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# Stability properties of coupled impedance passive LTI systems

Xiaowei Zhao and George Weiss

***Abstract*—We study the stability of the feedback interconnection of two impedance passive linear time-invariant systems, of which one is finite-dimensional. The closed-loop system is well known to be impedance passive, but no stability properties follow from this alone. We are interested in two main issues: (1) the strong stability of the operator semigroup associated with the closed-loop system, (2) the input-output stability (meaning transfer function in $H^\infty$) of the closed-loop system. Our results are illustrated with the system obtained from the non-uniform SCOLE (NASA Spacecraft Control Laboratory Experiment) model representing a vertical beam clamped at the bottom, with a rigid body having a large mass on top, connected with a trolley mounted on top of the rigid body, via a spring and a damper. Such an arrangement called a tuned mass damper (TMD), is used to stabilize tall buildings. We show that the SCOLE-TMD system is strongly stable on the energy state space and that the system is input-output stable from the horizontal force input to the horizontal velocity output.**

***Index Terms*—coupled system, impedance passivity, well-posed linear system, strong stability, input-output stability, SCOLE model, tuned mass damper**

## I. Introduction

THE study of passive systems, largely initiated by Jan Willems [50], is now a central topic in systems and control theory, see for instance van der Schaft [42], Ortega *et al* [27], Khalil [21] and Jacob and Zwart [20]. It is well known that under suitable assumptions passivity is preserved by interconnections, for instance, impedance passivity is preserved by the standard feedback connection [21], [42].

The aim of this paper is to investigate the well-posedness and stability properties of two impedance passive linear systems connected in feedback, of which one is finite-dimensional, with an extensive engineering application. Two classes of such systems are investigated. The first class concerns the feedback interconnection of a well-posed and strictly proper linear system $\Sigma_d$ and a finite-dimensional linear system $\Sigma_f$ as shown in Figure 2. We show that these coupled systems are well-posed, regular, and (under suitable assumptions including impedance passivity of the subsystems) they are strongly stable in their natural state space (the product of the natural state space of $\Sigma_d$ and of $\Sigma_f$). It is well known that the transfer function of an impedance passive system is positive. The second class of systems considered here are feedback interconnections of systems with positive transfer functions, as shown in Figure 3. Under some technical assumptions, we show the input-output stability of this class of systems. We apply our theory to show the well-posedness and strong as well as input-output stability of the SCOLE (NASA Spacecraft Control Laboratory Experiment) model coupled with a tuned mass damper (TMD). This application is motivated by the problem of suppressing the vibrations of wind turbine towers to extend their life-expectancy, using passive mechanical devices.

This paper is a continuation of our recent work [48], [51] where we have developed a theory for the well-posedness, regularity, controllability and observability of several classes of coupled systems that are a feedback connection of an infinite-dimensional subsystem $\Sigma_d$ and a finite-dimensional subsystem $\Sigma_f$. In [48] we assume that $\Sigma_d$ is such that it becomes well-posed and strictly proper when connected in cascade with an integrator and that the feedthrough matrix of $\Sigma_f$ is zero. We have shown that such a coupled system is well-posed and regular when the state space is chosen to be a certain subspace of the product of the state spaces of the subsystems. Under suitable assumptions, including the exact controllability of the subsystems, the coupled system is exactly controllable. In [51] the well-posedness and controllability results are simpler and neater since the assumptions allow us to work with the natural product state space, where $\Sigma_d$ is assumed to be well-posed and strictly proper. In [51] we have also obtained analogous observability results. We did not have impedance passivity assumptions in [48], [51]. The first class of systems in the present paper are the same as in [51] but with the additional impedance passivity assumption, and will use results from [51].

We mention some other works on the stability of coupled infinite-dimensional systems. Ahmadi *et al* [2] formulates small gain conditions for stability of the interconnections of a class of infinite-dimensional systems using dissipation inequalities and passivity concepts. Dáger and Zuazua [12] mainly talks about flexible strings connected to form a planar graph. The stabilization of strings and beams connected in graphs has been investigated in a series of papers by K. Ammari and co-authors, see [5] and the references therein. Lagnese *et al* [22] deals with elastic multi-link flexible structures, each component system being infinite-dimensional (such as a beam, string, plate or shell), possibly nonlinear. The theses Villegas [43] and Pasumarthy [28] are about the power-preserving interconnection of several port-Hamiltonian systems using the formalism of Dirac structures. Lasiecka [23] mainly introduces the structural acoustic model, where a plate and a wave equa-

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tion are coupled to create a model of an aircraft cockpit. The same coupled system has been investigated in [6] and other papers. The stability of various specific infinite-dimensional coupled systems has been investigated in a series of papers by J.M. Wang and co-authors, see [16], [45], [46], also by F. Alabau-Boussouira and co-authors [3], [4], S. Hansen and co-authors [17], [18], and our paper [56]. Many further references on coupled systems can be found in these works. We mention the books Bensoussan *et al* [7], Lasiecka and Triggiani [24], Staffans [34], and Jacob and Zwart [20] as valuable references on infinite-dimensional linear systems, dealing in particular with stability properties, but very little with coupled systems.

As an application of the theory in this paper, we show that a TMD can be used to stabilize the SCOLE model. The SCOLE system is a well known model for a flexible beam with one end clamped and the other end connected to a rigid body. Originally it has been developed to model a mast carrying an antenna on a satellite, see Littman and Markus [25], [26]. It is suitable also to model wind turbine towers, since these are typically clamped at the bottom while the upper end of the tower is linked to the heavy nacelle, which together with the rotor, plays the role of the rigid body.

There is a large literature on the SCOLE model. The approximate/exact controllability/observability of the non-uniform SCOLE model on the energy state space on certain smoother state spaces has been investigated for either force control or torque control in Guo [14] and in Guo and Ivanov [15]. The exponential stabilization of the non-uniform SCOLE model on the energy state space has been achieved via high order feedback in [15]. The largest state space in which the uniform SCOLE model is well-posed and exactly controllable has been found in our paper [52]. The non-uniform SCOLE model has been used as a wind turbine tower model in the plane of the turbine axis and as one subsystem of the wind turbine tower model in the plane of the turbine blades in our work [54], [55], where the wind turbine tower model's well-posedness, regularity, controllability and stabilization have been investigated. The global asymptotic stabilization of the SCOLE model by nonlinear colocated static output feedback is considered in the very recent paper Curtain and Zwart [11].

To approach the practical engineering problem of stabilizing a wind-turbine tower, we should use the natural energy state space for the SCOLE model, where only strong stability can be achieved (because the control and observation operators are bounded). (To understand why bounded control and observation operators make it difficult to obtain exponential stabilization, we refer to the well-known Russell's principle and to [29] and its references.) If we try to use static state or output feedback for strong stabilization, then mathematically this may work very well, but it is not practical, because it requires to apply an external feedback force or a torque on the nacelle, which is difficult to achieve (we would need extra cables or rods between the nacelle and the ground). We can get an effect resembling an external torque in the plane of the turbine blades by modulating the generator torque, and we can get an effect resembling an external force acting vertically to this plane by modulating the pitch angle of the turbine blades, see our paper [56], but this type of control action is likely to interfere with the proper functioning of the wind turbine.

In this paper we design and analyze a TMD control system to reduce the vibrations of the SCOLE model. TMDs have been successfully used in the vibration reduction of tall buildings, they can decrease the worst-case wind-induced motion of the building by about $40\%$, see for instance Connor [10] or the well presented Wikipedia article [49]. It has been proposed to use TMDs in wind turbines as well, see Stewart and Lackner [37], Tong *et al* [39].

A TMD consists of a large mass, connected to the top of the structure to be stabilized via springs and dampers. The idea is that the TMD is tuned to a particular structural frequency and thus will resonate and dissipate input energy via the dampers when the structure is excited at that particular frequency. In this paper we analyze this technique in one plane only, so that there is only one spring and one damper, connected in parallel between the rigid body of the SCOLE system and the mass component of the TMD. In a real application, the mass component would be able to move in two orthogonal horizontal directions. This mass component can be either put on a trolley, or it can be hanged via cables. Although mathematically there is no difference between these cases (at least in the linear approximation), we have in mind the first case. A similar TMD system has been used in the John Hancock Tower in Boston and the Citicorp Center in Manhattan, see Figure 1.

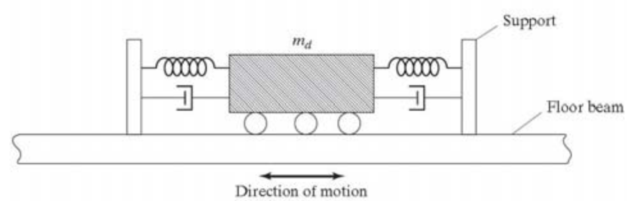


Fig. 1. Schematic diagram of the TMD systems used in the John Hancock Tower in Boston and the Citicorp Center in Manhattan, taken from [10].

