Proceedings Paper:

Zhao, S. orcid.org/0000-0003-3098-8059 and Sun, Z. (2017) Defend the practicality of single-integrator models in multi-robot coordination control. In: 13th IEEE International Conference on Control and Automation. 13th IEEE International Conference on Control and Automation, 03/07/2017 - 06/07/2017, Ohrid, Macedonia. Institute of Electrical and Electronics Engineers .

10.1109/ICCA.2017.8003139

Defend the Practicality of Single-Integrator Models in Multi-Robot Coordination Control

Shiyu Zhao and Zhiyong Sun

***Abstract*— Single-integrator models have been widely used to model robot kinematics in multi-robot coordination control problems. However, it is also widely believed that this model is too simple to lead to practically useful control laws. In this paper, we prove that if a gradient-descent distributed control law designed for single integrators has been proved to be convergent for a given coordination task, then the control law can be readily modified to adapt for various motion constraints including velocity saturation, obstacle avoidance, and nonholonomic models. This result is valid for a wide range of coordination tasks. It defends the practical usefulness of many existing coordination control laws designed based on single-integrator models and suggests a new methodology to design coordination control laws subject motion constraints.**

I. Introduction

The single-integrator model is the simplest model to characterize the motion of a mobile robot. This model has been widely used in multi-robot coordination control problems such as consensus and formation control. However, it is also believed that this model is too simple to give practically useful coordination control laws. That is because the velocity of a single-integrator robot can be arbitrarily assigned whereas both the direction and magnitude of the velocity of a real robot are constrained. As a result, even if a control law designed for single integrators has been proved to be convergent, the constraints may undermine the convergence of the control law when applied in practice and consequently cause potential safety risks. Motivated by this, many researchers have studied multi-robot coordination control with motion constraints such as nonholonomic dynamics [1], [2], velocity saturation [3], [4], and obstacle avoidance [5]–[7]. However, when motion constraints are considered, the coordination control systems are usually highly nonlinear and very challenging to analyze. The existing results are mainly restricted to specific types of coordination tasks or motion constraints. General approaches that can simultaneously guarantee system convergence and handle multiple motion constraints for a wide range of coordination tasks are highly desirable.

In this paper, we suppose a gradient distributed coordination control law has been obtained and proved to be convergent for a given coordination task. Our objective is to generalize the gradient control law so that the convergence is preserved and in the meantime various motion constraints can be fulfilled. The basic idea of our approach is to introduce an orthogonal projection matrix into the gradient control law. With a carefully designed projection matrix, the magnitude and direction of the velocity of each robot can be adjusted as required in a distributed manner to handle velocity saturation, obstacle avoidance, and unicycle constraints. This idea is motivated by the recent work in [8], where the authors use a time-varying rotation matrix to adjust the velocity of each robot to realize obstacle and collision avoidance. Compared to [8], our approach is more flexible since it is able to adjust both of the velocity direction and magnitude and is applicable to a wide range of coordination control problems and motion constraints.

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II. Problem Setup

Consider n robots in $\mathbb{R}^d$ ($n \geq 1$, $d = 2, 3$). Let $p_i \in \mathbb{R}^d$ be the position of robot i and $p = [p_1^T, \dots, p_n^T]^T \in \mathbb{R}^{dn}$. The interaction among the robots is described by a graph $\mathcal{G} = (\mathcal{E}, \mathcal{V})$, which consists of a vertex set $\mathcal{V} = \{1, \dots, n\}$ and an edge set $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$. If $(i, j) \in \mathcal{E}$, robot j is an neighbor of robot i and robot i receives the information of robot j. The set of neighbors for robot i is $\mathcal{N}_i = \{j \in \mathcal{V} : (i, j) \in \mathcal{E}\}$.

Given a coordination task, let the vector $e(p)$ of appropriate dimension be the error state so that $e(p) = 0$ if and only if the coordination task is achieved. Let $V(e)$ be a positive definite Lyapunov function. The gradient control law

$$\dot{p}_i = -\frac{\partial V(e)}{\partial p_i} := f_i(e, p), \quad i \in \mathcal{V}, \tag{1}$$

is usually a good candidate to solve the given coordination task because $\dot{V}(e) = \sum_{i \in \mathcal{V}} -f_i^T f_i \leq 0$. If $f_i(e, p)$ merely depends on the states of robot i and its neighbors, then the gradient control is distributed. By denoting $f = [f_1^T, \dots, f_n^T]^T \in \mathbb{R}^{dn}$, we have the error dynamics under the gradient control as $\dot{e} = (\partial e/\partial p) f(e, p)$.

