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C_0 -SEMIGROUPS OF 2-ISOMETRIES AND DIRICHLET SPACES

EVA A. GALLARDO-GUTIÉRREZ AND JONATHAN R. PARTINGTON

ABSTRACT. In the context of a theorem of Richter, we establish a similarity between C_0 -semigroups of analytic 2-isometries $\{T(t)\}_{t \geq 0}$ acting on a Hilbert space \mathcal{H} and the multiplication operator semigroup $\{M_{\phi_t}\}_{t \geq 0}$ induced by $\phi_t(s) = \exp(-st)$ for s in the right-half plane \mathbb{C}_+ acting boundedly on weighted Dirichlet spaces on \mathbb{C}_+ . As a consequence, we derive a connection with the right shift semigroup $\{S_t\}_{t \geq 0}$

$$S_t f(x) = \begin{cases} 0 & \text{if } 0 \leq t \leq x, \\ f(x-t) & \text{if } x > t, \end{cases}$$

acting on a weighted Lebesgue space on the half line \mathbb{R}_+ and address some applications regarding the study of the invariant subspaces of C_0 -semigroups of analytic 2-isometries.

1. INTRODUCTION

The concept of a 2-isometry was introduced by Agler in the early eighties (cf. [1]); this is related to notions due to J. W. Helton (see [8] and [9]) and characterized in terms of their extension properties (see [2]). Recall that a bounded linear operator T on a separable, infinite dimensional complex Hilbert space \mathcal{H} is called a *2-isometry* if it satisfies

$$T^{*2}T^2 - 2T^*T + I = 0,$$

where I denotes the identity operator. In addition, such operators are called *analytic* if no nonzero vector is in the range of every power of T . It turns out that M_z , i.e. the multiplication operator by z , acting on the classical Dirichlet space, is a cyclic analytic 2-isometry. But, moreover, in [15] (see also [14]) Richter proved that any cyclic analytic 2-isometry is unitarily equivalent to M_z acting on a generalized Dirichlet space $D(\mu)$.

More precisely, let μ be a finite non-negative Borel measure on the unit circle \mathbb{T} and $D(\mu)$ the *generalized Dirichlet space* associated to μ , that is, the Hilbert space consisting of analytic functions on the unit disc \mathbb{D} such that the integral

$$\int_{\mathbb{D}} |f'(z)|^2 \left(\int_{|\xi|=1} \frac{1-|z|^2}{|\xi-z|^2} d\mu(\xi) \right) \frac{dm(z)}{\pi}$$

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is finite (here $dm(z)$ denotes the Lebesgue area measure in \mathbb{D}). Note that if $\mu = 0$, the space $D(\mu)$ is defined to be the classical Hardy space H^2 and for non-zero, finite, non-negative Borel measures μ on \mathbb{T} , the space $D(\mu)$ is contained in the Hardy space (see [7, Chapter 7]). Then Richter's Theorem reads as follows:

Theorem (Richter). *Let T be a bounded linear operator on an infinite dimensional complex Hilbert space \mathcal{H} . Then the following condition are equivalent:*

- (i) T is an analytic 2-isometry with $\dim \text{Ker } T^* = 1$,
- (ii) T is unitarily equivalent to $(M_z, D(\mu))$ for some finite non-negative Borel measure on \mathbb{T} , where

$$\|f\|_{D(\mu)}^2 = \|f\|_{H^2}^2 + \int_{\mathbb{D}} |f'(z)|^2 \left(\int_{|\xi|=1} \frac{1-|z|^2}{|\xi-z|^2} d\mu(\xi) \right) \frac{dm(z)}{\pi}.$$

