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Recovering the initial distribution for strongly damped wave equation

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Abstract

We study for the first time the inverse backward problem for the strongly damped wave equation. First, we show that the problem is severely ill-posed in the sense of Hadamard. Then, under the *a priori* assumption on the exact solution belonging to a Gevrey space, we propose the Fourier truncation method for stabilising the ill-posed problem. A stability estimate of logarithmic type is established.

Keywords and phrases: Fourier regularization method; final value problem; strongly damped wave equation.
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1. Introduction

The strongly damped wave equation (SDWE), see (1.1) below, occurs in a wide range of applications modelling the motion of viscoelastic materials [3, 7, 9, 10]. From both the theoretical and numerical point of view, the initial value problem for this equation has been extensively studied (e.g., [4, 5, 11]). However, to the best of our knowledge, the final value (backward) problem has not been solved yet (though it is worth mentioning that in [6], Lattes and Lions introduced the problem (1.1)-(1.2) but they did not regularize it). Our major objective is to provide a regularization method for solving the ill-posed nonlinear problem (1.1)-(1.2).

Let T be a positive number and Ω be an open, bounded and connected domain in $\mathbb{R}^n$, $n \geq 1$ with a smooth boundary $\partial\Omega$. We are interested in the following inverse backward problem: Find the initial data $u(x,0)$ for $x \in \Omega$, where $u(x,t)$ satisfies the following semilinear SDWE:

$$u_{tt} - \alpha \mathcal{A} u_t - \mathcal{A} u = F(x,t,u(x,t)), \quad (x,t) \in \Omega \times (0,T), \tag{1.1}$$

subject to the conditions

$$\begin{cases} u = 0, & (x,t) \in \partial\Omega \times (0,T), \\ u(x,T) = g(x), & (x,t) \in \Omega \times (0,T), \\ u_t(x,T) = h(x), & (x,t) \in \Omega \times (0,T), \end{cases} \tag{1.2}$$

where $\alpha > 0$ is a given damping constant, $g(x)$ and $h(x)$ are given functions, and the source function F will be defined later. Here, the operator $\mathcal{A} : D(\mathcal{A}) \subset L^2(\Omega) \to L^2(\Omega)$ is a linear, positive-definite, self-adjoint operator with compact inverse in $L^2(\Omega)$. For instance, $\mathcal{A} := \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}(l_{ij}(x)\frac{\partial}{\partial x_j})$, $x \in \Omega$ is a linear second-order elliptic operator with smooth coefficients $\{l_{ij}\}_{i,j=1}^{d}$ being symmetric and uniformly positive definite. Then, the Dirichlet eigenvalues $(\lambda_j)_{j\in\mathbb{N}^*}$ of $-\mathcal{A}$ satisfy, see Chapter 6.5 in [2], $0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq ... \leq \lambda_j \leq ...$ and $\lim_{j\to\infty} \lambda_j = \infty$.

In practice, the data (g,h) is obtained by measurement contaminated with noise. Hence, instead of (g,h), we have the observation data $(g^\epsilon, h^\epsilon) \in L^2(\Omega) \times L^2(\Omega)$ satisfying

$$\|g^\epsilon - g\|_{L^2(\Omega)} + \|h^\epsilon - h\|_{L^2(\Omega)} \leq \epsilon, \tag{1.3}$$

where the constant $\epsilon > 0$ represents a bound on the measurement error. We will show that the inverse backward problem (1.1)-(1.2) is ill-posed in the sense of Hadamard in Section 2. In order to stabilise the solution, in Section 3 we develop a regularization method based on the truncated Fourier method for which a stability estimate of logarithmic type is established.

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2. Ill-posedness of the inverse problem (1.1)-(1.2)

Assume that the problem (1.1)-(1.2) has a unique solution in the series form

$$u(x,t) = \sum_{j=1}^{\infty} u_j(t)\phi_j(x), \tag{2.4}$$

where ϕ_j denotes the eigenfunction corresponding to the eigenvalue λ_j and

$$\begin{cases} u_j''(t) + \alpha\lambda_j u_j'(t) + \lambda_j u_j(t) = F_j(u)(t), & t \in (0,T), \\ u_j(T) = g_j, \quad u_j'(T) = h_j, \end{cases} \tag{2.5}$$

where $F_j(u)(t) = \int_\Omega F(x,t,u(x,t))\phi_j(x)dx$, $g_j = \int_\Omega g(x)\phi_j(x)dx$ and $h_j = \int_\Omega h(x)\phi_j(x)dx$.

