WARWICK
THE UNIVERSITY OF WARWICK

Original citation:
Carmesin, J., Federici, B. and Georgakopoulos, Agelos. (2017) A Liouville hyperbolic souvlaki. Electronic Journal of Probability, 22. 23.

Permanent WRAP URL:
http://wrap.warwick.ac.uk/86749

Copyright and reuse:
The Warwick Research Archive Portal (WRAP) makes this work of researchers of the University of Warwick available open access under the following conditions.

This article is made available under the Creative Commons Attribution 4.0 International license (CC BY 4.0) and may be reused according to the conditions of the license. For more details see: http://creativecommons.org/licenses/by/4.0/

A note on versions:
The version presented in WRAP is the published version, or, version of record, and may be cited as it appears here.

For more information, please contact the WRAP Team at: wrap@warwick.ac.uk

warwick.ac.uk/lib-publications

A Liouville hyperbolic souvlaki[*]

Johannes Carmesin[†]   Bruno Federici[‡]   Agelos Georgakopoulos[‡]

Abstract

We construct a transient bounded-degree graph no transient subgraph of which embeds in any surface of finite genus.

Moreover, we construct a transient, Liouville, bounded-degree, Gromov–hyperbolic graph with trivial hyperbolic boundary that has no transient subtree. This answers a question of Benjamini. This graph also yields a (further) counterexample to a conjecture of Benjamini and Schramm. In an appendix by Gábor Pete and Gourab Ray, our construction is extended to yield a unimodular graph with the above properties.

Keywords: Liouville property; hyperbolic graph; infinite graph; amenability; transience; flow; harmonic function.
AMS MSC 2010: 57M15; 05C63; 05C81; 31C20.
Submitted to EJP on April 14, 2016, final version accepted on March 5, 2017.

1 Introduction

A well-known result of Benjamini & Schramm [6] states that every non-amenable graph contains a non-ambenable tree. This naturally motivates seeking for other properties that imply a subtree with the same property. However, there is a simple example of a transient graph that does not contain a transient tree [6] (such a graph had previously also been obtained by McGuinness [23]). We improve this by constructing —in Section 7— a transient bounded-degree graph no transient subgraph of which embeds in any surface of finite genus (even worse, every transient subgraph has the complete graph K^r as a minor for every r). This answers a question of I. Benjamini (private communication).

Given these examples, it is natural to ask for conditions on a transient graph that would imply a transient subtree. In this spirit, Benjamini [4, Open Problem 1.62] asks whether hyperbolicity is such a condition. We answer this in the negative by constructing —in Section 6— a transient hyperbolic (bounded-degree) graph that has no transient subtree. While preparing this manuscript, T. Hutchcroft and A. Nachmias (private communication) provided a simpler example with these properties, which we sketch in

[†]University of Cambridge. E-mail: j.carmesin@dpmms.cam.ac.uk
[‡]University of Warwick

Section 6.1. In an appendix by Gábor Pete and Gourab Ray, our construction is extended to yield a unimodular graph with the above properties.

A related result of Thomassen states that if a graph satisfies a certain isoperimetric inequality, then it must have a transient subtree [27].

The starting point for this paper was the following problem of Benjamini and Schramm

Conjecture 1.1 ([7, 1.11. Conjecture]). *Let M be a connected, transient, Gromov-hyperbolic, Riemannian manifold with bounded local geometry, with the property that the union of all bi-infinite geodesics meets every ball of sufficiently large radius. Then M admits non constant bounded harmonic functions. Similarly, a Gromov-hyperbolic bounded valence, transient graph, with C-dense bi-infinite geodesics has non constant bounded harmonic functions.*

The term C-dense here means that every vertex of the graph is at distance at most some constant C from a bi-infinite geodesic. We remark that in order to disprove —the second assertion of— this, it suffices to find a transient, Gromov-hyperbolic bounded valence (aka. degree) graph with the Liouville propertyinfinite geodesics (rather than bi-infinite ones) having the remaining properties ; for given such a graph G, one can attach a disjoint 1-way infinite path to each vertex of G, to obtain a graph having 1-dense bi-infinite geodesics while preserving all other properties. As pointed out by I. Benjamini (private communication), it is not hard to prove that any 'lattice' in a horoball in 4-dimensional hyperbolic space has these properties. We prove that our example also has these properties, thus providing a further counterexample to Conjecture 1.1. A perhaps surpising aspect of our example is that all of its geodesics eventually coincide despite its transience; see Section 2.

In Section 2.1 we provide a sketch of this construction, from which the expert reader might be able to deduce the details.

Although we do not formally provide a counterexample to the first assertion of Conjecture 1.1, we believe it is easy to obtain one by blowing up the edges of our graph into tubes.

2 The hyperbolic Souvlaki

In this section we construct a bounded-degree graph Ψ with the following properties

1. it is hyperbolic, and its hyperbolic boundary consists of a single point;
2. for every vertex x of Ψ, there is a unique infinite geodesic starting at x, and any two 1-way infinite geodesics of Ψ eventually coincide;
3. it is transient;
4. every subtree of Ψ is recurrent;
5. it has the Liouville property.

This graph thus yields a counterexample to [4, Open Problem 1.62] and Conjecture 1.1 as mentioned in the Introduction.

2.1 Sketch of construction

Let us sketch the construction of this graph Ψ, and outline the reasons why it has the above properties. It consists of an 1-way infinite path $S = s_0 s_1 \ldots$, on which we glue a sequence M_i of finite increasing subgraphs of an infinite '3-dimensional' hyperbolic graph H_3. For example, H_3 could be the 1-skeleton of a regular tiling of 3-dimensional hyperbolic space, and the M_i could be taken to be copies of balls of increasing radii around some origin in H_3, although it was more convenient for our proofs to construct different H_3 and M_i.

