma_2^4(\neg P(a_1) \wedge P(a_2) \wedge P(a_3)) &= P(a_1) \wedge P(a_2) \wedge \neg P(a_3)
\end{aligned}$$

Note that in general, some of the σ_x^i will differ and some will be the same permutation. The only requirement is that the resulting $\sigma_x^i(\theta_i)$ are mutually exclusive and jointly exhaustive.

We then have that

$$\bigvee_{x=1}^{n}\bigvee_{i=1}^{r}\sigma_x^i(\theta_i) \equiv \top$$

And so, since ϕ is quantifier free and has no predicates or constants in common with θ,

$$\bigvee_{x=1}^{n}\bigvee_{i=1}^{r}\sigma_x^i(\theta_i) \wedge \phi \equiv \phi$$

Now for any probability function w we have

$$w\left(\bigvee_{i=1}^{r}(\theta_i \wedge \phi)\right) = \sum_{i=1}^{r} w\left(\theta_i \wedge \phi\right) = m^{-1}\sum_{\theta_i \models \theta} w\left(\theta_i \wedge \phi\right) = m^{-1}w(\theta \wedge \phi)$$

Moreover for any w satisfying Ex + Ax and any x,

$$w(\sigma_x^i(\theta_i) \wedge \phi) = w(\theta_i \wedge \phi)$$

So for each x,

$$w\left(\bigvee_{i=1}^{r}(\sigma_x^i(\theta_i) \wedge \phi)\right) = m^{-1}w(\theta \wedge \phi)$$

Hence

$$\begin{aligned} w\left(\bigvee_{x=1}^{n}\bigvee_{i=1}^{r}(\sigma_x^i(\theta_i) \wedge \phi)\right) &= \sum_{x=1}^{n} w\left(\bigvee_{i=1}^{r}(\sigma_x^i(\theta_i) \wedge \phi)\right) \\ &= nm^{-1}w(\theta \wedge \phi) \end{aligned}$$

as required. □

So in particular we have the following.

Corollary 47. *For any quantifier free sentence $\theta(a_1, ..., a_k)$ such that $w(\theta)$ is constant for all probability functions w satisfying Ax, and any $\theta'(a_{k+1}, ..., a_{2k})$ obtained by replacing all predicate and constant symbols in θ by new ones,*

$$w(\theta \,|\, \theta') = w(\theta)$$

for all w satisfying Ax.

The converse to Corollary 47 is easily shown.

Proposition 48. *Suppose that for θ, θ' as above,*

$$w(\theta \,|\, \theta') = w(\theta)$$

for all w satisfying Ax. Then $w(\theta)$ is constant for all w satisfying Ax.

Proof. Let w_1, w_2 be distinct probability functions satisfying Ax. Then $2^{-1}(w_1 +$

w_2) also satisfies Ax and so by assumption,

$$\begin{aligned}2^{-1}(w_1+w_2)(\theta\wedge\theta') &= 2^{-1}(w_1+w_2)(\theta)2^{-1}(w_1+w_2)(\theta')\\ &= (2^{-1}(w_1+w_2)(\theta))^2\end{aligned}$$

Multiplying out and re-arranging we get

$$2w_1(\theta\wedge\theta')+2w_2(\theta\wedge\theta') = w_1(\theta)^2+2w_1(\theta)w_2(\theta)+w_2(\theta)^2$$

And then, since w_1, w_2 satisfy Ax, by the assumption we have

$$2w_1(\theta)^2+2w_2(\theta)^2 = w_1(\theta)^2+2w_1(\theta)w_2(\theta)+w_2(\theta)^2$$

and by re-arranging

$$(w_1(\theta)-w_2(\theta))^2 = 0.$$

Hence $w_1(\theta)=w_2(\theta)$ as required. □

Having seen a class of sentences for which equality always holds in the statement of CP, we turn to consider a case in which strict inequality holds for all non-constant $\theta \in QFSL$. In order to do so we recall the following special case of a theorem (Theorem 1) from [20].

Theorem 49. *Any probability function w on L satisfying ULi with Ax can be represented as an integral*

$$w = \int_{\mathbb{B}} u^{\bar{p},L}d\mu \tag{6.2}$$

for some measure μ on the Borel subsets of $\mathbb{B}$.

Conversely any such function defined in this way satisfies ULi with Ax.[4]

Theorem 50. *For a probability function $w=\int_{\mathbb{B}} u^{\bar{p},L}d\mu$, if every point in $\mathbb{B}$ is a support*[5] *point of μ then strict inequality holds in CP whenever θ is not of the constant type.*

[4]Notice that the 'building block functions' here, i.e. the $u^{\bar{p},L}$, are precisely the probability functions satisfying ULi with Ax and WIP. An exactly analogous result holds if we drop Ax here, see [18].

[5]Recall that a point $\vec{e}\in\mathbb{B}$ is in the *support* of μ if $\mu(B)>0$ for all open subsets B of $\mathbb{B}$ containing $\vec{e}$.

Proof. Assume that w can be expressed in this way and let θ, θ' be as in the statement of CP. Then since the $u^{\bar{p},L}$ satisfy WIP,

$$\begin{aligned} w(\theta \wedge \theta') - w(\theta)^2 &= \int_{\mathbb{B}} u^{\bar{p},L}(\theta \wedge \theta')\, d\mu(\bar{p}) - \left(\int_{\mathbb{B}} u^{\bar{q},L}(\theta)\, d\mu(\bar{q}) \right)^2 \\ &= \int_{\mathbb{B}} u^{\bar{p},L}(\theta)^2\, d\mu(\bar{p}) - \left(\int_{\mathbb{B}} u^{\bar{q},L}(\theta)\, d\mu(\bar{q}) \right)^2 \\ &= \int_{\mathbb{B}} \left(u^{\bar{p},L}(\theta) - \int_{\mathbb{B}} u^{\bar{q},L}(\theta)\, d\mu(\bar{q}) \right)^2 d\mu(\bar{p}) \ \geq 0. \end{aligned}$$

Since the support of μ is all of $\mathbb{B}$, and $u^{\bar{p},L}(\theta)$ is continuous (see [27]) the only way we can have $w(\theta \,|\, \theta') = w(\theta)$ is if

$$u^{\bar{p},L}(\theta) = \int_{\mathbb{B}} u^{\bar{q},L}(\theta)\, d\mu(\bar{q})$$

for all $\bar{p} \in \mathbb{B}$. In other words $u^{\bar{p},L}(\theta)$ must be constant for all $\bar{p} \in \mathbb{B}$.