One of the main applications of Richter's Theorem concerns the study of the invariant subspaces for the multiplication operator M_z in the spaces $D(\mu)$ and its relationship with the classical Beurling Theorem for the Hardy space H^2 (see [3]). For instance, regarding the Dirichlet space $D = D\left(\frac{|d\xi|}{2\pi}\right)$, Richter and Sundberg [16] proved that any closed, invariant subspace \mathcal{M} under M_z satisfies that $\dim \mathcal{M} \ominus z\mathcal{M} = 1$. Moreover, if $\varphi \in \mathcal{M} \ominus z\mathcal{M}$ with $\|\varphi\|_D = 1$, then $|\varphi(z)| \leq 1$ for $|z| \leq 1$ and $\mathcal{M} = \varphi D(m_\varphi)$, where dm_φ is the measure on \mathbb{T} given by $dm_\varphi(\xi) = |\varphi(\xi)|^2 \frac{|d\xi|}{2\pi}$. For general $D(\mu)$ spaces, an analogous result holds. We refer the reader to Chapters 7 and 8 in the recent monograph "A primer on the Dirichlet space" [7] for more on the subject.

Motivated by the Beurling-Lax Theorem and the work carried out by Richter, the aim of this work is taking further the study of the 2-isometries and considering C_0 -semigroups of 2-isometric operators. In particular, we will establish a similarity between C_0 -semigroups of analytic 2-isometries $\{T(t)\}_{t \geq 0}$ acting on a Hilbert space \mathcal{H} and the multiplication operator semigroup $\{M_{\phi_t}\}_{t \geq 0}$ induced by $\phi_t(s) = \exp(-st)$ for s in the right-half plane \mathbb{C}_+ acting boundedly on weighted Dirichlet spaces $\tilde{\mathcal{D}}_{\mathbb{C}_+}(\nu)$ on \mathbb{C}_+ (see Definition 2.3). As a consequence, by means of the Laplace transform, we derive a connection with the right shift semigroup $\{S_t\}_{t \geq 0}$

$$S_t f(x) = \begin{cases} 0 & \text{if } 0 \leq t \leq x, \\ f(x-t) & \text{if } x > t, \end{cases}$$

acting on a weighted Lebesgue space on the half line \mathbb{R}_+ . Finally, some applications regarding the study of the invariant subspaces of C_0 -semigroups of analytic 2-isometries are also discussed in Section 3.

2. C_0 -SEMIGROUPS OF ANALYTIC 2-ISOMETRIES

First, we introduce some basic concepts and terminology regarding C_0 -semigroups of bounded linear operators. For more on this topic, we refer the reader to the Engel–Nagel monograph [6].

A C_0 -semigroup $\{T(t)\}_{t \geq 0}$ of operators on a Hilbert space \mathcal{H} is a family of bounded linear operators on \mathcal{H} satisfying the functional equation

$$\begin{cases} T(t+s) = T(t)T(s) & \text{for all } t, s \geq 0, \\ T(0) = I, \end{cases}$$

and such that $T(t) \rightarrow I$ in the strong operator topology as $t \rightarrow 0^+$. Given a C_0 -semigroup $\{T(t)\}_{t \geq 0}$, there exists a closed and densely defined linear operator A that determines the semigroup uniquely, called the generator of $\{T(t)\}_{t \geq 0}$, defined by means of

$$Ax := \lim_{t \rightarrow 0^+} \frac{T(t)x - x}{t},$$

where the domain $D(A)$ of A consists of all $x \in \mathcal{H}$ for which this limit exists (see [6, Chapter II], for instance). Although the generator is, in general, an unbounded operator, it plays an important role in the study of a C_0 -semigroup, reflecting many of its properties.

However, if 1 is in the resolvent of A , that is, in the set

$$\rho(A) = \{\lambda \in \mathbb{C} : (A - \lambda I) : D(A) \subset \mathcal{H} \rightarrow \mathcal{H} \text{ is bijective}\},$$

then $(A - I)^{-1}$ is a bounded operator on \mathcal{H} by the Closed Graph Theorem, and the Cayley transform of A defined by

$$V := (A + I)(A - I)^{-1}$$

is a bounded operator on \mathcal{H} , since $V - I = 2(A - I)^{-1}$. Therefore V determines the semigroup uniquely, since A does. This operator is called the *cogenerator* of the C_0 -semigroup $\{T(t)\}_{t \geq 0}$. Observe that 1 is not an eigenvalue of V .