In order to glue M_i on S, we identify the subpath $s_{2^i} \dots s_{2^{i+2}-1}$ with a geodesic of the same length in M_i. Thus M_i intersects M_{i-1} and M_{i+1} but no other M_j, and this intersection is a subpath of S; see Figure 4. (Our graph can be quasi-isometrically embedded in $\mathbb{H}^5$, but probably not in $\mathbb{H}^4$.) We call this graph a *hyperbolic souvlaki*, with *skewer* S and *meatballs* M_i. We detail its construction in Section 2.

To prove that this graph is transient, we construct a flow of finite energy from s_0 to infinity (Section 4). This flow carries a current of strength 2^{-i} inside M_i out of each vertex in $s_{2^i} \dots s_{2^{i+1}-1}$, and distributes it evenly to the vertices in $s_{2^{i+1}} \dots s_{2^{i+2}}$ for every i. These currents can be thought of as flowing on spheres of varying radii inside M_i, avoiding each other, and it was important to have at least three dimensions for this to be possible while keeping the energy dissipated under control.

To prove that our graph has the Liouville property, we observe that random walk has to visit S infinitely often, and has enough time to 'mix' inside the M_i between subsequent visits to S (Section 5).

2.2 Formal construction

We now explain our precise construction, which is similar but not identical to the above sketch. We start by constructing a hyperbolic graph H_3 which we will use as a model for the 'meatballs' M_i; more precisely, the M_i will be chosen to be increasing subgraphs of H_3.

Let T_3 denote the infinite tree with one vertex r, which we call the *root*, of degree 3 and all other vertices of degree 4. For $n = 1, 2, \dots$, we put a cycle —of length 3^n— on the vertices of T_3 that are at distance n from r in such a way that the resulting graph is planar[1]; see Figure 1. We denote this graph by H_2. It is not hard to see that H_2 is hyperbolic, for instance by checking that any two infinite geodesics starting at r either stay at bounded distance or diverge exponentially, and using [26, Section 2.20].
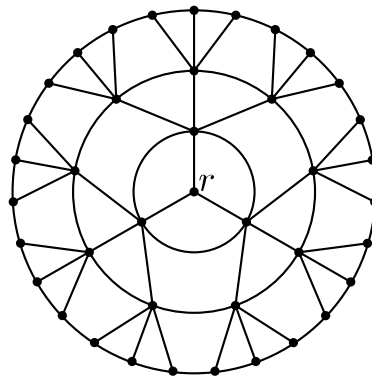

Figure 1: The ball of radius 3 around the root of H_2.

Recall that a *ray* is a 1-way infinite path. We will now turn H_2 into a '3-dimensional' hyperbolic graph H_3, in such a way that each ray inside T_3 (or H_2) starting at r gives rise to a subgraph of H_3 isomorphic to the graph W of Figure 2, which is a subgraph of the Cayley graph of the Baumslag-Solitar group $BS(1, 2)$. Formally, we construct W from

---

[1]Formally, we pick a cyclic ordering on the neighbours of r and a linear ordering on the outer neighbours of every other vertex of T_3. Given a cyclic ordering on the vertices at level n of T_3, we get a cyclic ordering at level $n + 1$ by replacing each vertex by the linear ordering on its outer neighbours. Now we add edges between any two vertices that are adjacent in any of these cyclic orderings.

infinitely many vertex disjoint double rays[2] $D_0, D_1, D_2, ..$, where $D_i = ...r_i^{-2}r_i^{-1}r_i^0 r_i^1 r_i^2 ....$
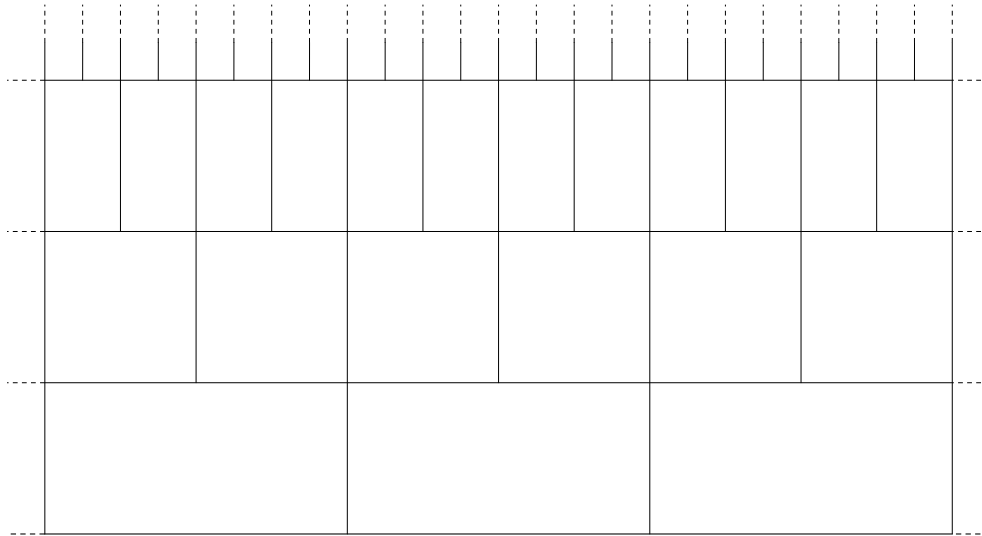Then we add all edges of the form $r_i^k r_{i+1}^{2k}$.

Figure 2: The graph W: a subgraph of the standard Cayley graph of the Baumslag-Solitar group $BS(1,2)$. It is a plane hyperbolic graph.

To define H_3, we let the *height* $h(t)$ of a vertex $t \in V(H_2)$ be its distance $d(r,t)$ from the root r. For a vertex w of W, we say that its height $h(w)$ is n if w lies in D_n, the nth horizontal double ray in Figure 2.