For given fixed damping $\alpha > 0$, consider the decomposition

$$\mathbb{N}^* = \left(\mathbb{D}_1 = \left\{j \in \mathbb{N}^* | \lambda_j > \frac{4}{\alpha^2}\right\}\right) \cup \left(\mathbb{D}_2 = \left\{j \in \mathbb{N}^* | \lambda_j = \frac{4}{\alpha^2}\right\}\right) \cup \left(\mathbb{D}_3 = \left\{j \in \mathbb{N}^* | \lambda_j < \frac{4}{\alpha^2}\right\}\right).$$

Then, the solution of (2.5) is given by:

(i) if $j \in \mathbb{D}_1$ then

$$u_j(t) = \frac{\lambda_j^+ e^{(T-t)\lambda_j^-} - \lambda_j^- e^{(T-t)\lambda_j^+}}{\sqrt{\Lambda_j}} g_j - \frac{e^{(T-t)\lambda_j^+} - e^{(T-t)\lambda_j^-}}{\sqrt{\Lambda_j}} h_j + \int_t^T \frac{e^{(s-t)\lambda_j^+} - e^{(s-t)\lambda_j^-}}{\sqrt{\Lambda_j}} F_j(u)(s)ds, \tag{2.6}$$

(ii) if $j \in \mathbb{D}_2$ then

$$u_j(t) = e^{\frac{2}{\alpha}(T-t)}\left[1 - \frac{2}{\alpha}(T-t)\right] g_j - e^{\frac{2}{\alpha}(T-t)}(T-t)h_j + \int_t^T (s-t)e^{\frac{2}{\alpha}(s-t)} F_j(u)(s)ds,$$

(iii) if $j \in \mathbb{D}_3$ then

$$\begin{aligned} u_j(t) &= \frac{2e^{\frac{\alpha\lambda_j}{2}(T-t)}}{\sqrt{-\Lambda_j}}\left[\frac{\sqrt{-\Lambda_j}}{2}\cos\left(\frac{\sqrt{-\Lambda_j}}{2}(T-t)\right) + \frac{\alpha\lambda_j}{2}\sin\left(\frac{\sqrt{-\Lambda_j}}{2}(T-t)\right)\right] g_j \\ &- \frac{2e^{\frac{\alpha\lambda_j}{2}(T-t)}}{\sqrt{-\Lambda_j}}\sin\left(\frac{\sqrt{-\Lambda_j}}{2}(T-t)\right) h_j + \int_t^T \frac{2e^{\frac{\alpha\lambda_j}{2}(s-t)}}{\sqrt{-\Lambda_j}}\sin\left(\frac{\sqrt{-\Lambda_j}}{2}(s-t)\right) F_j(u)(s)ds, \end{aligned}$$

where

$$\lambda_j^+ = \frac{\alpha\lambda_j + \sqrt{\alpha^2\lambda_j^2 - 4\lambda_j}}{2}, \quad \lambda_j^- = \frac{\alpha\lambda_j - \sqrt{\alpha^2\lambda_j^2 - 4\lambda_j}}{2}, \quad \Lambda_j = \alpha^2\lambda_j^2 - 4\lambda_j. \tag{2.7}$$

Hence, the solution (2.4) can be represented as

$$u(x,t) = \sum_{j\in\mathbb{D}_1} u_j(t)\phi_j + \sum_{j\in\mathbb{D}_2} u_j(t)\phi_j + \sum_{j\in\mathbb{D}_3} u_j(t)\phi_j.$$

From above observations, we can show that the term $\sum_{j\in\mathbb{D}_2} u_j(t)\phi_j + \sum_{j\in\mathbb{D}_3} u_j(t)\phi_j$ containing sin and cos trigonometric functions is bounded and stable, and no regularization is need it for it. We only need to regularize the first term $\sum_{j\in\mathbb{D}_1} u_j(t)\phi_j$ which contains the exponential terms in (2.6). Alternatively, we can take $\mathbb{D}_1 = \mathbb{D}_2 = \emptyset$ by assuming that $\alpha^2\lambda_1 > 4$, as will adopted in the remanding of the paper. Note that this also implies $\alpha^2\lambda_j^2 - 4\lambda_j > 0$ for all $j \in \mathbb{N}^*$, and hence the roots in (2.7) are real and distinct.