Recall that if A is a closed operator, then the *spectral bound* $s(A)$ of A is defined by

$$s(A) := \sup\{\operatorname{Re} \lambda : \lambda \in \sigma(A)\},$$

where $\sigma(A) = \mathbb{C} \setminus \rho(A)$ is the spectrum of A , and in case that A is the generator of a C_0 -semigroup, then $s(A)$ is always dominated by the *growth bound of the semigroup*, that is,

$$-\infty \leq s(A) \leq w_0 = \inf \left\{ w \in \mathbb{R} : \begin{array}{l} \text{there exists } M_w \geq 1 \text{ such that} \\ \|T(t)\| \leq M_w e^{wt} \text{ for all } t \geq 0 \end{array} \right\}.$$

Indeed, if $r(T(t))$ denotes the spectral radius of $T(t)$, it follows that $w_0 = \frac{1}{t} \log r(T(t))$ for each $t > 0$ (see [6, Section 2, Chapter IV], for instance). The following lemma will be useful in the context of our main result later.

Lemma 2.1. *Let $\{T(t)\}_{t \geq 0}$ be a C_0 -semigroup on a separable, infinite dimensional complex Hilbert space \mathcal{H} consisting of 2-isometries and A its generator. Then $1 \in \rho(A)$ and therefore, the cogenerator V of $\{T(t)\}_{t \geq 0}$ is well-defined.*

Proof. By induction it follows that, for any $n \geq 1$ and $t \geq 0$, $T(t)$ satisfies

$$T(t)^{*n}T(t)^n - nT(t)^*T(t) + (n-1)I = 0,$$

and so

$$\|T(t)^n x\|^2 = n\|T(t)x\|^2 - (n-1)\|x\|^2$$

for $x \in \mathcal{H}$. From here, it follows that $\|T(t)^n\| \leq Cn$, where C is a constant independent of n , and therefore the spectral radius $r(T(t)) \leq 1$ for any t . Therefore, $s(A) \leq 0$; and therefore $1 \in \rho(A)$. \square

The next result consists of a particular instance of [11, Theorem 1], where C_0 -semigroups of hypercontractions are considered. We state it for C_0 -semigroups of 2-isometries and include its proof for the sake of completeness.

Proposition 2.2. *Let $\{T(t)\}_{t \geq 0}$ be a C_0 -semigroup on a separable, infinite dimensional complex Hilbert space \mathcal{H} . Then the following conditions are equivalent:*

- (i) $T(t)$ is a 2-isometry for every $t \geq 0$.
- (ii) The mapping $t \in \mathbb{R}_+ \mapsto \|T(t)x\|^2$ is affine for each $x \in \mathcal{H}$.
- (iii) $\operatorname{Re}\langle A^2y, y \rangle + \|Ay\|^2 = 0$ ($y \in \mathcal{D}(A^2)$).
- (iv) The cogenerator V of $\{T(t)\}_{t \geq 0}$ exists and is a 2-isometry.

Proof. (i) \iff (ii): If each $T(t)$ is a 2-isometry, then for $t \geq 0$ and $\tau > 0$ we have

$$\langle T(t+2\tau)x, T(t+2\tau)x \rangle - 2\langle T(t+\tau)x, T(t+\tau)x \rangle + \langle T(t)x, T(t)x \rangle = 0,$$

so that

$$(1) \quad \|T(t+\tau)x\|^2 = \frac{1}{2}(\|T(t)x\|^2 + \|T(t+2\tau)x\|^2).$$

Since $t \in \mathbb{R}_+ \rightarrow \|T(t)x\|^2$ is continuous, the mapping is affine.