Instead of considering any specific coordination task, we consider general tasks that satisfy the following conditions. Let $\|\cdot\|$ be the Euclidian norm of a vector.

Assumption 1. *For the given coordination control task, $V(e)$ and $e(p)$ satisfy*

(a) $V(e)$ is positive definite and continuously differentiable;
(b) The level set $\Omega(r) = \{e : V(e) \leq r\}$ with any $r \geq 0$ is compact;
(c) There exists $r_0 > 0$ so that $f = 0 \Leftrightarrow e = 0$ on $\Omega(r_0)$;
(d) $\|\partial e(p)/\partial p\|$ and $\|f(e, p)\|$ are bounded when $\|e\|$ is bounded;

(e) $f(e,p)$ *is continuous in* e *and uniformly continuous*[1] *in* p.

Remarks on Assumption 1 are given below. (i) Assumption 1 is mild since it is satisfied by a wide range of coordination tasks including, but not limited to, consensus, relative-position-based formation control, distance-based formation control, and bearing-based formation control (examples will be given later). (ii) Under Assumption 1, it follows from the invariance principle [9, Theorem 4.4] that $e = 0$ is asymptotically stable and the set $\Omega(r_0)$ is the *attraction region*, which means any trajectory of the error dynamics starting from $\Omega(r_0)$ converges to $e = 0$. For many linear coordination control problems, the attraction region is the entire space $\mathbb{R}^{\dim(e)}$. For nonlinear coordination tasks such as distance-based formation control, the attraction region may be a sufficiently small neighborhood of the origin $e = 0$. (iii) Condition (c) indicates that $e = 0$ if and only if $f = 0$. In other words, $e = 0$ is the unique critical point where the gradient flow vanishes in $\Omega(r_0)$. This condition usually requires some graphical conditions. For example, for the consensus problem as shown in Example 1, this condition holds if and only if the undirected graph is connected. (iv) For many linear coordination control problems, the function f merely depends on e; for some nonlinear coordination control problems such as distance-based formation control, f depends both on e and p.

To illustrate, we show some examples of coordination control problems that satisfy Assumption 1. The results presented in the following sections will be applicable to these examples. For the sake of simplicity, the underlying graphs are assumed to be undirected and connected in the following examples. Let $m = |\mathcal{E}|/2$ be the number of undirected edges. Let I_d be the $d \times d$ identity matrix and $\otimes$ be the Kronecker product.

Example 1 (**Consensus**). *The objective of consensus is to steer the robots from some initial positions to a common position. The Lyapunov function is*

$$V = \frac{1}{4} \sum_{(i,j)\in\mathcal{E}} \|p_i - p_j\|^2.$$

Then $V = 0$ *if and only if consensus is achieved. The corresponding gradient control law*

$$\dot{p}_i = f_i = \sum_{j\in\mathcal{N}_i} (p_j - p_i)$$

is the consensus protocol proposed in [10], [11]. Here the weight for each edge is set to be one. The error state can be defined as $e_k = p_i - p_j$ *for* $(i,j) \in \mathcal{E}$ *and* $e = [e_1^T, \ldots, e_m^T]^T \in \mathbb{R}^{m\times n}$. *Then we have* $e = (H \otimes I_d)p$ *where* $H \in \mathbb{R}^{m\times n}$ *is the incidence matrix [12]. Consequently,* $V(e) = 1/2 \sum_{k=1}^m \|e_k\|^2$, $\partial e/\partial p = H \otimes I_d$ *is constant,* f *is continuous in* e, *and* $\|f\|$ *is bounded when* $\|e\|$ *is bounded. Condition (c) in Assumption 1 is satisfied since the graph is connected and then the attraction region* $\Omega(r_0)$ *is the entire space* $\mathbb{R}^{dm}$.