By a result in [27, Chapter 34] any probability function w on L satisfying Ax is of the form

$$w = (\lambda + 1) \int_{\mathbb{B}} u^{\bar{p},L}\, d\mu_1(\bar{p}) \ - \ \lambda \int_{\mathbb{B}} u^{\bar{p},L}\, d\mu_2(\bar{p})$$

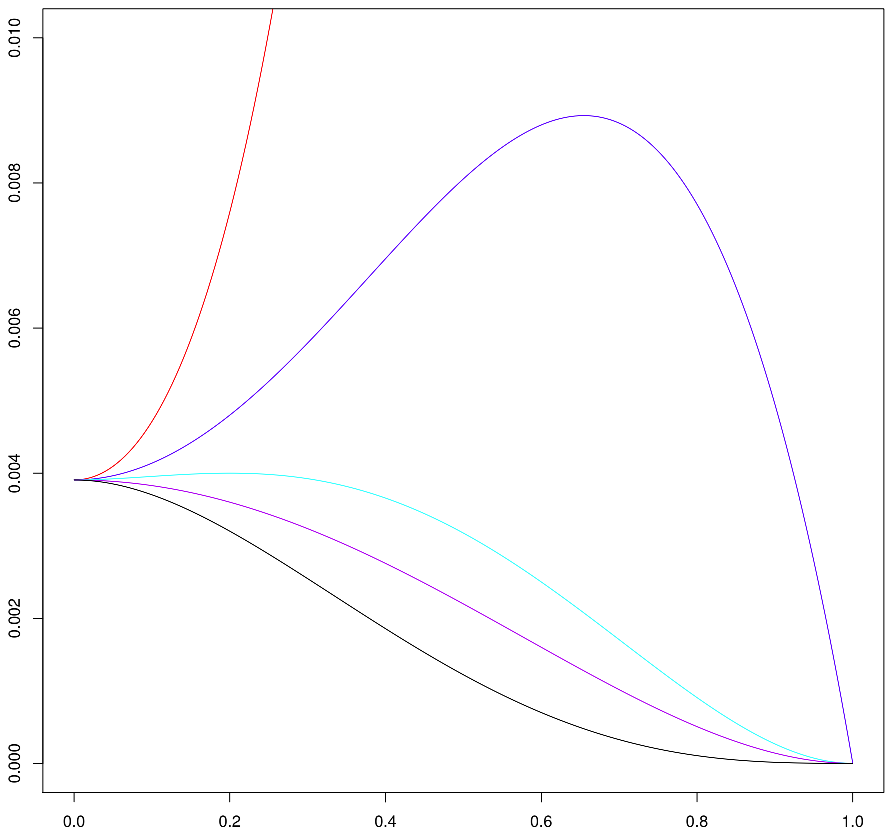
for some $0 \leq \lambda$ and measures μ_1, μ_2 on $\mathbb{B}$. Hence if all the $u^{\bar{p},L}(\theta)$ are constant then so too are all $w(\theta)$ for w satisfying Ax. In other words θ is of the constant type. □

It might have been hoped at this point that any probability function w satisfying ULi with Ax would either satisfy WIP, and so *never* give strict inequality in CP, or else not satisfy WIP and *always* give strict inequality in CP whenever θ was not of the constant type. Unfortunately as the following example shows the situation is not as simple as that.

Let $q = 2$ and define $w_L^\delta = u^{\bar{p},L}$ where $\bar{p} = \langle 1 - \delta, \delta, 0, 0, 0, ... \rangle$. Then for a state description $\theta \in QFSL$ with spectrum $\langle 3, 1, 0, 0 \rangle$ or $\langle 2, 2, 0, 0 \rangle$ the mapping $\delta \mapsto w^\delta(\theta)$ has a maximum point in $(0, 1)$.

This can be seen in the following figure, which shows the probability given to the possible L_2 state descriptions of four objects by the function w^δ as δ varies between 0 and 1. The x-axis represents the value of δ. The coloured lines are the graphs of $w^\delta(\theta)$ for state descriptions θ with spectra as follows:

red: $\langle 4,0,0,0\rangle$ blue: $\langle 3,1,0,0\rangle$ cyan: $\langle 2,2,0,0\rangle$
purple: $\langle 2,1,1,0\rangle$ black: $\langle 1,1,1,1\rangle$



So for a state description θ with spectrum $\langle 3,1,0,0\rangle$ or $\langle 2,2,0,0\rangle$, there are $0<\nu<\tau<1$ such that $w^{\nu}(\theta)=w^{\tau}(\theta)$. Recall that w^{ν} and w^{τ} satisfy WIP; hence if we define $w:=2^{-1}(w^{\nu}+w^{\tau})$ we have that $w(\theta\,|\,\theta')=w(\theta)$. However θ is not of the constant type.

The conditions given by Theorem 50 which ensures that w satisfies CP with strict inequality for all non constant type sentences can be shown to hold for Carnap's Continuum c_λ when $0<\lambda<\infty$, thus ensuring that these c_λ satisfy this strong version of CP. However showing this appears to be quite involved and in general we currently have little insight into when these conditions hold for particular probability functions (unlike the situation with the de Finetti's Representation). We note that the significance of this result depends on the plausibility of the Counterpart Principle as a rational principle. It is either a success for Carnap's Continuum that the c_λ satisfy a form of reasoning by analogy or, if CP is unacceptable, a decisive failure.