Conversely, taking $t = 0$ we see that (1) implies that $T(\tau)$ is a 2-isometry.

(ii) \iff (iii): For $t > 0$ we calculate the second derivative of the function $g : t \mapsto \|T(t)y\|^2$ for $y \in \mathcal{D}(A^2)$. We have

$$\begin{aligned} g''(t) &= \frac{d^2}{dt^2} \langle T(t)y, T(t)y \rangle \\ &= \langle A^2T(t)y, T(t)y \rangle + 2\langle AT(t)y, AT(t)y \rangle + \langle T(t)y, A^2T(t)y \rangle. \end{aligned}$$

For g affine, g'' is zero, and Condition (iii) follows on letting $t \rightarrow 0$. Conversely, Condition (iii) implies Condition (ii) for $y \in \mathcal{D}(A^2)$, and hence for all y by density.

(iii) \iff (iv): We calculate

$$\langle (I - 2V^*V + V^{*2}V^2)x, x \rangle$$

for $x = (A - I)^2 y$ (note that $(A - I)^{-2} : H \rightarrow H$ is defined everywhere and has dense range). We obtain

$$\begin{aligned} & \langle (A - I)^2 y, (A - I)^2 y \rangle - 2\langle (A^2 - I)y, (A^2 - I)y \rangle + \langle (A + I)^2 y, (A + I)^2 y \rangle \\ & = 4\langle A^2 y, y \rangle + 8\langle Ay, Ay \rangle + 4\langle y, A^2 y \rangle. \end{aligned}$$

Thus V is a 2-isometry if and only if Condition (iii) holds. \square

Before stating the main result of the section, let us introduce the following definition.

Definition 2.3. *Let ν be a finite positive Borel measure supported on the imaginary axis. The Dirichlet space $\tilde{\mathcal{D}}_{\mathbb{C}_+}(\nu)$ is defined as the space of analytic functions F on right half-plane \mathbb{C}_+ such that*

$$\|F\|^2 = |F(1)|^2 + \frac{1}{\pi} \int_{\mathbb{C}_+} |F'(s)|^2 \left(x + \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{x}{x^2 + (y - \tau)^2} d\nu(\tau) \right) dx dy < \infty$$

where $s = x + iy$.

The spaces $\tilde{\mathcal{D}}_{\mathbb{C}_+}(\nu)$ arise, in a natural way, when we analyze C_0 -semigroups of analytic 2-isometries in Hilbert spaces, as it is stated in our main result:

Theorem 2.4. *Let $\{T(t)\}_{t \geq 0}$ be a C_0 -semigroup on a separable, infinite dimensional complex Hilbert space \mathcal{H} consisting of analytic 2-isometries for every $t > 0$ such that*

$$(2) \quad \dim \bigcap_{t > 0} \ker (T^*(t) - e^{-t} I) = 1.$$

Then there exists a finite positive Borel measure ν supported on the imaginary axis such that $\{T(t)\}_{t \geq 0}$ is similar to the semigroup of multiplication operators induced by $\exp(-ts)$ acting on the space $\tilde{\mathcal{D}}_{\mathbb{C}_+}(\nu)$. Moreover, if the multiplication operators induced by $\exp(-ts)$ act continuously for every $t > 0$ on a Dirichlet space $\tilde{\mathcal{D}}_{\mathbb{C}_+}(\tilde{\nu})$ where $\tilde{\nu}$ is a finite positive Borel measure supported on the imaginary axis, then the corresponding semigroup consists of analytic 2-isometries and satisfies (2).

Before proceeding further, let us remark that our main result yields similarity for the semigroup $\{T(t)\}_{t \geq 0}$ because of the definition of the norm in $\tilde{\mathcal{D}}_{\mathbb{C}_+}(\nu)$. In addition, as we shall see later, condition (2) is a way of expressing the property that $\dim \ker V^* = 1$, where V is the cogenerator of the semigroup $\{T(t)\}_{t \geq 0}$.