Example 2 (**Relative-Position-Based Formation control**). *The objective of relative-position-based formation control is to steer the robots from some initial positions to converge to a desired geometric pattern defined by relative positions* $\{p_i^* - p_j^*\}_{(i,j)\in\mathcal{E}}$. *The Lyapunov function is*

$$V = \frac{1}{4} \sum_{(i,j)\in\mathcal{E}} \left\|(p_i - p_j) - (p_i^* - p_j^*)\right\|^2.$$

Then $V = 0$ *if and only if the target formation is achieved. The gradient control law*

$$\dot{p}_i = f_i = \sum_{j\in\mathcal{N}_i} \Big[(p_j - p_i) - (p_j^* - p_i^*)\Big]$$

is the relative-position-based formation control law [13], [14]. The error state is defined as $e_k = p_i - p_j - (p_i^* - p_j^*)$ *for* $(i,j) \in \mathcal{E}$ *and* $e = (H \otimes I_d)(p - p^*)$. *Then,* $V(e) = 1/2 \sum_{k=1}^m \|e_k\|^2$, $\partial e/\partial p = H \otimes I_d$ *is constant,* f *is continuous in* e, *and* $\|f\|$ *is bounded when* $\|e\|$ *is bounded. Condition (c) is satisfied when the graph is connected and then* $\Omega(r_0)$ *is the entire space* $\mathbb{R}^{dm}$.

Example 3 (**Distance-Based Formation Control**). *The objective of distance-based formation control is to steer the robots from some initial positions to converge to a desired geometric pattern defined by inter-neighbor distances* $\{\ell_{ij}\}_{(i,j)\in\mathcal{E}}$. *Consider*

$$V = \frac{1}{8} \sum_{(i,j)\in\mathcal{E}} \Big(\|p_i - p_j\|^2 - \ell_{ij}^2\Big)^2.$$

Then $V = 0$ *if and only if the inter-neighbor distances satisfy the constraints. The gradient control law*

$$\dot{p}_i = f_i = \sum_{j\in\mathcal{N}_i} \Big(\|p_i - p_j\|^2 - \ell_{ij}^2\Big)(p_j - p_i)$$

is the distance-based formation control law [12], [14], [15]. The error state is defined as $e_k = \|p_i - p_j\|^2 - \ell_{ij}^2 := \|\delta_k\|^2 - \ell_k^2$ *for* $(i,j) \in \mathcal{E}$. *Then,*$V(e) = 1/4 \sum_{k=1}^m e_k^2$, $\partial e/\partial p = 2\mathrm{diag}(\delta_1^T, \ldots, \delta_m^T)(H \otimes I_d)$ *is bounded when* e *is bounded,* f *is uniformly continuous in both* e *and* p, *and* $\|f_i\|$ *is bounded when* $\|e\|$ *is bounded. Condition (c) is satisfied when the distance constraints correspond to an infinitesimally distance rigid formation and the attraction region* $\Omega(r_0)$ *is a sufficiently small neighborhood of* $e = 0$. *Note that distance rigidity is merely sufficient but not necessary to have condition (c).*

Example 4 (**Bearing-Based Formation Control**). *The objective of bearing-based formation control is to steer the robots from some initial positions to converge to a desired*

[1] A function $f(x)$ is uniformly continuous in x if for any $\epsilon > 0$ there exists $\delta > 0$ such that $\|f(x_1) - f(x_2)\| < \epsilon$ for every pair of x_1 and x_2 satisfying $\|x_1 - x_2\| < \delta$. For a differentiable function, if its derivative is bounded then the function is uniformly continuous. Note that this condition is sufficient but not necessary because a uniformly continuous function may not be differentiable.

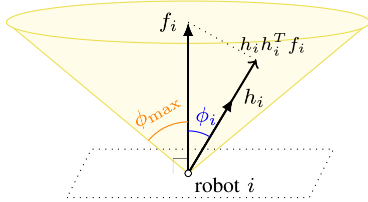


Fig. 1: An illustration of control law (2) and Theorem 1. The direction of h_i must be inside the cone.

geometric pattern defined by constant inter-neighbor bearings $\{g_{ij}^*\}_{(i,j)\in\mathcal{E}}$. Consider

$$V = \frac{1}{4}\sum_{(i,j)\in\mathcal{E}} \|P_{g_{ij}^*}(p_i - p_j)\|^2,$$

where $P_{g_{ij}^*} = I_d - g_{ij}^*(g_{ij}^*)^T$. The gradient control law

$$\dot{p}_i = f_i = \sum_{j\in\mathcal{N}_i} P_{g_{ij}^*}(p_j - p_i)$$

is the bearing-based formation control law [16], [17]. The error state can be defined as $e_k = P_{g_{ij}^*}(p_i - p_j)$ for $(i,j) \in \mathcal{E}$. Then, $V(e) = 1/2\sum_{k=1}^m \|e_k\|^2$, $\partial e/\partial p = \operatorname{diag}(P_{g_1^*}, \ldots, P_{g_m^*})(H \otimes I_d)$ is constant, f is uniformly continuous in e, and $\|f_i\|$ is bounded when $\|e\|$ is bounded. Condition (c) is satisfied if the bearing constraints correspond to an infinitesimally bearing rigid formation and $\Omega(r_0)$ is the entire space $\mathbb{R}^{dm}$.

