Real Rainbow Options in Commodity Applications
Valuing Multi-Factor Output Options under Uncertainty

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Abstract

Name of the University:	The University of Manchester
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This thesis focuses on the valuation of real options when there is flexibility given by the choice between two risky outputs. We develop models to value these rainbow options and to determine optimal operating and investment policies. These models are studied in the context of commodity applications because output flexibility is particularly relevant in volatile commodity markets. We provide insights into the behaviour and sensitivities of option values and operating policies and discuss implications for decision-making.

In the early stages of real options theory, research centred on basic options with closed-form solutions, modelling single uncertainty in most cases. The challenge is now to incorporate more complexities in the models in order to further bridge the gap between theoretical models and reality, thereby promoting the widespread application of real options theory in corporate finance.

The new option models developed in this thesis are organised in three self-contained research papers to address specific research problems. The first research paper studies an asset with flexibility to continuously choose the best of two risky commodity outputs by switching between them. We develop quasi-analytical and numerical lattice solutions for this real option model, taking into account operating and switching costs. An empirical application to a flexible fertilizer plant shows that the value of flexibility between the two outputs, ammonia and urea, exceeds the required additional investment cost (given the parameter values) despite the high correlation between the commodities. Implications are derived for investors and policy makers. The real asset value is mainly driven by non-stationary commodity prices in combination with constant operating costs. In the second research paper, we study an asset with flexibility to continuously choose the best of two co-integrated commodities. The uncertainty in two commodity prices is reduced to only one source of uncertainty by modelling the spread, which is mean-reverting in the case of co-integration. Our quasi-analytical solution distinguishes between different risk and discount factors which are shown to be particularly relevant in the context of mean-reversion. In an empirical application, a polyethylene plant is valued and it is found that the value of flexibility is reduced by strong mean-reversion in the spread between the commodities. Hence, operating flexibility is higher when the commodities are less co-integrated. In the third research paper, we develop real option models to value European sequential rainbow options, first on the best of two correlated stochastic assets and then on the spread between two stochastic co-integrated assets. We present finite difference and Monte Carlo simulation results for both, and additionally a closed-form solution for the latter. Interestingly, the sequential option value is negatively correlated with the volatility of one of the two assets in the special case when the volatility of that asset is low and the option is in-the-money. Also, the sequential option on the mean-reverting spread does not necessarily increase in value with a longer time to maturity.

Declaration

No portion of the work referred to in the thesis has been submitted in support of an application for another degree or qualification of this or any other university or other institute of learning.

Copyright Statement

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The contribution and support of certain people was instrumental in completing this thesis. First and foremost, I would like to express my thanks to Dean Paxson who supervised this thesis. His guidance and directions ensured the continuous progression of the thesis and the further development of my research skills. I highly value his advice, his research experience and his insights in the field of real options and beyond. My acknowledgements also include the valuable contributions from Dean Paxson's real options research group, in particular the discussions with Roger Adkins. Finally, I thank my family for their outstanding support.

1 Introduction

This thesis focuses on the valuation of real options when there is flexibility given by the choice between two risky outputs. We develop valuation models for these rainbow options, mostly in the context of commodity applications. The high volatility in commodity markets in recent years increases the relevance of flexibly choosing the output product. We identify the factors driving the option value, give guidance on decision-making with regard to the exercise of options and convey the implications of the new models. The notion of rainbow option was first introduced by Rubinstein (1991b). According to his definition and the definition in Hull (2006), rainbow options are options on two or more risky assets. Rubinstein provides examples for two-colour rainbow options including options delivering the best of two risky assets and cash, calls and puts on the maximum/minimum of two risky assets, spread options, portfolio options or dual-strike options. While the rainbow options in this thesis concentrate on choosing the best of two risky assets, the specific context requires various rainbow option models to value the opportunity.

In the past, the development of real options theory centred around closed-form solutions for basic options, modelling single uncertainty in most cases. The models now need to incorporate more complexities in order to further bridge the gap between theory and practice, thereby promoting the widespread application of real options theory in corporate finance. More recent research confirms this trend, with real option models considering multiple uncertainties, increasing flexibility, and complex decision situations. The availability of abundant computer power facilitates this development since the more complex problems frequently require numerical solutions.

The new option models developed in this thesis are organised in three self-contained research papers to address specific research problems. The first research paper considers an American perpetual rainbow option to switch between two risky and correlated assets which can be exercised at any time repeatedly. We apply the model to value a flexible fertilizer plant. The second paper develops a rainbow option model to switch between two risky assets under the special circumstance of co-integration between these assets, so that the option can be valued based on the mean-reverting price spread between them. This model is applied to a polyethylene plant. The third paper develops a model to value a European sequential rainbow option to obtain the best of two risky assets, and another model to value a European sequential rainbow option to exchange one risky asset for another when the spread between them is mean-reverting (special case of co-integration). Table 1 summarises which option models are developed in each research paper.

Table 1. Overview of the Rainbow Option Models developed in this Thesis

Option model / *Uncertainties*	**Continuous Rainbow Option (American Perpetual Switching)**	**European Sequential Rainbow**
Two-factor model with correlated **gBm** assets	Quasi-analytical Numerical *(Lattice)* **Research paper #1**	Numerical *(Finite difference, Monte Carlo)* **Research paper #3**
One-factor model with **mean-reverting** spread	Quasi-analytical **Research paper #2**	Analytical Numerical *(Finite difference, Monte Carlo)* **Research paper #3**

Building on existing real options theory and models, the option models developed in this thesis subsequently add complexities to the problem specification in order to further narrow the gap between abstract valuation and complex reality. By doing so, the portfolio of real option models available to the research community is increased and their application to practical valuation cases is demonstrated. Moreover, developing more realistic models is a key success factor in enhancing the acceptance of real options as a valuation method in a business environment.

The alternative thesis format based on individual research papers in a format suitable for submission for publication in a peer-reviewed journal is particularly appropriate for this thesis because specific research problems can be studied which are also relevant to business. Furthermore, the relevance of real options theory to valuation issues in the commodity context is stressed by developing research solutions to different problems and applications, which are best presented in individual research papers. This format is also in line with how academic research is conducted and disseminated/communicated and therefore allows a broader reach of the results. The author of this thesis has conceived these research papers, developed the main ideas, the context and the real option models and solutions, identified appropriate applications and reasoned the relevant implications. In short, he carried out the entire research process leading to the research results as presented herein. The research papers have been co-authored by the thesis supervisor who provided valuable ideas and feedback, thought-provoking insights and general guidance.

1.1 Research Objectives and Questions

The objective of the research is to develop valuation models for specific real rainbow options which are likely to be encountered in commodity applications. This is done by using real options theory to account for the flexibility of an active management to react to a changing environment and to incorporate new information. The outcome of the research answers the following research questions:

1. What is the value of an asset with operating flexibility between two correlated stochastic cash flows, when switching is continuously possible by incurring a switching cost, and operating costs are taken into account? At which combinations of cash flow levels is switching optimal?
2. What is the value of flexibility between two co-integrated stochastic cash flows with a mean-reverting spread and when switching is continuously possible by incurring a switching cost? At which spread level is switching optimal?
3. What is the value of a European sequential option on an option on the best of two correlated stochastic assets? And what is the value of a European sequential option on an option on the mean-reverting spread between two stochastic co-integrated assets?

Each paper presented in this thesis addresses one of the research questions above.

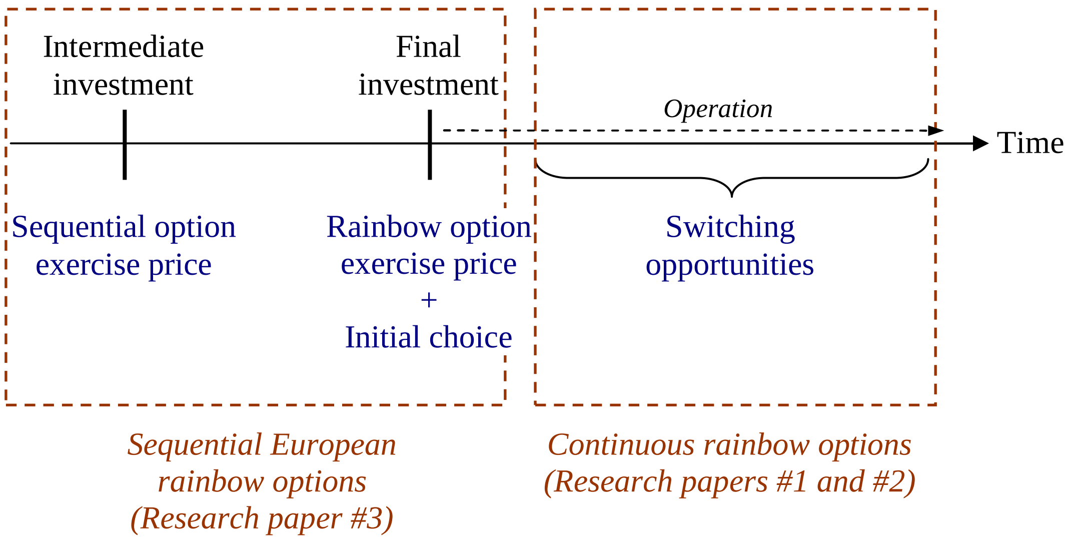
1.2 Contributions to Knowledge

The main contribution to knowledge and to the research community is that the real options discussed in the research papers included in this thesis can now be valued and optimal operating policies determined. These real option models are designed to solve specific valuation problems found in business and therefore incorporate complexities so that reality can be replicated more closely. This in turn increases decision-makers' acceptance of real options theory as the preferred valuation technique under uncertainty and further bridges the gap between theory and practice, thereby contributing to the widespread application and relevance of real options theory. Besides enhancing the understanding of real options involving the choice between two stochastic outputs, we also discuss strategic and policy implications for stakeholders in flexible assets based on the modelling results. The specific research findings are presented in the three research papers individually and discussed in the concluding section of this thesis.

1.3 Thesis Overview

Before presenting the detailed structure of the thesis, it makes sense to describe our real rainbow options along some classifying features. It should be noted that these rainbow options are all directed towards choosing the best of two commodity outputs. The Figure below indicates where the models developed in this thesis fit on a timeline, ranging from investment opportunity to 'in operation'.

Figure 1. The Rainbow Option Models on a Timeline



In all of the above models, uncertainty is given by stochastic output prices. For the continuous rainbow options, which refer to an installed project in operation, this is reflected in stochastic instantaneous cash flows generated from the outputs. Slightly different from that, the stochastic variables in the sequential option models are the present values of the cash flows generated from the outputs. For the sequential option, we are only interested in the present value from the output because there is no output switching subsequent to the initial choice. Table 2 below details the various option characteristics.

Table 2. Characteristics of the Rainbow Option Models

	Commodity price modelling	Commodity prices as gBm	Mean-reverting price spread	**Option type**	Operating flexibility: American perpetual switching option	Sequential European option on the best of two commodities	Sequential European option on the spread between two commodities	**'Best-of' choice**	Initial choice	Single switching	Continuous switching	**Other features**	Switching cost	Operating cost	Temporary suspension	**Solution method**	Analytical	Quasi-analytical	Numerical - Trees	Numerical - Finite differences	Numerical - Monte Carlo simulation
Continuous Rainbow Option on Commodity Outputs (Research Paper #1)																					
a) Continuous switching option		x			x						x		x	(x)				x			
b) One-way switching option		x			x					x			x	x				x			
c) Continuous switching with suspension option		x			x						x		x	x	x				x		
Continuous Rainbow Option in Co-integrated Market (Research Paper #2)			x		x						x		x	x				x			
Sequential Real Rainbow Option (Research Paper #3)																					
a) on two gBm assets		x				x			x											x	x
b) on mean-reverting spread			x				x		x								x			x	x

The remaining part of this thesis is structured as follows. Chapter 2 reviews seminal work on rainbow options. A structured overview of the literature relevant to the models developed in this thesis is presented. We discuss alternative approaches to modelling stochastic commodity prices and their implications on real option valuation. Furthermore, numerical solution techniques are discussed and compared because the complexity of our rainbow option models frequently requires numerical solution techniques. Chapters 3, 4 and 5 present three self-contained research papers on real rainbow options with a focus on commodity applications, each addressing one of the research questions. Chapter 3 (Research paper #1) develops a continuous rainbow option on commodity outputs. This real option model to choose the best of two commodity outputs allows for continuous switching and is presented in the form of a quasi-analytical solution. An alternative numerical lattice solution is developed which is less restrictive on operating costs and further takes the possibility to temporarily suspend operations into account. This model is applied to the valuation of a fertilizer plant, which is flexible to produce ammonia or urea. Chapter 4 (Research paper #2) presents a model for a continuous rainbow option in co-integrated markets. This is an option to choose the best of two commodities when their price spread is mean-reverting. We apply the theoretical model to value a polyethylene plant based on the spread between polyethylene and ethylene. Chapter 5 (Research paper #3) develops sequential real rainbow options, first on two stochastic assets represented by correlated geometric Brownian motion processes, and second, on the mean-reverting spread between two co-integrated stochastic variables. The sequential option follows the logic of the Geske compound option but with the underlying option now being a rainbow option. Chapter 6 concludes by summarising the main findings, discusses assumptions and the applied theory with a critical view, and provides an outlook on potential future research.

2 Review of Rainbow Options and the Commodity Context

Real options analysis is a strategy and valuation approach which allows one to "capitalize on good fortune or to mitigate loss" (Brealey and Myers, 2003, p. 617). Uncertainty, and thereby risk, is considered an opportunity for real options because of asymmetric payoffs (see Kester, 2004). This partly reverses the traditional convention in corporate finance that higher risk reduces the value of an asset. The overall effect of risk can now be interpreted in the framework of Trigeorgis and Mason (1987) who consider an asset value to be the sum of static NPV and option premium. Higher uncertainty reduces the static NPV but increases the option premium.

It should be noted explicitly that real options analysis is not only about getting a number but provides a powerful strategic decision-making framework. It assists in identifying opportunities, limiting risks, and challenging or supporting intuition. The usefulness of real options thinking is also evident from the fact that contingent claims analysis is an economically corrected version of decision tree analysis, as Trigeorgis and Mason (1987) explain.

2.1 Rainbow Options

The term rainbow option was coined by Rubinstein (1991b) as “an option on two or more risky underlying assets“. A two-colour rainbow option is an option on two risky assets, a three-colour rainbow is an option on three risky assets, and so on. Rubinstein imposes the additional condition for a multi-factor option to be a rainbow option that it shall not be possible to transform the problem analytically so that it could be valued as if it was an option on only one underlying risky asset. In other words, Rubinstein assumes that the option value is not homogeneous of degree one in the underlying assets. Homogeneity of degree one is given if multiplying the values of the underlying assets by a factor (λ) gives the same result as multiplying the option value by λ. In general, a function is said to be homogeneous of degree k if multiplying the independent variables by a factor (λ) is equivalent to multiplying the dependent variable by λ to the power of k:

$$V(\lambda\, x, \lambda\, y) = \lambda^k\ V(x, y).$$

An example for an analytical dimension-reducing case is the pure exchange option as modelled by McDonald and Siegel (1986) where homogeneity of degree one allows one to replace the ratio between the exchange variables by a new variable, thereby reducing uncertainty to one dimension. Subsequent authors have largely ignored this additional restriction in the definition of a rainbow option (see Hull, 2006). Examples of rainbow options therefore include options on the best of several risky assets and cash, options on the maximum or minimum of several risky assets, spread options, portfolio options, switching options and exchange options.

The particular challenge in valuing rainbow options is the mathematical complexity introduced by multiple uncertainties in the form of stochastic processes. The partial differential equations describing the option behaviour seldom have analytical solutions, at best quasi-analytical solutions. Seminal work on multi-factor options include Margrabe (1978) and McDonald and Siegel (1986) who value European and American perpetual exchange options, respectively. They make use of the homogeneity of degree one property of the pure exchange option and substitute the ratio of one stochastic asset to the other by a single stochastic variable, reducing the two-factor to a one-factor problem. Rubinstein (1991a) also uses relative prices in his binomial approach to valuing European and American exchange options. Moreover, he shows that exchange options can be used to value the option to choose the best among two assets. The best of two is the sum of one asset plus the option to exchange that asset for the other. Stulz (1982) develops an analytical formula for European options on the minimum or maximum of two assets. He does so by transforming the double integral over the bivariate density function into the cumulative bivariate normal distribution. Johnson (1987) extends this type of option to several underlying assets by using an intuitive approach based on the interpretation of the Black-Scholes formula. Rubinstein (1991b) provides an intuitive, though mathematically precise and analytical approach to some European two-colour rainbow options such as the option delivering the best of two risky assets and cash or the option on the minimum or maximum of two risky assets. For more complex rainbow options, such as spread options or dual-strike options, he approximates the continuous bivariate normal density function with a discrete bivariate binomial density. This can also be extended to account for American-style two-colour rainbow options by setting up a binomial pyramid which is like a three-dimensional tree and models the underlying asset values and their probabilities at each time step.

In the context of forward contracts, Boyle (1989) develops an approach to evaluate the quality option (option to deliver one out of several assets). Pearson (1995) determines the value of a European spread option on two correlated log-normal random walk variables by simplifying the double integration problem to a one factor integration problem which can be approximated with a piecewise linear function. This is done by factoring the joint density function into the product of conditional density function of one asset, given the terminal price of the other asset, and marginal density function of the other asset. The approach is also applicable to other two-factor European options when an analytic expression of the conditional density function is available. Zhang (1998) summarises a number of multi-factor options and acknowledges that "many problems in finance can be converted to options written on the maximum or minimum of two assets" (p. 469). He also studies spread options, which are options on the difference between two prices or indices, and the possibility to model these either as a two-factor option based on the two underlying assets or alternatively as a one-factor option with the spread as the underlying variable. The latter approach has the disadvantage that "the correlation coefficient between the two assets involved does not play any explicit role in the pricing formula" (p. 479) and the sensitivity of the option with respect to the two underlying assets cannot be derived. However, the sensitivity of the option with respect to the spread can be derived in a one-factor model.

Geltner et al. (1996) model the opportunity to develop land for one of two possible uses and determine the appropriate timing. Childs et al. (1996) extend this framework to include the possibility of redevelopment, i.e. they consider switching between alternative uses. Trigeorgis and Mason (1987) discuss the general concept of operating flexibility and in particular the option to switch to a more profitable use (output product) if product flexibility is available. Kulatilaka and Trigeorgis (2004) note that in the presence of switching costs, the value of operating flexibility needs to be determined simultaneously with the optimal switching policy and that multiple switching is actually a complex compound exchange option so that simple option additivity no longer holds.

Real rainbow options play an important role in commodity applications because often there is the option to choose between different inputs or to choose between different outputs. The petrochemical industry offers numerous examples where the producer has the option to sell product A or to process it further and sell product B. If the facilities for the production of B are in place, the producer has the option to sell the best of A and B. Furthermore, commodity prices can be observed as actual market prices or as futures prices, which provides crucial information for modelling the prices in the form of stochastic processes.

Table 3 provides an overview of the literature relevant to the research problems in this thesis.

Table 3. Literature Overview on Rainbow Options

Author	Year	Title	**Stochastic processes**	geometric Brownian motion	arithmetic Brownian motion	Mean-reverting	**Option type**	Call option	Exit/Entry option	Spread option	Exchange option	Switching option	Other Rainbow option	Compound option	**Non-stochastic factors**	Operating cost (Variable)	Switching cost	**Maturity**	European	American finite	American perpetual	**Solution method**	Analytical	Quasi-analytical	Numerical
Abadie and Chamorro	2008	Valuing flexibility: The Case of an Integrated Gasification Combined Cycle power plant		x		x						x		x			x			x	x				x
Adkins and Paxson	2010	Renewing Assets with Uncertain Revenues and Operating Costs		x									x			x	x				x			x	
Adkins and Paxson	2010	Reciprocal Energy Switching Options		x								x				x	x				x		x	x	
Andricopoulos, Widdicks, Duck, Newton	2003	Universal option valuation using quadrature methods		x				x						x					x	x					x
Andricopoulos, Widdicks, Newton, Duck	2007	Extending quadrature methods to value multi-asset and complex path dependent options		x				x					x						x	x					x
Bastian-Pinto, Brandao, Hahn	2009	Flexibility as a Source of Value in the Production of Alternative Fuels: The Ethanol Case		x		x						x									x				x
Bhattacharya	1978	Project Valuation with Mean-Reverting Cash Flow Streams				x																	x		
Bjerksund and Ekern	1995	Contingent Claims Evaluation of Mean-Reverting Cash Flows in Shipping				x		x											x				x		
Bjerksund and Stensland	2006	Closed Form Spread Option Valuation		x						x									x				x		
Brekke and Schieldrop	2000	Investment in Flexible Technologies under Uncertainty		x									x								x				x
Brennan	1979	The Pricing of Contingent Claims in Discrete Time Models		x	x			x											x				x		
Brennan and Schwartz	1985	Evaluating Natural Resource Investments		x				x	x								x				x		x		
Carr	1988	The Valuation of Sequential Exchange Opportunities		x							x			x					x	x			x		
Childs, Ott, Triantis	1998	Capital Budgeting for Interrelated Projects: A Real Options Approach			x			x						x					x				x		
Childs, Riddiough, Triantis	1996	Mixed Uses and the Redevelopment Option		x								x	x				x				x				x

Author	Year	Title	**Stochastic processes**	geometric Brownian motion	arithmetic Brownian motion	Mean-reverting	**Option type**	Call option	Exit/Entry option	Spread option	Exchange option	Switching option	Other Rainbow option	Compound option	**Non-stochastic factors**	Operating cost (Variable)	Switching cost	**Maturity**	European	American finite	American perpetual	**Solution method**	Analytical	Quasi-analytical	Numerical
Cox and Ross	1976	The Valuation of Options for Alternative Stochastic Processes		x	x	x		x											x				x		
Dickey and Fuller	1979	Distribution of the Estimators for Autoregressive Time Series with a Unit Root																							
Dixit and Pindyck	1994	Investment under Uncertainty		x	x	x		x	x		x			x		x	x				x		x	x	x
Duan and Pliska	2004	Option Valuation with Co-integrated Asset Prices		x						x									x						x
Fleten and Näsäkkälä	2010	Gas-fired Power Plants: Investment Timing, Operating Flexibility and CO_2 Capture			x	x			x	x						x	x				x			x	
Geltner, Riddiough, Stojanovic	1996	The Perpetual Option on the Best of Two Underlying Assets		x									x								x				x
Geman	2007	Mean Reversion Versus Random Walk in Oil and Natural Gas Prices				x																			
Geske	1979	The Valuation of Compound Options		x				x						x					x				x		
He and Pindyck	1992	Investments in Flexible Production Capacity		x									x								x				x
Hopp and Tsolakis	2004	Investment Applications in the Shipping Industry		x					x		x								x	x					x
Hull and White	1990	Valuing Derivative Securities Using the Explicit Finite Difference Method		x																					x
Hull	2006	Options, Futures and Other Derivatives		x				x			x			x					x	x			x		x
Johnson	1987	Options on the Maximum or the Minimum of Several Assets		x									x						x				x		
Kirk	1995	Correlation in the energy markets		x						x									x				x		
Kulatilaka and Trigeorgis	2004	The General Flexibility to Switch: Real Options Revisited										x					x			x	x				x
Laughton and Jacoby	1993	Reversion, Timing Options, and Long-Term Decision-Making		x		x		x											x	x					x

Author	Year	Title	**Stochastic processes**	geometric Brownian motion	arithmetic Brownian motion	Mean-reverting	**Option type**	Call option	Exit/Entry option	Spread option	Exchange option	Switching option	Other Rainbow option	Compound option	**Non-stochastic factors**	Operating cost (Variable)	Switching cost	**Maturity**	European	American finite	American perpetual	**Solution method**	Analytical	Quasi-analytical	Numerical
Liu	2007	Options' Prices under Arithmetic Brownian Motion and their Implication for Modern Derivatives Pricing			x			x											x				x		
Margrabe	1978	The value of an option to exchange one asset for another		x							x								x				x		
McDonald and Siegel	1986	The value of waiting to invest		x							x										x		x		
Merton	1973	Theory of Rational Option Pricing		x				x											x	x			x		
Näsäkkälä and Fleten	2005	Flexibility and Technology Choice in Gas Fired Power Plant Investments			x	x		x		x						x					x			x	
Paxson	2007	Sequential American Exchange Property Option		x							x			x						x				x	
Paxson and Pinto	2005	Rivalry under Price and Quantity Uncertainty		x																	x		x		
Rubinstein	1991	Somewhere Over the Rainbow		x						x			x						x				x		x
Schwartz	1997	The Stochastic Behaviour of Commodity Prices: Implications for Valuation and Hedging				x																			
Shimko	1994	Options on Futures Spreads: Hedging, Speculation, and Valuation		x		x				x									x					x	
Smith and McCardle	1999	Options in the Real World: Lessons Learned in Evaluating Oil and Gas Investments		x		x								x					x						x
Sodal, Koekebakker, Aaland	2006	Value Based Trading of Real Assets in Shipping under Stochastic Freight Rates				x				x		x					x				x		x		
Sodal, Koekebakker, Aaland	2007	Market Switching in Shipping – A Real Option Model Applied to the Valuation of Combination Carriers				x				x		x				x	x				x		x		
Song, Zhao, Swinton	2010	Switching to Perennial Energy Crops under Uncertainty and Costly Reversibility		x		x						x					x				x				x

Author	Year	Title	**Stochastic processes**	geometric Brownian motion	arithmetic Brownian motion	Mean-reverting		**Option type**	Call option	Exit/Entry option	Spread option	Exchange option	Switching option	Other Rainbow option	Compound option		**Non-stochastic factors**	Operating cost (Variable)	Switching cost		**Maturity**	European	American finite	American perpetual		**Solution method**	Analytical	Quasi-analytical	Numerical
Stulz	1982	Options on the Minimum or the Maximum of two risky Assets		x										x								x					x		
Triantis and Hodder	1990	Valuing Flexibility as a Complex Option			x					x				x								x					x		
Trigeorgis and Mason	1987	Valuing Managerial Flexibility											x										x						x
Tvedt	2000	The Ship Lay-Up Option and Equilibrium Freight Rates		x						x								x	x										
Tvedt	2003	Shipping Market Models and the Specification of the Freight Rate Process				x																					x		
Zhang	1998	Exotic Options: A Guide to Second Generation Options		x					x		x	x		x	x							x	x				x		

2.2 Stochastic Processes for Commodity Prices

Option pricing models have been mostly developed around financial assets, in particular common stock. In the majority of cases, common stock is modelled by a geometric Brownian motion process. In real option applications, the uncertainty in the underlying assets is typically more difficult to observe than is the case for financial options on common stock. In this section, we discuss different stochastic processes used to model commodity prices and their advantages/disadvantages in the context of real options valuation.

Stochastic processes can be classified into continuous (diffusion) and discontinuous (jump) processes. It can be argued that jumps play a role in the spot price behaviour of commodities, such as sudden demand or supply changes influence electricity prices or political crises impact on the oil price. So far, however, commodity prices have been modelled mostly as continuous processes because of an improved analytical tractability in the mathematical operations on these processes. This is also why our focus is on continuous processes. The stochastic element is modelled as a Wiener process which is a random variable with a standard normal distribution multiplied by the square root of time. Diffusion processes can be either non-stationary or stationary, where the basic model for the former is the Brownian motion with drift, and the basic model for the latter is the first-order autoregressive process, also called the Ornstein-Uhlenbeck mean-reverting process. Brownian motion and mean-reversion both have the Markov property, meaning that future values depend only on the current value and not on past values. Both processes can be arithmetic or geometric. An arithmetic process is normally distributed and can take any value, positive or negative. This is why arithmetic processes are suitable to model profits and spreads. A geometric process is log-normally distributed and therefore takes positive values only, which is why it is commonly used to model prices.

The geometric Brownian motion is the most frequently used stochastic process to model asset prices and has the advantage that closed-form solutions for valuations dependent on these prices can be obtained in many cases, such as the Black-Scholes formula for European call and put options. Closed-form solutions improve tractability and transparency and thereby provide an important pillar for further theory development. Brennan and Schwartz (1985) model commodity prices as random walk described by geometric Brownian motion in order to value an investment opportunity in natural resources. They also discuss the concept of convenience yield applied to commodities as "the flow of services that accrues to an owner of the physical commodity but not to the owner of a contract for future delivery of the commodity" (p. 139). The convenience yield is mathematically equivalent to the dividend yield (payout ratio) on common stock, and therefore describes the shortfall of the price change in relation to the required return on the price. Paddock et al. (1988) also use geometric Brownian motion to model asset values based on oil prices and determine the option value to develop oil reserves. They argue that the value of an oil reserve follows geometric Brownian motion because the owner requires a return on that asset just as if it was common stock. Dixit and Pindyck (1994) develop most of their generic real option models based on the assumption that the underlying assets follow geometric Brownian motion. They also do so when applying real options theory to value oil investments.

One might argue that commodity prices are determined primarily by supply and demand economics and that the long-term equilibrium price is therefore bound by the long-term production cost. If prices exceed the long-term equilibrium price, above-average profits can be made inducing more producers to enter the market which increases the downward pressure on the market price. On the other hand, if prices fall below the variable cost, some high-cost producers drop out of the market and thereby tighten supply, which stabilises the market price. These dynamics might be better represented by a price following a mean-reverting process. Bessembinder et al. (1995) test whether investors expect prices to revert. On the basis of regression analyses between the change in spot prices and associated changes in the futures term structure, they find that equilibrium prices of agricultural commodities and crude oil exhibit strong mean-reversion. Mean-reversion in equilibrium prices of metals is much less but still statistically significant. Since their analysis is limited to maturities of up to two years, conclusions on the price behaviour in the medium and long-term cannot be inferred from this. According to Smith and McCardle (1999), managers in oil and gas companies believe that oil prices revert to some long-run average. Mean-reversion in oil prices is supported when using empirical data spanning a period of 94 years. Smith and McCardle (1999) value investment opportunities when the underlying commodity (oil) is modelled as geometric mean-reversion as compared to geometric Brownian motion. They find that the option of flexibility is valued lower when the underlying variable is mean-reverting because of the reduced uncertainty, and point out that the methodology and findings could also be valid in other commodity applications. This effect is consistent with the Laughton and Jacoby (1993) explanation that mean-reversion reduces uncertainty which leads to a lower discount factor and higher present value (discounting effect) but at the same time reduces the option value (variance effect).

Alternative models for commodity prices have been developed, most of them based on the basic processes explained above. Schwartz (1997) analyses three different models reflecting mean-reversion in commodity prices: A one-factor model with a mean-reverting price process, a two-factor model with random walk in spot prices and a stochastic convenience yield, and a three-factor model with a stochastic interest rate in addition. He finds empirical evidence that copper and oil prices exhibit mean-reversion, but not gold prices. Furthermore, he notes that the real options approach leads to investments being made too late if mean-reversion in the commodity prices is ignored. This is also what Abadie and Chamorro (2008) find when they analyse investment opportunities on flexible power plants based on a fuel price following inhomogeneous geometric Brownian motion (IGBM). IGBM is a mean-reverting stochastic process where the uncertainty is log-normally distributed. Schwartz and Smith (2000) develop a two-factor model for commodity prices with mean-reversion in short-term prices and random walk in the long-term (equilibrium) prices. Their model is equivalent to the Gibson and Schwartz (1990) model of random walk prices and stochastic convenience yields because the short-term price deviation is a linear function of the convenience yield. Based on empirical comparisons for crude oil, they find that this model outperforms simple geometric Brownian motion and mean-reversion models. Cortazar and Schwartz (1994) establish a direct link between the stochastic processes of commodity futures and the corresponding spot prices and use principal components analysis to develop a three-factor model of copper prices from empirical data.

Several authors discuss the behaviour of commodity prices and compare the suitability of different stochastic models. Baker et al. (1998) take a critical stance towards the assumption that commodity prices follow a pure random walk process. However, while noting that "the underlying economics of the marketplace constrains the rise or fall of a commodity's price" (p. 122) and thereby implying that not all price changes are permanent, they also acknowledge that not all price changes are temporary. Hence, a model combining mean-reversion with random walk in the long-term seems appropriate from their point of view. In their concluding remarks, Baker et al. (1998) point out that the more complex models for commodity prices increase the fidelity in the quality of the underlying "engine" but this "extra complexity can impose severe burdens on the computations required for many valuations" (p. 145), thus decreasing the fidelity in the valuation results. Pindyck (2001) stresses the validity of two-factor models for commodity prices with short-term variations and random walk in the long-term, but concludes by saying that valuation errors in real option applications have been shown to be relatively small if a pure random walk is assumed instead.

Song et al. (2010) analyse returns to land use and stipulate that agricultural returns can be theoretically justified by both geometric Brownian motion and mean-reversion. This is supported by statistical tests which are equivocal regarding the stationarity or non-stationarity of agricultural returns. Stationarity of a stochastic variable is typically tested with an Augmented Dickey Fuller (ADF) test which assumes under the null hypothesis that the time series has a unit root, implying that the series is non-stationary. One issue here is that the null hypothesis might also be rejected due to insufficient information, so that the conclusion regarding non-stationary behaviour might be erroneous. It is therefore advisable to perform an additional test examining the opposite null hypothesis, i.e. that the series is stationary, which the Kwiatkowski-Phillips-Schmidt-Shin (KPSS) test does. While the choice of the stochastic model influences the real option value, the optimal investment timing, the (a)symmetry of switching boundaries and the effect of uncertainty over time (short-term vs. long-term variance), the exact effect often depends on the specific problem and parameters. Dixit and Pindyck (1994) note that performing statistical tests, in particular the unit root test, is a suitable approach to determine whether the price of a commodity follows random walk or mean-reversion. The results depend, however, to a large degree on the data sampling period because mean-reversion for commodities can be very slow. Empirical analysis on crude oil and copper suggests that the random walk hypothesis cannot be rejected for a period of 30-40 years. For longer time periods, mean-reversion becomes more significant. They note that besides statistical tests, theoretical considerations and the requirement for analytical tractability should play a role in modelling commodity prices when used for valuation purposes. In other words, when commodity prices represent the underlying uncertainty in real option problems, one needs to balance the need for realistic stochastic models and the need for developing a transparent valuation framework and solution.

In our econometric analyses of empirical commodity data, such as regression analysis or testing for stationarity (unit root test, KPSS test) or other diagnostic tests, we use the EViews 6 Student Version.

2.3 Overview of Solution Techniques

Pindyck (1991) demonstrates that investment opportunities can be valued both by contingent claims analysis and by dynamic programming. Contingent claims analysis uses a portfolio of assets which includes the derivative (e.g. investment opportunity) and the underlying assets in such proportions that all risk is eliminated, so that the contingent claim can be described by a differential equation. Dynamic programming is based on the fundamental equation of optimality (Bellman equation) which stipulates that the required return on an asset must equal the immediate payout plus the expected change in the asset value. It also leads to a differential equation but uses a discount rate equal to the required return on the derivative which is not known and dependent on non-linear cash flows. This discount factor problem can be solved by using the risk-free interest rate provided that the underlying stochastic process has been transformed into its risk-neutral form, which is effectively the risk-neutral valuation applied to contingent claims. Once the differential equation is specified together with the boundary conditions, it is desirable to obtain an analytical solution for the value of the contingent claim. However, in most real options applications, analytical solutions cannot be obtained due to the complexity involved in the models. The best alternative is a quasi-analytical solution which requires a set of simultaneous equations. Although the solution is then not available in explicit form, thereby compromising transparency and efficiency of computation, a quasi-analytical solution is still preferable to solutions by pure numerical techniques because the value matching and smooth pasting conditions can be verified transparently, more general findings can be produced and the computations are typically less onerous. If neither analytical nor quasi-analytical solutions are available, one needs to refer to numerical solution methods.

There are numerous numerical solution techniques available. We discuss trees/lattices, finite differences and Monte Carlo simulation since they are most relevant in the valuation of contingent claims. An overview and comparison of these techniques is given by Geske and Shastri (1985) and Hull (2006). They can be classified into forward induction (Monte Carlo simulation) and backward induction procedures (trees, finite difference) according to the direction in time in which the iteration problem is solved. The backward induction procedures rely on some kind of grid spanned by discrete values of the underlying variables at discrete time steps while the Monte Carlo simulation generates random paths of the underlying variables.

The Cox, Ross and Rubinstein (1979) approach proposes to use binomial trees to discretise the stochastic process. Uncertainty in the state variable is then represented by two alternative states, as pointed out by Cortazar (2004). At each time step, the current value of the state variable (S) can increase to $u \cdot S$ with a probability (p) or decrease to $d \cdot S$ with a probability (1-p). It is important to note that the probability distribution of the state variable must be consistent with the risk-neutral form of the stochastic process. The option valuation starts at the time of maturity, where option values are known, and iterates backwards. At each node, the value of the option is determined by discounting the expected value from the two possible future states at the risk-free rate until the option value at the current time is determined. When early exercise is taken into account, the option value at each node is the maximum of continuation value and immediate exercise. Alternative tree approaches are possible, for example binomial trees with equal up and down probabilities or trinomial trees. Boyle (1988) transforms the binomial tree to a trinomial tree and extends it to account for two underlying variables which have a bivariate lognormal distribution. Boyle et al. (1989) extend the binomial tree approach to a generalised approximation framework for options on several underlying variables.

The finite difference method solves the partial differential equation by transforming it into a difference equation and iterating backwards from the terminal boundary condition. For this purpose, a grid of values of the underlying variable is constructed for each time step, extending up to the theoretical boundaries defined by the boundary conditions. Schwartz (1977) applies this method to a call option on dividend paying stock and finds that values are very close to market values and the

Black-Scholes approach. Two variations of the finite difference method can be distinguished. The explicit method determines the option value at each node as a function of future values. Since the finite difference is a backward induction procedure, the option values at each node can be readily determined, iterating backwards. In the implicit method, however, the option value at each node is given by values one step back in time. This generates a set of simultaneous equations which can only be solved once all nodes have been defined. Although the latter is more complex to implement, it is more robust than the explicit method. Hull and White (1990) suggest some modifications to the explicit finite difference approach to ensure convergence. Both the finite difference framework and trees can be used to value European and American options but not for path-dependent options. While trees provide the option value only for a specific starting point, the finite difference approach provides option values for a set of different starting points. Brennan and Schwartz (1978) establish a connection between finite differences and trees. The explicit finite difference method is shown to be equivalent to a trinomial tree while the implicit finite difference is equivalent to a multinomial tree.

Monte Carlo simulation relies on the Cox and Ross (1976) risk-neutral valuation approach insofar as many sample paths of the risk-neutral stochastic process of the underlying variable are generated and the average payoff (expected payoff) is discounted at the risk-free rate. Boyle (1977) applies the Monte Carlo simulation to value European options on dividend-paying stock. He stresses that this solution technique is particularly flexible with regard to the form of the underlying stochastic process since "the distribution used to generate returns on the underlying stock need not have a closed form analytic expression." (p. 334). With regard to efficiency, it should be noted that the standard error of the estimate is inversely proportional only to the square root of the number of simulation runs. However, efficiency can be improved by using variance reduction techniques such as control variate and antithetic variates. Boyle et al. (1997) discuss further techniques to improve efficiency of Monte Carlo simulations. They also present algorithms to allow the valuation of options with early exercise opportunities. As Cortazar (2004) explains "these methods attempt to combine the simplicity of forward induction with the ability of determining the optimal option exercise of backward induction" (p. 612).

The Least Squares Monte Carlo simulation (LSMC) was developed by Longstaff and Schwartz (2001), based on Carriere (1996), specifically to value American options by simulation. This method simulates the underlying asset paths, approximates the conditional expectations at each time step ex-post in order to determine the optimal exercise policy, and finally discounts the cash flows according to this optimal exercise policy. The conditional expectation is approximated by a regression function involving the level of the underlying asset as the independent variable and the discounted cash flows from continuation as the dependent variable. For multi-factor American options, the LSMC tends to be relatively more efficient and more readily applicable than other numerical methods such as trees or finite differences. While Monte Carlo simulations are able to determine option values with complex, path-dependent payoff structures, the simulation is conditional on the starting point and needs to be repeated if a different starting point is assumed.

The above mentioned techniques can all be used for diffusion, jump or combined jump-diffusion stochastic processes, as Geske and Shastri (1985) point out, although with different degrees of implementation complexity. The eventual choice of the numerical solution method depends on the type of option in the specific valuation setting. Table 4 summarises the comparison of the presented numerical solution techniques.

Table 4. Comparison of Numerical Solution Methods

	Trees	**Finite Difference**	**Monte Carlo Simulation**	**Least Squares Monte Carlo Simulation**
Approximation approach	Discretise the risk-neutral stochastic process and discount at the risk-free rate	Discretise the (partial) differential equation and boundary conditions	Simulate the risk-neutral stochastic process and discount at the risk-free rate	Simulate the risk-neutral stochastic process, determine optimal exercise policy ex-post, and discount at the risk-free rate
Direction of induction	Backward induction	Backward induction	Forward induction	Backward and Forward induction
Typical applications	European options American options	European options American options	European options Path-dependent options	American options Path-dependent options
Limitations	Increased complexity when used for Path-dependent options; Valuation conditional on starting point	Increased complexity when used for Path-dependent options	Difficult to accommodate for American options; Valuation conditional on starting point	Valuation conditional on starting point
Efficiency / Computation time	Efficient for one-factor options, less efficient for options with three or more variables or when there are dividends	Efficient for one-factor options, less efficient for options with three or more variables	Relatively efficient for multi-factor options	Relatively efficient for multi-factor American options

3 Research Paper #1 – Continuous Rainbow Options on Commodity Outputs: What is the Real Value of Switching Facilities?

Continuous Rainbow Options on Commodity Outputs: What is the Real Value of Switching Facilities?

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We develop real rainbow option models to value an operating asset with the flexibility to choose between two commodity outputs. We provide a quasi-analytical solution to a model with continuous switching opportunities between two commodity outputs, taking into account operating and switching costs. A specific version of the former is a quasi-analytical solution for one-way switching. In addition, a model with continuous switching and temporary suspension options is conceived and solved by a numerical lattice approach. The models are applied to an illustrative case, demonstrating that the quasi-analytical solution and the lattice approach provide near identical results for the asset valuation and optimal switching boundaries. We find that the switching boundaries generally narrow as prices decline. In the presence of operating costs and temporary suspension, however, the thresholds diverge for low enough prices. A fertilizer plant with flexibility between selling ammonia and urea is valued in an empirical section using our real option models. Despite the high correlation between the two alternative commodities, ammonia and urea, there is significant value in the flexibility to choose between the two. The results are highly sensitive to changes in expected volatilities and correlation, as shown by using different sampling periods for the estimation of volatility and correlation. Both strategic and policy implications for stakeholders in flexible assets are discussed, with some generalisations outside the fertilizer industry.

JEL Classifications: D81, G31

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1 Introduction

When is the right time for an operator of a flexible facility to switch back and forth between two possible commodity outputs in order to maximise value when operating and switching costs are taken into account? Which factors should be monitored in making these decisions? How much should an investor pay for such a flexible operating asset? What are the strategy implications for the operator, investor and possibly policy makers?

Flexible production and processing facilities are typically more expensive to operate, with a higher initial investment cost, than inflexible facilities. One problem is that one part of the flexible facility, which requires an additional investment cost, might be idle at times. Investing in a facility which is not productive all the time seems counter-intuitive at first glance. What is frequently misunderstood is that the additional option value through "operating flexibility" (according to Trigeorgis and Mason, 1987) may have significant value in uncertain markets when input or output factors have different volatilities. Examples of flexible assets include shipping (combination carriers), the chemical industry (flexible fertilizer plants), electricity generation (combined cycle: natural gas/ coal gasification), and real estate (multiple property uses).

The traditional approach to determine switching boundaries between two operating modes is to discount future cash flows and use Marshallian triggers. This methodology does not fully capture the option value which may arise due to the uncertainty in future input or output prices. The value of waiting to gain more information on future price developments, and consequently on the optimal switching triggers can be best viewed in a real options framework.

Conceptually, the switch between two volatile assets or commodities can be modelled as an exchange option. Margrabe (1978) and McDonald and Siegel (1986) model European and American perpetual exchange options, respectively, which are linear homogeneous in the underlying stochastic variables. Hopp and Tsolakis (2004) use an American exchange option based on a two-state variable binomial tree to value the option on the best of two assets where the asset choice is a one-time decision. An analytical model for flexible production capacity is presented by He and Pindyck (1992) where switching costs and product-specific operating costs are ignored, thereby eliminating the components which would lead to a non-linearity of the value function in the underlying processes. Brekke and Schieldrop (2000) also assume costless switching in their study on the value of operating flexibility between two stochastic input factors, in which they determine the optimal investment timing for a flexible technology in comparison to a technology that does not allow input switching. Adkins and Paxson (2010) present quasi-analytical solutions to input switching options, where two-factor functions are not homogeneous of degree one, and thus dimension reducing techniques used in McDonald and Siegel (1986) and Paxson and Pinto (2005) are not available. Other approaches model the profits or returns in the respective operating state to be stochastic, such as Song et al. (2010), and Triantis and Hodder (1990) who assume profits follow arithmetic Brownian motion and switching is costless.

Geltner, Riddiough and Stojanovic (1996) develop a framework for a perpetual option on the best of two underlying assets, applied to the case of two alternative uses for properties, and provide a comprehensive discussion of relevant assumptions for such a contingent-claims problem. Childs, Riddiough and Triantis

(1996) extend the aforementioned model to allow for redevelopment or switching between alternative uses.

Sodal et al. (2006, 2007) develop a framework for an option to operate in the best of dry or wet shipping markets, where switching between the two is possible by incurring switching costs. These authors assume that a mean-reverting stochastic process is appropriate for the difference between the two types of freight rates. Bastian-Pinto et al. (2009) focus on the Brazilian sugar industry and model the price of the two possible output commodities, sugar and ethanol, as mean-reverting. A bivariate lattice is used to replicate the discrete and correlated development of the two commodity prices. However, they allow for switching at no cost. Abadie and Chamorro (2008) apply numerical solution techniques to value input flexibility between two fuels following inhomogeneous Brownian motion in the presence of switching costs.

We develop three real option models for an asset with switching opportunities between two commodity outputs, taking into account switching costs and operating costs. The first one is a quasi-analytical solution for continuous switching, the second for one-way switching, and the third is a numerical lattice solution for continuous switching with suspension option. The rest of this paper is organised in five sections. Section 2 defines the framework, develops the real option models and provides the solution methods. Section 3 provides a numerical illustration and compares the results of the different models. Section 4 introduces empirics of the fertilizer industry, including commodity price behaviour and parameter estimation, and values a flexible fertilizer plant based on the new real option models. Section 5 discusses policy and strategy implications. Section 6 concludes and discusses issues for future research.

2 Modelling the Real Rainbow Options on Commodity Outputs

2.1 Assumptions

The asset to be considered is flexible to produce two different commodity outputs by switching between operating modes. We assume the prices of the two commodities, x and y, are stochastic and correlated and follow geometric Brownian motion (gBm):

$$dx = (\mu_x - \delta_x)x\,dt + \sigma_x\,x\,dz_x \tag{1}$$

$$dy = (\mu_y - \delta_y)y\,dt + \sigma_y\,y\,dz_y \tag{2}$$

with the notations:

μ Required return on the commodity

δ Convenience yield of the commodity

σ Volatility of the commodity

dz Wiener process (stochastic element)

ρ Correlation between the two commodities: $dz_x\,dz_y$ / dt

The instantaneous cash flow in each operating mode is the respective commodity price of the output product less unit operating cost, multiplied by the production units, i.e. p_1 (x – c_x) in operating mode ‘1’ and p_2 (y – c_y) in operating mode ‘2’. The parameters p_1 and p_2 represent the production per time unit (year) and c_x and c_y are operating costs per unit produced. A switching cost of S_{12} is incurred when switching from operating mode ‘1’ to ‘2’, and S_{21} for switching vice versa.

Definitions

Variable operating cost	c_X, c_Y
Capacity	p_1, p_2

Switching cost	S_{12}, S_{21}
Risk-free interest rate	r

Further assumptions are that the operating cost is deterministic and constant, the lifetime of the asset is infinite, and the company is not restricted in the product mix choice because of selling commitments. Moreover, the typical assumptions of real options theory apply, with interest rates, yields, risk premium, volatilities and correlation constant over time.