Our first main theoretical result is Theorem 3.1, which is the well-posedness and strong stability result for coupled impedance passive systems interconnected as in Figure 2. The second main theoretical result is Theorem 4.2, which is an input-output stability result for coupled systems as in Figure 3 with positive transfer functions. Our main application results are Theorem 5.5 and Theorem 5.6, which state that the SCOLE-TMD system $\Sigma_c$ is well-posed, regular, strongly stable on its energy state space and input-output stable. We prove them using Theorems 3.1 and 4.2. We mention that a spatially discretized version of this SCOLE-TMD model $\Sigma_c$ has been derived in [39], using the spectral element method, to model a monopile wind turbine tower. They have derived the optimal parameters of the TMD system using $H_2$ optimization. Simulation studies have shown that this optimal TMD system achieves substantial vibration reduction when subject to random wind and wave loading.

The structure of the paper is as follows. In Section II we introduce some background on well-posed systems including approximate observability, well-posedness and regularity. In addition, we recall various stability concepts, and impedance passivity. In Section III we show the well-posedness and strong stability of a class of coupled impedance passive linear systems shown in Figure 2. Section IV is devoted to the input-output stability of coupled systems as shown in Figure 3, with each subsystem having a positive transfer function. Section V is about the application of the two theoretical results to the vibration reduction of the SCOLE model using TMD - we show the well-posedness, strong stability and input-output stability of the resulting coupled system.

## II. Some background on well-posed systems

In this section we briefly introduce some concepts and results on well-posed linear systems. A more detailed introduction and relevant references can be found in Staffans [34] and in Tucsnak and Weiss [40], [41].

Let $A$ be the generator of a strongly continuous semigroup $\mathbb{T}$ on a Hilbert space $X$. Then $\mathbb{T}$ and $X$ determine two additional Hilbert spaces: $X_1$ is $\mathcal{D}(A)$ with the norm $\|z\|_1 = \|(\beta I - A)z\|$, and $X_{-1}$ is the completion of $X$ with respect to the norm $\|z\|_{-1} = \|(\beta I - A)^{-1}z\|$, where $\beta \in \rho(A)$ is fixed. We have $X_1 \subset X \subset X_{-1}$ densely and with continuous embeddings. We can extend $A$ to a bounded operator from $X$ to $X_{-1}$, still denoted by $A$. The semigroup generated by this extended $A$ is the continuous extension of $\mathbb{T}$ to $X_{-1}$, which is still denoted by $\mathbb{T}$.

In this section, $U$ and $Y$ are Hilbert spaces, $B \in \mathcal{L}(U, X_{-1})$ and $C \in \mathcal{L}(X_1, Y)$. We use the following notation for half-planes: for any $\alpha \in \mathbb{R}$,

$$\mathbb{C}_\alpha = \{s \in \mathbb{C} \mid \operatorname{Re} s > \alpha\}.$$

For any operator $T$ acting on a Hilbert space, we denote

$$\operatorname{Re} T = \frac{1}{2}(T + T^*).$$

*Definition 2.1:* $B$ is said to be an *admissible control operator* for $\mathbb{T}$ if for some (hence, for every) $\tau > 0$,

$$\int_0^\tau \mathbb{T}_{\tau-t} B u(t) \mathrm{d}t \in X \qquad \forall\, u \in L^2([0,\infty), U).$$

The admissibility of $B$ is equivalent to the fact that the solutions $z(\cdot)$ (in the sense of [40, Section 4.1]) of

$$\dot{z}(t) = Az(t) + Bu(t), \tag{2.1}$$

with initial state $z(0) = z_0 \in X$ and with $u \in L^2_{loc}([0,\infty), U)$ remain in $X$. The operator $B$ is said to be *bounded* if $B \in \mathcal{L}(U, X)$ and *unbounded* otherwise.

*Definition 2.2:* $C$ is said to be an *admissible observation operator* for the semigroup $\mathbb{T}$, if for some (hence, for every) $\tau > 0$ there exists a constant $\kappa_\tau > 0$ such that

$$\int_0^\tau \|C\mathbb{T}_t x\|^2 \mathrm{d}t \le \kappa_\tau \|x\|^2 \qquad \forall\, x \in \mathcal{D}(A).$$

$C$ is said to be *infinite-time admissible* if the above inequality holds for $\tau = \infty$ .

$C$ is said to be *bounded* if it can be extended such that $C \in \mathcal{L}(X, Y)$ and *unbounded* otherwise. The $\Lambda$-extension of $C$, denoted by $C_\Lambda$, is defined by

$$C_\Lambda z_0 = \lim_{\lambda \to +\infty} C\lambda(\lambda I - A)^{-1} z_0,$$

and its domain $\mathcal{D}(C_\Lambda)$ consists of all $z_0 \in X$ for which the limit exists. If $C$ is admissible, then for any $\tau > 0$ we can define a bounded operator $\Psi_\tau : X \to L^2([0,\tau], Y)$ such that for any $z_0 \in \mathcal{D}(A)$ we have $(\Psi_\tau z_0)(t) = C\mathbb{T}_t z_0$. The operator $C_\Lambda$ has the following remarkable property: for every $z_0 \in X$ we have $\mathbb{T}_t z_0 \in \mathcal{D}(C_\Lambda)$ for almost every $t \ge 0$ and $(\Psi_\tau z_0)(t) = C_\Lambda \mathbb{T}_t z_0$ holds for almost every $t \in [0,\tau]$.

*Definition 2.3:* Let $A, C$ be as in the previous definition. We say that $(A, C)$ is approximately observable in time $\tau > 0$ if for every $x \in X$ the following holds: If $C_\Lambda \mathbb{T}_t x = 0$ for almost every $t \in [0,\tau]$, then $x = 0$.

A *well-posed linear system* with *input space* $U$, *state space* $X$ and *output space* $Y$ is a family of bounded linear operators (parametrized by $\tau \ge 0$) that associates to every initial state $z_0 \in X$ and every input signal $u \in L^2([0,\tau], U)$ a final state $z(\tau)$ and an output signal $y \in L^2([0,\tau], Y)$. These operators have to satisfy certain natural functional equations, for the formal definition we refer to Salamon [31] or Staffans [34], [35] or Weiss [47] or Tucsnak and Weiss [41]. We recall some facts about well-posed linear systems following [47].

Let $\Sigma$ be a well-posed system with input space $U$, state space $X$ and output space $Y$. Then $\Sigma$ is completely determined by its *generating triple* $(A, B, C)$ and its transfer function $\mathbf{G}$. Here, $A$ is the *semigroup generator* of $\Sigma$, which generates a strongly continuous semigroup $\mathbb{T}$ on $X$, $B \in \mathcal{L}(U, X_{-1})$ is the *control operator* of $\Sigma$ and $C \in \mathcal{L}(X_1, Y)$ is its *observation operator*. The state trajectories of $\Sigma$ are solutions of (2.1). The transfer function $\mathbf{G}$ satisfies for all $s, \beta \in \rho(A)$,

$$\mathbf{G}(s) - \mathbf{G}(\beta) = C[(sI - A)^{-1} - (\beta I - A)^{-1}]B.$$

If $u \in L^2_{loc}([0,\infty), U)$ is the input function of $\Sigma$ and its initial state is zero, then its output function $y \in L^2_{loc}([0,\infty), Y)$ is given by $y = \mathbb{F}u$, where $\mathbb{F} : L^2_{loc}([0,\infty), U) \to L^2_{loc}([0,\infty), Y)$ is the input-output map of $\Sigma$. $\mathbb{F}$ is easiest to represent using Laplace transforms, as follows: if $u \in L^2([0,\infty), U)$ and $y = \mathbb{F}u$, then $y$ has a Laplace transform $\hat{y}$ and

$$\hat{y}(s) = \mathbf{G}(s)\hat{u}(s) \tag{2.2}$$

for all $s \in \mathbb{C}$ with $\operatorname{Re} s$ sufficiently large. $\mathbf{G}$ is *proper*, which means that its domain contains a right half-plane $\mathbb{C}_\alpha$ such that $\mathbf{G}$ is uniformly bounded on $\mathbb{C}_\alpha$. A classical solution of (2.1) exists if $u \in \mathcal{H}^1_{\mathrm{loc}}(0,\infty; U)$ (i.e., $u(t) - u(0) = \int_0^t v(\sigma)\mathrm{d}\sigma$, where $v \in L^2_{\mathrm{loc}}([0,\infty), U)$) and $Az(0) + Bu(0) \in X$, and in this case the corresponding output function satisfies $y \in \mathcal{H}^1_{\mathrm{loc}}(0,\infty; Y)$, see for instance [33], [36, Proposition 3.5].

*Definition 2.4:* The well-posed linear system $\Sigma$ with transfer function $\mathbf{G}$ is called *regular* if for every $\mathrm{v} \in U$, the limit

$$\lim_{s \to +\infty} \mathbf{G}(s)\mathrm{v} = D\mathrm{v}$$

exists, where $s$ is real. In this case, the operator $D \in \mathcal{L}(U, Y)$ is called the *feedthrough operator* of $\Sigma$.