III. A Modified Gradient Control Law

In this section, we propose a flexible modified gradient control law,

$$\dot{p}_i = \kappa_i(t)h_i(t)h_i^T(t)f_i(e,p), \quad i \in \mathcal{V}, \tag{2}$$

where $\kappa_i(t) > 0$ is a time-varying scalar and $h_i(t) \in \mathbb{R}^d$ is a unit vector whose direction may be time-varying. Since $h_i h_i^T$ is an orthogonal projection matrix, the direction of the velocity $\dot{p}_i$ is parallel to h_i and the magnitude of the velocity is $\kappa_i|h_i^T f_i|$. We may design appropriate $\kappa_i(t)$ and $h_i(t)$ to adjust the velocity of each robot so as to fulfil motion constraints. The design will be given in the following sections. Since robot i may choose $\kappa_i(t)$ and $h_i(t)$ based on its local information, control law (2) remains distributed if the original gradient control is distributed.

We now give the first result in this paper which shows that the proposed control law (2) preserves system convergence under some mild conditions.

Theorem 1 (Flexible Control of Single Integrators). *Given a coordination control problem, if the gradient control* (1) *solves the coordination problem as stated in Assumption 1, then the modified gradient control law* (2) *also solves the coordination task with the same attraction region guaranteed if the following conditions are satisfied:*

(a) $\kappa_i(t)$ is bounded as $0 < \kappa_{\min} \le \kappa_i(t) \le \kappa_{\max}$ for all i and all t;

(b) $\phi_i(t)$ is bounded as $0 \le \phi_i(t) \le \phi_{\max} < \pi/2$ for all i and all t where $\phi_i(t)$ is the angle between h_i and f_i;

(c) both $\kappa_i(t)$ and $h_i(t)$ are uniformly continuous in t for all i.

Proof. Since (2) is a nonautonomous system, the invariance principle for autonomous systems is inapplicable. We use the Barbalat's Lemma to prove the convergence [9, Lemma 8.2].

Consider the same Lyapunov function $V(e)$ as used for the autonomous system (1). The derivative of $V(e)$ along system (2) is

$$\begin{aligned}\dot{V} &= -\sum_{i\in\mathcal{V}} f_i^T \dot{p}_i \\ &= -\sum_{i\in\mathcal{V}} \kappa_i f_i^T h_i h_i^T f_i \\ &= -\sum_{i\in\mathcal{V}} \kappa_i \|f_i\|^2 \cos^2\phi_i \\ &\le -\kappa_{\min}\cos^2\phi_{\max}\sum_{i\in\mathcal{V}}\|f_i\|^2 \le 0.\end{aligned} \tag{3}$$

Since $V(t)$ is nonincreasing and bounded from below, it converges as $t \to \infty$. We next show $\dot{V}(t)$ is uniformly continuous. First of all, since $\dot{V} \le 0$, the set $\Omega(V(e_0)) \subseteq \Omega(r_0)$ is compact and invariant with respect to the error dynamics. On one hand, since $f(e,p)$ is continuous in e, it is uniformly continuous in e over the compact set $\Omega(V(e_0))$. Since it is also uniformly continuous in p, $f(e,p)$ is uniformly continuous in both e and p. On the other hand, by letting $H = \operatorname{blkdiag}(\kappa_1 h_1 h_1^T, \ldots, \kappa_n h_n h_n^T)$, we have $\dot{p} = H(t)f(e,p)$ and hence $\dot{e} = (\partial e/\partial p)H(t)f(e,p)$. According to condition (d) in Assumption 1 and the fact that $\kappa_i \le \kappa_{\max}$, both $\dot{e}$ and $\dot{p}$ are bounded and consequently $e(t)$ and $p(t)$ are uniformly continuous in t (a differentiable function is uniformly continuous if its derivative is bounded). Now we conclude $f(e(t),p(t))$ is uniformly continuous in t. Therefore, we know $\dot{V}$ is uniformly continuous in t because $\kappa_i(t)$ and $h_i(t)$ are also uniformly continuous. It then follows from Barbalat's Lemma [9, Lemma 8.2] that $\dot{V}$ converges to zero as $t \to \infty$. By (3), we have $\|f_i\|$ converges to zero for all i. It then follows from condition (c) in Assumption 1 that the error e converges to zero. □