6.3 CP and Px

In Theorem 42 we showed that CP follows from ULi, and we then went on to restrict our attention to those probability functions satisfying ULi with Ax, since these are of particular interest. However we might consider what happens when we leave out the requirement of Ax. In this case we have the following:

Proposition 51. *For any $0 < k < 1$ there is no θ such that $w(\theta) = k$ for all w satisfying Ex + Px.*

To show this, we introduce some notation. For $a_1, ..., a_m$ let $\{\phi_1, \phi_2, ...\phi_t\}$ be a maximal set of state descriptions in L appearing in the Disjunctive Normal Form of $\top(a_1, ..., a_m)$ such that no two ϕ_i, ϕ_j can be generated from one another by permutating predicates. For any $\theta \in QFSL$ let $f_\theta(\phi_i)$ be the number of times that a state description obtainable from ϕ_i by permutations of predicates appears in the DNF of θ. Note that for any probability function w satisfying Px and any sentence θ,

$$w(\theta) = \sum_{i=1}^{t} f_\theta(\phi_i) w(\phi_i).$$

For example, for state descriptions of a_1 in L_2 set $\phi_1 = \alpha_1$, $\phi_2 = \alpha_2$, $\phi_3 = \alpha_4$. This is a maximal set as α_3 is obtainable from α_2 by the permutation that swaps P_1 and P_2. Now suppose $\theta = P_1(a_1) \vee P_2(a_1)$. We have $\theta \equiv \alpha_1(a_1) \vee \alpha_2(a_1) \vee \alpha_3(a_1)$, so $f_\theta(\phi_1) = 1$, $f_\theta(\phi_2) = 2$, $f_\theta(\phi_3) = 0$ and $w(\theta) = w(\alpha_1) + 2w(\alpha_2)$.

We can now give the following Lemma.

Lemma 52. *Let $\theta(a_1, ..., a_m)$ be a sentence such that for any probability function w satisfying Ex and Px,*

$$\sum_{i=1}^{t} f_\theta(\phi_i) w(\phi_i) = k \tag{6.3}$$

for some constant k. Then for each ϕ_i, $f_\theta(\phi_i) = k f_\top(\phi_i)$.

Proof. Given reals $s_1, s_2, ...s_{2^q} \geq 0$ and not all zero let $w^{\vec{s}}$ be the probability function on L such that

$$w^{\vec{s}}(\phi_i) = q!^{-1} \sum_\sigma s_{\sigma(1)}^{n_1} s_{\sigma(2)}^{n_2} ... s_{\sigma(2^q)}^{n_{2^q}} (s_1 + s_2 + ... + s_{2^q})^{-m}$$

where $\langle n_1, ..., n_{2^q} \rangle$ is the spectrum of ϕ_i (so $\sum_i n_i = m$) and σ ranges over the permutations of $1, 2, .., 2^q$ that correspond to those permutations of atoms that Px licenses. Then $w^{\vec{s}}$ satisfies Px and (6.3) together with the fact that $w^{\vec{s}}(\top) = 1$ gives that

$$\begin{aligned}\sum_{i=1}^t f_\theta(\phi_i) q!^{-1} \sum_\sigma s_{\sigma(1)}^{n_1} s_{\sigma(2)}^{n_2} ... s_{\sigma(2^q)}^{n_{2^q}} &= k w_{\vec{s}}(\top)(s_1 + s_2 + ... + s_{2^q})^m \\ &= k \sum_{i=1}^t f_\top(\phi_i) q!^{-1} \sum_\sigma s_{\sigma(1)}^{n_1} s_{\sigma(2)}^{n_2} ... s_{\sigma(2^q)}^{n_{2^q}}.\end{aligned}$$

Since we can take each s_i to be algebraically independent this is only possible if the coefficients of $s_1^{n_1} s_2^{n_2} ... s_{2^q}^{n_{2^q}}$ on both sides agree, from which the result follows. □

We are now in a position to prove Proposition 51.

Proof. Suppose that for some $0 < k < 1$ there did exist $\theta(a_1, ..., a_m)$ such that $w(\theta) = k$ for all w satisfying Px. Then $f_\theta(\phi_i) = k f_\top(\phi_i)$ for all ϕ_i. So in particular,

$$f_\theta\left(\bigwedge_{j=1}^m \bigwedge_{i=1}^q P_i(a_j)\right) = k f_\top\left(\bigwedge_{j=1}^m \bigwedge_{i=1}^q P_i(a_j)\right).$$

But

$$f_\top\left(\bigwedge_{j=1}^m \bigwedge_{i=1}^q P_i(a_j)\right) = 1,$$

so since

$$f_\theta\left(\bigwedge_{j=1}^m \bigwedge_{i=1}^q P_i(a_j)\right)$$

must be an integer, we must have that $k = 1$, contradicting the assumption. □

Proposition 51 makes clear that the analogue of Theorem 46 for all functions satisfying Px is satisfied trivially. That is, since the only 'constant' sentences for Px are tautologies, the proposition reduces to the obvious fact that

$$w(\top \mid \phi) = w(\top).$$

The analogue of Proposition 48 is the following:

Proposition 53. *If $\theta \in QFSL$ is such that for all w satisfying Ex and Px,*

$$w(\theta \,|\, \theta') = w(\theta)$$

then $w(\theta)$ is constant for all w satisfying Ex and Px.

Proof. Let w_1, w_2 be any two probability functions satisfying Px. Then $1/2(w_1 + w_2)$ also satisfies Px, hence

$$1/2(w_1 + w_2)(\theta \wedge \theta') = (1/2(w_1(\theta) + w_2(\theta)))^2$$

hence

$$w_1(\theta)^2 + w_2(\theta)^2 - 2w_1(\theta)w_2(\theta) = 0$$

so $w_1(\theta) = w_2(\theta)$ as required. □

This means that there are no non-trivial θ that give $w(\theta \,|\, \theta') = w(\theta)$ for all w satisfying Px.

If we weaken the condition to $w(\theta) = k$ for all w satisfying both Px and SN, then we do get a non-trivial result. Take $\{\phi_1, ..., \phi_t\}$ this time to be a maximal set of state descriptions not inter-derivable from the permutations of atoms (and constants) licensed by both Px and SN (+ Ex) and let $f_\theta(\phi_i)$ be the number of times that a state description obtainable from ϕ_i by permutations or negations of predicates appears in the DNF of θ. Then the obvious modification of the proof of Lemma 52 gives that $f_\theta(\phi_i) = kf_\top(\phi_i)$ for all i. We can then modify the proof of Theorem 46 in the obvious way to give the following.

Theorem 54. *Let $\theta(a_1, ..., a_k)$ and $\chi(a_{k+1}, ..., a_{k+r})$ be quantifier free sentences with no predicate or constant symbols in common, and define L_θ and L_χ to be the sets of all predicates occurring in θ and χ respectively. Suppose also that $w(\theta)$ is constant for all probability functions w on L_θ satisfying Ex + Px + SN. Then*

$$w(\theta \,|\, \chi\,) = w(\theta)$$

for all w on $L_\theta \cup L_\phi$ satisfying Ex + Px + SN.