From now on, we denote by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ the inner product and norm in $L^2(\Omega)$, respectively. For $\varphi \in L^2(\Omega)$, defining

$$\mathcal{S}(t)\varphi = \sum_{j=1}^{\infty} \frac{\lambda_j^+ e^{t\lambda_j^-} - \lambda_j^- e^{t\lambda_j^+}}{\lambda_j^+ - \lambda_j^-}\left\langle\varphi,\phi_j\right\rangle\phi_j, \quad \mathcal{P}(t)\varphi = \sum_{j=1}^{\infty} \frac{e^{t\lambda_j^-} - e^{t\lambda_j^+}}{\lambda_j^+ - \lambda_j^-}\left\langle\varphi,\phi_j\right\rangle\phi_j, \tag{2.8}$$

we can recast (2.4) in the form

$$u(x,t) = \mathcal{S}(T-t)g(x) + \mathcal{P}(T-t)h(x) - \int_t^T \mathcal{P}(s-t)F(u(s))ds. \tag{2.9}$$

Next, we give an example which shows that the solution $u_m(x,t)$ (for any $m \in \mathbb{N}^*$) of problem (1.1)-(1.2) (if it exists) is not stable. Let $u_{T,m}$ and $v_{T,m}$ and F_0 be defined as follows:

$$u_m(x,T) = u_{T,m}(x) = 0, \quad \partial_t u_m(x,T) = v_{T,m}(x) = \frac{\phi_m(x)}{\sqrt{\lambda_m}} \text{ and } F_0(w) = \sum_{j=1}^{\infty} \frac{e^{-\alpha T \lambda_j}}{2T^2} \left\langle w(x,t), \phi_j \right\rangle \phi_j, \quad \forall w \in L^2(\Omega). \tag{2.10}$$

Let u_m satisfy the integral equation

$$u_m(x,t) = \mathcal{P}(T-t)v_{T,m} - \int_t^T \mathcal{P}(s-t)F_0(u_m(s))ds. \tag{2.11}$$

First, we show that (2.11) has a unique solution $u_m \in C([0,T]; L^2(\Omega))$. Indeed, we consider the function

$$\mathcal{H}(w)(t) = \mathcal{P}(T-t)v_{T,m}(x) - \int_t^T \mathcal{P}(s-t)F_0(w(s))ds. \tag{2.12}$$

Then, for any $v_1, v_2 \in C([0,T]; L^2(\Omega))$, we have

$$\begin{aligned} \|\mathcal{H}(v_1)(t) - \mathcal{H}(v_2)(t)\| &\le \int_t^T \left\| \mathcal{P}(s-t)\left(F_0(v_1(s)) - F_0(v_2(s))\right) \right\| ds \\ &= \int_t^T \sqrt{\sum_{j=1}^{\infty} \left[\frac{e^{(s-t)\lambda_j^-} - e^{(s-t)\lambda_j^+}}{\lambda_j^+ - \lambda_j^-} \right]^2 \left\langle F_0(v_1(s)) - F_0(v_2(s)), \phi_j \right\rangle^2 } ds \\ &= \frac{1}{2} \int_t^T \sqrt{\sum_{j=1}^{\infty} \left[\frac{e^{(s-t)\lambda_j^-} - e^{(s-t)\lambda_j^+}}{\lambda_j^+ - \lambda_j^-} \right]^2 \frac{e^{-2\alpha T \lambda_j}}{T^4} \left\langle v_1(s) - v_2(s), \phi_j \right\rangle^2 } ds. \end{aligned} \tag{2.13}$$

Using the the inequality $|e^{-a} - e^{-b}| \le |a-b|$ for $a, b > 0$, we have

$$\left[\frac{e^{(s-t)\lambda_j^-} - e^{(s-t)\lambda_j^+}}{\lambda_j^+ - \lambda_j^-} \right]^2 \frac{e^{-2\alpha T\lambda_j}}{T^4} = e^{2(s-t)(\lambda_j^+ + \lambda_j^-)} \left[\frac{e^{-(s-t)\lambda_j^+} - e^{-(s-t)\lambda_j^-}}{\lambda_j^+ - \lambda_j^-} \right]^2 \frac{e^{-2\alpha T\lambda_j}}{T^4} \le (s-t)^2 e^{2\alpha(s-t)\lambda_j} \frac{e^{-2\alpha T\lambda_j}}{T^4} \le \frac{1}{T^2}. \tag{2.14}$$

From (2.13) and (2.14) we deduce that

$$\|\mathcal{H}(v_1)(t) - \mathcal{H}(v_2)(t)\| \le \frac{1}{2} \int_t^T \frac{1}{T} \|v_1(s) - v_2(s)\| ds \le \frac{1}{2} \|v_1 - v_2\|_{C([0,T];L^2(\Omega))}, \quad \forall t \in [0,T]. \tag{2.15}$$

This implies that

$$\|\mathcal{H}(v_1) - \mathcal{H}(v_2)\|_{C([0,T];L^2(\Omega))} \le \frac{1}{2} \|v_1 - v_2\|_{C([0,T];L^2(\Omega))}. \tag{2.16}$$

Hence, $\mathcal{H}$ is a contraction. Using the Banach fixed-point theorem, we conclude that $\mathcal{H}(w) = w$ has a unique solution $u_m \in C([0,T]; L^2(\Omega))$.

