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THE SUPPORT OF LOCAL COHOMOLOGY MODULES

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ABSTRACT. We describe the support of F-finite F-modules over a polynomial ring R of prime characteristic. Our description yields an algorithm to compute the support of such local cohomology modules of R; the complexity of our algorithm is also analyzed. To the best of our knowledge, this is the first practical algorithm regarding local cohomology modules in prime characteristic. We also use the idea behind this algorithm to prove that the support of $H^j_I(S)$ is Zariski closed for each ideal I of S where R is noetherian commutative ring of prime characteristic with finitely many isolated singular points and $S = R/gR$ ($g \in R$).

1. Introduction

Local cohomology is a powerful tool introduced by Alexander Grothendieck in the 1960's ([Har67]) and it has since yielded many geometric and algebraic insights. From an algebraic point of view, given an ideal I in a commutative ring R, local cohomology modules $\mathrm{H}^i_I(-)$ $(i \geq 0)$ arise as right-derived functors of the torsion functor on R-modules given by $\Gamma_I(M) = \{a \in M \mid I^k a = 0 \text{ for some } k \geq 0\}$. A central question in the theory of local cohomology is to determine for which values of i does the local cohomology module $\mathrm{H}^i_I(M)$ vanish. This question is both useful and difficult even in the case where R is a regular local ring and $M = R$, and this case has been studied intensely since the introduction of local cohomology (e.g., cf. [Har68], [PS73] and [Ogu73]).

The aim of this paper is to describe the support of local cohomology modules in prime characteristic. Specifically, we first study the support of F-finite F-modules over a regular ring R and show a computationally feasible method for computing these without the need to compute generating roots. To the best of our knowledge, this is the first computationally feasible algorithm for calculating the support of these modules in prime characteristic. We then apply this to the calculation of supports of local cohomology modules and of iterated local cohomology modules $\mathrm{H}^{i_1}_{I_1}\left(\mathrm{H}^{i_2}_{I_2}\left(\dots \mathrm{H}^{i_n}_{I_n}(R)\dots\right)\right)$ thus, for example, giving an effective method for determining the vanishing of Lyubeznik numbers.

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Our methods are interesting both from theoretical and practical points of view. A careful analysis of the algorithms resulting from these methods (see Section 4 below) shows that

(a) the degrees of the polynomials appearing in the calculations have a low upper bound, and, furthermore,
(b) when the method is applied to the calculation of supports of local cohomology modules, if the input is given by polynomials with integer coefficients, then the calculation of supports modulo different primes p involves polynomials *whose degrees can be bounded from above by a constant times* p*, that constant being independent of* p.

In [Lyu97]) Gennady Lyubeznik described an algorithm for computing the support of F-finite F-modules. That algorithm requires the calculation for roots of these modules, and this relies on the calculation of Grobner bases; these are often too complex to be computed in practice. Crucially, *our algorithm does not involve Gröbner bases,* and consists essentially of matrix multiplications together with the listing of terms of polynomials of degrees of order p. It is this that makes our algorithm a practical tool for computing F-finite F-modules.[1]

The reason why we are able to compute and analyze in characteristic p the support of F-finite F-modules is the existence of the *eth iterated Frobenius endomorphism* $f^e : R \to R$, taking $a \in R$ to a^{p^e} $(e \geq 0)$. The usefulness of these lies in the fact that given an R-module M, we may endow it with a new R-module structure via f^e: let $F^e_* M$ denote the additive Abelian group M denoting its elements $\{F^e_* m \,|\, m \in M\}$, and endow $F^e_* M$ with the R-module structure is given by $aF^e_* m = F^e_* a^{p^e} m$ for all $a \in R$ and $m \in M$.

This also allows us to define the *eth Frobenius functors* from the category of R-modules to itself given by $F^e_R(M) = F^e_* R \otimes_R M$ and viewing this as a R-module via the identification of $F^e_* R$ with R: the resulting R-module structure on $F^e_R(M)$ satisfies $a(F^e_* b \otimes m) = F^e_* ab \otimes m$ and $F^e_* a^p b \otimes m = F^e_* a \otimes bm$ for al $a, b \in R$ and $m \in M$.

We will be interested in this construction mainly for regular rings and henceforth in this paper R will denote a regular ring of prime characteristic p.

Recall that an F-finite F_R-module $\mathcal{M}$ is an R-module obtained as a direct limit of a direct limit system of the form

$$M \xrightarrow{U} F^1_R(M) \xrightarrow{F^1_R(U)} F^2_R(M) \xrightarrow{F^2_R(U)} \dots$$

where M is a finitely generated module and U is an R-linear map (cf. [Lyu97]). The main interest in F-finite F-modules follows from the fact that local cohomology modules are F-finite F-modules, as we now explain.

[1]The various algorithms in this paper have been incorporated in the "FSing" package of Macaulay 2[GS].

The *jth local cohomology module of M with support on an ideal $I \subset R$* is defined as

$$\mathrm{H}^j_I(M) = \varinjlim_e \mathrm{Ext}^j_R(R/I^{[p^e]}, M) \tag{1}$$

where maps in the direct limit system are induced by the surjections $R/I^{[p^{e+1}]} \to R/I^{[p^e]}$. If we apply this with $M = R$, we obtain

$$\begin{aligned} \mathrm{H}^j_I(R) &= \varinjlim_e \mathrm{Ext}^j_R(R/I^{[p^e]}, R) \\ &\cong \varinjlim_e \mathrm{Ext}^j_R(F^e_R(R/I), F^e_R R) \\ &\cong \varinjlim_e F^e_R\left(\mathrm{Ext}^j_R(R/I, R)\right) \end{aligned}$$

were we use the facts that $F^e_R(R) \cong R$, $F^e_R(R/I) \cong R/I^{[p^e]}$, and that, since R is regular, the Frobenius functor $F^e_R(-)$ is exact and thus commutes with the computation of cohomology. This shows that $\mathrm{H}^j_I(R)$ are F-finite F-modules, and we may apply our F-finite F-module machinery to them.