2.2 Quasi-analytical Solution for Continuous Switching

The asset value with opportunities to continuously switch between the two operating modes is given by the present value of perpetual cash flows in the current operating mode plus the option to switch to the alternative mode. Let V_1 be the asset value in operating mode '1', producing commodity x, and V_2 the asset value in operating mode '2', producing commodity y accordingly. The switching options depend on the two correlated stochastic variables x and y, and so do the asset value functions which are defined by partial differential equations:

$$\frac{1}{2}\sigma_x^2 x^2 \frac{\partial^2 V_1}{\partial x^2} + \frac{1}{2}\sigma_y^2 y^2 \frac{\partial^2 V_1}{\partial y^2} + \rho\sigma_x\sigma_y xy \frac{\partial^2 V_1}{\partial x \partial y} + (r-\delta_x)x\frac{\partial V_1}{\partial x} + (r-\delta_y)y\frac{\partial V_1}{\partial y} - rV_1 + p_1(x-c_x) = 0 \tag{3}$$

$$\frac{1}{2}\sigma_x^2 x^2 \frac{\partial^2 V_2}{\partial x^2} + \frac{1}{2}\sigma_y^2 y^2 \frac{\partial^2 V_2}{\partial y^2} + \rho\sigma_x\sigma_y xy \frac{\partial^2 V_2}{\partial x \partial y} + (r-\delta_x)x\frac{\partial V_2}{\partial x} + (r-\delta_y)y\frac{\partial V_2}{\partial y} - rV_2 + p_2(y-c_y) = 0 \tag{4}$$

Two-factor problems which are linear homogeneous, i.e. $V(\lambda \cdot x; \lambda \cdot y) = \lambda \cdot V(x; y)$, can typically be solved analytically by substitution of variables so that the partial differential equation can be simplified into a one-factor differential equation. An example of this is the perpetual American exchange option in McDonald and Siegel (1986). Our continuous rainbow option encompasses a number of complexities, such as switching cost, operating cost and multiple switching, in order to make it more realistic. As a consequence, the problem is no longer homogenous of degree one and the dimension reducing technique cannot be used. This makes an analytical solution practically unavailable. Based on the approach of Adkins and Paxson (2010), we derive a quasi-analytical solution for this kind of two-factor non-homogeneous problem. The partial differential equations are satisfied by the following general solutions:

$$V_1(x, y) = \frac{p_1 x}{\delta_x} - \frac{p_1 c_x}{r} + A\, x^{\beta_{11}}\, y^{\beta_{12}} \tag{5}$$

where β_{11} and β_{12} satisfy the characteristic root equation

$$\tfrac{1}{2}\sigma_x^2\beta_{11}(\beta_{11}-1) + \tfrac{1}{2}\sigma_y^2\beta_{12}(\beta_{12}-1) + \rho\sigma_x\sigma_y\beta_{11}\beta_{12} + \beta_{11}(r-\delta_x) + \beta_{12}(r-\delta_y) - r = 0, \tag{6}$$

and

$$V_2(x, y) = \frac{p_2 y}{\delta_y} - \frac{p_2 c_y}{r} + B\, x^{\beta_{21}}\, y^{\beta_{22}} \tag{7}$$

where β_{21} and β_{22} satisfy the characteristic root equation

$$\tfrac{1}{2}\sigma_x^2\beta_{21}(\beta_{21}-1) + \tfrac{1}{2}\sigma_y^2\beta_{22}(\beta_{22}-1) + \rho\sigma_x\sigma_y\beta_{21}\beta_{22} + \beta_{21}(r-\delta_x) + \beta_{22}(r-\delta_y) - r = 0 \tag{8}$$

A, B, β_{11}, β_{12}, β_{21} and β_{22} are assumed constant with respect to x and y. The characteristic root equation (6) is solved by combinations of β_{11} and β_{12} forming an ellipse of such form that β_{11} could be positive or negative and β_{12} could be positive or negative. The same is true for equation (8). Since the option to switch from x to y decreases with x and increases with y, β_{11} must be negative and β_{12} positive. Likewise, β_{21} must be positive and β_{22} negative. Switching between the operating modes always depends on the level of both x and y. At the switching points (x_{12}, y_{12}) and (x_{21}, y_{21}), the asset value in the current operating mode must be equal to the asset value in the alternative operating mode net of switching cost. These value matching conditions are stated formally below:

$$A\,x_{12}^{\beta_{11}}\,y_{12}^{\beta_{12}} + \frac{p_1 x_{12}}{\delta_x} - \frac{p_1 c_x}{r} = B\,x_{12}^{\beta_{21}}\,y_{12}^{\beta_{22}} + \frac{p_2 y_{12}}{\delta_y} - \frac{p_2 c_y}{r} - S_{12} \tag{9}$$

$$A\,x_{21}^{\beta_{11}}\,y_{21}^{\beta_{12}} + \frac{p_1 x_{21}}{\delta_x} - \frac{p_1 c_x}{r} - S_{21} = B\,x_{21}^{\beta_{21}}\,y_{21}^{\beta_{22}} + \frac{p_2 y_{21}}{\delta_y} - \frac{p_2 c_y}{r} \tag{10}$$

Furthermore, smooth pasting conditions hold at the boundaries:

$$\beta_{11}\,A\,x_{12}^{\beta_{11}-1}\,y_{12}^{\beta_{12}} + \frac{p_1}{\delta_x} = \beta_{21}\,B\,x_{12}^{\beta_{21}-1}\,y_{12}^{\beta_{22}} \tag{11}$$

$$\beta_{12}\,A\,x_{12}^{\beta_{11}}\,y_{12}^{\beta_{12}-1} = \beta_{22}\,B\,x_{12}^{\beta_{21}}\,y_{12}^{\beta_{22}-1} + \frac{p_2}{\delta_y} \tag{12}$$

$$\beta_{11}\,A\,x_{21}^{\beta_{11}-1}\,y_{21}^{\beta_{12}} + \frac{p_1}{\delta_x} = \beta_{21}\,B\,x_{21}^{\beta_{21}-1}\,y_{21}^{\beta_{22}} \tag{13}$$

$$\beta_{12}\,A\,x_{21}^{\beta_{11}}\,y_{21}^{\beta_{12}-1} = \beta_{22}\,B\,x_{21}^{\beta_{21}}\,y_{21}^{\beta_{22}-1} + \frac{p_2}{\delta_y} \tag{14}$$

There are only 8 equations, (6) and (8) - (14), for 10 unknowns, β_{11}, β_{12}, β_{21}, β_{22}, A, B, x_{12}, y_{12}, x_{21}, y_{21}, so there is no completely analytical solution. Yet, for every value of x, there has to be a corresponding value of y when switching should occur, (x_{12}, y_{12}) and (x_{21}, y_{21}). So a quasi-analytical solution can be found by assuming values for x, which then solves the set of simultaneous equations for all remaining variables, given that $x = x_{12} = x_{21}$. This procedure is repeated for many values of x, providing the corresponding option values and the switching boundaries. This quasi-analytical solution implies, however, that the values of A, B, β_{11}, β_{12}, β_{21} and β_{22} change when a different x is chosen. Hence, the initial assumption of these parameters being constant over the whole range of x and y is now relaxed insofar as these parameters are constant only locally. The local solution must then be considered an approximation because the derivatives at the chosen point are calculated with constant parameter values.

From (9) and (10), it can be seen by rearranging that the total cost of exercising the switching option is the sum of the switching cost and the difference in the present value of operating costs, $S_{12} + \left(\frac{p_2 c_y}{r} - \frac{p_1 c_x}{r} \right)$ and $S_{21} - \left(\frac{p_2 c_y}{r} - \frac{p_1 c_x}{r} \right)$, respectively. The optimal switching policy and thus the option value can only be computed if the exercise cost is a positive number. That means the continuous switching option can only be valued if the present value of the difference in operating costs does not exceed the switching cost. If this premise does not hold, the value of flexibility can only be determined on the basis of a one-way switch with positive exercise cost.

2.3 *Quasi-analytical Solution for One-Way Switching*

Deriving a solution for the asset value with a one-way switching option from the above model with continuous switching is straight-forward. Assuming $p_2c_y>p_1c_x$, the American perpetual option to switch from x to y can be determined. The switching option vice versa is ignored because the exercise cost $S_{21} - \left(\frac{p_2 c_y}{r} - \frac{p_1 c_x}{r} \right)$ might be negative. The asset value V_1 is given by (5) with its characteristic root equation (6), and V_2 is given by (7) with B=0, thereby eliminating the option to switch back. The characteristic root equation (6) together with value matching condition (9) and smooth pasting conditions (11) and (12) represents the system of 4 equations, while there are 5 unknowns, β_{11}, β_{12}, A, x_{12}, y_{12}. Applying the same solution procedure as explained above, a quasi-analytical solution is obtained. It is noted again, that this procedure is an approximation because the constants A, β_{11} and β_{12} are constant only locally.

2.4 Numerical Solution for Continuous Switching with Suspension Option

The continuous switching option between the two operating modes with the purpose of choosing the best of two output products can also be valued with a numerical lattice. While this approach is less transparent than the quasi-analytical solution and the computations are more onerous, it overcomes the restriction that the present value of the difference in operating costs in the two operating modes must be smaller than the switching cost. It requires, however, that the option to suspend operations to avoid net losses is taken into account so that the fixed boundaries of the lattice can be determined. The asset value functions at the fixed boundaries x=0 and y=0 can be reduced to known one-factor functions only if the suspension option is taken into account. Without suspension option, the asset value functions at these fixed boundaries would depend on both variables, x and y, with switching options between them, so that the functions could therefore not be determined. It is assumed that costs are incurred neither for suspension nor during suspension of the asset operation. The results from the numerical lattice approach and the quasi-analytical solution as developed before can be compared on a like-for-like basis for the case of zero operating costs because suspension is then irrelevant.

Both the techniques of constructing trees (lattice) and of finite differences are appropriate for this kind of valuation problem because it involves American-style options and only two stochastic factors. We use a lattice approach based on the modified explicit finite difference method. The backward iteration requires the specification of the switching boundary conditions and the fixed boundaries. With the switching triggers (x_{12},y_{12}) and (x_{21},y_{21}) as before, the general value matching and smooth pasting conditions are:

$$V_1(x_{12}, y_{12}) = V_2(x_{12}, y_{12}) - S_{12} \tag{15}$$

$$V_1(y_{21}, x_{21}) - S_{21} = V_2(y_{21}, x_{21}) \tag{16}$$

$$\left.\frac{\partial V_1(x,y)}{\partial x}\right|_{x=x_{12},y=y_{12}} = \left.\frac{\partial V_2(x,y)}{\partial x}\right|_{x=x_{12},y=y_{12}} \tag{17}$$

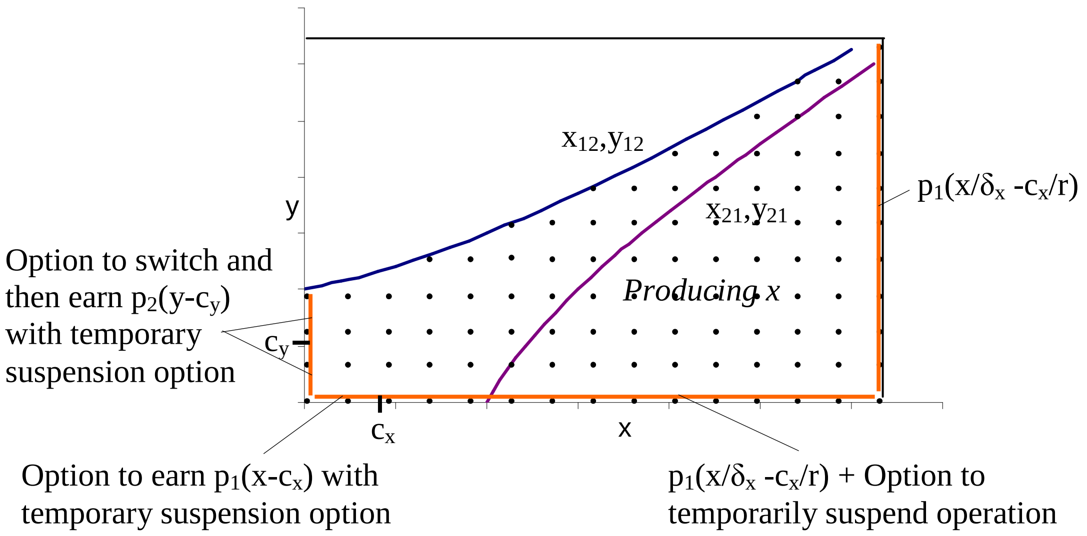
$$\left.\frac{\partial V_1(x,y)}{\partial y}\right|_{x=x_{12},y=y_{12}} = \left.\frac{\partial V_2(x,y)}{\partial y}\right|_{x=x_{12},y=y_{12}} \tag{18}$$

$$\left.\frac{\partial V_1(x,y)}{\partial x}\right|_{x=x_{21},y=y_{21}} = \left.\frac{\partial V_2(x,y)}{\partial x}\right|_{x=x_{21},y=y_{21}} \tag{19}$$

$$\left.\frac{\partial V_1(x,y)}{\partial y}\right|_{x=x_{21},y=y_{21}} = \left.\frac{\partial V_2(x,y)}{\partial y}\right|_{x=x_{21},y=y_{21}} \tag{20}$$

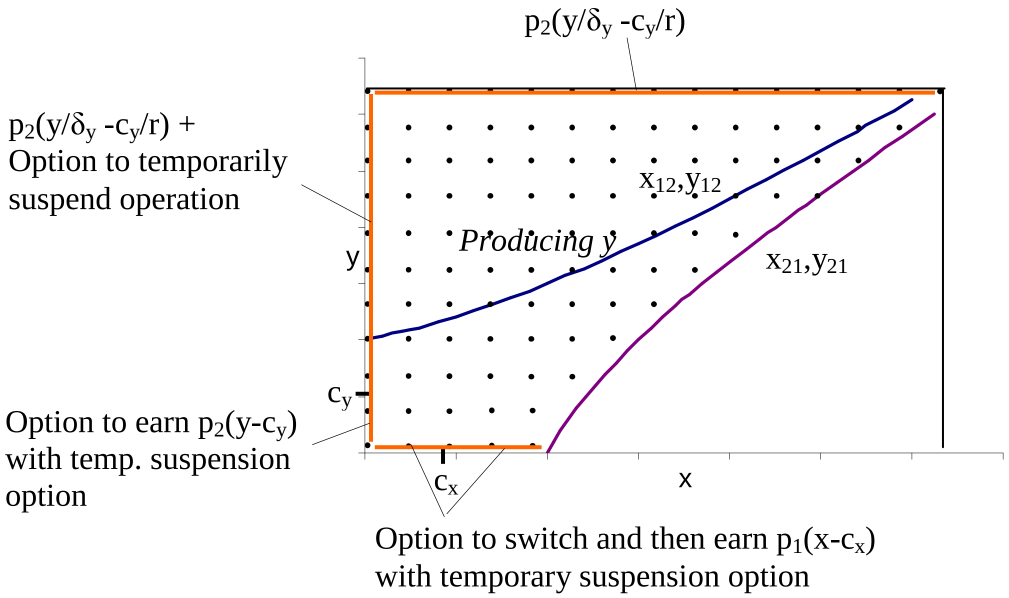
On the fixed boundaries, the problem reduces from two stochastic factors to a single stochastic factor. Figures 1 and 2 show the boundaries for both operating modes. The corresponding value functions are developed below.

Figure 1. Generalised x-y-Grid with Asset Value V_1 Defined on the Fixed Boundaries



Two-factor valuation problem with stochastic variables x and y and with the option to earn either as cash flow. c_x and c_y are the respective operating costs. p_1 and p_2 are annual capacities, δ_x and δ_y convenience yields of x and y respectively. Switching boundaries are shown schematically, with (x_{12},y_{12}) the boundary to switch from x to y, and (x_{21},y_{21}) to switch back. Asset value V_1 is given for the fixed boundaries when currently producing x.

Figure 2. Generalised x-y-Grid with Asset Value V_2 Defined on the Fixed Boundaries



Two-factor valuation problem with stochastic variables x and y and with the option to earn either as cash flow. c_x and c_y are the respective operating costs. p_1 and p_2 are annual capacities, δ_x and δ_y convenience yields of x and y respectively. Switching boundaries are shown schematically, with (x_{12},y_{12}) the boundary to switch from x to y, and (x_{21},y_{21}) to switch back. Asset value V_2 is given for the fixed boundaries when currently producing y.

Case: x=0 and y=0

When both product prices fall to zero, gBm implies that they will remain zero forever. The asset would then be suspended forever and its value is nil.

$$V_1(x = 0; y = 0) = V_2(0;0) = 0 \tag{21}$$

Case: y=0

If y is zero and the asset is currently in operating mode '1', switching can be ignored. The asset will be operated when the revenue from x exceeds the operating cost, and will be suspended otherwise, with the option to resume operation. Brennan and Schwartz (1985) provide models to value an asset based on a single underlying stochastic cash flow, where there are operating costs and temporary suspension is possible.

$$V_1(x; y = 0) = \begin{cases} A_1 (p_1 x)^{\beta_{x1}} & \text{if } x < c_x \\ B_1 (p_1 x)^{\beta_{x2}} + (p_1 x)/\delta_x - (p_1 c_x)/r & \text{if } x > c_x \end{cases} \tag{22}$$

where
$$A_1 = \frac{(p_1 c_x)^{1-\beta_{x1}}}{\beta_{x1} - \beta_{x2}} \left(\frac{\beta_{x2}}{r} - \frac{\beta_{x2} - 1}{\delta_x} \right), \tag{23}$$

$$B_1 = \frac{(p_1 c_x)^{1-\beta_{x2}}}{\beta_{x1} - \beta_{x2}} \left(\frac{\beta_{x1}}{r} - \frac{\beta_{x1} - 1}{\delta_x} \right), \tag{24}$$

$$\beta_{x1/x2} = \tfrac{1}{2} - (r - \delta_x)/\sigma_x^{\,2} \pm \sqrt{\left((r - \delta_x)/\sigma_x^{\,2} - \tfrac{1}{2}\right)^2 + 2r/\sigma_x^{\,2}}\ . \tag{25}$$

β_{x1} and β_{x2} are defined in (25) and represent the positive and negative solution to the fundamental quadratic equation of a one-factor American perpetual option on x. The statement that the switching option can be ignored when y=0 is only valid because we take the suspension option into account. If the suspension option is ignored, switching from x to y might still be a valid strategy even when y=0 as long as operating losses can be reduced this way. The asset value would then depend on both x and y even on the fixed boundaries, for which values could therefore not be determined. This is why the suspension option is incorporated into this numerical solution for continuous switching.

When the current operating mode is ‘2’, no value is gained from that operating mode since y=0 so that the asset value is given by the option to switch to the alternative mode. The option to invest in an asset with operating costs and the possibility of temporary suspension is:

$$V_2(x; y=0) = D_1 (p_1 x)^{\beta_{x1}} \tag{26}$$

with
$$D_1 = \frac{B_1 (p_1 x^*)^{\beta_{x2}} + p_1 x^* / \delta_x - p_1 c_x / r - S_{21}}{(p_1 x^*)^{\beta_{x1}}} \tag{27}$$

where x* is the switching boundary x_{21}(y=0) and satisfies the equation:

$$(\beta_{x1} - \beta_{x2}) B_1 (p_1 x^*)^{\beta_{x2}} + (\beta_{x1} - 1)(p_1 x^*)/\delta_x - \beta_{x1}\left((p_1 c_x)/r + S_{21}\right) = 0 \tag{28}$$

Case: x=0

The same logic applies as for the case of y=0.

$$V_2(x=0; y) = \begin{cases} A_2 (p_2 y)^{\beta_{y1}} & \text{if } y < c_y \\ B_2 (p_2 y)^{\beta_{y2}} + (p_2 y)/\delta_y - (p_2 c_y)/r & \text{if } y > c_y \end{cases} \tag{29}$$

where
$$A_2 = \frac{(p_2 c_y)^{1-\beta_{y1}}}{\beta_{y1} - \beta_{y2}} \left(\frac{\beta_{y2}}{r} - \frac{\beta_{y2} - 1}{\delta_y} \right), \tag{30}$$

$$B_2 = \frac{(p_2 c_y)^{1-\beta_{y2}}}{\beta_{y1} - \beta_{y2}} \left(\frac{\beta_{y1}}{r} - \frac{\beta_{y1} - 1}{\delta_y} \right), \tag{31}$$

$$\beta_{y1/y2} = \tfrac{1}{2} - \left(r - \delta_y\right) \big/ \sigma_y{}^2 \pm \sqrt{\left(\left(r - \delta_y\right) \big/ \sigma_y{}^2 - \tfrac{1}{2}\right)^2 + 2r \big/ \sigma_y{}^2} \,. \tag{32}$$

β_{y1} and β_{y2} are defined in (32) and represent the positive and negative solution to the fundamental quadratic equation of a one-factor American perpetual option on y. The value V_1 when x=0 is given by:

$$V_1(x = 0; y) = D_2 \, (p_2 y)^{\beta_{y1}} \tag{33}$$

with
$$D_2 = \frac{B_2 (p_2 y^*)^{\beta_{y2}} + p_2 y^* / \delta_y - p_2 c_y / r - S_{12}}{(p_2 y^*)^{\beta_{y1}}} \tag{34}$$

where y* is the switching boundary y_{12}(x=0) and satisfies the equation:

$$(\beta_{y1} - \beta_{y2}) B_2 (p_2 y^*)^{\beta_{y2}} + (\beta_{y1} - 1)(p_2 y^*)/\delta_y - \beta_{y1}\big((p_2 c_y)/r + S_{12}\big) = 0 \tag{35}$$

Case: x → ∞, or y → ∞

When one of the two prices approaches infinity, no more switching to the other product will take place and the suspension option becomes irrelevant.

$$V_1(x \to \infty; y) = p_1(x/\delta_x - c_x/r) \tag{36}$$

$$V_2(x; y \to \infty) = p_2(y/\delta_y - c_y/r) \tag{37}$$

2.5 Implementation of the Numerical Solution Method

Childs et al. (1996) solve a valuation problem similar to the one presented above. They use a trinomial lattice to approximate the value of the redevelopment option for property uses. The main difference is that their model ignores operating costs (and the suspension option as a consequence).

Definition of the Lattice

Childs et al. (1996) provide a framework for the lattice numerical solution which is based on the Hull and White (1990) modification of the explicit finite difference method. The lattice is spanned by x and y-values and time (t) as follows:

$$x_{i,j,k} = x_{min}\, e^{i\sigma_x\sqrt{3\Delta t}}\, e^{k\left(r-\delta_x-1/2\,\sigma_x^2\right)\Delta t}, \tag{38}$$

$$y_{i,j,k} = y_{min}\, e^{i\rho\sigma_y\sqrt{3\Delta t}}\, e^{j\sigma_y\sqrt{3\Delta t\left(1-\rho^2\right)}}\, e^{k\left(r-\delta_y-1/2\,\sigma_y^2\right)\Delta t}, \tag{39}$$

where (i,j,k) defines a point in the three-dimensional x-y-t grid by indicating the number of increments in the respective variable. From the above functions, it can be observed that the x-values in the grid depend on time, while y depends both on time and on x. This interdependency is required in order to map the correlation between the two variables.

The lattice also defines the marginal probabilities, i.e. the probabilities of up or down movements of x and y within an increment of time:

$$\begin{array}{ccc}
 & & \begin{array}{ccc} & i & \\ \hline \text{up} & \text{over} & \text{down} \end{array} \\
j & \begin{array}{r} \text{up} \\ \text{over} \\ \text{down} \end{array} & \begin{bmatrix} 1/36 & 1/9 & 1/36 \\ 1/9 & 4/9 & 1/9 \\ 1/36 & 1/9 & 1/36 \end{bmatrix}
\end{array}$$

Terminal values are required in order to solve the system by backward iteration. Distant cash flows do not affect the present value significantly. So we make the assumption that the switching option is no longer available beyond a distant point in time, e.g. beyond fifty years of operation, which allows us to determine the value of the asset in fifty years time as the terminal value. The terminal values are given by equation (22) for V_1 and by equation (29) for V_2.

Model Implementation

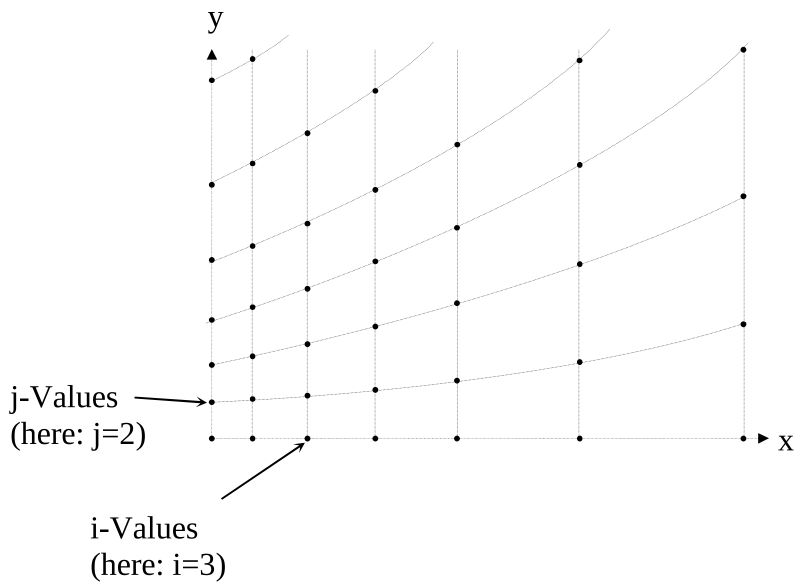
According to the x-y-t grid, asset values are determined at each node. This results in a value grid for each of V_1 and V_2. First, all fixed boundaries are computed and then the terminal values. At the terminal boundary, switching is no longer possible, so the asset value is given by the static value plus suspension option.

Starting from the terminal values backwards, V_1 and V_2 are determined at every point in time. $V1_{i,j,k}$ is the value at (x=iΔx, y=jΔy, t=kΔt), assuming the current operating mode is '1'. It is equal to the sum of instantaneous cash flow and discounted value of the higher of V_1 and V_2–S_{12} at the time t+1 according to the marginal probabilities.

$$V1_{i,j,k} = \mathrm{Max}\left[p_1\left(x - c_x\right)\Delta t;0\right] + e^{-r\Delta t}\left(\tfrac{4}{9}\mathrm{Max}\left[V1_{i,j,k+1}, V2_{i,j,k+1} - S_{12}\right] + \ldots\right) \qquad (40)$$

The asset value at the present time (t=0) can be represented as a surface spanned over the x-y-area.

Figure 3. Grid of correlated Variables x and y for the Numerical Solution



Two-factor valuation problem with stochastic variables x and y and with the option to earn either as cash flow. Pairs of x and y in a valuation grid which maps the correlation between the two.

Determination of Asset Values at any x-y-point within the Value Grid

The value V_1 or V_2 can be determined at any point within the value grid by interpolating between known values. It has to be taken into account that the grid is not straight due to the dependence of y-values upon x, as is illustrated in Figure 3. The Mathematica code of the numerical lattice solution is available in the Appendix to the Thesis.

3 Numerical Illustration

The three models developed in the previous section are now applied to an illustrative numerical case in order to compare the valuation results and switching behaviour. We present three scenarios for operating costs:

a) Zero operating costs in both operating modes

b) Non-zero but equal operating costs in both operating modes

c) Different operating costs in the two operating modes

Scenario (a) has the advantage that the suspension option in the numerical lattice solution is irrelevant so that this model can be compared with the quasi-analytical solution. Scenario (b) can be used to identify the value of the suspension option by comparing the numerical lattice solution with the quasi-analytical solution. Scenario (c) is the most general one but is not applicable to the quasi-analytical solution for continuous switching because of the aforementioned restrictions. Table 1 shows the parameters used for the illustration and also presents the quasi-analytical solution for continuous switching when operating costs are zero.

Table 1. Quasi-analytical Solution for Continuous Switching with No Operating Costs

Parameters		**Base case**	**Sensitivities**	
Commodity price	x	100	90	110
Commodity price	y	100	100	100
Convenience yield of x	δ_x	0.03	0.03	0.03
Convenience yield of y	δ_y	0.03	0.03	0.03
Volatility of x	σ_x	0.40	0.40	0.40
Volatility of y	σ_y	0.30	0.30	0.30
Correlation between x and y	ρ	0.50	0.50	0.50
Risk-free interest rate	r	0.05	0.05	0.05
Operating cost for x	c_x	0	0	0
Operating cost for y	c_y	0	0	0
Capacity of x	p_1	1	1	1
Capacity of y	p_2	1	1	1
Switching cost from x to y	S_{12}	50	50	50
Switching cost from y to x	S_{21}	70	70	70
Switching boundary x to y	x_{12}	100	90	110
Switching boundary y to x	x_{21}	100	90	110
Solution				
Asset value in operating mode '1'	$V_1(x,y)$	5,254	4,990	5,526
Asset value in operating mode '2'	$V_2(x,y)$	5,255	5,010	5,510
	A	16.17	15.99	16.31
	B	15.72	15.53	15.90
Switching boundary x to y	$y_{12}(x)$	150	137	163
Switching boundary x to y	$y_{21}(x)$	53	46	60
	β_{11}	-0.317	-0.315	-0.319
	β_{12}	1.355	1.355	1.354
	β_{21}	1.333	1.332	1.334
	β_{22}	-0.289	-0.285	-0.293
Equations				
Value matching condition	EQ 9	0.000	0.000	0.000
Value matching condition	EQ 10	0.000	0.000	0.000
Smooth pasting condition	EQ 11	0.000	0.000	0.000
Smooth pasting condition	EQ 12	0.000	0.000	0.000
Smooth pasting condition	EQ 13	0.000	0.000	0.000
Smooth pasting condition	EQ 14	0.000	0.000	0.000
Characteristic root equation	EQ 6	0.000	0.000	0.000
Characteristic root equation	EQ 8	0.000	0.000	0.000
	Sum	0.000	0.000	0.000

Asset values (V_1, V_2) and switching boundaries (y_{12}, y_{21}) are obtained from (5) and (7) and the simultaneous solution of (6) and (8) - (14).

It can be seen that the option factors A and B are positive, β_{11} and β_{22} are negative and β_{12} and β_{21} are positive, thereby fulfilling the requirements from the theoretical model. The system of value matching conditions, smooth pasting conditions and characteristic root equations is fully satisfied. The Table also shows that when x is increased by 10% (from 100 to 110), the maximum change in any of the parameters A, B, β_{11}, β_{12}, β_{21} and β_{22} is only 1.3%. This comparatively small change suggests that the approximation error by assuming these parameters constant in an infinitesimal small area around x is not significant.

Table 2. Comparison of Switching models for Illustrative Cases

$x = 100$; $y = 100$; $p_1 = 1$; $p_2 = 1$

$\delta_x = \delta_y = 0.03$; $\sigma_x = 0.40$; $\sigma_y = 0.30$; $\rho = 0.50$; $r = 0.05$; $S_{12} = 50$; $S_{21} = 70$

Scenarios		No switching	Quasi-analytical solution for continuous switching	Quasi-analytical solution for one-way switching	Numerical solution for continuous switching with suspension option
$c_x = c_y = 0$	V_1	3,333	5,254	4,875	5,143
	V_2	3,333	5,255	3,333	5,142
	y_{12}	– n.a. –	150	396	158
	y_{21}		53	– n.a. –	55
$c_x = c_y = 25$	V_1	2,833	4,754	4,375	4,688
	V_2	2,833	4,755	2,833	4,686
	y_{12}	– n.a. –	150	396	155
	y_{21}		53	– n.a. –	55
$c_x = 0$, $c_y = 25$	V_1	3,333	– n.a. –	4,775	4,912
	V_2	2,833		2,833	4,864
	y_{12}	– n.a. –		442	184
	y_{21}			– n.a. –	79

V_1 is the asset value when currently producing x, V_2 is the asset value when currently producing y. y_{12} is the level of y (at the given level of x) when it is optimal to switch from x to y, and y_{21} is the level of y for switching vice versa. c_x and c_y are the operating costs and p_1 and p_2 the annual capacities of x and y, respectively. σ_x and σ_y are the volatilities, δ_x and δ_y the convenience yields, ρ is the correlation between x and y. r is the risk-free rate. S_{12} is the switching cost from x to y and S_{21} for switching vice versa.
Solution with no switching is obtained from (5) and (7) with A=B=0. Quasi-analytical solution for continuous switching is obtained from (5) and (7) and the simultaneous solution of (6) and (8) - (14). Quasi-analytical solution for one-way switching is obtained from (5) and (7) with B=0 and the simultaneous solution of (6), (9), (11), (12). Numerical solution for

continuous switching with suspension option is obtained from the lattice approach based on (21) - (40) and grid spacing of i=50, j=50, k=250, Δt=0.2. Parameter values are from Table 1.

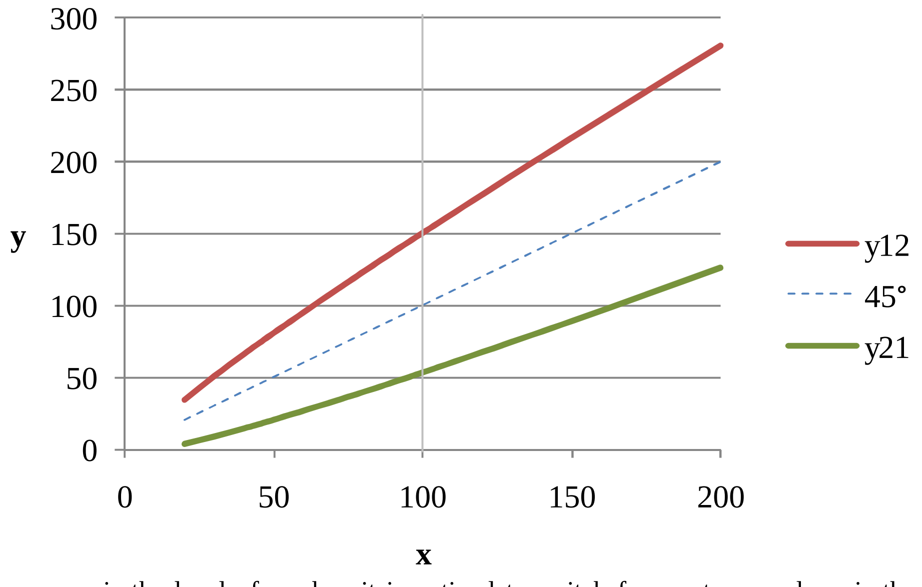
Table 2 presents the numerical results in the three scenarios for our three switching option models and for the case of no switching option. The asset values are given in both operating modes, V_1 and V_2, and the level of y is indicated when it is optimal to switch from x to y (y_{12}) and vice versa (y_{21}). With x and y having the same initial values and the same yields, the asset value with no switching is identical in both operating modes when the operating cost is the same. Higher operating costs reduce the asset value. When operating costs are nil, the asset value V_1 with continuous switching opportunities is valued at 5,254 according to the quasi-analytical solution and at 5,143 according to the numerical lattice solution, which is a difference of only 2.1% between the two models. The switching option value is the difference between the asset value and the value with no switching option, i.e. 5,254-3,333=1,921. Hence, the option to continuously switch between the two operating modes adds about 55% to the inflexible asset value. Given the current level of x of 100, the switching boundary y_{12} is 150 (numerical lattice model: 158) and y_{21} is 53 (55). The spread between y_{12} and y_{21} is caused by switching costs and increases with high volatilities and low correlation, following real options theory. It should be noted that changing x also changes the switching boundaries y_{12} and y_{21}, and that the switching boundaries x_{12} and x_{21} for a given level of y can be determined in a similar way. The fact that y_{12} and y_{21} are not symmetrical to x = 100 is primarily due to the log-normality of the commodity prices, and further due to $S_{12} \neq S_{21}$ and $\sigma_x \neq \sigma_y$. The small differences between the quasi-analytical solution and the numerical lattice solution are due to the fact that both approaches are approximations. Thus, there is uncertainty as to whether the quasi-analytical solution or the numerical lattice solution is more accurate. The imprecision in the numerical lattice solution is introduced by the limited number of grid points and the interpolation between grid points as a consequence. When the number of grid points is doubled, the numerical lattice solution values the asset at 5,165 which differs by only 1.7% from the quasi-analytical solution. Hence, the precision of the lattice solution can be increased by refining the grid. Moreover, the lattice approach assumes a terminal boundary of 50 years beyond which switching is no longer possible. Choosing a terminal boundary of 25 years with the same time increment as before results in asset values which are 3.1% lower, while a terminal boundary of 100 years results in asset values which are 0.3% higher. This suggests that increasing the terminal boundary beyond 50 years hardly impacts on the asset value. When switching is only possible from x to y but not vice versa, the switching level y_{12} is much higher because the decision cannot be reversed. This is also why the asset value V_1 is lower compared to continuous switching. The asset value V_2 for the one-way switching model is 3,333, just the same as the asset with no switching.

When operating costs are non-zero but equal in both operating modes, the asset values decline generally. While V_1 of the numerical lattice solution was 2.1% less than V_1 of the quasi-analytical solution when operating costs were nil, it is only 1.4% less in the presence of operating costs ($c_x = c_y = 25$). This may be an indication of the positive value of the suspension option which is part of the numerical lattice solution only, but also shows that the value of the suspension option is not significant for the given parameters. For the two quasi-analytical models, which do not take temporary suspension into account, the switching boundaries are unchanged compared to the case of zero operating costs because the total operating costs cannot be reduced by switching. Since the suspension option is not significant at the given parameters, the switching boundaries of the numerical lattice solution are also similar to the ones in scenario (a).

The quasi-analytical solution for continuous switching is not available for scenario (c) because it requires that the present value of the difference in operating costs in the two operating modes is smaller than the switching cost, which is not the case here. When operating costs are incurred in one operating mode but not in the other, intuition is confirmed that the asset value is lower compared to the case of no operating costs and higher compared to the case of operating costs in both operating modes. Since scenario (c) assumes operating costs are only incurred for y, switching from x to y is delayed more and switching from y to x takes place earlier.

Figure 4. Switching Boundaries of the Quasi-analytical Solution for Continuous Switching

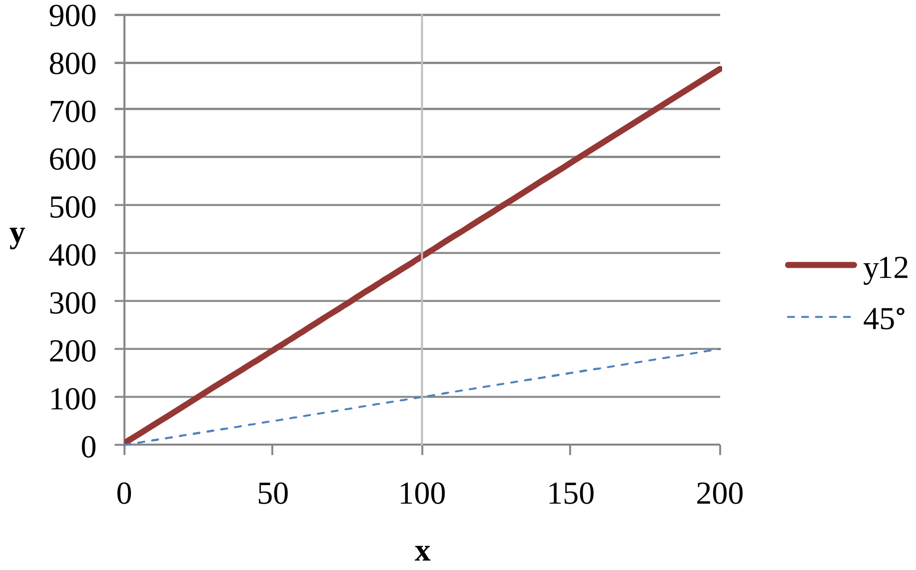
Scenario $c_x = c_y = 0$ and Scenario $c_x = c_y = 25$



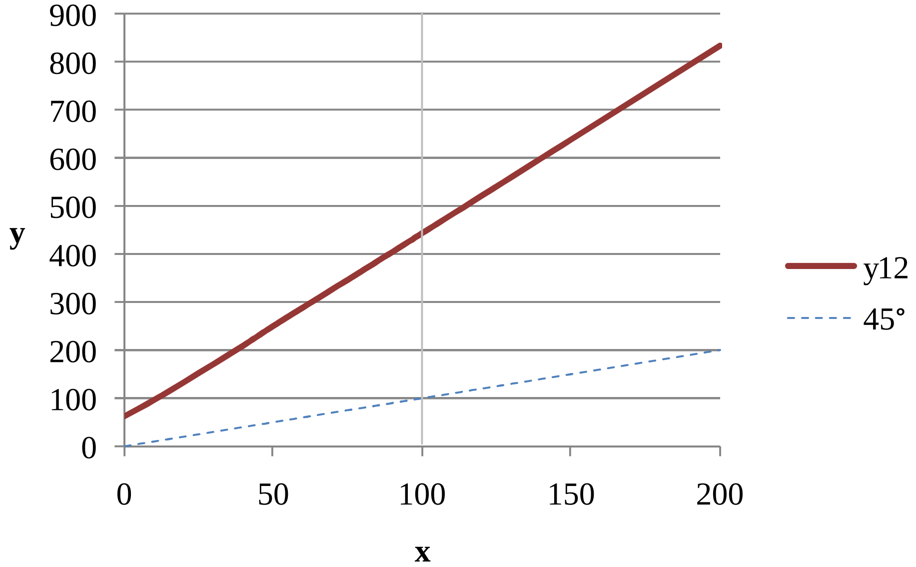
y_{12} is the level of y when it is optimal to switch from x to y, and y_{21} is the level of y for switching vice versa. Switching boundaries are obtained from solving (6) and (8) - (14) simultaneously for parameter values from Table 1 except for c_x and c_y.

Figure 5. Switching Boundary of the Quasi-analytical Solution for One-Way Switching

Scenario $c_x = c_y = 0$ and Scenario $c_x = c_y = 25$



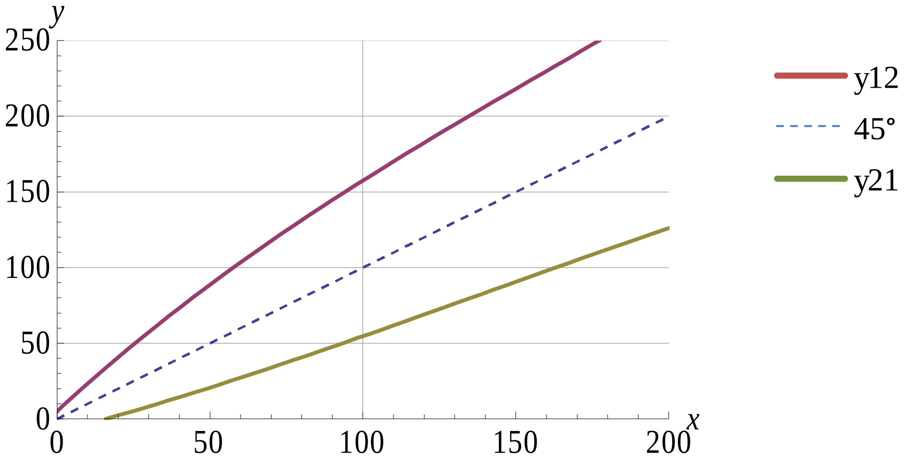
Scenario $c_x = 0$, $c_y = 25$



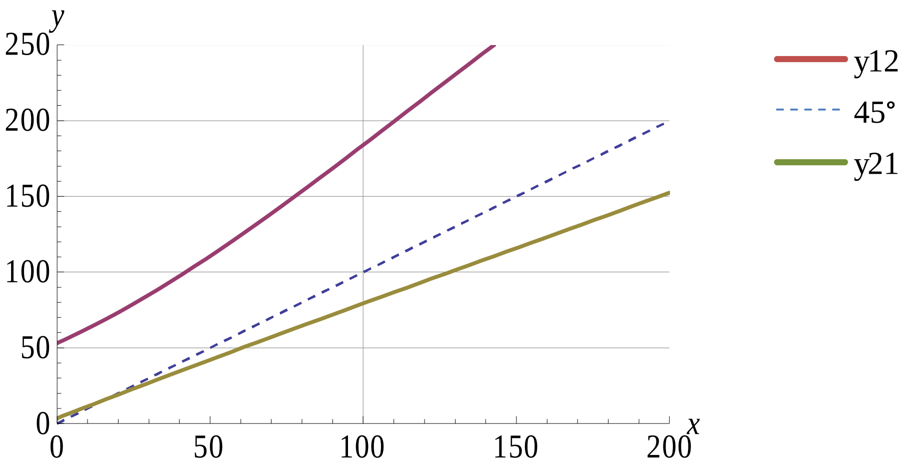
y_{12} is the level of y when it is optimal to switch from x to y, and y_{21} is the level of y for switching vice versa. Switching boundaries are obtained from the simultaneous solution of (6), (9), (11), (12), with B=0, for parameter values from Table 1 except for c_x and c_y.

Figure 6. Switching Boundaries of the Numerical Solution for Continuous Switching and Suspension Options

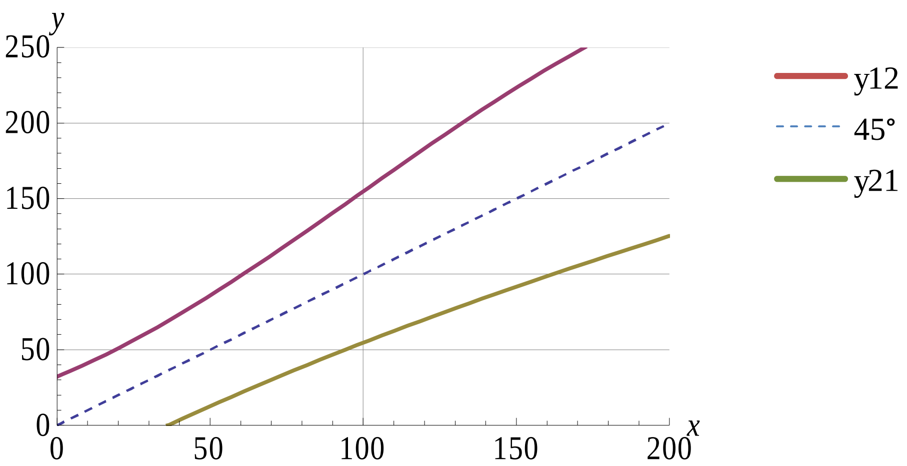
Scenario $c_x = c_y = 0$



Scenario $c_x = 0$, $c_y = 25$



Scenario $c_x = c_y = 25$

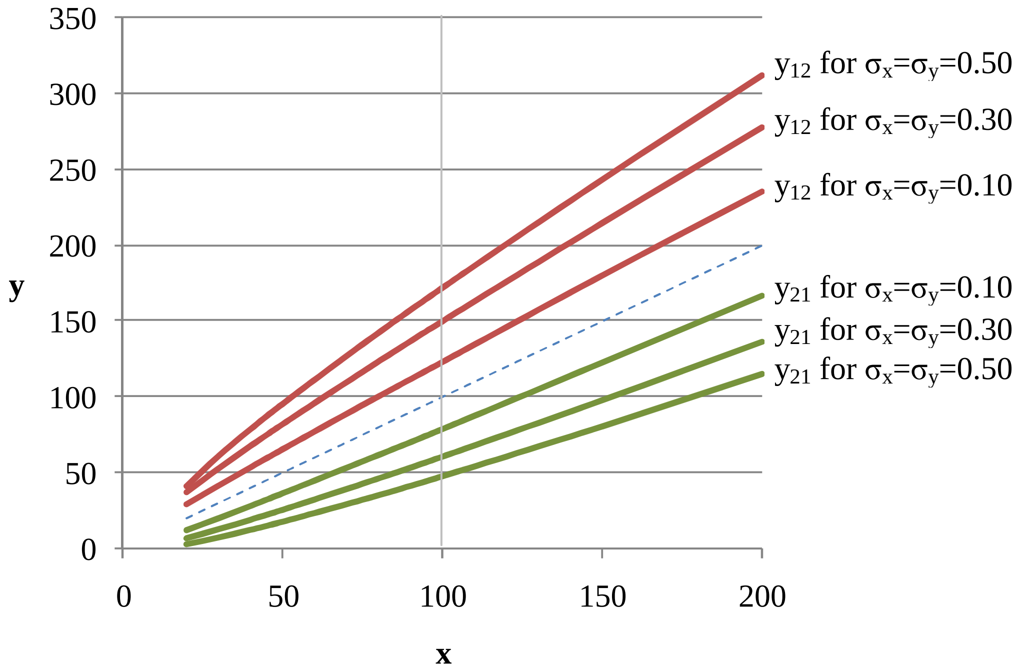


y_{12} is the level of y when it is optimal to switch from x to y, and y_{21} is the level of y for switching vice versa. Switching boundaries are from lattice solution based on (21) - (40) for parameter values from Table 1 except for c_x and c_y, grid spacing of i=50, j=50, k=250, Δt=0.2.

The Figures above depict the shape of the switching boundaries y_{12} and y_{21} according to each model for different levels of x or y. The switching boundaries of the two models with continuous switching opportunities (Figure 4 and Figure 6) are almost identical for the scenario of zero operating costs, confirming again that the models are equivalent in terms of the results except for the somewhat lower precision of the numerical lattice solution. The log-normal characteristic of the underlying commodity prices leads to higher variances in absolute terms when prices increase. This causes the switching boundaries y_{12} and y_{21} to spread further apart when the commodity prices x and y increase. In the limit of x and y approaching zero, the switching boundaries come close together but keep a minimum distance as a result of switching costs. In the presence of operating costs and suspension options (Figure 6), the switching boundaries take a different shape for low levels of the underlying commodity prices. They are spread further apart for low levels of x and y because the asset operation can be suspended so that switching occurs only when there is a high enough positive net cash flow in the alternative operating mode. In the absence of the suspension option, switching can occur even when the net cash flow in the alternative mode is negative as long as losses can be reduced. When operating costs are different in the two operating modes, the switching boundaries move in such a way as to delay switching to the mode of higher operating costs and to accelerate switching to the mode of lower operating costs. Finally, Figure 5 describes the switching boundary y_{12} for one-way switching and shows that y needs to be about four times as high as x to trigger a switch if switching back is not possible.

The preceding comparison of the three models reveals that it is desirable to obtain a (quasi-)analytical solution for the sake of transparency and accuracy, but that the numerical lattice solution approaches the quasi-analytical solution if the grid spacing is further refined (which increases computation time). Numerical approaches used to value real options on two stochastic outputs with continuous switching, such as in Childs et al. (1996) and Song (2010), might therefore be compared to results obtained from the quasi-analytical framework provided. For instance, we have used in our lattice solution a time interval of 0.2 and the model inaccuracy was 2.1%. Childs et al. (1996) have used the same time interval. Other things being equal, their results might be expected to have a similar degree of inaccuracy and be improved by refining the grid spacing or, better, using a quasi-analytical solution.

Figure 7. Switching Boundaries of the Quasi-analytical Solution for Continuous Switching for Different Volatilities



y_{12} is the level of y when it is optimal to switch from x to y, and y_{21} is the level of y for switching vice versa. σ_x and σ_y are volatilities of x and y. Switching boundaries are obtained from solving (6) and (8) - (14) simultaneously for parameter values from Table 1 except for σ_x and σ_y.