Although $\kappa_i(t)$ and $h_i(t)$ must be bounded, they may vary within sufficiently large intervals. For instance, $\kappa_{\min}$ can be chosen to be sufficiently small, $\kappa_{\max}$ sufficiently large, and $\phi_{\max}$ sufficiently close to $\pi/2$. In addition, although $\kappa_i(t)$ and $h_i(t)$ must be uniformly continuous, the varying rate may be sufficiently large as long as it is finite. Therefore, $\kappa_i(t)$ and $h_i(t)$ can be designed flexibly.

Theorem 1 indicates that the attraction region of $e = 0$ does not shrink under the modified gradient control. More specifically, if $\Omega(r_0)$ is the attraction region for the gradient system (1), then it is still an attraction region for the modified gradient system (2). As a result, if the gradient control is globally (respectively, locally) stable, then the modified one is also globally (respectively, locally) stable.

The following result shows if the original gradient control system is exponentially stable, then the system under the action of (2) is also exponentially stable.

Corollary 1 (Exponential Stability). *Under Assumption 1, if the gradient control system* (1) *further satisfies the following two conditions:*

(a) there exists $c > 0$ such that $\sum_{i\in\mathcal{V}} \|f_i\|^2 \geq cV$ for all $e \in \Omega(r_0)$;

(b) V is a quadratic function of e;

then $e = 0$ is exponentially stable under the modified gradient control law in (2).

Proof. Under the modified gradient control law in (2), we have (3) and consequently $\dot{V} \leq -\kappa_{\min}\cos^2\phi_{\max}\sum_{i\in\mathcal{V}}\|f_i\|^2 \leq -c\kappa_{\min}\cos^2\phi_{\max}V$. As a result, V converges to zero exponentially fast. Since V is a quadratic function of e, the error e also converges to zero exponentially fast. □

Corollary 1 is applicable to all the four examples in Section II.

IV. How to Handle Velocity Saturation and Obstacle Avoidance

In this section, we show how to design $\kappa_i(t)$ and $h_i(t)$ to preserve the system convergence and in the meantime fulfill the motion constraints on velocity saturation and obstacle avoidance.

A. Velocity Saturation

Under control law (2), we have the velocity magnitude as $v_i = \kappa_i h_i^T f_i > 0$. The reason why $v_i > 0$ for all t is the angle between h_i and f_i is always less than $\pi/2$. Suppose the velocity is constrained by $v_i \leq \nu_{\max}$ where $\nu_{\max}$ is the maximum speed. In order to handle this saturation constraint, we design

$$\kappa_i(t) = \begin{cases} 1, & h_i^T f_i \leq \nu_{\max}, \\ \dfrac{\nu_{\max}}{h_i^T(t) f_i(t)}, & h_i^T f_i > \nu_{\max}. \end{cases} \tag{4}$$

It follows from (4) that $\kappa_i h_i^T f_i = \mathrm{sat}(h_i^T f_i)$ where $\mathrm{sat}(\cdot)$ is the saturation function,

$$\mathrm{sat}(x) = \begin{cases} \nu_{\max}, & x > \nu_{\max}, \\ x, & 0 \leq x \leq \nu_{\max}. \end{cases}$$

As a result, control law (2) becomes

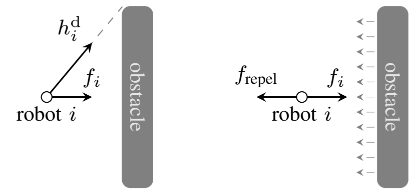
$$\dot{p}_i = h_i \mathrm{sat}(h_i^T f_i). \tag{5}$$

We next prove that the saturation constraint does not jeopardize the system convergence.