We mention a few facts about regular systems, following [47]. Regularity is equivalent to the fact that the product $C_\Lambda(sI - A)^{-1}B$ makes sense, for some (hence for every) $s \in \rho(A)$. If $\Sigma$ is regular then for every initial state $z_0 \in X$ and every $u \in L^2_{loc}([0,\infty), U)$, the solution of $\dot{z} = Az + Bu$

with $z(0) = z_0$ satisfies $z(t) \in \mathcal{D}(C_\Lambda)$ for almost every $t \geq 0$ and the corresponding output function is given by

$$y(t) = C_\Lambda z(t) + Du(t) \quad \text{for almost every} \quad t \geq 0. \qquad (2.3)$$

The transfer function of the regular system $\Sigma$ is

$$\mathbf{G}(s) = C_\Lambda(sI - A)^{-1}B + D \qquad \forall\, s \in \rho(A). \qquad (2.4)$$

The operators $A, B, C, D$ are called the *generating operators* of $\Sigma$. This is because they determine $\Sigma$ via (2.1) and (2.3). If $C$ is bounded, then $C$ replaces $C_\Lambda$ in (2.3) and (2.4), and (2.3) holds for every $t \geq 0$.

Let $\mathbf{H}$ be a function defined on some domain in $\mathbb{C}$ that contains a right half-plane, with values in a normed space. We say that $\mathbf{H}$ is *strictly proper* if

$$\lim_{\operatorname{Re} s \to \infty} \|\mathbf{H}(s)\| = 0, \quad \text{uniformly with respect to} \quad \operatorname{Im} s.$$

A linear system is called *strictly proper* if its transfer function is strictly proper. Clearly this implies $D = 0$ in (2.4).

*Definition 2.5:* Let $\mathbb{T}$ be an semigroup generator of an infinite-dimensional system $\Sigma$ on a Hilbert space $X$. The system $\Sigma$ (or the semigroup $\mathbb{T}$) is called *weakly stable* if $\langle \mathbb{T}_t z, y\rangle \to 0$ as $t \to \infty$, for all $z, y \in X$. The system $\Sigma$ (or the semigroup $\mathbb{T}$) is called *strongly stable* if $\mathbb{T}_t z \to 0$ as $t \to \infty$, for all $z \in X$. The system $\Sigma$ (or the semigroup $\mathbb{T}$) is called *exponentially stable* if its growth bound is negative.

The following is the dual version of Theorem 3.1 and Proposition 3.2 together with Remark 3.1 of Hansen and Weiss [19], see also Theorem 5.1.1 in [40].

*Theorem 2.6:* Let $A$ be the generator of a strongly continuous semigroup $\mathbb{T}$ on a Hilbert space $X$. Let $Y$ be a Hilbert space and consider the observation operator $C \in \mathcal{L}(X_1, Y)$. The following two statements are equivalent:

(i) $C$ is infinite-time admissible for $\mathbb{T}$.

(ii) There exists an operator $P \in \mathcal{L}(X)$ such that

$$Px = \lim_{\tau \to \infty} \int_0^\tau \mathbb{T}_t^* C^* C \mathbb{T}_t x \,\mathrm{d}t \quad \forall x \in \mathcal{D}(A). \qquad (2.5)$$

($P$ is called the observability Gramian of $(A, C)$.)

In addition, if $P > 0$ (meaning that $P \geq 0$ and $\operatorname{Ker} P = \{0\}$) and $\mathbb{T}$ is uniformly bounded, then $\mathbb{T}$ is weakly stable. Furthermore if $\sigma(A) \cap i\mathbb{R}$ is at most countable, then $\mathbb{T}$ is strongly stable.

We remark that if $(A, C)$ is approximately observable in some time, then $P$ in (2.5) satisfies $P > 0$.

Passive systems are a class of dynamical systems with an energy functional defined on the state space and a supply rate defined on the input $u$ and output $y$, such that on any time interval, the change of energy is less or equal the integral of the supply rate, see for instance [42], [50]. Here we are only interested in the most frequently used supply rate $\operatorname{Re}\langle u(t), y(t)\rangle$, in which case the system is called impedance passive, and we deal only with linear time invariant systems. For more details about linear impedance passive systems we refer to Staffans [32], [33], Staffans and Weiss [36].

Let $H$ be a Hilbert space, $P \in \mathcal{L}(H)$ and $P > 0$. We define an inner product by $\langle q, \vartheta\rangle_P = \langle Pq, \vartheta\rangle$ $(\forall q, \vartheta \in H)$ which induces the norm $\|q\|_P = \sqrt{\langle q, q\rangle_P}$.

*Definition 2.7:* The well-posed system $\Sigma$ is *impedance $P$-passive* if $Y = U$ and for any input signal $u \in L^2_{loc}([0,\infty), U)$, any initial state $z(0) \in X$ and any time $\tau \geq 0$, the following inequality holds

$$\|z(\tau)\|_P^2 - \|z(0)\|_P^2 \leq 2\int_0^\tau \operatorname{Re}\langle u(t), y(t)\rangle \mathrm{d}t, \qquad (2.6)$$

where $y$ is the output function corresponding to $z$ and $u$.

This is equivalent to the fact that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|z(t)\|_P^2 \leq 2\operatorname{Re}\langle u(t), y(t)\rangle$$

holds when $z$ is a classical solution of (2.1) and $y$ is the corresponding output function, see for instance [32], [33] or [41, Section 6]. If $P = I$, then we say that the system is *impedance passive* instead of impedance $I$-passive. From the energy point of view, (2.6) is an energy balance inequality, $E(t) = \frac{1}{2}\|z(t)\|_P^2 = \frac{1}{2}\langle Pz(t), z(t)\rangle$ is the energy functional, and $\operatorname{Re}\langle u(t), y(t)\rangle$ is the supply rate.

It is well known that any impedance passive system has a *positive transfer function* $\mathbf{P}$, which means that $\mathbf{P}$ is analytic on $\mathbb{C}_0$ and $\operatorname{Re}\mathbf{P}(s) \geq 0$ for all $s \in \mathbb{C}_0$, see for instance [32]–[34], [41]. Positive transfer functions were introduced in Brune [9] (based on his PhD thesis). He also introduced the slightly more restrictive (but extremely useful and much used) concept of positive real transfer function (not used in this paper).

## III. The well-posedness and strong stability of a class of coupled impedance passive linear systems

In this section we derive some well-posedness and strong stabilization results for a class of coupled impedance passive linear time-invariant systems.

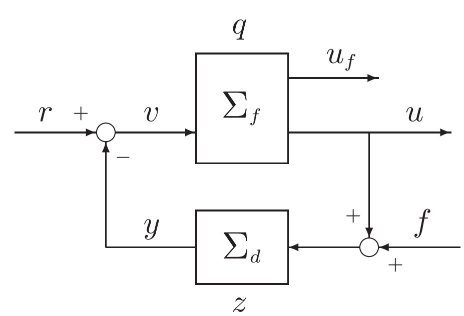


Fig. 2. A coupled system $\Sigma_c$ consisting of a well-posed and strictly proper system $\Sigma_d$ and a finite-dimensional system $\Sigma_f = (a, b, [c; c_f], [d; d_f])$, connected in feedback.

Consider a coupled system $\Sigma_c$ consisting of a well-posed system $\Sigma_d$ and a finite-dimensional system $\Sigma_f$ connected in feedback as shown in Figure 2. The subsystem $\Sigma_f$ has input $v = r - y$, where $y$ is the output of $\Sigma_d$ while $r$ is the first external input of $\Sigma_c$. $\Sigma_f$ has two outputs denoted $u$ and $u_f$. The output $u$ added to the second external input $f$ is the input signal $u + f$ of $\Sigma_d$. The other output $u_f$ appears in the dissipation inequality of $\Sigma_f$, stated in assumption (ii) of Theorem 3.1. The system $\Sigma_f$ is described by

$$\begin{cases} \dot{q}(t) = aq(t) + bv(t), & (3.1)\\ u(t) = cq(t) + dv(t), & (3.2)\\ u_f(t) = c_f q(t) + d_f v(t), & (3.3)\end{cases}$$

where $a \in \mathbb{C}^{n\times n}$, $b \in \mathbb{C}^{n\times m}$, $c \in \mathbb{C}^{m\times n}$, $c_f \in \mathbb{C}^{j\times n}$, $d \in \mathbb{C}^{m\times m}$, $d_f \in \mathbb{C}^{j\times m}$ and $q(t) \in \mathbb{C}^n$ is the state of $\Sigma_f$. We denote by $\mathbf{g}$ the transfer function of $\Sigma_f$, from $v$ to $u$:

$$\mathbf{g}(s) = c(sI - a)^{-1}b + d.$$

The well-posed linear system $\Sigma_d$ is assumed to be strictly proper (hence regular with $D = 0$), and it is determined by its generating triple $(A, B, C)$ via

$$\dot{z}(t) = Az(t) + B\big(u(t) + f(t)\big), \qquad y(t) = C_\Lambda z, \quad (3.4)$$

where $z$, $u+f$ and $y$ are the state trajectory, the input function and the output function of $\Sigma_d$, with values in the state space $X$ (a Hilbert space), the input space $\mathbb{C}^m$ and the output space $\mathbb{C}^m$, respectively. $A$ is the semigroup generator of $\Sigma_d$, so that it generates a strongly continuous semigroup $\mathbb{T}$ on the state space $X$, $B \in \mathcal{L}(\mathbb{C}^m, X_{-1})$ and $C \in \mathcal{L}(X_1, \mathbb{C}^m)$ are the control operator and the observation operator respectively, and $C_\Lambda$ is the $\Lambda$-extension of $C$. The transfer function of $\Sigma_d$ is