It is easy to see that (here, noting that $F_0(0) = 0$)

$$\left\| \int_t^T \mathcal{P}(s-t)F_0(u_m(s))ds \right\| = \|\mathcal{H}(u_m)(t) - \mathcal{H}(0)(t)\| \le \frac{1}{2} \|u_m\|_{C([0,T];L^2(\Omega))}. \tag{2.17}$$

Hence,

$$\|u_m(t)\| \ge \left\| \mathcal{P}(T-t)v_{T,m} \right\| - \left\| \int_t^T \mathcal{P}(s-t)F_0(u_m(s))ds \right\| \ge \left\| \mathcal{P}(T-t)v_{T,m} \right\| - \frac{1}{2} \|u_m\|_{C([0,T];L^2(\Omega))}. \tag{2.18}$$

This leads to

$$\|u_m\|_{C([0,T];L^2(\Omega))} \ge \frac{2}{3} \sup_{0 \le t \le T} \left\| \mathcal{P}(T-t)v_{T,m} \right\|. \tag{2.19}$$

We continue to estimate the right hand side of this inequality. We have

$$\left\| \mathcal{P}(T-t)v_{T,m} \right\|^2 = \left[\frac{e^{(T-t)\lambda_m^-} - e^{(T-t)\lambda_m^+}}{\lambda_m^+ - \lambda_m^-} \right]^2 \frac{1}{\lambda_m} = \frac{e^{2(T-t)\lambda_m^+} \left(1 - e^{-(T-t)\sqrt{\alpha^2\lambda_m^2 - 4\lambda_m}} \right)^2}{\lambda_m \left(\lambda_m^+ - \lambda_m^-\right)^2} \ge \frac{e^{2(T-t)\lambda_m^+} \left(1 - e^{-(T-t)\sqrt{\alpha^2\lambda_1^2 - 4\lambda_1}} \right)^2}{\lambda_m \left(\alpha^2\lambda_m^2 - 4\lambda_m\right)}.$$

Since the function $\Phi(t) = e^{(T-t)\lambda_m^+}\left(1 - e^{-(T-t)\sqrt{\alpha^2\lambda_1^2 - 4\lambda_1}}\right)$ is a decreasing function, we deduce that

$$\sup_{0\le t\le T}\left\|\mathcal{P}(T-t)v_{T,m}\right\| \ge \sup_{0\le t\le T}\frac{e^{(T-t)\lambda_m^+}\left(1 - e^{-(T-t)\sqrt{\alpha^2\lambda_1^2-4\lambda_1}}\right)}{\sqrt{\lambda_m(\alpha^2\lambda_m^2 - 4\lambda_m)}} = \frac{e^{T\lambda_m^+}\left(1 - e^{-T\sqrt{\alpha^2\lambda_1^2-4\lambda_1}}\right)}{\sqrt{\lambda_m(\alpha^2\lambda_m^2 - 4\lambda_m)}} \ge \frac{e^{\alpha T\lambda_m/2}\left(1 - e^{-T\sqrt{\alpha^2\lambda_1^2-4\lambda_1}}\right)}{\sqrt{\lambda_m(\alpha^2\lambda_m^2 - 4\lambda_m)}}. \tag{2.20}$$

Combining (2.19) and (2.20) yields

$$\|u_m\|_{C([0,T];L^2(\Omega))} \ge \frac{2}{3}\frac{e^{\alpha T\lambda_m/2}\left(1 - e^{-T\sqrt{\alpha^2\lambda_1^2-4\lambda_1}}\right)}{\sqrt{\lambda_m(\alpha^2\lambda_m^2 - 4\lambda_m)}}. \tag{2.21}$$

As $m \to +\infty$, we see that

$$\begin{aligned}\lim_{m\to+\infty}(\|u_{T,m}\| + \|v_{T,m}\|) &= \lim_{m\to+\infty}\frac{1}{\sqrt{\lambda_m}} = 0,\\ \lim_{m\to+\infty}\|u_m\|_{C([0,T];L^2(\Omega))} &\ge \lim_{m\to+\infty}\frac{2}{3}\frac{e^{\alpha T\lambda_m/2}\left(1 - e^{-T\sqrt{\alpha^2\lambda_1^2-4\lambda_1}}\right)}{\sqrt{\lambda_m(\alpha^2\lambda_m^2 - 4\lambda_m)}} = +\infty.\end{aligned} \tag{2.22}$$

This shows that problem (1.1)-(1.2) is ill-posed in the sense of Hadamard in the L^2-norm.