We define the vertex set of H_3 to consist of all ordered pairs (t,w) where t is a vertex of H_2 and w is a vertex of W and $h(w) = h(t)$. The edge set of H_3 consists of all pairs of pairs $(t,w)(t',w')$ such that either

- $tt' \in E(H_2)$ and $ww' \in E(W)$, or
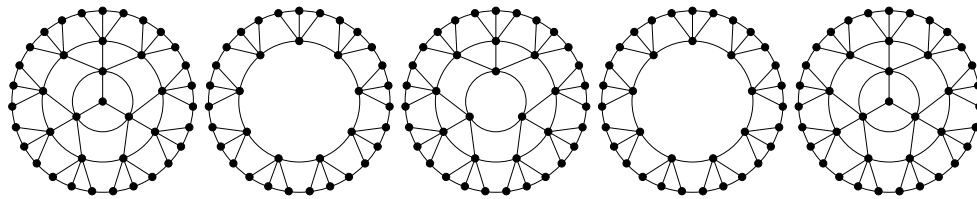- $tt' \in E(H_2)$ and $w = w'$, or
- $t = t'$ and $ww' \in E(W)$.

Figure 3: A subgraph of H_3. Edges of the form $(t,w)(t',w')$ with $t = t'$ and $ww' \in E(W)$ are missing from the figure: these are all the edges joining corresponding vertices in consecutive components of the figure.

Thus every vertex t of H_2 gives rise to a double ray in H_3, which consists of those vertices of H_3 that have t as their first coordinate. Similarly, every vertex w of W gives rise to a cycle in H_3, the length of which depends on $h(w)$. We call the vertices on any such cycle *cocircular*. Every ray of T_2 starting at r gives rise to a copy of W, and if two such paths share their first k vertices, then the corresponding copies of W share their

---

[2]A *double ray* is a 2-way infinite path.

first k levels of h. It is not hard to prove that H_3 is a hyperbolic graph, but we will omit the proof as we will not use this fact.

We next construct Ψ by glueing a sequence of finite subgraphs M_n of H_3 along a ray S. We could choose the subgraph M_n to be a ball in H_3, but we found it more convenient to work with somewhat different subgraphs of H_3: we let M_n be the finite subgraph of H_3 spanned by those vertices (t, w) such that w lies in a certain *box* $B_n \subseteq W$ of W defined as follows. Consider a subpath P_n of the bottom double-ray of W of length $3 \cdot 2^n$, and let B_n consist of those vertices w that lie in or above P_n (as drawn in Figure 2) and satisfy $h(w) \leq 2^{n+1}$.

This completes the definition of M_n. We let S_n denote the vertices of M_n corresponding to P_n, and we index the vertices of S_n as $\{r(x), 0 \leq x \leq 3 \cdot 2^n\}$. Note that S_n is a geodesic of M_n. We subdivide S_n into three parts: $L_n := \{r(x), 0 \leq x < 2^n\}, m_n := r(2^n)$ and $R_n := \{r(x), 2^n < x \leq 3 \cdot 2^n\}$. We define the *ceiling* F_n of M_n to be its vertices of maximum height, i.e. the vertices $(t, w) \in V(M_n)$ with $h(w) = 2^{n+1}$.

Finally, it remains to describe how to glue the M_n together to form Ψ. We start with a ray S, the first vertex of which we denote by o and call the *root* of Ψ. We glue M_1 on S by identifying S_1 with the initial subpath of S of length $|S_1|$. Then, for $n = 2, 3, \ldots$, we glue M_n on S in such a way that L_n is identified with R_{n-1} (where we used the fact that $|L_n| = |R_{n-1}| = 2^n$ by construction), m_n is identified with the following vertex of S, and R_n is identified with the subpath of S following that vertex and having length $|R_n| = 2^{n+1}$. Of course, we perform this identification in such a way that the linear orderings of L_n and R_n are given by the induced linear ordering of S. We let Ψ denote the resulting graph. We think of M_n as a subgraph of Ψ.

2.3 Properties of Ψ

By construction, for $j > i$ we have $M_i \cap M_j = \emptyset$ unless $j = i + 1$, in which case $M_i \cap M_j = R_i = L_j \subset S$. The following fact is easy to see.

For every n, R_n separates L_n (and o) from infinity. (2.1)

The following assertion will be important for the proof of the Liouville property.

There is a uniform lower bound $p > 0$ for the probability $\mathbb{P}_v [\tau_{F_n} < \tau_{S_n}]$ that random walk in Ψ from any vertex of L_n will visit the ceiling F_n before returning to S_n. (2.2)

Indeed, we can let p be the probability for random walk on H_2 starting at the root o to never visit o again; this is positive because H_2 is transient. Then (2.2) holds because in a random walk from S_n on M_n, any steps inside the copies of H_2 behave like random walk on H_2 until hitting F_n, and the steps 'parallel' to S_n do not have any influence.

3 Hyperbolicity

In this section we prove that Ψ is hyperbolic in the sense of Gromov [16].

Lemma 3.1. *The graph Ψ is hyperbolic, and has a one-point hyperbolic boundary.*

Proof. We claim that for every vertex $x \in V(\Psi)$, there is a unique 1-way infinite geodesic starting at x. Indeed, this geodesic $x_0 x_1 \ldots$, takes a step from x_i towards the root of T_3 inside the copy of H_2 corresponding x_i whenever such an edge exists in Ψ, and it takes a horizontal step in the direction of infinity whenever such an edge does not exist. To see that γ is the unique infinite geodesic starting at x, suppose there is a second such geodesic δ. Clearly, δ has infinitely many vertices on the skewer S as all components of $\Psi \setminus S$ are finite. In fact, it is not hard to see that δ eventually coincides with S as the latter contains the unique geodesic between any two of its vertices. Thus γ and δ meet,

and we can let y be their first common vertex. Now consider their subpaths $x\gamma y$ and $x\delta y$ from x to y. Note that Ψ has two types of edges: those that lie in a copy of H_2, and horizontal ones. It is easy to see that any x-y path must have at least as many edges of each type as $x\gamma y$. Moreover, by considering the first edge e at which $x\delta y$ deviates from $x\gamma y$, it is not hard to check that $x\delta y$ has more edges of the same type as e as $x\gamma y$, which leads to a contradiction.