Figure 7 illustrates the sensitivity of the switching boundaries to changes in volatility according to the quasi-analytical solution. It is clearly evident that switching boundaries are further apart when volatilities are higher. This is consistent with the general real option theory because uncertainty is taken into account which delays switching in order to gain more information. In contrast to this, the Marshallian rule stipulates that switching is justified as soon as the difference between present value of expected cash flows in the new operating mode and present value of expected cash flows in the incumbent operating mode exceeds the switching cost. To see how the spread between the two switching boundaries is different when using our real option model compared to the Marshallian rule, we use a similar approach to Adkins and Paxson (2010) and define:

$$\frac{p_2\, y_{12}}{\delta_y} \cdot \Omega_{y_{12}} - \frac{p_1\, x_{12}}{\delta_x} \cdot \Omega_{x_{12}} = S_{12} \tag{41}$$

$$\frac{p_1\, x_{21}}{\delta_x} \cdot \Omega_{x_{21}} - \frac{p_2\, y_{21}}{\delta_y} \cdot \Omega_{y_{21}} = S_{21} \tag{42}$$

The Marshallian rule is satisfied when all wedges (Ω_{x12}, Ω_{x21}, Ω_{y12}, Ω_{y21}) are equal to one. To determine the wedges for the real option model, we transform equations (11) - (12) into:

$$A\, x_{12}^{\beta_{11}}\, y_{12}^{\beta_{12}} = \frac{1}{\beta_{12}\,\beta_{21} - \beta_{11}\,\beta_{22}} \left(\beta_{22}\, \frac{p_1\, x_{12}}{\delta_x} + \beta_{21}\, \frac{p_2\, y_{12}}{\delta_y} \right) \tag{43}$$

$$B\, x_{12}^{\beta_{21}}\, y_{12}^{\beta_{22}} = \frac{1}{\beta_{12}\,\beta_{21} - \beta_{11}\,\beta_{22}} \left(\beta_{12}\, \frac{p_1\, x_{12}}{\delta_x} + \beta_{11}\, \frac{p_2\, y_{12}}{\delta_y} \right) \tag{44}$$

and (13) - (14) into:

$$A\, x_{21}^{\beta_{11}}\, y_{21}^{\beta_{12}} = \frac{1}{\beta_{12}\,\beta_{21} - \beta_{11}\,\beta_{22}} \left(\beta_{22}\, \frac{p_1\, x_{21}}{\delta_x} + \beta_{21}\, \frac{p_2\, y_{21}}{\delta_y} \right) \tag{45}$$

$$B\, x_{21}^{\beta_{21}}\, y_{21}^{\beta_{22}} = \frac{1}{\beta_{12}\beta_{21} - \beta_{11}\beta_{22}} \left(\beta_{12} \frac{p_1\, x_{21}}{\delta_x} + \beta_{11} \frac{p_2\, y_{21}}{\delta_y} \right) \tag{46}$$

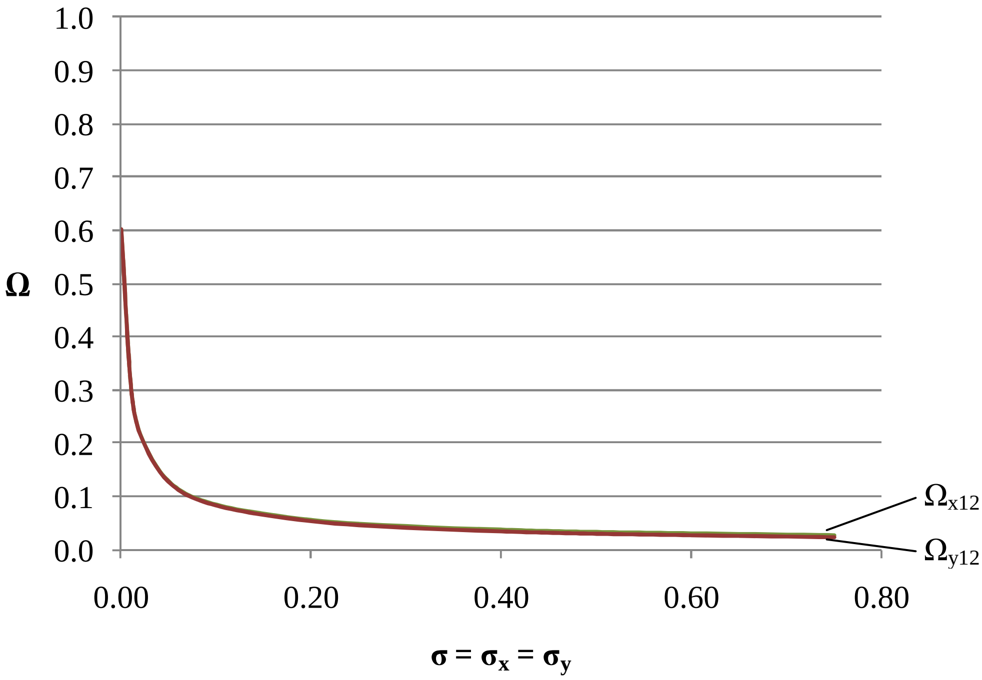
Substituting (43) - (44) into (9) and (45) - (46) into (10) yields the wedges:

$$\Omega_{x_{12}} = \Omega_{x_{21}} = 1 - \frac{\beta_{12} - \beta_{22}}{\beta_{12}\beta_{21} - \beta_{11}\beta_{22}} \tag{47}$$

$$\Omega_{y_{12}} = \Omega_{y_{21}} = 1 - \frac{\beta_{21} - \beta_{11}}{\beta_{12}\beta_{21} - \beta_{11}\beta_{22}} \tag{48}$$

Since β_{12} and β_{21} are positive and β_{11} and β_{22} are negative and the denominator of (47) and (48) needs to be positive to justify the option values in (43) - (46), the wedges are less than one. This demonstrates that the switching hysteresis is larger than suggested by the Marshallian rule.

Figure 8. Wedges of the Quasi-analytical Solution for Continuous Switching

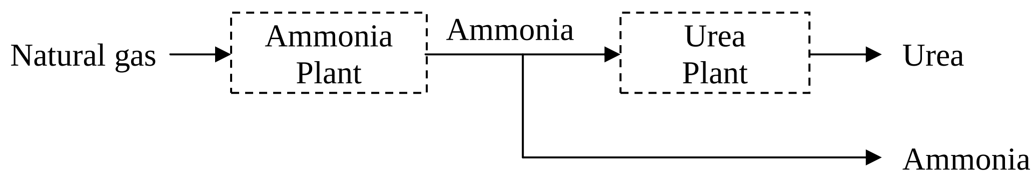


Wedges (Ω) lower than one indicate a switching hysteresis that is larger than the one suggested by Marshallian triggers. The lower the wedges, the larger the switching hysteresis. σ_x and σ_y are volatilities of x and y. Wedges are obtained from (47) and (48) for parameter values from Table 1.

Figure 8 shows the wedges as a function of volatilities. It is evident that the wedges decrease rapidly when uncertainty is introduced. The lower the wedges, the larger the switching hysteresis. When there is no volatility, the wedges are 0.6, and not 1.0 as suggested by the Marshallian rule. This is interesting because it underpins the value of waiting even when there is no uncertainty. Dixit and Pindyck (1994) provide the optimal investment threshold as r/δ times the investment cost when the underlying variable has a deterministic growth rate. The wedge of 0.6 in the deterministic case is therefore $\delta / r = 0.03/0.05$.

4 Empirical Application

In this section, the real option approach is applied to the valuation of a fertilizer plant with switching opportunities between two output products, ammonia and urea. The following simplistic scheme depicts the required input, the basic transformation and the outputs of such a facility with production mix flexibility:



Operating mode '1' of our model corresponds to operating only the ammonia plant and selling ammonia. Operating mode '2' corresponds to operating both plants, because the production of urea requires ammonia as a raw material, but selling only urea. V_1 is then the value of the total fertilizer plant when currently selling ammonia, and V_2 the value of the total fertilizer plant when currently selling urea. Switching between the two products is done by ramping up or down the downstream urea plant.

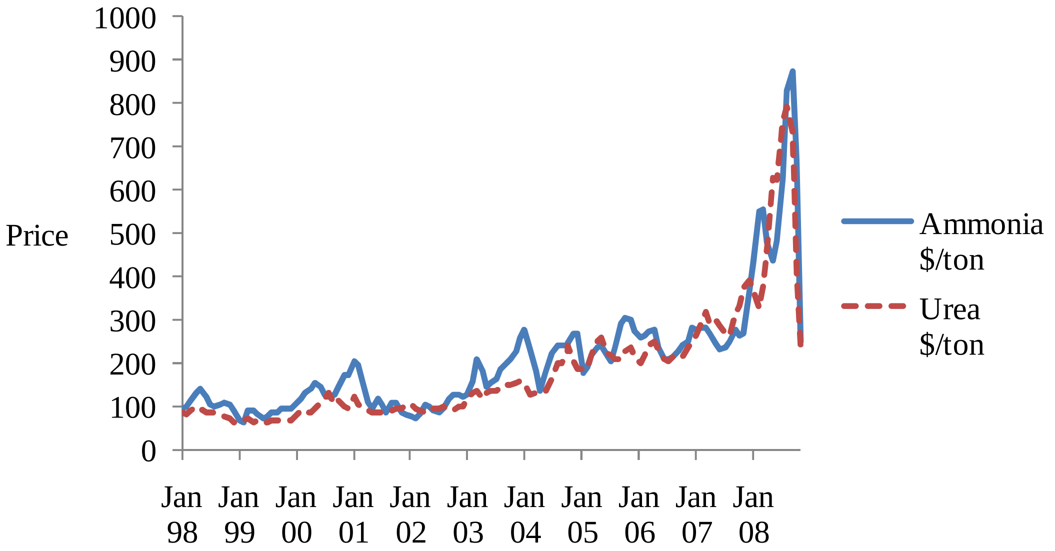
The investment cost for an ammonia plant with a capacity of 677,440 mt per year (p_1) is estimated by industry experts to be around $550 m. Since the production of one metric tonne of urea requires only 0.58 mt of ammonia, the corresponding capacity of the urea plant is 1,168,000 mt per year (p_2). The investment cost for such a urea plant is around $340 m, so that the investment cost for the total fertilizer plant sums up to $890 m. As is illustrated above, the production of urea requires both the ammonia and urea plant.

The industry dynamics are such that in times of low demand for fertilizer, the equilibrium price is supply-driven. The marginal producers with the highest cost base – typically based in regions of high gas prices (US, Western Europe) or inefficient facilities (e.g. Eastern Europe) – drop out until the prices have been stabilised. Estimates indicate that about 10% of the global urea capacity was closed in January 2009 (Yara, 2009). In times of high demand on the other hand, prices are no longer determined by the cost base but by the marginal value for the customer at full capacity of the industry.

4.1 Econometric Analysis of Commodity Prices

As discussed in the previous section, the real option model is based on the assumption that commodity prices follow geometric Brownian motion. We assume the historic volatility of the commodity prices is a reasonable estimate of the future volatility. An analysis of the time series month-by-month over the last decade reveals an annual volatility of 57% for ammonia and 40% for urea. It can also be seen from Figure 9 that the price movements are slightly more marked for ammonia. Furthermore, the figure suggests a high correlation between the two types of fertilizer. Numerical analysis confirms this with a correlation between ammonia and urea of 0.92.

Figure 9. Prices of Ammonia and Urea



Average of prices indicated in industry publications for the applicable month (Source: Yara). Ammonia fob Black Sea. Urea bulk Black Sea.

The graph also suggests that the volatility was higher in the last two years of the data sampling period compared to the years before. During the period 1998-2006, the annualised volatilities of ammonia and urea were 48% and 30% respectively while the price volatility has been significantly higher in the years 2007-2008, with 88% for ammonia and 71% for urea. During the period 1998-2006, ammonia and urea were closely correlated at 0.90 and slightly less correlated (0.82) during the years 2007-2008.

There is only limited evidence for estimating the convenience yields of the fertilizer prices, since futures or forward prices are not publicly available. Our assumption for these consumption commodities therefore is that the convenience yields are positive and at the same level as the risk-free interest rate so that the expected growth rate in the risk-neutral setting is nil. The main reason for choosing the convenience yields to equal the risk-free interest rate is that operating costs are assumed constant and a positive growth rate of the commodity prices would increase the net cash flow as time passes, thereby producing increasingly higher profits which could not be economically justified.

Table 3. Econometric Analysis of Commodity Prices

	Ammonia	**Urea**
Parameter estimation		
Volatility (σ)	0.57	0.40
Correlation (ρ)	0.92	
Volatility (σ) in the period 1998-2006	0.48	0.30
Correlation (ρ) in the period 1998-2006	0.90	
Volatility (σ) in the period 2007-2008	0.88	0.71
Correlation (ρ) in the period 2007-2008	0.82	
Testing for (non-)stationarity		
Augmented Dickey Fuller test		
Null-hypothesis: Series is non-stationary Settings: Include intercept; Include 12 lags to account for autocorrelation		
p-value	0.9203	0.9977
Conclusion	Hypothesis of non-stationarity can clearly not be rejected	Hypothesis of non-stationarity can clearly not be rejected
KPSS test		
Null-hypothesis: Series is stationary Settings: Include intercept		
p-value	0.0000	0.0000
Conclusion	Hypothesis of stationarity can be rejected with certainty	Hypothesis of stationarity can be rejected with certainty

Underlying data: Monthly prices as an average of prices indicated in industry publications (Source: Yara). Ammonia fob Black Sea. Urea bulk Black Sea.

We have determined above the parameters for the geometric Brownian motion processes of the commodity prices. In order to test our assumption of random walk ammonia and urea prices for basic validity, we test these commodity prices for stationarity. The Augmented Dickey-Fuller (ADF) test examines the null hypothesis that the time series has a unit root, which means the series is non-stationary, while the Kwiatkowski-Phillips-Schmidt-Shin (KPSS) test examines the opposite hypothesis, that the series is stationary. Table 3 summarises the test results and conclusions. The ADF tests finds that the null hypothesis of units roots cannot be rejected for both, ammonia and urea, which is confirmed by the KPSS test results stating that both variables are non-stationary. A stationary process such as the Ornstein-Uhlenbeck process would therefore not be appropriate to model ammonia and urea prices. It might well be that more complex stochastic processes would improve the commodity price modelling, such as introducing a stochastic convenience yield or stochastic volatility. This is however beyond the scope of this real options valuation.

The empirical application uses the following set of commodity parameters.

Overview of Commodity Parameters

Current ammonia price[1]	x	\$251	/ mt
Current urea price[1]	y	\$243	/ mt
Ammonia convenience yield	δ_x	5.0	%
Urea convenience yield	δ_y	5.0	%
Ammonia volatility	σ_x	57	%
Urea volatility	σ_y	40	%
Correlation ammonia/urea	ρ	0.92	
Risk-free interest rate	r	5.0	%

1. Average of prices in November 2008

4.2 Plant-specific Parameters and NPV Analysis of the Fertilizer Plant

Natural gas is the main raw material in the production of nitrogen-based fertilizers. At current market prices (Nov 2008) in the US, natural gas represents about 90% of the operating cost of both ammonia and urea. The production of one metric ton (mt) of ammonia requires 36 mm Btu of natural gas. The remaining production cost amounts to $26/mt. Assuming a gas price of $7 per mm Btu, this amounts to a total operating cost for ammonia (c_x) of $278 /mt. In some countries, fertilizer companies actually have fixed-price gas contracts with state-owned suppliers while companies in other places are exposed to the volatility of the (spot) natural gas market. We assume a constant natural gas price in our analysis. For the production of one metric tonne of urea, 0.58 mt of ammonia are required (0.58 x 36 mm Btu x gas price + 0.58 x $26). The conversion of ammonia to urea further requires 5.2 mm Btu of gas and other processing costs of $22/mt. Based on the gas price of $7/mm Btu, this comes to a total operating cost for urea (c_y) of $220/mt. The assumption here is that the ammonia required for the production of urea cannot be imported but is supplied by the own plant. The following parameters are used for the further analyses:

Overview of Plant Parameters

Operating cost ammonia production	c_X	$278	/mt ammonia
Operating cost urea production	c_Y	$220	/mt urea
Capacity ammonia plant	p_1	677,440	mt ammonia per year
Capacity urea plant	p_2	1,168,000	mt urea per year
Switching cost[1] ammonia → urea	S_{12}	$150,000	
Switching cost[1] urea → ammonia	S_{21}	$150,000	

1. The switching cost has been estimated to correspond approximately to the lost profit on selling urea (assumed gross margin of $60/mt urea) over a twelve hours non-productive time plus 50% in addition for inefficient use of materials and energy during the switching process. This is a "best guess" using

opinions of participants in the fertilizer industry since precise calculations of switching costs are not available.

The net cash flow (NCF) can be calculated as commodity price less operating cost and is the net profit/loss per metric ton. The present value of the fertilizer plant selling ammonia in perpetuity is given by:

$$V_1(x;y) = p_1(x/\delta_x - c_x/r) \quad (49)$$

With a current net cash flow of -\$27 per ton (=251-278), the value is negative at -\$366 m. If the option to suspend the asset operation to avoid net losses is taken into account, using equations (22) - (24), the asset value is \$2,220 m. The suspension option is very valuable (\$2,586 m) at the current price of ammonia and even justifies an investment of \$550 m for the ammonia plant. The present value of the fertilizer plant selling urea only is:

$$V_2(x;y) = p_2(y/\delta_y - c_y/r) \quad (50)$$

At the current price of urea, a net cash flow of \$23 per ton (=243-220) is earned and the present value of selling urea is \$537 m. With the suspension option, the asset value increases to \$3,168 m, which compares to an investment cost of \$890 m. Hence, a static NPV analysis, ignoring the suspension option, suggests that both ammonia and urea are not worthwhile investments at current prices. When the suspension option is taken into account, however, each investment is highly valuable.

4.3 Fertilizer Plant with One-Way Switching Option

Let us now compare this to the hypothetical case of a fertilizer plant which is flexible to switch from ammonia to urea but not vice versa. The quasi-analytical solution for one-way switching as developed in Section 2.3 is applied to the fertilizer plant parameters. The asset value V_1, currently selling ammonia but with the option to switch to selling urea, amounts to \$1,450 m. This is equal to the sum of perpetually selling ammonia (-\$366 m) and the switching option (\$1,816), given as F_1 in Table 4.

Table 4. Value and Switching Boundary of Fertilizer Plant with One-Way Switching

V_1 [m USD]

y \ x	100	200	251	300	400
100	-1,772	-649	-20	602	1,900
200	-283	409	896	1,414	2,564
243	571	1,043	**1,450**	1,908	2,968
300	- switch to y -	2,042	2,326	2,689	4,659
400	- switch to y -	- switch to y -	4,241	4,403	5,019

F_1 [m USD]

y \ x	100	200	251	300	400
100	640	407	346	304	247
200	2,128	1,465	1,262	1,116	911
243	2,983	2,100	**1,816**	1,610	1,315
300	- switch to y -	3,098	2,692	2,391	3,006
400	- switch to y -	- switch to y -	4,607	4,105	3,366

Switching boundary

x_{12}	**100**	**200**	**251**	**300**	**400**
y_{12}	276	381	440	498	619

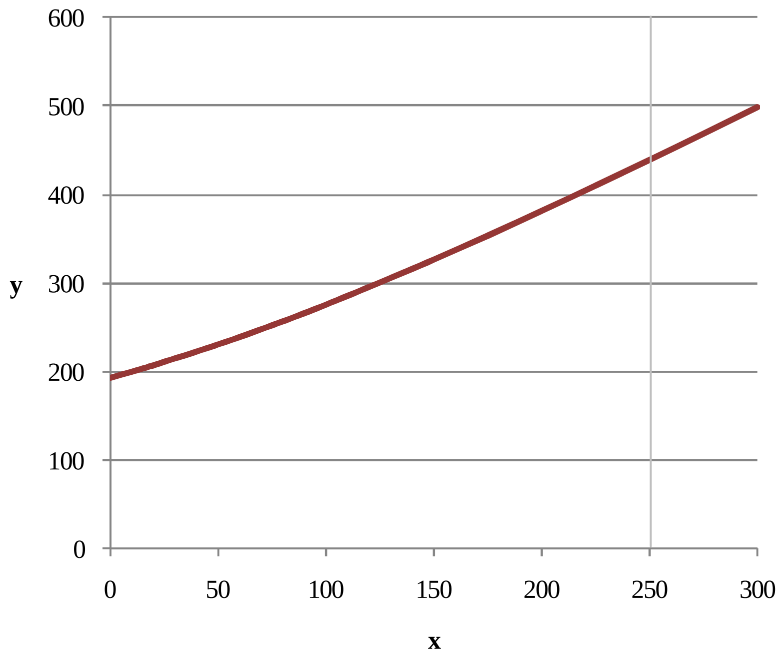
V_1 is the asset value when currently selling ammonia (x). F_1 is the option to switch from ammonia (x) to urea (y), given by the last term in (5). Switching boundary (x_{12},y_{12}) indicates when to switch from ammonia (x) to urea (y). Asset values and switching boundaries are obtained from (5) and the simultaneous solution of (6), (9), (11), (12), with B=0. Prices of x and y are in \$/mt.

The option to switch can be obtained by investing \$340 m for the urea plant which brings in the flexibility. It increases in value with higher urea prices because switching becomes more attractive and decreases with higher ammonia prices because switching becomes less attractive. The switching option is always positive and exceeds the total asset value when ammonia prices are low. The asset value V_2, selling urea, is given by equation (50) because we assume no opportunity to switch from urea to ammonia in this model, and amounts to $537 m at current prices.

At the current ammonia price of $251/mt, switching to urea is recommended when urea surpasses a level of $440/mt. As was mentioned earlier, the switching cost makes the option non-homogeneous of degree one in the commodity prices which results in a switching boundary where y/x is not constant, as is evident in Figure 10. There is a minimum level of y, below which switching is never optimal. This minimum is defined by the case x=0 with $y_{12}(x=0)=\frac{\beta_{y1}}{\beta_{y1}-1}\left(S_{12}+\frac{p_2 c_2}{r}-\frac{p_1 c_1}{r}\right)\frac{\delta_y}{p_2}$

where β_{y1} is the solution to the quadratic equation of a one-factor perpetual call option on y.

Figure 10. Switching Boundary for the Fertilizer Plant with One-Way Switching



Switching boundary (x_{12},y_{12}) indicates when to switch from ammonia (x) to urea (y). Prices are in $/mt. Switching boundary is obtained from the simultaneous solution of (6), (9), (11) (12), with B=0, for parameter values of the fertilizer plant.

4.4 *Fertilizer Plant with Continuous Switching and Suspension Options*

The same parameters are now applied to the model with continuous switching and suspension options. As defined above, V_1 is the value for the total fertilizer plant assuming the current operating mode '1', selling ammonia. Correspondingly, V_2 is the value of the total fertilizer plant assuming that currently urea is sold. Using the numerical solution procedure based on a trinomial lattice as presented earlier, the asset values for different combinations of commodity prices and the switching boundaries are shown in Table 5.

Table 5. Values and Switching Boundaries of Fertilizer Plant with Continuous Switching and Suspension Options

V_1 [m USD]

y \ x	100	200	251	300	400
100	1,196	1,968	2,433	2,944	4,003
200	2,558	3,032	3,374	3,795	4,748
243	3,297	3,694	**3,972**	4,332	5,180
300	4,361	4,685	4,907	5,202	5,907
400	6,373	6,612	6,780	7,004	7,527

V_2 [m USD]

y \ x	100	200	251	300	400
100	1,197	1,968	2,433	2,939	3,994
200	2,558	3,034	3,374	3,792	4,731
243	3,304	3,701	**3,978**	4,334	5,169
300	4,379	4,704	4,926	5,215	5,910
400	6,415	6,654	6,822	7,041	7,553

F_1 [m USD]

y \ x	100	200	251	300	400
100	3,608	3,025	2,799	2,646	2,350
200	4,970	4,089	3,740	3,497	3,095
243	5,709	4,751	**4,338**	4,034	3,527
300	6,773	5,742	5,273	4,904	4,254
400	8,785	7,669	7,146	6,706	5,874

F_2 [m USD]

y \ x	100	200	251	300	400
100	4,000	4,771	5,236	5,742	6,797
200	3,025	3,501	3,841	4,259	5,198
243	2,767	3,164	**3,441**	3,797	4,632
300	3,842	2,835	3,057	3,346	4,041
400	2,210	2,449	2,617	2,836	3,348

Switching boundary

x_{12}	100	200	251	300	400
y_{12}	234	250	263	279	317

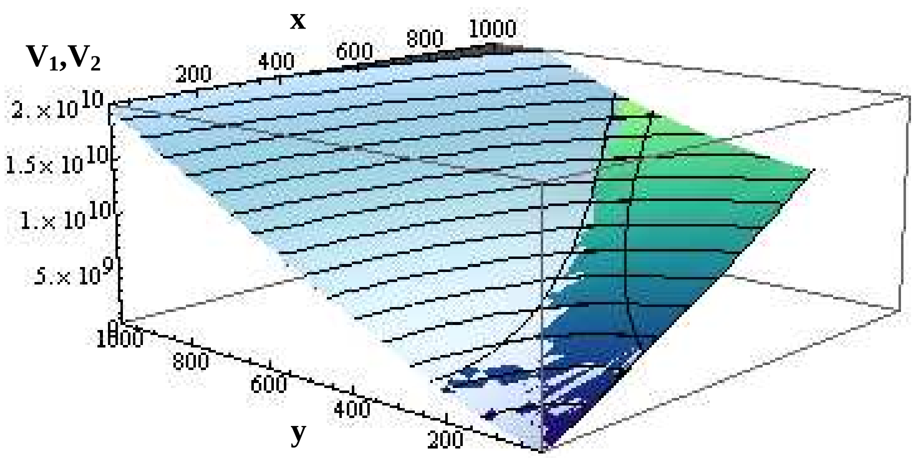
Switching boundary

x_{21}	348	442	491	563	706
y_{21}	100	200	243	300	400

V_1 is the asset value when currently selling ammonia (x) and V_2 the asset value when currently selling urea (y). F_1 and F_2 are the values exceeding the static present value of the respective operating mode, i.e. the total value of flexibility (switching and suspension).

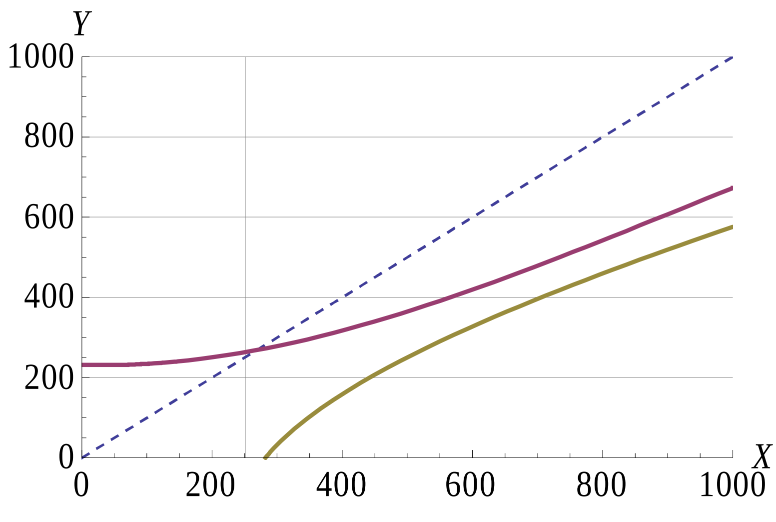
Switching boundary (x_{12}, y_{12}) indicates when to switch from ammonia (x) to urea (y) and boundary (x_{21}, y_{21}) indicates when to switch vice versa. Asset values and switching boundaries are obtained from the lattice approach based on (21) - (40) and grid spacing of i=50, j=50, k=250, Δt=0.2. Prices of x and y are in \$/mt.

Figure 11. Value Surface of the Fertilizer Plant with Continuous Switching and Suspension Options



Asset values in \$ as a function of x and y. V_1 (dark colour) is the asset value when currently selling ammonia (x), and V_2 (light colour) the asset value when currently selling urea (y). Switching boundaries are indicated as lines where upper line is the boundary (x_{12}, y_{12}) to switch from x to y and the lower line the boundary (x_{21}, y_{21}) to switch from y to x. Results are from lattice solution based on (21) - (40) for parameter values of the fertilizer plant and grid spacing of i=50, j=50, k=250, Δt=0.2.

Figure 12. Switching Boundaries for the Fertilizer Plant with Continuous Switching and Suspension Options



Straight (dotted) line as reference for x=y. Upper line represents boundary (x_{12},y_{12}) to switch from ammonia (x) to urea (y), lower line represents boundary (x_{21},y_{21}) to switch vice versa.

Switching boundaries are from lattice solution based on (21) - (40) for parameter values of the fertilizer plant and grid spacing: i=50, j=50, k=250, $\Delta t=0.2$. Prices are in \$/mt.

The results above can be shown graphically in the form of value surfaces for V_1 and V_2 as a function of the commodity prices, together with the switching boundaries. The asset value increases both with x and y, as should be the case for an option to choose the best among two alternatives. The value surface represents the expected "options-like" shape, with smooth transitions between operation and suspension as well as between the two alternative operating modes, separated at the switching boundaries only by the switching cost. V_1 and V_2 should not be different by more than the switching cost because otherwise switching would take place immediately. The fact that V_1 and V_2 differ by more than the switching cost for some combinations of x and y provided in the table is caused by the imprecision introduced by interpolation within the numerical solution grid which is necessary to retrieve asset values for any x-y-combination.

At current prices of \$251 and \$243 for ammonia and urea, respectively, the flexible fertilizer plant is valued at \$3,972 m. This compares to a present value of the inflexible fertilizer plant of -\$366 m when selling ammonia alone, or \$537 m when selling urea alone, so that the total value of flexibility (switching and suspension) is worth \$4,338 m for an ammonia producer and \$3,441 m for a urea producer. The asset value with continuous switching and suspension options also exceeds the asset value where only one-way switching is possible and no suspension option is available, as calculated before (\$1,450 m). Thus, the options of unlimited switching and temporary suspension add about 270%. Moreover, the real asset value significantly exceeds the investment cost of \$890 m for the total fertilizer plant.

An analysis of the switching boundaries (x_{12},y_{12}) and (x_{21},y_{21}) reveals that the boundaries are far apart when commodity prices are low, then narrow with increasing prices and finally drift again apart for very high prices. Generally, the boundaries diverge with increasing commodity prices because of the increasing variance of the log-normal prices, reflecting the higher uncertainty. In the presence of operating costs and a suspension option, however, a positive net cash flow in the alternative operating mode is a necessary requirement to trigger switching, i.e. the price level triggering the switch must at least exceed the operating cost. This effect is highly relevant for low prices and less relevant for higher prices. Furthermore, different capacities in the two operating modes cause the boundaries to not move along the 45° line. The combined effect of the aforementioned phenomena produces the shape of the boundaries as illustrated. Comparing the optimal switching behaviour with that of the asset with a one-way switching option reveals that switching occurs at significantly lower price levels. Hence, the flexibility of unlimited switching is valuable and the switching triggers indicate more frequent switching.

4.5 Sensitivities and Discussion

We compare the valuation results for the fertilizer plant with no flexibility, with suspension option but no switching option, with one-way switching but no suspension option, and with continuous switching and suspension options. These asset values are given in Table 6 for three scenarios reflecting different volatilities and correlation because ammonia and urea were much more volatile and less correlated in the years 2007-2008 compared to the period 1998-2006.

Table 6. Comparison of Valuation Models and Sensitivity Analysis

x = 251 USD/ton; cx = 278 USD/ton; p_1 = 677,440 tons p.a.
y = 243 USD/ton; cy = 220 USD/ton; p_2 = 1,168,000 tons p.a.
$\delta_x = \delta_y = 0.05$; r = 0.05; $S_{12} = S_{21}$ = 150,000 USD

Sensitivities		**Inflexible asset**	**Inflexible asset with suspension option**	**Asset with one-way switching option**	**Asset with continuous switching and suspension option**
σ_x=0.57, σ_y=0.40, ρ=0.92 (Period 1998-2008)	V_1 [m USD]	-366	2,220	1,450	3,972
	V_2 [m USD]	537	3,168	537	3,978
σ_x=0.48, σ_y=0.30, ρ=0.90 (Period 1998-2006)	V_1 [m USD]	-366	1,988	1,262	3,397
	V_2 [m USD]	537	2,599	537	3,401
σ_x=0.88, σ_y=0.71, ρ=0.82 (Period 2007-2008)	V_1 [m USD]	-366	2,729	2,719	5,899
	V_2 [m USD]	537	4,311	537	5,906

V_1 is the asset value when currently selling ammonia and V_2 is the asset value when currently selling urea. x is the current ammonia price and y is the current urea price. c_x and c_y are the respective operating costs, p_1 and p_2 the annual capacities, σ_x and σ_y the volatilities, ρ the correlation, r the risk-free rate, δ_x and δ_y the convenience yields. S_{12} is the switching cost from ammonia (x) to urea (y) and S_{21} for switching vice versa.
Values of inflexible asset are obtained from (5) and (7) with A=B=0. Values of inflexible asset with suspension option are obtained from (22) and (29). Values of asset with one-way switching option are obtained from (5) and (7) with B=0 and the simultaneous solution of (6), (9), (11), (12). Values of asset with continuous switching and suspension option are obtained from the lattice approach based on (21) - (40) and grid spacing of i=50, j=50, k=250, Δt=0.2.

We find that the option to suspend operations adds $2,586 m to the value of a fertilizer plant selling ammonia only (-$366 m), while this option is worth $2,631 m for a fertilizer plant selling urea only ($537 m). The option to temporarily suspend operations is a practical management tool if partner contracts are conceived intelligently. Here, the real options approach provides an asset value which incorporates more realistic management behaviour than assumed in a DCF approach. When continuous switching opportunities between ammonia and urea are available in addition to the suspension option, the fertilizer plant is valued at about $3,972 m which is a surplus of about 80% on ammonia only and about 25% on urea only.

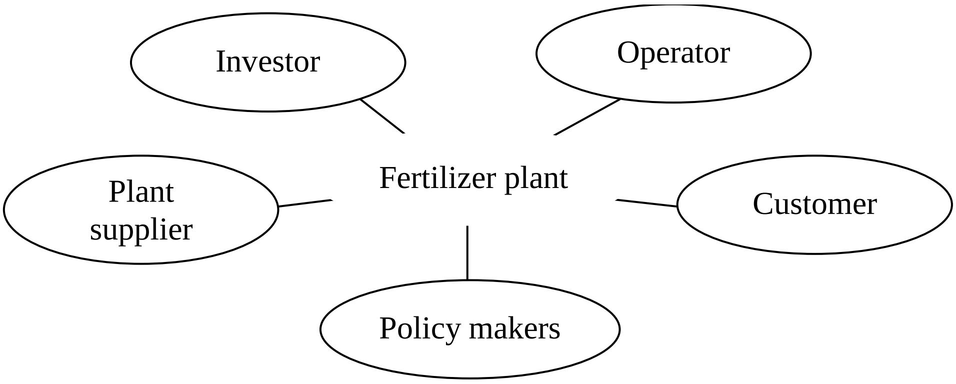
When the sampling period for the commodity price parameters is split into a period 1998-2006 and a period 2007-2008, uncertainty is lower in the former and markedly higher in the latter period. As a consequence, the asset values with flexibility are lower in the former period and significantly higher in the latter. The combined effect of higher volatility and lower correlation makes the flexible asset comparatively much more attractive than the inflexible one. The asset value with continuous switching and suspension option is valued at about $3,400 m if the volatilities and correlation are based on 1998-2006 data, and at about $5,900 m based on 2007-2008 data, which is a difference of $2,500 m. In this comparison, only volatilities and correlation are changed, not the current price level. This huge difference emphasizes how sensitive these option-based values are to changes in the measures of uncertainty. An investor has to critically evaluate whether these volatilities are expected to persist in the future or whether this is a temporary phenomenon based on market turbulences. The estimation of expected future volatility and correlation is critical and should ideally combine insights from historic data with the expected commodity-specific market dynamics. It is also interesting to observe that when uncertainty is comparatively low, the fertilizer plant with suspension option and selling ammonia only is worth more than the one with a one-way switching option to urea and no suspension option. But when uncertainty is high, the value of the latter grows faster than the former. The explanation is that when uncertainty is low and the current net cash flow is negative, significant value is gained from avoiding losses on selling ammonia, while the upside potential from switching to urea grows fast when uncertainty is high. So the choice between a suspension option and the option to switch from ammonia to urea depends on the expected volatilities and correlation. The value of the inflexible asset is indifferent to changes in volatility and correlation because no action can be taken in response to price movements.

The asset value with continuous switching and suspension options is about four times the investment cost. Even the asset values without switching but with a suspension option are over three times the investment cost. We identify the non-stationary characteristic of the two commodity prices in combination with the constant operating cost as one main reason for these high values because the upside potential is huge while the downside is limited by the suspension option. As stated earlier, we refer in our example to a fertilizer plant in an environment where natural gas is sourced on a long-term contract at constant prices which justifies the constant operating cost. The price of natural gas on the (spot) market, on the other hand, is volatile and fertilizer prices are driven by the price of natural gas in the medium- to long-term. Hence, if natural gas is not available at constant prices, the operating cost would be stochastic and correlated with the fertilizer prices, thereby introducing a third source of uncertainty and possibly changing the option values. One conclusion from this is that the availability of a gas supply at constant prices makes an investment in a fertilizer plant highly attractive because it offers significant upside potential (increasing fertilizer prices at constant operating cost) and the option to limit the downside risk (suspend if fertilizer prices fall below the constant operating cost). The assumption of no operating and maintenance costs during suspension, and ignoring competitive and customer behaviour, further contribute to the high asset values. Lattice inaccuracies certainly exist, but can be expected to be small, based on our analysis in Section 3. Although we have shown statistically that the commodity prices are non-stationary based on the 1998-2008 data, the comparatively high asset values could be an indication that decision-makers might actually have a different perception on the commodity price behaviour.

5 Policy and Strategy Implications

There are a number of stakeholders whose decisions and behaviour might potentially be influenced by the research results.

Figure 13. Stakeholders in the Project



Parties having an interest in the way a fertilizer plant is set up or operated.

Investors

The numerical results have shown that at current prices it is worth supplementing an existing ammonia plant by a downstream urea plant in order to have the flexibility to switch between the two products. For higher prices this will hold true even more. Only if the commodity prices go down so that a profitable operation is no longer possible should the investor not add a urea plant. It is important for the investor to keep in mind that ammonia and urea prices are highly correlated so that their diversion in the long-term is limited. The main drivers to reap the benefits of the product choice are low switching costs and flexible supply contracts, meaning that the company should not be stuck in rigid contracts forcing it to supply a specific product to contract customers, if it is better to sell the other product.

Operators

An obvious task of the operators is to minimise the operating costs as well as the switching costs in order to make the most of the available flexibility. Furthermore, the current supply commitments and inventory levels should be monitored continuously in order to assess the practical level of flexibility. The operational management should be aware of the current market prices and regularly update expected future prices. Similarly, expected volatilities should be updated in regular intervals.

Plant Suppliers

The above results prove a real opportunity for plant suppliers, because it supports the idea of more sophisticated (and expensive) assets. The strategy implication here is to aggressively market more flexible assets, focusing on the financial benefits to the investor. Internally, the asset could be optimised for flexibility, that is focusing design on minimising downtime and costs of switching.

Customers/Commodity Traders

A commodity trader focused on arbitrage is not interested in long-term contracts and therefore is not in conflict with the increased flexibility request of the fertilizer supplier. Other traders might have long-term customer agreements which they need to back up by long-term supply agreements with the producers. Therefore they might insist on long-term supply commitments for a specific product or otherwise might turn to single-product producers.

Policy Makers

The interest of the policy makers in this context can be considered to be the functioning of markets. Let us consider the example of a shift in fertilizer demand from ammonia to urea. The end users and therefore the policy makers would welcome a quick response in supply in order to gain from lower prices. This will happen much faster if the assets are capable of multi-product operation. The political support might be put into place for instance by giving preference in permitting processes to extending current facilities to incorporate flexibility over new inflexible investments.

6 Conclusion and Areas for Future Research

Flexibility between output products is particularly relevant in volatile commodity markets. In this paper, we value an operating asset with the option to choose the best of two commodity outputs. We develop three output switching models in the presence of operating and switching costs, first a quasi-analytical solution for continuous switching, second, a special case of the former for one-way switching, and third, a numerical lattice solution for continuous switching with suspension options. Illustrative numerical cases demonstrate that the quasi-analytical solution and the lattice approximation provide near identical results for the asset valuation and optimal switching boundaries in a comparable setting. While the switching boundaries are found to narrow as prices decline, this is different in the presence of operating costs and temporary suspension when the thresholds diverge for low enough prices.

Applying these models to a fertilizer plant with output flexibility between ammonia and urea, the value of flexibility is significant despite the high correlation between the two alternative commodities and also exceeds the required investment cost for the specified parameters. The asset value is highly sensitive to volatilities and correlation, and therefore depends on the data used for estimating these parameters. The results also demonstrate that the possibility of temporary suspension shapes the asset value surface for low spreads between commodity prices and operating cost, and this option is a practical, valuable management tool. An important implication for policy-makers is that flexible assets contribute to a fast reaction of markets to changes in demand and therefore constitutes a strategy which seems to be worthwhile supporting.

The results and interpretation also raise some further research questions. In particular, the overall asset value seems to be rather high compared to the investment cost, driven by non-stationary commodity prices in combination with constant operating costs. In particular, the price of natural gas as the input to the process and main cost driver was assumed constant. A stochastic gas price can be expected to have a positive correlation with fertilizer prices, possibly reducing the asset value. Future research might also relax the assumption of no maintenance costs during suspension, and consider possible reactions of customers and/or competitors to product switches. Further applications of the continuous rainbow option model include alternative uses of other facilities, such as multiuse sports or entertainment or educational facilities, transportation vehicles for passengers or cargo, rotating agricultural crops, and solar energy for electricity or water desalination. Another topic for future research is to extend the quasi-analytical solution for continuous switching to incorporate the suspension option.

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4 Research Paper #2 – Continuous Rainbow Options in Co-integrated Markets

Continuous Rainbow Options in Co-integrated Markets

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We develop a real option model to continuously choose between the best of two commodities when their price spread follows an arithmetic mean-reverting process. The co-integration of the two variables effectively allows the complexity to be reduced from two sources of uncertainty to only one by focusing on the spread, which is why the model can also be applied to continuous entry/exit problems on a single mean-reverting variable. We provide a quasi-analytical solution to value this real rainbow option and determine the trigger levels which justify switching between the two operating modes by incurring a switching cost. The risk-neutral valuation approach distinguishes between the different risk and discount factors which proves to be particularly critical in the context of mean-reversion. All parameters of our solution are estimated from empirical data. We apply the theoretical model to value a polyethylene plant based on the spread between polyethylene and ethylene. The spread is shown to be stationary and the parameters of the stochastic process are estimated by OLS regression and tested for validity. The sensitivity analysis reveals important implications for the valuation and operating decisions of investors in a flexible plant, both depending on the extent of mean-reversion in the value-driver.

JEL Classifications: D81, G31, L72

Keywords: Uncertainty, Flexibility, Real rainbow option, Continuous switching, Mean-reversion, Co-integration, Commodities

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1 Introduction

Industrial and agricultural applications frequently exhibit inherent options to choose between the best of two commodities. If these options are well established, the probability is high that the markets of these commodities are co-integrated to some degree. If prices drift apart, suppliers would exercise the option and switch from the less favourable to the more favourable product, typically by incurring a switching cost, until the equilibrium is re-established. This market behaviour is reflected in a mean-reverting price spread. In these cases, a two-factor valuation problem can be reduced to a problem with a single stochastic factor. Co-integrated markets are found particularly when commodities have similar applications and can be substituted rather easily for one another or when the production cost of one commodity is heavily influenced by another commodity. Examples include dry (bulk) and wet (oil) markets in the shipping industry (see Sodal et al., 2007), commercial and residential uses of real estate, industrial plants with flexibility on the product mix, refining margins and other conversion processes in the chemical industry, such as the production of polyethylene which is created by polymerisation of ethylene. Both ethylene and polyethylene are traded products, so that the conversion can be considered a real rainbow option. The valuation of this rainbow option based on the conversion spread will be the subject of the empirical application.

Kulatilaka and Trigeorgis (2004) discuss the general approach of valuing switching options, including options additivity and asymmetric switching costs. Stulz (1982) and Johnson (1987) develop closed-form solutions for a European option on the maximum or minimum of two or more assets. A quasi-analytical solution to a two-factor problem, where the option is not homogenous of degree one in the stochastic variables, is provided by Adkins and Paxson (2010a) and extended to a general switching model for two alternative energy inputs (2010b). Dockendorf and Paxson (2009) develop a real option model on the best of two commodity outputs with continuous switching, including the option of temporary suspension, and apply the model to value a flexible fertilizer plant. All of the above mentioned models are based on uncertainty represented by geometric Brownian motion.

The Schwartz (1997) analysis on the behaviour of commodity prices reveals that many commodity prices exhibit strong mean reversion. Also, Geman (2007) tests energy commodity prices for mean reversion and finds that oil and natural gas prices are mean-reverting during one period and random walk during another. Tvedt (2000) values a vessel with lay-up option in a shipping market with freight rate equilibrium and acknowledges in his conclusion that mean-reversion should be considered in the freight rate dynamics to improve the model for practical valuation. Tvedt (2003) develops an equilibrium model for freight rates and suggests mean-reversion as the underlying stochastic process.

The option pricing theory on co-integrated assets has been explored by Duan and Pliska (2004), who value finite spread options on stock indices subject to time-varying volatility by means of Monte Carlo simulations. Dixit and Pindyck (1994) provide a solution to the investment problem on an asset which follows a geometric mean-reverting stochastic process, i.e. where the variable has an absorbing barrier at zero. Option valuation on mean-reverting assets is applied by Bastian-Pinto et al. (2009) to the Brazilian sugar industry by approximating the prices of sugar and ethanol as discrete binomial mean-reverting processes and determining the value of switching between the two commodities within a bivariate lattice option framework. Näsäkkälä and Fleten (2005) and Fleten and Näsäkkälä (2010) value gas-fired power plants on the basis of the spark spread with mean-reverting variations in the short term and an arithmetic equilibrium price in the long-term. They model the spread directly from empirical data instead of considering electricity and natural gas price separately, because the spread is considered the driver of uncertainty.

Sodal et al. (2007) value the switching option for combination carriers between the co-integrated dry and wet bulk markets by modelling the price spread as mean-reverting. The approach is based on the Bellman equation which uses for the solution of the maximisation problem a rate ρ to discount the future option values. However, such a discount rate cannot be reasonably estimated because of the specific risk characteristics of the options. Sodal's empirical application confirms that the option value is highly sensitive to this discount rate ρ. The option value almost triples if ρ is reduced from 0.15 to 0.05. Furthermore, the cash flows of the static project with no switching option, which include non-stochastic cash flows, have been discounted at the same rate ρ. We develop an option model based on contingent-claims and the risk-neutral valuation approach, which only includes parameters that can be estimated from empirical data.

The remaining part of this paper is organised as follows: Section 2 introduces the characteristics of the mean-reverting spread, provides the present value of perpetual cash flows without a switching option, and then develops a model for the continuous rainbow option. It also includes a comparison to the Sodal model and demonstrates the advantages of the new model. Section 3 applies the continuous rainbow option to the valuation of a polyethylene plant based on an econometric model of the polyethylene-ethylene conversion spread. Specific and general implications are discussed in Section 4. Section 5 concludes and raises issues for further research.

2 Valuing the Switching Opportunity

2.1 Modelling the mean-reverting spread

We assume that the asset can be operated in two different modes where each operating mode is associated with one commodity. The flexibility to switch between two operating modes – the base mode (denoted by '0') and the alternative mode (denoted by '1') – means that we are faced with two underlying uncertainties, which are the prices of the two commodities. In co-integrated markets, however, the prices of the two commodities are bound to one another by economic reasons, so that the complexity can be reduced to only one underlying uncertainty by modelling the difference between the two commodity prices as a mean-reverting stochastic process. Let (p) be the weighted spread of the commodity prices,

$$p = p_1 - \frac{k_0}{k_1} p_0 \ , \tag{1}$$

where p_0 and p_1 are the commodity prices in the base and alternative mode, respectively, and k_0 and k_1 the capacities. The capacities enter into the equation in order to account for the fact that product units and output capacities of the asset may be different in the two operating modes. Hence, we unitise the spread with regard to the product sold in the alternative mode. The spread of two co-integrated commodities can be both positive or negative which is why the mean-reverting process is modelled as an arithmetic Ornstein-Uhlenbeck process:

$$dp = \eta(m - p)dt + \sigma dz \, , \tag{2}$$

where η is the speed of reversion, m the long-run mean of the spread, σ the standard deviation and dz a standardised Wiener process. We avoid the notion of volatility because volatility is commonly used to describe the standard deviation of percentage changes in stock while σ for the arithmetic Ornstein-Uhlenbeck process is the standard deviation in absolute terms. The expected value of p at time t is given by:

$$E[p]_t = m + (p - m)\,e^{-\eta t} \tag{3}$$

and the variance of p_t:

$$\operatorname{Var}[p]_t = \frac{\sigma^2}{2\eta}\left(1 - e^{-2\eta t}\right). \tag{4}$$

Under risk-neutrality, the expected growth of the stochastic process given in (2) must be $(r - \delta)p\,dt$, where r is the risk-free interest rate and δ is the convenience yield. The risk-neutral process of p, denoted by p^*, is then determined by:

$$dp^* = (r - \delta)p^*\,dt + \sigma\,dz^*, \tag{5}$$

Let μ be the instantaneous expected return of p, which is assumed constant, and α the expected increase in the level of p. The convenience yield is then defined by:

$$\delta = \mu - \alpha = \mu - \frac{\eta(m - p)}{p}. \tag{6}$$

Inserting the convenience yield from (6) into (5) provides the stochastic process of p under risk-neutrality:

$$dp^* = \left[\eta m - (\mu - r + \eta)\,p^*\right]\,dt + \sigma\,dz^*. \tag{7}$$

2.2 Discounted cash flow with no flexibility

Assuming no operating flexibility, the asset value is determined separately for each of the two operating modes as the discounted respective cash flow. The cash flow and therefore the value of the asset in the base operating mode is nil by definition because the base mode is considered the reference point for valuing the operating flexibility. The cash flow in the alternative operating mode is given by the spread less the additional operating cost, $k_1(p\text{-}c)$. The additional operating cost (c) is the weighted difference in variable operating cost between the two operating modes:

$$c = c_1 - \frac{k_0}{k_1} c_0 \ . \tag{8}$$

The discount factor for stochastic variables consists of a risk component and a time component. For a geometric Brownian motion process, the variance over time is not bounded and the risk discount factor is compounded in the same way as the time discount factor. This is different from the mean-reverting process, where the variance over time is bounded and the applicable risk discount factor cannot be compounded in the same way as the time discount factor. Therefore, mean-reverting cash flows are best discounted by discounting the equivalent risk-neutral cash flows by the risk-free interest rate, as suggested by Bhattacharya (1978). Let $M = \frac{m\eta}{\eta + \mu - r}$, then from equation (7):

$$dp^* = (\eta + \mu - r)(M - p)dt + \sigma\, dz^* . \tag{9}$$

In analogy to equation (3), the expected value of p in the risk-neutral scenario is then given by:

$$E\left[p^*\right]_t = M + (p - M)e^{-(\eta + \mu - r)t} \tag{10}$$

The risk-neutral cash-flow could either be discounted at the risk-free rate of return for an asset lifetime of T years, or be discounted in perpetuity at a higher rate taking into account the physical deterioration of the asset in the form of exponential decay. We take the latter approach since we also need to consider technological, political and environmental risk. Let λ be the arrival rate of a Poisson event which incorporates both physical deterioration and technological risk, so that the risk-neutral cash-flow is discounted at the rate ($r+\lambda$). The asset value in the alternative mode without switching opportunity is then:

$$PV_1(p) = k_1 \int_0^\infty \left(E\left[p^*\right]_t - c \right) e^{-(r+\lambda)t} dt = k_1 \left(\frac{m}{(r+\lambda)\left(1+\dfrac{\mu+\lambda}{\eta}\right)} + \frac{p}{\mu+\eta+\lambda} - \frac{c}{r+\lambda} \right). \quad (11)$$

The discounted cash flow consists of three parts. Firstly, the long-term average (m) is discounted at a rate of $(r+\lambda)(1+(\mu+\lambda)/\eta)$. This discount rate increases with the systematic risk in the stochastic fluctuations of p, represented by μ, and decreases with the speed of mean-reversion (η), because the faster p returns to its long-run average the faster the risk dissipates. With $\eta>>(\mu+\lambda)$, the discount rate is hardly affected by μ. Secondly, the current value of p is discounted at $(\mu+\eta+\lambda)$ which corresponds to discounting the η-decaying exponential function of p at the discount rate μ and accounting for deterioration and political/technical risk. Thirdly, the additional operating cost is discounted at the risk-free rate augmented by the Poisson probability.