$$\mathbf{G}(s) = C_\Lambda(sI - A)^{-1}B \qquad \forall\, s \in \rho(A).$$

By setting $v = r - y$, as in Figure 2, we obtain the coupled system $\Sigma_c$ that has the state $z^c = \left[\begin{smallmatrix} z \\ q \end{smallmatrix}\right]$, state space $X \times \mathbb{C}^n$, input $\left[\begin{smallmatrix} r \\ f \end{smallmatrix}\right]$ and output $\left[\begin{smallmatrix} u \\ u_f \\ y \end{smallmatrix}\right]$, and has the generating operators $(A^c, B^c, C^c, D^c)$ given by

$$A^c = \begin{bmatrix} A - BdC_\Lambda & Bc \\ -bC_\Lambda & a \end{bmatrix}, \qquad B^c = \begin{bmatrix} Bd & B \\ b & 0 \end{bmatrix}, \quad (3.5)$$

$$\mathcal{D}(A^c) = \left\{ \begin{bmatrix} z \\ q \end{bmatrix} \in X \times \mathbb{C}^n \;\middle|\; A^c \begin{bmatrix} z \\ q \end{bmatrix} \in X \times \mathbb{C}^n \right\}, \quad (3.6)$$

$$C^c = \begin{bmatrix} -dC_\Lambda & c \\ -d_f C_\Lambda & c_f \\ C_\Lambda & 0 \end{bmatrix}, \qquad D^c = \begin{bmatrix} d & 0 \\ d_f & 0 \\ 0 & 0 \end{bmatrix}. \quad (3.7)$$

The transfer function of $\Sigma_c$ from $\left[\begin{smallmatrix} r \\ f \end{smallmatrix}\right]$ to $\left[\begin{smallmatrix} u \\ y \end{smallmatrix}\right]$ is

$$\mathbf{G}^c = \begin{bmatrix} \mathbf{g}(I + \mathbf{G}\mathbf{g})^{-1} & \mathbf{g}\mathbf{G}(I + \mathbf{g}\mathbf{G})^{-1} \\ \mathbf{G}\mathbf{g}(I + \mathbf{G}\mathbf{g})^{-1} & \mathbf{G}(I + \mathbf{g}\mathbf{G})^{-1} \end{bmatrix}. \quad (3.8)$$

If $B$ is bounded (i.e., $B \in \mathcal{L}(\mathbb{C}^m, X)$), then $\mathcal{D}(A^c) = \mathcal{D}(A) \times \mathbb{C}^n$. All this can be derived as in [51, Sect. 4].

We denote by $\rho_\infty(A)$ the connected component of $\rho(A)$ containing some right half-plane.

For the well-posedness and strong stability of the coupled system $\Sigma_c$ we have the following theorem:

*Theorem 3.1:* Let $\Sigma_d$ be a well-posed and strictly proper (hence regular) system with state space $X$ (a Hilbert space), input space $\mathbb{C}^m$, output space $\mathbb{C}^m$, semigroup $\mathbb{T}$ and generating triple $(A, B, C)$. Let $a, b, c, c_f, d, d_f$ be matrices as in (3.1)–(3.3). Then the coupled system $\Sigma_c$ from Figure 2 described by (3.1)–(3.4) (or equivalently by (3.5)–(3.7)), with state $z^c = \left[\begin{smallmatrix} z \\ q \end{smallmatrix}\right]$, input $\left[\begin{smallmatrix} r \\ f \end{smallmatrix}\right]$ and output $\left[\begin{smallmatrix} u \\ u_f \\ y \end{smallmatrix}\right]$, is well-posed and regular with the state space $X^c = X \times \mathbb{C}^n$. We denote by $A^c$ the semigroup generator of $\Sigma_c$.

Now we assume additionally the following:

(i) $\Sigma_d$ is impedance passive, i.e., along classical solutions of (2.1) with $y$ given by (2.3),

$$\frac{\mathrm{d}}{\mathrm{d}t}\|z(t)\|^2 \le 2\mathrm{Re}\,\langle u(t) + f(t), y(t)\rangle;$$

(ii) $\Sigma_f$ satisfies the following requirement (that resembles strict output passivity): there exists $\gamma > 0$ such that along any classical solution of (3.1)-(3.3),

$$\frac{\mathrm{d}}{\mathrm{d}t}\|q(t)\|^2 \le 2\mathrm{Re}\,\langle v(t), u(t)\rangle - \gamma\|u_f(t)\|^2;$$

(iii) $(A, C)$ is approximately observable in some time;

(iv) $(a, c)$ is observable;

(v) $d \in \mathbb{C}^{m\times m}$ is invertible;

(vi) Denote $a^\times = a - bd^{-1}c$. We have $\sigma(a^\times) \subset \rho_\infty(A)$;

(vii) $\Sigma_f$ has the following property: If $u_f = 0$ (for all $t \ge 0$) and $v$ has finite power (meaning that $\limsup_{\tau\to\infty} \frac{1}{\tau}\int_0^\tau \|v\|^2 \mathrm{d}t < \infty$), then $u = 0$.

Then $\Sigma_c$ is weakly stable. Furthermore if $\sigma(A^c) \cap i\mathbb{R}$ is at most countable, then $\Sigma_c$ is strongly stable.

*Remark 3.2:* The assumption (vii) in the above theorem is rather unusual, as far as we know. We give a simple example of a finite-dimensional system $\Sigma_f$ that satisfies (vii), to show that this assumption makes sense. Consider $\Sigma_f$ described by

$$\begin{cases} \dot{q}_1(t) = q_2(t), \quad \dot{q}_2(t) = 0, & (3.9) \\ u(t) = q_1(t) + q_2(t) + v(t), & (3.10) \\ u_f(t) = q_1(t) + v(t). & (3.11) \end{cases}$$

We see that $q_2$ is constant and $q_1(t) = q_1(0) + q_2 t$. If $u_f = 0$ then $v(t) = -q_1(t) = -q_1(0) - q_2 t$. In general, this does not imply that $u = 0$, but if $v$ has finite power, then we must have $q_2 = 0$, so that $u(t) = q_1(t) + v(t) = u_f(t) = 0$. Notice that this system satisfies also the assumptions (iv) and (v) in the theorem. Another example is in Section V.

*Remark 3.3:* We can formulate a particular case of Theorem 3.1 by taking $u_f = u$ (so that $\Sigma_f$ has only one output $u$, possibly a vector). Now the assumption (ii) means that $\Sigma_f$ is strictly output passive, and assumption (vii) is trivially satisfied. We think that this particular case of the theorem is also of interest. This particular case of Theorem 3.1, when restricted to finite-dimensional linear systems, is related to the linear version of Theorem 6.3 in [21].

*Proof of Theorem 3.1.* The well-posedness and regularity results from the input $r$ to the output $\left[\begin{smallmatrix} u \\ y \end{smallmatrix}\right]$ follow from Proposition I.4 in our paper [51]. Now we show these properties for the case from $f$ to $\left[\begin{smallmatrix} u \\ y \end{smallmatrix}\right]$. We denote the transfer functions of $\Sigma_d$ and $\Sigma_f$ (from $v$ to $u$) by $\mathbf{G}$ and $\mathbf{g}$ respectively. Then the transfer functions from the input $f$ to the outputs $u$ and $y$ are $\mathbf{G}^c_u$ and $\mathbf{G}^c_y$, as seen in the second column of the matrix in (3.8). Because $\mathbf{G}$ is strictly proper and $\mathbf{g}$ is proper, it is easy to see that $\mathbf{G}^c_u$ and $\mathbf{G}^c_y$ are strictly proper. Thus the well-posedness and regularity of $\Sigma_c$ from the input $f$ to the output $\left[\begin{smallmatrix} u \\ y \end{smallmatrix}\right]$ hold. The well-posedness from the input $f$ to the state $\left[\begin{smallmatrix} z \\ q \end{smallmatrix}\right]$ is very easy to see: if $f \in L^2([0,\tau], \mathbb{C}^m)$ then (due to the preceding argument) we have $u \in L^2([0,\tau], \mathbb{C}^m)$, hence the input $u + f$ of $\Sigma_d$ is in $L^2([0,\tau], \mathbb{C}^m)$, and since $\Sigma_d$ is well-posed, $z(\tau) \in X$. The fact that $q(\tau) \in \mathbb{C}^n$ is obvious.