The hyperbolicity of Ψ now follows from a well-known fact saying that a space is hyperbolic if and only if any two geodesics with a common starting point are either at bounded distance or diverge exponentially in a certain sense; see [26, Section 2.20]. We skip the details here as in our case the condition is trivially satisfied due to the above claim—namely, any two geodesics from a given point are at bounded distance since they coincide.

As all infinite geodesics eventually coincide with S, we also immediately have that the hyperbolic boundary of G consists of just one point. $\qquad\square$

4 Transience

In this section we prove that Ψ is transient. We do so by displaying a flow from o to infinity having finite Dirichlet energy; transience then follows from Lyons' criterion:

Theorem 4.1 (T. Lyons' criterion (see [20] or [21])). *A graph G is transient, if and only if G admits a flow of finite energy from a vertex to infinity.*

We refer the reader to [21] or [15] for the basics of electrical networks needed to understand this theorem.
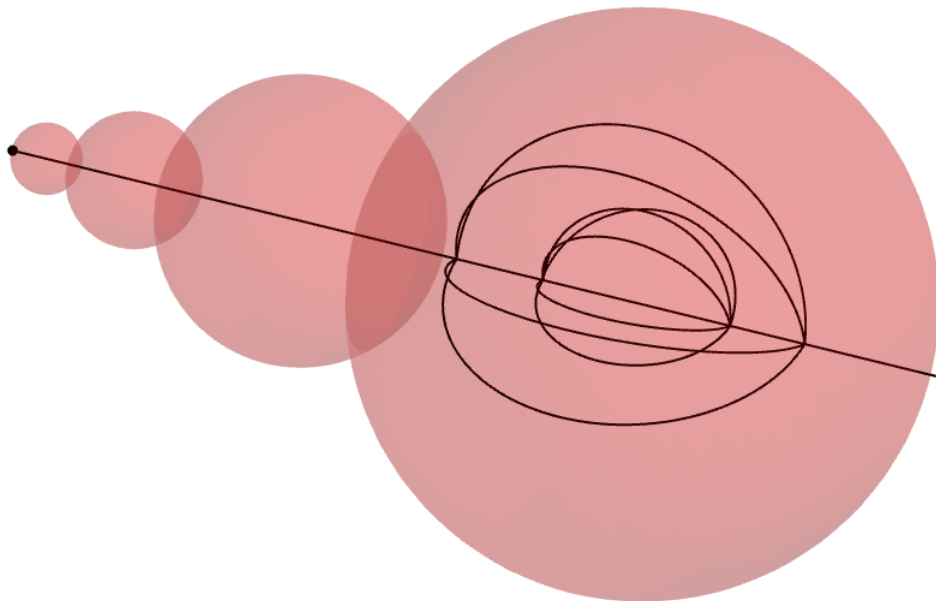

Figure 4: The structure of the graph Ψ, with the 'balls' intersecting along the ray and the flow inside the ball.

To construct this flow f, we start with the flow t on the tree $T_3 \subset H_2$ which sends the amount 3^{-n} through each directed edge of T_3 from a vertex of distance $n-1$ from the root to a vertex of distance n from the root. Note that t has finite Dirichlet energy.

Our flow f will be as described in the introduction, that is, it is composed of flows $g(n)$ in M_n. These flows flow from L_n to R_n. The flow $g(n)$ in turn is composed of 'atomic'

flows, one for each $v \in L_n$. Roughly, these atomic flows imitate t from above for some levels, then use the edges parallel to S_n to bring it 'above' R_n, and then collect it back to (two vertices of) S_n imitating t in the inverse direction. A key idea here is that although the energy dissipated along the long paths parallel to S_n is proportional to their length, by going up enough levels with the t-part of these flows, we can ensure that the flow i carried by each such path is very small compared to its length ℓ. Thus its contribution $i^2\ell$ to the Dirichlet energy can be controlled: although going up one level doubles ℓ, and triples the number of long paths we have, each of them now carries $1/3$ of the flow, and so its contribution to the energy is multiplied by a factor of $1/9$. Thus all in all, we save a factor of $6/9$ by going up one more level – and we have made the M_i high enough that we can go up enough levels.

We now describe $g(n)$ precisely. For every $n \in \mathbb{N}$, let us first enumerate the vertices of L_n as $l^j = l_n^j$, with j ranging from 1 to $|L_n| = 2^n$, in the order they appear on S_n as we move from the midpoint m_n towards the root o. Likewise, we enumerate the vertices of R_n as $r^j = r_n^j$, with j ranging from 1 to $|R_n| = 2|L_n|$, in the order they appear on S_n as we move from the midpoint m_n towards infinity. Thus r^1, l^1 are the two neighbours of m_n on S. We will let $g(n)$ be the union of $|L_n|$ subflows $g^j = g_n^j$, where g^j flows from l^j into r^{2j} and r^{2j-1}. More precisely, g^j sends $1/|L_n| = 2^{-n}$ units of current out of l^j, and half as many units of current into each of r^{2j} and r^{2j-1}.

We define g^j as follows. In the copy of H_2 containing the source l^j of g^j, we multiply the flow t from above by the factor 2^{-n}, and truncate it after j layers; we call this the out-part of g^j. Then, from each endpoint x of that flow, we send the amount of flow that x receives from l^j, which equals $2^{-n}3^{-j}$, along the horizontal path P_x joining x to the copy C_1 of H_2 containing r^{2j-1}. We let half of that flow continue horizontally to reach the copy C_2 of H_2 containing r^{2j}; call this the middle-part of g^j. Finally, inside each of C_1, C_2, we put a copy of the out-part of g^j multiplied by $1/2$ and with directions inverted; this is called the in-part of g^j. Note that the union of these three parts is a flow of intensity 2^{-n} from l^j to r^{2j} and r^{2j-1}, each of the latter receiving 2^{-n-1} units of current.