Theorem 2 (Linear Velocity Saturation). *Under Assumption 1, control law* (5) *solves the given coordination task with the same attraction region guaranteed if h_i satisfies the conditions in Theorem 1.*

Proof. Since $\dot{p}_i = h_i\mathrm{sat}(h_i^T f_i)$ can be rewritten as $\dot{p}_i = \kappa_i h_i h_i^T f_i$ with κ_i given in (4), we only need to show that κ_i is uniformly continuous and bounded from both below and above. Then the convergence follows directly from Theorem 1.

First, it is easy to verify that $\kappa_i \leq 1 = \kappa_{\max}$ and κ_i in (4) is uniformly continuous in $h_i^T f_i$ (here $h_i^T f_i$ is viewed as a single variable). Similar to the proof of Theorem 1, we know f_i and h_i are both uniformly continuous in t. Thus, κ_i is uniformly continuous in t. Second, since $h_i^T f_i \leq \|f_i\|$ and f_i is bounded over the compact set $\Omega(V(e_0))$, we know for arbitrary initial condition there exists a constant γ such that $\|f_i\| \leq \gamma$ and hence $h_i^T f_i \leq \gamma$ for all t. As a result, $\kappa_i \geq \nu_{\max}/\gamma = \kappa_{\min}$. Therefore, κ_i is bounded from both below and above and uniformly continuous. □



(a) The proposed approach

(b) Artificial potential approach

Fig. 2: An illustration of the proposed approach to obstacle avoidance and an comparison with the approach based on artificial potential.

B. Obstacle Avoidance

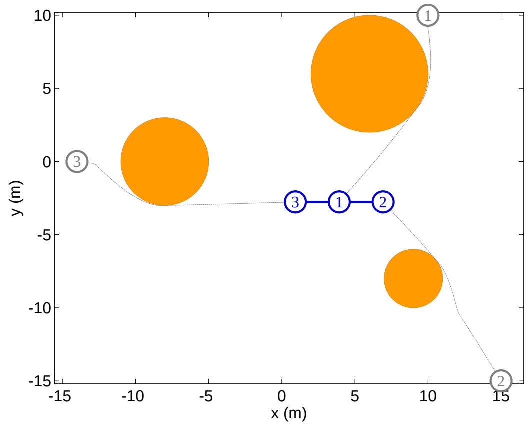
In order to achieve obstacle avoidance, we propose the following strategy to design $h_i(t)$. When there are no obstacles, let $h_i = f_i/\|f_i\|$ so that the $\dot{p}_i = h_i\mathrm{sat}(h_i^T f_i) = f_i/\|f_i\|\mathrm{sat}(\|f_i\|)$. In order to eliminate the singularity of $\|f_i\| = 0$, we consider two cases. In the case of $0 \leq \|f_i\| \leq \nu_{max}$, we have $\dot{p}_i = f_i$; in the case of $\|f_i\| > \nu_{max}$, we have $\dot{p}_i = f_i/\|f_i\|\nu_{max}$.

When there is an obstacle, let h_i change continuously from $f_i/\|f_i\|$ to h_i^{d}, where h_i^{d} may be any unit vector that does not point to the obstacle. Here we design h_i^{d} as the unit vector pointing from the robot to an edge point on the obstacle (see Figure 2(a)). Under some mild assumptions such as the obstacle is a sphere, h_i^{d} would vary uniformly continuously. In practice, the vector h_i^{d} may be easily measured by onboard sensors such as cameras or laser scanners. In order to have a continuous switch from $f_i/\|f_i\|$ to h_i^{d}, we design

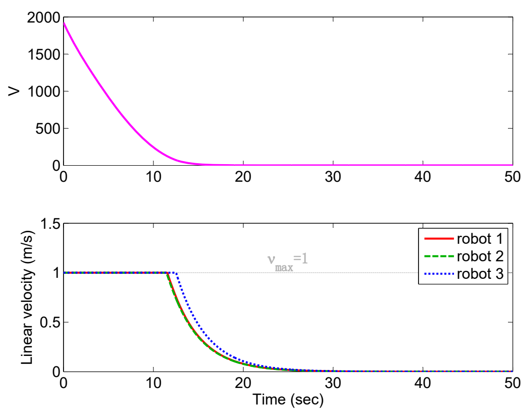
$$h_i(t) = \frac{c_i(t) f_i/\|f_i\| + [1 - c_i(t)]h_i^{\mathrm{d}}(t)}{\|c_i(t) f_i/\|f_i\| + [1 - c_i(t)]h_i^{\mathrm{d}}(t)\|}, \tag{6}$$

where $c_i(t)$ can be any uniformly continuous function varying from 1 to 0 within finite or infinite time. One simple choice is $c_i(t) = e^{k_i(t-t_o)}$ where k_i is positive constant and t_o is the time instance when the obstacle avoidance mechanism is triggered. With $h_i(t)$ in (6), the angle between h_i and f_i is less than $\phi_{\max}$ as long as the angle between h_i^{d} and f_i is less than $\phi_{\max}$.