Proof. Suppose that $\theta(a_1, ..., a_k)$ and $\chi(a_{k+1}, ..., a_{k+r})$ are as in the statement of the theorem, and that

$$w(\theta) = m/n$$

for all probability functions w satisfying Ex, Px and SN.

By putting θ in Disjunctive Normal Form (DNF) we can express θ as a disjunction of state descriptions from L_θ. By Lemma 52,

$$f_\theta(\phi_i) = m/nf_\top(\phi_i).$$

For each ϕ_i from the DNF of θ, take just $n^{-1}f_\top(\phi_i)$ state descriptions from θ that can be obtained from ϕ_i by permutations and negations of predicates, to give some subset $\{\phi_1, \phi_2, ..., \phi_r\}$ of the state descriptions from the DNF of θ. We then have that

$$w\left(\bigvee_{i=1}^{r} \phi_i\right) = \sum_{\phi_i} n^{-1}f_\top(\phi_i)w(\theta_i) = m^{-1}w(\theta).$$

For a given state description ϕ_i, it is possible to generate any of the other $f_\top(\phi_i)$ state descriptions obtainable by permuting and negating predicates by applying such permutations. So for each ϕ_i, $i = 1, ...r$, we can choose n permutations of state descriptions (given by permuting and negating predicates), $\sigma_1^i, ..., \sigma_n^i$ such that

$$\{\, \sigma_1^i(\phi_i), ..., \sigma_n^i(\phi_i),\ | \, i \in \{1, ..., r\}\}$$

is the set of all state descriptions that receive the same probability as ϕ_i from any w satisfying Ex, Px and SN.

For example, if $L_\theta = \{P, \neg P\}$ and $\theta = (P(a_1) \wedge P(a_2) \wedge P(a_3)) \vee (P(a_1) \wedge \neg P(a_2) \wedge P(a_3)) \vee (P(a_1) \wedge \neg P_1(a_2) \wedge \neg P(a_3)) \vee (\neg P(a_1) \wedge \neg P(a_2) \wedge P(a_3))$ (so in this case $m = 1$), then we could choose σ_x^i such that σ_1^i is the identity, for all i, and

$$\begin{aligned}
\sigma_2^1(P(a_1) \wedge P(a_2) \wedge P(a_3)) &= \neg P(a_1) \wedge \neg P(a_2) \wedge \neg P(a_3)\\
\sigma_2^2(P(a_1) \wedge \neg P(a_2) \wedge P(a_3)) &= \neg P(a_1) \wedge P(a_2) \wedge \neg P(a_3)\\
\sigma_2^3(P(a_1) \wedge \neg P(a_2) \wedge \neg P(a_3)) &= \neg P(a_1) \wedge \neg P(a_2) \wedge P(a_3)\\
\sigma_2^4(\neg P(a_1) \wedge \neg P(a_2) \wedge P(a_3)) &= P(a_1) \wedge P(a_2) \wedge \neg P(a_3)
\end{aligned}$$

Note that in general, some of the σ_x^i will differ and some will be the same permutation. The only requirement is that the resulting $\sigma_x^i(\phi_i)$ are mutually exclusive and jointly exhaustive.

We then have that

$$\bigvee_{x=1}^{n}\bigvee_{i=1}^{r}\sigma_x^i(\phi_i) \equiv \top$$

And so

$$\bigvee_{x=1}^{n}\bigvee_{i=1}^{r}\sigma_x^i(\phi_i) \wedge \chi \equiv \chi$$

Now for any probability function w we have

$$w\left(\bigvee_{i=1}^{r}(\phi_i \wedge \chi)\right) = \sum_{i=1}^{r} w\left(\phi_i \wedge \chi\right) = m^{-1} \sum_{\phi_i \models \theta} w\left(\phi_i \wedge \chi\right) = m^{-1}w(\theta \wedge \chi)$$

Moreover for any w satisfying Ex + Px + SN and any x,

$$w(\sigma_x^i(\phi_i) \wedge \chi) = w(\phi_i \wedge \chi)$$

So for each x,

$$w\left(\bigvee_{i=1}^{r}(\sigma_x^i(\phi_i) \wedge \chi)\right) = m^{-1}w(\theta \wedge \chi)$$

Hence

$$\begin{aligned} w\left(\bigvee_{x=1}^{n}\bigvee_{i=1}^{r}(\sigma_x^i(\phi_i) \wedge \chi)\right) &= \sum_{x=1}^{n} w\left(\bigvee_{i=1}^{r}(\sigma_x^i(\phi_i) \wedge \chi)\right) \\ &= nm^{-1}w(\theta \wedge \chi) \end{aligned}$$

as required. □

6.4 Variants on CP

Having looked at CP, the question naturally arises: what happens if instead of changing every symbol occurring in θ we just change some? When we look at the proof of Theorem 42, we can see that the reasoning used there can in fact justify a family of principles which we might call CPi, $i \in \mathbb{N}$.

Counterpart Principle for i Symbols (CPi)
For any $\theta \in QFSL$, let θ' be the result of swapping exactly i of the non-logical

symbols in θ for distinct symbols not already occurring in θ. Then

$$w(\theta \,|\, \theta') \geq w(\theta).$$

We have

Proposition 55. *Let w satisfy Ex, Px and ULi. Then w satisfies CPi for any i.*

Proof. The proof proceeds in the same way as that for Theorem 42, but for completeness we include it here.

Assume that w satisfies ULi and let w^+ be a probability function on the infinite (unary) language $L^+ = \{P_1, P_2, P_3, \ldots\}$ extending w and satisfying Ex + Px. Let θ, θ' be as in the statement of CPi. Without loss of generality let $a_1, ..., a_n$ be the constant symbols and $P_1, ..., P_k$ the predicate symbols that appear in both θ and θ'. Abbreviate $a_1, ..., a_n$ by $\vec{a}$, and $P_1, ..., P_k$ by $\vec{P}$.

Again without loss of generality let $a_{n+1}, ..., a_{n+r}$ and $P_{k+1}, ..., P_{k+s}$ be the subset of constant and predicate symbols appearing in θ that have been changed in θ' to $a_{n+r+1}, \ldots, a_{n+2r}$, P_{k+s+1}, $P_{s+2}, \ldots, P_{k+2s}$, where $r = s = i$.