2.3 Continuous rainbow option

We now allow for flexibility between the two operating modes. In the base mode, the commodity spread is foregone (zero cash flow). In the alternative mode, the spread is earned and variable operating costs are incurred. $V_0(p)$ and $V_1(p)$ represent the values of being in the respective operating mode, each with the option to switch to the other mode. The no-arbitrage approach can be used to set up partial differential equations describing the value functions. For this purpose, a portfolio comprising the asset and quantities of the underlying process is shown to be free of stochastic components which therefore needs to earn the risk-free rate of return. For V_0, the portfolio π is established as:

$$\pi = V_0 - \frac{\partial V_0}{\partial p} p\,. \tag{12}$$

The change in the portfolio value is given by:

$$d\pi = dV_0 - \frac{\partial V_0}{\partial p} dp - \delta\, p \frac{\partial V_0}{\partial p} dt\,, \tag{13}$$

with

$$dV_0 = \frac{\partial V_0}{\partial p} dp + \frac{1}{2}\frac{\partial^2 V_0}{\partial p^2} dp^2 = \frac{\partial V_0}{\partial p} dp + \frac{1}{2}\sigma^2 \frac{\partial^2 V_0}{\partial p^2} dt\,. \tag{14}$$

Inserting (14) into (13) shows that no stochastic components (dz) are left in the equation, so the portfolio must earn the risk-free rate of return (the asset V must earn the deterioration risk λ in addition):

$$d\pi = \frac{1}{2}\sigma^2 \frac{\partial^2 V_0}{\partial p^2} dt - \delta\, p \frac{\partial V_0}{\partial p} dt = (r+\lambda) V_0\, dt - r\, p \frac{\partial V_0}{\partial p} dt\,. \tag{15}$$

Regrouping (15) provides the partial differential equation for V_0:

$$\frac{1}{2}\sigma^2 \frac{\partial^2 V_0}{\partial p^2} + (r-\delta) p \frac{\partial V_0}{\partial p} - (r+\lambda) V_0 = 0\,, \tag{16}$$

and using the convenience yield from equation (6):

$$\frac{1}{2}\sigma^2 \frac{\partial^2 V_0}{\partial p^2} + [\eta m - (\mu - r + \eta) p] \frac{\partial V_0}{\partial p} - (r+\lambda) V_0 = 0\,. \tag{17}$$

The same procedure is used to determine the partial differential equation for V_1, the asset value in the alternative operating mode. It needs to be considered that a cash flow of $k_1(p-c)$ is earned:

$$\frac{1}{2}\sigma^2 \frac{\partial^2 V_1}{\partial p^2} + [\eta m - (\mu - r + \eta) p] \frac{\partial V_1}{\partial p} - (r+\lambda) V_1 + k_1 (p-c) = 0\,. \tag{18}$$

A more general form of the above PDEs is obtained by substituting $a = 2(r - \mu - \eta)/\sigma^2$, $b = 2\eta m/\sigma^2$ and $d = -2(r + \lambda)/\sigma^2$. Equation (17) becomes:

$$\frac{\partial^2 V_0}{\partial p^2} + (a\,p + b)\frac{\partial V_0}{\partial p} + d \cdot V_0 = 0. \tag{19}$$

With μ>r and η>0, parameter (a) will always be negative. For a<0, Kampke (1956, p. 416) suggests substituting $V_0 = F(x)$ and $x = \sqrt{|a|}\left(p + \frac{b}{a}\right)$ to obtain:

$$\frac{\partial^2 F}{\partial x^2} - x\frac{\partial F}{\partial x} - \frac{d}{a}F = 0. \tag{20}$$

Appendix A demonstrates in more detail how the above equation is obtained and how it can be further transformed into the following Weber equation by substituting $F = G(x)e^{\frac{1}{4}x^2}$ (see also Kampke, 1956, p. 414):

$$\frac{\partial^2 G}{\partial x^2} = \left(\frac{x^2}{4} + \frac{d}{a} - \frac{1}{2}\right)G. \tag{21}$$

Spanier and Oldham (1987, p. 447) establish that the above Weber differential equation is satisfied by the parabolic cylinder function of order (-d/a) and argument (x) and (-x), represented by $D_{-d/a}(x)$ and $D_{-d/a}(-x)$, so that G(x) is determined by:

$$G(x) = A \cdot D_{-d/a}(x) + B \cdot D_{-d/a}(-x), \tag{22}$$

where A and B are constant parameters and the parabolic cylinder function is defined by:

$$D_v(x) = \frac{1}{2} \cdot \frac{\sqrt{2^{v+2}\pi}}{\Gamma\left(\frac{1-v}{2}\right)} e^{-\frac{x^2}{4}} M\left(-\frac{v}{2}, \frac{1}{2}, \frac{x^2}{2}\right) + \frac{1}{2} \cdot \frac{-\sqrt{2^{v+3}\pi}}{\Gamma\left(\frac{-v}{2}\right)} x\, e^{-\frac{x^2}{4}} M\left(\frac{1-v}{2}, \frac{3}{2}, \frac{x^2}{2}\right), \tag{23}$$

with M the Kummer function:

$$M(a,b,z) = 1 + \frac{a}{b}z + \frac{a(a+1)}{b(b+1)}\frac{z^2}{2!} + \ldots = \sum_{k=0}^{\infty}\frac{\Gamma(a+k)}{\Gamma(b+k)}\frac{\Gamma(b)}{\Gamma(a)}\frac{z^k}{k!}. \tag{24}$$

The asset in the base mode is therefore valued as:

$$V_0(p) = \left(A_0 \cdot D_{-d/a}\left(\sqrt{|a|}\left(p+\frac{b}{a}\right)\right) + B_0 \cdot D_{-d/a}\left(-\sqrt{|a|}\left(p+\frac{b}{a}\right)\right)\right)e^{\frac{1}{4}\left(\sqrt{|a|}\left(p+\frac{b}{a}\right)\right)^2} \tag{25}$$

The value of the asset in the alternative operating mode is determined by the non-homogenous partial differential equation (18), the solution to which consists of the sum of the general solution to the homogeneous PDE and a particular solution to the non-homogeneous PDE. A particular solution to the non-homogeneous equation is the present value of the perpetual cash flow k_1(p-c) which has already been determined in equation (11), and is repeated below:

$$V_{1P}(p) = k_1\left(\frac{p}{\mu+\eta+\lambda} + \frac{m\eta}{(r+\lambda)(\mu+\eta+\lambda)} - \frac{c}{r+\lambda}\right). \tag{26}$$

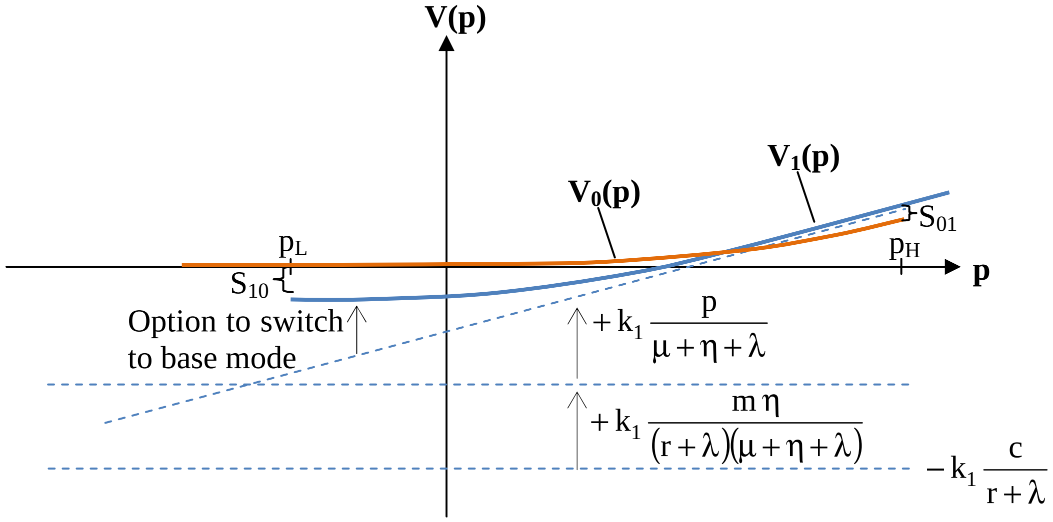
With the substitutions $u = \frac{k_1}{\mu+\eta+\lambda}$ and $w = k_1\left(\frac{m\eta}{(r+\lambda)(\mu+\eta+\lambda)} - \frac{c}{r+\lambda}\right)$, the value of the asset operating in the alternative mode is determined by the function below:

$$V_1(p) = \left(A_1 \cdot D_{-d/a}\left(\sqrt{|a|}\left(p+\frac{b}{a}\right)\right) + B_1 \cdot D_{-d/a}\left(-\sqrt{|a|}\left(p+\frac{b}{a}\right)\right)\right)e^{\frac{1}{4}\left(\sqrt{|a|}\left(p+\frac{b}{a}\right)\right)^2} + u \cdot p + w \tag{27}$$

The reader can verify that the solution to the homogenous partial differential equation based on a Bellman equation with the unspecified discount rate ρ, as provided by Sodal et al. (2007), can be transformed into the above equation by substituting $\eta \to \eta + \mu - r$, $m \to \frac{\eta m}{\eta + \mu - r}$ and $\rho \to r$ in the former, where notations apply as used in this paper. This shows that our model carefully distinguishes between the different sources of risk and that these can be determined from empirical data. These advantages are illustrated in more detail in a numerical application in the next sub-section. It should be noted that the Sodal solution to the non-homogenous differential equation cannot be transformed in a similar way to our non-homogeneous solution since all stochastic and non-stochastic components of the perpetual cash flows have been discounted uniformly at the rate ρ in the former solution. We have solved equation (20) by transforming it into the Weber equation, as suggested by Kampke (1956). Kampke also provides a direct solution to (20) in the form of a series function. In Appendix B, we show that this alternative provides the same valuation result but appears to be less transparent and straightforward.

As Kulatilaka and Trigeorgis (2004, p. 195) state, the "valuation of the flexible project must be determined simultaneously with the optimal operating policy". So we can expect the coefficients A and B to depend on the switching boundaries given by the spread levels of p_H and p_L, where p_H triggers a switch from the base operating mode to the alternative operating mode and p_L vice versa. In order to determine the coefficients, we investigate the general form of the value functions. The option value of switching from the base mode to the alternative mode needs to increase with the spread, since the spread can only be earned in the alternative mode, and tends towards zero for large negative spreads. When operating in the alternative mode and earning the cash flow p-c, the option to switch and forego the cash flow needs to increase in value with lower (more negative) p-values and should be almost worthless for very high values of p. Figure 1 below depicts the general form of the value functions.

Figure 1. General Shape of the Value Functions V_0 and V_1



Asset values as a function of the spread (p). V_0 is the asset value in the base mode, V_1 the asset value in the alternative mode. Switching from base mode to alternative mode at p_H for a switching cost of S_{01}, reverse switching at p_L for S_{10}. Slope of V_1 obtained from equation (26).

The parabolic cylinder function $D_v(x)$ tends towards infinity for large negative values of x and towards zero for large positive x for all v<0. It is a monotonically decreasing function in x for (v < -0.20494) and has one local maximum for (-0.20494 < v < 0). The exponential multiplier term in the option value in V_0 and V_1 makes the option values monotonically increasing and decreasing respectively for all v<0. For V_0, the option value of switching increases with p and becomes negligible for large negative values of p. Hence A_0 must be zero and B_0 positive. For V_1, it is the other way round, so that A_1 must be positive and B_1 zero.

Switching between operating modes occurs when the value in the new operating mode exceeds the value in the current mode by the switching cost. These rules are formalised by two boundary conditions,

$$V_0(p_H) = V_1(p_H) - S_{01}, \tag{28}$$

$$V_1(p_L) = V_0(p_L) - S_{10}, \tag{29}$$

where S_{01} and S_{10} are the respective switching costs. V_0 and V_1 must also comply with the smooth pasting conditions at the trigger levels, p_H and p_L:

$$\frac{\partial V_0(p_H)}{\partial p} = \frac{\partial V_1(p_H)}{\partial p}, \tag{30}$$

$$\frac{\partial V_0(p_L)}{\partial p} = \frac{\partial V_1(p_L)}{\partial p}. \tag{31}$$

The four equations, (28), (29), (30), (31), enable us to determine the four unknown parameters, B_0, A_1, p_H and p_L. This system of simultaneous equations can be solved directly with appropriate software (e.g. Excel). Since the simultaneous equations are solved numerically, we have a quasi-analytical solution. Appendix C provides the detailed equations.

2.4 Comparison to the Sodal et al. (2007) model

We now compare our model in a numerical application with the model by Sodal et al. (2007) who values the flexibility to switch between dry and wet shipping markets based on a mean-reverting freight rate spread. Sodal discounts all future cash flows at a constant rate ρ which is the rate of return required on the asset. Since the asset incorporates option characteristics, this discount rate can, however, not be specified empirically and would not be constant over time. Moreover, this general discount rate ρ has been universally applied to all cash flows, irrespective of whether these are stochastic, deterministic or constant. In contrast to this, we have used the risk-neutral approach in deriving the option model and are therefore able to differentiate between risky and risk-free cash flows. More precisely, we explicitly take into account the risk-free interest rate (r), the required instantaneous return on the stochastic variable p (μ), and technological risk or deterioration (λ).

When Sodal et al. (2007) argues that the discount rate ρ "can be seen as the sum of a real interest rate, r, a rate of depreciation, λ, and a possible adjustment for risk" (p. 187), it is implied that $\rho = r + \lambda + (\mu - r) = \mu + \lambda$. Our purpose here is to demonstrate that it is important to differentiate between the different discount rates and risk factors. Therefore, we use the same empirical data as Sodal et al. (2007) and vary μ, r and λ in such a way that they are always compatible with their base case assumption of ρ=0.10.

Table 1. Comparison with the Sodal model

			Sodal model Base case	New model for cases compatible with ρ=0.1 I	II	III
Parameters						
General discount rate	ρ		0.10			
Required return	μ			0.10	0.10	0.08
Risk-free interest rate	r			0.10	0.05	0.05
Deterioration	λ			0.00	0.00	0.02
Solution						
Upper switching trigger	p_H	[$ per day]	4,969	4,969	4,968	4,968
Lower switching trigger	p_L	[$ per day]	-4,230	-4,230	-4,228	-4,228
Value of flexibility	$V_0(p)$	[$ million]	5.432	5.432	10.771	7.720

Empirics based on Sodal et al. (2007) who value the flexibility to switch between dry and wet shipping markets where the freight rate spread ($p = p_{dry} - p_{wet}$) is modelled as an Ornstein-Uhlenbeck mean-reverting process.
Option values and switching boundaries are obtained from equations (25) and (27) with A_0=0 and B_1=0 and the simultaneous solution of equations (28), (29), (30) and (31) for the respective parameter values. Parameters: Current value of the spread: p = $0 per day; Long-run mean of p: m = -$5,400 per day (annualised by multiplying by 330 days); Speed of mean-reversion of p: η=2.4; Standard deviation of p: σ = $22,600 per day (to have the same annualised basis as Sodal, we also multiply σ by 330 days although we would prefer to multiply by the square root of 330); Difference in operating cost: c = $0 per day; Capacity of p: k1 = 1; Cost for switching from wet to dry: S_{01} = €40,000; Cost for switching from dry to wet: S_{10} = €40,000.

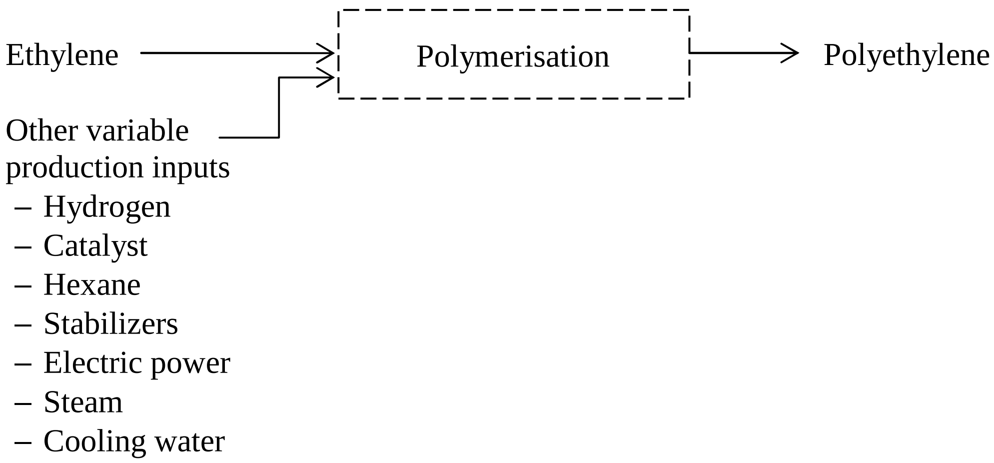
Table 1 shows that our model exactly replicates their results when we choose μ=r=0.10, so the underlying assumption in the Sodal model must be ρ=μ=r. In reality, however, the required return on the stochastic process of p must be higher than on a risk-free cash flow (μ>r). We then choose μ=0.10 and r=0.05 and find that the switching triggers hardly change but the asset value almost doubles. This huge difference is mainly due to the different approaches of discounting the mean-reverting spread. The Sodal model implies that the risky cash flows from the spread are discounted at the rate ρ, which represents the required return according to the fundamental equation of optimality (Bellman equation), times the discounting time period. In the long-run, the mean-reverting spread tends towards its long-run mean and the total variance is bounded, approaching $\sigma^2/2\eta$. Hence, the total risk from the stochastic fluctuations in the spread is not proportional to time. Compounding the risk discount factor in the same manner as the interest rate is not correct. The future cash flows have thus been discounted too heavily in the Sodal model. We have shown in equations (5) - (7) how to transform the risky spread to the risk-neutral form which can then be discounted at the risk-free rate r over time.

If we now choose μ=0.08 and λ=0.02, so that ρ=0.10 still holds, the asset value is about 30% lower compared to the case of μ=0.10 and λ=0.00. In this example, physical deterioration weighs heavier than risk-adjustment. Our conclusion from the comparison with the Sodal model is that it matters for real options with mean-reverting stochastic variables how interest rate, risk-adjustment and deterioration contribute to the total discount rate, and that a universal discount factor might produce misleading results.

3 Empirical Application: Valuing a Polyethylene Plant

In this empirical section, the continuous rainbow option is applied to determine the market value of a polyethylene plant which converts ethylene into polyethylene. The latter product is a plastic which is widely used in pipes, film, blow and injection moulding applications and fibres, while ethylene is the main product from the petrochemical naphtha cracking process. We assume that the operating company disposes of a continuous supply of ethylene, either by way of own upstream facilities or by buying from the open market. Our purpose is to determine how much value can be generated over and above the present value of ethylene sales by having the opportunity to convert ethylene into polyethylene. At first glance, this seems to be an input/output option rather than an option on the best of two outputs (rainbow option). However, both commodities are traded and ethylene could be sold to the market instead of converting it to polyethylene. In that sense, the polyethylene plant can be considered a rainbow option on ethylene and polyethylene, where ethylene is chosen by suspending the plant operation (base mode) and polyethylene is chosen by operating the plant (alternative mode). Figure 2 below depicts a simplified scheme of the transformation.

Figure 2. Simplified Scheme of Inputs and Outputs of a Polyethylene Plant (HDPE)



Feed components and output of the slurry polymerisation process of ethylene to high-density polyethylene (HDPE)

While various patented polyethylene processes are used in industry, we focus on the slurry process for the production of high-density polyethylene (HDPE). The asset under consideration is assumed to be in Europe with an annual production capacity of

250,000 tons of HDPE and an initial investment of €200 million. Meyers (2004) provides specific consumption data for the slurry process which requires about 1,017 kg of ethylene for the production of 1,000 kg of polyethylene. The conversion spread is therefore defined as:

$$p = p_{polyethylene} - 1.017 \cdot p_{ethylene} \tag{32}$$

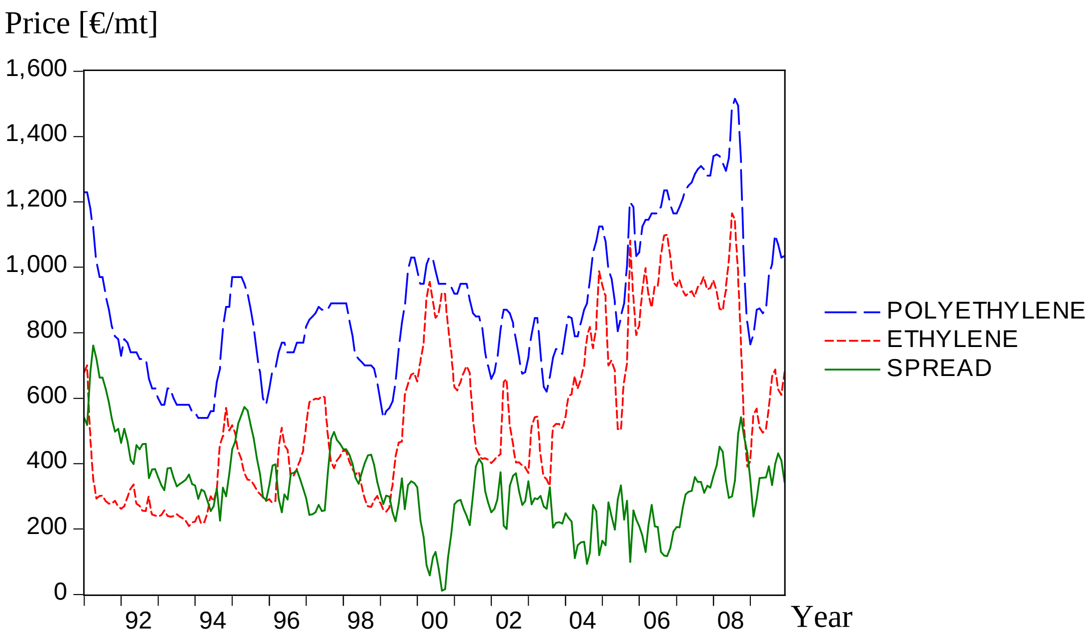
Although other materials are required for the chemical transformation, prices of polyethylene are largely determined by ethylene as the dominant feedstock. This suggests that both prices are co-integrated, i.e. they are bound in the longer term and the difference between the two tends to revert to a long-term average which should cover operating costs of converting ethylene to polyethylene, capital costs and profit. To further explain this mechanism, consider the following scenarios. An increase in ethylene prices means higher production costs of polyethylene which will eventually lead to an increase in the market price of the latter. The extent of this price increase depends on whether the market price is more cost-driven or demand driven at that time. A cost-driven market price is much more responsive to a change of production costs than a demand-driven market price. This relationship is inverse for a change in demand of polyethylene. A change of demand will lead to significant adjustments in polyethylene prices in a demand-driven market but less so in a cost-driven market. Furthermore, a polyethylene demand change will also impact on the prices of ethylene since about 60% of the global ethylene production output is used to produce polyethylene, according to estimates of Deutsche Bank (2009). While most of the remaining share is used to produce other chemical products, ethylene also has some direct applications (e.g. fuel gas for special applications or ripening of fruit).

3.1 Econometric model for the stochastic spread

As Dixit and Pindyck (1994) acknowledge, both theoretical considerations and statistical tests are important to determine whether a variable follows a mean-reverting stochastic process. Following the discussion on equilibrium mechanisms above, this section intends to econometrically test the spread for mean-reversion and then to estimate the parameters of this stochastic process. According to Brooks (2008) and Duan and Pliska (2004), a linear combination of non-stationary variables of integration order one will be stationary if the variables are co-integrated. In other words, the spread of polyethylene and ethylene prices is stationary and can be modelled as an autoregressive mean-reverting process if the two commodity prices are co-integrated. Hence, we first test the commodity prices for co-integration and the spread for stationarity. If these tests confirm the mean-reverting nature of the spread, the parameters of the Ornstein-Uhlenbeck process are determined by means of an Ordinary Least Squares regression and statistical tests are performed on the validity of the regression.

Time series with monthly data for ethylene and polyethylene prices from Jan 1991 to Dec 2009 are the basis for the empirical analysis. These prices are for delivery within Europe, i.e. gross transaction prices. Figure 3 gives a graphical representation of the historical commodity prices as well as the conversion spread. It can be seen from the figure that the two commodity prices tend to move together and the spread is more stationary, though volatile.

Figure 3. Time Series of Commodity Prices and of the Spread



Monthly data, prices in € per metric ton and for delivery within Europe. Ethylene: spot prices. Polyethylene: HDPE quality (high-density polyethylene). Spread defined as polyethylene price less 1.017 times ethylene price.

3.1.1 Test for mean-reversion

The purpose of this chapter is to test whether the spread follows a mean-reverting/ stationary process. This can be done either directly by demonstrating that the spread is stationary or indirectly by showing that ethylene and polyethylene prices are co-integrated, because according to the Granger representation theorem, this implies that a linear combination of the two (such as the conversion spread) is stationary.

Two variables are co-integrated if their levels are non-stationary and the 1st difference in levels is stationary. An Augmented Dickey-Fuller (ADF) unit root test assumes that the series is non-stationary under the null hypothesis. Hence, the two variables are co-integrated if the ADF test statistic for each variable is not rejected on the levels but rejected on the 1st difference in levels. Co-integration is confirmed for ethylene and polyethylene prices by considering the probabilities of making an error when rejecting the null hypothesis of unit roots, as shown in Table 2. When the p-value is below 5%, the null hypothesis can be rejected with a confidence level of more than 95%. The null hypothesis of unit roots cannot be rejected at the 1% level and at the 5% level for polyethylene. For ethylene, it can also not be rejected at the 1% level but it is rejected at the 5% level with a p-value of 0.043. We consider this confidence level as good enough and accept the null hypothesis of unit roots for both ethylene and polyethylene. The hypothesis of unit roots in the 1st difference of the two commodity prices can be rejected with certainty. This means that the commodity prices tend to be non-stationary, but the spread as a linear combination of ethylene and polyethylene prices is stationary.

Table 2. Augmented Dickey-Fuller Test for Unit Roots in Time Series

	Probability of unit roots on prices	Probability of unit roots on 1st difference of prices
Ethylene	0.043	0.000
Polyethylene	0.057	0.000
Spread	0.005	0.000

MacKinnon one-sided p-values give the probability of making an error when rejecting the null hypothesis that unit roots exit. Unit roots are present if the regression $y_t = \sum_{i=1}^{12} \phi_i y_{t-i} + u_t$ yields $\Phi_i \geq 1$ for any i, where y_t is the dependent variable at time t and u_t the residual at time t. The presence of unit roots indicates that the process is non-stationary. Maximum number of lags to account for autocorrelation: 12 months.

The same table also provides the ADF statistic for the spread, for which the null hypothesis of non-stationarity is strongly rejected. Because there is the possibility that the null hypothesis might be rejected due to insufficient information, we also perform a stationarity test to confirm the above analysis. A KPSS test assumes the series is stationary under the null hypothesis. The KPSS test statistic for the spread series is 0.72, which means that the null hypothesis of stationarity is rejected at the 5% level (critical value: ≥ 0.46) but not rejected at the 1% level (critical value: ≥ 0.74).

3.1.2 *Regression model*

The Ornstein-Uhlenbeck process for the spread (p) is specified in continuous time. In order to estimate the parameters (η, m, σ), the model needs to be converted to its discrete time equivalent. The corresponding discrete-time process of the spread is a standard first-order autoregressive times series, AR(1), as expected from the Ornstein-Uhlenbeck equation:

$$p_t = m\left(1 - e^{-\eta}\right) + e^{-\eta} p_{t-1} + \varepsilon_t , \tag{33}$$

where ε_t is normally distributed with mean zero and standard deviation σ_ε:

$$\sigma_\varepsilon^2 = \frac{\sigma^2}{2\eta}\left(1 - e^{-2\eta}\right). \tag{34}$$

It should be noted, that the parameters η and σ depend on the chosen time interval Δt which is one month. The regression is then run on:

$$p_t = \alpha + \beta p_{t-1} + \varepsilon_t , \tag{35}$$

with

$$\hat{\eta} = -\log \hat{\beta} , \tag{36}$$

$$\hat{m} = \frac{\hat{\alpha}}{1 - \hat{\beta}} , \tag{37}$$

$$\hat{\sigma} = \hat{\sigma}_\varepsilon \sqrt{2 \frac{\log \hat{\beta}}{\hat{\beta}^2 - 1}} . \tag{38}$$

To transform the parameters η and σ from a monthly to an annual scale, multiply the mean-reversion rate by twelve and the standard deviation by the square root of twelve. Table 3 provides the parameter estimates of the regression model, based on the 1991-2009 monthly data of the spread, as well as the transformed parameters for the Ornstein-Uhlenbeck process. Both parameters, α and β, are statistically significant (p-values: 0.00), thereby confirming that the model is auto-regressive. The regression estimates the mean of the spread (m) at €317/mt, the annual standard deviation (σ) at €198 and the mean-reversion rate (η) at 1.35 on an annual basis. This mean-reversion rate implies that the difference between p and m is expected to halve within 0.51 years (=ln2/η).

Table 3. Regression Model for the Ornstein-Uhlenbeck Process of the Spread

Regression parameter	Value	Std. Error	p-value
α	33.62	10.06	0.00
β	0.894	0.029	0.00
σ_{ε}	54.11		

Parameters of the Spread	Value	Unit
m	316.8	EUR/t
η_{month}	0.11	per month
η_{year}	1.35	per year
σ_{month}	57.2	EUR/t per month
σ_{year}	198.0	EUR/t per year

Ordinary least squares (OLS) regression model of the polyethylene-ethylene spread (p): $p_t = \alpha + \beta p_{t-1} + \varepsilon_t$. P-values give the probability of making an error when rejecting the null hypothesis that the respective parameter is zero.

3.1.3 Statistical tests

The above regression model needs to undergo a number of diagnostic tests in order to verify its validity. The residuals of the regression should be homoscedastic, not autocorrelated and normally distributed. Further tests on the stability of the parameters and the linearity in the functional form are performed. The results of these tests are given in Table 4 and are discussed below.

Table 4. Diagnostic Tests on Regression Model

Test	p-value	Interpretation
Heteroskedasticity test of residuals		
White Test		
Probability F-distribution	0.49	Do not reject the null hypothesis of homoscedasticity
Autocorrelation test of residuals		
Breusch-Godfrey		
Probability F-distribution	0.21	Do not reject the null hypothesis of no autocorrelation
Normality test of residuals		
Bera-Jarque		
Probability Chi^2-distribution	0.01	Reject the null hypothesis of normality
Test for misspecification of functional form		
Ramsey's RESET test		
Probability F-distribution	0.09	Do not reject the null hypothesis of the functional form being linear

Regression model: $p_t = \alpha + \beta p_{t-1} + \varepsilon_t$.
The White test yields the probability according to an F-distribution for the joint null hypothesis that ρ_1=0, ρ_2=0 and ρ_3=0 in the auxiliary regression of the residuals $\hat{\varepsilon}_t^{\,2} = \rho_1 + \rho_2 p_{t-1} + \rho_3 p_{t-1}^{\,2} + u_t$ where u_t is a normally distributed disturbance term. Squared terms are included.
The Breusch-Godfrey test yields the probability according to an F-distribution for the joint null hypothesis that ρ_i=0 for i=1..12 in the auxiliary regression of the residuals $\hat{\varepsilon}_t = \gamma_1 + \gamma_2 p_{t-1} + \sum_{i=1}^{12} \rho_i \hat{\varepsilon}_{t-i} + u_t$ where u_t is a normally distributed disturbance term. To account for autocorrelation covering 12 months, 12 lagged terms are included.
The Bera-Jarque test statistic is given by $\frac{N}{6}\left(S^2 + \frac{(K-3)^3}{4}\right)$, where $S = \frac{E[\varepsilon^3]}{\sigma^3}$ is the skewness and $K = \frac{E[\varepsilon^4]}{\sigma^4}$ the kurtosis of the residuals distribution. The Bera-Jarque statistic is distributed as a Chi-square with 2 degrees of freedom.
Ramsey's RESET test yields the probability according to an F-distribution for the null hypothesis that ρ_1=0 in the auxiliary regression of the residuals $p_t = \gamma_1 + \gamma_2 p_{t-1} + \rho_1 p_t^{\,2} + u_t$ where u_t is a normally distributed disturbance term.

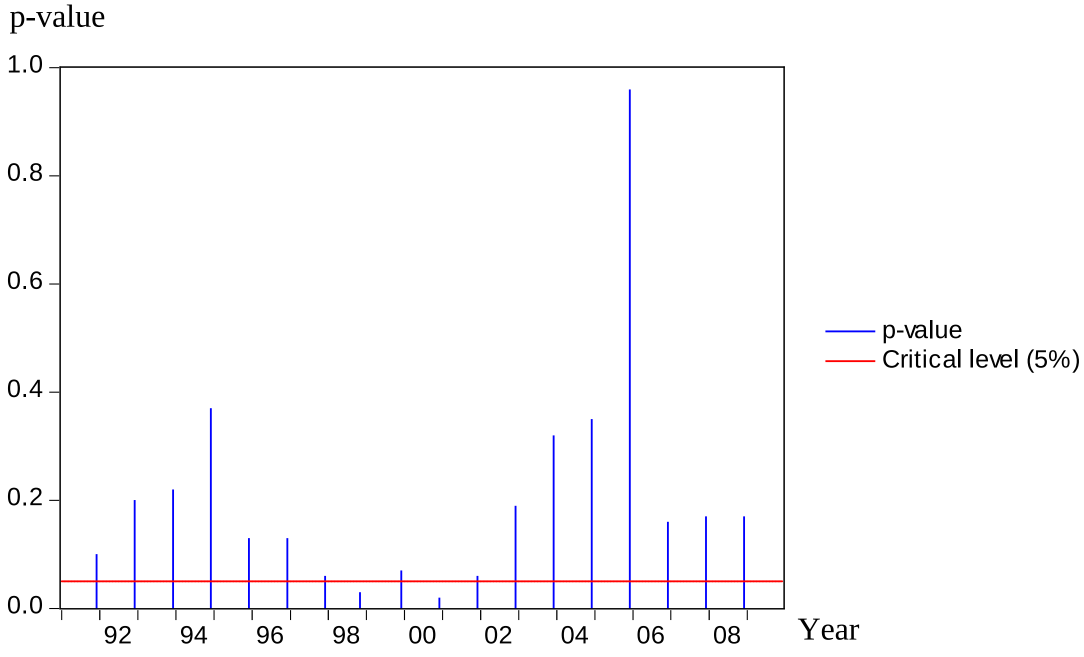
The distribution of the residuals ought to be of constant variance over time, i.e. homoscedastic. If this is not given, the standard error of the parameter estimates would be flawed and so would be any inference on the significance of the parameters.

However, the parameter values would be unbiased even in the presence of heteroscedasticity. The White test indicates that the probability of making an error when rejecting the null hypothesis of homoscedasticity is 0.49. We adopt the 0.05 probability level as the threshold between rejection and non-rejection. Hence, the residuals are not heteroscedastic. The autoregressive regression model already takes into account autocorrelation in the spread. We still need to test whether the model covers all of the autocorrelation. The consequences of ignoring autocorrelation in the residuals are the same as for heteroscedasticity, i.e. the parameters would be inefficient but unbiased. The Breusch-Godfrey test confirms that the residuals are not correlated. The Bera-Jarque test for normal distribution of the residuals rejects the hypothesis of normality at the 1% significance level, meaning the residuals are not normally distributed. While the residuals distribution is not skewed, it is leptokurtic (peaked relative to the normal) with a kurtosis of 4.07 (3.0 for a normal distribution). Since the kurtosis does not impact on the mean of the residuals distribution, this non-normality has no practical consequences for the validity of the regression model.

The functional form of the chosen regression model is linear. The appropriateness of this form can be tested by means of Ramsey's RESET test which adds exponential terms of the dependent variable to the regression model. With one fitted term (square of the dependent variable), the alternative hypothesis of a non-linear functional form can be rejected at the 0.05 significance level so that our chosen linear functional model is appropriate.

Parameter stability tests intend to verify if the parameter estimates are stable over time or whether they change significantly. Performing a series of Chow tests with different breakpoints over the sampling period suggests that there might be breakpoints at the end of 1998 and 2000, as can be seen from Figure 4. Hence, parameter estimates based on data before the breakpoint would be significantly different from estimates thereafter. In the long-run the polyethylene-ethylene conversion spread depends on the conversion ratio and needs to cover operating and fixed/capital costs. With existing plants being distributed globally, any changes in these factors would happen slowly which is why there seems to be no economic justification for a sudden change in the long-term behaviour of the spread. Recursive coefficient estimates show that both α and β converge to stable values (see Figure 5) which might be an indication of parameter stability or simply a result of the power of averaging. A CUSUM test also shows that the cumulative sum of the recursive residuals is within the 0.05 significance range at all times, suggesting that the parameters are stable.

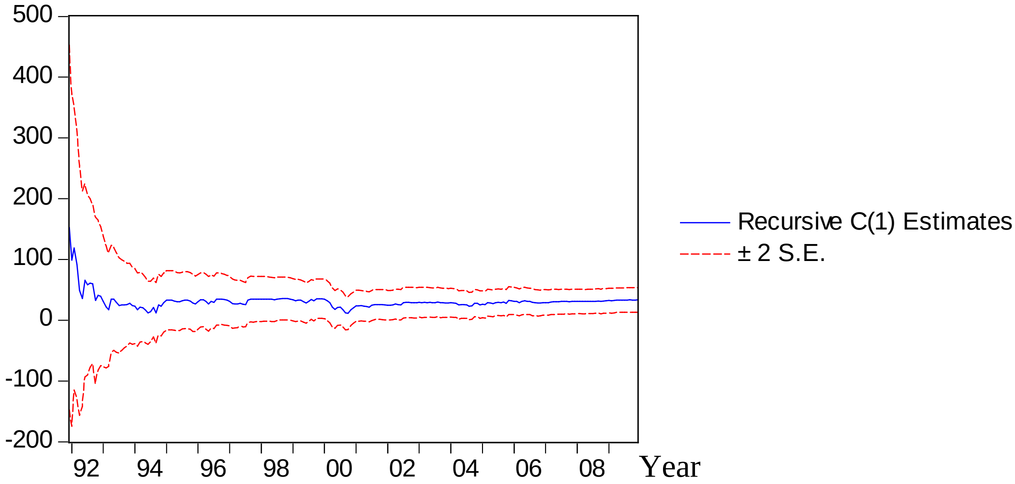
Figure 4. Chow Tests on Parameter Stability with respect to Particular Breakpoints



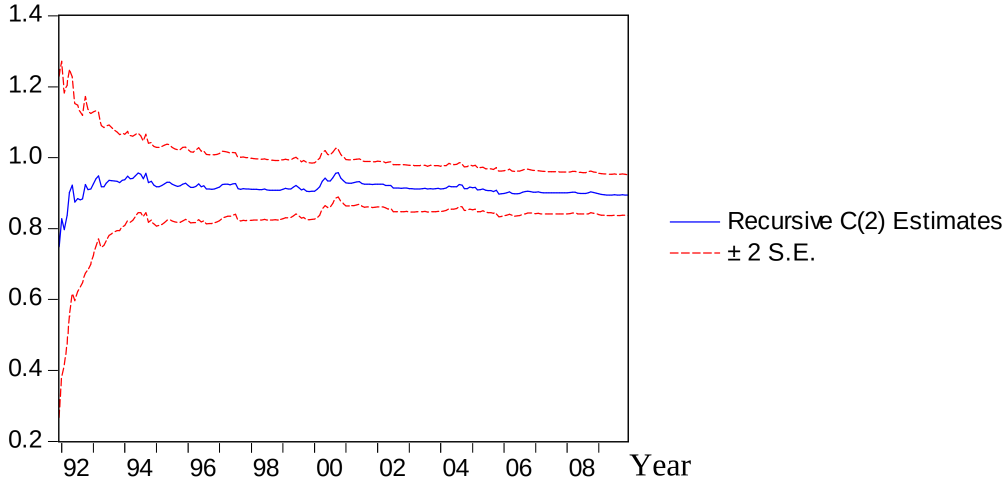
P-value gives the probability of making an error when rejecting the null hypothesis of no breakpoint. Chow test splits the sample data into two periods divided by the breakpoint and compares the residual sums of the regressions from these sub-samples with the residual sum of the regression over the whole period. Ordinary least squares regression for the Ornstein-Uhlenbeck process: $p_t = \alpha + \beta p_{t-1} + \varepsilon_t$

Figure 5. Recursive Coefficient Estimates

Coefficient C(1)



Coefficient C(2)



Regression model: $p_t = \alpha + \beta p_{t-1} + \varepsilon_t$. C(1) corresponds to α, C(2) to β. Parameter estimates start from Jan-1991 and subsequently add more data points until all data up to Dec 2009 is considered. Convergence towards a stable value is supposed to indicate parameter stability but interpretation is difficult since stable values might also be a result of the power of averaging.

3.2 Asset-specific parameters

The key characteristics of the polyethylene plant are given in Table 5 together with the calculation of the operating margin based on the spread as of December 2009.

Table 5. Overview of Parameters for the Polyethylene Plant

Capacity polyethylene	k_1	250,000	mt per year
Feedstock ethylene	k_0	254,250	mt per year
Ramp-up cost	S_{01}	40	‘000 €
Ramp-down cost	S_{10}	20	‘000 €
Current spread	*p*	*340.0*	*€/mt polyethylene*
Logistics cost		50.0	€/mt polyethylene
Consumption materials		74.5	€/mt polyethylene
Personnel cost		4.0	€/mt polyethylene
Variable operating cost	c	128.5	€/mt polyethylene
Current margin	*p-c*	*211.5*	*€/mt polyethylene*

During the ramp-up phase the process stability is not given at all times so that the polyethylene produced is of lower quality. The ramp-up cost is then the lost income based on an estimated price reduction of €20/mt for the lower grade and a ramp-up time of 24h up to 3 days. When suspending the operations temporarily, the variable personnel costs cannot be eliminated immediately, assuming that one week's salaries will be incurred for non-productive time following a ramp-down.
As quoted commodity prices refer to delivered products, logistics cost refer to delivery of polyethylene within Europe. Current spread as of December 2009.
Source: Meyers (2004), ICIS website and discussion, Interviews with industry experts

Table 6. Cost of Consumption Materials for the HDPE Slurry Process

Production inputs	Consumption for 1,000 kg of HDPE		Unit prices		Cost for 1,000 kg of HDPE
Catalyst					€4
Hydrogen	0.7	kg	2.4	€/kg	€1.7
Hexan	7	kg	650	€/t	€4.5
Stabilisers					€20
Steam	500	kg	25	€/t	€12.5
Electric power	600	kWh	45	€/MWh	€27.0
Cooling water	200	m^3	2.4	€ ct/m^3	€4.8
					€74.5

Main production inputs to the HDPE slurry process other than ethylene. Consumption data based on Meyers (2004). Electric power and cooling water consumption data adjusted to account for the extruder. Estimate for cost of hydrogen on natural gas basis from FVS (2004). Prices of hexan, steam and cooling water based on industry experts interview. Electric power based on average spot electricity prices at European Energy Exchange.
Source: Meyers (2004), ICIS website, Interviews with industry experts

The variable cost of production is composed of consumption material cost (see cost-breakdown provided in Table 6), logistics cost for the delivery of the final product within Europe, and personnel cost. According to industry experts from a supplier of chemical plants, about 30 people are required to operate the shifts next to a management team of about 4-6. This is under the assumption that the plant is part of a larger petrochemical complex, so that general services can be shared. Assuming an annual personnel cost of €50,000 per employee, the total personnel cost amounts to €1.75 m. When the plant is not operated, a fire-and-hire strategy would reduce the cost but endanger the know-how base. Many European countries provide for some flexibility with regard to personnel deployment, such as flexible working-time accounts and short-time allowance. Therefore, we consider 2/3 of the shift personnel cost to be variable (€1 million) so that the variable personnel cost per ton of polyethylene produced is €4. Annual maintenance cost for this kind of chemical plant is estimated at 1.5% of the investment cost (€3 million). Together with the fixed personnel cost, the total fixed operating cost amounts to €3.75 million.

As was said earlier in this paper, limited lifetime of the asset (deterioration) and specific technological and political risks associated with the investment are accounted for by a Poisson event with the arrival rate λ. The limited lifetime is modelled in the form of an exponential decay, where $\phi_T = \int_{t=0}^{T} \lambda e^{-\lambda t} dt$ is the probability that the asset has reached the end of its lifetime before T. Assuming an expected lifetime of 20 years, use $T = 20$ and $\phi_{20} = 0.5$ to get the corresponding arrival rate for deterioration as $\lambda_D = 0.035$. Investing in, owning and operating a chemical plant is associated with significant technological risks, ranging from non-compliance of the chemical processes, patent conflicts, to product obsolescence. Furthermore, political risks persist over the asset lifetime, such as terrorist attacks, environmental issues or health concerns. We choose $\lambda_T = 0.045$, and get the Poisson arrival rate for the asset as $\lambda = \lambda_D + \lambda_T$.

3.3 Asset valuation

The theoretical model developed in Section 2 is now applied to value a polyethylene plant with the empirical data from above. As an extension, we introduce a hypothetical tax rate γ on the cash flow, so that the cash flow in the alternative operating mode becomes $(1-\gamma)k_1(p-c)$. The total asset value in the respective operating mode is then given by AV_0 and AV_1, according to $AV_{0/1} = V_{0/1} - PV(c_{fix}) + PV(tax)$, where $PV(c_{fix}) = \frac{c_{fix}}{r+\lambda}$ is the present value of the annual fixed operating cost and PV(tax) the present value of the tax break. We assume the investment cost (I) is linearly depreciated over the depreciation period (T) for tax accounting purposes, which is the case in Germany for example, so the annual tax break during T years is given by $\gamma I/T$ and its present value is $PV(tax) = \sum_{t=1}^{T} \frac{I}{T}\gamma\frac{1}{(1+r)^t}$. The asset value and switching boundaries for the specified parameters are given in Table 7 and represented in graphical form as a function of the spread in Figure 6.

Table 7. Value of the Polyethylene Plant and Switching Boundaries

Parameters		
Spread	p	340
Standard deviation of p	σ	198
Long-run mean	m	317
Rate of mean-reversion	η	1.35
Required return	μ	0.10
Risk-free interest rate	r	0.05
Technical/political risk	λ	0.08
Cost difference	c	128.5
Capacity in base operating mode	k_0	254,250
Capacity in alternative operating mode	k_1	250,000
Switching cost from '0' to '1'	S_{01}	40,000
Switching cost from '1' to '0'	S_{10}	20,000
Tax rate	γ	0.30
Fixed annual operating cost	c_{fix}	3,750,000
Investment	I	200,000,000

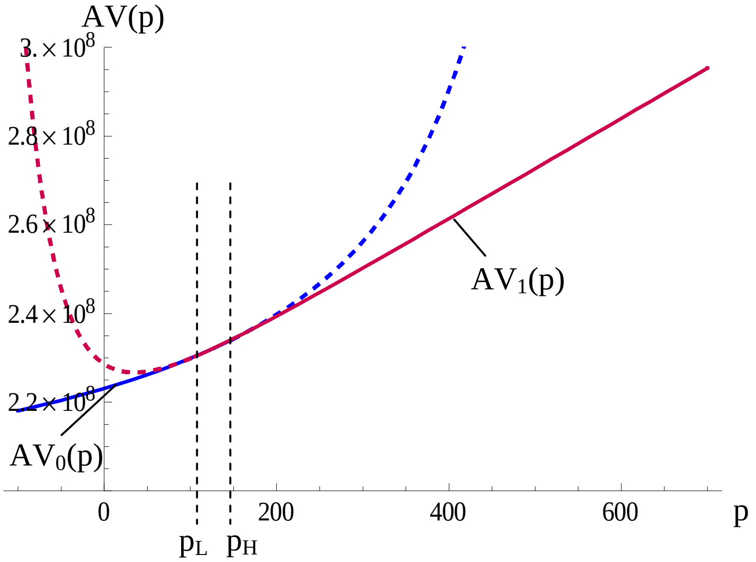
Substitution of variables		
	a	-7.142E-05
	b	2.182E-02
	d	-6.632E-06
	u	1.144E+05
	w	2.033E+08

Solution		
Value in base operating mode	$V_0(p)$	see V1(p)
Value in alternative operating mode	$V_1(p)$	246,130,750
Upper switching boundary	p_H	148.22
Lower switching boundary	p_L	104.83
Coefficient	B_0	2.358E+08
Coefficient	A_1	3.836E+06

Equations		
Value matching condition 1	EQ 24	0.000
Value matching condition 2	EQ 25	0.000
Smooth pasting condition 1	EQ 26	0.000
Smooth pasting condition 2	EQ 27	0.000
	Sum	*0.000*

Asset Value		
Value in base/alternative operating mode	$V(p)$	246,130,750
PV of fixed operating cost	$PV(c_{fix})$	28,846,154
PV of tax break	$PV(_{tax})$	37,386,631
Total asset value	$\mathbf{AV(p) = V(p) - PV_{fix} + PV_{tax}}$	**254,671,227**

Figure 6. Value of the Polyethylene Plant as a Function of the Spread



Values of the polyethylene plant (AV_0, AV_1) in € as a function of the spread (p) in €/mt. Switching from not operating to operating the asset at p_H, vice versa at p_L. Asset values on dashed lines not applicable because switching of operating mode is triggered.
Option values and switching boundaries are obtained from equations (25) and (27) with A_0=0 and B_1=0 and the simultaneous solution of equations (28), (29), (30), (31), taking into account the tax rate γ, for the following parameter values. Long-run mean of p: m = €316.8/mt; Speed of mean-reversion of p: η = 1.35; Standard deviation of p: σ = €198 p.a.; Variable operating cost: c = €128.5/mt; Capacity of p: k_1 = 250,000 mt p.a.; Switching cost for resuming operation: S_{01} = €40,000; Switching cost for suspending operation: S_{10} = €20,000; Required return: μ = 0.10; Risk-free rate of return: r = 0.05; Exponential decay and technological/political risk: λ = 0.08; Tax rate γ = 0.3; Fixed operating cost: c_{fix} = €3.75 million p.a.; Annual depreciation: €10 million for 20 years.

Considering first the alternative operating state, when the plant is operated and the spread is earned, it can be seen that the asset value (AV_1) increases linearly in p for very high levels of p while the function is convex for lower levels of p. This is explained by the option to switch to the base operating mode which is relevant for lower p-values and negligible for high p-values. The value function AV_1 increases steeply to the left of the switching boundary p_L because the switching option would largely exceed the discounted cash flows. However, the function AV_1 is not relevant for $p<p_L$ since the operating mode is changed at p_L. The asset value in the base mode (AV_0) increases gradually until the option to switch and earn the spread reaches V_1-S_{01} at the switching boundary p_H. Even for highly negative p-values, it is expected

that p will eventually revert to the long-run mean (m) so that the option on the spread declines only slowly towards zero for negative spread levels.