One interesting feature of the obstacle avoidance approach proposed above is that it merely relies on the bearing information $h_i^{\mathrm{d}}(t)$ of the obstacle. Although distance information is also required to trigger the obstacle avoidance mechanism, it is not required to be accurate because it is not used in



(a) Trajectory



(b) Lyapunov function and velocity magnitude

Fig. 3: Relative-position-based formation control with velocity saturation and obstacle avoidance.

the obstacle avoidance algorithm. Finally, it must be noted that the proposed obstacle avoidance strategy may only be applicable to simple cases where there are not too many obstacles; otherwise, the strategy may fail to work.

C. Simulation

To demonstrate, we apply the proposed control law to relative-position-based formation control. The formation control law and Lyapunov function are given in Example 2. In the simulation example, there are three robots and the underlying graph is complete. In the target formation, the three robots should be distributed evenly on a line segment. The control law is $\dot{p}_i = h_i \text{sat}(h_i^T f_i)$ where f_i is the relative-position-based formation control law given in Example 2. Here h_i is designed in the previous subsection for obstacle avoidance and the velocity saturation is $\nu_{\max} = 1$. As shown in Figure 3, the convergence is achieved because the Lyapunov function converges to zero. In the meantime, the velocity saturation and obstacle avoidance are both realized.

V. How to Handle Unicycle Constraints

In this section we apply the modified gradient control in (2) to handle unicycle models while preserving the system convergence.

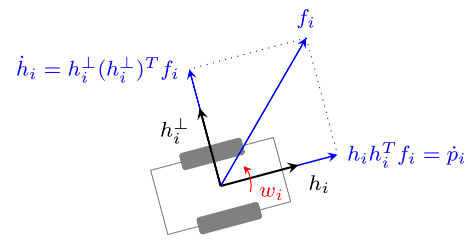


Fig. 4: The geometric meaning of the unicycle control laws in (9) and (10).

Consider a group of unicycle robots in $\mathbb{R}^2$. Let $p_i = [x_i, y_i]^T \in \mathbb{R}^2$ and $\theta_i \in \mathbb{R}$ denote the position coordinate and heading angle of robot i, respectively. The motion of robot i is governed by

$$\begin{aligned} \dot{x}_i &= v_i \cos\theta_i, \\ \dot{y}_i &= v_i \sin\theta_i, \\ \dot{\theta}_i &= w_i, \end{aligned} \tag{7}$$

where $v_i \in \mathbb{R}$ and $w_i \in \mathbb{R}$ are the linear and angular velocities to be designed.

Consider the modified gradient control law $\dot{p}_i = h_i h_i^T f_i$ where we set $\kappa_i = 1$. The heading vector of a unicycle robot is physically constrained as

$$h_i = \begin{bmatrix} \cos\theta_i \\ \sin\theta_i \end{bmatrix}. \tag{8}$$

Then, the modified control law becomes

$$\dot{p}_i = h_i h_i^T f_i = (h_i^T f_i) \begin{bmatrix} \cos\theta_i \\ \sin\theta_i \end{bmatrix}.$$

By comparing it with the unicycle model (7), we design the linear velocity as

$$v_i = h_i^T f_i. \tag{9}$$

The design of the angular velocity w_i can be very flexible. We give the following specific control law:

$$w_i = (h_i^{\perp})^T f_i \tag{10}$$

where

$$h_i^{\perp} = \begin{bmatrix} -\sin\theta_i \\ \cos\theta_i \end{bmatrix}. \tag{11}$$

Note $h_i^{\perp}$ is orthogonal to h_i. The angular velocity has a clear geometric interpretation. That is w_i aims at rotating the heading vector h_i of the robot to align with the gradient flow f_i (see Figure 4). The convergence of the proposed unicycle control law is proved below.