In order to prove Theorem 2.4, we need the following auxiliary results.

Proposition 2.5. *Let $\{T(t)\}_{t \geq 0}$ be a C_0 -semigroup on a separable, infinite dimensional complex Hilbert space \mathcal{H} consisting of analytic 2-isometries. Then the cogenerator V is an analytic 2-isometry.*

Proof. First, we observe that V is well-defined by Lemma 2.1 and, it is a 2-isometry by Proposition 2.2. So, we are required to show that V is analytic.

The Wold Decomposition Theorem for 2-isometries (see [12], for instance), yields that V can be decomposed as $V = S \oplus U$ with respect to $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$, where U is the unitary part on $\mathcal{H}_2 = \bigcap_n V^n \mathcal{H}$ and S is an analytic 2-isometry. We will show that $U = 0$.

Let us assume, on the contrary, that $U \neq 0$.

First, we observe that since 1 is not an eigenvalue of V , the generator A of the semigroup $\{T(t)\}_{t \geq 0}$ may be expressed as the (possibly) unbounded operator

$$(V + I)(V - I)^{-1}.$$

Moreover, since $T(t)$ commutes with $(A - I)^{-1}$ and hence with V , it holds that \mathcal{H}_2 is invariant under $T(t)$ for every $t \geq 0$. In addition, the generator B of the restricted semigroup $\{T(t)|_{\mathcal{H}_2}\}_{t \geq 0}$ is the restriction of A to the $D(A) \cap \mathcal{H}_2$ (see [6, Ch. 2, Sec. 2], for instance); and the cogenerator is U .

Now, taking into account the fact that U is unitary, one deduces that B is skew-adjoint (i. e., $B^* = -B$). Then the restriction of $T(t)$ to \mathcal{H}_2 is unitary for every $t \geq 0$ and, therefore, every vector in \mathcal{H}^2 is in the range of (powers of) $T(t)$. Since $T(t)$ is analytic, it follows that $\mathcal{H}^2 = \{0\}$. Hence, $U = 0$ and the proof is completed. \square

Lemma 2.6. *Let $\{T(t)\}_{t \geq 0}$ be a C_0 -semigroup on a separable, infinite dimensional complex Hilbert space \mathcal{H} and A its generator. The following conditions are equivalent:*

- (1) $Ax_0 = -x_0$ for some $x_0 \in D(A)$.
- (2) $T(t)x_0 = e^{-t}x_0$ for all $t \geq 0$ and $x_0 \in D(A)$.

In addition, if $1 \in \rho(A)$ and V is the cogenerator, any of the previous conditions is equivalent to

- (3) $Vx_0 = 0$ for some $x_0 \in D(A)$.

Note that the equivalence between (1) and (2) in Lemma 2.6 just follows from the relationship between the eigenspaces of A and the semigroup $\{T(t)\}_{t \geq 0}$, that is,

$$\text{Ker}(\mu I - A) = \bigcap_{t \geq 0} \text{Ker}(e^{\mu t} - T(t)),$$

with $\mu \in \mathbb{C}$ (see [6, Corollary 3.8, Section IV], for instance). The last statement follows from the definition of V .

We are now in position to prove Theorem 2.4.

Proof of Theorem 2.4 Assume that $\{T(t)\}_{t > 0}$ consists of analytic 2-isometries. Let V denote its cogenerator; this is well-defined by Lemma 2.1, and it is an analytic 2-isometry by Proposition 2.5.

In addition, the hypotheses $\dim \bigcap_{t > 0} \text{ker}(T^*(t) - e^{-t}I) = 1$ along with Lemma 2.6 applied to the adjoint semigroup $\{T^*(t)\}_{t \geq 0}$, yields that $\dim \text{Ker } V^* = 1$.