Table 8 provides the value of the asset in the alternative mode, currently operating the plant, with and without operating flexibility, together with the switching boundaries for the standard parameters and various scenarios in order to test the sensitivity to changing parameters. It should be noted here, that the physical asset is basically the same in the flexible and inflexible case. However, relations with business partners and employees can be set up and managed in a way that takes flexibility into account or not. In particular, the company is not obliged to deliver certain quantities of either ethylene or polyethylene over a longer period of time since this would restrict the product choice and therefore the operating flexibility. Furthermore, the labour contracts allow the reduction of shift personnel during times when the asset is not operated.

For the standard parameters, we find a value of the operated plant with no operating flexibility of €251 million compared to an asset value with operating flexibility of €255 million, which is a 2% premium. The investment cost for a polyethylene plant without flexibility is about €200 million. The value of flexibility (€4 million with standard parameters) needs to be compared to the cost of providing this flexibility. Assuming the shift personnel requires a 10% higher income as a compensation for the higher employment risk (due to increased flexibility for the employer), the discounted value of this additional cost in perpetuity amounts to about €1 million (10% on the cost of variable shift personnel of €1 million, discounted at $r+\lambda=0.13$). Hence, incorporating flexibility increases the net value by about €3 million.

Table 8. Sensitivity Analysis of Switching Boundaries and Polyethylene Plant Values

Sensitivities	p_L [€/mt]	p_H [€/mt]	**Flexible asset** $AV_1 = V_1 - PV(c_{fix}) + PV(tax)$ [million €]	**Non-flexible asset** $PV_1 - PV(c_{fix}) + PV(tax)$ [million €]
Standard parameters	104.83	148.22	255	251
Sensitivity to volatility				
$\sigma = 0$	115.16	128.53	251	251
Sensitivity to mean-reversion (η)				
$\eta = 0$	106.95	150.17	315	166
Operating cost sensitivity				
c = €100/mt	76.00	119.41	291	289
c = €150/mt	126.58	169.95	228	222
Switching cost sensitivity				
$S_{01} = S_{10}$ = €0	128.50	128.50	255	251
$S_{01} = S_{10}$ = €200,000	80.01	162.62	254	251
Sensitivity to current spread (p_0)				
$p_0 = 500$	104.83	148.22	273	269
$p_0 = 150$	104.83	148.22	234	229

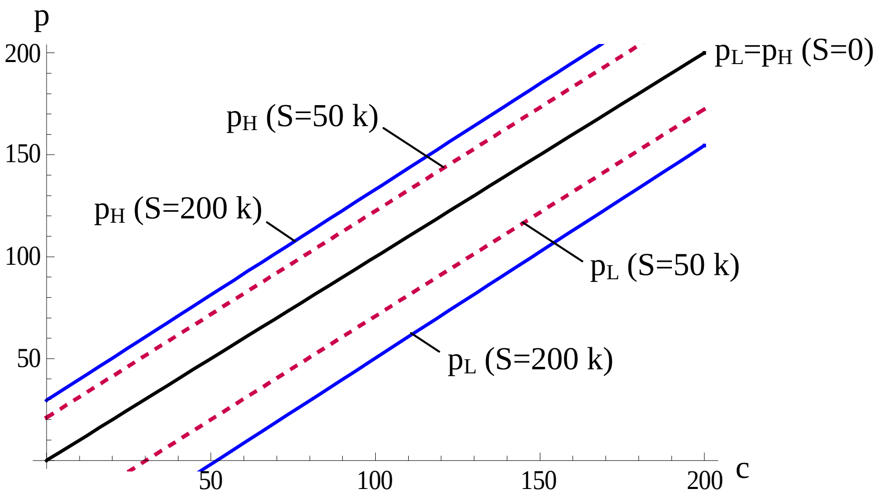
Option values (V_1) and switching boundaries (p_L, p_H) are obtained from equations (25) and (27) with A_0=0 and B_1=0 and the simultaneous solution of equations (28), (29), (30), (31), taking into account the tax rate γ, for the respective parameter values. PV_1 is obtained from equation (11). Standard parameters: Current value of the spread: p = €340/mt; Long-run mean of p: m = €316.8/mt; Speed of mean-reversion of p: η=1.35; Standard deviation of p: σ = €198 p.a.; Variable operating cost: c = €128.5/mt; Fixed cost: c_{fix} = €3.75 million p.a.; Capacity of p: k_1 = 250,000 mt p.a.; Switching cost for resuming operation: S_{01} = €40,000; Switching cost for suspending operation: S_{10} = €20,000; Required return: μ = 0.10; Risk-free rate of return: r = 0.05; Exponential decay and technological/political risk: λ = 0.08; Tax rate γ = 0.3; Fixed operating cost: c_{fix} = €3.75 million p.a.; Annual depreciation: €10 million for 20 years.

The switching boundaries p_L and p_H lie on either side of the variable operating cost (c), but not symmetrically. Suspending the operations is recommended at a net cash flow (p-c) of -€23.67/mt compared to restarting at €19.72/mt. This asymmetry is explained by the long-run mean of p which is significantly above the operating cost. Suspension is delayed more than resumption. The switching boundaries are distributed symmetrically around the variable operating cost if the long-run mean of the spread is identical to the variable operating cost or if the switching cost is zero, so $p_L=p_H=c$.

Let us first validate the behaviour of the value function with regard to the parameters of the underlying uncertainty and then with regard to asset-specific parameters. When testing for zero standard deviation, the spread will tend towards its long-run mean (m) in a deterministic way. With m>c and all stochastic elements eliminated, the plant would always be operated and the option to switch to the base mode and thereby foregoing the cash flow (p-c) becomes irrelevant, so the operating flexible asset is valued exactly the same as the inflexible one. However, if the plant was not operated for some reason, operation would be resumed as soon as the spread exceeds the variable operating cost because with m>c, the net cash flows (p-c) are positive from that time on and the present value of those net cash flows exceeds the switching cost (S_{01}). Now, let the speed of mean-reversion (η) be zero so that the Ornstein-Uhlenbeck process simplifies to a Brownian motion process with no drift, $dp = \sigma\, dz$. For the inflexible plant, the present value declines when mean-reversion is relaxed because the risk increases with time (standard deviation is proportional to the square root of time). This is reflected in a higher discount rate for the spread in equation (11). As a result, the present value of €166 million is significantly lower compared to the mean-reversion case and would not justify the investment. In contrast, the value of the flexible asset increases significantly by about 25% to €315 million when relaxing mean-reversion, which is a 90% premium on the inflexible asset. This is consistent with real options theory because the lower the speed of mean-reversion the higher the variance and the higher the option values.

Assuming different variable operating costs, the option premium increases with higher operating cost because the probability of exercising the switching option increases. However, as long as the cash flow is vastly positive, $(p-c) \gg 0$, the premium is rather small. The option model confirms the intuition that in the absence of switching cost, switching is optimal as soon as the spread crosses the operating cost, so that the cash flow is given by Max[p-c;0]. Although the switching cost significantly influences the switching boundaries, its effect on the asset value is minor because the current and long-run expected spread is far above the operating cost and hence the probability of incurring switching costs and not operating the plant is low.

Figure 7. Switching Boundaries as a Function of Variable Operating Cost

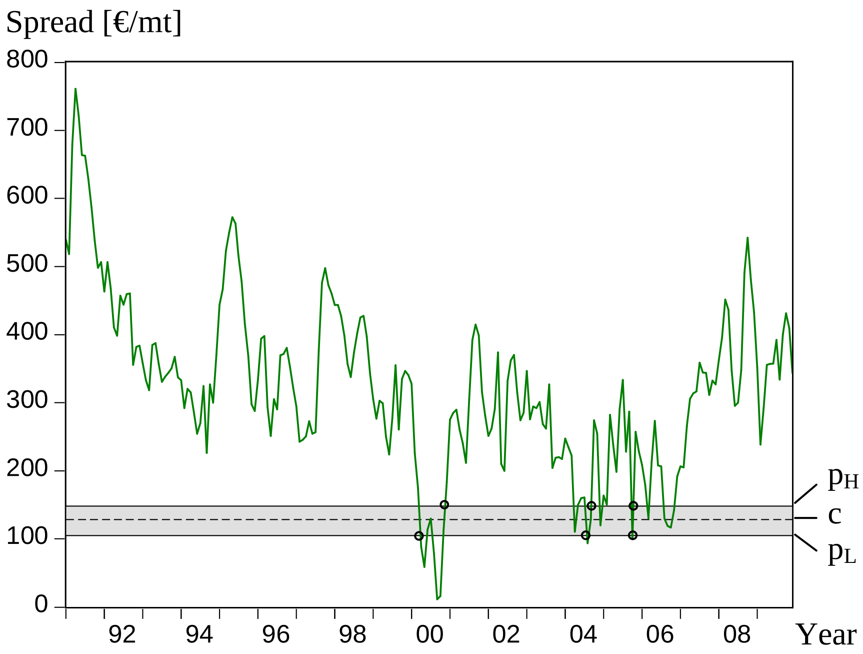


Switching boundaries p_H and p_L in €/mt as a function of variable operating cost (c) in €/mt and for different switching costs for resuming and suspending operation ($S_{01} = S_{10}$) of €0, €50,000 and €200,000.
Switching boundaries are obtained from the simultaneous solution of equations (28), (29), (30), (31), taking into account the tax rate γ, for the respective parameter values. Standard parameters: Current value of the spread: p = €340/mt; Long-run mean of p: m = €316.8/mt; Speed of mean-reversion of p: η=1.35; Standard deviation of p: σ = €198 p.a.; Capacity of p:

k_1 = 250,000 mt p.a.; Required return: μ = 0.10; Risk-free rate of return: r = 0.05; Exponential decay and technological/political risk: λ = 0.08; Tax rate γ = 0.3; Fixed operating cost: c_{fix} = €3.75 million p.a.; Annual depreciation: €10 million for 20 years.

Figure 7 illustrates the sensitivity of the switching boundaries to the variable operating cost and to the switching cost. It can be seen that while p_H and p_L move in line with the operating cost, the switching boundaries are not symmetrically distributed around the operating cost because m≠c. The final analysis in Table 8 compares the case of an initial spread of €500/mt vs. a spread of €150/mt, which results in a difference in the asset value of about €40 million.

Figure 8. Time Series of the Spread and Optimal Switching Points



Monthly prices for the polyethylene/ethylene spread (polyethylene price less 1.017 times ethylene price) in € per metric ton of polyethylene and for delivery within Europe. Suspend conversion of ethylene to polyethylene if the spread falls below p_L = 104.8 €/mt and resume at p_L = 148.2 €/mt.
Switching boundaries are obtained from the simultaneous solution of equations (28), (29), (30), (31), taking into account the tax rate γ, for the following parameter values. Variable operating cost: c = €128.5/mt; Capacity of p: k_1 = 250,000 mt p.a.; Switching cost for resuming operation: S_{01} = €40,000; Switching cost for suspending operation: S_{10} = €20,000; Required return: μ = 0.10; Risk-free rate of return: r = 0.05; Exponential decay and technological/political risk: λ = 0.08; Tax rate γ = 0.3; Fixed operating cost: c_{fix} = €3.75 million p.a.; Annual depreciation: €10 million for 20 years.

It is now interesting to simulate the asset operation on the basis of historical commodity prices. Figure 8 shows the development of the polyethylene/ethylene spread over the last two decades and the switching boundaries of the conversion plant, assuming constant operating costs. It can be seen that the plant should have been idle most of the year 2000 and in 2004 and 2005 for about one month each time. In these cases, ethylene should be sold in the market instead of being used to produce polyethylene. Most of the time, however, the spread level exceeds the variable cost by far, so that producing and selling polyethylene was the better choice.

We are also interested in how the switching boundaries change if the variance in the spread was different. Figure 9 illustrates the switching boundaries p_H and p_L for the case of mean-reversion (η=1.35) and for the case of no mean-reversion (η=0). The boundaries p_H and p_L diverge when the standard deviation of the spread increases because increased uncertainty generally delays switching. In the case of no mean-reversion, the switching boundaries are spread almost symmetrically around the sides of the operating cost (c) while they are "pulled down" in the case of mean-reversion, because the long run mean (m) of the spread is significantly higher than the operating cost. Hence, with m>c, the expectation of the spread returning to its long-run mean accelerates switching to earn the spread and delays switching to forego the spread. It can also be stated that the effect of mean-reversion on the switching boundaries is stronger when the variance is smaller.

When the standard deviation is very low, the solution procedure is less precise because of the high exponentials in the equations (e.g. in the parabolic cylinder functions). For the case of σ=0, p_H and p_L can also be determined by the following argument. When there is no uncertainty and m>c, switching to earn the spread occurs as soon as the spread exceeds the operating cost, hence p_H=c=128.5. The lower boundary p_L must satisfy the condition that the discounted expected cash flow, incurred during the time when the spread moves from p_L to p_H, equals the sum of the appropriately discounted switching costs:

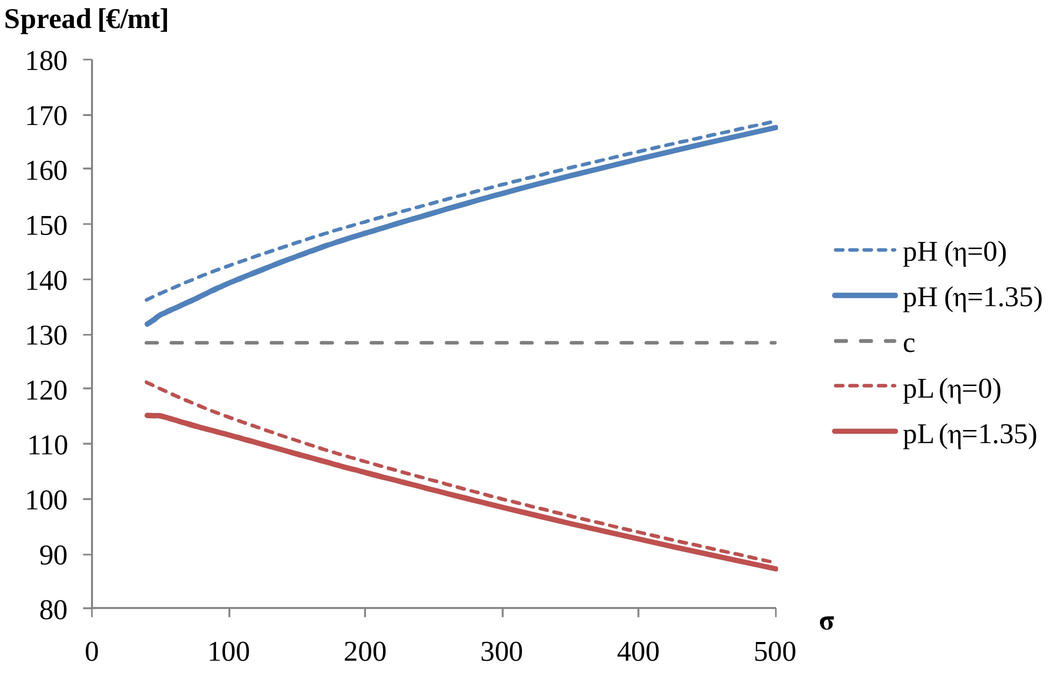
$$k_1 \int_{t=0}^{T} \left(E[p_t]^* - c\right) e^{-(r+\lambda)t} dt = S_{10} + e^{-rT} S_{01}, \tag{39}$$

where T is the expected time for the spread to move from p_L to p_H, given by:

$$T = -\frac{1}{\eta} \ln\left(\frac{p_H - m}{p_L - m}\right). \tag{40}$$

Inserting (40) in (39) and solving for p_L provides p_L=117.4, i.e. even in the absence of uncertainty, switching to forego the spread occurs only once the net cash flow is as low as -11.1.

Figure 9. Switching Boundaries as a Function of the Standard Deviation



Switching boundaries p_H and p_L in €/mt as a function of the standard deviation (σ) in €/mt p.a. for the case of mean-reversion (η=1.35) and the case of no mean-reversion (η=0).
Switching boundaries are obtained from the simultaneous solution of equations (28), (29), (30), (31), taking into account the tax rate γ, for the respective parameter values. Standard parameters: Long-run mean of p: m = €316.8/mt; Variable operating cost: c = €128.5/mt; Capacity of p: k_1 = 250,000 mt p.a.; Switching cost for resuming operation: S_{01} = €40,000;

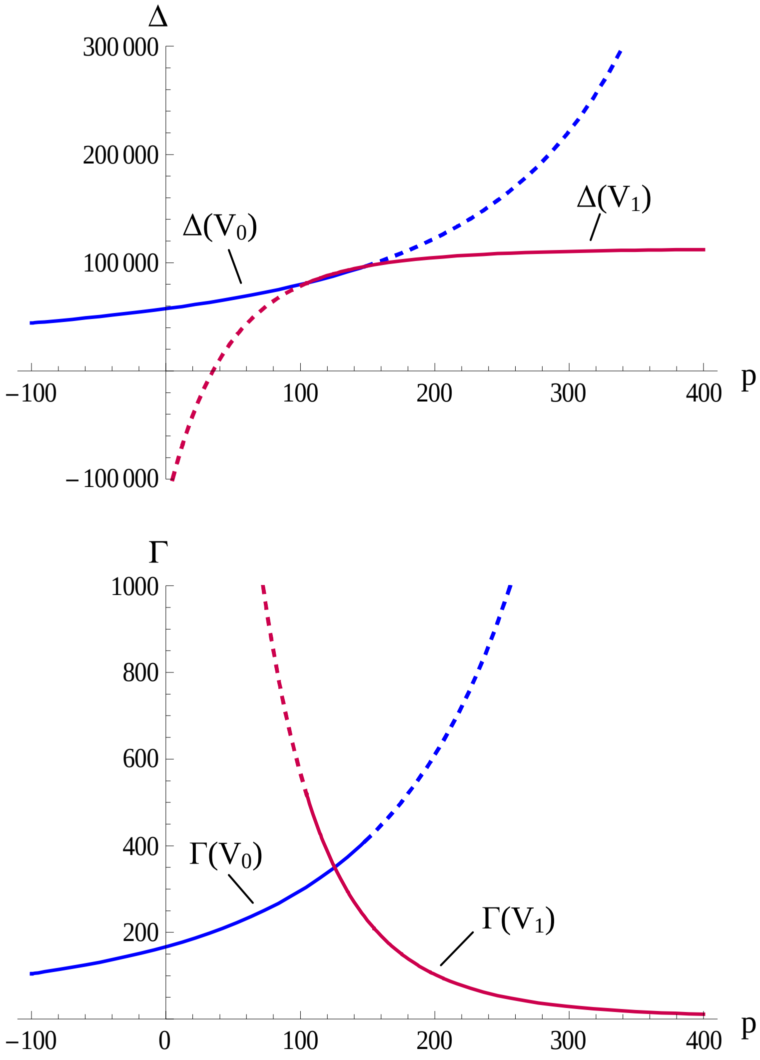
Switching cost for suspending operation: S_{10} = €20,000; Required return: $\mu = 0.10$; Risk-free rate of return: $r = 0.05$; Exponential decay and technological/political risk: $\lambda = 0.08$; Tax rate $\gamma = 0.3$; Fixed operating cost: c_{fix} = €3.75 million p.a.; Annual depreciation: €10 million for 20 years.

3.4 The Greek Letters

The risk measures Delta and Gamma of the asset value are provided in Figure 10. Delta is defined as the change of the asset value with changes in the spread (p), and Gamma is the change of Delta with changes in the spread. It can be seen that both Delta in the base operating mode, $\Delta(V_0)$, and Delta in the alternative operating mode, $\Delta(V_1)$, are increasing functions of the spread because a higher spread increases the immediate cash flow or the option on the spread. They intersect twice, at p_L and at p_H, satisfying the smooth pasting conditions. The mean-reverting characteristic of the spread causes the Delta to level off when the spread is very high or very low (negative). But this levelling-off is slower when the spread is low (negative) because the long-run mean of the spread exceeds the operating cost (c) and the switching boundary p_H, so that a positive cash flow can be expected at some point.

These Delta and Gamma functions are relevant for hedging if one intends to reduce the risk from variations in the stochastic spread. The fact that the Delta functions level off due to mean-reversion in the underlying process is positive for hedging because the required adjustments in the hedge positions are smaller compared to the case where the underlying process is non-stationary. It would, however, be difficult to implement this hedging since ethylene and polyethylene are not futures traded commodities.

Figure 10. Delta and Gamma for Asset Values depending on the Operating Mode



Delta (top) and Gamma (bottom) for the asset value in the base operating mode and in the alternative operating mode as a function of the spread (p in €/mt). $\Delta = \delta V / \delta p$ and $\Gamma = \delta^2 V / \delta p^2$. Dotted line indicates that function is only hypothetical because it is beyond the switching boundary (p_H and p_L respectively).
Parameters: Long-run mean of p: m = €316.8/mt; Speed of mean-reversion of p: $\eta = 1.35$; Standard deviation of p: σ = €198 p.a.; Variable operating cost: c = €128.5/mt; Capacity of p: k_1 = 250,000 mt p.a.; Switching cost for resuming operation: S_{01} = €40,000; Switching cost for suspending operation: S_{10} = €20,000; Required return: $\mu = 0.10$; Risk-free rate of return: $r = 0.05$; Exponential decay and technological/political risk: $\lambda = 0.08$; Tax rate $\gamma = 0.3$; Fixed operating cost: c_{fix} = €3.75 million p.a.; Annual depreciation: €10 million for 20 years.

4 Implications

4.1 *Implications for participants in the polyethylene industry*

Three generic strategies are available to companies involved in the production of polyethylene: investing in a polyethylene plant by building a new one or buying an existing one, optimising the operations, or divesting. The model and the results from the previous section enable us to evaluate these strategies and to point out opportunities and pitfalls.

Both investment and divestment decisions require transparency on the value of the transaction asset to determine an appropriate transaction price or to compare to the investment cost. When setting up a new plant, the investment is supposed to add value and the project should be implemented at the right time to maximise the value. The polyethylene plant is valued at €255 million which compares to the investment cost of about €200 million. Hence, the investment would be positive in the current set of circumstances. We have seen that the asset value would vary by about 15% (or €40 million in absolute terms) if we vary the initial spread level between the extreme levels of €150/mt and €500/mt. Ceteris paribus, the investment is more valuable if the current spread is high. With regard to taking the actual investment decision, this needs to be interpreted in combination with the time to build (about two years) and the correlation between the spread and a possibly stochastic investment cost.

In the design phase of the project, decisions are taken regarding the degree of operating flexibility to be incorporated, both physically into the asset and structurally in relations with business partners. This flexibility has been shown to be worth about €4 million. Thus, the contracts with business partners and employees should be designed to allow the required operating flexibility (free choice between operating the plant or suspending) as long as the additional cost incurred from these more flexible contracts are less than €4 million. Furthermore, a trade-off between reduced operating cost and higher investment cost is commonly encountered. For instance, if the variable operating cost of the polyethylene plant could be reduced from €128.5/mt to €100/mt, this would justify a €36 million higher investment cost.

Transparency on the spread levels triggering switches between operating and not operating is essential for the management team of the plant so that these critical decisions can be prepared in good time. One needs to be aware that switching boundaries change when variable operating cost (e.g. logistics cost) or the cost of ramping up or down change.

4.2 General implications

The application of the continuous rainbow option has shown that the flexible asset increases in value when relaxing the mean-reversion (η) in the underlying uncertainty. This is consistent with the Smith and McCardle (1999) conclusion that the option of flexibility is worth less when the underlying variable is mean-reverting instead of random walk. For $\eta=0$, the stochastic process simplifies to a Brownian motion with no drift. Real options theory suggests that the value of real options increases with volatility, and a non-stationary Brownian motion is more volatile than a stationary mean-reversion process. On the contrary, the inflexible asset decreases significantly in value if mean-reversion is relaxed which is due to the higher discount rate. Laughton and Jacoby (1993) call these two opposing phenomena the variance and discounting effects. From this can be concluded that incorporating flexibility in assets is more valuable when the value drivers are non-stationary, whereas additional value needs to be carefully weighed against the extra cost for flexibility when the value-drivers are stationary.

The results also highlight the relevance of assessing the degree of co-integration of markets. If two co-integrated variables are modelled as geometric Brownian motion with the appropriate correlation, their spread would not necessarily be bounded and asset values based on these variables might be overstated if a long time horizon is considered. As long as the spread of two co-integrated variables is the value-driver, the spread should be modelled directly as a stationary process.

5 Conclusion

This paper presents a real option model to continuously choose between the best of two co-integrated commodities. Since the spread of two co-integrated variables can be modelled as arithmetic mean-reversion, the complexity is reduced from two-factor to one-factor. This real rainbow option can also be interpreted as an entry/exit valuation problem on a mean-reverting stochastic variable. We develop a quasi-analytical solution for which all parameters can therefore be estimated from empirical data. A comparison with the Sodal et al. (2007) model demonstrates how important it is for a real option model with a mean-reverting stochastic process to distinguish between the different risk and discount factors instead of using a single general discount rate. Based on the risk-neutral valuation approach, we explicitly consider the risk-free interest rate, the risk-adjusted instantaneous required return on the commodity spread, and technological/political risk and physical deterioration.

An application of the model to value a polyethylene plant based on the spread between polyethylene and ethylene demonstrates that the option to switch between the two commodities increases when there is no mean-reversion. For the empirical data, the premium of the continuous rainbow option over the operation with no switching flexibility amounts to 2% which is influenced by the mean-reverting characteristic of the spread and the on average large positive net cash flow, resulting in a low probability of switching. However, the net value of flexibility is shown to be noticeable in absolute terms. When simulating zero mean-reversion, the flexible asset is twice as valuable as the inflexible asset. This confirms the intuition that incorporating flexibility into assets seems more promising when the value-drivers are non-stationary, while the value of flexibility in co-integrated markets is more limited. On the other hand, opportunities are found in anti-cyclical investing when the value-driver is stationary because the investment can be made when prices and initial costs are low, with prices expected to revert back to their long-run mean by the time the benefits are realised. An interesting extension to the model would therefore be to determine the optimal investment timing based on a fixed investment cost and then on a stochastic investment cost correlated with the spread.

Appendix A. Proof of the transformation of the PDE into Weber's equation

Kampke (1956, p. 416, Equation 2.54) studies the differential equation:

$$\frac{\partial^2 V_0}{\partial p^2} + (a \cdot p + b)\frac{\partial V_0}{\partial p} + (c \cdot p + d) \cdot V_0 = 0$$

where a, b, c, d are constants. He provides the following solution:

$$V_0(p) = F(x) e^{-\frac{c}{a}x}$$

$$\text{where } x = \sqrt{|a|}\left(p + \frac{a\,b - 2c}{a^2}\right)$$

$$\text{and } \frac{\partial^2 F}{\partial x^2} \pm x \frac{\partial F}{\partial x} \pm \frac{c^2 - a\,b\,c + a^2 d}{a^3} F = 0$$

(upper sign if a>0 and lower sign if a<0).

In our case, according to equation (19) repeated below, c=0 and a<0:

$$\frac{\partial^2 V_0}{\partial p^2} + (a \cdot p + b)\frac{\partial V_0}{\partial p} + d \cdot V_0 = 0 \tag{19}$$

Hence, the above solution simplifies to:

$$V_0(p) = F(x)$$

$$\text{where } x = \sqrt{|a|}\left(p + \frac{b}{a}\right)$$

$$\text{and } \frac{\partial^2 F}{\partial x^2} - x \frac{\partial F}{\partial x} - \frac{d}{a} F = 0\,. \tag{20}$$

The derivation of equation (20) can be proved as follows. Let $V_0 = F(x)$ and $x = \sqrt{|a|}\left(p + \frac{b}{a}\right)$, then the first and second derivative functions of V are:

$$\frac{\partial V}{\partial p} = \frac{\partial F}{\partial x}\frac{\partial x}{\partial p} = \frac{\partial F}{\partial x}\sqrt{|a|}\,,$$

$$\frac{\partial^2 V}{\partial p^2}=\frac{\partial\left(\frac{\partial F}{\partial x}\frac{\partial x}{\partial p}\right)}{\partial p}=\frac{\partial\left(\frac{\partial F}{\partial x}\sqrt{|a|}\right)}{\partial p}=\frac{\partial^2 F}{\partial x^2}|a|\,.$$

Applying these derivatives to equation (19) and taking into consideration that a<0, we obtain (20):

$$\frac{\partial^2 V}{\partial p^2}+(a\ p+b)\frac{\partial V}{\partial p}+d\cdot V=\frac{\partial^2 F}{\partial x^2}|a|+\left(a\left(\frac{x}{\sqrt{|a|}}-\frac{b}{a}\right)+b\right)\frac{\partial F}{\partial x}\sqrt{|a|}+d\cdot F$$

$$=\frac{\partial^2 F}{\partial x^2}-x\frac{\partial F}{\partial x}-\frac{d}{a}\cdot F=0\,.$$

Now substitute $F=G(x)e^{\frac{1}{4}x^2}$, as suggested by Kampke (1956, p. 414, Equation 2.44), and take the derivatives:

$$\frac{\partial F}{\partial x}=\left(\frac{\partial G}{\partial x}+\frac{1}{2}x\cdot G\right)e^{\frac{1}{4}x^2}\,,$$

$$\frac{\partial^2 F}{\partial x^2}=\left(\frac{\partial^2 G}{\partial x^2}+x\frac{\partial G}{\partial x}+\left(\frac{1}{2}+\frac{1}{4}x^2\right)G\right)e^{\frac{1}{4}x^2}\,.$$

Inserting the above functions into (20) provides the Weber equation:

$$\frac{\partial^2 G}{\partial x^2}=\left(\frac{x^2}{4}+\frac{d}{a}-\frac{1}{2}\right)G\,. \qquad (21)$$

Appendix B. Alternative direct solution to the PDE

A direct solution to equation (20) is provided by Kampke (1956, p. 414, Equation 2.44) as the following series function:

$$F(x) = C_1\left(1 + \frac{\frac{d}{a}}{2!}x^2 + \frac{\frac{d}{a}\left(\frac{d}{a}+2\right)}{4!}x^4 + \frac{\frac{d}{a}\left(\frac{d}{a}+2\right)\left(\frac{d}{a}+4\right)}{6!}x^6 + \ldots\right)$$

$$+ C_2\left(x + \frac{\frac{d}{a}+1}{3!}x^3 + \frac{\left(\frac{d}{a}+1\right)\left(\frac{d}{a}+3\right)}{5!}x^5 + \frac{\left(\frac{d}{a}+1\right)\left(\frac{d}{a}+3\right)\left(\frac{d}{a}+5\right)}{7!}x^7 + \ldots\right) \quad (41)$$

In the following, we demonstrate that the above solution is equivalent to:

$$F(x) = C_1\, M\left(\frac{d/a}{2}, \frac{1}{2}, \frac{x^2}{2}\right) + C_2\, x\, M\left(\frac{d/a+1}{2}, \frac{3}{2}, \frac{x^2}{2}\right), \quad (42)$$

where M is the Kummer function as defined in equation (24). Writing (42) in the series form provides:

$$F(x) = C_1\left(1 + \frac{\frac{d}{a}}{1}\frac{x^2}{2} + \frac{\frac{d}{a}\left(\frac{d}{a}+2\right)}{1\cdot 3}\frac{x^4}{2!\cdot 2^2} + \frac{\frac{d}{a}\left(\frac{d}{a}+2\right)\left(\frac{d}{a}+4\right)}{1\cdot 3\cdot 5}\frac{x^6}{3!\cdot 2^3} + \ldots\right)$$

$$+ C_2\, x\left(1 + \frac{\frac{d}{a}+1}{1\cdot 3}\frac{x^2}{2} + \frac{\left(\frac{d}{a}+1\right)\left(\frac{d}{a}+3\right)}{1\cdot 3\cdot 5}\frac{x^4}{2!\cdot 2^2} + \frac{\left(\frac{d}{a}+1\right)\left(\frac{d}{a}+3\right)\left(\frac{d}{a}+5\right)}{1\cdot 3\cdot 5\cdot 7}\frac{x^6}{3!\cdot 2^3} + \ldots\right) \quad (43)$$

The numerators of the series in (43) and (41) are obviously the same. To show that the denominators are also identical, it needs to be demonstrated that:

$1\cdot 3$	$\cdot 2!$	$\cdot 2^2$	$= 4!$
$1\cdot 3\cdot 5$	$\cdot 3!$	$\cdot 2^3$	$= 6!$
$1\cdot 3\cdot 5\cdot 7$	$\cdot 4!$	$\cdot 2^4$	$= 8!$
$1\cdot 3\cdot 5\cdot 7\cdot 9$	$\cdot 5!$	$\cdot 2^5$	$= 10!$
…			

and

$1\cdot 3$	$\cdot 1!$	$\cdot 2^1$	$= 3!$
$1\cdot 3\cdot 5$	$\cdot 2!$	$\cdot 2^2$	$= 5!$
$1\cdot 3\cdot 5\cdot 7$	$\cdot 3!$	$\cdot 2^3$	$= 7!$
$1\cdot 3\cdot 5\cdot 7\cdot 9$	$\cdot 4!$	$\cdot 2^4$	$= 9!$
…			

Considering that $n!\cdot 2^n = 2\cdot 4\cdot 6\cdot 8\cdot ...\cdot 2n$, it can be seen that the above transformations hold. Hence, when the constants C_1 and C_2 are chosen appropriately, the following relationship holds, as can be verified with the definition of the parabolic cylinder function in (23):

$$\left(A\cdot D_{-d/a}(x)+B\cdot D_{-d/a}(-x)\right)e^{\frac{1}{4}x^2} = C_1\, M\left(\frac{d/a}{2},\frac{1}{2},\frac{x^2}{2}\right)+C_2\, x\, M\left(\frac{d/a+1}{2},\frac{3}{2},\frac{x^2}{2}\right). \quad (44)$$

This shows that the above procedure results in the same value function as the one provided in the text. It should be noted, however, that its derivation is built on information obtained from the solution of the Weber equation, namely the parameters to be used in the Kummer functions in order to obtain the required series function.

Appendix C. System of equations

With A_0=0 and B_1=0, equations (25) and (27) simplify to

$$V_0(p) = B_0 \cdot D_{-d/a}\left(-\sqrt{|a|}\left(p + \frac{b}{a}\right)\right) e^{\frac{1}{4}\left(\sqrt{|a|}\left(p+\frac{b}{a}\right)\right)^2} \quad \text{and}$$

$$V_1(p) = A_1 \cdot D_{-d/a}\left(\sqrt{|a|}\left(p + \frac{b}{a}\right)\right) e^{\frac{1}{4}\left(\sqrt{|a|}\left(p+\frac{b}{a}\right)\right)^2} + u \cdot p + w\,.$$

With the value functions above, the two boundary conditions, (28) and (29), can be evaluated:

$$\left(B_0 \cdot D_{-d/a}\left(-\sqrt{|a|}\left(p_H + \frac{b}{a}\right)\right) - A_1 \cdot D_{-d/a}\left(\sqrt{|a|}\left(p_H + \frac{b}{a}\right)\right)\right) e^{\frac{1}{4}\left(\sqrt{|a|}\left(p_H+\frac{b}{a}\right)\right)^2}$$

$$-u \cdot p_H - w + S_{01} = 0\,, \qquad \text{(28 operational)}$$

$$\left(B_0 \cdot D_{-d/a}\left(-\sqrt{|a|}\left(p_L + \frac{b}{a}\right)\right) - A_1 \cdot D_{-d/a}\left(\sqrt{|a|}\left(p_L + \frac{b}{a}\right)\right)\right) e^{\frac{1}{4}\left(\sqrt{|a|}\left(p_L+\frac{b}{a}\right)\right)^2}$$

$$-u \cdot p_L - w - S_{10} = 0\,. \qquad \text{(29 operational)}$$

For the evaluation of the smooth pasting conditions, the derivative function of the parabolic cylinder function is used:

$$\frac{d}{dx} D_v(f(x)) = \left(-\frac{1}{2} f(x) D_v(f(x)) + v\, D_{v-1}(f(x))\right) \frac{d}{dx} f(x)\,.$$

(30) can then be assessed and simplified:

$$\frac{d\sqrt{|a|}}{a} e^{\frac{1}{4}\left(\sqrt{|a|}\left(p_H+\frac{b}{a}\right)\right)^2} \left[B_0 \cdot D_{-1-d/a}\left(-\sqrt{|a|}\left(p_H + \frac{b}{a}\right)\right) + A_1 \cdot D_{-1-d/a}\left(\sqrt{|a|}\left(p_H + \frac{b}{a}\right)\right)\right] - u = 0$$

Similarly, from (31):

$$\frac{d\sqrt{|a|}}{a} e^{\frac{1}{4}\left(\sqrt{|a|}\left(p_L+\frac{b}{a}\right)\right)^2} \left[B_0 \cdot D_{-1-d/a}\left(-\sqrt{|a|}\left(p_L + \frac{b}{a}\right)\right) + A_1 \cdot D_{-1-d/a}\left(\sqrt{|a|}\left(p_L + \frac{b}{a}\right)\right)\right] - u = 0$$

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5 Research Paper #3 – Sequential Real Rainbow Options

Sequential Real Rainbow Options

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We develop two models to value European sequential rainbow options. The first model is a sequential option on the best of two stochastic assets where these assets follow correlated geometric Brownian motion processes. The second model is a sequential option on the mean-reverting spread between two assets, which is applicable if the assets are co-integrated. We provide numerical solutions in the form of finite difference frameworks and compare these with Monte Carlo simulations. For the sequential option on a mean-reverting spread, we also provide a closed-form solution. Sensitivity analysis provides the interesting results that in particular circumstances, the sequential rainbow option value is negatively correlated with the volatility of one of the two assets, and that the sequential option on the spread does not necessarily increase in value with a longer time to maturity. With given maturity dates, it is preferable to have less time until expiry of the sequential option if the current spread level is way above the long-run mean.

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1 Introduction

In many cases, the process of investing is best represented by a sequence of investments where the benefits are realised only when the final investment is completed. While academic work has so far concentrated on sequential options on vanilla call options on Markovian assets, we develop two models to value sequential investment opportunities on the option to choose the best of two stochastic assets. The first model is a sequential option on the best of two assets following geometric Brownian motion (gBm) processes. The second model is a sequential option on the spread between two stochastic co-integrated assets, where the spread is mean-reverting. While the former is a two-factor and the latter a one-factor model, we consider both as rainbow options because both options are about the choice between two assets. Moreover, the options are European-style, i.e. assuming the investment opportunities are available only at specified dates.

An example of the first model is found in the commodity industry when investing in a fertiliser plant. An investor considers bidding for a long-term take-or-pay gas supply contract, which is scheduled for auction, in order to use the gas to produce fertiliser. Having secured the gas supply, the investor needs to decide on the design of the fertiliser plant, i.e. whether the plant is supposed to produce ammonia or urea, and actually build the plant within a certain time. The present value of the gas cost is the intermediate investment (sequential option exercise price), the sum of investment cost for the plant and present value of the operating cost is the final investment (rainbow option exercise price), and the expected discounted cash flows from selling either ammonia or urea are the two underlying stochastic assets. Another example for this type of option is found in agriculture where a farmer has the option to prepare land in the autumn season in order to grow wheat or maize in the following spring. The cost for preparing the land is the intermediate investment. The rainbow option is then exercised in the spring season by growing one of the two commodities. The total cost of growing and harvesting is therefore regarded as the final investment, and the present value of revenues from selling wheat or maize are the two underlying stochastic assets of the rainbow option.

The second model, a sequential option on the spread between two assets, is relevant to a similar application when a piece of arable land worn out by growing maize is kept idle in order for the land to recover and thereby create the option to grow wheat instead of maize in the year thereafter, i.e. to change the use of land. The foregone profit from keeping the land idle is the intermediate investment, the extra cost of growing wheat instead of maize is the final investment and the present value of the difference between wheat revenues and maize revenues is the underlying stochastic spread. A sequential option to build a plant which processes a lower value product into a higher value product is another example of this type of option. Assuming an ethylene producer considers building a polyethylene plant but needs to acquire and prepare the land and secure the necessary permits which is the intermediate investment. Building the plant is the final investment and the present value of the difference between expected polyethylene and ethylene revenues is the underlying stochastic spread.

Geske (1978) was the first to develop the classical European compound option on a European call option where the underlying stochastic asset is a log-normal Markovian variable. This compound option is also the basis for the Carr (1988) compound option on a Margrabe (1978) European exchange option. Carr imposes assumptions on the proportionality of the interim expenditure and the (stochastic) exercise cost of the exchange option, which makes the compound option homogenous in the underlying assets and thereby allows for an analytical solution. Paxson (2007) provides a closed-form approximation to a compound exchange option where the exercise date of the compound option is fixed (European) while the exercise date of the underlying exchange option is American finite. Childs et al. (1998) compare several alternatives of sequential investment schemes with parallel investments.

Stulz (1982) and Zhang (1998) provide valuation formulae for European rainbow options. Johnson (1987) uses an alternative way to determine the value of an option on the maximum of several assets, including the two-colour rainbow option as a special case. Dockendorf and Paxson (2009) develop an option model on the best of two commodity outputs of correlated geometric Brownian motion processes with continuous switching opportunities. They incorporate the possibility of temporary suspension and apply the models to a flexible fertilizer plant. Dockendorf and Paxson (2010) further provide a quasi-analytical model to value the continuous option to operate in the best of two co-integrated markets, which implies that the spread follows a mean-reverting stochastic process, and allow switching between the two operating modes at any time by incurring fixed switching costs. While Rubinstein (1991) and Shimko (1994) derive double and single integral solutions to European spread options, and Kirk (1995) and Bjerksund and Stensland (2006) develop closed-form approximations, Andricopoulos et al. (2003) apply quadrature methods to the valuation of complex options and extend this approach to multi-asset options (2007).

The remaining part of this paper is structured as follows. Section 2 presents the general sequential investment model with the options and timing. Section 3 develops the valuation framework for a sequential rainbow option on two correlated assets following geometric Brownian motion. Section 4 develops the valuation framework for a sequential rainbow option on the mean-reverting spread of two co-integrated assets and derives a closed-form solution. The finite difference solution method applied to the valuation frameworks is outlined in Section 5. Section 6 shows sensitivities using the previous models and discusses some practical implications. Section 7 concludes and discusses issues for further research.

2 General Sequential Investment Model

2.1 Investment Sequence

We develop valuation models for a European call option on a two asset European rainbow option. In other words, this option consists of a sequential (or compound) option and an "inner" rainbow option. The rainbow option can only be exercised once, i.e. the choice between the two underlying assets is made at maturity, and no further switches are possible. The investment sequence is shown graphically in Figure 1. The motivation for this type of option is based on the nature of many real investment opportunities, which effectively leads to investments being carried out in sequential phases.

2.2 Definitions and Assumptions

The definitions below will be used throughout this paper:

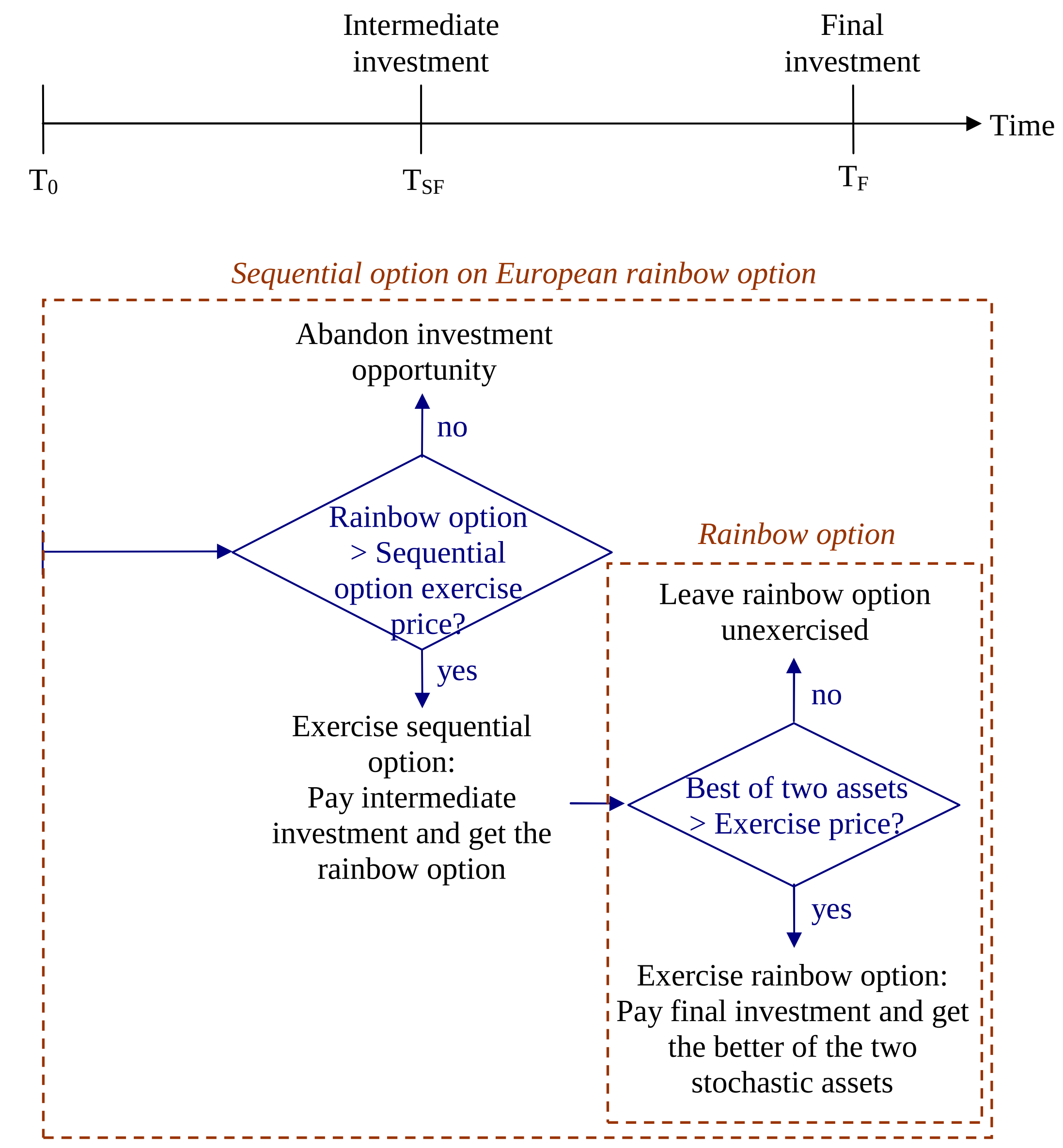
SF	Value of the sequential rainbow option
F	Value of the ("inner") rainbow option
X	Value of asset one
Y	Value of asset two
P	Spread between asset values Y and X, so P=Y-X
K_{SF}	Exercise price for the sequential option

K_F Exercise price for the rainbow option

τ_{SF} Time to expiration of the sequential option: T_{SF}-t

τ_F Time to expiration of the rainbow option: T_F-t

Figure 1. Investment Sequence



We set up models for two sequential rainbow options: first, a sequential option on the best of the stochastic assets X and Y, and second, a sequential option on the spread (P) between Y and X. The intrinsic values of the former are

$F_{intrinsic} = Max[Max[X, Y] - K_F, 0]$ and $SF_{intrinsic} = Max[F(X, Y) - K_{SF}, 0]$, and $F_{intrinsic} = Max[P - K_F, 0]$ and $SF_{intrinsic} = Max[F(P) - K_{SF}, 0]$ for the latter. In our approach, the typical assumptions of real options theory apply with interest rates, yields, risk premium, correlations and volatilities constant over time. The financial markets are perfect with no transaction costs. The stochastic variables X and Y are assumed to follow geometric Brownian motion with a correlation denoted by ρ. For the alternative model, X and Y are assumed to be co-integrated so that the spread (P) between the two is mean-reverting.

3 Sequential Rainbow Option on Two Correlated gBm Assets

3.1 Partial Differential Equation

The values of two stochastic assets, X and Y, are modelled separately by geometric Brownian motion with correlation ρ. The behaviour of the variables is defined by:

$$dX = \alpha_X X\, dt + \sigma_X X\, dz_X , \tag{1}$$

$$dY = \alpha_Y Y dt + \sigma_Y Y\, dz_Y , \tag{2}$$

where α is the respective drift rate, σ the volatility and dz a Wiener process. The sequential rainbow option is a function of X, Y and time t. Its incremental change is determined by Itô's Lemma:

$$dSF = \frac{\partial SF}{\partial t} dt + \frac{\partial SF}{\partial X} dX + \frac{\partial SF}{\partial Y} dY + \frac{1}{2}\frac{\partial^2 SF}{\partial X^2} dX^2 + \frac{1}{2}\frac{\partial^2 SF}{\partial Y^2} dY^2 + \frac{\partial^2 SF}{\partial X \partial Y} dX\, dY \tag{3}$$

The incremental asset changes dX and dY comprise the Wiener processes dz_X and dz_Y. Let π be a portfolio containing SF and short positions of X and Y:

$$\pi = SF - \frac{\partial SF}{\partial X} X - \frac{\partial SF}{\partial Y} Y \tag{4}$$

Considering convenience yields δ_X and δ_Y, the incremental change in π is given by:

$$d\pi = dSF - \frac{\partial SF}{\partial X} dX - \frac{\partial SF}{\partial Y} dY - \delta_X X \frac{\partial SF}{\partial X} dt - \delta_Y Y \frac{\partial SF}{\partial Y} dt \tag{5}$$

This portfolio does not contain any stochastic elements and must therefore earn the risk-free rate of return on the invested portfolio value π. The partial differential equation (PDE) for the value function of the sequential rainbow option then follows:

$$\frac{\partial SF}{\partial t} + \frac{1}{2}\sigma_X^2 X^2 \frac{\partial^2 SF}{\partial X^2} + \frac{1}{2}\sigma_Y^2 Y^2 \frac{\partial^2 SF}{\partial Y^2} + \rho\sigma_X\sigma_Y XY \frac{\partial^2 SF}{\partial X \partial Y}$$

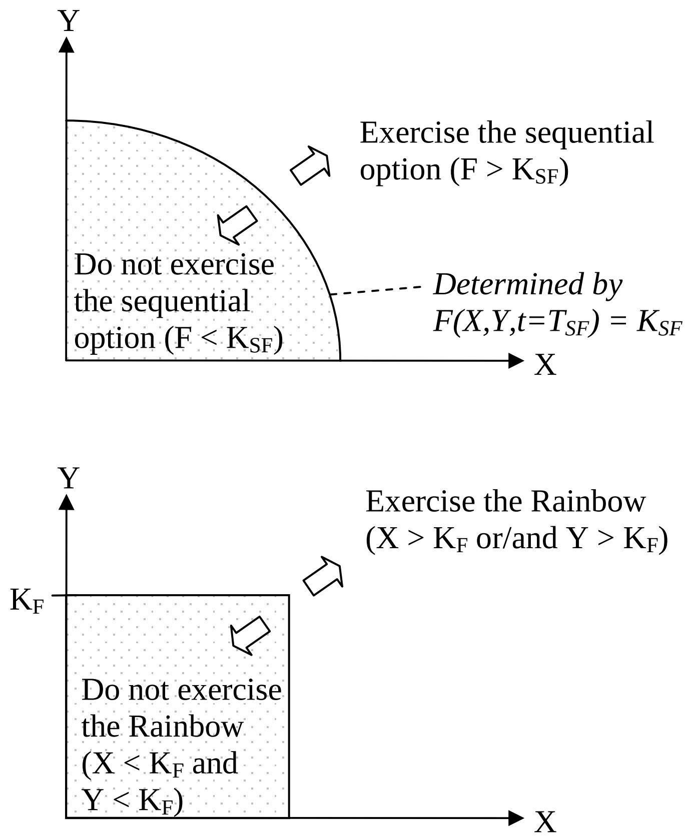
$$+ (r - \delta_X) X \frac{\partial SF}{\partial X} + (r - \delta_Y) Y \frac{\partial SF}{\partial Y} - r\,SF = 0 \tag{6}$$

There are two particular difficulties with the above equation. First, SF is a function of three variables (X, Y, t), and second, SF is not linear homogenous in X and Y, because X and Y are part of the “inner” Rainbow option. Therefore, X and Y cannot be substituted by a single variable. Equation (6) contains six derivative functions of SF and SF itself. Hence, in order to fully describe SF, a system of seven equations has to be established.