Theorem 3 (Control of Unicycle Robots). *Given a coordination control problem, if the gradient control* (1) *solves the coordination problem as stated in Assumption 1, then the unicycle control law in* (9)-(10) *also solves the coordination problem with the same attraction region guaranteed.*

Proof. With h_i given in (8), we have

$$\dot{h}_i = \begin{bmatrix} -\sin\theta_i \\ \cos\theta_i \end{bmatrix} \dot{\theta}_i = h_i^{\perp} w_i.$$

Substituting (10) into $\dot{h}_i$ gives the closed-loop system as

$$\begin{aligned} \dot{p}_i &= h_i h_i^T f_i, \\ \dot{h}_i &= h_i^{\perp} (h_i^{\perp})^T f_i. \end{aligned} \tag{12}$$

It is notable that the convergence of the closed-loop system (12) does not simply follow from Theorem 1 because h_i in (12) may be orthogonal to f_i, which is not allowed in Theorem 1. Since the closed-loop system is an autonomous system, we can use the invariance principle [9, Theorem 4.4] to prove its stability.

We first examine the equilibrium of the closed-loop system. By letting $\dot{p}_i = 0$ and $\dot{h}_i = 0$, we have $h_i h_i^T f_i = 0$ and $h_i^{\perp}(h_i^{\perp})^T f_i = 0$, which imply $f_i = 0$. Therefore, the system has a unique equilibrium at $f_i = 0$ for all i. This equilibrium is the origin $e = 0$ according to condition (c) in Assumption 1. The time derivative of V along the trajectory of (12) is

$$\dot{V} = -\sum_{i \in \mathcal{V}} f_i^T h_i h_i^T f_i \leq 0.$$

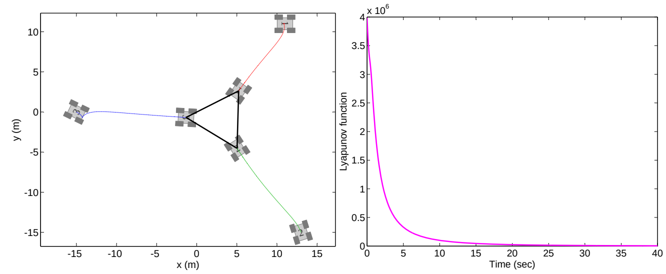
Thus, the set $\Omega(r_0)$ as defined in Assumption 1 is a positive invariant set. Let $E = \{e : \dot{V}(e) = 0\}$. Then, the system trajectory starting from any point in $\Omega(r_0)$ converges to the largest invariant set in E according to the invariance principle [9, Theorem 4.4]. For any point in E, we have $h_i^T f_i = 0$ which means either $h_i \perp f_i$ or $f_i = 0$. Assume $h_i \perp f_i = 0$ but $f_i \neq 0$, then we have $\dot{h}_i = h_i^{\perp}(h_i^{\perp})^T f_i = f_i \neq 0$ and consequently the system trajectory will escape from the point and hence E. As a result, if a point is in the invariant set in E, it must satisfy $f_i = 0$ for all i, which means $e = 0$. □

Theorem 3 indicates that the original gradient control law (1) can be immediately generalized to the unicycle control law in (9)-(10) while the convergence is preserved. If the gradient control is globally (respectively, locally) stable, then the unicycle control law is also globally (respectively, locally) stable.

To demonstrate, we apply the proposed control law in (9)-(10) to distance-based formation control of unicycle robots. The gradient control law and Lyapunov function are given in Example 3. Figure 5 shows the simulation results. In this simulation example, there are three robots and the underlying graph is complete. The target formation is an equilateral triangle with each side length as five meters. As can be seen, under the proposed control law, the formation control target is achieved because the Lyapunov function converges to zero. Since the initial error is large, $\|f_i\|$ and v_i may reach 10^4, which is unrealistic in practice. Motivated by this, we naively introduce a small control gain as 0.0001 into v_i and w_i to achieve smaller linear and angular velocities. A systematic way to simultaneously handle velocity saturation and unicycle models will be studied in the future.

VI. Conclusion

This paper proposed a new modified gradient control approach to multi-robot coordination control. It was shown that the adjustment of the velocity of each robot may preserve the



(a) Trajectory

(b) Lyapunov function

Fig. 5: Distance-based formation control of unicycle robots.

system stability under mild conditions and, in the meantime, fulfill various motion constraints such as velocity saturation, obstacle avoidance, and unicycle models. In the future, how to simultaneously handle unicycle models and linear and angular velocity saturation is an important research topic.

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