3. Fourier's truncation method

For $\varphi \in L^2(\Omega)$, let us define the truncated version of (2.8) as

$$\mathcal{S}_N(t)\varphi = \sum_{j=1}^{N}\frac{\lambda_j^+e^{t\lambda_j^-} - \lambda_j^-e^{t\lambda_j^+}}{\lambda_j^+ - \lambda_j^-}\left\langle\varphi,\phi_j\right\rangle\phi_j, \qquad \mathcal{P}_N(t)\varphi = \sum_{j=1}^{N}\frac{e^{t\lambda_j^-} - e^{t\lambda_j^+}}{\lambda_j^+ - \lambda_j^-}\left\langle\varphi,\phi_j\right\rangle\phi_j. \tag{3.23}$$

Let us define the regularized solution by Fourier's truncation method as follows:

$$u^{N,\epsilon}(x,t) = \mathcal{S}_N(T-t)g^\epsilon(x) + \mathcal{P}_N(T-t)h^\epsilon(x) - \int_t^T \mathcal{P}_N(s-t)F(u^{N,\epsilon})(x,s)ds, \tag{3.24}$$

where N is a parameter regularization to be prescribed.

Lemma 3.1. *The following estimates hold:*

$$\|\mathcal{S}_N(t)\|_{\mathcal{L}(L^2(\Omega))} \le \sqrt{\frac{2\alpha^2 + 8T^2}{\alpha^2}}e^{\alpha t\lambda_N}, \qquad \|\mathcal{P}_N(t)\|_{\mathcal{L}(L^2(\Omega))} \le Te^{\alpha t\lambda_N}, \quad \forall t\in[0,T]. \tag{3.25}$$

Proof. Let $\varphi \in L^2(\Omega)$. From (3.23) we have

$$\|\mathcal{S}_N(t)\varphi\|^2 = \sum_{j=1}^{N}\left[\frac{\lambda_j^+e^{t\lambda_j^-} - \lambda_j^-e^{t\lambda_j^+}}{\lambda_j^+ - \lambda_j^-}\right]^2\left\langle\varphi,\phi_j\right\rangle^2 = \sum_{j=1}^{N}e^{2t(\lambda_j^+ + \lambda_j^-)}\left[\frac{\lambda_j^+e^{-t\lambda_j^+} - \lambda_j^-e^{-t\lambda_j^-}}{\lambda_j^+ - \lambda_j^-}\right]^2\left\langle\varphi,\phi_j\right\rangle^2. \tag{3.26}$$

Using the inequality $(a+b)^2 \le 2a^2 + 2b^2$ for $a, b \in \mathbb{R}$ and the inequality $|e^{-a} - e^{-b}| \le |a-b|$ for $a, b > 0$, we obtain

$$\begin{aligned}\left[\frac{\lambda_j^+e^{-t\lambda_j^+} - \lambda_j^-e^{-t\lambda_j^-}}{\lambda_j^+ - \lambda_j^-}\right]^2 &= \left(e^{-t\lambda_j^+} + \lambda_j^-\frac{e^{-t\lambda_j^+} - e^{-t\lambda_j^-}}{\lambda_j^+ - \lambda_j^-}\right)^2 \le 2e^{-2t\lambda_j^+} + 2|\lambda_j^-|^2t^2\\ &\le 2 + \left(\frac{2t\lambda_j}{\alpha\lambda_j + \sqrt{\alpha^2\lambda_j^2 - 4\lambda_j}}\right)^2 \le \frac{2\alpha^2 + 8T^2}{\alpha^2}.\end{aligned} \tag{3.27}$$

It follows from (3.26) and (3.27) that

$$\|\mathcal{S}_N(t)\varphi\|^2 \le \frac{2\alpha^2 + 8T^2}{\alpha^2}\sum_{j=1}^{N}e^{2t(\lambda_j^+ + \lambda_j^-)}\left\langle\varphi,\phi_j\right\rangle^2 \le \frac{2\alpha^2 + 8T^2}{\alpha^2}\sum_{j=1}^{N}e^{2\alpha t\lambda_j}\left\langle\varphi,\phi_j\right\rangle^2 \le \frac{2\alpha^2 + 8T^2}{\alpha^2}e^{2\alpha t\lambda_N}\|\varphi\|^2. \tag{3.28}$$