By means of Richter's Theorem, it follows that V is similar to M_z acting on the space $D(\mu)$ for some finite non-negative Borel measure μ on \mathbb{T} considered with the equivalent

norm

$$\begin{aligned}
 \|f\|_{D(\mu)}^2 &\approx |f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 \left(\int_{|\xi|=1} \frac{1-|z|^2}{|\xi-z|^2} d\mu(\xi) \right) \frac{dm(z)}{\pi} \\
 (3) \qquad &= |f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 P_\mu(z) \frac{dm(z)}{\pi}.
 \end{aligned}$$

Observe that the similarity is the price paid when we consider the equivalent norm. Hence, for any $t \geq 0$, it follows that $T(t)$ is unitarily equivalent to the multiplication operator induced by $\exp(-t(1+z)/(1-z))$ on $D(\mu)$. Now, we migrate to the right half-plane $\mathbb{C}_+ = \{\operatorname{Re} s > 0\}$ applying the change of variables $s = (1+z)/(1-z)$, or $z = (s-1)/(s+1)$.

First, we observe that

$$P_\mu \left(\frac{s-1}{s+1} \right) = \int_{|\xi|=1} \frac{1 - \left| \frac{s-1}{s+1} \right|^2}{\left| \xi - \frac{s-1}{s+1} \right|^2} d\mu(\xi) \quad (s \in \mathbb{C}_+)$$

is a positive harmonic function in \mathbb{C}_+ ; so there exists a non-negative constant ρ and a finite positive Borel measure ν supported on the imaginary axis such that

$$(4) \qquad P_\mu \left(\frac{s-1}{s+1} \right) = \rho x + \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{x}{x^2 + (y-\tau)^2} d\nu(\tau), \quad (s = x + iy)$$

(see [10, Exercise 6, p. 134], for instance).

We can express ν in terms of μ , since with $\xi = (u-1)/(u+1)$ for $u = i\tau \in i\mathbb{R}$, we have

$$\begin{aligned}
 P_\mu \left(\frac{s-1}{s+1} \right) &= \mu(1) \frac{|s+1|^2 - |s-1|^2}{|(s+1) - (s-1)|^2} + \int_{\xi \in \mathbb{T} \setminus \{1\}} \frac{(|s+1|^2 - |s-1|^2)|u+1|^2}{|(u-1)(s+1) - (u+1)(s-1)|^2} d\mu(\xi) \\
 &= \mu(1)x + \int_{\xi \in \mathbb{T} \setminus \{1\}} \frac{x|u+1|^2}{|u-s|^2} d\mu(\xi), \\
 &= \mu(1)x + \int_{\xi \in \mathbb{T} \setminus \{1\}} \frac{x(1+\tau^2)}{x^2 + (y-\tau)^2} d\mu(\xi),
 \end{aligned}$$

where $s = x + iy \in \mathbb{C}_+$. So in (4) we have

$$(5) \qquad \rho = \mu(1) \quad \text{and} \quad \frac{d\nu(\tau)}{\pi(1+\tau^2)} = d\mu(\xi).$$

Then, upon applying the change of variables $s = (1+z)/(1-z)$ in (3), we deduce that $T(t)$ is similar to the multiplication operator induced by $\exp(-ts)$ acting on the space $\tilde{D}_{\mathbb{C}_+}(\nu)$ consisting of analytic functions F on \mathbb{C}_+ such that

$$(6) \qquad \frac{1}{\pi} \int_{\mathbb{C}_+} |F'(s)|^2 \left(x + \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{x}{x^2 + (y-\tau)^2} d\nu(\tau) \right) dx dy < \infty,$$

where $s = x + iy$ and $F(s) = f(z)$. This proves the first half of Theorem 2.4.