Now with the obvious notation we can write

$$\theta = \theta(\vec{a}, a_{n+1}, a_{n+2}, \ldots, a_{n+r}, \vec{P}, P_{k+1}, P_{k+2}, \ldots, P_{k+s}),$$

$$\theta' = \theta(\vec{a}, a_{n+r+1}, a_{n+r+2}, \ldots, a_{n+2r}, \vec{P}, P_{k+s+1}, P_{k+s+2}, \ldots, P_{k+2s})$$

With this notation let

$$\theta_{i+1} = \theta(\vec{a}, a_{n+ir+1}, a_{n+ir+2}, \ldots, a_{n+(i+1)r}, \vec{P}, P_{k+is+1}, P_{k+is+2}, \ldots, P_{k+(i+1)s})$$

so $\theta_1 = \theta$, $\theta_2 = \theta'$. Let $\mathbb{L}$ be the unary language with a single unary relation symbol R and define $\tau : QFS\mathbb{L} \to QFSL^+$ by

$$\tau(R(a_i)) = \theta_i,$$

$$\tau(\neg\phi) = \neg\tau(\phi),$$

$$\tau(\phi \wedge \psi) = \tau(\phi) \wedge \tau(\psi), \qquad \text{etc.}$$

for $\phi, \psi \in QFS\mathbb{L}$.

Now set $v : QFS\mathbb{L} \to [0, 1]$ by

$$v(\phi) = w^+(\tau(\phi)).$$

Then since w^+ satisfies (P1-2) (on $QFSL$) so does v (on $QFS\mathbb{L}$). Also since w^+ satisfies Ex + Px, for $\phi \in QFS\mathbb{L}$, permuting the θ_i in $w(\tau(\phi))$ will leave this value unchanged so permuting the a_i in ϕ with leave $v(\phi)$ unchanged. Hence v satisfies Ex.

By Gaifman's Theorem v has an extension to a probability function on $S\mathbb{L}$ satisfying Ex and hence satisfying PIR. In particular then

$$v(R(a_1) \,|\, R(a_2)) \geq v(R(a_1)).$$

But since $\tau(R(a_1)) = \theta$, $\tau(R(a_2)) = \theta'$ this amounts to the Chinese Principle for i symbols.

$$w(\theta \,|\, \theta') \geq w(\theta).$$

□

Notice that the union of all the CPi for $i \in \mathbb{N}$ entails the original Counterpart Principle. For any failure of CP is actually a failure of some CPi. For example, if CP fails for sentences $\theta(P_1, ..., P_n, a_1, ..., a_r)$, $\theta'(P_{n+1}, ..., P_{2n}, a_{r+1}, ..., a_{2r})$, then CP$(n+r)$ fails. The converse does not hold, that is, the Counterpart Principle does not entail the union of all CPi. This can be seen with the following example.

Let $\vec{b} = \langle 0, 2/3, 1/3, 0 \rangle$, so $w_{\vec{b}}$ is a probability function on L_2 satisfying WIP. Any state description θ on L_k can be written as the conjunction of a state description θ_1 on L_2 and another state description θ_2 on $L_k \setminus L_2$. Now define a probability function u_k on L_k by

$$u_k(\theta) = w_{\vec{b}}(\theta_1) \cdot c_\infty(\theta_2)$$

Any $\theta \in QFSL^+$ must belong to $QFSL_k$ for some k, so we can define a probability function u on L^+ by setting

$$u(\theta) = \sum_{\theta_i \models \theta} u_k(\theta_i)$$

Since $w_{\vec{b}}$ and c_∞ both satisfy WIP, u will satisfy WIP, and hence CP.

However, we can show that u does not satisfy CPi for any i. Consider the sentences $P_1(a_1)$ and $P_2(a_1)$, and choose θ, θ' to be consistent sentences not containing a_1, P_1 or P_2 and with $i-1$ transformations of symbols needed to turn one into the other.

Since $\theta \wedge P_1(a_1) \wedge \theta' \wedge P_2(a_1) \models P_1(a_1) \wedge P_2(a_1)$, we have that

$$\begin{aligned} u(\theta \wedge P_1(a_1) \wedge \theta' \wedge P_2(a_1)) &\leq u(P_1(a_1) \wedge P_2(a_1)) \\ &= w_{\vec{b}}(P_1(a_1) \wedge P_2(a_1)) \\ &= 0. \end{aligned}$$

However

$$\begin{aligned} u(\theta \wedge P(a_1)) \cdot u(\theta' \wedge P_2(a_1)) &= w_{\vec{b}}(P_1(a_1)) \cdot c_\infty(\theta) \cdot w_{\vec{b}}(P_2(a_1)) \cdot c_\infty(\theta') \\ &= 2/3 \cdot c_\infty(\theta) \cdot 1/3 \cdot c_\infty(\theta') \\ &> 0 \end{aligned}$$

and so

$$u(\theta \wedge P_1(a_1) \wedge \theta' \wedge P_2(a_1)) < u(\theta \wedge P_1(a_1)) \cdot u(\theta' \wedge P_2(a_1)),$$

contradicting CPi.

We saw in Proposition 43 that the Counterpart Principle does not entail Unary Language Invariance. Since the union of all the CPi is strictly stronger than the Counterpart Principle, a question for future research is whether the union of all the CPi might entail Unary Language Invariance.

Chapter 7

Conclusions

Firstly some remarks on Theorem 42. In Chapter 2 we stated that the object of this thesis was not to argue for a single rational probability function, but to investigate logical dependencies between rational rules. An often found theme is this: a principle which seeks to preserve uniformity in our treatment of the language will logically entail a principle which captures a form of inductive inference. The derivation of PIR from Ex is one example of this. Jeff Paris and Peter Waterhouse have proved (see [37]) that Ax entails another form of singular enumerative induction:

Unary Principle of Induction

$$w(\alpha_i \,|\, \alpha_1^{m_1}\alpha_2^{m_2}...\alpha_{2^q}^{m_{2^q}}) \geq w(\alpha_j \,|\, \alpha_1^{m_1}\alpha_2^{m_2}...\alpha_{2^q}^{m_{2^q}})$$

whenever $m_i > m_j$.

Theorem 42 can be seen as fitting into this pattern: a rule about symmetry (ULi) entails an ampliative rule (CP). One idea often found in the philosophical literature is that induction is justified by a presumption of the uniformity of nature. These results show that induction may be justified, even mandated, by the presumption of uniformity of language.