Let us calculate the energy $E(g^j)$. The contribution to $E(g^j)$ by its out-part is bounded above by $2^{-2n}E(t)$ because that part is contained in the flow $2^{-n}t$. Similarly, the contribution of the in-part is half of the contribution of the out-part. The contribution of the middle-part is $3^j \cdot (2j+1)2^j \cdot (2^{-n}3^{-j})^2$: the factor 3^j counts the number of horizontal paths used by the flow, each of which has length $(2j+1)2^j$, and carries $2^{-n}3^{-j}$ units of current (except for its last 2^j edges, from C_1 to C_2, which carry half as much, but we can afford to be generous). Note that this expression equals $2^{-2n}(2j+1)(6/9)^j$, which is upper bounded by $k2^{-2n}$ for some constant k.

Adding up these contributions, we see that $E(g^j) \le K2^{-2n}$ for some constant K (which depends on neither n nor j).

Now let $g(n)$ be the union of the 2^n flows g^j. Note that g^j, g^i are disjoint for $i \ne j$, and therefore the energy $E(g(n))$ of g is just the sum $\sum_{j<2^n} E(g^j)$. By the above bound, this yields $E(g(n)) \le K2^{-n}$.

Now let $f = \bigcup_{n\in\mathbb{N}} g(n)$ be the union of all the flows $g(n)$. Then $g(n), g(m)$ are disjoint for $n \ne m$, because they are in different $M_i's$. Thus $E(f) = \sum_n E(g(n)) \le K$ is finite. Since $g(n)$ removes as much current from each vertex of L_n as $g(n-1)$ inputs, f is a flow from o to infinity. Hence Ψ is transient by Lyons' criterion (Theorem 4.1).

5 Liouville property

In this section we prove that Ψ is *Liouville*, i.e. it admits no bounded non-constant harmonic functions.

We remark that a well-known theorem of Ancona [2] states that in any non-amenable hyperbolic graph the hyperbolic boundary coincides with the Martin boundary. We cannot apply this fact to our case in order to deduce the Liouville property from the fact that our hyperbolic boundary is trivial, because our graph turns out to be amenable.

We will use some elementary facts about harmonic functions that can be found e.g. in [14].

Let h be a bounded non-constant harmonic functions on a graph G. We may assume that the range of h is contained in $[0, 1]$. Recall that, by the bounded martingale convergence theorem, if $(X_n)_{n \in \mathbb{N}}$ is a simple random walk on G, then $h(X_n)$ converges almost surely. We call such a function h *sharp*, if this limit $\lim_n h(X_n)$ is either 0 or 1 almost surely. It is well-known that if a graph admits a bounded non-constant harmonic function, then it admits a sharp harmonic function, see [14, Section 4].

So let us assume from now on that $h : V(\Psi) \to [0, 1]$ is a sharp bounded harmonic function on Ψ.

We first recall some basic facts from [14, Section 7]; we repeat some of the proofs for the convenience of the reader.

Lemma 5.1. *If h is a sharp harmonic function, then $h(z) = \mathbb{P}_z \left[\lim h(Z_n) = 1 \right]$ for every vertex z, where Z_n denotes a random walk from z.*
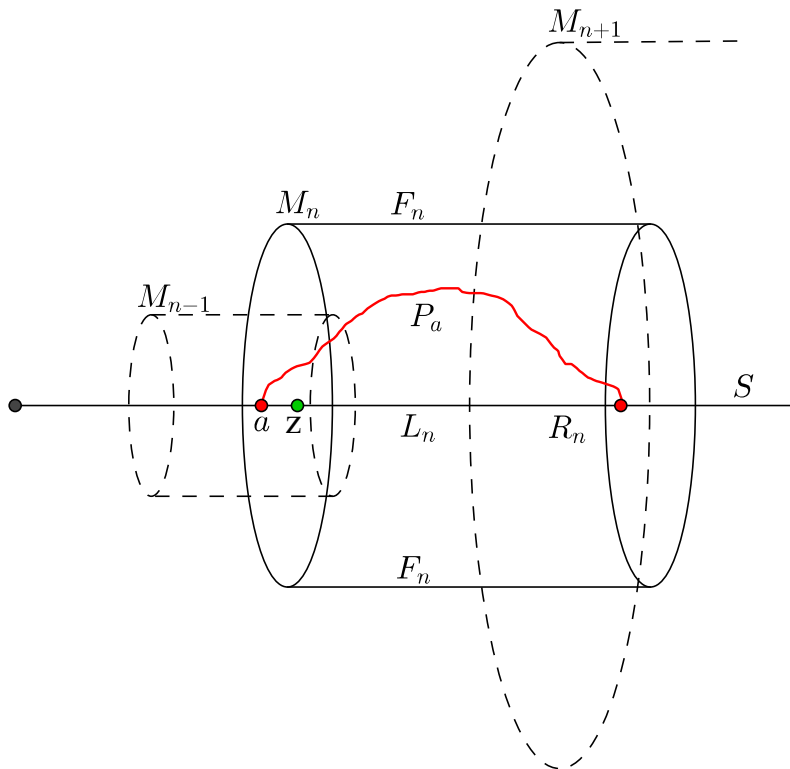

Figure 5: The path P_a in the proof of the Liouville property.