3.2 Boundary Conditions

The sequential rainbow option is exercised at maturity (T_{SF}) if the value of the rainbow option to be received is higher than the exercise price of the sequential option (K_{SF}). If the European rainbow option is acquired, it is exercised at its maturity (T_F) if at least one of the assets X or Y exceeds the exercise price K_F. Figure 2 illustrates the exercise boundaries of both the sequential rainbow option and the rainbow option.

Figure 2. Exercise Boundaries of Sequential Option and Inner Rainbow Option



Above: Exercise boundary for the sequential rainbow option at maturity T_{SF}
Below: Exercise boundary for the rainbow option at maturity T_F

The value of the sequential option at maturity can be formalised as follows.

$$SF(X,Y,t=T_{SF}) = \begin{cases} F(X,Y,t) - K_{SF} & \text{if} \quad F > K_{SF} \\ 0 & \text{if} \quad F < K_{SF} \end{cases} \tag{7}$$

where the European rainbow option is given by Stulz (1982) as:

$$F(X,Y,t) = X\,e^{-\delta_X \tau_F} N_2(d_{11}, -d_{12}, -\rho_1) + Y\,e^{-\delta_Y \tau_F} N_2(d_{22}, -d_{21}, -\rho_2)$$
$$- K_F\,e^{-r\tau_F}\left[1 - N_2(-d_1, -d_2, \rho)\right] \tag{8}$$

with

$$d_1 = \left(\ln \frac{X}{K_F} + (r - \delta_X - \frac{1}{2}\sigma_X^2)\tau_F\right) \Big/ (\sigma_X \sqrt{\tau_F})$$

$$d_{11} = d_1 + \sigma_X \sqrt{\tau_F}$$

$$d_2 = \left(\ln \frac{Y}{K_F} + (r - \delta_Y - \frac{1}{2}\sigma_Y^2)\tau_F \right) \Big/ (\sigma_Y \sqrt{\tau_F})$$

$$d_{22} = d_2 + \sigma_Y \sqrt{\tau_F}$$

$$d_{12} = \left(\ln \frac{Y}{X} + (\delta_X - \delta_Y - \frac{1}{2}\sigma_a^2)\tau_F \right) \Big/ (\sigma_a \sqrt{\tau_F})$$

$$d_{21} = \left(\ln \frac{X}{Y} + (\delta_Y - \delta_X - \frac{1}{2}\sigma_a^2)\tau_F \right) \Big/ (\sigma_a \sqrt{\tau_F})$$

$$\rho_1 = \frac{\rho\sigma_Y - \sigma_X}{\sigma_a}, \ \rho_2 = \frac{\rho\sigma_X - \sigma_Y}{\sigma_a}$$

$$\sigma_a = \sqrt{\sigma_X^2 - 2\rho\sigma_X\sigma_Y + \sigma_Y^2}$$

The stochastic variables are log-normally distributed, i.e. they have an absorbing barrier at zero. Hence, when either variable is zero, it will stay so forever, so the rainbow option simplifies to a vanilla call option and the sequential rainbow option reduces to a Geske (1978) compound option on the remaining stochastic variable. With N_2 the bivariate cumulative normal function (see Kotz et al., 2000), the sequential rainbow option value for X=0 is given below. The case for Y=0 is constructed similarly by substituting Y, σ_Y and δ_Y for X, σ_X and δ_X.

$$SF(X=0, Y, t) = Y e^{-\delta_Y \tau_{SF}} N_2(h + \sigma_Y \sqrt{\tau_{SF}}, k + \sigma_Y \sqrt{\tau_F}; \sqrt{\tau_{SF} / \tau_F})$$

$$- K_F e^{-r\tau_F} N_2(h, k; \sqrt{\tau_{SF}/\tau_F}) - K_{SF} e^{-r\tau_{SF}} N_1(h) \quad (9)$$

where

$$h = \frac{\ln(Y/\overline{Y}) + (r - \delta_Y - \frac{1}{2}\sigma_Y{}^2)\tau_{SF}}{\sigma_Y \sqrt{\tau_{SF}}}$$

$$k = \frac{\ln(Y/K_F) + (r - \delta_Y - \frac{1}{2}\sigma_Y{}^2)\tau_F}{\sigma_Y \sqrt{\tau_F}}$$

$$\overline{Y}\, e^{-\delta_Y(\tau_F - \tau_{SF})} N_1(k + \sigma_Y \sqrt{\tau_F - \tau_{SF}}) - K_F\, e^{-r(\tau_F - \tau_{SF})} N_1(k) - K_{SF} = 0$$

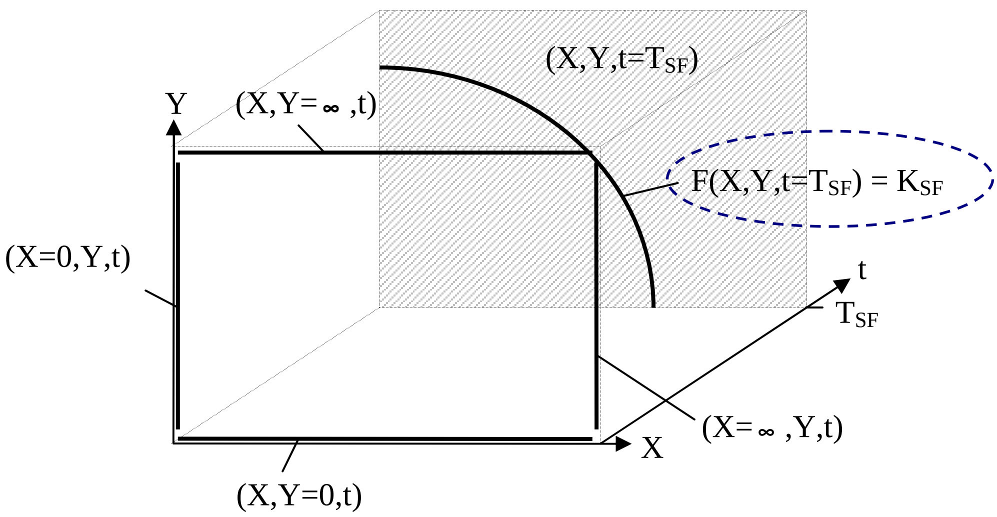
If one of the asset values tends towards infinity, the rainbow option will definitively be exercised to obtain that asset, and the sequential option will also be exercised:

$$SF(X = \infty, Y, t) = X\, e^{-\delta_X \tau_F} - K_F\, e^{-r\tau_F} - K_{SF}\, e^{-r\tau_{SF}} \tag{10}$$

$$SF(X, Y = \infty, t) = Y\, e^{-\delta_Y \tau_F} - K_F\, e^{-r\tau_F} - K_{SF}\, e^{-r\tau_{SF}} \tag{11}$$

An additional condition is required describing the boundary between exercising the sequential real rainbow option and not exercising, as shown in Figure 3. An analytical description of this boundary is not readily deduced which is why the value of the sequential rainbow option needs to be determined by numerical methods.

Figure 3. Boundary Conditions for the Sequential Rainbow Option on Two Assets



Boundary conditions for the sequential rainbow option as a function of the underlying variables X and Y, between current time and maturity of the sequential option (T_{SF}). K_{SF} is the exercise price of the sequential option and F is the value of the inner rainbow option.

4 Sequential Rainbow Option on the Mean-Reverting Spread

4.1 Partial Differential Equation

If the asset values X and Y are co-integrated, the spread (P) between the two is stationary and its stochastic process can be modelled as arithmetic mean-reversion:

$$dP = \eta(m - P)dt + \sigma dz \tag{12}$$

where m is the long-run mean of the spread, η the speed of mean-reversion, σ the standard deviation and dz a standardised Wiener process. We use the notation of standard deviation for arithmetic processes since the term volatility is commonly used to describe the standard deviation of percentage changes in stock (see Liu, 2007). Dixit and Pindyck (1994) provide the expected value of P at time T for the above Ornstein-Uhlenbeck process as:

$$E[P]_T = P \cdot e^{-\eta T} + m\left(1 - e^{-\eta T}\right), \tag{13}$$

and the variance by time T as:

$$\nu_T^2 = \frac{\sigma^2}{2\eta}\left(1 - e^{-2\eta T}\right). \tag{14}$$

Dockendorf and Paxson (2009) derive the risk-neutral equivalent (P^*) to the above arithmetic mean-reverting process:

$$dP^* = \kappa\left(\theta - P^*\right)dt + \sigma\, dz\,, \tag{15}$$

with $\kappa = \eta + \mu - r$ the speed of reversion and $\theta = \dfrac{\eta}{\eta + \mu - r} m$ the long-run mean and the notation (*) referring to a setting under risk-neutrality.

We assume that asset X is owned in the base case, so the rainbow option to choose the best of X and Y is then essentially a call option on the spread (P). Hence, the sequential rainbow option is then the option to acquire the call option on the spread. We construct a framework for the sequential rainbow option (SF) on a mean-reverting spread, following the same methodology as in the previous section on two gBm assets. The partial differential equation for the option on a single stochastic variable of the general Itô process is given by:

$$\frac{\partial SF}{\partial t} + \tfrac{1}{2}\sigma(P)^2 \frac{\partial^2 SF}{\partial P^2} + (r - \delta)P\frac{\partial SF}{\partial P} - rSF = 0 \tag{16}$$

With the yield of the arithmetic mean-reversion being $\delta = \mu - \eta(m - P)/P$, the sequential option is defined by:

$$\frac{\partial SF}{\partial t} + \frac{1}{2}\sigma^2 \frac{\partial^2 SF}{\partial P^2} + [\eta m - (\mu - r + \eta)P]\frac{\partial SF}{\partial P} - rSF = 0 \tag{17}$$

The pay-off function of the sequential option, Max[F-K_{SF},0], is not linear in the underlying variable (P). The exercise boundary of the sequential option cannot be determined analytically.

4.2 Boundary Conditions

Boundary conditions are specified for extreme values of the underlying asset and time of maturity of the sequential option. At the time of maturity of the sequential option, T_{SF}, its value is determined by the greater of the call option on P less exercise price K_{SF} and zero.

$$SF(P, t = T_{SF}) = \begin{cases} F(P,t) - K_{SF} & \text{if} \quad F > K_{SF} \\ 0 & \text{if} \quad F < K_{SF} \end{cases} \tag{18}$$

To determine the value of the call option (F) on the normal mean-reverting variable P, let (S) be a variable following arithmetic Brownian motion with a proportional drift rate μ:

$$dS = \mu S dt + \sigma dz. \tag{19}$$

Brennan (1979) derives the value of the call option on this arithmetic stochastic variable with exercise price K and time to maturity τ in a one-period framework. In order to allow a more general interpretation, we reorganise his results to show that the call option is a function of the expected value of the underlying variable and of its standard deviation by the time to maturity (ν_τ):

$$C(S,t) = e^{-r\tau}\left[\left(E\left[S^*\right]_\tau - K\right)N(d) + \nu_\tau\, n(d)\right] \tag{20}$$

$$d = \frac{E\left[S^*\right]_\tau - K}{\nu_\tau},$$

$$E\left[S^*\right]_\tau = S\, e^{r\tau},$$

$$\nu_\tau^2 = \frac{\sigma^2}{2r}\left(e^{2r\tau} - 1\right),$$

with N the cumulative standard normal distribution and n the standard normal distribution. The standard deviation ν_τ of the arithmetic Brownian motion with proportional drift rate as given above approaches σ for $\tau \rightarrow 0$ which is used in the Brennan one-period model. Cox and Ross (1976) come to the same result and point out in addition that the arithmetic Brownian motion with proportional drift is a special case of the arithmetic mean-reverting process. In fact, the former is obtained by choosing $\eta = -\mu$ and $m = 0$ in the mean-reverting process, as we demonstrate in Appendix A. This relationship can now be used to derive the call option on the arithmetic mean-reverting variable (P) by replacing $E[S^*]_\tau$ and ν_τ with the applicable (risk-neutral) expressions:

$$F(P,t) = e^{-r\tau}\left[\left(E\left[P^*\right]_\tau - K\right)N(d) + \nu_\tau\, n(d)\right] \tag{21}$$

$$d = \frac{E\left[P^*\right]_\tau - K}{\nu_\tau},$$

$$E\left[P^*\right]_\tau = P \cdot e^{-\kappa\tau} + \theta\left(1 - e^{-\kappa\tau}\right),$$

$$\nu_\tau^2 = \frac{\sigma^2}{2\kappa}\left(1 - e^{-2\kappa\tau}\right).$$

The above call option is hence the discounted value of the expected value of the underlying (risk-neutral) asset less the exercise cost, both multiplied by the probability that the option will be exercised, plus a term accounting for the positive option value effect from the variance. Taking a different approach to deriving the value of a European call option on an Ornstein-Uhlenbeck variable, Bjerksund and Ekern (1995) come to the same structure as in (21) but with different expected value and variance of the underlying variable in the risk-neutral setting because they have assumed the market price of risk of the underlying arithmetic mean-reverting variable (p) to be independent of the level of p. However, since the market price of risk relates the risk premium to the relative volatility, it must be a function of the underlying variable when that one is arithmetic.

If P tends towards infinity, both the sequential option and the call option on P will definitely be exercised:

$$SF(P \to \infty, t) = E\left[P^*\right]_{\tau_F} e^{-r\tau_F} - K_{SF}\, e^{-r\tau_{SF}} - K_F\, e^{-r\tau_F}\,. \tag{22}$$

On the other hand, if P tends towards minus infinity, the options are worthless because they will definitely not be exercised:

$$SF(P \to -\infty, t) = 0\,. \tag{23}$$

4.3 Closed-form Solution

The sequential option on the mean-reverting spread is given by the discounted expected value of the maximum of inner call option less exercise price and zero, evaluated in a risk-neutral setting:

$$SF(P,t) = e^{-r\tau_{SF}} \int_{\overline{P}}^{\infty} \left(F\left(P^*,t\right) - K_{SF}\right) g_{SF}\left(P^*\right) dP^* \qquad (24)$$

where P^* is again the risk-neutral form of the spread P, $g_{SF}(P^*)$ the density function of P^* at t_{SF}, and $\overline{P}$ the minimum level of P^* required to exercise the sequential option at t_{SF}. The particular difficulty with this equation is to find a solution to the integral over the call option F. For this purpose, we use the relationship between the arithmetic Brownian motion with proportional drift and the mean-reverting process, as established before. Childs et al. (1998) consider sequential investment opportunities involving two different assets, both following arithmetic Brownian motion. We build on their results to derive the sequential option on an arithmetic mean-reverting variable.

Let $C_Z(S)$ be a call option on S with exercise price uZ+K, where both S and Z are stochastic and follow arithmetic Brownian motion as defined in equation (19), correlated by ρ. Hence, the exercise price consists of a stochastic component (u times Z) and a fixed component (K). Assuming this option is only available if the value of the stochastic variable Z at maturity is between two arbitrary constants, a and b, then the value of that opportunity is $\int_{Z=a}^{b} C_Z(S,t)\, g(Z)\, dZ$, where g(Z) is the probability distribution of Z at maturity. Childs et al. (1998) derive the value of this integral. Choosing u=0 eliminates the stochastic component in the exercise price so that the call option becomes a plain vanilla call option with a constant exercise price K. A sequential call option on S can now be valued on the basis of the above integral expression by choosing u=0 and Z to replicate the probability distribution of S in a risk-neutral setting (S^*) at time t_{SF}, and the lower and upper integration limits to represent the range of S^* where the sequential option is exercised:

$$SC(S,t) = e^{-r\,\tau_{SF}} \int_{\overline{S}}^{\infty} \left(C\left(S^*,t\right) - K_{SF}\right) g_{SF}\left(S^*\right) dS^* \,, \tag{25}$$

with $\overline{S}$ the minimum level of S^* required to exercise the sequential option, i.e. where the call option value equals K_{SF}. Appendix B shows in more detail, how to derive the explicit form for equation (25). The sequential option on the arithmetic mean-reverting variable P can now be valued when the expected values and volatilities of P are used instead of those of the arithmetic Brownian motion variable S, based on the relationship between these stochastic processes as outlined before:

$$\begin{aligned} SF(P,t) = {} & e^{-r\tau_F} \left(E\left[P^*\right]_{\tau_F} - K_F \right) N_2\left(d_{\tau_F}, d_{\tau_{SF}}, \rho\right) \\ & + e^{-r\,\tau_F} \nu_{\tau_F}\, \rho\, n\left(d_{\tau_{SF}}\right) N_1\left(\frac{d_{\tau_F} - \rho\, d_{\tau_{SF}}}{\sqrt{1-\rho^2}} \right) \\ & + e^{-r\,\tau_F} \nu_{\tau_F}\, n\left(d_{\tau_F}\right) N_1\left(\frac{d_{\tau_{SF}} - \rho\, d_{\tau_F}}{\sqrt{1-\rho^2}} \right) \\ & - e^{-r\,\tau_{SF}}\, K_{SF}\, N_1\left(d_{\tau_{SF}}\right), \end{aligned} \tag{26}$$

where N_1 is the univariate and N_2 the bivariate cumulative normal distribution, n is the normal density function, and

$$d_{\tau_F} = \frac{E\left[P^*\right]_{\tau_F} - K_F}{\nu_{\tau_F}} \,,$$

$$d_{\tau_{SF}} = \frac{E\left[P^*\right]_{\tau_{SF}} - \overline{P}}{\nu_{\tau_{SF}}} \,,$$

$$E\left[P^*\right]_\tau = P \cdot e^{-\kappa\tau} + \theta\left(1 - e^{-\kappa\tau}\right),$$

$$\nu_\tau^2 = \frac{\sigma^2}{2\kappa}\left(1 - e^{-2\kappa\tau}\right),$$

$$\rho = \frac{\sigma^2}{2\kappa}\left(e^{-\kappa(\tau_F - \tau_{SF})} - e^{-\kappa(\tau_F + \tau_{SF})}\right) \Big/ \left(\nu_{\tau_{SF}}\, \nu_{\tau_F}\right),$$

where $\overline{P}$ is the minimum level of the risk-neutral P required to exercise the sequential option, given by:

$$e^{-r(\tau_F - \tau_{SF})}\left[\left(E\left[\overline{P}^*\right]_{\tau_F - \tau_{SF}} - K_F\right) N_1\left(d_{\tau_F - \tau_{SF}}\right) + \nu_{\tau_F - \tau_{SF}}\; n\left(d_{\tau_F - \tau_{SF}}\right)\right] = K_{SF},$$

$$d_{\tau_F - \tau_{SF}} = \frac{E\left[\overline{P}^*\right]_{\tau_F - \tau_{SF}} - K_F}{\nu_{\tau_F - \tau_{SF}}}.$$

The first term of equation (26) is the discounted value of the risk-neutral expected asset value at final maturity net of exercise cost, multiplied by the probability that both the sequential and the inner option will be exercised. The second and third terms are the value contributions from the variance of the underlying asset to the sequential option and the inner option, respectively. Finally, the last term represents the discounted value of the sequential option exercise price, multiplied by the probability that it will be exercised. The correlation used in equation (26) needs some explanation. Assuming a stochastic process with independent increments, the correlation of its distribution by time τ_1 with the distribution by time τ_2 is given by $\sqrt{\tau_1/\tau_2}$, where $\tau_1<\tau_2$. This is the case in the Geske (1978) compound option model, for instance, where the normalised process of the underlying gBm variable has independent increments. The arithmetic mean-reverting process does not have independent increments, however. Doob (1942) provides the correlation function for this stochastic process between two different points in time.

Two limiting cases can be considered which are when the sequential option matures either immediately or at the same time as the inner option. In the special case of $\tau_{SF}=\tau_F$, the variance and expected value of the underlying asset at maturity of the sequential option are the same as at maturity of the inner option, hence the correlation is perfect ($\rho=1$). With $\overline{P} > K_F$, it follows that $d_{\tau SF}<d_{\tau F}$, so the cumulative normal distribution in the second term in equation (26) is one, and nil in the third term. Furthermore, $N_2(d_{\tau F},d_{\tau SF},\rho)$ can be simplified to $N_1(d_{\tau SF})$ because the bivariate normal distribution has the shape of a univariate normal distribution for $\rho=1$, and the cumulative probability function is then determined by the lower of $d_{\tau SF}$ and $d_{\tau F}$, which is $d_{\tau SF}$. The sequential option then simplifies to a simple call option on the spread with exercise price $(K_{SF}+K_F)$:

$$SF(P,t)=e^{-r\tau_F}\left[\left(E\left[P^*\right]_{\tau_F}-K_F-K_{SF}\right)N_1\left(d_{\tau_{SF}}\right)+\nu_{\tau_F}\, n\left(d_{\tau_{SF}}\right)\right] \qquad (27)$$

In the special case of $\tau_{SF}=0$, the sequential option needs to be exercised either immediately by paying K_{SF} or not at all, so that the value is $\mathrm{Max}[F(P,t)-K_{SF}, 0]$. This case implies $\rho=0$ and $\nu_{\tau SF}=0$. If the initial level of P is lower than $\overline{P}$, $d_{\tau SF}$ is $-\infty$ and all terms in equation (26) vanish which represents the case where the sequential option remains unexercised. If the initial level of P justifies exercising the sequential option, $d_{\tau SF}$ is $+\infty$. As a consequence, $n(d_{\tau SF})=0$, $N_1(d_{\tau SF})=1$ and $N_2(d_{\tau F},d_{\tau SF},0)$ can be simplified to $N_1(d_{\tau F})$. The sequential option for the case of $\tau_{SF}=0$ is therefore given by:

$$SF(P,t)=\mathrm{Max}\left[e^{-r\tau_F}\left[\left(E\left[P^*\right]_{\tau_F}-K_F\right)N_1\left(d_{\tau_F}\right)+\nu_{\tau_F}\, n\left(d_{\tau_F}\right)\right]-K_{SF}, 0\right]. \qquad (28)$$

Comparing our sequential option with the Geske compound option shows that the basic structure of these value functions is comparable, however with different factors in the normal density and cumulative functions and a different correlation coefficient and additional terms in our sequential option representing the value contribution from the variance of the underlying arithmetic variable.

These differences are due to the different underlying stochastic processes, i.e. arithmetic mean-reversion (Ornstein-Uhlenbeck) vs. geometric Brownian motion (gBm). Merton (1973) establishes that a European call option on a non-dividend paying stock following a gBm cannot be greater than the stock itself. The logic is that an American option on this common stock with zero exercise price has the same value as the stock itself, and the American option is at least as valuable as its European counterpart. The assumption of perfect financial markets rules out riskless arbitrage opportunities. If the option value exceeded the underlying stock, one could sell the option, buy the stock and earn a risk-free return on the remaining proceeds. Hence, the value of a call option on a gBm asset cannot exceed the asset value even if there is infinite volatility. The Black-Scholes formula respects this premise by incorporating volatility as a parameter to the cumulative standard normal distribution, so that volatility affects the option value only indirectly through the probability of exercising the option. In contrast, the call option on an asset following an arithmetic Itô process is not bounded by the underlying asset value because the option value is always positive or at least nil while the underlying asset can take negative values. This is represented in the value functions of the simple call option (21) and the sequential call option (26) by the fact that the variance is a direct term instead of appearing only in the probability terms as is the case in the Black-Scholes and Geske formulas on a gBm asset.

5 Finite Difference Solution Framework

The rainbow options presented in the preceding sections have been defined by a partial differential equation and boundary conditions. The finite difference method solves these valuation problems by spanning a grid over the stochastic variable(s) and time and determining the option values by iteration. We apply the implicit finite difference method as outlined in Hull (2006) to value the sequential rainbow options because it ensures convergence. The Visual Basic code for the implementation of the finite difference methods is provided in the Appendix to the Thesis.

5.1 Finite Differences of the Sequential Rainbow Option on Two Assets

Equation (6) is to be transformed into a difference equation. The three-dimensional space of X, Y and time t is divided into small increments or cubes. The time horizon until the expiry date of the sequential option (T_{SF}) is divided into increments of Δt. Similarly, the variables X and Y are divided into increments of ΔX and ΔY between their minimum and maximum values. Minimum and maximum values are chosen in such way that the boundary conditions (9) to (11) are satisfied. Since X and Y are log-normal variables and the increments are constant, the finite difference procedure is more precise and efficient when using the normalised variables $x = \ln X$ and $y = \ln Y$ instead. Applying the specified boundary conditions yields the option value for each point on the boundaries. The option values for the remaining points, which are not described by a boundary condition, are determined by the partial differential equation, starting from the terminal boundary backwards to the present time. For this purpose, the partial differential equation needs to be expressed in terms of the finite difference equivalents. Define $SF_{i,j,k}$ as the value of the sequential option at $x=i\Delta x$, $y=j\Delta y$, $t=k\Delta t$, then Appendix C provides the finite difference expressions for the derivatives which results in the finite difference equation:

$$a_{i,j}SF_{i-1,j,k}+b_{i,j}SF_{i,j-1,k}+c_{i,j}SF_{i,j,k}+d_{i,j}SF_{i+1,j,k}+e_{i,j}SF_{i,j+1,k}$$

$$+f_{i,j}SF_{i+1,j+1,k}+g_{i,j}SF_{i-1,j-1,k}+h_{i,j}SF_{i+1,j-1,k}+m_{i,j}SF_{i-1,j+1,k}=SF_{i,j,k+1} \tag{29}$$

where

$$a_{i,j}=\frac{1}{2}\frac{\left(r-\delta_X-\sigma_X^2/2\right)}{\Delta x}\Delta t-\frac{1}{2}\frac{\sigma_X^2}{\Delta x^2}\Delta t\,,$$

$$b_{i,j}=\frac{1}{2}\frac{\left(r-\delta_Y-\sigma_Y^2/2\right)}{\Delta y}\Delta t-\frac{1}{2}\frac{\sigma_Y^2}{\Delta y^2}\Delta t\,,$$

$$c_{i,j}=1+\frac{\sigma_X^2}{\Delta x^2}\Delta t+\frac{\sigma_Y^2}{\Delta y^2}\Delta t+r\Delta t\,,$$

$$d_{i,j}=-\frac{1}{2}\frac{\sigma_X^2}{\Delta x^2}\Delta t-\frac{1}{2}\frac{\left(r-\delta_X-\sigma_X^2/2\right)}{\Delta x}\Delta t\,,$$

$$e_{i,j}=-\frac{1}{2}\frac{\sigma_Y^2}{\Delta y^2}\Delta t-\frac{1}{2}\frac{\left(r-\delta_Y-\sigma_Y^2/2\right)}{\Delta y}\Delta t\,,$$

$$f_{i,j}=-\frac{\rho\sigma_X\sigma_Y}{4\,\Delta x\,\Delta y}\Delta t\,,$$

$$g_{i,j}=-\frac{\rho\sigma_X\sigma_Y}{4\,\Delta x\,\Delta y}\Delta t\,,$$

$$h_{i,j}=\frac{\rho\sigma_X\sigma_Y}{4\,\Delta x\,\Delta y}\Delta t\,,$$

$$m_{i,j}=\frac{\rho\sigma_X\sigma_Y}{4\,\Delta x\,\Delta y}\Delta t\,.$$

Appendix D describes the detailed procedure to solve this finite difference valuation problem with two stochastic variables.

5.2 *Finite Differences of the Sequential Rainbow Option on the Spread*

The procedure of setting up the finite difference framework for the sequential option on a mean-reverting spread (P) is similar to the one presented above, though simpler because only one stochastic factor needs to be considered. Equation (17) is transformed into the finite difference equation:

$$a_i SF_{i-1,k} + b_i SF_{i,k} + c_i SF_{i+1,k} = SF_{i,k+1} \tag{30}$$

where

$$a_i = \frac{1}{2}\left(\frac{[\eta m - (\mu - r + \eta) i \Delta P]}{\Delta P} - \frac{\sigma^2}{\Delta P^2}\right)\Delta t\,,$$

$$b_i = 1 + \left(\frac{\sigma^2}{\Delta P^2} + r\right)\Delta t\,,$$

$$c_i = -\frac{1}{2}\left(\frac{[\eta m - (\mu - r + \eta) i \Delta P]}{\Delta P} + \frac{\sigma^2}{\Delta P^2}\right)\Delta t\,,$$

and the solution methodology is then applied according to Appendix D.

6 Real Option Sensitivities and Practical Implications

In this section, the two models are tested with regard to their behaviour to changing parameters. Furthermore, the option valuation is compared to the alternative valuation technique Monte Carlo simulation. In a Monte Carlo simulation, random paths of the underlying stochastic variable in its risk-neutral form are generated, the pay-offs at maturity are calculated and discounted at the risk-free interest rate to obtain the present values, which are then averaged over the number of simulation runs. The Visual Basic code for the implementation of the Monte Carlo simulations is also provided in the Appendix to the Thesis.

It should be noted that the two sequential option models are not compared directly because the second model implicitly assumes that one asset is already held when the sequential option becomes available. This sequential option then refers to the spread between the two underlying asset values. In contrast, the holder of a sequential option on two gBm assets might end up with none of the two underlying assets if they are out of the money at maturity. Therefore, a direct link between the two cannot be established. In addition, different exercise prices are chosen for the two models in order to make clear that the investments required to acquire one of the two underlying assets would be different from the investments required to acquire the spread between the two assets.

6.1 Sensitivities of the Sequential Rainbow Option on Two Assets

We apply the sequential rainbow option model on two correlated stochastic assets, as developed in Section 3, with the finite difference solution method from Section 5, to a set of numerical parameters in order to interpret the option behaviour and its sensitivities. With the parameters provided in Table 1, the sequential rainbow option is valued at 21.53 which is only about 0.2% less than the value indicated by Monte Carlo simulation (21.58). When the number of intervals in the finite difference grid is increased, the valuation result approaches the Monte Carlo simulation result. When assuming $K_{SF} = 0$, the sequential rainbow option simplifies to a simple rainbow option which is valued at 51.37. If the exercise price of the sequential option was not optional, the NPV would be 14.26 which is the rainbow option value less the discounted K_{SF} of 40. Hence, the additional value from the sequential option is about 51% (21.53 vs. 14.26).

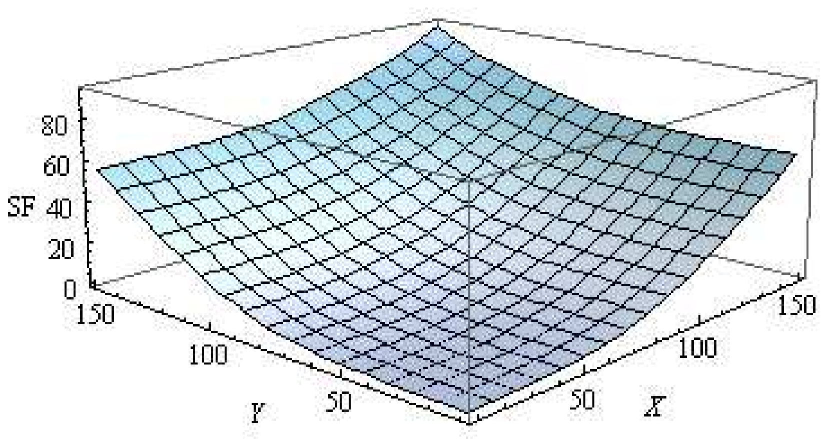
Table 1. Parameters and Valuation Results for the Sequential Rainbow Option on Two Assets

Parameters		
Value of asset X	X	70
Value of asset Y	Y	100
Volatility of X	σ_X	0.40
Volatility of Y	σ_Y	0.30
Convenience yield of asset X	δ_X	0.00
Convenience yield of asset Y	δ_Y	0.00
Correlation between X and Y	ρ	0.50
Risk-free interest rate	r	0.05
Time to maturity of the sequential option	τ_{SF}	1.5
Time to maturity of the rainbow option	τ_F	3.0
Exercise price for the sequential option	K_{SF}	40
Exercise price for the rainbow option	K_F	70
Rainbow Option (given by K_{SF}=0)	**F**	**51.37**
Finite Difference Solution		
Maximum value of X	X_{max}	400
Maximum value of Y	Y_{max}	400
Nr of increments in X	$X_{max}/\Delta X$	20
Nr of increments in Y	$Y_{max}/\Delta Y$	20
Nr of time increments	$\tau_{SF}/\Delta T$	5
Sequential Rainbow Option	**SF**	**21.53**
Monte Carlo Solution		
Nr of simulations	N_{Sim}	50000
Sequential Rainbow Option	**SF**	**21.58**

Rainbow Option value is obtained from equation (8). Finite difference solution of Sequential Rainbow Option is based on equation (29) and boundary conditions (7), (9), (10), (11).

While the stochastic assets X and Y are assumed to take the initial values of 70 and 100 respectively according to Table 1, they are varied in Table 2 to obtain the option values for different combinations of X and Y. Figure 4 illustrates the sequential option value surface as a function of X and Y. The value surface is in line with expectations from real options theory because the option value increases with both X and Y, the shape is convex in the area of being at-the money and more linear in the area of being far in-the-money. The option value drops to zero only when both assets are worthless.

Figure 4. Sequential Rainbow Option on Two Assets as a Function of these Assets



Value surface of sequential rainbow option on stochastic assets X and Y. Implicit finite difference solution is based on equation (29) and boundary conditions (7), (9), (10), (11), with Maximum value of X: X_{max} = 400; Maximum value of Y: Y_{max} = 400; Finite difference grid set up on ln(X) and ln(Y) with 20 intervals for each stochastic variable and 5 time intervals; Linear interpolation between grid points.
Option parameters: Time to maturity of the sequential option: τ_{SF} = 1.5 years; Time to maturity of the rainbow option: τ_F = 3.0 years; Exercise price for the sequential option: K_{SF} = 40; Exercise price for the rainbow option: K_F = 70; Volatility of X: σ_X = 0.40 p.a.; Volatility of Y: σ_Y = 0.30 p.a.; Convenience yield of X: δ_X = 0.00; Convenience yield of Y: δ_Y = 0.00; Correlation between X and Y: ρ = 0.50; Risk-free interest rate: r = 0.05 p.a.

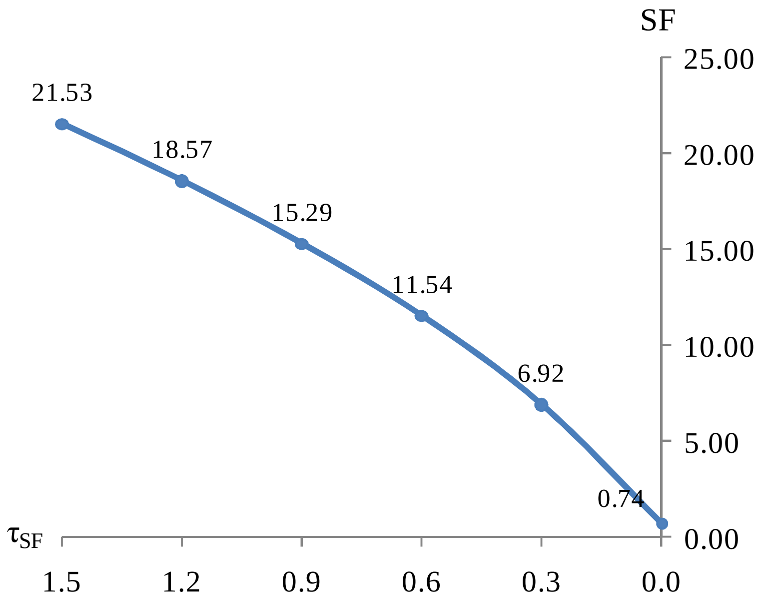
Table 2. Sequential Rainbow Option on Two Assets as a Function of these Assets

Y \ X	10	20	30	40	50	60	70	80	90	100	110	120	130	140	150
10	0.00	0.03	0.20	0.64	1.64	3.56	6.49	9.69	15.19	20.11	27.65	34.70	42.29	51.33	59.74
20	0.00	0.03	0.20	0.65	1.65	3.57	6.49	9.70	15.19	20.10	27.65	34.69	42.29	51.32	59.73
30	0.04	0.07	0.25	0.70	1.71	3.62	6.55	9.75	15.24	20.15	27.69	34.73	42.31	51.34	59.75
40	0.18	0.22	0.40	0.87	1.90	3.83	6.75	9.95	15.41	20.30	27.82	34.84	42.41	51.43	59.82
50	0.60	0.65	0.84	1.35	2.37	4.33	7.29	10.49	15.96	20.85	28.30	35.32	42.76	51.77	60.16
60	1.67	1.72	1.92	2.44	3.51	5.49	8.47	11.67	17.10	21.96	29.32	36.29	43.56	52.53	60.88
70	3.69	3.73	3.94	4.46	5.56	7.56	10.51	13.67	19.04	23.83	31.02	37.86	44.96	53.80	62.04
80	6.01	6.05	6.26	6.77	7.88	9.87	12.78	15.93	21.12	25.77	32.80	39.37	46.50	55.05	63.02
90	10.92	10.93	11.12	11.58	12.69	14.61	17.40	20.32	25.51	30.16	36.75	43.32	49.70	58.26	66.22
100	15.31	15.29	15.47	15.89	16.99	18.86	21.53	24.24	29.44	34.08	40.28	46.85	52.57	61.12	69.09
110	22.97	22.97	23.12	23.48	24.46	26.15	28.58	31.03	35.78	40.03	45.78	51.80	57.37	65.35	72.78
120	30.17	30.16	30.30	30.59	31.57	33.15	35.35	37.38	42.13	46.38	51.45	57.47	62.07	70.05	77.48
130	38.01	38.00	38.10	38.37	39.06	40.36	42.32	44.40	48.23	51.65	56.79	61.84	67.06	74.04	80.53
140	47.47	47.46	47.55	47.77	48.46	49.66	51.40	53.05	56.88	60.30	64.71	69.76	73.89	80.86	87.36
150	56.27	56.28	56.36	56.52	57.21	58.31	59.85	61.10	64.93	68.36	72.08	77.13	80.24	87.22	93.71

Value surface of sequential rainbow option on stochastic assets X and Y. Implicit finite difference solution is based on equation (29) and boundary conditions (7), (9), (10), (11), with Maximum value of X: $X_{max} = 400$; Maximum value of Y: $Y_{max} = 400$; Finite difference grid set up on ln(X) and ln(Y) with 20 intervals for each stochastic variable and 5 time intervals; Linear interpolation between grid points.
Option parameters: Time to maturity of the sequential option: $\tau_{SF} = 1.5$ years; Time to maturity of the rainbow option: $\tau_F = 3.0$ years; Exercise price for the sequential option: $K_{SF} = 40$; Exercise price for the rainbow option: $K_F = 70$; Volatility of X: $\sigma_X = 0.40$ p.a.; Volatility of Y: $\sigma_Y = 0.30$ p.a.; Convenience yield of X: $\delta_X = 0.00$; Convenience yield of Y: $\delta_Y = 0.00$; Correlation between X and Y: $\rho = 0.50$; Risk-free interest rate: $r = 0.05$ p.a.

An option on a stochastic asset with a yield of less than the risk-free rate and fixed exercise price will be the more valuable, the longer the remaining time to maturity, as shown in Figure 5. At maturity, the sequential rainbow option (SF) is the maximum of the rainbow option (F = 40.74) less exercise price of the sequential option (K_{SF} = 40) and zero.

Figure 5. Sequential Rainbow Option on Two Assets as a Function of the Maturity



Sequential rainbow option value (SF) as a function of the time to maturity of the sequential option (τ_{SF}) in years. Implicit finite difference solution is based on equation (29) and boundary conditions (7), (9), (10), (11), with Maximum value of X: X_{max} = 400; Maximum value of Y: Y_{max} = 400; Finite difference grid set up on ln(X) and ln(Y) with 20 intervals for each stochastic variable and 5 time intervals; Linear interpolation between grid points.
Option parameters: Current asset value X = 70; Current asset value Y = 100; Time between maturities of the sequential option and the rainbow option: τ_F-τ_{SF} = 1.5 years; Exercise price for the sequential option: K_{SF} = 40; Exercise price for the rainbow option: K_F = 70; Volatility of X: σ_X = 0.40 p.a.; Volatility of Y: σ_Y = 0.30 p.a.; Convenience yield of X: δ_X = 0.00; Convenience yield of Y: δ_Y = 0.00; Correlation between X and Y: ρ = 0.50; Risk-free interest rate: r = 0.05 p.a.

The sensitivity of the option to the volatilities of the underlying assets exhibits two phenomena, according to the sensitivity analysis in Table 3.

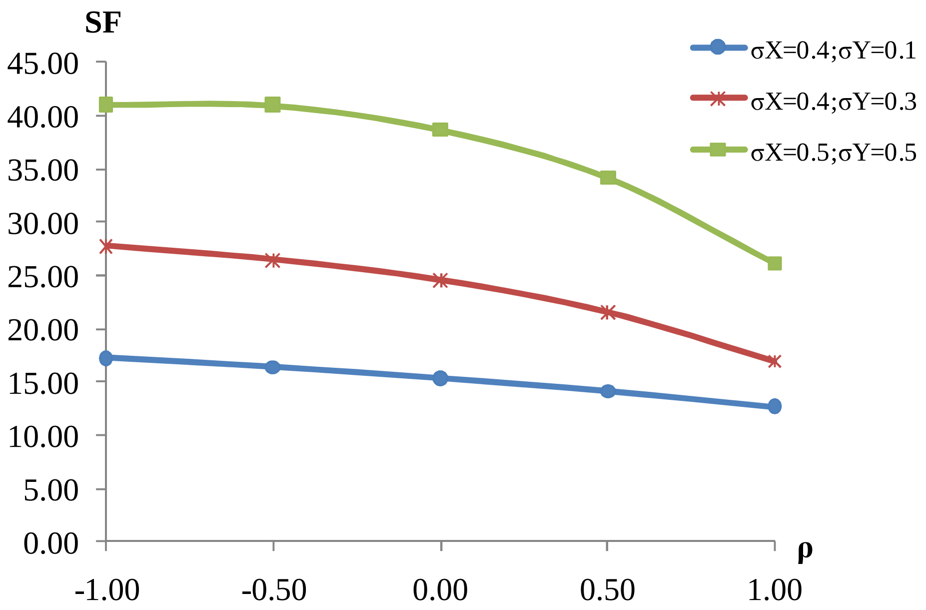
Table 3. Sequential Rainbow Option on Two Assets as a Function of Volatilities

σ_Y \ σ_X	0.00	0.05	0.10	0.20	0.30	0.40	0.50
0.00	4.10	4.16	4.39	5.96	9.09	13.41	18.07
0.05	4.69	4.74	4.93	6.34	9.15	13.35	17.86
0.10	6.26	6.29	6.47	7.71	10.27	14.16	18.48
0.20	10.79	10.82	10.98	11.95	14.04	17.36	21.25
0.30	15.83	15.78	15.94	16.83	18.73	21.53	25.04
0.40	21.18	21.15	21.28	22.14	23.83	26.30	29.39
0.50	26.59	26.59	26.76	27.64	29.21	31.42	34.17

Sequential rainbow option values for different volatilities (σ) of the stochastic assets X and Y. Implicit finite difference solution is based on equation (29) and boundary conditions (7), (9), (10), (11), with Maximum value of X: X_{max} = 400; Maximum value of Y: Y_{max} = 400; Finite difference grid set up on ln(X) and ln(Y) with 20 intervals for each stochastic variable and 5 time intervals; Linear interpolation between grid points.
Option parameters: Current asset value X = 70; Current asset value Y = 100; Time to maturity of the sequential option: τ_{SF} = 1.5 years; Time to maturity of the rainbow option: τ_F = 3.0 years; Exercise price for the sequential option: K_{SF} = 40; Exercise price for the rainbow option: K_F = 70; Convenience yield of X: δ_X = 0.00; Convenience yield of Y: δ_Y = 0.00; Correlation between X and Y: ρ = 0.50; Risk-free interest rate: r = 0.05 p.a.

The sequential rainbow option behaves as a typical call option, i.e. the option value increases with volatility. However, there is one exception when the volatility of Y is low and the volatility of X is high. For $\sigma_X = 0.50$, the option value decreases when σ_Y increases from 0 to 0.05, and increases for $\sigma_Y \geq 0.10$. This seems to be counter-intuitive at first glance. The initial values of X and Y are 70 and 100 respectively. The exercise prices of the sequential option and of the rainbow option are 40 and 70, respectively. With yields of zero and assuming the volatility of both assets is zero, the NPV is the higher of X and Y less the discounted exercise prices. Y is expected to be slightly in-the-money at maturity with an NPV of 2.64. While a higher volatility of X only increases the upside potential, a higher volatility of Y means upside and downside potential at the same time because a positive NPV might be increased or lost altogether. Three factors determine whether an increase in the volatility of Y further increases or decreases the option value. The first factor is whether Y is expected to be in-the-money or out-of-the-money at maturity. If it is in-the-money, a low volatility might be beneficial in case the other (more volatile) asset drops below the exercise trigger. If it is currently out-of-the-money, there is not much to lose, so high volatility is favourable. The second factor is the correlation between the underlying assets.

Figure 6. Sequential Rainbow Option on Two Assets as a Function of the Correlation



ρ	-1.00	-0.50	0.00	0.50	1.00
σ_X=0.4;σ_Y=0.1	17.26	16.39	15.35	14.16	12.67
σ_X=0.4;σ_Y=0.3	27.72	26.44	24.53	21.53	16.95
σ_X=0.5;σ_Y=0.5	41.07	40.98	38.68	34.17	26.10

Sequential rainbow option values as a function of the correlation between the stochastic assets X and Y. Implicit finite difference solution is based on equation (29) and boundary conditions (7), (9), (10), (11), with Maximum value of X: X_{max} = 400; Maximum value of Y: Y_{max} = 400; Finite difference grid set up on ln(X) and ln(Y) with 20 intervals for each stochastic variable and 5 time intervals; Linear interpolation between grid points.
Option parameters: Current asset value X = 70; Current asset value Y = 100; Time to maturity of the sequential option: τ_{SF} = 1.5 years; Time to maturity of the rainbow option: τ_F = 3.0 years; Exercise price for the sequential option: K_{SF} = 40; Exercise price for the rainbow option: K_F = 70; Convenience yield of X: δ_X = 0.00; Convenience yield of Y: δ_Y = 0.00; Risk-free interest rate: r = 0.05 p.a.

If X and Y are highly correlated and only Y is in-the-money, a low volatility of Y tends to be preferable in order to ensure a positive pay-off. When the correlation is low or negative, however, high volatility tends to be desirable for both assets because one of the two would probably increase. This is also evident from Figure 6, showing that the option value is significantly higher for high volatilities unless the correlation is high. The third factor is the magnitude of the volatility of Y. As soon as the volatility of Y surpasses a certain level, its effect is again positive, except for almost perfect correlation.

Considering the practical examples described in the introductory section, the above results allow to value the opportunity to bid for a long-term take-or-pay gas supply contract, which is scheduled for auction, in order to use the gas to produce fertiliser (ammonia or urea). This valuation is relevant for instance, if the local investor (opportunity holder) needs to raise money or intends to sell shares. In the case, where a farmer has the option to prepare land in the autumn season in order to grow wheat or maize in the following spring, our preceding analysis suggests that high volatilities of both wheat and maize are generally favourable to the option value but that a low volatility of one crop (say wheat) is desirable if the expected value of only that crop is in the money and the correlation with the other crop (maize) is high. If the circumstances allow, this insight might even induce the decision-maker to consider other alternative crops with more suitable combinations of volatilities, correlation and expected values, thereby creating more valuable options.

6.2 Sensitivities of the Sequential Rainbow Option on the Spread

The sequential option on the spread between the two underlying assets, as developed in Section 4, is now valued using the parameter values in Table 4.