Finally, in section 6 we turn our attention to hypersurfaces and describe the support of their local cohomology modules, which turn out to be closed.[2] Given a fixed $g \in R$, one can ask for the locus of primes $P \subseteq R$ for which the multiplication by g map $g : \mathrm{H}^i_I(R_P) \to \mathrm{H}^i_I(R_P)$ is injective and the locus of primes for which this is surjective. We show that these two loci are Zariski closed by describing explicitly the defining ideals of these loci, and we use these to describe the defining ideal of the (Zariski closed) support for $\mathrm{H}^i_I(R/gR)$. We also extend the Zariski-closedness of $\mathrm{H}^i_I(R/gR)$ to the case when R has finitely many isolated singular points.

The methods used for the various calculations in this paper are described in section 2.

2. Prime characteristic tools

Definition 2.1. Let $e \geq 0$. Let T be a commutative ring of prime characteristic p.

(a) Given any matrix (or vector) A with entries in T, we define $A^{[p^e]}$ to be the matrix obtained from A by raising its entries to the p^eth power.
(b) Given any submodule $K \subseteq T^\alpha$, we define $K^{[p^e]}$ to be the R-submodule of T^α generated by $\{v^{[p^e]} \mid v \in K\}$.

Henceforth in this section, T will denote a regular ring with the property that $F^e_* T$ are *intersection flat T-modules* for all $e \geq 0$, i.e., for any family of

[2] The fact that the support is closed was simultaneously and independently also discovered by Mel Hochster and Luis Núñez-Betancourt in [HNB] using a different method.

T-modules $\{M_\lambda\}_{\lambda\in\Lambda}$,

$$F_*^e T \otimes_T \bigcap_{\lambda\in\Lambda} M_\lambda = \bigcap_{\lambda\in\Lambda} F_*^e T \otimes_T M_\lambda.$$

These include rings T for which $F_*^e T$ are free T-modules (e. g. , polynomial rings and power series rings with F-finite coefficient rings,) and also all complete regular rings (cf. [Kat08, Proposition 5.3]). These rings have that property that for any collection of submodules $\{L_\lambda\}_{\lambda\in\Lambda}$ of T^α, $\left(\bigcap_{\lambda\in\Lambda} L_\lambda\right)^{[p^e]} = \bigcap_{\lambda\in\Lambda} L_\lambda^{[p^e]}$: indeed, the regularity of T implies that for any submodule $L \subseteq T^\alpha$, $L^{[p^e]}$ can be identified with $F_T^e(L)$ and and the intersection-flatness of $F_*^e T$ implies

$$F_T^e(\bigcap_{\lambda\in\Lambda} L_\lambda) = F_*^e T \otimes_T \bigcap_{\lambda\in\Lambda} L_\lambda = \bigcap_{\lambda\in\Lambda} F_*^e T \otimes_T L_\lambda = \bigcap_{\lambda\in\Lambda} F_T^e(L_\lambda).$$

The theorem below extends the $I_e(-)$ operation defined on ideals in [Kat08, Section 5] and in [BMS08, Definition 2.2] (where it is denoted $(-)^{[1/p^e]}$) to submodules of free R-modules.

Theorem 2.2. *Let $e \geq 1$. Given a submodule $K \subseteq T^\alpha$ there exists a minimal submodule $L \subseteq T^\alpha$ for which $K \subseteq L^{[p^e]}$. We denote this minimal submodule $I_e(K)$.*

Proof. Let L be the intersection of all submodules $M \subseteq T^\alpha$ for which $K \subseteq M^{[p^e]}$. The intersection-flatness of T implies that $K \subseteq L^{[p^e]}$ and clearly, L is minimal with this property.

□

When $F_*^e T$ is T-free, this is a straightforward generalization of the calculation of I_e for ideals. To do so, fix a free basis $\mathcal{B}$ for $F_*^e T$ and note that every element $v \in T^\alpha$ can be expressed uniquely in the form $v = \sum_{b\in\mathcal{B}} u_b^{[p^e]} b$ where $u_b \in T^\alpha$ for all $b \in \mathcal{B}$.

Proposition 2.3. *Let $e \geq 1$.*

(a) *For any submodules $V_1, \ldots, V_\ell \subseteq R^n$, $I_e(V_1 + \cdots + V_\ell) = I_e(V_1) + \cdots + I_e(V_\ell)$.*

(b) *Let $\mathcal{B}$ be a free basis for $F_*^e T$. Let $v \in R^\alpha$ and let*

$$v = \sum_{b\in\mathcal{B}} u_b^{[p^e]} b$$

be the unique expression for v where $u_b \in T^\alpha$ for all $b \in \mathcal{B}$. Then $I_e(Tv)$ is the submodule W of T^α generated by $\{u_b \mid b \in \mathcal{B}\}$.

Proof. The proof of this proposition is a straightforward modification of the proofs of propositions 5.2 and 5.6 in [Kat08] and Lemma 2.4 in [BMS08].

Clearly, $I_e(V_1+\cdots+V_\ell) \supseteq I_e(V_i)$ for all $1 \leq i \leq \ell$, hence $I_e(V_1+\cdots+V_\ell) \supseteq I_e(V_1) + \cdots + I_e(V_\ell)$. On the other hand

$$(I_e(V_1) + \cdots + I_e(V_\ell))^{[p^e]} = I_e(V_1)^{[p^e]} + \cdots + I_e(V_\ell)^{[p^e]} \supseteq V_1 + \cdots + V_\ell$$

and the minimality of $I_e(V_1+\cdots+V_\ell)$ implies that $I_e(V_1+\cdots+V_\ell) \subseteq I_e(V_1)+\cdots+I_e(V_\ell)$ and (a) follows.

Clearly $v \in W^{[p^e]}$, and so $I_e(Tv) \subseteq W$. On the other hand, let W be a submodule of T^α such that $v \in W^{[p^e]}$. Write $v=\sum_{i=1}^s r_i w_i^{[p^e]}$ for $r_i \in T$ and $w_i \in W$ for all $1 \le i \le s$, and for each such i write $r_i=\sum_{b\in\mathcal{B}} r_{bi}^{p^e} b$ where $r_{bi}\in T$ for all $b \in \mathcal{B}$. Now

$$\sum_{b\in\mathcal{B}} u_b^{[p^e]} b = v = \sum_{b\in\mathcal{B}} \left(\sum_{i=1}^s r_{bi}^{p^e} w_i^{[p^e]}\right) b$$

and since these are direct sums, we compare coefficients and obtain $u_b^{[p^e]} = \left(\sum_{i=1}^s r_{bi}^{p^e} w_i^{[p^e]}\right)$ for all $b \in \mathcal{B}$ and so $u_b = (\sum_{i=1}^s r_{bi} w_i)$ for all $b \in \mathcal{B}$ hence $u_b \in W$ for all $b \in \mathcal{B}$. □

The behavior of the I_e operation under localization and completion will be crucial for obtaining the results of this paper. To investigate this we need the following generalization of [LS01, Lemma 6.6].