Lemma 5.2. *If h is a sharp harmonic function that is not constant, then for every $\epsilon > 0$ there are $a, z \in V$ with $h(a) < \epsilon$ and $h(z) > 1 - \epsilon$.*

Let $\mathcal{A}$ be a *shift-invariant event* of our random walk, i.e. an event not depending on the first n steps for every n. (The probability space we work with here is the space of 1-way infinite walks, endowed with the natural probability measure induced by simple

random walk. The only kind of event we will later consider is the event 1^s that $s(Z_n)$ converges to 1, where s is our fixed sharp harmonic function.) By an *event* here we mean a measurable subset of the space of 1-way infinite walks in our fixed graph; the starting vertex of the walks is not fixed, it can be an arbitrary vertex of our graph. A *walk* here is a sequence $v_0, v_1, \dots$ of vertices of the graph such that v_i is joined with an edge to v_{i+1} for every $i \geq 0$. As usual, we consider the σ-algebra generated by sets of walks that start with a fixed finite sequence of steps and are arbitrary after those steps.

For $r \in (0, 1/2]$, let

$$A_r := \{v \in V \mid \mathbb{P}_v[\mathcal{A}] > 1 - r\} \text{ and }$$
$$Z_r := \{v \in V \mid \mathbb{P}_v[\mathcal{A}] < r\},$$

where $\mathbb{P}_v[\cdot]$ denotes the law of random walk from a vertex v. Note that $A_r \cap Z_r = \emptyset$ for every such r.

By Lemma 5.1, if we let $\mathcal{A} := 1^s$ then we have $A_r = \{v \in V \mid s(v) > 1 - r\}$ and $Z_r = \{v \in V \mid s(v) < r\}$.

Lemma 5.3. *For every $\epsilon, \delta \in (0, 1/2]$, and every $v \in A_\epsilon$, we have*
$\mathbb{P}_v[\text{visit } V \setminus A_\delta] < \epsilon/\delta$. *Similarly, for every $v \in Z_\epsilon$, we have*
$\mathbb{P}_v[\text{visit } V \setminus Z_\delta] < \epsilon/\delta$.

Proof. Start a random walk (Z_n) at v, and consider a stopping time τ at the first visit to $V \setminus A_\delta$. If τ is finite, let $z = Z_\tau$ be the first vertex of random walk outside A_δ. Since $z \notin A_\delta$, the probability that $s(X_n)$ converges to 0 for a random walk (X_n) starting from z is at least δ by the definition of A_δ. Thus, conditioning on ever visiting $V \setminus A_\delta$, the event $\mathcal{A}$ fails with probability at least δ since $\mathcal{A}$ is a shift-invariant event and our random walk has the Markov property. But $\mathcal{A}$ fails with probability less than ϵ because $v \in A_\epsilon$, and so $\mathbb{P}_v[\text{visit } V \setminus A_\delta] < \epsilon/\delta$ as claimed.

The second assertion follows by the same arguments applied to the complement of $\mathcal{A}$. $\qquad\square$

Corollary 5.4. *If a random walk from $v \in A_\epsilon$ (respectively, $v \in Z_\epsilon$) visits a set $W \subset V$ with probability at least κ, then there is a v–W path all vertices of which lie in $A_{\epsilon/\kappa}$ (resp. $Z_{\epsilon/\kappa}$).*

Proof. Apply Lemma 5.3 with $\delta = \epsilon/\kappa$. Then the probability that random walk always stays within $A_{\epsilon/\kappa}$ is larger than $1 - \kappa$. Hence there is a nonzero probability that random walk meets W and along its trace only has vertices from $A_{\epsilon/\kappa}$. $\qquad\square$

Easily, h is uniquely determined by its values on the skewer S. Indeed, for every other vertex v, note that random walk X_n from v visits S almost surely, and so $h(v) = \mathbb{E}h(X_{\tau(R)})$, where $\tau(S)$ denotes the first hitting time of S by X_n. The same argument implies that

> h is radially symmetric, i.e. for every two cocircular vertices v, w, we have $h(v) = h(w)$. $\qquad(5.1)$

Indeed, this follows from the fact that cocircular vertices have the same hitting distribution to S, which is easy to see (for any vertex on a circle, random walk has the same probability to move to some other circle).

We claim that, given an arbitrarily small $\epsilon > 0$, all but finitely many of the L_n contain a vertex in A_ϵ.

Indeed if not, then since random walk from o has to visit all L_n by transience and (2.1) (where we use the fact that $L_n = R_{n-1}$), we would have $\mathbb{P}[\lim h(X_n) = 1] = 0$ for random walk X_n from o by Lemma 5.1, because if our random walk visits infinitely many vertices y such that $h(y) < 1 - \epsilon$ then $h(X_n)$ cannot converge to 1. But that probability is

equal to $h(o)$ by Lemma 5.1, and if it is zero, then using Lemma 5.1 again easily implies that h is identically zero, contrary to our assumption that it is not constant.

Similarly, all but finitely many of the L_n contain a vertex in Z_ϵ, because as h is sharp, $h(X_n)$ must converge to either 0 or 1. Thus we can find a late enough M_n such that L_n contains a vertex $a \in A_\epsilon$ as well as a vertex $z \in Z_\epsilon$. We assume that a and z are the last vertices of L_n (in the ordering of L_n induced by the well-ordering of S) that are in A_ϵ and Z_ϵ respectively. Assume without loss of generality that a appears before z in the ordering of L_n.

Note that, since R_n separates a from infinity (2.1), random walk from a visits R_n almost surely. Thus we can apply Corollary 5.4 with $W := R_n$ and $\kappa = 1$ to obtain an a–R_n path P_a with all its vertices in A_ϵ. We may assume that $P_a \subset M_n$ by taking a subpath contained in M_n if needed. Indeed, P_a can meet L_n only in vertices that are not past a in the linear ordering of L_n.

Let O_a denote the set of vertices $\{x = (t, w) \in M_n \mid \text{ there is } (t', w') \in V(P_a) \text{ with } w' = w\}$ obtained by 'rotating' P_a around S. By (5.1), we have $O_a \subset A_\epsilon$ since $P_a \subset A_\epsilon$. Note that O_a separates z from the ceiling F_n of M_n. But as random walk from $z \in Z_\epsilon$ visits F_n before returning to S with probability uniformly bounded below by (2.2), we obtain a contradiction to Lemma 5.3 with $\delta = 1/2$ for ϵ small enough compared to that bound.