Following assumptions (iii), (iv), (v), (vi) and Proposition I.4 of [51], we know that $\Sigma_c$ is approximately observable in some time with the output $u$. The next step is to show that $\Sigma_c$

is approximately observable in some time with the output $u_f$. By assumptions (i) and (ii), we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|z^c(t)\|^2 &= \frac{\mathrm{d}}{\mathrm{d}t}\|z(t)\|^2 + \frac{\mathrm{d}}{\mathrm{d}t}\|q(t)\|^2 \\ &\leq 2\mathrm{Re}\,\langle f(t)+u(t), y(t)\rangle + 2\mathrm{Re}\,\langle v(t), u(t)\rangle - \gamma\|u_f(t)\|^2 \\ &= 2\mathrm{Re}\,\langle f(t), y(t)\rangle + 2\mathrm{Re}\,\langle r(t), u(t)\rangle - \gamma\|u_f(t)\|^2, \quad (3.12)\end{aligned}$$

which implies that $\Sigma_c$ is impedance passive on the state space $X^c$ with input $\left[\begin{smallmatrix} r \\ f \end{smallmatrix}\right]$ and output $\left[\begin{smallmatrix} u \\ y \end{smallmatrix}\right]$. Thus the semigroup $\mathbb{T}^c$ of $\Sigma_c$ is a contraction semigroup on the state space $X^c$. Now let $r = 0$ and $f = 0$ because for the study of the stability of $\Sigma_c$ they are not relevant, hence $v = -y$. Due to Proposition 4.3.3 in [40] (the case $\omega = 0$), $y$ has finite power. Based on this fact and the assumption (vii), we know that if $u_f = 0$, then $u = 0$. This implies that the initial state of $\Sigma_c$ is zero, because $\Sigma_c$ is approximately observable in some time with the output $u$. Therefore the approximate observability of $\Sigma_c$, with $u_f$ as its output, holds.

The dynamic equations of the coupled system $\Sigma_c$, with $r = f = 0$ and with the output $u_f$, are

$$\begin{cases} \dot{z}^c(t) &= A^c z^c(t), \\ u_f(t) &= C^c_{f\Lambda}\, z^c(t). \end{cases} \qquad (3.13)$$

From (3.12) with $r = 0$ and $f = 0$ we know that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|z^c(t)\|^2 \leq -\gamma\|u_f(t)\|^2. \qquad (3.14)$$

For any $z^c(0) = z^c_0 \in \mathcal{D}(A^c)$ for (3.13), we have

$$u_f(t) = C^c_f \mathbb{T}^c_t z^c_0.$$

If we substitute this $u_f$ into (3.14) and integrate both sides, we obtain

$$\int_0^\infty \|C^c_f \mathbb{T}_t z^c_0\|^2 \mathrm{d}t \leq \frac{1}{\gamma}\|z^c_0\|^2,$$

which shows that $C^c_f$ is infinite-time admissible. By Theorem 2.6, the observability Gramian $P^c_f$ of $(A^c, C^c_f)$ exists, as defined in (2.5). Due to the approximate observability of $(A^c, C^c_f)$, we have $P^c_f > 0$ (see the remark after Theorem 2.6). This fact together with $\mathbb{T}^c$ being a contraction semigroup (thus uniformly bounded) and Theorem 2.6 show the weak and strong stabilities of $\Sigma_c$. ■

We remark that $\sigma(A^c) \cap i\mathbb{R}$ of measure zero would be enough in the last part of the theorem to conclude strong stability, thanks to Corollary 2 in Boyadzhiev and Levan [8], which is based on Proposition 6.7 in Sz.-Nagy *et al* [38].

## IV. Input-output stability of coupled systems with positive transfer functions

Recall the well known fact that any impedance passive system has a positive transfer function $\mathbf{P}$. It is also well known and easy to see that the feedback interconnection of two positive transfer functions, as shown in Figure 3, gives rise to a positive closed-loop transfer function from $\left[\begin{smallmatrix} r \\ f \end{smallmatrix}\right]$ to $\left[\begin{smallmatrix} v \\ h \end{smallmatrix}\right]$, see for instance [21], [42] for various generalizations of this fact. A (possibly operator-valued) transfer function $\mathbf{P}$ is called *input-output stable* if it is a bounded and analytic function on $\mathbb{C}_0$, which we write as $\mathbf{P} \in \mathcal{H}^\infty(\mathbb{C}_0)$.

Positive transfer functions need not be input-output stable, as the trivial example $\mathbf{P}(s) = 1/s$ shows. However, it is often of great interest to establish if a positive transfer function obtained by a feedback interconnection is input-output stable. Various results in this direction can be found in [33], [42], Vidyasagar [44], and many others. We shall need the following technical result that appears as Lemma 3.3 in [30].

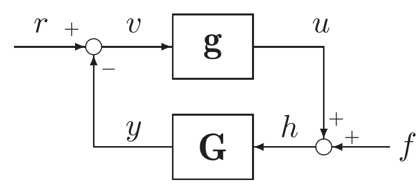


Fig. 3. The standard feedback connection of two transfer functions. In this section we are interested in the case when both $\mathbf{G}$ and $\mathbf{g}$ are positive, and $\mathbf{g}$ is rational and stable.

*Lemma 4.1:* Let $H$ be a Hilbert space and let $\mathbf{T} \in \mathcal{L}(H)$. Then we have

$$\mathrm{Re}\,\mathbf{T} \geq \frac{1}{2}I$$

if and only if there exits $\mathbf{Q} \in \mathcal{L}(H)$ such that

$$\mathbf{T} = (I - \mathbf{Q})^{-1}, \quad \text{where} \quad \|\mathbf{Q}\| \leq 1.$$

In addition, if these conditions are satisfied, then we have

$$\mathrm{Re}\,\mathbf{Q} \leq \left(1 - \frac{1}{2M^2}\right)I, \qquad \text{where} \quad M = \|\mathbf{T}\|.$$

*Theorem 4.2:* Let $U$ be a finite-dimensional inner product space and let $\mathbf{G}$ and $\mathbf{g}$ be $\mathcal{L}(U)$-valued positive transfer functions. Suppose in addition that:

(i) $\mathbf{g}$ is rational and input-output stable, with $\mathrm{Re}\,\mathbf{g}(\infty) > 0$.

(ii) $\mathbf{G}$ is bounded around the imaginary zeros of $\mathbf{g}$, i.e.,

$$\limsup_{s \to i\omega_0} \|\mathbf{G}(s)\| < \infty,$$

for all points $\omega_0 \in \mathbb{R}$ where $\mathbf{g}(i\omega_0)$ is not invertible. Moreover, for the same points $\omega_0$ we also have

$$\limsup_{s \to i\omega_0} \|\mathbf{g}(s)\mathbf{G}(s)\| < 1. \qquad (4.1)$$

(iii) For any $\omega \in \mathbb{R}$, if $\mathbf{g}(i\omega)$ is invertible, then also $\mathrm{Re}\,\mathbf{g}(i\omega)$ is invertible.

Then the feedback connection of $\mathbf{G}$ and $\mathbf{g}$ as in Figure 3 is input-output stable from $\left[\begin{smallmatrix} r \\ f \end{smallmatrix}\right]$ to $\left[\begin{smallmatrix} v \\ h \end{smallmatrix}\right]$.

This theorem is somewhat related to Theorem 6.2 in [21]. The stated stability property is also called internal stability of the feedback system.

*Proof of Theorem 4.2.* For any $\delta > 0$, we denote by $Z$ the set of zeros of $\mathbf{g}$ on the imaginary axis, i.e., the points $i\omega$ where $\omega \in \mathbb{R}$ and $\mathbf{g}(i\omega)$ is not invertible. Since $\mathbf{g}(\infty)$ is invertible, $Z$ is finite. For any $\delta > 0$ we denote by $Z_\delta$ the subset of $\mathbb{C}$ consisting of points that are at a distance $< \delta$ from $Z$ (a union of open disks).

We claim that for every $\delta > 0$ there exists $\varepsilon > 0$ such that $\mathrm{Re}\,\mathbf{g}(s) \geq \varepsilon I$ for all $s \in \mathbb{C}_0 \setminus Z_\delta$. To prove this claim, let us denote by $\mathrm{clos}\,\mathbb{C}_0$ the closed right half-plane. We have $\mathrm{Re}\,\mathbf{g}(s) > 0$ for every $s \in \mathrm{clos}\,\mathbb{C}_0 \setminus Z_\delta$. Indeed, for $\mathrm{Re}\,s = 0$

this follows from assumption (iii). Using this, for $\operatorname{Re} s > 0$ the fact that $\operatorname{Re}\mathbf{g}(s) > 0$ follows from the Poisson integral representation of the harmonic function $\operatorname{Re}\mathbf{g}$, which we write in weak form: for any $\mathrm{v} \in \mathbb{C}^m$ and for any $\alpha > 0$, $\beta \in \mathbb{R}$,

$$\langle \operatorname{Re}\mathbf{g}(\alpha + i\beta)\mathrm{v}, \mathrm{v}\rangle = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\alpha\langle \operatorname{Re}\mathbf{g}(i\omega)\mathrm{v}, \mathrm{v}\rangle}{\alpha^2 + (\beta - \omega)^2}\,\mathrm{d}\omega,$$

see for instance Ahlfors [1, p. 171] or Garnett [13, p. 12].