Lemma 2.4. *Let $\mathcal{T}$ be a completion of T at a prime ideal P. Let $\alpha \ge 0$ and let W be a submodule of $\mathcal{T}^\alpha$. For all $e \ge 0$, $W^{[p^e]} \cap T = (W \cap T)^{[p^e]}$.*

Proof. If T is local with maximal ideal P, the result follows from a straightforward modification of the proof of [LS01, Lemma 6.6].

We now reduce the general case to the previous case which implies that $W^{[p^e]} \cap T_P = (W \cap T_P)^{[p^e]}$. Intersecting with T now gives

$$W^{[p^e]} \cap T = (W\cap T_P)^{[p^e]} \cap T = (W \cap T_P \cap T)^{[p^e]} = (W\cap T)^{[p^e]}.$$

□

Lemma 2.5 (cf. [Mur13])**.** *Let $\mathcal{T}$ be a localization of T or a completion at a prime ideal.*

For all $e \ge 1$, and all submodules $V \subseteq T^\alpha$, $I_e(V\otimes_T \mathcal{T})$ exists and equals $I_e(V)\otimes_T \mathcal{T}$.

Proof. Let $L \subseteq \mathcal{T}^\alpha$ be a submodule, such that $L^{[p^e]} \supseteq V \otimes_T \mathcal{T}$. We clearly have $L^{[p^e]} \cap T^\alpha = (L \cap T^\alpha)^{[p^e]}$ when $\mathcal{T}$ is a localization of T and when $\mathcal{T}$ is a completion of T this follows from the previous Lemma. We deduce that $(L\cap T^\alpha) \supseteq I_e(V\otimes_T \mathcal{T} \cap T^\alpha)$ and hence $L \supseteq (L\cap T^\alpha)\otimes_T \mathcal{T} \supseteq I_e(V \otimes_T \mathcal{T} \cap T^\alpha)\otimes_T \mathcal{T}$.

But since $I_e(V\otimes_T \mathcal{T} \cap T^\alpha)\otimes_T \mathcal{T}$ satisfies

$$(I_e(V\otimes_T \mathcal{T} \cap^\alpha)\otimes_T \mathcal{T})^{[p^e]} = I_e(V\otimes_T\mathcal{T}\cap T^\alpha)^{[p^e]}\otimes_T\mathcal{T} \supseteq (V\otimes_T\mathcal{T}\cap T^\alpha)\otimes_T\mathcal{T} \supseteq V\otimes_T\mathcal{T}$$

we deduce that $I_e(V\otimes_T \mathcal{T}\cap T^\alpha)\otimes_T \mathcal{T}$ is the smallest submodule $K \subseteq \mathcal{T}^\alpha$ for which $K^{[p^e]} \supseteq V \otimes_T \mathcal{T}$. We conclude that $I_e(V\otimes_T \mathcal{T})$ equals $I_e(V\otimes_T \mathcal{T} \cap T^\alpha)\otimes_T \mathcal{T}$.

We always have

$$I_e(V\otimes_T\mathcal{T}) = I_e(V\otimes_T \mathcal{T}\cap T^\alpha)\otimes_T \mathcal{T} \supseteq I_e(V)\otimes_T \mathcal{T}.$$

On the other hand

$$(I_e(V) \otimes_T \mathcal{T})^{[p^e]} = I_e(V)^{[p^e]} \otimes_T \mathcal{T} \supseteq V \otimes_T \mathcal{T}$$

hence $I_e(V \otimes_T \mathcal{T}) \subseteq I_e(V) \otimes_T \mathcal{T}$ and thus $I_e(V \otimes_T \mathcal{T}) = I_e(V) \otimes_T \mathcal{T}$.

□

3. Calculation of supports of F_R-finite F_R-modules

We begin by recalling the following result from [Lyu97, Proposition 2.3].

Remark 3.1 (Vanishing of γ_t). Let $\mathcal{M}$ be an F_R-finite F_R-module with a generating homomorphsim $\gamma : M \to F_R(M)$. Let γ_t denote the composition $M \to F_R(M) \to \cdots \to F_R^t(M)$. We may assume that M has a presentation: $R^\alpha \xrightarrow{A} R^\beta \to M \to 0$ and write the generating homomorphism as $\operatorname{Coker}(A) \xrightarrow{U} \operatorname{Coker}(A^{[p]})$. Then γ_t is the composition of $\operatorname{Coker}(A) \to \cdots \to \operatorname{Coker}(A^{[p^t]})$. Note that $\mathcal{M} = 0$ if and only if there is a t such that $\gamma_t = 0$. We have

$$\begin{aligned}
\gamma_t = 0 &\Leftrightarrow \operatorname{Im}(U^{[p^{t-1}]} \circ \cdots \circ U^{[p]} \circ U) \subseteq \operatorname{Im}(A^{[p^t]}) \\
&\Leftrightarrow I_t(\operatorname{Im} U^{[p^{t-1}]} \circ \cdots \circ U^{[p]} \circ U) \subseteq \operatorname{Im} A \\
&\Leftrightarrow I_1(UI_1(\cdots I_1(UI_1(\operatorname{Im} U))) \subseteq \operatorname{Im} A \\
&\Leftrightarrow \frac{I_1(UI_1(\cdots I_1(UI_1(\operatorname{Im} U))) + \operatorname{Im} A}{\operatorname{Im} A} = 0
\end{aligned}$$

where we made repeated use of the facts that for any submodule $W \subseteq R^\beta$ we have $I_{\ell+1}(M) = I_1(I_\ell(M))$ and also $I_\ell(U^{[p^\ell]}M) = UI_\ell(M)$.

Theorem 3.2.