6 A transient hyperbolic graph with no transient subtree

In this section we explain how our souvlaki construction can be slightly modified so that it does not contain any transient subtrees but remains transient and hyperbolic (and Liouville). This answers a question of I. Benjamini (private communication). The question is motivated by the fact that it is not too easy to come up with transient graphs that do not have transient subtrees [6].

We start with a very fast growing function $f : \mathbb{N} \to \mathbb{N}$, whose precise definition we reveal at the end of the proof. Roughly speaking, we will attach a sequence of finite graphs $(M_{f(n)})_{n \in \mathbb{N}}$ similar to the 'meatballs' from above to a ray S (the 'skewer') in such a way that most of the intersection of S with a fixed meatball is not contained in any other meatball. Formally, we let P_m be the 'bottom path' of M_m as defined in Section 2, and we tripartition $P_{f(n)}$ as follows: Let L_n consist of the first 2^n vertices on $P_{f(n)}$, and R_n consist of its last 2^{n+1} vertices. The set of the remaining vertices of $P_{f(n)}$ we denote by Z_n, which by our choice of f will be much larger than R_n. As before, we glue the $M_{f(n)}$ on S by identifying $P_{f(n)}$ with a subpath of S. We start by glueing $M_{f(1)}$ on the initial segment of S of the appropriate length. Then we recursively glue the other $M_{f(n)}$ in such a way that L_n is identified with R_{n-1}. We call the resulting graph $\bar{\Psi}$.

Theorem 6.1. $\bar{\Psi}$ *is a bounded degree transient gromov-hyperbolic graph that does not contain a transient subtree.*

Proof. The hyperbolicity of $\bar{\Psi}$ can be proved by the arguments we used for the original souvlaki Ψ. Also $\bar{\Psi}$ is transient by an argument analogue given to the one for Ψ: the obvious analogue of the flow f described in Section 4 is in $M_{f(n)}$ a flow of intensity one from L_n to R_n of energy at most constant times 2^{-n}. The computation is analogue to the one given above.[3] So it remains to show that $\bar{\Psi}$ does not have a transient subtree.

Let T be any subtree of $\bar{\Psi}$. We want to prove that T is not transient. Easily, we may assume that T does not have any degree 1 vertices. We will show that the following quotient Q of T is not transient: for each n, we identify all vertices in L_n to a new vertex v_n.

---

[3]Although Z_n gets larger if $f(n)$ increases, the flow f then branches more before 'traversing' Z_n. Since the increase of Z_n has an additive effect on the energy while the branching has a multiplicative effect, the effect due to branching dominates, hence the energy remains bounded.

Note that the vertices v_n and v_{n+1} are cut-vertices of Q; let Q_n be the union of those components of $Q - v_n - v_{n+1}$ that send edges to both vertices v_n and v_{n+1}. We will show that in Q_n the effective resistance from v_n to v_{n+1} is bounded away from 0, from which the recurrence of T will follow using Lyons' criterion.

Let $d = |L_{n+1}|$. We claim that there is some constant $c = c(d)$ only depending on d such that there are at most c vertices of Q_n with a degree greater than 2: indeed, $Q_n \setminus \{v_n, v_{n+1}\}$ is a forest with at most $d(v_n) + d(v_{n+1})$ leaves. Since these degrees are bounded also the number of leaves is bounded. Hence all but boundedly many vertices of Q_n have degree two.

Next, we observe that Q_n has maximum degree at most d. Furthermore, the distance between v_n and v_{n+1} in Q_n is at least Z_n, which —by the choice of f— is huge compared to d and so also compared to c. Hence it remains to prove the following:

Lemma 6.2. *For every constant C and every m there is some $s = s(m, C)$, such that for every finite graph K with maximum degree at most C and at most C vertices of degree greater than 2, and for any two vertices x, y of K with distance at least s, the effective resistance between x and y in K is at least m.*

Proof. We start with a large natural number R the value of which we reveal later, and set $s = R \cdot C$.

Let K' be the graph obtained from K by *suppressing* all vertices of degree 2; suppressing a vertex x of degree 2 means replacing x and its two incident edges with a single edge between the neighbours of x. The *length* of an edge of K' is the number of times it is subdivided in K. Let N' be the electrical network with underlying graph K', where the resistance of an edge of K' is its length. Clearly, the effective resistance between x and y in the graph K is equal to the effective resistance between x and y in the network K'. Hence it suffices to show that the effective resistance between x and y in K' is at least m.

We colour an edge of K' black if it has length at least R. Note that K' has at most C vertices. Thus every x-y-path in K' has length at most C, but in K any such path has length at least s. Therefore each x-y-path in K' contains a black edge. Hence in K' there is an x-y-cut consisting of black edges only. This cut has at most C^2 edges. Thus by Rayleigh's monotonicity law [21] the effective resistance in K' between x and y is at least the one of that cut, which is as large as we want: indeed, we can pick R so large that the latter resistance exceeds m. □

Now we reveal how large we have picked $f(n)$: recall that $d = 2^{n+1}$ and that $|Z_n| = f(n) - 3 \cdot 2^n$. We pick $f(n)$ large enough that $|Z_n| \geq s(1, max(c(d), d))$, where s is as given by the last lemma. With these choices the effective resistance between v_n and v_{n+1} in Q_n is at least 1. So Q cannot be transient by Lyons' criterion (Theorem 4.1) as the Q_n are disjoint and any flow to infinity has to traverse all but finitely many of them with a constant intensity. By Rayleigh's monotonicity law [21], T is recurrent too. □

6.1 Another transient hyperbolic graph with no transient subtree

We now sketch another construction of a transient hyperbolic graph with no transient subtree, provided by Tom Hutchcroft and Asaf Nachmias (private communication).