The continuous function $s \to \|[\operatorname{Re}\mathbf{g}(s)]^{-1}\|$, which is continuous at $\infty$ thanks to assumption (i), must achieve a finite maximum on the compact set obtained as the one-point compactification of $\operatorname{clos}\mathbb{C}_0 \setminus Z_\delta$. We denote this maximum with $1/\varepsilon$, then it follows that $\|\operatorname{Re}\mathbf{g}(s)\mathrm{v}\| \geq \varepsilon\|\mathrm{v}\|$ for all $s \in \mathbb{C}_0 \setminus Z_\delta$ and all $\mathrm{v} \in U$. Since $\operatorname{Re}\mathbf{g}(s) \geq 0$, this implies our claim, or equivalently, that

$$\operatorname{Re}\frac{\mathbf{g}(s)}{2\varepsilon} \geq \frac{1}{2}I \qquad \forall\, s \in \mathbb{C}_0 \setminus Z_\delta. \tag{4.2}$$

According to Lemma 4.1 we have

$$\frac{\mathbf{g}(s)}{2\varepsilon} = \big[I - \mathbf{Q}(s)\big]^{-1} \text{ where } \|\mathbf{Q}(s)\| \leq 1\ \forall s \in \mathbb{C}_0 \setminus Z_\delta$$

and

$$\operatorname{Re}\mathbf{Q}(s) \leq \Big[1 - \frac{1}{2M^2}\Big]I, \tag{4.3}$$

where $M = \sup_{s\in\mathbb{C}_0}\|\frac{\mathbf{g}(s)}{2\varepsilon}\| > 0$. Thus we have

$$\begin{aligned} I + \mathbf{g}(s)\mathbf{G}(s) &= I + 2\varepsilon\frac{\mathbf{g}(s)}{2\varepsilon}\mathbf{G}(s) \\ &= I + 2\varepsilon\big[I - \mathbf{Q}(s)\big]^{-1}\mathbf{G}(s) \\ &= \big[I - \mathbf{Q}(s)\big]^{-1}\big[I - \mathbf{Q}(s) + 2\varepsilon\mathbf{G}(s)\big] \\ &= \frac{\mathbf{g}(s)}{2\varepsilon}\big[I - \mathbf{Q}(s) + 2\varepsilon\mathbf{G}(s)\big], \end{aligned} \tag{4.4}$$

for all $s \in \mathbb{C}_0 \setminus Z_\delta$. Using (4.3), we have

$$\operatorname{Re}\big[I - \mathbf{Q}(s)\big] \geq \frac{1}{2M^2}.$$

It follows that

$$\|\big[I - \mathbf{Q}(s) + 2\varepsilon\mathbf{G}(s)\big]\mathrm{v}\| \geq \frac{1}{2M^2}\|\mathrm{v}\| \qquad \forall\, \mathrm{v} \in U.$$

Combing this with (4.4) and (4.2), we get that for $s \in \mathbb{C}_0 \setminus Z_\delta$

$$\|\big[I + \mathbf{g}(s)\mathbf{G}(s)\big]\mathrm{v}\| \geq \frac{1}{4M^2}\|\mathrm{v}\| \qquad \forall\, \mathrm{v} \in U. \tag{4.5}$$

Thus for any $\delta > 0$, $I + \mathbf{g}\mathbf{G}$ is bounded from below on $\mathbb{C}_0 \setminus Z_\delta$.

Now we look at the case $s \in \mathbb{C}_0 \cap Z_\delta$. It follows from assumption (ii) that for $\delta > 0$ sufficiently small,

$$\sup_{s\in\mathbb{C}_0\cap Z_\delta}\|\mathbf{g}(s)\mathbf{G}(s)\| = \beta < 1.$$

It follows that for such $\delta$, and for all $s \in \mathbb{C}_0 \cap Z_\delta$,

$$\|\big[I + \mathbf{g}(s)\mathbf{G}(s)\big]\mathrm{v}\| \geq (1-\beta)\|\mathrm{v}\| \qquad \forall\, \mathrm{v} \in U. \tag{4.6}$$

This estimate, together with (4.5) shows that $I + \mathbf{g}\mathbf{G}$ is bounded from below on $\mathbb{C}_0$.

Now we show that the four transfer functions from the inputs $f$, $r$ to the outputs $h$, $v$ in Figure 3 are all in $\mathcal{H}^\infty(\mathbb{C}_0)$. It is clear from the boundedness from below of $I + \mathbf{g}\mathbf{G}$ on $\mathbb{C}_0$ that $(I + \mathbf{g}\mathbf{G})^{-1}$ (the transfer function from $f$ to $h$) is in $\mathcal{H}^\infty(\mathbb{C}_0)$. Now we consider $\mathbf{H} = -\mathbf{G}(I + \mathbf{g}\mathbf{G})^{-1}$, the transfer function from $f$ to $v$. As we have already seen, for any $\delta > 0$, $\mathbf{g}^{-1}$ is bounded on $\mathbb{C}_0 \setminus Z_\delta$. Since $\mathbf{H} = -\mathbf{g}^{-1}\mathbf{g}\mathbf{G}(I + \mathbf{g}\mathbf{G})^{-1}$, and we have seen earlier that

$$\mathbf{g}\mathbf{G}(I + \mathbf{g}\mathbf{G})^{-1} = I - (I + \mathbf{g}\mathbf{G})^{-1} \in \mathcal{H}^\infty(\mathbb{C}_0),$$

it follows that $\mathbf{H}$ is bounded on $\mathbb{C}_0 \setminus Z_\delta$. It follows from assumption (ii) that for $\delta > 0$ sufficiently small, $\mathbf{G}$ is bounded on $\mathbb{C}_0 \cap Z_\delta$, so that also $\mathbf{H}$ is bounded there. Thus, $\mathbf{H}$ is bounded on $\mathbb{C}_0$. Due to this fact, we see from $(I + \mathbf{G}\mathbf{g})^{-1} = I - \mathbf{G}(I + \mathbf{g}\mathbf{G})^{-1}\mathbf{g}$ that also $(I + \mathbf{G}\mathbf{g})^{-1}$ (the transfer function from $r$ to $v$) is in $\mathcal{H}^\infty(\mathbb{C}_0)$. Due to assumption (ii), the same is true for $\mathbf{g}(I + \mathbf{G}\mathbf{g})^{-1}$, the transfer function from $r$ to $h$. ■

*Remark 4.3:* The assumption (4.1) in Theorem 4.2 is used solely to derive (4.6). It is possible to replace the assumption (4.1) with a weaker one: there exists $\delta > 0$ and $\eta > 0$ such that $\sup_{s\in\mathbb{C}_0\cap Z_\delta}\|\big[I + \mathbf{g}(s)\mathbf{G}(s)\big]\mathrm{v}\| \geq \eta\|\mathrm{v}\|$ for all $\mathrm{v} \in U$.

*Remark 4.4:* The assumption (iii) cannot be eliminated from Theorem 4.2. A very simple scalar counterexample is

$$\mathbf{g}(s) = \frac{s}{s+1} + i, \qquad \mathbf{G}(s) = \frac{1}{s - i}.$$

We have $\mathbf{g}(0) = \mathbf{G}(0) = i$ so that $1 + \mathbf{g}(0)\mathbf{G}(0) = 0$, hence the closed-loop system is not stable.

The assumption (iii) can be eliminated from Theorem 4.2 if we modify assumption (ii) as follows: $\mathbf{G}$ is bounded around the imaginary zeros of $\operatorname{Re}\mathbf{g}$ and (4.1) holds around these points. The proof is similar, but now $Z$ is the set of zeros of $\operatorname{Re}\mathbf{g}$ on the imaginary axis. This version of the theorem can be further modified along the lines of Remark 4.3.

## V. Well-posedness and stability properties of the SCOLE-TMD system

The mathematical model of the SCOLE system with a TMD moving along one direction is the system $\Sigma_c$ described by the following set of equations:

$$\begin{cases} \rho(x)w_{tt}(x,t) + (EI(x)w_{xx}(x,t))_{xx} = 0, & (5.1) \\ (x,t) \in (0,l) \times [0,\infty), & \\ w(0,t) = 0,\ w_x(0,t) = 0, & (5.2) \\ mw_{tt}(l,t) - (EIw_{xx})_x(l,t) = F_e(t) + u(t), & (5.3) \\ Jw_{xtt}(l,t) + EI(l)w_{xx}(l,t) = T_e(t), & (5.4) \\ m_1 p_{tt}(t) = -u(t), & (5.5) \\ u(t) = k_1[p(t) - w(l,t)] + d_1[p_t(t) - w_t(l,t)], & (5.6) \end{cases}$$

where the subscripts $t$ and $x$ denote derivatives with respect to the time $t$ and the position $x$. The equations (5.1)-(5.4) are the non-uniform SCOLE model $\Sigma_d$ with force input $F_e + u$ ($F_e$ is the external force) and torque input $T_e$ acting on the rigid body, while the equations (5.5)-(5.6) are the TMD system $\Sigma_f$. $\Sigma_d$ and $\Sigma_f$ are interconnected through the velocity of the rigid body

$$y(t) = w_t(l,t) \tag{5.7}$$

and the force $u$ generated by the spring and damper in the TMD system, as shown in Figure 1.