(a) *If*

$$I_e(\operatorname{Im} U^{[p^{e-1}]} \circ \cdots \circ U^{[p]} \circ U) = I_{e+1}(\operatorname{Im} U^{[p^e]} \circ \cdots \circ U^{[p]} \circ U)$$

then

$$I_e(\operatorname{Im} U^{[p^{e-1}]} \circ \cdots \circ U^{[p]} \circ U) = I_{e+j}(\operatorname{Im} U^{[p^{e+j-1}]} \circ \cdots \circ U^{[p]} \circ U) \tag{2}$$

for all $j \geq 0$.

(b) *There exists an integer* e *such that (2) holds.*

Proof. Write $V_e = I_e(\operatorname{Im} U^{[p^{e-1}]} \circ \cdots \circ U^{[p]} \circ U)$. First we claim that if $V_e = V_{e+1}$ then $V_e = V_{e+j}$ for all $j \geq 0$; we proceed by induction on $j \geq 0$. Using again the facts that for any submodule $W \subseteq R^\beta$ we have $I_{\ell+1}(M) = I_1(I_\ell(M))$ and that $I_\ell(U^{[p^\ell]}M) = UI_\ell(M)$, we deduce that, if $j \geq 1$, then $V_{e+j} = I_1(I_{e+(j-1)}(\operatorname{Im} U^{[p^{e+j-1}]} \circ \cdots \circ U^{[p]} \circ U)) = I_1(UV_{e+j-1})$ and this, by the induction hypothesis, equals $I_1(UV_e) = V_{e+1}$.

Next, we wish to show that for each prime ideal $\mathfrak{p}$ there exists an integer $e_\mathfrak{p}$ such that

$$V_{e_\mathfrak{p}} R_\mathfrak{p} = V_{e_\mathfrak{p}+1} R_\mathfrak{p}. \tag{3}$$

and that for this $e_{\mathfrak{p}}$, $V_{e_{\mathfrak{p}}+j}R_{\mathfrak{p}} = V_{e_{\mathfrak{p}}}R_{\mathfrak{p}}$ for all $j \geq 0$. After completing at $\mathfrak{p}$, we have assume that our ring is a complete regular local ring, and we let E denote the injective hull of the residue field of $\widehat{R_{\mathfrak{p}}}$. As this ring is complete and regular, there is a natural Frobenius map on E which we denote T, which can be extended to a Frobenius map on direct sums of E by letting T act coordinate-wise; we denote these Frobenius maps also with T.

We now consider the Frobenius map $\Theta = U^t T$ on E^β; in [KZ14, Lemma 3.6] it is shown that $\operatorname{ann}_{E^\beta} I_e(\operatorname{Im} U^{[p^{e-1}]} \circ \cdots \circ U^{[p]} \circ U R_{\mathfrak{p}})^t \subseteq E^\beta$ consists of all elements killed by Θ^e. Now (cf. [HS77, Proposition 1.11] and [Lyu97, Proposition 4.4]) show that there is an integer $e_{\mathfrak{p}}$ such that $I_{e_{\mathfrak{p}}}(\operatorname{Im} U^{[p^{e-1}]} \circ \cdots \circ U^{[p]} \circ U R_{\mathfrak{p}}) = I_{e_{\mathfrak{p}}+1}(\operatorname{Im} U^{[p^{e_{\mathfrak{p}}}]} \circ \cdots \circ U^{[p]} \circ U R_{\mathfrak{p}})$ and $I_{e_{\mathfrak{p}}}(\operatorname{Im} U^{[p^{e_{\mathfrak{p}}}-1]} \circ \cdots \circ U^{[p]} \circ U R_{\mathfrak{p}}) = I_{e_{\mathfrak{p}}+j}(\operatorname{Im} U^{[p^{e_{\mathfrak{p}}+j-1}]} \circ \cdots \circ U^{[p]} \circ U R_{\mathfrak{p}})$ for all $j \geq 0$. Crucially, Lemma 2.5 implies that $I_e\left(\left(\operatorname{Im} U^{[p^{e-1}]} \circ \cdots \circ U^{[p]} \circ U\right) R_{\mathfrak{p}}\right) = V_e R_{\mathfrak{p}}$ for all $e \geq e_{\mathfrak{p}}$ and so (3) holds.

Consider the following subsets of $\operatorname{Spec}(R)$:

$$\mathcal{P}_t = \{\mathfrak{p} \in \operatorname{Spec}(R) \mid V_t R_{\mathfrak{p}} = V_{t+1} R_{\mathfrak{p}}\} = \operatorname{Spec} R \setminus \operatorname{Supp} \frac{V_t}{V_{t+1}}.$$

These form an increasing sequence of open subsets of $\operatorname{Spec} R$, and since for each prime ideal $\mathfrak{p}$ there is an integer $t_{\mathfrak{p}}$ such that

$$V_{t_{\mathfrak{p}}} R_{\mathfrak{p}} = V_{t_{\mathfrak{p}}+1} R_{\mathfrak{p}},$$

we have $\bigcup_t \mathcal{P}_t = \operatorname{Spec}(R)$. Now the quasicompactness of $\operatorname{Spec} R$, guarantees the existence of an integer e such that $\mathcal{P}_e = \operatorname{Spec}(R)$; clearly that e satisfies (2). $\square$

Corollary 3.3. *If $I_e(\operatorname{Im} U^{[p^{e-1}]} \circ \cdots \circ U^{[p]} \circ U) = I_{e+1}(\operatorname{Im} U^{[p^e]} \circ \cdots \circ U^{[p]} \circ U)$, then*

$$\operatorname{Supp}_R\Big(\frac{\operatorname{Im} I_e(U^{[p^{e-1}]} \circ \cdots \circ U^{[p]} \circ U) + \operatorname{Im} A}{\operatorname{Im} A}\Big) = \operatorname{Supp}_R(\mathcal{M}).$$

4. Our algorithm and its complexity

Let be given a matrix A, which gives a presentation $\operatorname{Coker} A \cong M$, and a $\beta \times \beta$ matrix U, for which the the map $\operatorname{Coker} A \xrightarrow{U} \operatorname{Coker} A^{[p]}$ is isomorphic to a generating morphism $M \to F_R(M)$ of an F-finite F-module $\mathcal{M}$. We compute the support of $\mathcal{M}$ as follows.

Define $L_0 = R^\beta$ and for all $i \geq 0$, $L_{i+1} = I_1(UL_i)$. Note that $I_e(\operatorname{Im} U^{[p^{e-1}]} \circ \cdots \circ U^{[p]} \circ U) = L_e$. The output of the algorithm is the stable value of $\operatorname{Supp}(\frac{L_e + \operatorname{Im} A}{\operatorname{Im} A})$.