The above observation, that Theorem 42 is one instance of a more general pattern of results, actually provides a rather nice example of the intuition behind the Counterpart Principle. That is, Theorem 42 is itself a counterpart of Theorem 1.

CP will be of particular interest to philosophers as it is so widely satisfied, including by Carnap's c_λ probability functions. If the underlying language can be freely interpreted, it seems a very strong requirement that a fact involving one set of properties will give inductive support for a proposition involving an entirely different set; as a result, CP raises some interesting issues for the relationship between Pure and Applied Inductive Logic. Pinning down the relationship between Language Invariance and the collection of Counterpart Principles (CPi) is one aim for future research. Another is to extend the results of Chapter 6 to include quantified sentences.

More generally, I hope that this thesis can provide a useful overview of the issues surrounding the representation of reasoning by analogy in PIL. It seems that the analogy principles that have received most attention thus far - those based on similarity between atoms or primitive predicates - create a great many constraints and are not easily satisfied, especially given the popular background conditions derived from symmetry considerations. This is surprising given that some measures of similarity, notably Hamming Distance, seem so naturally aligned with the requirements of Px and SN.

Moving away from Hamming Distance and indeed from simple distance functions altogether changes the situation. Our notion of further-away-ness gives rise to principles (S)AP$_F$ satisfiable on languages of all sizes, even given the combined background conditions of Ex + Px + SN. We can in fact classify these completely, as demonstrated by Theorem 40. If we restrict the conditioning evidence to state descriptions we know that there are even more probability functions on languages of all sizes satisfying the corresponding principle SDAP$_F$. Classifying these completely would be a next step.

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Appendix A

Proof of Theorem 22

In the proof of Theorem 22 we had that

$$\langle x_1 + x_2, x_3 + x_4, x_5 + x_6, x_7 + x_8\rangle = \langle a, b, b, a^{-1}b^2\rangle$$

and

$$\left.\begin{array}{l}\langle x_1 + x_3, x_2 + x_4, x_5 + x_7, x_6 + x_8\rangle,\\ \langle x_1 + x_5, x_3 + x_7, x_2 + x_6, x_4 + x_8\rangle\end{array}\right\} = \left\{\begin{array}{l}\langle a, b, b, a^{-1}b^2\rangle,\\ \text{or } \langle b, a, a^{-1}b^2, b\rangle,\\ \text{or } \langle b, a^{-1}b^2, a, b\rangle,\\ \text{or } \langle a^{-1}b^2, b, b, a\rangle,\end{array}\right.$$

By checking cases, we can show that the only solutions to such a set are the points $\langle x_1, a - x_1, b - x_1, x_1, b - x_1, x_1, a^{-1}b^2 - b + x_1, b - x_1\rangle$ and $\langle x_1, a - x_1, a - x_1, b - a + x_1, a - x_1, b - a + x_1, b - a + x_1, a^{-1}b^2 + a - b - x_1\rangle$. The details of this are as follows.

1. $\langle x_1 + x_3, x_2 + x_4, x_5 + x_7, x_6 + x_8\rangle = \langle c, b, b, a\rangle$. Then $x_2 + x_4 = b = x_3 + x_4$, so $x_2 = x_3$ and hence $c = x_1 + x_3 = x_1 + x_2 = a$, which is a contradiction.

2. $\langle x_1 + x_3, x_2 + x_4, x_5 + x_7, x_6 + x_8\rangle = \langle b, c, a, b\rangle$. So $x_1 + x_3 = b = x_3 + x_4$ so $x_1 = x_4$. But then $c = x_2 + x_4 = x_2 + x_1 = a$, which is a contradiction.

3. $\langle x_1+x_3, x_2+x_4, x_5+x_7, x_6+x_8\rangle = \langle b, a, c, b\rangle$. So as in 2., $x_1+x_3 = b = x_3+x_4$ so $x_1 = x_4$ and $x_5 = x_8$. Now $\langle x_1 + x_5, x_3 + x_7, x_2 + x_6, x_4 + x_8\rangle = \langle x_1 + x_5, x_3 + x_7, x_2 + x_6, x_1 + x_5$, so we have two options:

   (a) $\langle x_1 + x_5, x_3 + x_7, x_2 + x_6, x_1 + x_5 = \langle b, a, c, b\rangle$. So $x_1 + x_2 = x_3 + x_7 = a > c = x_2 + x_6 = x_7 + x_8$, hence $x_1 > x_6$ and $x_3 > x_8$. But

$x_1 + x_3 = x_6 + x_8 = b$, so we have a contradiction.

(b) $\langle x_1+x_5, x_3+x_7, x_2+x_6, x_1+x_5 = \langle b, c, a, b\rangle$. Then $x_3+x_7 = x_7+x_8 = c$ and $x_2 + x_6 = x_1 + x_2 = a$, hence $x_3 = x_8$ and $x_1 = x_6$. So the original point must be $\langle x_1, a - x_1, b - x_1, x_1, b - x_1, x_1, a^{-1}b^2 - b + x_1, b - x_1\rangle$, as required.

4. $\langle x_1+x_3, x_2+x_4, x_5+x_7, x_6+x_8\rangle = \langle a, b, b, c\rangle$. So as in 1., $x_2 = x_3$ and $x_6 = x_7$. Then $\langle x_1+x_5, x_3+x_7, x_2+x_6, x_4+x_8\rangle = \langle x_1+x_5, x_2+x_6, x_2+x_6, x_4+x_8\rangle$, so we have two options:

   (a) $\langle x_1+x_5, x_2+x_6, x_2+x_6, x_4+x_8\rangle = \langle c, b, b, a\rangle$. Then $x_1+x_5 = x_7+x_8 = c < a = x_1 + x_2 = x_4 + x_8$, hence $x_5 < x_2$ and $x_7 < x_4$. But $x_2 + x_4 = x_3 + x_4 = b$ and $x_5 + x_7 = x_5 + x_6 = b$, so we have a contradiction.

   (b) $\langle x_1 + x_5, x_2 + x_6, x_2 + x_6, x_4 + x_8\rangle = \langle a, b, b, c\rangle$. So $x_5 = x_2$, $x_4 = x_7$, and the original point must be $\langle x_1, a - x_1, a - x_1, b - a + x_1, a - x_1, b - a + x_1, b - a + x_1, a^{-1}b^2 + a - b - x_1\rangle$ as required.