This completes the proof of the first part of (3.25). Also, using $|e^{-a} - e^{-b}| \le |a-b|$ for $a, b > 0$, we obtain

$$\|\mathcal{P}_N(t)\varphi\|^2 = \sum_{j=1}^{N}\left[\frac{e^{t\lambda_j^-} - e^{t\lambda_j^+}}{\lambda_j^+ - \lambda_j^-}\right]^2\left\langle\varphi,\phi_j\right\rangle^2 = \sum_{j=1}^{N}e^{2t(\lambda_j^+ + \lambda_j^-)}\left[\frac{e^{-t\lambda_j^+} - e^{-t\lambda_j^-}}{\lambda_j^+ - \lambda_j^-}\right]^2\left\langle\varphi,\phi_j\right\rangle^2 \le T^2\sum_{j=1}^{N}e^{2t(\lambda_j^+ + \lambda_j^-)}\left\langle\varphi,\phi_j\right\rangle^2, \tag{3.29}$$

which completes the proof of the second part of (3.25). □

At this stage, in order to obtain the convergence rate (3.31) given in the following theorem, we introduce the abstract Gevrey class of functions of order $\gamma > 0$ and index $\sigma > 0$, *e.g.*, [1], defined by

$$\mathbb{G}_{\gamma,\sigma} = \left\{ v \in L^2(\Omega) : \sum_{j=1}^{\infty} \lambda_j^{2\gamma} e^{2\sigma\lambda_j} \left| \left\langle v, \phi_j(x) \right\rangle \right|^2 < \infty \right\},$$

which is a Hilbert space equipped with the inner product

$$\langle v_1, v_2 \rangle_{\mathbb{G}_{\gamma,\sigma}} := \left\langle (-\Delta)^\gamma e^{\sigma\sqrt{-\Delta}} v_1, (-\Delta)^\gamma e^{\sigma\sqrt{-\Delta}} v_2 \right\rangle, \quad \forall v_1, v_2 \in \mathbb{G}_{\gamma,\sigma},$$

and the corresponding norm $\|v\|_{\mathbb{G}_{\gamma,\sigma}} = \sqrt{\sum_{j=1}^{\infty} \lambda_j^{2\gamma} e^{2\sigma\lambda_j} \left| \left\langle v, \phi_j(x) \right\rangle \right|^2} < \infty$.

We also assume that F is globally Lipschitz, i.e. there exists a constant $K > 0$ such that

$$\|F(x,t,u) - F(x,t,v)\| \le K\|u - v\|, \quad \forall u, v \in L^2(\Omega),\ \forall (x,t) \in \Omega \times (0,T). \tag{3.30}$$

Theorem 3.1. *The nonlinear integral equation* (3.24) *has a unique solution* $u^{N,\epsilon} \in C([0,T]; L^2(\Omega))$. *Assuming further that* $u \in L^\infty(0,T;\mathbb{G}_{\gamma,\alpha T})$ *for some* $\gamma > 0$, *then we have the following estimate:*

$$\|u^{N,\epsilon}(\cdot,t) - u(\cdot,t)\| \le e^{-\alpha t\lambda_N} e^{KT^2} \left(\left(\sqrt{\frac{2\alpha^2 + 8T^2}{\alpha^2}} + T \right) e^{\alpha T\lambda_N} \epsilon + \lambda_N^{-\gamma} \|u\|_{L^\infty(0,T;\mathbb{G}_{\gamma,\alpha T})} \right), \quad \forall t \in [0,T]. \tag{3.31}$$

Remark 3.1. *If we choose* $N = N(\epsilon)$ *such that* $\lambda_N \le \frac{\delta}{\alpha T} \ln\left(\frac{1}{\epsilon}\right)$ *for some* $\delta \in (0,1)$, *then the error* $\|u^{N,\epsilon}(\cdot,t) - u(\cdot,t)\|$ *is of logarithmic order* $\left[\ln(\frac{1}{\epsilon})\right]^{-\gamma}$. *Also, if* $F(x,t,u) = F(x,t)$ *does not depend on* u *then we do not need to employ the Gevrey space but only require that* $u \in L^\infty(0,T;L^2(\Omega))$. *To remove the assumption on* u *belonging to* $L^\infty(0,T;\mathbb{G}_{\gamma,\alpha T})$, *we can employ a new regularization method described in [8] for the Cauchy problem for semilinear elliptic equations, but its extension to the present damped semilinear wave equation* (1.1) *is deferred to a future work.*

Proof. **Part 1.** The existence and uniqueness solution of the nonlinear integral equation (3.24).