In the rest of this section we discuss the complexity of this algorithm for computing supports of F-finite F-modules.

Let δ be the largest degree of an entry in U and for any $j \geq 0$ let δ_j be the largest degree of a polynomial in a generator of L_j. The calculation of

$I_1(-)$ as described in Proposition 2.3 implies that $\delta_{j+1} \leq (\delta_j + \delta)/p$ hence

$$\delta_e \leq \frac{\delta_0}{p^e} + \delta(\frac{1}{p} + \cdots + \frac{1}{p^e}) \leq \frac{\delta}{p-1}.$$

Let $R = \mathbb{Z}[x_1, \ldots, x_n]$ and let $J \subseteq R$ be an ideal. For any prime p let J_p denote the image of J in $R_p = \mathbb{Z}/p\mathbb{Z}[x_1, \ldots, x_n]$. An interesting and natural question arising in this context is the description of the properties of the local cohomology module $\mathrm{H}^j_{J_p}(R_p)$ as p ranges over all primes and we now turn our attention to these.

For different choices of prime p the matrices A and U above will be different and this could result in different values of δ_e which are unbounded as p ranges over all primes. We now show that this is not the case. Let U_p denote the square matrix that induces the map $\mathrm{Ext}^j(R_p/J_p, R_p) \to \mathrm{Ext}^j(R_p/J_p^{[p]}, R_p)$ and δ_p to denote the maximal degree of entries in U_p. We also denote $L_{0,p} = R_p$ and $L_{i+1,p} = I_1(U_p L_{i,p})$ and use $\delta_{e,p}$ to denote the largest degree of a polynomial in a generator of $L_{e,p}$.

Theorem 4.1. *Let* $0 \to R^{b_s} \xrightarrow{A_s} \cdots \xrightarrow{A_2} R^{b_1} \xrightarrow{A_1} R \to R/J \to 0$ *be a free resolution of* R/J. *Let* Δ *denote the maximal degree of any entry in* $A_1, \ldots, A_s$. *Let* p *be a prime integer which is also a regular element on* R/J. *Then* $\delta_{e,p} \leq 2j\Delta$ *for all integers* $e \geq 1$.

Proof. Since p is a regular element on R/J, tensoring the free resolution of R/J with R/pR produces a free resolution of R_p/J_p. Hence the maximal degree of entries in the maps of this free resolution of R_p/J_p is at most Δ. Let θ_j denote the map $R_p^{b_j} \to R_p^{b_j}$ in the following commutative diagram induced by $R_p/J_p^{[p]} \to R_p/J_p$

$$\begin{tikzcd}
0 \arrow[r] & R_p^{b_s} \arrow[r, "A_{s,p}"] & \cdots \arrow[r] & R_p^{b_j} \arrow[r, "A_{j,p}"] & \cdots \arrow[r] & R_p^{b_1} \arrow[r, "A_{1,p}"] & R_p \arrow[r] & R_p/J_p \arrow[r] & 0 \\
0 \arrow[r] & R_p^{b_s} \arrow[r, "A_{s,p}^{[p]}"] \arrow[u, "\theta_s"] & \cdots \arrow[r] & R_p^{b_j} \arrow[r, "A_{j,p}^{[p]}"] \arrow[u, "\theta_j"] & \cdots \arrow[r] & R_p^{b_1} \arrow[r, "A_{1,p}^{[p]}"] & R_p \arrow[r] \arrow[u, "="] & R_p/J_p^{[p]} \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

An easy induction on j shows that the maximal degree of entries in θ_j is at most $jp\Delta$. The map $U_p : \mathrm{Ext}^j(R_p/J_p, R_p) \to \mathrm{Ext}^j(R_p/J_p^{[p]}, R_p)$ is induced by the transpose of θ_j and hence the maximal degree in an entry of U_p is also bounded by $jp\Delta$. Now

$$\delta_{e,p} \leq \frac{jp\Delta}{p-1} \leq 2j\Delta,$$

for all $e \geq 1$. $\square$

Corollary 4.2. *There is an integer* N, *independent of* p, *such that* $\delta_{e,p} \leq N$ *for all* e *and all* p.

In particular, there is an integer N', independent of p, such that $\min\{e \mid L_{e,p} = L_{e+1,p}\} \leq N'$ *for all p, i.e. for each prime integer p, the number of steps required to compute the stable value $L_{e,p}$ is bounded by N'.*

Proof. The second statement follows immediately from the first since, once the degree is bounded, the number of steps will be bounded by the number of monomials with the bonded degree.

To prove the first statement, it suffices to note there are only finitely many associated prime ideals of R/J in R and hence p is a regular element on R/J for almost all p. □

The complexity of our algorithm lies in the calculation $L_{i+1} = I_1(UL_i)$ which involves

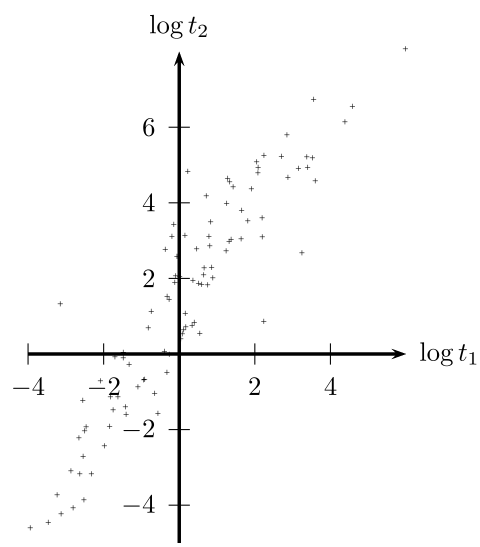
(a) the size β of U, which is an input to the algorithm and does not depend on p,
(b) the total number of terms occurring in each of the coordinates of a set of generators of L_i.

In the worst case scenario, if the maximal degree of an entry in U_p is Cp, the total number of terms in (b) is bounded by

$$\left(\binom{Cp+n-1}{n-1}\right)^{\beta} = \mathcal{O}\left(p^{\beta(n-1)}\right).$$

In practice, the number of terms is much lower than this worst case.