We also look at the point

$$\langle x_1 + x_2, x_3 + x_4, x_5 + x_6, x_7 + x_8\rangle = \langle a, a, b, b\rangle$$

together with the system of equations

$$\left.\begin{array}{l}\langle x_1 + x_3, x_2 + x_4, x_5 + x_7, x_6 + x_8\rangle,\\ \langle x_1 + x_5, x_3 + x_7, x_2 + x_6, x_4 + x_8\rangle\end{array}\right\} = \begin{cases}\langle a, a, b, b\rangle,\\ \text{or } \langle a, b, a, b\rangle,\\ \text{or } \langle b, a, b, a\rangle,\\ \text{or } \langle b, b, a, a\rangle,\\ \text{or } \langle \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\rangle,\end{cases}$$

By looking at cases we can show that the only solution is the point

$$\langle x_1, a - x_1, a - x_1, x_1, \tfrac{1}{4} - x_1, \tfrac{1}{4} - a + x_1, \tfrac{1}{4} - a + x_1, \tfrac{1}{4} - x_1\rangle.$$

1. $\langle x_1 + x_3, x_2 + x_4, x_5 + x_7, x_6 + x_8\rangle = \langle \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\rangle$. Then $x_1 + x_2 = a > \frac{1}{4} = x_1 + x_3$, so $x_2 > x_3$. But also, $x_2 + x_4 = \frac{1}{4} < a = x_3 + x_4$, so $x_2 < x_3$, which is a contradiction.

2. $\langle x_1+x_3, x_2+x_4, x_5+x_7, x_6+x_8\rangle = \langle a, b, a, b\rangle$. Then $x_1+x_3 = a = x_1+x_2$, hence $x_2 = x_3$. But then $b = x_2+x_4 = x_3+x_4 = a$, contradicting $a > b$.

3. $\langle x_1+x_3, x_2+x_4, x_5+x_7, x_6+x_8\rangle = \langle b, a, b, a\rangle$. Then $x_2+x_4 = a = x_3+x_4$, so $x_2 = x_3$, but then $b = x_1+x_3 = x_1+x_2 = a$, contradicting $a > b$.

4. $\langle x_1+x_3, x_2+x_4, x_5+x_7, x_6+x_8\rangle = \langle b, b, a, a\rangle$. Recall that by assumption $x_2 = a - x_1$ and $x_4 = a - x_3$. So in this case we have $x_1 + x_3 = b = 2a - (x_1+x_3)$, contradicting $a > b$.

5. $\langle x_1+x_3, x_2+x_4, x_5+x_7, x_6+x_8\rangle = \langle a, a, b, b\rangle$. Then $x_2 = x_3$, $x_6 = x_7$, and the original point is $\langle x_1, a-x_1, a-x_1, x_1, \frac{1}{4}-x_1, \frac{1}{4}-a+x_1, \frac{1}{4}-a+x_1, \frac{1}{4}-x_1\rangle$, as required.

Appendix B

Proof of Theorem 25

We look at each of the options (i) - (iv) in conjunction with the specified constraints to show that only one of two cases can obtain.

(i) $a > d$, $b = c$, $\alpha = \delta$, $\beta > \gamma$.

Assume firstly that $\alpha \geq a$. Then $\delta = \alpha \geq a > d$. The constraints include $(\delta - d)(a\gamma - \alpha b) \geq 0$ and $(a - \alpha)(a\gamma - \alpha b) \geq 0$, so we must have $a\gamma = \alpha b$ or $\alpha = a$. In the first case, we then have $a\beta > \alpha b = \alpha c$ since $\beta > \gamma$, $b = c$, contradicting the requirement that $(a\beta - \alpha c)(a - \alpha) \geq 0$. In the second case, $\alpha = a$, $(\delta - d)(a\gamma - \alpha b)$ becomes $a(\delta - d)(\gamma - b)$, and the further constraint $(\delta - d)(a\beta - \alpha c) \geq 0$ becomes $a(\delta - d)(\beta - c) \geq 0$. But then $\alpha = a$, $\delta > d$, and $\beta > \gamma \geq b = c$, contradicting the fact that $a + b + c + d = 1 = \alpha + \beta + \gamma + \delta$.

So we must have that $a > \alpha = \delta$. Now, from $(a - \delta)(\alpha c - \beta d) \geq 0$ we must have that $\delta b = \alpha c \geq \beta d > \gamma d$. Hence from $(b - \beta)(\gamma d - \alpha c) \geq 0$ we must have $\beta \geq b$, from $(\delta - d)(b\delta - \gamma d) \geq 0$ we must have $\delta \geq d$, and from $(\gamma - c)(\gamma d - \alpha c) \geq 0$ we must have $b = c \geq \gamma$. If $\beta = b$, then $b = c = \beta > \gamma$. Then from $(b - \gamma)(\beta d - c\delta) \geq 0$ we get that $0 \leq \beta d - c\delta = b(d - \delta)$, and so $d \geq \delta$ and in fact $d = \delta$. But then $a > \alpha, b = \beta, c > \gamma$ and $d = \delta$, contradicting $a + b + c + d = 1 = \alpha + \beta + \gamma + \delta$.

So we must have that $\beta > b$. Since we require that $(a - \delta)(\alpha b - \beta d) \geq 0$ and $(b - \beta)(\alpha b - \beta d) \geq 0$, we have that $\delta b = \alpha b = \beta d$; if $\delta = d$ we then get $\beta = b$, contradicting $\beta > b$.

So we must have that $\delta > d$. From $(\beta - c)(b\delta - a\gamma) \geq 0$, if $\gamma = c = b$, then $b(\beta - c)(\delta - a) \geq 0$, which contradicts $a > \alpha = \delta$ and $\beta > b = c$. So we must have $c > \gamma$.

As already shown, we must have $b\delta = \beta d$. We also have, from $(a-\alpha)(a\gamma - \alpha b) \geq 0$ and $(b-\beta)(a\gamma - c\delta) \geq 0$, that $a\gamma = \alpha b = \delta c$.

To summarise then, we must have $a > \alpha = \delta > d$, $\beta > b = c > \gamma$, and $\beta d = \alpha b = a\gamma$. Is it easy to check that under these conditions, each of the constraints is satisfied.