Let us examine the SCOLE subsystem $\Sigma_d$. In the equations (5.1)-(5.4) $l$, $\rho$ and $EI$ denote the beam's height, mass density function and flexural rigidity function, while $w$ denotes its transverse displacement. $\rho, EI \in C^4[0,l]$ are assumed to be strictly positive functions. The parameters $m>0$ and $J>0$ are the mass and the moment of inertia of the rigid body. The state of $\Sigma_d$ at the time $t$ is

$$z(t) = \begin{bmatrix} z_1(t) \\ z_2(t) \\ z_3(t) \\ z_4(t) \end{bmatrix} = \begin{bmatrix} w(\cdot,t) \\ w_t(\cdot,t) \\ w_t(l,t) \\ w_{xt}(l,t) \end{bmatrix}. \tag{5.8}$$

Its natural energy state space is

$$X = \mathcal{H}_l^2(0,l) \times L^2[0,l] \times \mathbb{C}^2, \tag{5.9}$$

where

$$\mathcal{H}_l^2(0,l) = \{h \in \mathcal{H}^2(0,l) \mid h(0)=0,\ h_x(0)=0\}.$$

Here $\mathcal{H}^n$ ($n \in \mathbb{N}$) denote the usual Sobolev spaces. The natural norm on $X$ is defined via its square as

$$\|z\|^2 = \int_0^l EI(x)|z_{1xx}(x)|^2 \mathrm{d}x + \int_0^l \rho(x)|z_2(x)|^2 \mathrm{d}x + m|z_3|^2 + J|z_4|^2, \tag{5.10}$$

for all $z \in X$, which represents twice the physical energy.

In the TMD system $\Sigma_f$, $m_1>0$, $k_1>0$ and $d_1>0$ are the mass, spring constant and damping coefficient. $p$ and $p_t$ are the position and transverse velocity of the TMD. The state of $\Sigma_f$ is defined as

$$q(t) = \begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix} = \begin{bmatrix} p(t) - w(l,t) \\ p_t(t) \end{bmatrix}, \tag{5.11}$$

with state space

$$X^f = \mathbb{C}^2, \tag{5.12}$$

on which the norm is defined via

$$\|q(t)\|^2 = k_1|q_1(t)|^2 + m_1|q_2(t)|^2, \tag{5.13}$$

which represents twice of the physical energy. Note that $q_1$ is the displacement of the TMD with respect to the rigid body.

The state of the SCOLE-TMD coupled system $\Sigma_c$ from (5.1)-(5.6) is

$$z^c(t) = \begin{bmatrix} z(t) \\ q(t) \end{bmatrix}, \tag{5.14}$$

where $z$ is as in (5.8) (the state of the SCOLE subsystem) and $q$ is as in (5.11) (the state of the TMD subsystem). The energy state space of $\Sigma_c$ is

$$X^c = X \times X^f = \mathcal{H}_l^2(0,l) \times L^2[0,l] \times \mathbb{C}^4.$$

The natural norm on $X^c$ is defined via its square,

$$\begin{aligned} \|z^c(t)\|^2 &= \|z\|^2 + \|q\|^2 \\ &= \int_0^l EI(x)|z_{1xx}(x,t)|^2 \mathrm{d}x \\ &\quad + \int_0^l \rho(x)|z_2(x,t)|^2 \mathrm{d}x + m|z_3(t)|^2 \\ &\quad + J|z_4(t)|^2 + k_1|q_1(t)|^2 + m_1|q_2(t)|^2, \end{aligned}$$

which again represents twice the physical energy.

Now we show the well-posedness, regularity and strong stabilization of the SCOLE-TMD system $\Sigma_c$ described by (5.1)–(5.6), based on Theorem 3.1. Following Figure 2 with $r=0$ and $f=F_e$, we decompose $\Sigma_c$ into a nonuniform SCOLE subsystem $\Sigma_d$ (described by (5.1)–(5.4)) with force input $u+F_e$ and velocity (of the rigid body) output $y$, and a TMD subsystem $\Sigma_f$ with input $-y$, and outputs $u$ and $u_f$, where $u$ and $y$ are defined in (5.6) and (5.7), and

$$u_f(t) = \sqrt{d_1}(p_t(t) - w_t(l,t)). \tag{5.15}$$

We analyze the SCOLE subsystem $\Sigma_d$ first. With the state $z$ as in (5.8) and the natural state space $X$ as in (5.9), the state space realization of $\Sigma_d$ (described by (5.1)–(5.4)) is

$$\begin{cases} \dot{z}(t) = Az(t) + B\big(u(t) + F_e(t)\big), \\ y(t) = Cz(t), \end{cases} \tag{5.16}$$

where, for every $\xi \in \mathcal{D}(A)$,

$$A\xi = \begin{bmatrix} \xi_2 \\ -\rho^{-1}(x)\big(EI(x)\xi_{1xx}(x)\big)_{xx} \\ m^{-1}\big(EI\xi_{1xx}\big)_x(l) \\ -J^{-1}EI(l)\xi_{1xx}(l) \end{bmatrix},$$

$$\mathcal{D}(A) = \left\{ \xi \in \left[\mathcal{H}^4 \cap \mathcal{H}_l^2\right] \times \mathcal{H}_l^2 \times \mathbb{C}^2 \ \middle|\ \begin{matrix} \xi_3 = \xi_2(l) \\ \xi_4 = \xi_{2x}(l) \end{matrix} \right\},$$

$$B = \begin{bmatrix} 0 & 0 & \frac{1}{m} & 0 \end{bmatrix}^T, \qquad C = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}.$$

The natural norm on $X$ is as in (5.10).

We cite several results about the nonuniform SCOLE model $\Sigma_d$ described by (5.16) from literature, which will be used in proving our main results later.

*Proposition 5.1:* [15, Propositions 1.1 and 1.2] $A$ is skew-adjoint on $X$. Its spectrum $\sigma(A)$ consists of imaginary, non-zero, simple and isolated eigenvalues.

*Proposition 5.2:* [15, Corollary 3.1] $(A, C)$ is approximately observable in any time $T_0 > 0$ (on $X$).

*Proposition 5.3:* [53, Propositions 16 and 17] or [56] $\Sigma_d$ described by (5.16) is impedance passive on the state space $X$ and its generator $A$ has compact resolvents.

*Proposition 5.4:* $\Sigma_d$ described by (5.16) is well-posed, regular and strictly proper with the state space $X$. These properties still hold with extra input $T_e$ (external torque acting on the rigid body) and extra output $w_{xt}(l,\cdot)$ (angular velocity of the rigid body) as in (5.4).

*Proof.* This proposition follows from the facts that $A$ is skew-adjoint on $X$ by Proposition 5.1, and that the control and observation operators $B$ and $C$ are bounded on $X$ and the feedthrough operator is zero. ∎

With the state $q$ as in (5.11), the state space $X^f$ as in (5.12) and its norm as in (5.13), the TMD subsystem $\Sigma_f$ extracted from the SCOLE-TMD coupled model $\Sigma_c$ (5.1)–(5.6) is

$$\begin{cases} \dot{q}(t) &= a\,q(t) - by(t), \\ u(t) &= cq(t) - dy(t), \\ u_f(t) &= c_f\,q(t) - d_f y(t), \end{cases} \tag{5.17}$$

where

$$a = \begin{bmatrix} 0 & 1 \\ -\frac{k_1}{m_1} & -\frac{d_1}{m_1} \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ -\frac{d_1}{m_1} \end{bmatrix},$$

$$c=\begin{bmatrix}k_1 & d_1\end{bmatrix}, \quad c_f=\begin{bmatrix}0 & \sqrt{d_1}\end{bmatrix}, \quad d=d_1, \quad d_f=\sqrt{d_1}.$$

Here $-y$ is the input of $\Sigma_f$ and $u$ is its output. Note that $u_f$ (also expressed in (5.15)) is $\sqrt{d_1}$ times the velocity of the TMD with respect to the rigid body; thus $\|u_f\|^2$ is the power dissipation of the TMD. The above equations correspond to $r=0$ and $f=F_e$ in Figure 2.

*Theorem 5.5:* The SCOLE-TMD coupled system $\Sigma_c$ described by (5.1)–(5.6), with input $F_e$, state $z^c$ (as defined in (5.14)) and output $\left[\begin{smallmatrix}u\\ y\end{smallmatrix}\right]$ (as defined in (5.6) and (5.7)) is well-posed and regular with the state space

$$X^c=\mathcal{H}_l^2(0,l)\times L^2[0,l]\times\mathbb{C}^4.$$

These properties still hold with the extra input $T_e$ (the torque acting on the rigid body) and the extra output $w_{xt}(l,\cdot)$ (the angular velocity of the rigid body).

Moreover, $\Sigma_c$ is strongly stable (on $X^c$).

*Proof.* First we prove the well-posedness and regularity. We have seen earlier that $\Sigma_c$ can be decomposed into $\Sigma_d$ described by (5.16) and $\Sigma_f$ described by (5.17), interconnected as in Figure 2 with $r=0$ and $f=F_e$. We shall use the first part of Theorem 3.1 - actually we only need a very simple version of this result, corresponding to bounded $B$ and $C$.