In order to assess the practical advantage of our algorithm, we computed the support of 100 F-finite F-modules with randomly generated generating morphism $C \to F_R^1(C)$ where C is a quotient of R^2 and $R = \mathbb{Z}/2\mathbb{Z}[x_1, \dots, x_5]$. We demote t_1 the time required by our algorithm to compute the support and t_2 the time required to compute a root using Grobner bases. The following is a plot of $\log t_2$ as a function of $\log t_1$



This suggests that for this characteristic and rank, t_2 is approximately t_1^2.[3]

To further illustrate the effectiveness of our algorithm we compute the following example.

Example 4.3. Consider three generic degree-2 polynomials in t: $F_1(t) = x_0 + x_1 t + x_2 t^2$, $F_2(t) = y_0 + y_1 t + y_2 t^2$, $F_1(t) = z_0 + z_1 t + z_2 t^2$. For any two polynomials $F(t), G(t)$ let $\operatorname{Res}(F, G)$ denote their Sylvester resultant, e. g. ,

$$\operatorname{Res}(F_1, F_2) = \det \begin{bmatrix} x_0 & x_1 & x_2 & 0 \\ 0 & x_0 & x_1 & x_2 \\ y_0 & y_1 & y_2 & 0 \\ 0 & y_0 & y_1 & y_2 \end{bmatrix}$$

Let I denote the ideal generated by $\operatorname{Res}(F_1, F_2), \operatorname{Res}(F_1, F_3), \operatorname{Res}(F_2, F_3), \operatorname{Res}(F_1 + F_2, F_3)$ in the polynomial ring R over a field k whose variables are the x,y, and zs above. In [Lyu95] it was asked whether $\mathrm{H}_I^4(R) = 0$ and this was settled in prime characteristic $p > 2$ (cf. [Kat97]) and in characteristic zero (cf. [Yan99, Theorem 3].) We used an implementation of our algorithm [KZ] with Macaulay2 ([GS]) to settle the remaining case of characteristic 2: a 20-second run calculated the support of $\mathrm{H}_I^4(R)$ to be empty.

5. Iterated local cohomology modules

Let $f_1, \ldots, f_m$ be a sequence of elements in R and let N be an R-module. We will write $K_i := \bigoplus_{1 \le j_1 < \cdots < j_i \le m} N_{j_1 \cdots j_i}$ to denote the i-th term of the Koszul (co)complex $\mathcal{K}^\bullet(M; \underline{f})$ (where each $N_{j_1 \cdots j_i} = N$), and we will use $H^i(N; \underline{f})$ to denote the i-th Koszul (co)homology.

Proposition 5.1. *Let $\mathcal{M}$ be an F_R-finite F_R-module with a generating homomorphism $M \xrightarrow{\varphi} F_R(M)$ and let $I = (f_1, \ldots, f_m)$ be an ideal of R. Then $H_I^i(\mathcal{M})$ admits a generating homomorphism*

$$H^i(M; \underline{f}) \to F_R(H^i(M; \underline{f})).$$

Proof. Consider the following commutative diagram:

$$\begin{tikzcd}
 & \vdots & & \vdots & & \vdots & \\
0 \arrow[r] & F_R(K^1) \arrow[u, "F_R(\phi_1)"] \arrow[r, "F_R(\delta^1)"'] & \cdots \arrow[r] & F_R(K^i) \arrow[u, "F_R(\phi_i)"] \arrow[r, "F_R(\delta^i)"'] & \cdots \arrow[r] & F_R(K^m) = F_R(M) \arrow[u, "F_R(\phi_m)"] \arrow[r] & 0 \\
0 \arrow[r] & K^1 \arrow[u, "\phi_1"] \arrow[r, "\delta^1"'] & \cdots \arrow[r] & K^i \arrow[u, "\phi_i"] \arrow[r, "\delta^i"'] & \cdots \arrow[r] & K^m = M \arrow[u, "\phi_m"] \arrow[r] & {}
\end{tikzcd}$$

[3] The Macaulay2 code used to produce this data and the data itself is available at [KZ].

where the bottom row is the Koszul (co)complex of M on $\underline{f}$ and

$$\phi_i : \bigoplus_{1\le j_1<\cdots<j_i\le m} M_{j_1\cdots j_i} \xrightarrow{\oplus_{1\le j_1<\cdots<j_i\le m}\varphi\circ(f_{j_1}\cdots f_{j_i})^{p-1}} F_R(\bigoplus_{1\le j_1<\cdots<j_i\le m} M_{j_1\cdots j_i}).$$

It follows from [Lyu97, 1.10(c)] that the ϕ_i are generating morphisms of $\mathcal{M}_{f_{j_1}\cdots f_{j_i}}$. Therefore taking direct limit of each row of the diagram produces the Čech complex $\check{C}(\mathcal{M};\underline{f})$. Since taking direct limits preserves exactness, $\varinjlim(H^i(M;\underline{f}) \to F_R(H^i(M;\underline{f})) \to \cdots) = H^i_I(\mathcal{M})$. Our conclusion follows. $\square$

Combining what we have so far in this section, we now have an algorithm to compute the support of $H^{i_1}_{I_1}\cdots H^{i_s}_{I_s}(R)$. For example, the case $s=2$ relevant to the calculation of Lyubeznik numbers in handled as follows. Start with a generating morphism $\operatorname{Ext}^{i_2}(R/I_2,R) \to F_R(\operatorname{Ext}^{i_2}(R/I_2,R))$. Using Proposition 5.1, we know that the Koszul cohomology $H^{i_1}(\operatorname{Ext}^{i_2}(R/I_2,R);\underline{f})$ (with $I_1=(\underline{f})$) is a generating homomorphism of $H^{i_1}_{I_1}H^{i_2}_{I_2}(R)$. We may then apply Corollary 3.3 to compute the support of $H^{i_1}_{I_1}H^{i_2}_{I_2}(R)$.

6. The support of local cohomology of hypersurfaces

Throughout this section R denotes a regular ring of prime characteristic p, $I\subseteq R$ an ideal, and $g\in R$ some fixed element.