Table 4. Parameters and Valuation Results for the Sequential Rainbow Option on the Spread

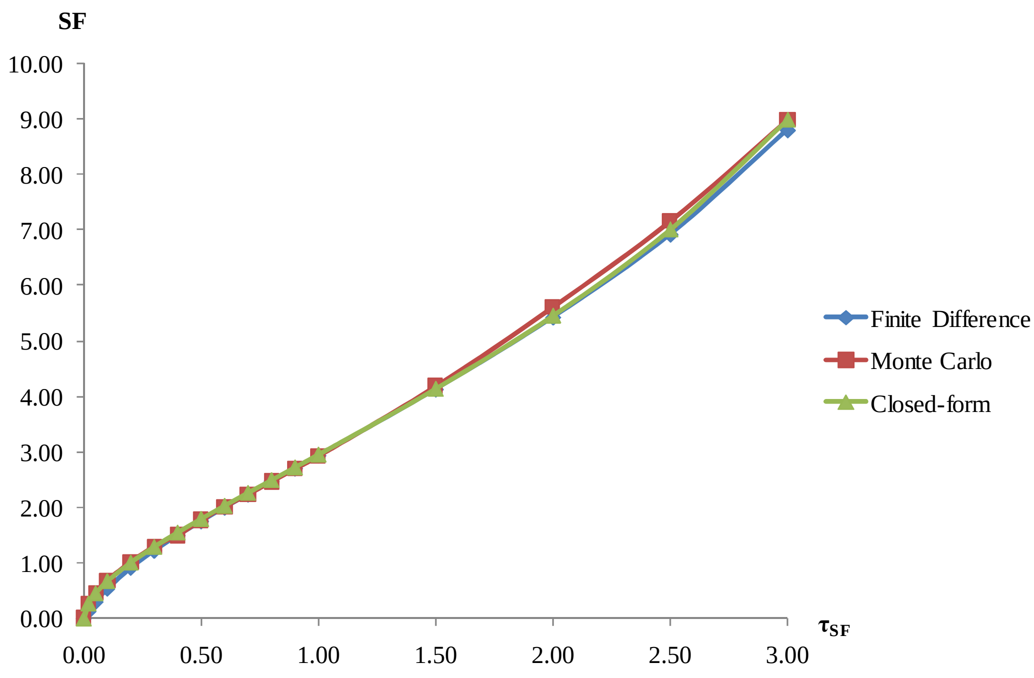
Parameters		
Spread between Y and X	P	30
Long-run mean of P	m	50
Volatility of P	σ	25
Speed of mean-reversion of P	η	0.35
Required return on P	μ	0.10
Risk-free interest rate	r	0.05
Time to maturity of the sequential option	τ_{SF}	1.5
Time to maturity of the inner (rainbow) option	τ_F	3.0
Exercise price for the sequential option	K_{SF}	20
Exercise price for the rainbow option	K_F	20
Rainbow Option (given by K_{SF}=0)	**F**	**19.96**
Finite Difference Solution		
Maximum value of P	P_{max}	1000
Minimum value of P	P_{min}	-1000
Nr of increments in P	$P_{max}/\Delta P$	200
Nr of time increments	$\tau_{SF}/\Delta T$	30
Sequential Rainbow Option	**SF**	**4.13**
Monte Carlo Solution		
Nr of time increments	$\tau_{SF}/\Delta T$	100
Nr of simulations	N_{Sim}	6000
Sequential Rainbow Option	**SF**	**4.18**
Closed-form Solution		
Risk-neutral P at T_{SF}	$E[P]_{\tau SF}$	36.20
Risk-neutral P at T_F	$E[P]_{\tau F}$	39.61
Critical P for exercising the sequential option	P_{crit}	34.67
Risk-neutral P_{crit} at (T_F-T_{SF})	$E[P_{crit}]_{\tau F-\tau SF}$	38.76
Volatility of P by T_{SF}	$v_{\tau SF}$	23.37
Volatility of P by T_F	$v_{\tau F}$	26.65
Volatility of P by (T_F-T_{SF})	$v_{\tau F-\tau SF}$	23.37
	$d_{\tau SF}$	0.07
	$d_{\tau F}$	0.74
	$d_{\tau F-\tau SF}$	0.80
Correlation	ρ	0.48
	$n(d_{\tau SF})$	0.40
	$n(d_{\tau F})$	0.30
Equation determining P_{crit}	$F(P_{crit},T_F-T_{SF})-K_{SF}$ =	0.00
Sequential Rainbow Option	**SF**	**4.14**

Closed-form solution is obtained from equation (26). Finite difference solution is based on equation (30) and boundary conditions (18), (22), (23).

With these parameters, the sequential option is valued at 4.13 according to the finite difference procedure, at 4.18 according to a Monte Carlo simulation and at 4.14 based on the closed-form solution. Hence, both approximating methods, finite differences and Monte Carlo simulation, deviate by less than 1% from the exact solution. Comparing these values with the NPV of 1.41 – given by the value of the rainbow option (19.96) less the discounted exercise price of the sequential option (K_{SF}=20) – shows that the sequential option adds significant value here.

Figure 7 keeps the time to maturity of the inner option fixed (τ_{SF} = 3.0 years) and varies the time to maturity of the sequential option (τ_{SF}).

Figure 7. Sequential Rainbow Option on the Spread as a function of the Time to Maturity



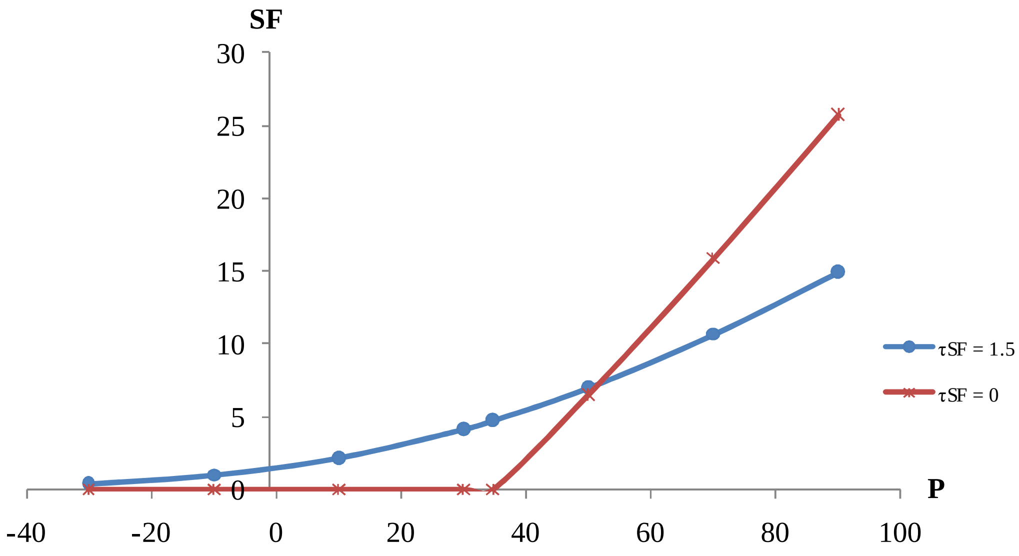
Sequential rainbow option value (SF) as a function of the time to maturity of the sequential option (τ_{SF}). Closed-form solution is obtained from equation (26). Implicit finite difference solution is based on equation (30) and boundary conditions (18), (22), (23), with Maximum value of P: P_{max} = 1000; Minimum value of P: P_{min} = -1000; Finite difference grid divided into 200 intervals in P and 30 time intervals; Linear interpolation between grid points.
Option parameters: Current spread: P = 30; Time to maturity of the inner option: τ_F = 3.0 years; Exercise price for the sequential option: K_{SF} = 20; Exercise price for the rainbow

option: K_F = 20; Long-run mean of P: m = 50; Standard deviation of P: σ = 25 p.a.; Speed of mean-reversion: η = 0.35; Required return on P: μ = 0.10; Risk-free interest rate: r = 0.05 p.a.

First of all, Figure 7 confirms our conclusion that the three solution methods provide near-identical results. If the exercise price K_{SF} is due immediately (τ_{SF}=0), one would leave the option unexercised because the rainbow option value is less than K_{SF}. The sequential option value increases with the time to maturity, due to several effects. First and foremost, the variance increases with the time to maturity τ_{SF} and thereby increases the option value. More precisely, the variance relevant for the exercise of the sequential option is $\nu_{\tau F}$ times the correlation ρ, as can be seen in the second term in equation (26). While $\nu_{\tau F}$ is independent of τ_{SF}, ρ increases with τ_{SF} in a non-monotonous way for the specified parameters. Secondly, the discounted exercise price K_{SF} decreases with τ_{SF}. And thirdly, the probability that both sequential option and inner option are exercised increases with a longer time to maturity of the sequential option because the probability distributions of P at τ_{SF} and of P at τ_F are then more closely correlated. While these effects depend in their magnitude on the parameters, especially the speed of mean-reversion and the initial level of P, they are all positively correlated with τ_{SF}. Hence, when the time to maturity of the inner option (τ_F) is fixed, the sequential option is the more valuable the later the sequential option expires.

The following analysis discusses the effect of the passage of time on the sequential option value. For this purpose, both maturity dates are fixed (τ_F-τ_{SF} = 1.5 years) and we consider the case where there is still time left until expiry of the sequential option (τ_{SF}=1.5) and the case where the sequential option expires immediately (τ_{SF}=0).

Figure 8. Sequential Rainbow Option on the Spread as a Function of the Spread (P)

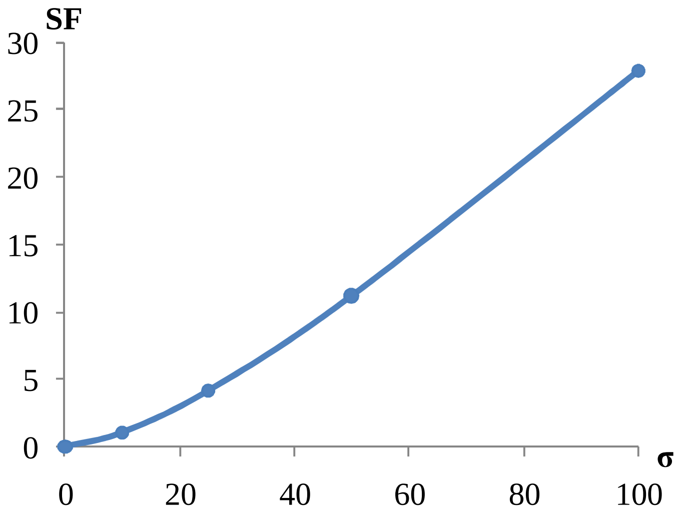


P / τ_{SF}	-30.00	-10.00	10.00	30.00	34.70	50.00	70.00	90.00
τ_{SF} = 1.5	0.39	0.99	2.17	4.14	4.73	6.99	10.66	14.94
τ_{SF} = 0	0.00	0.00	0.00	0.00	0.00	6.53	15.87	25.72

Sequential rainbow option value (SF) as a function of the initial level of the stochastic spread (P) for different times until expiry of the sequential option (τ_{SF}) in years. Option values are obtained from equation (26).
Option parameters: Time between maturities of the rainbow option and the sequential option: τ_F-τ_{SF} = 1.5 years; Exercise price for the sequential option: K_{SF} = 20; Exercise price for the rainbow option: K_F = 20; Long-run mean of P: m = 50; Standard deviation of P: σ = 25 p.a.; Speed of mean-reversion: η = 0.35; Required return on P: μ = 0.10; Risk-free interest rate: r = 0.05 p.a.

Figure 8 illustrates the sequential option value as a function of the initial spread level (P) for these two cases. While the functions take the general form expected for an option, with zero as the lower boundary for negative spread values and option values increasing in P, the slope is always significantly less than one. The reason for this is the mean-reverting property of the spread, which pulls the spread towards the long-run mean (m=50). The general intuition is that the option value is worth more, the more time is left until expiry. In our case, we need to consider three factors. First, the spread level tends towards its long-run mean. Hence, an initial spread higher than the long-run mean is pulled down while an initial spread lower than the long-run mean is expected to increase. Second, the variance until maturity increases the option value. Third, the more time is left until expiry, the lower the present value of the exercise price. The combination of these factors results in the effect that for low initial values of P, it is preferable to have more time left until expiry, while immediate expiry is preferable for high values of P. This is why the value functions for maturities of τ_{SF}=1.5 and τ_{SF}=0 intersect, in our case at P=51.

Figure 9. Sequential Rainbow Option on the Spread as a function of the Standard Deviation



σ	0	10	25	50	100
	0.00	1.05	4.14	11.18	27.90

Sequential rainbow option value (SF) as a function of the spread standard deviation (σ). Option values are obtained from equation (26).
Option parameters: Current spread: P = 30; Time to maturity of the sequential option: τ_{SF} = 1.5 years; Time to maturity of the inner option: τ_F = 3.0 years; Exercise price for the sequential option: K_{SF} = 20; Exercise price for the rainbow option: K_F = 20; Long-run mean of P: m = 50; Speed of mean-reversion: η = 0.35; Required return on P: μ = 0.10; Risk-free interest rate: r = 0.05 p.a.

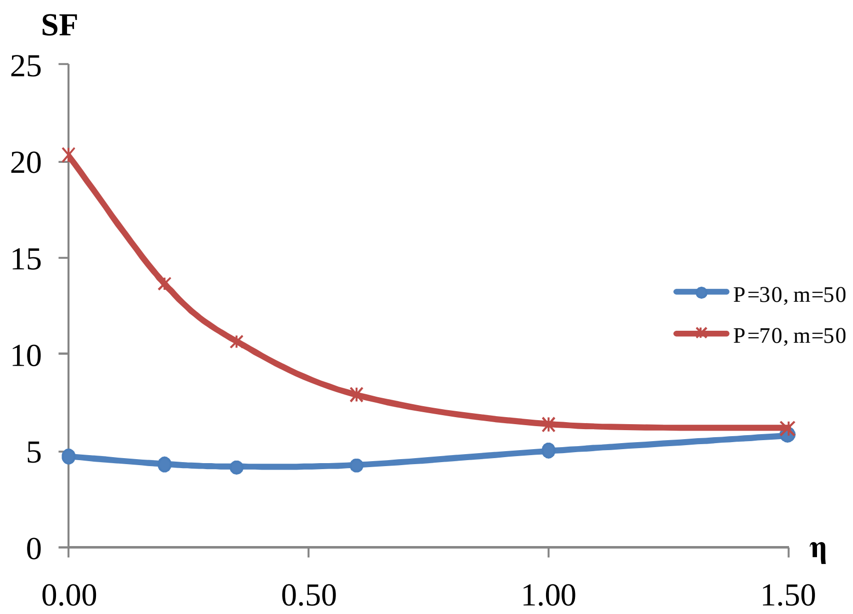
The magnitude of the value obtained from the option features can be assessed by comparing the option value with the case of zero standard deviation (σ=0), where the sequential option simplifies to the maximum of zero and expected value of the risk-neutral spread at final maturity less exercise prices, discounted with the applicable time periods, as shown below:

$$SF(P, \sigma = 0) = \text{Max}\left[e^{-r\,\tau_F}\left(E[P^*]_{\tau_F} - K_F\right) - e^{-r\,\tau_{SF}} K_{SF}, 0\right] \tag{31}$$

For the specified parameters, the option with zero standard deviation is then Max[-1.68,0]=0, which is 4.14 less than the option value with σ=25. Figure 9 illustrates the sequential option value as a function of the standard deviation and shows that it is a convex function, approaching linearity when the standard deviation is very high.

Next, we investigate the influence of the speed of mean-reversion on the option value. Figure 10 shows that mean-reversion can either increase or decrease the value of a European sequential option depending on the current level of the underlying asset. Assuming zero mean-reversion, the option value highly depends on the initial spread level (P). For P=70, the option value is 20.31 while it is only 4.69 for P=30. When introducing mean-reversion, two effects come into play. First, the expected spread is pulled towards its long-run mean with a positive speed, and second, the variance is reduced. Hence, the option value decreases with the speed of mean-reversion if the current spread is above the long-run (risk-neutral) mean. If the spread is below the long-run mean, the total effect on the sequential option value depends on the net effect from increased option value by an expected increase in P and a reduced option value due to lower variance. It is interesting to see that this net effect is negative for a range of $0 < \eta < 0.50$ and positive for $\eta > 0.50$ when an initial spread of 30 is assumed. That means, variance reduction is an important factor when mean-reversion is introduced to a random walk process, while the effect of further increases in the speed of mean-reversion on the option value is dominated by the expected increase in P. For fast mean-reversion, the option values based on different initial spread levels converge since the initial difference from the long-run mean is almost immediately absorbed.

Figure 10. Sequential Rainbow Option on the Spread as a Function of the Mean-Reversion Speed

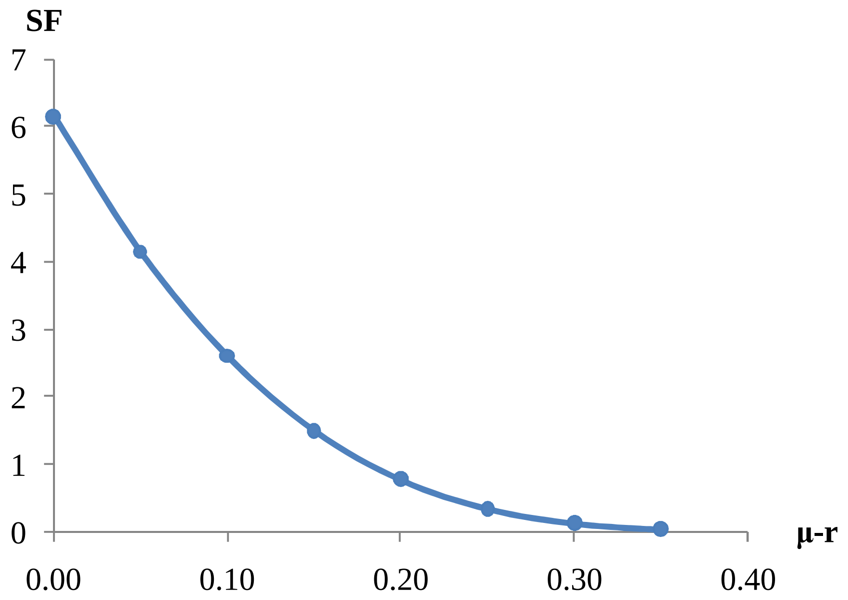


η	0.00	0.20	0.35	0.60	1.00	1.50
P=30, m=50	4.69	4.27	4.14	4.23	4.99	5.83
P=70, m=50	20.31	13.65	10.66	7.87	6.35	6.16

Sequential rainbow option value (SF) as a function of the mean-reversion speed (η) of the spread for different initial spread levels (P). Option values are obtained from equation (26). Option parameters: Time to maturity of the sequential option: τ_{SF} = 1.5 years; Time to maturity of the inner option: τ_F = 3.0 years; Exercise price for the sequential option: K_{SF} = 20; Exercise price for the rainbow option: K_F = 20; Long-run mean of P: m = 50; Standard deviation of P: σ = 25 p.a.; Required return on P: μ = 0.10; Risk-free interest rate: r = 0.05 p.a.

Figure 11 illustrates that the sequential option value is a decreasing function in the risk-premium (μ-r). It should be noted that changes in the risk-premium are not linked to changes the in instantaneous standard deviation (σ) in this analysis. In a risk-neutral setting, a higher risk-premium reduces the variance and the long-run mean while increasing the speed of mean-reversion, as can be seen from equation (21). The cumulative effect is that the option value converges to zero for a very high risk-premium and finds its maximum value when the risk-premium is zero.

Figure 11. Sequential Rainbow Option on the Spread as a Function of the Risk-Premium



μ-r	0.00	0.05	0.10	0.15	0.20	0.25	0.30	0.35
	6.15	4.14	2.60	1.50	0.77	0.34	0.13	0.04

Sequential rainbow option value (SF) as a function of the risk premium (μ-r). Option values are obtained from equation (26).
Option parameters: Current spread: P = 30; Time to maturity of the sequential option: τ_{SF} = 1.5 years; Time to maturity of the inner option: τ_F = 3.0 years; Exercise price for the sequential option: K_{SF} = 20; Exercise price for the rainbow option: K_F = 20; Long-run mean of P: m = 50; Standard deviation of P: σ = 25 p.a.; Speed of mean-reversion: η = 0.35; Risk-free interest rate: r = 0.05 p.a.

Some of the findings above are best discussed in the context of a practical example. An ethylene producer considering building a plant to process ethylene into polyethylene would try to accelerate the intermediate investment (acquisition and preparation of land, permissions) if the current level of the spread between polyethylene and ethylene is higher than its long-run mean, so that the investment is already productive when the spread is still above-average. On the other hand, if the current level of the spread is below the long-run mean, the investor might try to delay the investment. The value of this investment opportunity also depends on the speed of mean-reversion. Strong mean-reversion in the spread reduces the variance until maturity and thereby decreases the option value. However, if the current level of the spread is low, mean-reversion can be positive because the spread approaches the long-run mean faster. Hence, the current level of the spread compared to its long-run mean, the speed of mean-reversion and the time to maturity are important factors to be considered in the assessment and design of this kind of sequential option.

7 Conclusion

We have developed a model for a European sequential rainbow option on the best of two stochastic assets following geometric Brownian motion processes, and another model for a European sequential rainbow option on the mean-reverting spread between two co-integrated assets. The real option valuation is developed for each model and numerical solutions are provided based on the finite difference method and compared to Monte Carlo simulation. For the sequential option on a mean-reverting spread, we develop a closed-form solution. Both option models are tested extensively for various sensitivities, providing important insights into the value behaviour. We find the interesting result, that in particular circumstances, the value of the sequential rainbow option on two assets is negatively correlated with the volatility of one of the two assets. Also, the sequential option on the mean-reverting spread does not necessarily increase in value with a longer time to maturity. With given maturity dates, it is preferable to have less time until expiry of the sequential option if the current spread level is high enough above the long-run mean. This is exemplified by an application from the commodity industry. Assuming the commodities ethylene and polyethylene are co-integrated, a producer of ethylene, considering investing into a facility to convert ethylene into polyethylene, would value a longer time to maturity of the investment opportunity if the price spread between the two products is below its long-run mean and therefore expected to increase. However, if the spread is significantly above its long-run mean, the value of waiting is lower than the lost profits of delaying the investment. Further research might focus on extending the new sequential option models to American (finite) options because real investment opportunities can often be exercised over a period of time instead of at a fixed maturity date only.

Appendix A. Proof of Arithmetic Brownian Motion as Special Case of Arithmetic Mean-Reversion

The arithmetic mean-reverting process P is defined together with its expected value and variance by equations (12) - (14), and restated below:

$$dP = \eta(m - P)\,dt + \sigma\,dz\,,$$

$$E[P]_T = P \cdot e^{-\eta T} + m\left(1 - e^{-\eta T}\right),$$

$$\nu_T^2 = \frac{\sigma^2}{2\eta}\left(1 - e^{-2\eta T}\right).$$

Choosing $\eta = -\mu$ and $m = 0$ provides the stochastic process below, which is the arithmetic Brownian motion with proportional drift:

$$dP = \mu\,P\,dt + \sigma\,dz\,,$$

$$E[P]_T = P \cdot e^{\mu T}\,,$$

$$\nu_T^2 = \frac{\sigma^2}{2\mu}\left(e^{2\mu T} - 1\right).$$

The relationship between arithmetic mean-reversion and arithmetic Brownian motion with proportional drift can also be demonstrated with the risk-neutral stochastic processes. The risk-neutral form of the former is:

$$dP^* = \kappa(\theta - P^*)\,dt + \sigma\,dz\,,$$

$$E[P^*]_T = P^* \cdot e^{-\kappa T} + \theta\left(1 - e^{-\kappa T}\right),$$

$$\nu_T^2 = \frac{\sigma^2}{2\kappa}\left(1 - e^{-2\kappa T}\right),$$

where $\kappa = \eta + \mu - r$ and $\theta = \dfrac{\eta}{\eta + \mu - r}m$.

Choosing $\eta = -\mu$ and $m = 0$ implies $\kappa = -r$ and $\theta = 0$, so that the risk-neutral stochastic process of the arithmetic Brownian motion with proportional drift becomes:

$$dP^* = r\,P^*dt + \sigma dz\,,$$

$$E\left[P^*\right]_T = P^* \cdot e^{rT}\,,$$

$$\nu_T^2 = \frac{\sigma^2}{2r}\left(e^{2rT} - 1\right).$$

Appendix B. Derivation of the Closed-Form Solution for the Sequential Option on the Mean-Reverting Spread

As defined in the main text, $C_Z(S)$ is a call option on S with exercise price uZ+K, where both S and Z follow arithmetic Brownian motion, correlated by ρ. Childs et al. (1998) derive the following:

$$\int_{Z=a}^{b} C_Z(S,t)g(Z)dZ = \nu_{SZ}(u)e^{-r\tau}\{-h_{SZ}(K,u)[N_2(-h_{SZ}(K,u),h_Z(b),\alpha_{SZ}(u))$$

$$-N_2(-h_{SZ}(K,u),h_Z(a),\alpha_{SZ})]$$

$$-\alpha_{SZ}(u)\left[n(h_Z(a))N_1\left(\frac{-h_{SZ}(K,u)-\alpha_{SZ}(u)h_Z(a)}{\sqrt{1-\alpha_{SZ}{}^2(u)}}\right)\right.$$

$$\left.-n(h_Z(b))N_1\left(\frac{-h_{SZ}(K,u)-\alpha_{SZ}(u)h_Z(b)}{\sqrt{1-\alpha_{SZ}{}^2(u)}}\right)\right]$$

$$+n(-h_{SZ}(K,u))\left[N_1\left(\frac{h_Z(b)+\alpha_{SZ}(u)h_{SZ}(K,u)}{\sqrt{1-\alpha_{SZ}{}^2(u)}}\right)\right.$$

$$\left.\left.-\left[N_1\left(\frac{h_Z(a)+\alpha_{SZ}(u)h_{SZ}(K,u)}{\sqrt{1-\alpha_{SZ}{}^2(u)}}\right)\right]\right]\right\}$$

where

$$h_Z(a)=\frac{a-E[Z]^*}{\nu_Z} \text{ and } h_Z(b)=\frac{b-E[Z]^*}{\nu_Z},$$

$$h_{SZ}(K,u)=\frac{K-E[S]^*+u\,E[Z]^*}{\nu_{SZ}(u)},$$

$$\nu_{SZ}(u)=\sqrt{\nu_S^2-2u\,\rho\,\nu_Z\,\nu_S+u^2\,\nu_Z^2},$$

$$\alpha_{SZ}(u)=-\frac{\rho\,\nu_S-u\,\sigma_Z}{\nu_{SZ}}.$$

We now choose u=0, a=$\overline{Z}$ and b=∞, which simplifies the expressions above to $h_Z(\overline{Z}) = \frac{\overline{Z} - E[Z]^*}{\nu_Z}$, $h_Z(\infty) = \frac{\infty - E[Z]^*}{\nu_Z} = \infty$, $h_{SZ}(K, u=0) = \frac{K - E[S]^*}{\nu_S} = h_S(K)$ and:

$$\int_{Z=\overline{Z}}^{\infty} C_Z(S,t)\, g(Z)\, dZ = \nu_S\, e^{-r\tau} \{ -h_S(K) [N_2(-h_S(K), \infty, -\rho) - N_2(-h_S(K), h_Z(\overline{Z}), -\rho)]$$

$$+ \rho \left[n(h_Z(\overline{Z})) N_1\left(\frac{-h_S(K) + \rho\, h_Z(\overline{Z})}{\sqrt{1-\rho^2}} \right) \right]$$

$$+ n(-h_S(K)) \left[1 - N_1\left(\frac{h_Z(\overline{Z}) - \rho\, h_S(K)}{\sqrt{1-\rho^2}} \right) \right] \Bigg\}$$

The above function can be further simplified with some basic transformations of the normal density and normal cumulative functions: $n(a) = n(-a)$, $N_1(a) = 1 - N_1(-a)$, $N_2(a, b, \rho) = N_2(a, \infty, -\rho) - N_2(a, -b, -\rho)$. When Z is chosen to replicate the probability distribution of S in a risk-neutral setting (S^*) at maturity of the sequential option, the explicit form of the integral over the call option can be used to evaluate the sequential option on S in equation (25). The sequential option on the arithmetic mean-reverting variable P according to equation (26) is then obtained by replacing the expected values, volatilities and correlation of S with those of P.

Appendix C. Finite Difference Expressions for the Sequential Rainbow Option on two Assets

Defining the normally distributed variables $x = \ln X$ and $y = \ln Y$, partial differential equation (6) becomes:

$$\frac{\partial SF}{\partial t} + \frac{1}{2}\sigma_X^2 \frac{\partial^2 SF}{\partial x^2} + \frac{1}{2}\sigma_Y^2 \frac{\partial^2 SF}{\partial y^2} + \rho\sigma_X\sigma_Y \frac{\partial^2 SF}{\partial x \partial y}$$

$$+\left(r - \delta_X - \frac{\sigma_X^2}{2}\right)\frac{\partial SF}{\partial x} + \left(r - \delta_Y - \frac{\sigma_Y^2}{2}\right)\frac{\partial SF}{\partial y} - r\,SF = 0$$

The derivatives of the sequential rainbow option according to the implicit finite difference method are:

$$\frac{\partial SF}{\partial t} = \frac{SF_{i,j,k+1} - SF_{i,j,k}}{\Delta t}$$

$$\frac{\partial SF}{\partial x} = \frac{SF_{i+1,j,k} - SF_{i-1,j,k}}{2\Delta x}$$

$$\frac{\partial SF}{\partial y} = \frac{SF_{i,j+1,k} - SF_{i,j-1,k}}{2\Delta y}$$

$$\frac{\partial^2 SF}{\partial x^2} = \frac{SF_{i+1,j,k} + SF_{i-1,j,k} - 2\,SF_{i,j,k}}{\Delta x^2}$$

$$\frac{\partial^2 SF}{\partial y^2} = \frac{SF_{i,j+1,k} + SF_{i,j-1,k} - 2\,SF_{i,j,k}}{\Delta y^2}$$

$$\frac{\partial^2 SF}{\partial x\,\partial y} = \frac{SF_{i+1,j+1,k} - SF_{i+1,j-1,k} - SF_{i-1,j+1,k} + SF_{i-1,j-1,k}}{4\,\Delta x\,\Delta y}$$

These derivative functions are inserted into the partial differential equation describing x and y to get the finite difference equation (29).

Appendix D. Solution Methodology of the Finite Difference Problem

The matrix of values of the sequential rainbow option is three-dimensional of the order $(i_{max}+1),(j_{max}+1),(k_{max}+1)$. For each point in time, there is an x-y-layer of option values, as exemplified below for the time of maturity and the present time.

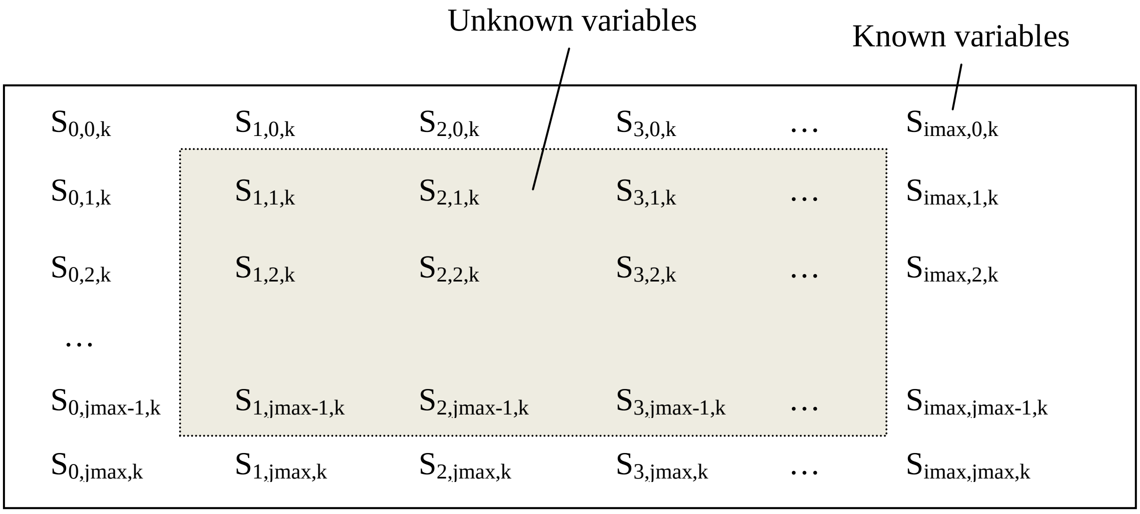
At t = T_{SF}:

		x				
	$S_{0,0,kmax}$	$S_{1,0,kmax}$	$S_{2,0,kmax}$	$S_{3,0,kmax}$	…	$S_{imax,0,kmax}$
	$S_{0,1,kmax}$	$S_{1,1,kmax}$	$S_{2,1,kmax}$	$S_{3,1,kmax}$	…	$S_{imax,1,kmax}$
y	$S_{0,2,kmax}$	$S_{1,2,kmax}$	$S_{2,2,kmax}$	$S_{3,2,kmax}$	…	$S_{imax,2,kmax}$
	…					
	$S_{0,jmax,kmax}$	$S_{1,jmax,kmax}$	$S_{2,jmax,kmax}$	$S_{3,jmax,kmax}$	…	$S_{imax,jmax,kmax}$

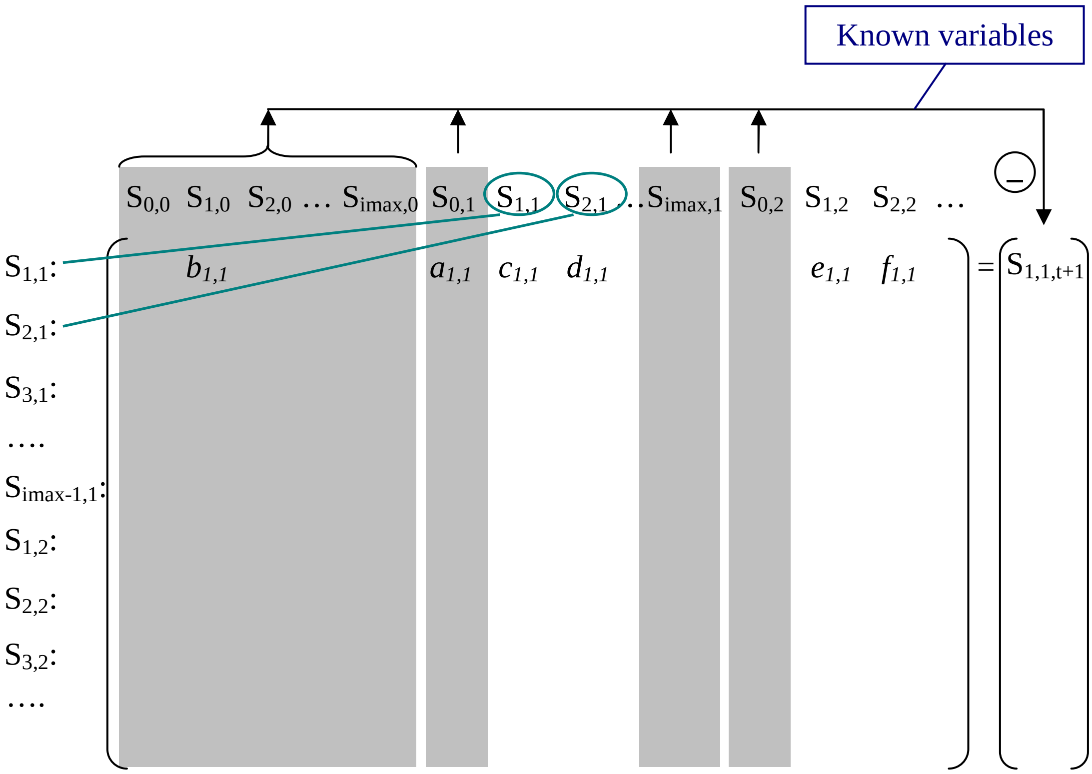
At t = 0:

$S_{0,0,0}$	$S_{1,0,0}$	$S_{2,0,0}$	$S_{3,0,0}$	…	$S_{imax,0,0}$
$S_{0,1,0}$	$S_{1,1,0}$	$S_{2,1,0}$	$S_{3,1,0}$	…	$S_{imax,1,0}$
$S_{0,2,0}$	$S_{1,2,0}$	$S_{2,2,0}$	$S_{3,2,0}$	…	$S_{imax,2,0}$
…					
$S_{0,jmax,0}$	$S_{1,jmax,0}$	$S_{2,\ jmax,0}$	$S_{3,\ jmax,0}$	…	$S_{imax,\ jmax,0}$

The option values for t=T_{SF} are determined by boundary condition (7). Furthermore, the values at X=0, X=X_{max} (assumed infinity) and Y=0, Y=Y_{max}, respectively, are given by boundary conditions (9) to (11). The illustration below shows that $(i_{max}-1)\cdot(j_{max}-1)\cdot(k_{max})$ unknown values are to be determined by means of the same number of finite difference equations.



This linear system is then solved by means of a Gaussian algorithm. For this purpose, a matrix of the order $(i_{max}-1)\cdot(j_{max}-1)$ is created for every time interval as follows:



Each row describes one unknown variable $S_{i,j,k}$ by its finite difference equation, e.g. the first row for the variable $S_{1,1,k}$. If any known variables are involved in this equation, they are multiplied by the respective factor, e.g. $S_{1,0}$ is multiplied by $b_{1,1}$, and subtracted from the right-hand side of the equation. This means that the left-hand side is a quadratic matrix of the order $(i_{max} - 1) \cdot (j_{max} - 1)$ of the unknown variables multiplied by the respective factors. The right-hand side is a vector of the same order with known values. This system of $(i_{max} - 1) \cdot (j_{max} - 1)$ equations and $(i_{max} - 1) \cdot (j_{max} - 1)$ unknowns is solved using the Gaussian algorithm.

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6 Conclusion

This Chapter summarises the main contributions and findings from the real option models developed in this thesis. It also includes a critical discussion of the option models and their assumptions to make transparent their limitations. Finally, we provide an outlook on potential areas of future research for real rainbow options in the context of commodities.

6.1 Findings

In the course of this thesis, we have developed models for three types of real rainbow options which are highly relevant to the valuation of commodity-related assets and investment opportunities. These models allow one to value the options and give guidance on optimal operating and investment policies. We provide insights into the behaviour and sensitivities of option values and operating policies and discuss implications for decision-making. Some general conclusions can be given. The option of choosing between two uncertain assets always has a positive value. Whether this value exceeds the additional cost for installing flexibility depends to a large extent on the underlying uncertainty and starting price levels. An investor in an asset with output flexibility should generally prefer high volatility, low correlation and a low degree of co-integration between the underlying variables. Exceptions to this rule have been identified and are discussed further below. An operator of a flexible asset needs to optimise the practical level of flexibility and monitor the main drivers of switching triggers in order to take the right operating decisions. Policy makers interested in the functioning of markets ought to promote flexibility in real assets, particularly with regard to commodity outputs, which would generally speed up adjustments to a shift in demand.

The first type of real rainbow option developed is a model with continuous choice on the best of two commodity outputs. This model is appropriate to value an asset with flexibility between two stochastic and correlated revenue flows, described by geometric Brownian motion, where operating costs are incurred and switching between the two operating modes is possible at any time by incurring a switching cost. Both quasi-analytical and numerical lattice solutions are presented, with the latter solution also taking into account that the asset operation can be suspended at no cost. A comparison of these two approaches reveals that the quasi-analytical solution is more transparent and accurate. The accuracy of the numerical lattice solution is improved by refining the grid spacing, which increases computation time on the other hand. It has the advantage, however, that it is more flexible on operating costs. Optimal switching policies are determined as a function of the underlying commodity prices. While the switching boundaries generally diverge when the underlying prices increase because of increasing variance, they also diverge for very low prices when operating costs and the option to suspend are taken into account. This is because net cash flows no longer justify the switching cost when prices are very low and the asset operation can be suspended instead. In an empirical application of the option model to the valuation of a flexible fertilizer plant, the value of flexibility between the two outputs, ammonia and urea, exceeds the required additional investment cost despite the high correlation between the commodities. The real value of the fertilizer plant is found to be high compared to the investment cost, mainly driven by non-stationary commodity prices in combination with constant operating costs and by the assumption of no operating and maintenance costs during suspension.

We have further developed a real option model to value the flexibility on the best of two co-integrated commodities under the premise of unlimited switching by incurring switching costs. The uncertainty in two commodity prices is reduced to only one source of uncertainty by modelling the spread which is mean-reverting in the case of co-integration. Optimal spread levels for switching are determined. Our solution is quasi-analytical and all parameters can be estimated from empirical data. Moreover, we distinguish between the different risk and discount factors – as compared to an existing model based a single general discount rate – and demonstrate how this leads to better results for a real option with a mean-reverting stochastic process. When the model is applied to the valuation of a polyethylene plant, based on the spread between polyethylene and ethylene, we find that the value of flexibility highly depends on the degree of mean-reversion in the spread between the underlying variables. When the spread is non-stationary, the potential pay-offs from switching are much higher compared to the case where the spread reverts to a long-run mean. Hence, operating flexibility is higher when commodity prices are not co-integrated. The valuation result of this rainbow option on a mean-reverting spread is conceptually equivalent to a continuous entry/exit option on a mean-reverting stochastic variable. This also leads to the insight that anti-cyclical investing is particularly attractive when the value-driver is a stationary variable.

The third type of option we have developed is a European sequential rainbow option, both on an option on the best of two correlated stochastic assets following geometric Brownian motion processes and on an option on the spread between two stochastic co-integrated assets. Numerical solutions have been provided based on a finite difference framework and Monte Carlo simulation. In addition, we have provided a full closed-form solution for the sequential option on the mean-reverting spread. It has been found that the sequential rainbow option value is negatively correlated with the volatility of one of the two assets in the special case when the volatility of that asset is low and the option is in-the-money only on that asset. This is because the chances are higher that at least one asset exceeds the exercise price at the time of maturity. If the volatilities are very high, however, the potential pay-offs overcompensate the positive effect of having one asset safely in-the-money. Another interesting result is that the sequential option on the mean-reverting spread does not necessarily increase in value with a longer time to maturity. Whether a longer time to maturity has a positive effect on the option value depends especially on the initial level of the spread in relation to the long-run mean, further on the speed of mean-reversion, the risk-free interest rate and the volatility. Furthermore, given a fixed maturity of the inner option, the sequential option value does not necessarily increase in a monotonous way with the expiry date of the sequential option because of the exponential correlation function of the mean-reverting stochastic process between the two maturities.

6.2 Critical Discussion

Developing several real rainbow option models offers the opportunity to compare different approaches and methods. We also need to distinguish the different problem settings and highlight the limitations of each model. In this section, we review these critical issues and draw some further conclusions from comparison.

6.2.1 Problem framing

The continuous rainbow options in research papers #1 and #2 assume that the asset is already in operation, producing one of the two alternative commodities. Guidance is given with regard to the optimal timing of switching between the operating modes but not with regard to the optimal timing for investing in the asset. The sequential options developed in research paper #3 also do not indicate the optimal investment timing because they are defined as European options.

The continuous rainbow option on the spread between two co-integrated commodities considers the asset operation in the base mode to be the base line (starting point for valuation). The model does not evaluate whether this is a profitable operation at all but rather focuses on the decision which of the two commodities is the better one at any point of time. The same is true for the sequential option on the spread where the underlying assumption is that one commodity asset (present value of the commodity revenues) is already held and the option focuses on exchanging it for the alternative commodity asset. This is also why the sequential option on two gBm assets and the sequential option on the spread cannot be compared directly.

The gap between the valuation given by the model and the true value is influenced by the extent to which the model ignores relevant restrictions that would be encountered in a realistic setting. The continuous rainbow option models oversimplify the flexibility insofar as some restrictions in a real life setting are ignored. We have mostly assumed unlimited switching opportunities between the commodity outputs. In reality, the commodities are sold to customers, often with contractual long-term supply arrangements. Switching to a different product might mean not being able to satisfy the customer requirements which is at least unfavourable to the business relationship. Personnel costs have been differentiated into fixed (management) and variable (operators). Although operators can probably be fired and hired more easily than management, qualified operators are crucial to ensure an efficient production process. It would be difficult to retain these people when the plant is under temporary suspension. While the model might overstate the value of flexibility as explained above, it should also be noted that not all options available in operating the asset have been considered. Examples include the option to reduce the throughput, to change input materials, to optimise the production processes and become more efficient, and many more.

The sequential option on the spread can be interpreted as a sequential exchange option. It is more general compared to the Carr (1988) sequential exchange option insofar as it considers exercise prices for the inner option and for the sequential option to be fixed and independent of the underlying assets. On the other hand, it is more restrictive insofar as it is only applicable in co-integrated markets, when the spread is characterised by mean-reversion. We have presented in research paper #3 a number of applications of the European sequential rainbow options. In other settings, American finite sequential options might be more appropriate and Least-squares Monte Carlo (LSMC) simulation would then be considered as a solution technique.

6.2.2 Critical Assumptions

An intuitive approach of critically reviewing the valuation models is to ask for concerns if one was confronted with taking investment decisions based on the approach presented. The stochastic processes chosen to model the uncertainties (commodity prices) have a significant effect on the valuation and Chapter 2 extensively discusses various approaches of modelling commodity prices. There is no general consensus on the best way of modelling commodities, first because there are an unlimited number of commodities and they exhibit huge differences in behaviour, and second because even for one commodity, different models exist, each with its advantages and disadvantages. However, there seems to be a tendency in the more recent literature suggesting that commodity prices are not completely random walk.

Demand/supply economics are at the root of this rationale. The short-term market equilibrium is established by incumbents suspending and resuming production based on their variable costs. The long-term equilibrium is established by investment/ divestment decisions based on long-run total costs. Capacity adjustments can take a long time in the commodity industry because required investments are frequently very large. Our analysis of urea and ammonia prices for the two-factor rainbow option covers a period of ten years of historical data. It is quite probable that the level of stationarity in the prices would be different (higher or lower) if a different period length was considered. Introducing a stochastic convenience yield according to the Gibson and Schwartz (1990) model would be equivalent to mean-reversion in short-term prices and random walk in the long-term prices. Stochastic convenience yields would be expected to improve the underlying "engine", but at the cost of complicating the solution framework from two-factor to four-factor which could hardly be handled by a lattice. The stochastic process of the polyethylene-ethylene spread has been clearly identified as mean-reverting and the valuation of the polyethylene plant also seems plausible when compared to the investment cost.

6.2.3 Theory

The stochastic spread in research paper #2 is modelled directly from empirical data instead of deriving its parameters from the stochastic processes of the two underlying commodities. There is no direct connection to their volatilities and correlation, so assumed changes in these factors cannot be linked to the spread and therefore are not assessed regarding the effect on the option. However, this lack of information does not compromise the results of the option model because the spread is the value-driver and capturing its empirical information directly is beneficial to the validity of the stochastic process parameters. Fleten and Näsäkkälä (2010), who also model the spread directly when valuing gas-fired power plants based on the spark spread, say that it "has the advantage of avoiding the need for explicitly specifying the correlation between electricity and natural gas prices" (p. 807). Moreover, in co-integrated markets, it is more promising to find the appropriate stochastic model for the spread (mean-reverting) than to find the appropriate stochastic models for the individual commodities.

We have used the contingent claims and risk-neutral approach to determine the continuous rainbow option on the mean-reverting spread, as compared to the dynamic programming approach used by Sodal et al. (2007) for a similar problem. Contingent claims analysis assumes that the stochastic changes of the underlying assets are spanned by other assets in the open market which can be justified for many commodities. Even if spanning did not hold in practice, one could argue that the theory still holds because if these assets were traded, the valuation would have to follow the approach to rule out arbitrage. As explained earlier, the dynamic programming approach does not require spanning of assets but is based on an arbitrary and constant discount rate. It leads to the contingent claims and risk-neutral valuation results if this arbitrary discount factor is replaced by the risk-free interest rate and the underlying stochastic process is transformed into its risk-neutral form. We have shown this connection between our solution and the one in Sodal et al. (2007). While the solution is determined analytically, we say that the solution is quasi-analytical because the system of equations needs to be solved simultaneously, due to the involvement of confluent hypergeometric functions, so that an analytical expression of the switching triggers and the constants is not available.

6.2.4 Numerical solution methods

In research paper #1 we have used a trinomial lattice (tree) based on the Hull and White (1990) modification to the explicit finite difference approach which ensures convergence. The trinomial lattice is equivalent to the explicit finite difference approach. Correlation is taken into account and influences the form of the grid insofar as the interval size for one of the two variables depends on the value of the other variable. When a long time horizon is mapped, as is the case in our application with 50 years, the non-linear form of the grid complicates the implementation of the lattice significantly because the number and size of intervals needs to be chosen in a way so that the relevant range of variable values is represented. The alternative implicit finite difference method would require setting up and solving a system of (i-1) times (j-1) times k equations, where i and j are the number of grid intervals of the two underlying variables and k the number of time intervals. With a time horizon of 50 years and time intervals of 3 months, k alone would be 250 and the number of intervals for each variable also needs to be quite high to reflect the range of possible values within this long time period, so the system of equations would become hugely complex.

Although the basic Monte Carlo simulation is not appropriate for American-style options, it can be adapted for this purpose, for instance as a Least Squares Monte Carlo simulation. However, when multiple switching between two underlying variables is possible and a long time horizon is taken into account (such as 50 years), even adaptations of Monte Carlo simulation are not appropriate. This is why a lattice or explicit finite difference approach is the preferred method for the continuous rainbow option on two stochastic variables.

In research paper #3, we have used the implicit finite difference method which is robust and always converges to the solution of the differential equations when the variable and time intervals approach zero. The implementation is rather complex because the difference equation needs to be set up for all points in time before this system of equations can be solved. Since the time horizon for the sequential option is limited (up to 3 years in our numerical application), the number of grid points and therefore the number of equations is limited. While the explicit finite difference method is easier to implement because the equations are solved for each point in time, working backwards until the starting point, precautions have to be taken to ensure convergence. Finite differences have proved to be slightly more efficient than Monte Carlo simulation when solving for the one-factor sequential options in research paper #3. This is shown by the results of the sequential option on the spread for which a closed-form solution is available. The option value given by the finite difference method is 4.13, almost exactly the same as the closed-form solution (4.14). The Monte Carlo simulation with a similar computing time indicates an option value of 4.18. The accuracy of both numerical approaches can be improved by refining the grid spacing and number of simulations, respectively. Another advantage of the finite difference approach is that option values can be determined for different starting values with only one computation run, while the Monte Carlo simulation needs to be repeated for different starting values.

6.3 Directions of Future Research

We consider the main areas of future research on real rainbow options to be the extension of the models to account for alternative stochastic commodity prices and to determine the optimal investment timing. The level of complexity/flexibility incorporated in the models in this thesis closes a significant part of the gap between traditional valuation models and reality. The focus should now be to develop these rainbow options for alternative stochastic processes of commodity prices in order to be able to apply them to a variety of different commodities of different behaviour. Promising approaches seem to be the Gibson and Schwartz (1990) and Schwartz and Smith (2000) two-factor commodity price models incorporating both random walk and mean-reversion, as well as a multi-factor stochastic process based on commodity futures pricing as outlined by Cortazar and Schwartz (1994). While the expectation would be to further improve the validity of the valuation results, the transparency and tractability would most certainly be reduced due to increased complexity.

As part of the option valuation, optimal switching policies have been determined for the new rainbow options. It would now be interesting to determine the optimal investment timing as a function of the underlying prices. In a next step, the investment cost could be considered stochastic and correlated with the commodity price which is especially interesting when the value driver (e.g. commodity price or price spread) is mean-reverting because it might present a strong case for anti-cyclical investment.

We have focused on modelling the commodity outputs. An interesting extension would be to model the input in addition. In a first step, this could mean allowing deterministic changes in the input prices. More advanced cases would introduce a stochastic process for the input, thereby linking the behaviour of revenues and costs. Future research might also relax some of the other assumptions such as neutral behaviour of competition and customers to output switching.