For $w \in C([0,T]; L^2(\Omega)))$, we define

$$\mathcal{G}(w)(x,t) = \mathcal{S}_N(T-t)g^\epsilon(x) + \mathcal{P}_N(T-t)h^\epsilon(x) - \int_t^T \mathcal{P}_N(s-t)F(w(x,s))ds. \tag{3.32}$$

We shall prove by induction that

$$\left\| \mathcal{G}^m(w_1) - \mathcal{G}^m(w_2) \right\|_{C([0,T];L^2(\Omega))} \le \frac{\left(KTe^{\alpha T\lambda_N}(T-t)\right)^m}{m!} \|w_1 - w_2\|_{C([0,T];L^2(\Omega))}, \quad \forall w_1, w_2 \in C([0,T]; L^2(\Omega)). \tag{3.33}$$

For $m = 1$, we have

$$\begin{aligned} \|\mathcal{G}(w_1) - \mathcal{G}(w_2)\| &= \left\| \int_t^T \mathcal{P}_N(s-t)\left(F(w_1(s)) - F(w_2(s))\right) ds \right\| \\ &\le \int_t^T Te^{\alpha(s-t)\lambda_N} \|F(w_1(s)) - F(w_2(s))\| ds \le KTe^{\alpha T\lambda_N}(T-t)\|w_1 - w_2\|_{C([0,T];L^2(\Omega))}. \end{aligned} \tag{3.34}$$

Assume that (3.33) holds for $m = p$. We show that (3.33) holds for $m = p+1$. Indeed, we have

$$\begin{aligned} \|\mathcal{G}^{p+1}(w_1) - \mathcal{G}^{p+1}(w_2)\| &= \left\| \int_t^T \mathcal{P}_N(s-t)\left(F(\mathcal{G}^p(w_1)(s)) - F(\mathcal{G}^p(w_1)(s))\right) ds \right\| \\ \le \int_t^T Te^{\alpha(s-t)\lambda_N} \left\| F(\mathcal{G}^p(w_1)(s)) - F(\mathcal{G}^p(w_1)(s)) \right\| ds &\le KTe^{\alpha T\lambda_N} \int_t^T \|\mathcal{G}^p(w_1(s)) - \mathcal{G}^p(w_2(s))\| ds \\ \le KTe^{\alpha T\lambda_N} \int_t^T \frac{\left(KTe^{\alpha T\lambda_N}(T-s)\right)^p}{p!} \|w_1 - w_2\|_{C([0,T];L^2(\Omega))} ds &\le \frac{\left(KTe^{\alpha T\lambda_N}(T-t)\right)^{p+1}}{(p+1)!} \|w_1 - w_2\|_{C([0,T];L^2(\Omega))}. \end{aligned} \tag{3.35}$$

Therefore, by the induction principle, we have that (3.33) holds. Since $\lim_{m\to+\infty} \frac{\left(KT^2e^{\alpha T\lambda_N}\right)^m}{m!} = 0$ there exists a positive integer m_0 such that $\mathcal{G}^{m_0}$ is a contraction. It follows that the equation $\mathcal{G}^{m_0}w = w$ has a unique solution $u^{N,\epsilon} \in C([0,T]; L^2(\Omega))$. We claim that $\mathcal{G}(u^{N,\epsilon}) = u^{N,\epsilon}$. Indeed, since $\mathcal{G}^{m_0}(u^{N,\epsilon}) = u^{N,\epsilon}$, we know that $\mathcal{G}\left(\mathcal{G}^{m_0}(u^{N,\epsilon})\right) = \mathcal{G}(u^{N,\epsilon})$. This is equivalent to $\mathcal{G}^{m_0}\left(\mathcal{G}(u^{N,\epsilon})\right) = \mathcal{G}(u^{N,\epsilon})$. Hence, $\mathcal{G}(u^{N,\epsilon})$ is a fixed point of $\mathcal{G}^{m_0}$. Moreover, as noted above, $u^{N,\epsilon}$ is a fixed point of $\mathcal{G}^{m_0}$.