Following [Lyu97, §2] we write

$$\mathrm{H}^i_I(R) = \lim_{\to}\left[\operatorname{Ext}^i_R(R/I,R) \xrightarrow{\phi} F^1_R\operatorname{Ext}^i_R(R/I,R) \xrightarrow{F^2_R\phi} F^2_R\operatorname{Ext}^i_R(R/I,R) \xrightarrow{F^3_R\phi} \dots\right]$$

where $F^e_R(-)$ denotes the eth Frobenius functor, and $\phi:\operatorname{Ext}^i_R(R/I,R)\xrightarrow{\phi} F^1_R\operatorname{Ext}^i_R(R/I,R)\cong\operatorname{Ext}^i_R(R/I^{[p]},R)$ is the R-linear map induced by the surjection $R/I^{[p]}\to R/I$. For all $i\ge 0$ we fix a presentation $R^{\alpha_i}\xrightarrow{A_i} R^{\beta_i}$ where A_i is a $\beta_i\times\alpha_i$ matrix with entries in R. We can now find a $\beta_i\times\beta_i$ matrix U_i with entries in R for which the map $\phi:\operatorname{Ext}^i_R(R/I,R)\xrightarrow{\phi} F^1_R\operatorname{Ext}^i_R(R/I,R)$ is isomorphic to the map $U_i:\operatorname{Coker}A_i\to F^1_R(\operatorname{Coker}A_i)=\operatorname{Coker}A_i^{[p]}$ given by multiplication by U_i.

Theorem 6.1. *For any $i\ge 0$ consider the map $g:\mathrm{H}^i_I(R_P)\to\mathrm{H}^i_I(R_P)$ given by multiplication by g. Let $\mathcal{I}^i$ denote the set of primes $P\subset R$ for which the map g is not injective and let $\mathcal{S}^i$ denote the set of primes $P\subset R$ for which the map g is not surjective. For $\ell,e,j\ge 0$ write*

$$V^{(\ell)}_{ej} = U_\ell^{[p^{e+j-1}]}U_\ell^{[p^{e+j-2}]}\cdots U_\ell^{[p^e]}.$$

Then

(a) *$\mathcal{I}^i$ is closed and equal to* $\operatorname{Supp}\dfrac{(\ker V^{(i)}_{0\eta} :_{R^\beta} g)}{\ker V^{(i)}_{0\eta}}$ *for some $\eta>0$,*

(b) $\mathcal{S}^i$ *is closed and equal to* $\operatorname{Supp} \dfrac{R^\beta}{\bigcup_{j\geq 0}\left(gR^\beta + \operatorname{Im} A^{[p^j]} :_{R^\beta} V_{0j}^{(i)}\right)}$, *and*

(c) *the support of* $\mathrm{H}_I^i(R/gR)$ *is closed and equal to* $\mathcal{I}^i \cup \mathcal{S}^i$.

Proof. Fix some $i \geq 0$ and write β, A and U for β_i, A_i and U_i. The map $g : \mathrm{H}_I^i(R) \to \mathrm{H}_I^i(R)$ can be described as a map of direct limit systems

$$\begin{tikzcd}
\operatorname{Coker} A \arrow[r, "U"] \arrow[d, "g"] & \operatorname{Coker} A^{[p]} \arrow[r, "U^{[p]}"] \arrow[d, "g"] & \dots & {} \arrow[r, "U^{[p^{e-1}]}"] & \operatorname{Coker} A^{[p^e]} \arrow[r, "U^{[p^e]}"] \arrow[d, "g"] & \dots \\
\operatorname{Coker} A \arrow[r, "U"] & \operatorname{Coker} A^{[p]} \arrow[r, "U^{[p]}"] & \dots & {} \arrow[r, "U^{[p^{e-1}]}"] & \operatorname{Coker} A^{[p^e]} \arrow[r, "U^{[p^e]}"] & \dots
\end{tikzcd} \tag{4}$$

For any $e, j \geq 0$ abbreviate $V_{ej} = V_{ej}^{(i)}$, and note that it is the matrix corresponding to the composition map $\operatorname{Coker} A^{[p^e]} \to \operatorname{Coker} A^{[p^{e+j}]}$ in the direct limits in (4). Any element in $\mathrm{H}_I^i(R_P)$ can be represented by an element $a \in \operatorname{Coker} A_P^{[p^e]}$ for some $e \geq 0$, and this element represents the zero element if and only if there exists a $j \geq e$ for which $V_{ej} a \in \operatorname{Im} A_P^{[p^{e+j}]}$, i.e., if and only if

$$a \in (\operatorname{Im} A^{[p^{e+j}]} :_{R^\beta} V_{ej})_P.$$

Consider the kernels K_j of the maps $V_{0j} : \operatorname{Coker} A \to \operatorname{Coker} A^{[p^j]}$; these form an ascending chain of submodules of $\operatorname{Coker} A$ and hence stabilize for all j beyond some $\eta \geq 0$. Note that the map $V_{ej} : \operatorname{Coker} A^{[p^e]} \to \operatorname{Coker} A^{[p^{e+j}]}$ is obtained by applying the exact functor $F_R^e(-)$ to the map V_{0j}, hence the kernels of the maps V_{ej} also stabilize for $j \geq \eta$.

To prove (a) we now note that an element in $\mathrm{H}_I^i(R_P)$ represented $a \in \operatorname{Coker} A_P^{[p^e]}$ is multiplied by g to zero if and only if $a \in (\ker V_{e\eta} :_{R^\beta} g)_P$ and so g is injective if and only iff $\left(\dfrac{(\ker V_{e\eta} :_{R^\beta} g)}{\ker V_{e\eta}}\right)_P = 0$, i.e., if g is not a zero divisor on $\left(R^\beta / \ker V_{e\eta}\right)_P$. But $R^\beta/\ker V_{e\eta} = F_R^e(R^\beta/\ker V_{0\eta})$ and, since R is regular, $F_R^e(R^\beta/\ker V_{0\eta})$ and $R^\beta/\ker V_{0\eta}$ have the same associated primes, so we deduce that multiplication by g is injective if and only if g is not a zero divisor on $\left(R^\beta/\ker V_{0\eta}\right)_P$. We deduce that for a prime $P \subset R$, multiplication by g on $\mathrm{H}_I^i(R_P)$ is injective if and only if

$$\left(\frac{(\ker V_{0\eta} :_{R^\beta} g)}{\ker V_{0\eta}}\right)_P = 0$$

so $\mathcal{I}^i = \operatorname{Supp} \frac{(\ker V_{0\eta} :_{R^\beta} g)}{\ker V_{0\eta}}$.