(ii) $a > d, b = c, \alpha > \delta, \beta > \gamma$ and $\alpha\gamma = \beta\delta$. Suppose firstly that $\alpha = a$. Then $a\gamma = \alpha\gamma = \beta\delta$. So $(b-\beta)(a\gamma - c\delta) \geq 0$ becomes $(b-\beta)(\beta\delta - b\delta) \geq 0$, hence $b = \beta$. Then $\alpha b = a\beta > \beta d$, hence from $(\gamma - c)(\alpha b - \beta d) \geq 0$ we get that $\gamma \geq c = b = \beta$, which contradicts $\beta > \gamma$.

Now suppose that $a > \alpha$. Then from $(a-\alpha)(a\gamma - \alpha b) \geq 0$ we must have that $a\gamma \geq \alpha b$, and hence $a\gamma > b\delta = c\delta$. Then from $(\gamma - c)(a\gamma - c\delta) \geq 0$ and $(b-\beta)(a\gamma - c\delta) \geq 0$ we have $\gamma \geq c = b \geq \beta$ which contradicts $\beta > \gamma$.

Finally, if $\alpha > a$, then from $(a-\alpha)(b\delta - \gamma d) \geq 0$ we have that $\gamma d \geq b\delta$. But then $a\gamma > \gamma d \geq b\delta$, so as before, $(\gamma - c)(a\gamma - c\delta) \geq 0$ and $(b-\beta)(a\gamma - c\delta) \geq 0$ generate a contradiction.

(iii) $a > d, b > c, ac = bd, \alpha = \delta$ and $\beta > \gamma$. The proof of the impossibility of this is exactly analogous to case (ii), with every instance of α, β, γ and δ replaced by b, a, d and c respectively, and vice versa.

(iv) $a > d, b > c, ac = bd, \alpha > \delta, \beta > \gamma, \alpha\gamma = \beta\delta$.

Without loss of generality we can suppose that $a \geq \alpha$, hence $a > \delta$. Then from $(a-\delta)(\alpha c - \beta d) \geq 0$ we have that $bd = ac \geq \alpha c \geq \beta d$, hence $b \geq \beta$. Then from $(b-\gamma)(b\delta - a\gamma) \geq 0$ and $(a-\delta)(a\gamma - b\delta) \geq 0$ we have that $b\delta = a\gamma$. From $(b-\gamma)(\beta d - \alpha c) \geq 0$ and $(a-\delta)(\alpha c - \beta d) \geq 0$ we have that $\beta d = \alpha c$. Then $\alpha c > \gamma d$, so from $(\gamma - c)(\gamma d - \alpha c) \geq 0$ we have $c \geq \gamma$. And $\beta d = \alpha c > \delta c$, so from $(\delta - d)(c\delta - \beta d) \geq 0$ we must have $d \geq \delta$. But then $a \geq \alpha, b \geq \beta, c \geq \gamma$ and $d \geq \delta$, so in fact we must have $a = \alpha, b = \beta, c = \gamma, d = \delta$.

Appendix C

R (GNU S) Code

For the calculation of counter-example 4.17 on page 80.

```
> wfun <- function(z,a) { (z[1]^a[1])*(z[2]^a[2])*(z[3]^a[3])*(z[4]^a[4]) }
>
> yfun <- function(p,a){0.125*(wfun(p,a)+wfun(p,c(a[1], a[3], a[2], a[4])) +
wfun(p,c(a[2], a[1], a[4], a[3]))+wfun(p,c(a[2], a[4], a[1], a[3]))+
wfun(p,c(a[3], a[1], a[4], a[2]))+wfun(p,c(a[3], a[4], a[1], a[2]))+
wfun(p,c(a[4], a[2], a[3], a[1])) + wfun(p,c(a[4], a[3], a[2], a[1])))}
>
> ufun<-function(p,q,a){0.5*(yfun(p,a)+yfun(q,a))}
> a<- c(1,6,0,0)
> b<- c(1,5,0,1)
> d<- c(0,6,0,0)
> e<- c(0,5,0,1)
> p<- c(1/49,6/49,6/49,36/49)
> q<- c(1/1681,40/1681,40/1681,1600/1681)
>
> ufun(p,q,a)/ufun(p,q,d)
[1] 0.041021
>
> ufun(p,q,b)/ufun(p,q,e)
[1] 0.08149527
>
```

For the calculation of counter-example 4.18 on page 85.

```
> wfun<-function(p,a){p^a[1]*(1-p)^a[2]}
> yfun<-function(p,a){0.5*(wfun(p,a)+wfun(p,c(a[2],a[1])))}
> ufun<-function(p,q,a){0.5*(yfun(p,a)+yfun(q,a))}
> left<-function(p,q,a,b){ufun(p,q,a)*wfun(0.5,b)}
> right<-function(p,q,a,b){wfun(0.5,a)*ufun(p,q,b)}
> umix<-function(p,q,a,b){0.5*(left(p,q,a,b)+right(p,q,a,b))}
>
> p=0.8
> q=0.9
>
> umix(p,q,c(11,0),c(8,3))/umix(p,q,c(10,0),c(7,3))
[1] 0.4380952
>
> umix(p,q,c(11,0),c(7,4))/umix(p,q,c(10,0),c(6,4))
[1] 0.4381741
>
```

To draw graph on page 127.

```
> d1 <-function(d,n){((1-d)/4)^(n[4]+n[2]+n[3])*((1+3*d)/4)^n[1]}
> d2 <-function(d,n){((1-d)/4)^(n[1]+n[4]+n[3])*((1+3*d)/4)^n[2]}
> d3 <-function(d,n){((1-d)/4)^(n[4]+n[2]+n[1])*((1+3*d)/4)^n[3]}
> d4 <-function(d,n){((1-d)/4)^(n[1]+n[2]+n[3])*((1+3*d)/4)^n[4]}
>
> dfun<-function(d,n){0.25*(d1(d,n)+d2(d,n)+d3(d,n)+d4(d,n))}
>
> x=seq(0,1,length=200)
> y1=dfun(x,c(4,0,0,0))
> y2=dfun(x,c(3,1,0,0))
> y3=dfun(x,c(2,2,0,0))
> y4=dfun(x,c(2,1,1,0))
> y5=dfun(x,c(1,1,1,1))
>
> plot(x,y1,type='l',ylim=c(0,0.01), col="red" xlab="", ylab="")
```

```
> lines(x,y2,col="blue")
> lines(x,y3,col="cyan")
> lines(x,y4,col="purple")
> lines(x,y5,col="black")
```