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Appendix A.1. Implementation of Numerical Lattice Solution

This Appendix lists the Mathematica code used to implement the numerical lattice solution for the valuation of the flexible fertilizer plant in research paper #1 ('Continuous Rainbow Options on Commodity Outputs').

```
Clear["Global`x"]
(*______________________ Parameter values ________________________*)
T = 50;
p1 = 677440;
p2 = 1168000;
S12 = 150000;
S21 = 150000;
r = 0.05;
δx = 0.05;
δy = 0.05;
σx = 0.57;
σy = 0.40;
ρ = 0.92;
cx = 278; cy = 220;
CurrentX = 251;
CurrentY = 243;

(*______________________ Grid definition ________________________*)
iSteps = jSteps = 50;
TimeSteps = 250;
```

```
xmin = 0.01; ymin = 0.1;

imax = iSteps + 1; jmax = jSteps + 1; tmax = TimeSteps + 1;

Δt = T/TimeSteps;

AnalysisRangeX = 1000;

AnalysisRangeY = 1000;


(*______________________ Modelling ________________________*)

β11 = 1/2 - (r - δx)/σx^2 + Sqrt[((r - δx)/σx^2 - 1/2)^2 + 2 r/σx^2];

β12 = 1/2 - (r - δx)/σx^2 - Sqrt[((r - δx)/σx^2 - 1/2)^2 + 2 r/σx^2];

β21 = 1/2 - (r - δy)/σy^2 + Sqrt[((r - δy)/σy^2 - 1/2)^2 + 2 r/σy^2];

β22 = 1/2 - (r - δy)/σy^2 - Sqrt[((r - δy)/σy^2 - 1/2)^2 + 2 r/σy^2];

A1 = (p1 cx)^(1 - β11)/(β11 - β12) (β12/r - (β12 - 1)/δx)

B1 = (p1 cx)^(1 - β12)/(β11 - β12) (β11/r - (β11 - 1)/δx)

A2 = (p2 cy)^(1 - β21)/(β21 - β22) (β22/r - (β22 - 1)/δy);

B2 = (p2 cy)^(1 - β22)/(β21 - β22) (β21/r - (β21 - 1)/δy);

xStar = xStar /. FindRoot[(β11 - β12) B1 (p1 xStar)^β12 + (β11 - 1) (p1 xStar)/δx -
β11 (p1 cx/r + S21) == 0, {xStar, 2 cx}]

D1 = (B1 (p1 xStar)^β12 + p1 xStar/δx - p1 cx/r - S21)/(p1 xStar)^β11

yStar = yStar /. FindRoot[(β21 - β22) B2 (p2 yStar)^β22 + (β21 - 1) (p2 yStar)/δy -
β21 (p2 cy/r + S12) == 0, {yStar, 2 cy}]

D2 = (B2 (p2 yStar)^β22 + p2 yStar/δy - p2 cy/r - S12)/(p2 yStar)^β21;

V1 = Table[0, {i, 1, imax}, {j, 1, jmax}, {k, 1, tmax}]; (* Asset values assuming
current operating state 1 *)

V2 = Table[0, {i, 1, imax}, {j, 1, jmax}, {k, 1, tmax}]; (* Asset values assuming
current operating state 2 *)

x[i_, j_, k_] := xmin Exp[σx Sqrt[3 Δt] (i - 1)] Exp[(r - δx - 1/2 σx^2) Δt (k - 1)]

y[i_, j_, k_] := ymin Exp[σx Sqrt[3 Δt] ρ σy/σx (i - 1)] Exp[σy Sqrt[3 Δt (1 - ρ^2)] (j -
1)] Exp[(r - δy - 1/2 σy^2) Δt (k - 1)]
```

```
For[i = 1, i <= imax, i++, For[j = 1, j <= jmax, j++, V1[[i, j, tmax]] = If[x[i, j, tmax] > cx, B1 (p1 x[i, j, tmax])^β12 + p1 (x[i, j, tmax]/δx - cx/r), A1 (p1 x[i, j, tmax])^β11]]] (* Terminal boundary for V1 *)

For[j = 1, j <= jmax, j++, For[i = 1, i <= imax, i++, V2[[i, j, tmax]] = If[y[i, j, tmax] > cy, B2 (p2 y[i, j, tmax])^β22 + p2 (y[i, j, tmax]/δy - cy/r), A2 (p2 y[i, j, tmax])^β21]]] (* Terminal boundary for V2 *)

j12 = Table[0, {i, 1, imax}, {k, 1, tmax}]; (* Switching boundary from state 1 to state 2 *)

i21 = Table[0, {j, 1, jmax}, {k, 1, tmax}]; (* Switching boundary from state 2 to state 1 *)

For[i = 1, i <= imax, i++, j12[[i, tmax]] = 1; For[j = 1, j < jmax && V1[[i, j, tmax]] > V2[[i, j, tmax]] - S12 Exp[-r Δt], j++, j12[[i, tmax]] = j + 1]]

For[j = 1, j <= jmax, j++, i21[[j, tmax]] = 1; For[i = 1, i < imax && V2[[i, j, tmax]] > V1[[i, j, tmax]] - S21 Exp[-r Δt], i++, i21[[j, tmax]] = i + 1]]


(* Filling of the grid *)

For[k = tmax - 1, k >= 1, k--,

  For[i = 1, i <= imax, i++, V1[[i, jmax, k]] = Max[p1 (x[i, jmax, k]/δx - cx/r), p2 (y[i, jmax, k]/δy - cy/r) - S12]];

  For[j = 1, j <= jmax, j++, V1[[imax, j, k]] = Max[p1 (x[imax, j, k]/δx - cx/r), p2 (y[imax, j, k]/δy - cy/r) - S12]];

  For[i = 1, i <= imax, i++, V1[[i, 1, k]] = If[x[i, 1, k] > cx, B1 (p1 x[i, 1, k])^β12 + p1 (x[i, 1, k]/δx - cx/r), A1 (p1 x[i, 1, k])^β11]];

  For[j = jmax, j >= 1, j--, V1[[1, j, k]] = If[y[1, j, k] >= yStar, B2 (p2 y[1, j, k])^β22 + p2 (y[1, j, k]/δy - cy/r) - S12, D2 (p2 y[1, j, k])^β21]];

  For[i = 2, i <= imax - 1, i++,

    For[j = 2, j <= jmax - 1, j++, V1[[i, j, k]] = Max[p1 (x[i, j, k] - cx) Δt, 0] +
    Exp[-r Δt] (4/9 If[j < j12[[i, k + 1]], V1[[i, j, k + 1]], V2[[i, j, k + 1]] - S12] +
    1/9 If[j + 1 < j12[[i, k + 1]], V1[[i, j + 1, k + 1]], V2[[i, j + 1, k + 1]] - S12] +
    1/9 If[j - 1 < j12[[i, k + 1]], V1[[i, j - 1, k + 1]], V2[[i, j - 1, k + 1]] - S12] +
    1/9 If[j < j12[[i + 1, k + 1]], V1[[i + 1, j, k + 1]], V2[[i + 1, j, k + 1]] - S12] +
    1/9 If[j < j12[[i - 1, k + 1]], V1[[i - 1, j, k + 1]], V2[[i - 1, j, k + 1]] - S12] +
    1/36 If[j + 1 < j12[[i + 1, k + 1]], V1[[i + 1, j + 1, k + 1]], V2[[i + 1, j + 1, k + 1]] - S12] +
    1/36 If[j + 1 < j12[[i - 1, k + 1]], V1[[i - 1, j + 1, k + 1]], V2[[i - 1, j + 1, k + 1]] - S12] +
    1/36 If[j - 1 < j12[[i + 1, k + 1]], V1[[i + 1, j - 1, k + 1]], V2[[i + 1, j - 1, k + 1]]
```

```
      - S12] +
      1/36 If[j - 1 < j12[[i - 1, k + 1]], V1[[i - 1, j - 1, k + 1]], V2[[i - 1, j - 1, k + 1]] -
      S12])]];

   For[i = 1, i <= imax, i++, V2[[i, jmax, k]] = Max[p1 (x[i, jmax, k]/δx - cx/r) - S21,
   p2 (y[i, jmax, k]/δy - cy/r)]];

   For[j = 1, j <= jmax, j++, V2[[imax, j, k]] = Max[p1 (x[imax, j, k]/δx - cx/r) - S21,
   p2 (y[imax, j, k]/δy - cy/r)]];

   For[j = 1, j <= jmax, j++, V2[[1, j, k]] = If[y[1, j, k] > cy, B2 (p2 y[1, j, k])^β22 +
   p2 (y[1, j, k]/δy - cy/r), A2 (p2 y[1, j, k])^β21]];

   For[i = imax, i >= 1, i--, V2[[i, 1, k]] = If[x[i, 1, k] >= xStar, B1 (p1 x[i, 1, k])^β12
   + p1 (x[i, 1, k]/δx - cx/r) - S21, D1 (p1 x[i, 1, k])^β11]];

   For[j = 2, j <= jmax - 1, j++,

      For[i = 2, i <= imax - 1, i++, V2[[i, j, k]] = Max[p2 (y[i, j, k] - cy) Δt, 0] +
      Exp[-r Δt] (4/9 If[i < i21[[j, k + 1]], V2[[i, j, k + 1]], V1[[i, j, k + 1]] - S21] +
      1/9 If[i + 1 < i21[[j, k + 1]], V2[[i + 1, j, k + 1]], V1[[i + 1, j, k + 1]] - S21] +
      1/9 If[i - 1 < i21[[j, k + 1]], V2[[i - 1, j, k + 1]], V1[[i - 1, j, k + 1]] - S21] +
      1/9 If[i < i21[[j + 1, k + 1]], V2[[i, j + 1, k + 1]], V1[[i, j + 1, k + 1]] - S21] +
      1/9 If[i < i21[[j - 1, k + 1]], V2[[i, j - 1, k + 1]], V1[[i, j - 1, k + 1]] - S21] +
      1/36 If[i + 1 < i21[[j + 1, k + 1]], V2[[i + 1, j + 1, k + 1]], V1[[i + 1, j + 1, k +
      1]] - S21] +
      1/36 If[i + 1 < i21[[j - 1, k + 1]], V2[[i + 1, j - 1, k + 1]], V1[[i + 1, j - 1, k + 1]]
      - S21] +
      1/36 If[i - 1 < i21[[j + 1, k + 1]], V2[[i - 1, j + 1, k + 1]], V1[[i - 1, j + 1, k + 1]]
      - S21] +
      1/36 If[i - 1 < i21[[j - 1, k + 1]], V2[[i - 1, j - 1, k + 1]], V1[[i - 1, j - 1, k + 1]] -
      S21])]];

   For[i = 1, i <= imax, i++, j12[[i, k]] = 1; For[j = 1, j < jmax && V1[[i, j, k]] >=
   V2[[i, j, k]] - S12, j++, j12[[i, k]] = j + 1]];

   For[j = 1, j <= jmax, j++, i21[[j, k]] = 1; For[i = 1, i < imax && V2[[i, j, k]] >=
   V1[[i, j, k]] - S21, i++, i21[[j, k]] = i + 1]]]

 (* ______________________ End of model ________________________*)



(*___ Determination of asset values at any x-y-point within the grid at t=0 ___*)

GetiSteps[xValue_, k_] := Module[{iRef1 = 1, iRef2 = 2}, If[xValue < x[1, 1, k],
iRef1 = iRef2 = 1, If[xValue > x[imax, 1, k], iRef1 = iRef2 = imax,For[i = 2, i <=
imax, i++, If[xValue > x[i, 1, k], iRef1 = i; iRef2 = i + 1, Break[]]]]]; {iRef1, iRef2}];

GetjSteps[xValue_, yValue_, k_, i_] := Module[{jRef1 = jmax, jRef2 = jmax, yRef},
For[j = 1, j <= jmax, j++, If[Part[i, 1] == Part[i, 2], yRef = y[Part[i, 1], j, k], yRef =
```

```
y[Part[i, 1], j, k] + (xValue - x[Part[i, 1], j, k])/(x[Part[i, 2], j, k] - x[Part[i, 1], j, k])
(y[Part[i, 2], j, k] - y[Part[i, 1], j, k])]; If[yValue < yRef, jRef2 = j; jRef1 = jRef2 - 1;
If[jRef2 == 1, jRef1 = 1]; Break[]]]; {jRef1, jRef2}]

GetValue[xValue_, yValue_, k_, V1orV2_String] :=
    Module[{Value, ValueArray, iRef1, iRef2, jRef1, jRef2, xDistance, xInterval,
    yDistance, yInterval}, If[V1orV2 == "V2", ValueArray = V2, ValueArray = V1];
    iRef1 = Part[GetiSteps[xValue, k], 1];
    iRef2 = Part[GetiSteps[xValue, k], 2];
    jRef1 = Part[GetjSteps[xValue, yValue, k, {iRef1, iRef2}], 1];
    jRef2 = Part[GetjSteps[xValue, yValue, k, {iRef1, iRef2}], 2];
    xDistance = xValue - x[iRef1, jRef1, k];
    xInterval = x[iRef2, jRef1, k] - x[iRef1, jRef1, k];
    If[xInterval == 0, yDistance = yValue - y[iRef1, jRef1, k]; yInterval = y[iRef1,
    jRef2, k] - y[iRef1, jRef1, k], yDistance = yValue - (y[iRef1, jRef1, k] + (y[iRef2,
    jRef1, k] - y[iRef1, jRef1, k]) xDistance/xInterval); yInterval = (y[iRef1, jRef2, k]
    + (y[iRef2, jRef2, k] - y[iRef1, jRef2, k]) xDistance/xInterval) - (y[iRef1, jRef1, k]
    + (y[iRef2, jRef1, k] - y[iRef1, jRef1, k]) xDistance/xInterval)];
    If[xInterval == 0 && yInterval == 0, Value = ValueArray[[iRef1, jRef1, k]],
    If[xInterval == 0 , Value = ValueArray[[iRef1, jRef1, k]] + (ValueArray[[iRef1,
    jRef2, k]] - ValueArray[[iRef1, jRef1, k]]) yDistance/yInterval, If[yInterval == 0 ,
    Value = ValueArray[[iRef1, jRef1, k]] + (ValueArray[[iRef2, jRef1, k]] -
    ValueArray[[iRef1, jRef1, k]]) xDistance/xInterval, Value = ValueArray[[iRef1,
    jRef1, k]] + xDistance/xInterval (ValueArray[[iRef2, jRef1, k]] -
    ValueArray[[iRef1, jRef1, k]]) + yDistance/yInterval (xDistance/xInterval
    (ValueArray[[iRef2, jRef2, k]] - ValueArray[[iRef2, jRef1, k]]) + (xInterval -
    xDistance)/xInterval (ValueArray[[iRef1, jRef2, k]] - ValueArray[[iRef1, jRef1,
    k]]))]]];
    Value]

GetV1Value[xValue_, yValue_, k_] := GetValue[xValue, yValue, k, "V1"]

GetV2Value[xValue_, yValue_, k_] := GetValue[xValue, yValue, k, "V2"]

GetV1Value[251, 243, 1]

GetV2Value[251, 243, 1]
```

Appendix A.2. Implementation of Finite Difference Solution for Sequential Rainbow Option on Two Assets

This Appendix provides the Visual Basic module used to implement the finite difference solution for the sequential rainbow option on two correlated gBm assets as developed in research paper #3 ('Sequential Real Rainbow Options').

```
Public VolX          ' Volatility of X
Public VolY          ' Volatility of Y
Public YieldX        ' Yield/ Dividend pay-out of X
Public YieldY        ' Yield/ Dividend pay-out of Y
Public Corr As Single   ' Correlation between X and Y
Public r             ' Interest rate
Public TK            ' Expiry Date of SERO
Public TM            ' Expiry Date of Rainbow
Public K             ' Exercise Price of SERO
Public M             ' Exercise Price of Rainbow
Public x_min         ' Lowest value of normal X
Public y_min         ' Lowest value of normal Y
Public x_steps As Long  ' Number of steps (intervals) in X; Number of values in X is x_steps+1
Public y_steps As Long  ' Number of steps (intervals) in Y; Number of values in Y is y_steps+1
Public t_steps As Long  ' Number of time steps (intervals); Number of points in time is t_steps+1
Public dx            ' Length of x-interval
Public dy            ' Length of y-interval
Public dt            ' Length of t-interval
```

```
Public xytCube() As Single  ' Three-dimensional system in x-y-time of SERO values

Public aArray          ' Two-dimensional matrix in x-y of the constants 'a'

Public bArray

Public cArray

Public dArray

Public eArray

Public fArray

Public gArray

Public hArray

Public mArray


Function SERO(ValueX, ValueY, MaxX, StepsInX, MaxY, StepsInY,
ExpDateSERO, ExpDateRainbow, TimeSteps, ExPriceSERO, ExPriceRainbow,
IntRate, VolatilityX, VolatilityY, PayoutX, PayoutY, Correlation)

' Main function, returns the value of the Sequential European Rainbow Option

   Dim i, j, t As Integer

   Dim i_fix, i_rest, j_fix, j_rest

   TK = ExpDateSERO

   TM = ExpDateRainbow

   K = ExPriceSERO

   M = ExPriceRainbow

   r = IntRate

   VolX = VolatilityX

   VolY = VolatilityY

   YieldX = PayoutX

   YieldY = PayoutY

   Corr = Correlation

   x_steps = StepsInX
```

```
    y_steps = StepsInY

    t_steps = TimeSteps

    x_min = Log(MaxX / 100)

    y_min = Log(MaxY / 100)

    dx = (Log(MaxX) - x_min) / x_steps

    dy = (Log(MaxY) - y_min) / y_steps

    dt = TK / t_steps

    ReDim xytCube(x_steps + 1, y_steps + 1, t_steps + 1)

    Call FillFactorArrays

    For t = t_steps To 0 Step -1

        FillXYArray (t)

    Next t

    i_fix = Fix((Log(ValueX) - x_min) / dx)

    i_rest = (Log(ValueX) - x_min) / dx - i_fix

    j_fix = Fix((Log(ValueY) - y_min) / dy)

    j_rest = (Log(ValueY) - y_min) / dy - j_fix

    SERO = xytCube(i_fix, j_fix, 0) + i_rest * (xytCube(i_fix + 1, j_fix, 0) - 
xytCube(i_fix, j_fix, 0)) + _
            j_rest * (xytCube(i_fix, j_fix + 1, 0) - xytCube(i_fix, j_fix, 0))

End Function


Function Geske(i, delta, Yield, Volatility, t, var_min)

    Dim iBar

    Dim iBarLow

    Dim iBarHigh

    Dim h

    Dim ka

    Dim Eq
```

```
    Dim UpperEndReached As Boolean

    Dim LowerEndReached As Boolean

    Dim result

    Dim Vol

    Vol = Volatility

    iBar = i

    iBarLow = i / 2

    iBarHigh = i * 2

    UpperEndReached = False

    LowerEndReached = False


    ' Approximation algorithm to determine level of the stochastic variable where the option should be exercised

    Do

        Eq = Exp(iBar * delta + var_min) * Exp(-Yield * (TM - TK)) * Application.NormSDist((Log(Exp(iBar * delta + var_min) / M) + (r - Yield - 1 / 2 * Volatility ^ 2) * (TM - t * dt)) / (Volatility * Math.Sqr(TM - t * dt)) + Volatility * Sqr(TM - TK)) - M * Exp(-r * (TM - TK)) * Application.NormSDist((Log(Exp(iBar * delta + var_min) / M) + (r - Yield - 1 / 2 * Volatility ^ 2) * (TM - t * dt)) / (Volatility * Math.Sqr(TM - t * dt))) - K

        If Eq < -0.001 And UpperEndReached = True Then

            LowerEndReached = True

            iBarLow = iBar

            iBar = iBarLow + (iBarHigh - iBarLow) / 2

        ElseIf Eq < -0.001 And UpperEndReached = False Then

            LowerEndReached = True

            iBarLow = iBar

            iBar = iBar * 2

        ElseIf Eq > 0.001 And LowerEndReached = True Then

            UpperEndReached = True
```

```
            iBarHigh = iBar
            iBar = iBarLow + (iBarHigh - iBarLow) / 2
        ElseIf Eq > 0.001 And LowerEndReached = False Then
            UpperEndReached = True
            iBarHigh = iBar
            iBar = iBar / 2
        Else
            Exit Do
        End If
    Loop
    h = (Log(Exp(i * delta + var_min) / Exp(iBar * delta + var_min)) + (r - Yield - 1 / 2 * Vol ^ 2) * (TK - t * dt)) / (Vol * Sqr(TK - t * dt))
    ka = (Log(Exp(i * delta + var_min) / M) + (r - Yield - 1 / 2 * Vol ^ 2) * (TM - t * dt)) / (Vol * Sqr(TM - t * dt))
    Geske = Exp(i * delta + var_min) * Exp(-Yield * (TM - t * dt)) * BiVariateNormalCDF(h + Vol * Sqr(TK - t * dt), ka + Vol * Sqr(TM - t * dt), Sqr((TK - t * dt) / (TM - t * dt))) - M * Exp(-r * (TM - t * dt)) * BiVariateNormalCDF(h, ka, Sqr((TK - t * dt) / (TM - t * dt))) - K * Exp(-r * (TK - t * dt)) * Application.NormSDist(h)
End Function


Function DefiniteExercise(maxSteps, delta, Yield, t, var_min)
' Assumes such a high level of X or Y respectively (infinity), that the SERO will be exercised with certainty
    DefiniteExercise = Exp(maxSteps * delta + var_min) * Exp(-Yield * (TM - t * dt)) - M * Exp(-r * (TM - t * dt)) - K * Exp(-r * (TK - t * dt))
End Function


Function Rainbow(i, j)
    Dim d1, d11, d2, d22, d12, d21, rho1, rho2
```

```
    d1 = (Log(Exp(i * dx + x_min) / M) + (r - YieldX - 1 / 2 * VolX ^ 2) * (TM - t_steps * dt)) / (VolX * Sqr(TM - t_steps * dt))

    d11 = d1 + VolX * Sqr(TM - t_steps * dt)

    d2 = (Log(Exp(j * dy + y_min) / M) + (r - YieldY - 1 / 2 * VolY ^ 2) * (TM - t_steps * dt)) / (VolY * Sqr(TM - t_steps * dt))

    d22 = d2 + VolY * Sqr(TM - t_steps * dt)

    d12 = (Log(Exp(j * dy + y_min) / Exp(i * dx + x_min)) + (YieldX - YieldY - 1 / 2 * (VolX ^ 2 - 2 * Corr * VolX * VolY + VolY ^ 2) * (TM - t_steps * dt))) / (Sqr(VolX ^ 2 - 2 * Corr * VolX * VolY + VolY ^ 2) * Sqr(TM - t_steps * dt))

    d21 = (Log(Exp(i * dx + x_min) / Exp(j * dy + y_min)) + (YieldY - YieldX - 1 / 2 * (VolX ^ 2 - 2 * Corr * VolX * VolY + VolY ^ 2) * (TM - t_steps * dt))) / (Sqr(VolX ^ 2 - 2 * Corr * VolX * VolY + VolY ^ 2) * Sqr(TM - t_steps * dt))

    rho1 = (Corr * VolY - VolX) / Sqr(VolX ^ 2 - 2 * Corr * VolX * VolY + VolY ^ 2)

    rho2 = (Corr * VolX - VolY) / Sqr(VolX ^ 2 - 2 * Corr * VolX * VolY + VolY ^ 2)

    Rainbow = Exp(i * dx + x_min) * Exp(-YieldX * (TM - t_steps * dt)) * BiVariateNormalCDF(d11, -d12, -rho1) + Exp(j * dy + y_min) * Exp(-YieldY * (TM - t_steps * dt)) * BiVariateNormalCDF(d22, -d21, -rho2) - M * Exp(-r * (TM - t_steps * dt)) * (1 - BiVariateNormalCDF(-d1, -d2, Corr))

End Function



Function EuroCall(i, delta, Yield, Volatility, var_min)

' Black-Scholes European Call option

    Dim d1, d2

    Dim Vol

    Vol = Volatility

    d1 = (Log(Exp(i * delta + var_min) / M) + (r - Yield + 1 / 2 * (Vol ^ 2)) * (TM - t_steps * dt)) / (Vol * Sqr(TM - t_steps * dt))

    d2 = d1 - Vol * Sqr(TM - t_steps * dt)

    EuroCall = Exp(i * delta + var_min) * Exp(-Yield * (TM - t_steps * dt)) * Application.NormSDist(d1) - M * Exp(-r * (TM - t_steps * dt)) * Application.NormSDist(d2)

End Function
```

```
Sub FillFactorArrays()

' Calculate the constants a to f for all x-y values

        aArray = 1 / 2 * (r - YieldX - 1 / 2 * VolX ^ 2) / dx * dt - 1 / 2 * VolX ^ 2 / dx
^ 2 * dt

        bArray = 1 / 2 * (r - YieldY - 1 / 2 * VolY ^ 2) / dy * dt - 1 / 2 * VolY ^ 2 / dy
^ 2 * dt

        cArray = 1 + VolX ^ 2 / dx ^ 2 * dt + VolY ^ 2 / dy ^ 2 * dt + r * dt

        dArray = -1 / 2 * VolX ^ 2 / dx ^ 2 * dt - 1 / 2 * (r - YieldX - 1 / 2 * VolX ^ 2)
/ dx * dt

        eArray = -1 / 2 * VolY ^ 2 / dy ^ 2 * dt - 1 / 2 * (r - YieldY - 1 / 2 * VolY ^ 2)
/ dy * dt

        fArray = -Corr * VolX * VolY / (4 * dx * dy) * dt

        gArray = -Corr * VolX * VolY / (4 * dx * dy) * dt

        hArray = Corr * VolX * VolY / (4 * dx * dy) * dt

        mArray = Corr * VolX * VolY / (4 * dx * dy) * dt

End Sub


Sub XYTArray()

' Determine the SERO-values for the x-y matrix at t=TK (Exercise Date of SERO)

  Dim i, j, z

  xytCube(0, 0, t_steps) = 0

  For i = 1 To x_steps

    ' Boundary condition at Y=0, t=TK: SERO = max(EuroCall(X) - K; 0)

    z = EuroCall(i, dx, YieldX, VolX, x_min)

    If z - K > 0 Then

      xytCube(i, 0, t_steps) = z - K

    Else

      xytCube(i, 0, t_steps) = 0

    End If
```

```
        ' Boundary condition at Y=Ymax, t=TK
        xytCube(i, y_steps, t_steps) = DefiniteExercise(y_steps, dy, YieldY, t_steps, y_min)
    Next i
    For j = 1 To y_steps
        ' Boundary condition at X=0, t=TK: SERO = max(EuroCall(Y) - K; 0)
        z = EuroCall(j, dy, YieldY, VolY, y_min)
        If z - K > 0 Then
            xytCube(0, j, t_steps) = z - K
        Else
            xytCube(0, j, t_steps) = 0
        End If
        ' Boundary condition at X=Xmax, t=TK
        xytCube(x_steps, j, t_steps) = DefiniteExercise(x_steps, dx, YieldX, t_steps, x_min)
    Next j
    ' Boundary condition at all other X,Y at t=TK: SERO = max(Rainbow - K; 0)
    For i = 1 To x_steps - 1
        For j = 1 To y_steps - 1
            z = Rainbow(i, j)
            If z - K > 0 Then
                xytCube(i, j, t_steps) = z - K
            Else
                xytCube(i, j, t_steps) = 0
            End If
        Next j
    Next i
End Sub
```

```
Sub FillXYArray(t)

   Dim i, j

   If t = t_steps Then

      Call XYTArray

   Else

      xytCube(0, 0, t) = 0

      For i = 1 To x_steps

         ' Boundary condition at Y=0: SERO = Geske option of X

         xytCube(i, 0, t) = Geske(i, dx, YieldX, VolX, t, x_min)

         ' Boundary condition at Y=Ymax

         xytCube(i, y_steps, t) = DefiniteExercise(y_steps, dy, YieldY, t, y_min)

      Next i

      For j = 1 To y_steps

         ' Boundary condition at X=0: SERO = Geske option of Y

         xytCube(0, j, t) = Geske(j, dy, YieldY, VolY, t, y_min)

         ' Boundary condition at X=Xmax

         xytCube(x_steps, j, t) = DefiniteExercise(x_steps, dx, YieldX, t, x_min)

      Next j

      ' For all t<TK: Numerical procedure required to determine SERO values for the
fields of x-y

      Call GaussAlgorithm(t)

   End If

End Sub



Sub GaussAlgorithm(t)

' Numerical procedure to determine the SERO values for the fields of x-y for t<TK
```

```
' using Gaussian Algorith to solve linear system of variables
    Dim LinSysOrder
    Dim LinSys() As Single
    Dim SArray() As Single
    Dim StoreVar
    Dim i
    Dim j
    Dim n
    Dim summe
    LinSysOrder = (x_steps - 1) * (y_steps - 1)
    ReDim LinSys(LinSysOrder + 1, LinSysOrder)
    ReDim SArray(LinSysOrder)
    For j = 0 To LinSysOrder - 1
        For i = 0 To LinSysOrder
            LinSys(i, j) = 0
        Next i
    Next j

    ' Fill matrix with constants a,b,c,d,e,f as well as the results column
LinSys(LinSysOrder, ..)
    For j = 0 To y_steps - 2
        For i = 0 To x_steps - 2
        LinSys(i + j * (x_steps - 1), i + j * (x_steps - 1)) = cArray
        LinSys(LinSysOrder, i + j * (x_steps - 1)) = LinSys(LinSysOrder, i + j *
(x_steps - 1)) + xytCube(i + 1, j + 1, t + 1)
         If i = 0 Then
            ' S(i-1,j) is known, therefore a*S(i-1,j) is added to the results column
```

```
            LinSys(LinSysOrder, i + j * (x_steps - 1)) = LinSys(LinSysOrder, i + j * (x_steps - 1)) - xytCube(i, j + 1, t) * aArray
        Else
            LinSys(i - 1 + j * (x_steps - 1), i + j * (x_steps - 1)) = aArray
        End If
        If j = 0 Then
          ' S(i,j-1) is known, therefore b*S(i,j-1) is added to the results column
            LinSys(LinSysOrder, i + j * (x_steps - 1)) = LinSys(LinSysOrder, i + j * (x_steps - 1)) - xytCube(i + 1, j, t) * bArray
        Else
            LinSys(i + (j - 1) * (x_steps - 1), i + j * (x_steps - 1)) = bArray
        End If
        If i = x_steps - 2 Then
          ' S(i+1,j) is known, therefore d*S(i+1,j) is added to the results column
            LinSys(LinSysOrder, i + j * (x_steps - 1)) = LinSys(LinSysOrder, i + j * (x_steps - 1)) - xytCube(i + 2, j + 1, t) * dArray
        Else
            LinSys(i + 1 + j * (x_steps - 1), i + j * (x_steps - 1)) = dArray
        End If
        If j = y_steps - 2 Then
          ' S(i,j+1) is known, therefore e*S(i,j+1) is added to the results column
            LinSys(LinSysOrder, i + j * (x_steps - 1)) = LinSys(LinSysOrder, i + j * (x_steps - 1)) - xytCube(i + 1, j + 2, t) * eArray
        Else
            LinSys(i + (j + 1) * (x_steps - 1), i + j * (x_steps - 1)) = eArray
        End If
        If (i = x_steps - 2) Or (j = y_steps - 2) Then
          ' S(i+1,j+1) is known, therefore f*S(i+1,j+1) is added to the results column
```

```
          LinSys(LinSysOrder, i + j * (x_steps - 1)) = LinSys(LinSysOrder, i + j *
(x_steps - 1)) - xytCube(i + 2, j + 2, t) * fArray
        Else
          LinSys(i + 1 + (j + 1) * (x_steps - 1), i + j * (x_steps - 1)) = fArray
        End If
        If (i = 0) Or (j = 0) Then
          ' S(i-1,j-1) is known, therefore g*S(i-1,j-1) is added to the results column
          LinSys(LinSysOrder, i + j * (x_steps - 1)) = LinSys(LinSysOrder, i + j *
(x_steps - 1)) - xytCube(i, j, t) * gArray
        Else
          LinSys(i - 1 + (j - 1) * (x_steps - 1), i + j * (x_steps - 1)) = gArray
        End If
        If (i = x_steps - 2) Or (j = 0) Then
          ' S(i+1,j-1) is known, therefore h*S(i+1,j-1) is added to the results column
          LinSys(LinSysOrder, i + j * (x_steps - 1)) = LinSys(LinSysOrder, i + j *
(x_steps - 1)) - xytCube(i + 2, j, t) * hArray
        Else
          LinSys(i + 1 + (j - 1) * (x_steps - 1), i + j * (x_steps - 1)) = hArray
        End If
        If (i = 0) Or (j = y_steps - 2) Then
          ' S(i-1,j+1) is known, therefore m*S(i-1,j+1) is added to the results column
          LinSys(LinSysOrder, i + j * (x_steps - 1)) = LinSys(LinSysOrder, i + j *
(x_steps - 1)) - xytCube(i, j + 2, t) * mArray
        Else
          LinSys(i - 1 + (j + 1) * (x_steps - 1), i + j * (x_steps - 1)) = mArray
        End If
      Next i
   Next j
```

```
' Gaussian algorithm: Variable reduction
n = 0
Do
   For j = n + 1 To LinSysOrder - 1
      StoreVar = LinSys(n, j)
      For i = 0 To LinSysOrder
         LinSys(i, j) = LinSys(i, j) - LinSys(i, n) * StoreVar / LinSys(n, n)
      Next i
   Next j
   n = n + 1
Loop While n < LinSysOrder - 1


' Gaussian algorithm: Determination of variables
SArray(LinSysOrder - 1) = LinSys(LinSysOrder, LinSysOrder - 1) /
LinSys(LinSysOrder - 1, LinSysOrder - 1)
For j = LinSysOrder - 2 To 0 Step -1
   summe = 0
   For i = j + 1 To LinSysOrder - 1
      summe = summe + LinSys(i, j) * SArray(i)
   Next i
   If LinSys(j, j) = 0 Then
      n = 10
   Else
      SArray(j) = (LinSys(LinSysOrder, j) - summe) / LinSys(j, j)
   End If
Next j
```

```
    ' Transferring the option values from the "LinSys"-matrix to the normal xytCube
    Dim rest As Integer
    Dim multiple As Integer
    For j = 0 To LinSysOrder - 1
        rest = j Mod (x_steps - 1)
        multiple = (j - rest) / (x_steps - 1)
        xytCube(rest + 1, multiple + 1, t) = SArray(j)
    Next j
End Sub
```

Appendix A.3. Implementation of Finite Difference Solution for Sequential Rainbow Option on Mean-reverting Asset

This Appendix provides the Visual Basic module used to implement the finite difference solution for the sequential rainbow option on a mean-reverting spread as developed in research paper #3 ('Sequential Real Rainbow Options').

```
Public Mean          ' Long-run mean of P
Public eta           ' Speed of mean reversion of P
Public Vol           ' Volatility of P
Public u             ' Expected return on P
Public r             ' Interest rate
Public TK            ' Expiry Date of SERO
Public TM            ' Expiry Date of Rainbow
Public K             ' Exercise Price of SERO
Public M             ' Exercise Price of Rainbow
Public p_min         ' Lowest value of P
Public p_steps As Long  ' Number of steps (intervals) in P; Number of values in P is
p_steps+1
Public t_steps As Long  ' Number of time steps (intervals); Number of points in time
is t_steps+1
Public dp            ' Length of p-interval
Public dt            ' Length of t-interval
Public ptArray() As Single  ' Two-dimensional system in p-time of SERO values
Public aArray() As Single   ' List of p-dependent constants 'a'
Public bArray() As Single
Public cArray() As Single
```

```
Function SERO(ValueP, MinP, MaxP, StepsInP, ExpDateSERO, ExpDateRainbow,
TimeSteps, ExPriceSERO, ExPriceRainbow, IntRate, MeanP, ReversionSpeedP,
ExpReturnP, Volatility)

' Main function, returns the value of the Sequential European Rainbow Option

    Dim i, t As Integer

    Dim i_fix, i_rest

    TK = ExpDateSERO

    TM = ExpDateRainbow

    K = ExPriceSERO

    M = ExPriceRainbow

    r = IntRate

    Mean = MeanP

    eta = ReversionSpeedP

    Vol = Volatility

    u = ExpReturnP

    p_steps = StepsInP

    t_steps = TimeSteps

    p_min = MinP

    dp = (MaxP - MinP) / p_steps

    dt = TK / t_steps

    ReDim ptArray(p_steps + 1, t_steps + 1)

    Call FillFactorArrays

    For t = t_steps To 0 Step -1

        FillXYArray (t)

    Next t

    i_fix = Fix((ValueP - p_min) / dp)

    i_rest = (ValueP - p_min) / dp - i_fix

    SERO = ptArray(i_fix, 0) + i_rest * (ptArray(i_fix + 1, 0) - ptArray(i_fix, 0))
```

```
End Function


Sub FillFactorArrays()
' Calculate the constants a to c for all p-values
   ReDim aArray(p_steps)
   ReDim bArray(p_steps)
   ReDim cArray(p_steps)
   Dim i
   For i = 0 To p_steps
        aArray(i) = 1 / 2 * ((eta * Mean - (u - r + eta) * (i * dp + p_min)) / dp - (Vol / dp) ^ 2) * dt
   Next i
   For i = 0 To p_steps
        bArray(i) = 1 + ((Vol / dp) ^ 2 + r) * dt
   Next i
   For i = 0 To p_steps
        cArray(i) = -1 / 2 * ((eta * Mean - (u - r + eta) * (i * dp + p_min)) / dp + (Vol / dp) ^ 2) * dt
   Next i
End Sub


Sub FillXYArray(t)
   Dim i
   If t = t_steps Then
      Call XYTArray
   Else
      ptArray(0, t) = 0
      ptArray(p_steps, t) = DefiniteExercise(t)
```

```
      ' For all t<TK: Numerical procedure required to determine SERO values
      Call GaussAlgorithm(t)
   End If
End Sub


Sub XYTArray()
' Determine the SERO-values for the t=TK (Exercise Date of SERO)
   Dim i, z
   ' Boundary condition at all other X,Y at t=TK: SERO = max(CallOnP - K; 0)
   For i = 0 To p_steps
         z = EuroCallMeanReversion(i * dp + p_min, TM - t_steps * dt)
         If z - K > 0 Then
            ptArray(i, t_steps) = z - K
         Else
            ptArray(i, t_steps) = 0
         End If
   Next i
   ptArray(0, t_steps) = 0
   ptArray(p_steps, t_steps) = DefiniteExercise(t_steps)
End Sub


Function DefiniteExercise(t)
' Assumes such a high level of P (infinity), that the SERO will be exercised with
certainty
   DefiniteExercise = ExpectedP(p_steps * dp + p_min, TM - t * dt) * Exp(-r * (TM -
t * dt)) - K * Exp(-r * (TK - t * dt)) - M * Exp(-r * (TM - t * dt))
End Function
```

```
Function EuroCallMeanReversion(P, Time)

' European Call option on mean-reverting variable

   Dim d

   d = (ExpectedP(P, Time) - M) / VolP(Time)

   EuroCallMeanReversion = Exp(-r * Time) * ((ExpectedP(P, Time) - M) *
Application.NormSDist(d) + VolP(Time) * Exp(-1 / 2 * d ^ 2) / Sqr(2 *
Application.Pi()))

End Function



Sub GaussAlgorithm(t)

' Numerical procedure to determine the SERO values for the p-values for t<TK

' using Gaussian Algorithm to solve linear system of variables

   Dim LinSysOrder

   Dim LinSys() As Single

   Dim SArray() As Single

   Dim StoreVar

   Dim i

   Dim j

   Dim n

   Dim summe

   LinSysOrder = (p_steps - 1)

   ReDim LinSys(LinSysOrder + 1, LinSysOrder)

   ReDim SArray(LinSysOrder)

   For j = 0 To LinSysOrder - 1

      For i = 0 To LinSysOrder

         LinSys(i, j) = 0

      Next i

   Next j
```

```
' Fill matrix with constants a,b,c as well as the results column LinSys(LinSysOrder, ..)

For i = 0 To p_steps - 2

    LinSys(i, i) = bArray(i + 1)

    LinSys(LinSysOrder, i) = LinSys(LinSysOrder, i) + ptArray(i + 1, t + 1)

     If i = 0 Then

        ' S(i-1,j) is known, therefore a*S(i-1,j) is added to the results column

         LinSys(LinSysOrder, i) = LinSys(LinSysOrder, i) - ptArray(i, t) * aArray(i + 1)

     Else

         LinSys(i - 1, i) = aArray(i + 1)

     End If

     If i = p_steps - 2 Then

        ' S(i+1,j) is known, therefore c*S(i+1,j) is added to the results column

         LinSys(LinSysOrder, i) = LinSys(LinSysOrder, i) - ptArray(i + 2, t) * cArray(i + 1)

     Else

         LinSys(i + 1, i) = cArray(i + 1)

     End If

Next i


' Gaussian algorithm: Variable reduction

n = 0

Do

    For j = n + 1 To LinSysOrder - 1

        StoreVar = LinSys(n, j)

        For i = 0 To LinSysOrder
```

```
            LinSys(i, j) = LinSys(i, j) - LinSys(i, n) * StoreVar / LinSys(n, n)
         Next i
      Next j
      n = n + 1
   Loop While n < LinSysOrder - 1


   ' Gaussian algorithm: Determination of variables
   SArray(LinSysOrder - 1) = LinSys(LinSysOrder, LinSysOrder - 1) /
LinSys(LinSysOrder - 1, LinSysOrder - 1)
   For j = LinSysOrder - 2 To 0 Step -1
      summe = 0
      For i = j + 1 To LinSysOrder - 1
         summe = summe + LinSys(i, j) * SArray(i)
      Next i
      If LinSys(j, j) = 0 Then
         n = 10
      Else
         SArray(j) = (LinSys(LinSysOrder, j) - summe) / LinSys(j, j)
      End If
   Next j


   ' Transferring the option values from the "LinSys"-matrix to the normal ptArray
   For j = 0 To LinSysOrder - 1
      ptArray(j + 1, t) = SArray(j)
   Next j
End Sub
```

```
Function Get_ptArray(P, t)

' For demonstration purposes in Excel

    'Get_ptArray = Get_ptArray(i, j, Math.Round(t / dt))

    'Get_ptArray = ptArray(Math.Round((P - p_min) / dp), Math.Round(t / dt))

    Dim i_fix, i_rest

    i_fix = Fix((P - p_min) / dp)

    i_rest = (P - p_min) / dp - i_fix

    Get_ptArray = ptArray(i_fix, Math.Round(t / dt)) + i_rest * (ptArray(i_fix + 1,
Math.Round(t / dt)) - ptArray(i_fix, Math.Round(t / dt)))

End Function


Function SequentialMeanReverting(P)

    Dim ePBar

    Dim ePBarLow

    Dim ePBarHigh

    Dim Eq

    Dim UpperEndReached As Boolean

    Dim LowerEndReached As Boolean

    Dim d1, d2

    ePBar = Exp(P)

    ePBarLow = ePBar / 2

    ePBarHigh = ePBar * 2

    UpperEndReached = False

    LowerEndReached = False


    ' Approximation algorithm to determine level of the stochastic variable where the
sequential option should be exercised

    Do
```

```
        Eq = EuroCallMeanReversion(Log(ePBar), TM - TK) - K
        If Eq < -0.1 And UpperEndReached = True Then
            LowerEndReached = True
            ePBarLow = ePBar
            ePBar = ePBarLow + (ePBarHigh - ePBarLow) / 2
        ElseIf Eq < -0.1 And UpperEndReached = False Then
            LowerEndReached = True
            ePBarLow = ePBar
            ePBar = ePBar * 2
        ElseIf Eq > 0.1 And LowerEndReached = True Then
            UpperEndReached = True
            ePBarHigh = ePBar
            ePBar = ePBarLow + (ePBarHigh - ePBarLow) / 2
        ElseIf Eq > 0.1 And LowerEndReached = False Then
            UpperEndReached = True
            ePBarHigh = ePBar
            ePBar = ePBar / 2
        Else
            Exit Do
        End If
    Loop
    d1 = (ExpectedP(P, TK) - Log(ePBar) + VolP(TK) ^ 2) / VolP(TK)
    d2 = (ExpectedP(P, TM) - M + VolP(TM) ^ 2) / VolP(TM)
    SequentialMeanReverting = ExpectedP(P, TM) * Exp(-r * TM) *
BiVariateNormalCDF(d1, d2, VolP(TK) / VolP(TM)) - M * Exp(-r * TM) *
BiVariateNormalCDF(d1 - VolP(TK), d2 - VolP(TM), VolP(TK) / VolP(TM)) - K *
Exp(-r * TK) * Application.NormSDist(d1 - VolP(TK))
End Function
```

```
Function ExpectedP(P, Time)

    ExpectedP = P * Exp(-(eta + u - r) * Time) + Mean * eta / (eta + u - r) * (1 - Exp(-(eta + u - r) * Time))

End Function


Function VolP(Time)

    VolP = Sqr(1 / 2 * Vol ^ 2 / (eta + u - r) * (1 - Exp(-2 * (eta + u - r) * Time)))

End Function
```

Appendix A.4. Implementation of Monte Carlo Simulation for Sequential Rainbow Option on Two Assets

This Appendix provides the Visual Basic module used to implement the Monte Carlo simulation for the sequential rainbow option on two correlated gBm assets as developed in the research paper #3 ('Sequential Real Rainbow Options').

```
Function MonteCarlo(ValueX, ValueY, ExpDateSERO, ExpDateRainbow, ExPriceSERO, ExPriceRainbow, IntRate, VolX, VolY, YieldX, YieldY, Corr, NrOfSimulations)
    Dim X As Single
    Dim Y As Single
    Dim RainbowOption As Single
    Dim SumOfOptionValues As Double
    Dim dt As Double
    Dim i, j
    Dim SBM1, SBM2, VolXCorr, VolYCorr
    SumOfOptionValues = 0
    For i = 0 To NrOfSimulations - 1
        SBM1 = WorksheetFunction.NormSInv(Rnd())
        SBM2 = WorksheetFunction.NormSInv(Rnd())
        VolXCorr = VolX * SBM1
        VolYCorr = VolY * Corr * SBM1 + VolY * Math.Sqr(1 - Corr ^ 2) * SBM2
        X = ValueX * Exp((IntRate - YieldX - VolX ^ 2 / 2) * ExpDateSERO + VolXCorr * Math.Sqr(ExpDateSERO))
```

```
        Y = ValueY * Exp((IntRate - YieldY - VolY ^ 2 / 2) * ExpDateSERO + VolYCorr * Math.Sqr(ExpDateSERO))
        RainbowOption = EuropeanRainbow(X, Y, ExpDateRainbow - ExpDateSERO, ExPriceRainbow, IntRate, VolX, VolY, YieldX, YieldY, Corr)
        If RainbowOption > ExPriceSERO Then
            SumOfOptionValues = SumOfOptionValues + RainbowOption - ExPriceSERO
        End If
    Next i
    MonteCarlo = Exp(-IntRate * ExpDateSERO) * SumOfOptionValues / NrOfSimulations
End Function

Function EuropeanRainbow(X, Y, TimeToMaturity, ExPriceRainbow, IntRate, VolX, VolY, YieldX, YieldY, Corr)
    M = ExPriceRainbow
    r = IntRate
    Dim d1, d11, d2, d22, d12, d21, rho1, rho2
    d1 = (Log(X / M) + (r - YieldX - 1 / 2 * VolX ^ 2) * TimeToMaturity) / (VolX * Sqr(TimeToMaturity))
    d11 = d1 + VolX * Sqr(TimeToMaturity)
    d2 = (Log(Y / M) + (r - YieldY - 1 / 2 * VolY ^ 2) * TimeToMaturity) / (VolY * Sqr(TimeToMaturity))
    d22 = d2 + VolY * Sqr(TimeToMaturity)
```

```
    d12 = (Log(Y / X) + (YieldX - YieldY - 1 / 2 * (VolX ^ 2 - 2 * Corr * VolX * VolY + VolY ^ 2) * TimeToMaturity)) / (Sqr(VolX ^ 2 - 2 * Corr * VolX * VolY + VolY ^ 2) * Sqr(TimeToMaturity))
    d21 = (Log(X / Y) + (YieldY - YieldX - 1 / 2 * (VolX ^ 2 - 2 * Corr * VolX * VolY + VolY ^ 2) * TimeToMaturity)) / (Sqr(VolX ^ 2 - 2 * Corr * VolX * VolY + VolY ^ 2) * Sqr(TimeToMaturity))
    rho1 = (Corr * VolY - VolX) / Sqr(VolX ^ 2 - 2 * Corr * VolX * VolY + VolY ^ 2)
    rho2 = (Corr * VolX - VolY) / Sqr(VolX ^ 2 - 2 * Corr * VolX * VolY + VolY ^ 2)
    EuropeanRainbow = X * Exp(-YieldX * TimeToMaturity) * BiVariateNormalCDF(d11, -d12, -rho1) + Y * Exp(-YieldY * TimeToMaturity) * BiVariateNormalCDF(d22, -d21, -rho2) - M * Exp(-r * TimeToMaturity) * (1 - BiVariateNormalCDF(-d1, -d2, Corr))
End Function
```

Appendix A.5. Implementation of Monte Carlo Simulation for Sequential Rainbow Option on Mean-reverting Asset

This Appendix provides the Visual Basic module used to implement the Monte Carlo simulation for the sequential rainbow option on a mean-reverting spread as developed in the research paper #3 ('Sequential Real Rainbow Options').

```
Function MonteCarlo(ValueP, Mean, VolatilityP, ReversionSpeed, ExpReturn,
IntRate, ExpDateSERO, ExpDateRainbow, TimeSteps, ExPriceSERO,
ExPriceRainbow, NrOfSimulations)

   Dim SimP As Single

   Dim RainbowOption As Single

   Dim SumOfOptionValues As Double

   Dim i

   SumOfOptionValues = 0

   For i = 0 To NrOfSimulations - 1

      SimP = SimulateAssetValue(ValueP, Mean, VolatilityP, ReversionSpeed,
ExpReturn, IntRate, ExpDateSERO, TimeSteps)

      RainbowOption = Rainbow(SimP, Mean, VolatilityP, ReversionSpeed,
ExpReturn, IntRate, ExPriceRainbow, ExpDateRainbow - ExpDateSERO)

      If RainbowOption > ExPriceSERO Then

         SumOfOptionValues = SumOfOptionValues + RainbowOption -
ExPriceSERO

      End If

   Next i

   MonteCarlo = Exp(-IntRate * ExpDateSERO) * SumOfOptionValues /
NrOfSimulations

End Function


Function Rainbow(P, Mean, Vol, eta, u, r, M, Time)
```

```
' European Call option on mean-reverting variable

    Dim d1, d2

    Dim ExpectedP

    Dim VolP

    ExpectedP = P * Exp(-(eta + u - r) * Time) + Mean * eta / (eta + u - r) * (1 - Exp(-
(eta + u - r) * Time))

    VolP = Sqr(1 / 2 * Vol ^ 2 / (eta + u - r) * (1 - Exp(-2 * (eta + u - r) * Time)))

    d = (ExpectedP - M) / VolP

    Rainbow = Exp(-r * Time) * ((ExpectedP - M) * Application.NormSDist(d) + VolP
* Exp(-1 / 2 * d ^ 2) / Sqr(2 * Application.Pi()))

End Function



Function SimulateAssetValue(CurrentAssetValue, Mean, Vol, eta, u, r,
TimeToMaturity, TimeSteps)

    Dim dt As Double

    Dim AssetValue

    Dim i

    Dim z

    dt = TimeToMaturity / TimeSteps

    AssetValue = CurrentAssetValue

    For i = 1 To TimeSteps

        z = Vol * Math.Sqr(dt) * WorksheetFunction.NormSInv(Rnd())

        AssetValue = AssetValue * (1 - (eta + u - r) * dt) + eta * Mean * dt + z

    Next i

    SimulateAssetValue = AssetValue

End Function
```