To prove (b) we now note that an element in $\mathrm{H}_I^i(R_P)$ represented $a \in \operatorname{Coker} A_P^{[p^e]}$ is in the image of g if and only if there exists a $j \geq 0$ such that

$V_{ej}a \in \left(gR^\beta + \operatorname{Im} A_P^{[p^{e+j}]}\right)$ hence g is surjective if for all $e \geq 0$,

$$\bigcup_{j\geq 0} \left(gR^\beta + \operatorname{Im} A^{[p^{e+j}]} :_{R^\beta} V_{ej}\right)_P = R^\beta{}_P.$$

Furthermore,

$$\begin{aligned}\left(gR^\beta + \operatorname{Im} A^{[p^{e+j}]} :_{R^\beta} V_{ej}\right)^{[p]} &= \left(g^p R^\beta + \operatorname{Im} A^{[p^{e+1+j}]} :_{R^\beta} V_{e+1,j}\right)\\ &\subseteq \left(gR^\beta + \operatorname{Im} A^{[p^{e+1+j}]} :_{R^\beta} V_{e+1,j}\right)\end{aligned}$$

so for for all $e \geq 0$,

$$\bigcup_{j\geq 0} \left(gR^\beta + \operatorname{Im} A^{[p^{e+j}]} :_{R^\beta} V_{ej}\right)_P = R^\beta{}_P$$

if and only if

$$\bigcup_{j\geq 0} \left(gR^\beta + \operatorname{Im} A^{[p^{j}]} :_{R^\beta} V_{0j}\right)_P = R^\beta{}_P.$$

We conclude that g is not surjective if and only if $P \in \operatorname{Supp} R^\beta / \bigcup_{j\geq 0}\left(gR^\beta + \operatorname{Im} A^{[p^j]} :_{R^\beta} V_{0j}\right)$.

To prove (c) consider the long exact sequence

$$\cdots \to \mathrm{H}_I^i(R) \xrightarrow{g} \mathrm{H}_I^i(R) \to \mathrm{H}_I^i(R/gR) \to \mathrm{H}_I^{i+1}(R) \xrightarrow{g} \mathrm{H}_I^{i+1}(R) \to \dots$$

induced by the short exact sequence $0 \to R \xrightarrow{g} R \to R/gR \to 0$. Note that $\mathrm{H}_I^i(R/gR)_P = 0$ if and only if both $\left(\mathrm{H}_I^i(R) \xrightarrow{g} \mathrm{H}_I^i(R)\right)_P$ is surjective and $\left(\mathrm{H}_I^{i+1}(R) \xrightarrow{g} \mathrm{H}_I^{i+1}(R)\right)_P$ and the result follows.

□

Question 6.2. Theorem 3.2 gives us an effective method for the calculation of $\mathcal{I}^i$. However, we do not know how to compute $\mathcal{S}^i$, hence we ask the following: is there an effective method to bound the value of e for which

$$\bigcup_{j\geq 0} \left(gR^\beta + \operatorname{Im} A^{[p^j]} :_{R^\beta} V_{0j}^{(i)}\right) = \left(gR^\beta + \operatorname{Im} A^{[p^e]} :_{R^\beta} V_{0e}^{(i)}\right)?$$

It turns out that part of our Theorem 6.1 can be extended to the case of isolated singular points.

Corollary 6.3. *Let R be a noetherian commutative ring of prime characteristic that has finitely many isolated singular points. Let $g \in R$ be a nonzerodivisor. Then* $\operatorname{Supp}(H_I^j(R/gR))$ *is Zariski-closed for each integer j and ideal I of R.*

Proof. Let $\{\mathfrak{m}_1, \dots, \mathfrak{m}_t\}$ denotes the set of isolated singular points of R. Set $\mathfrak{a} = \bigcap_{i=1}^t \mathfrak{m}_i$. Let $\{f_1, \dots, f_s\}$ be a set of generators of $\mathfrak{a}$. It follows from Theorem 6.1 that $\operatorname{Supp}_{R_{f_k}}(\mathrm{H}_I^j(R_{f_k}/gR_{f_k}))$ is closed, *i.e.* it has finitely many minimal associated primes. By the bijection between the set of associated primes of $\mathrm{H}^j_I(R/gR)$ that do not contain f_k and $\mathrm{Ass}(\mathrm{H}^j_I(R_{f_k}/gR_{f_k}))$, it follows that the minimal associated primes of $\mathrm{H}^j_I(R/gR)$ are contained in the union of $\{\mathfrak{m}_1, \ldots, \mathfrak{m}_t\}$ and the set of minimal associated primes of $\mathrm{H}^j_I(R_{f_k}/gR_{f_k})$ which is a finite set. □

The proof of Corollary 6.3 can also be used to prove the following result which is of independent interest.

Proposition 6.4. *Let R be either*

(1) *a noetherian commutative ring of prime characteristic, or*
(2) *of finite type over a field of characteristic 0.*

Suppose that R has finitely many isolated singular points. Then $\mathrm{H}^j_I(R)$ has only finitely many associated primes for each integer j and each ideal I of R.

Proof. Let $\{\mathfrak{m}_1, \ldots, \mathfrak{m}_t\}$ denotes the set of isolated singular points of R. Set $\mathfrak{a} = \bigcap_{i=1}^t \mathfrak{m}_i$. Let $\{f_1, \ldots, f_s\}$ be a set of generators of $\mathfrak{a}$. It follows from our assumptions on R that R_{f_k} is either a noetherian regular ring of prime characteristic or a regular ring of finite type over a field of characteristic 0 (cf. [Lyu93, Corollary 3.6]). Consequently, $\mathrm{Ass}(\mathrm{H}^j_I(R_{f_k}))$ is finite for each generator f_k. Since there is a bijection between the set of associated primes of $\mathrm{H}^j_I(R)$ that do not contain f_k and $\mathrm{Ass}(\mathrm{H}^j_I(R_{f_k}))$, it follows that

$$\mathrm{Ass}(\mathrm{H}^j_I(R)) \subseteq \bigcup_{k=1}^{s} \mathrm{Ass}(\mathrm{H}^j_I(R_{f_k})) \bigcup \{\mathfrak{m}_1, \ldots, \mathfrak{m}_t\}.$$

□

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