Part 2. Denote

$$U^N(x,t) = \mathcal{S}_N(T-t)g(x) + \mathcal{P}_N(T-t)h(x) - \int_t^T \mathcal{P}_N(s-t)F(u^N(x,s))ds. \quad (3.36)$$

Step 1. Firstly, we estimate $\|u^{N,\epsilon}(\cdot,t) - U^N(\cdot,t)\|$. Using Lemma 3.1, we have

$$\|u^{N,\epsilon}(\cdot,t) - U^N(\cdot,t)\| \le \left\|\mathcal{S}_N(T-t)\big(g-g^\epsilon\big)\right\| + \left\|\mathcal{P}_N(T-t)\big(h-h^\epsilon\big)\right\| + \left\|\int_t^T \mathcal{P}_N(s-t)\Big(F(u^{N,\epsilon}(\cdot,s)) - F(U^N(\cdot,s))\Big)ds\right\|$$
$$\le \sqrt{\frac{2\alpha^2+8T^2}{\alpha^2}}e^{\alpha(T-t)\lambda_N}\|g-g^\epsilon\| + Te^{\alpha(T-t)\lambda_N}\|h-h^\epsilon\| + T\int_t^T e^{\alpha(s-t)\lambda_N}\left\|F(u^{N,\epsilon}(\cdot,s)) - F(U^N(\cdot,s))\right\|ds \quad (3.37)$$

It follows from (1.3) and (3.30) that

$$\|u^{N,\epsilon}(\cdot,t) - U^N(\cdot,t)\| \le \left(\sqrt{\frac{2\alpha^2+8T^2}{\alpha^2}} + T\right)e^{\alpha(T-t)\lambda_N}\epsilon + KT\int_t^T e^{\alpha(s-t)\lambda_N}\|u^{N,\epsilon}(\cdot,s) - U^N(\cdot,s)\|ds. \quad (3.38)$$

Multiplying both sides by $e^{\alpha t\lambda_N}$ and applying Gronwall's inequality, we derive that

$$e^{\alpha t\lambda_N}\|u^{N,\epsilon}(\cdot,t) - U^N(\cdot,t)\| \le \left(\sqrt{\frac{2\alpha^2+8T^2}{\alpha^2}} + T\right)e^{\alpha T\lambda_N}\epsilon e^{K(T-t)T}.$$

Hence,

$$\|u^{N,\epsilon}(\cdot,t) - U^N(\cdot,t)\| \le e^{KT(T-t)}\left(\sqrt{\frac{2\alpha^2+8T^2}{\alpha^2}} + T\right)e^{\alpha(T-t)\lambda_N}\epsilon. \quad (3.39)$$

Step 2. Secondly, we estimate $\|u(\cdot,t) - U^N(\cdot,t)\|$. First, it is easy to see that

$$\sum_{j=1}^N u_j(t)\phi_j(x) = \mathcal{S}_N(T-t)g(x) + \mathcal{P}_N(T-t)h(x) - \int_t^T \mathcal{P}_N(s-t)F(u(x,s))ds.$$

Using Lemma 3.1, we obtain

$$\|u(\cdot,t) - U^N(\cdot,t)\| \le \left\|u(\cdot,t) - \sum_{j=1}^N u_j(t)\phi_j(\cdot)\right\| + \left\|\sum_{j=1}^N u_j(t)\phi_j(\cdot) - U^N(\cdot,t)\right\|$$
$$\le \sqrt{\sum_{j=N+1}^\infty \lambda_N^{-2\gamma}e^{-2\alpha t\lambda_N}\lambda_j^{2\gamma}e^{2\alpha t\lambda_j}|u_j(t)|^2} + \int_t^T \mathcal{P}_N(s-t)\left\|F(u(\cdot,s)) - F(U^N(\cdot,s))\right\|ds$$
$$\le \lambda_N^{-\gamma}e^{-\alpha t\lambda_N}\|u\|_{L^\infty(0,T;\mathbb{G}_{\gamma,\alpha T})} + KT\int_t^T e^{\alpha(s-t)\lambda_N}\|u(\cdot,s) - U^N(\cdot,s)\|ds. \quad (3.40)$$

Multiplying both sides by $e^{\alpha t\lambda_N}$ and applying Gronwall's inequality, we derive that

$$\|u(\cdot,t) - U^N(\cdot,t)\| \le e^{KT(T-t)}\|u\|_{L^\infty(0,T;\mathbb{G}_{\gamma,\alpha T})}\lambda_N^{-\gamma}e^{-\alpha t\lambda_N}. \quad (3.41)$$

Combining (3.39) and (3.41) we deduce (3.31). □

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