According to Proposition 5.4, $\Sigma_d$ is well-posed and strictly proper on the state space $X$. Thus according to Theorem 3.1, $\Sigma_c$ (with input $F_e$, state $\left[\begin{smallmatrix}z\\ q\end{smallmatrix}\right]$ and output $\left[\begin{smallmatrix}u\\ y\end{smallmatrix}\right]$) is well-posed and regular with the state space

$$X^c=X\times X^f=\mathcal{H}_l^2(0,l)\times L^2[0,l]\times\mathbb{C}^4. \tag{5.18}$$

According to Proposition 5.4, the SCOLE subsystem $\Sigma_d$ is still well-posed and regular with the extra torque input $T_e$ and the extra output $w_{xt}(l,\cdot)$. Thus it is easy to check that $\Sigma_c$ is still well-posed and regular on $X^c$ if we take into account this extra input and the extra output.

The next step is to show the strong stability of $\Sigma_c$ on $X^c$. From Proposition 5.3, we know that $\Sigma_d$ is impedance passive on $X$. Therefore assumption (i) of Theorem 3.1 is met. Now we show assumption (ii). From (5.13), we know that

$$\|q(t)\|^2=k_1|q_1(t)|^2+m_1|q_2(t)|^2.$$

Remembering that the signals may be complex, we have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\|q(t)\|^2 &= 2\mathrm{Re}\left[k_1q_1(t)\overline{\dot q_1(t)}+m_1q_2(t)\overline{\dot q_2(t)}\right]\\
&= 2\mathrm{Re}\Big[k_1q_1(t)\overline{\big(q_2(t)-y(t)\big)}+m_1q_2(t)\\
&\quad\times\overline{\Big(-\frac{k_1}{m_1}q_1(t)-\frac{d_1}{m_1}q_2(t)+\frac{d_1}{m_1}y(t)\Big)}\Big]\\
&= -2\mathrm{Re}\,k_1q_1(t)\overline{y(t)}-2\mathrm{Re}\,d_1q_2(t)\\
&\quad\times\overline{\big(q_2(t)-y(t)\big)}\\
&= -2\mathrm{Re}\,k_1q_1(t)\overline{y(t)}-2d_1\big|q_2(t)-y(t)\big|^2\\
&\quad-2\mathrm{Re}\,d_1y(t)\overline{\big(q_2(t)-y(t)\big)}\\
&= -2\mathrm{Re}\Big(k_1q_1(t)+d_1\big(q_2(t)-y(t)\big)\Big)\overline{y(t)}\\
&\quad-2d_1\big|q_2(t)-y(t)\big|^2\\
&= -2\mathrm{Re}\,u(t)\overline{y(t)}-2|u_f(t)|^2,
\end{aligned}$$

which is assumption (ii) of Theorem 3.1 with $\gamma=2$.

According to Proposition 5.2, $(A,C)$ is approximately observable in any time $T_0>0$ with the state space $X$, which is the assumption (iii) of Theorem 3.1. By a simple computation, we have

$$\begin{bmatrix}c\\ ca\end{bmatrix}=\begin{bmatrix}k_1 & d_1\\ -\frac{k_1d_1}{m_1} & k_1-\frac{d_1^2}{m_1}\end{bmatrix},$$

whose determinant is

$$\det\begin{bmatrix}c\\ ca\end{bmatrix}=k_1^2>0. \tag{5.19}$$

Thus $(a,c)$ is observable, as in assumption (iv) of the theorem.

Because $d=d_1>0$, assumption (v) is satisfied. It is easy to check that

$$a^\times=\begin{bmatrix}-\frac{k_1}{d_1} & 0\\ 0 & 0\end{bmatrix},$$

whose eigenvalues are $0$ and $-\frac{k_1}{d_1}<0$. From Proposition 5.1 we know that $\sigma(A)$ consists of purely imaginary and nonzero eigenvalues. Therefore $\sigma(a^\times)\subset\rho_\infty(A)$ follows, which is assumption (vi).

Now we are going to verify the assumption (vii). From (5.17) we know that $u_f=\sqrt{d_1}\dot q_1=\sqrt{d_1}(q_2-y)$, $u=k_1q_1+d_1\dot q_1$ and $m_1\dot q_2=-u$. If $u_f=0$ for all $t\geq 0$, then $\dot q_1=0$ and $q_2=y$ for all $t\geq 0$. It follows that $q_1$ is constant. Hence $u=k_1q_1+d_1\dot q_1$ is constant. Since $m_1\dot q_2=-u$, and $q_2=y$, and $y=-v$ in Figure 2, which has finite power by assumption, it follows that $u=0$ for all $t\geq 0$. Therefore assumptions (vii) of Theorem 3.1 is met. So far we have the conclusion that $\Sigma_c$ is weakly stable.

Now we show that the semigroup generator $A^c$ of $\Sigma_c$ has compact resolvents, which implies that $\sigma(A^c)\cap i\mathbb{R}$ is at most countable. By a simple computation or by Proposition IV.1 of [51], we have

$$A^c=\begin{bmatrix}A-BdC_\Lambda & Bc\\ -bC_\Lambda & a\end{bmatrix}=\begin{bmatrix}A & 0\\ 0 & 0\end{bmatrix}+\begin{bmatrix}-BdC_\Lambda & Bc\\ -bC_\Lambda & a\end{bmatrix}.$$

It is easy to check that $\left[\begin{smallmatrix}A & 0\\ 0 & 0\end{smallmatrix}\right]$ is skew-adjoint with compact resolvents on $X^c$, using Propositions 5.1 and 5.3, and that $\left[\begin{smallmatrix}-BdC_\Lambda & Bc\\ -bC_\Lambda & a\end{smallmatrix}\right]$ is bounded on $X^c$. It easily follows that $A^c$ has compact resolvents. Thus $\Sigma_c$ is strongly stable on $X^c$. ■

Now we turn our attention to input-output stability.

*Theorem 5.6:* The SCOLE-TMD system (5.1)–(5.7) is input-output stable from the input $F_e$ to the outputs $u,y$.

*Proof.* It is easy to compute that the transfer function of the TMD subsystem $\Sigma_f$ is

$$\mathbf{g}(s)=\frac{m_1d_1s^2+m_1k_1s}{m_1s^2+d_1s+k_1}. \tag{5.20}$$

In the proof of Theorem 5.5 we have shown that $\Sigma_f$ satisfies assumption (ii) of Theorem 3.1, which implies that $\mathbf{g}$ is positive. This function $\mathbf{g}$ has two poles,

$$\frac{-d_1\pm\sqrt{d_1^2-4m_1k_1}}{2m_1},$$

and two zeros, namely $0$ and $-\frac{k_1}{d_1}$. Thus $\mathbf{g}$ is rational and input-output stable with $\mathbf{g}(\infty)=d_1>0$ being invertible, which is the assumption (i) of Theorem 4.2.

From Proposition 5.3 we know that the SCOLE subsystem $\Sigma_d$ is impedance passive on $X$. Thus its transfer function $\mathbf{G}$ is positive. According to Proposition

5.1, we know that the corresponding semigroup generator $A$ of $\mathbf{G}$ is skew-adjoint on the state space $X$. Its spectrum $\sigma(A)$ consists of simple eigenvalues that are isolated, purely imaginary and non-zero. Thus $\mathbf{G}$ is bounded around $0$, which is the only imaginary zero of $\mathbf{g}$. Moreover,

$$\limsup_{s \to 0} \|\mathbf{g}(s)\mathbf{G}(s)\| = 0 .$$

Therefore assumption (ii) of Theorem 4.2 holds.

We see from (5.20) that

$$\mathrm{Re}\,\mathbf{g}(i\omega) = \frac{m_1^2 d_1 \omega^4}{(k_1 - m_1\omega^2)^2 + d_1^2\omega^2} .$$

For any $\omega \in \mathbb{R}$, $\mathrm{Re}\,\mathbf{g}(i\omega)$ is not invertible only in the case that $\omega = 0$, which is the only imaginary zero of $\mathbf{g}$. Thus the assumption (iii) of Theorem 4.2 is satisfied. Thus, all the three assumptions of Theorem 4.2 are satisfied.

Now Theorem 4.2 implies the conclusion of Theorem 5.6. In fact, we could artificially add another input $r$ to the SCOLE-TMD system, as in Figure 2, and the theorem would still hold, based on the same argument. ■

## VI. Conclusions

In this paper we have investigated the well-posedness and stability properties of coupled impedance passive systems where one subsystem is finite-dimensional. Our results about well-posendess and strong stability are summarized in Theorem 3.1 while the result about input-output stability is in Theorem 4.2. As an application, we have investigated the vibration reduction of the SCOLE model by using a TMD. Using Theorem 3.1 and Theorem 4.2, we have shown that the SCOLE-TMD coupled system is well-posed, regular and strongly stable on the energy state space $X^c = \mathcal{H}_l^2(0,l) \times L^2[0,l] \times \mathbb{C}^4$ and that it is input-output stable as well.

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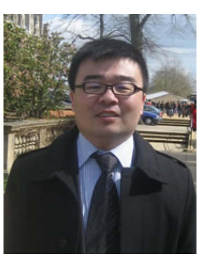

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