In order to conclude the proof, let us assume that the multiplication operators induced by $\exp(-ts)$ act continuously for every $t > 0$ on a Dirichlet space $\tilde{D}_{\mathbb{C}_+}(\tilde{\nu})$ where $\tilde{\nu}$ is a finite positive Borel measure supported on the imaginary axis. Reversing the steps above

and taking into account the fact that (5) defines a measure $\tilde{\mu}$ on \mathbb{T} , where $\tilde{\mu}(1) = \tilde{\nu}(0) = \rho$, we deduce that the given semigroup is similar to the semigroup of multiplication operators induced by $\phi_t(z) = \exp(-t(1+z)/(1-z))$ on $D(\tilde{\mu})$. Since the cogenerator of such a semigroup is M_z , which is a 2-isometry, it follows by Proposition 2.2 that $\{M_{\phi_t}\}_{t \geq 0}$ consists of 2-isometries.

It remains to show that M_{ϕ_t} is analytic for every $t > 0$. If not, then there are a $t_0 > 0$ and a $F \in \tilde{\mathcal{D}}_{\mathbb{C}_+}(\tilde{\nu})$ such that the function $s \mapsto e^{nt_0s}F(s)$ lies in $\tilde{\mathcal{D}}_{\mathbb{C}_+}(\tilde{\nu})$ for $n = 1, 2, 3, \dots$

In particular,

$$\int_{\mathbb{C}_+} |(F(s)e^{nt_0s})'|^2 x dx dy < \infty.$$

Transferring to the disc by letting $s = (1+z)/(1-z)$ and $F(s) = f(z)$, we have

$$\int_{\mathbb{D}} |[f(z) \exp(nt_0(1+z)/(1-z))]|^2 \frac{1-|z|^2}{|1-z|^2} dA(z) < \infty,$$

so that the function $z \mapsto f(z) \exp(nt_0(1+z)/(1-z))$ lies in the weighted Dirichlet space $D(\delta_1)$ corresponding to a Dirac measure at 1, and hence in $H^2(\mathbb{D})$, by [7, Thm. 7.1.2]. We conclude that f is identically zero, since no nontrivial H^2 function can be divisible by an arbitrarily large power of a nonconstant inner function. Hence the analyticity is also established. □

A connection with the right-shift semigroup in weighted $L^2(\mathbb{R}_+)$. Now, by means of the Laplace transform, we will establish a connection of C_0 -semigroups of analytic 2-isometries $\{T(t)\}_{t \geq 0}$ acting on a Hilbert space \mathcal{H} and the the right shift semigroup $\{S_t\}_{t \geq 0}$

$$S_t f(x) = \begin{cases} 0 & \text{if } 0 \leq t \leq x, \\ f(x-t) & \text{if } x > t, \end{cases}$$

acting on a weighted Lebesgue space on the half line \mathbb{R}_+ .

First, let us begin by recalling a result asserting that for each $\alpha > -1$, a function G analytic in \mathbb{C}_+ belongs to the *weighted Bergman space* $\mathcal{A}_\alpha^2(\mathbb{C}_+)$, that is, the space consisting of analytic functions on \mathbb{C}_+ for which

$$\|G\|_{\mathcal{A}_\alpha^2(\Pi^+)}^2 = \int_{\mathbb{C}_+} |G(x+iy)|^2 x^\alpha dx dy < \infty,$$

if and only if it has the form

$$G(s) := \mathcal{L}g(s) = \int_0^\infty e^{-st} g(t) dt, \quad s \in \mathbb{C}_+,$$

where g is a measurable function on \mathbb{R}^+ with

$$\int_0^\infty |g(t)|^2 t^{-1-\alpha} dt < \infty.$$

Moreover,

$$\|G\|_{\mathcal{A}_\alpha^2(\mathbb{C}_+)}^2 = \frac{\pi \Gamma(1+\alpha)}{2^\alpha} \int_0^\infty |g(t)|^2 t^{-1-\alpha} dt,$$

(see [4] or [5, Theorem 1], for instance). In other words, the Laplace transform is an isometric isomorphism between $\mathcal{A}_\alpha^2(\mathbb{C}_+)$ and $L^2(\mathbb{R}_+, \left(\frac{\pi \Gamma(1+\alpha)}{2^\alpha}\right)^{1/2} t^{-1-\alpha} dt)$. Hence, by means of a density argument, and taking $\alpha = 1$, it follows that for any $F \in \widetilde{\mathcal{D}}_{\mathbb{C}_+}(\nu)$, there exists $f \in L^2(\mathbb{R}_+)$ (unique in the usual sense of equivalence classes), such that

$$(i) \quad \mathcal{L}(tf(t)) = F'(s),$$

(ii)

$$\frac{1}{\pi} \int_{\mathbb{C}_+} |F'(s)|^2 x dx dy = \frac{1}{2} \int_0^\infty |f(t)|^2 dt, \quad (s = x + iy),$$

which corresponds to the first sum in (6); and

(iii)

$$\frac{1}{\pi^2} \int_{\mathbb{C}_+} |F'(s)|^2 \frac{x}{x^2 + (y - \tau)^2} dx dy = \frac{1}{2\pi} \int_0^\infty \left| \int_0^t u f(u) e^{-i\tau u} du \right|^2 \frac{dt}{t^2}, \quad (s = x + iy).$$

These three items along with the fact that for any $t \geq 0$, $T(t)$ is similar to the multiplication operator induced by $\exp(-ts)$ acting on the space $\widetilde{\mathcal{D}}_{\mathbb{C}_+}(\nu)$ yields, by means of the Laplace transform, that $\{T(t)\}_{t \geq 0}$ is transformed to the right-shift semigroup $\{S_t\}_{t \geq 0}$ acting on the Hilbert space \mathfrak{H} which consists of functions f defined on \mathbb{R}_+ such that

$$\int_0^\infty |f(t)|^2 dt + \int_0^\infty \int_{-\infty}^\infty \left| \int_0^t f(u) e^{-i\tau u} u du \right|^2 d\nu(\tau) \frac{dt}{t^2} < \infty.$$

3. A FINAL REMARK ON INVARIANT SUBSPACES OF C_0 -SEMIGROUPS OF ANALYTIC 2-ISOMETRIES

As an application of our main result, we deal with the study of the lattice of the closed invariant subspaces of a C_0 -semigroup $\{T(t)\}_{t \geq 0}$ of analytic 2-isometries.

Here we shall use the following result from [15, Thm. 7.1] and [16, Thm. 3.2].

Theorem 3.1. *Let \mathcal{M} be a non-zero invariant subspace of $(M_z, D(\mu))$. Then $\mathcal{M} = \phi D_{\mu_\phi}$ where $\phi \in \mathcal{M} \ominus z\mathcal{M}$ is a multiplier of $D(\mu)$ and $d\mu_\phi = |\phi|^2 d\mu$.*

In the continuous case we have the following result:

Theorem 3.2. *Let $\{T(t)\}_{t \geq 0}$ denotes the semigroup of multiplication operators induced by $\exp(-ts)$ on the space $\widetilde{\mathcal{D}}_{\mathbb{C}_+}(\nu)$, as in Theorem 2.4, and let \mathcal{M} be a non-zero closed subspace invariant under all the operators $T(t)$. Then there is a function $\psi \in \mathcal{M}$ such that $\mathcal{M} = \psi \widetilde{\mathcal{D}}_{\mathbb{C}_+}(\nu_\psi)$.*

Proof. If \mathcal{M} is invariant under the semigroup, then it is also invariant under the cogenerator V , and after transforming to the disc as in the proof of Theorem 2.4, we may apply Theorem 3.1.

Note that under the equivalence between $D(\mu)$ and $\tilde{\mathcal{D}}_{\mathbb{C}_+}(\nu)$, as detailed in (4) and (5), the subspace ϕD_{μ_ϕ} maps to a space $\psi \tilde{\mathcal{D}}_{\mathbb{C}_+}(\nu_\psi)$, where $\psi(s) = \phi((s-1)/(s+1))$ and $d\nu_\psi = |\psi|^2 d\nu$. \square

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