Let $[0, 1]^3$ be the unit cube. For each $n \geq 0$, let D_n be the set of closed dyadic subcubes of length 2^{-n}. For each $n \geq 0$, let G_n be the graph with vertex set $\bigcup_{i=0}^n D_i$, and where two cubes x and y are adjacent if and only if

- $x \supset y$, $x \in D_i$ and $y \in D_{i+1}$ for some $i \in \{0, \ldots n-1\}$,
- $y \supset x$, $y \in D_i$ and $x \in D_{i+1}$ for some $i \in \{0, \ldots n-1\}$, or
- $x, y \in D_i$ for some $i \in \{0, \ldots, n\}$ and $x \cap y$ is a square.

Then the graphs G_n are uniformly Gromov hyperbolic and, since the subgraph of G_n induced by D_n is a cube in $\mathbb{Z}^3$ (of size 4^n), the effective resistance between two corners this cube are bounded above uniformly in n. Moreover, the distance between these two points in G_n is at least n.

Let T be a binary tree, and let G be the graph formed by replacing each edge of T at height k from the root with a copy of G_{3^k}, so that the endpoints of each edge of T are identified with opposite corners in the corresponding copy of D_{3^k}. Since the graphs G_n are uniformly hyperbolic and T is a tree, it is easily verified that G is also hyperbolic. The effective resistance from the root to infinity in G is at most a constant multiple of the effective resistance to infinity of the root in T, so that G is transient. However, G does not contain a transient tree, since every tree contained in G is isomorphic to a binary tree in which each edge at height k from the root has been stretched by at least 3^k, plus some finite bushes.

7 A transient graph with no embeddable transient subgraph

We say that a graph H has a graph K as a *minor*, if K can be obtained from H by deleting vertices and edges and by contracting edges. Let K^r denote the complete graph on r vertices.

Proposition 7.1. *There is a transient bounded degree graph G such that every transient subgraph of G has a K^r minor for every $r \in \mathbb{N}$.*

In particular, G has no transient subgraph that embeds in any surface of finite genus.

We now construct this graph G. We will start with the infinite binary tree with root o, and replace each edge at distance r from o with a gadget D_{2^r} which we now define. Given $n \ (= 2^r)$, the vertices of D_n are organized in $2n + 1$ levels numbered $-n, \ldots, -1, 0, 1, \ldots, n$. Each level i has $2^{n-|i|}$ vertices, and two levels i, j form a complete bipartite graph whenever $|i - j| = 1$; otherwise there is no edge between levels i, j. Any edge of D_n from level $i \geq 0$ to level $i + 1$ or from level $-i$ to level $-(i + 1)$ is given a resistance equal to $2^{n-|i|}$ (we will later subdivide such edges into paths of that many edges each having resistance 1). With this choice, the effective resistance R_i between levels i and $i + 1$ of D_n is $2^{n-|i|}$ divided by the number of edges between those two levels, that is, $R_i = \frac{2^{n-|i|}}{2^{n-|i|}2^{n-|i|-1}} = 2^{-n+|i|+1}$, and so the effective resistance in D_n between its two vertices at levels n and $-n$ is $O(1)$

Let G' be the graph obtained from the infinite binary tree with root o by replacing each edge e at distance n from o with a disjoint copy of D_n, attaching the two vertices at levels n and $-n$ of D_n to the two end-vertices of e. We will later modify G' to obtain a bounded degree G with similar properties satisfying Proposition 7.1.

Note that as D_n has effective resistance $O(1)$, the graph G' is transient by Lyons' criterion.

We are claiming that if H is a transient subgraph of G', then H has a K^r minor for every $r \in \mathbb{N}$.

This will follow from the following basic fact of finite extremal graph theory [22, 18, 12]

Theorem 7.2. *For every $r \in \mathbb{N}$ there is a constant c_r such that every graph of average degree at least c_r has a K^r minor.*

Lemma 7.3. *If H is a transient subgraph of G', then H has a K^r minor for every $r \in \mathbb{N}$.*

Proof. Suppose that H has no K^r minor for some r, and fix any $m \in \mathbb{N}$. For every copy C of the gadget D_n in G' where $n > m$, consider the bipartite subgraph $G_m = G_m(C)$ of H spanned by levels m and $m + 1$ of $C \cap H$. By Theorem 7.2, the average degree of G_m is at most c_r. Thus, if we identify each of the partition classes of G_m into one vertex, we

obtain a graph with 2 vertices and at most $\frac{3}{2}2^{n-m}c_r$ parallel edges, each of resistance 2^{n-m}, so that the effective resistance of the contracted graph is at least $\frac{2}{3c_r} =: C_r$.

Now repeating this argument for $m+1, m+2, \ldots$, we see that the effective resistance between the two partition classes of G_{m+k} (which is edge-disjoint to G_m) is also at least the same constant C_r. This easily implies that the effective resistance between the two endvertices of $C \cap H$ for any copy C of D_n is $\Omega(n)$. Since G' has 2^r copies of D_{2^r} at each 'level' r, we obtain that the effective resistance from o (which we may assume without loss of generality to be contained in H) to infinity in H is $\Omega(\sum_r 2^r/2^r) = \infty$.

Thus H can have no electrical flow from a vertex to infinity, and by Lyons' criterion (Theorem 4.1) it is not transient. $\qquad\square$

Recall that the edges of G' had resistances greater than 1. By replacing each edge of resistance k by a path of length k with edges having resistance 1, we do not affect the transience of G'. We now modify G' further into a graph G of bounded degree, which will retain the desired property.

Let x be a vertex of some copy C of D_n, at some level $j \neq n, -n$ of C. Then x sends edges to the two neighbouring levels $j \pm 1$. Each of those levels L, L', sends $2^{k\pm1}$ edges to x for some k. Now disconnect all the edges from L to x, attach a binary tree T_L of depth $k \pm 1$ to x, and then reconnect those edges, one at each leaf of T_L.

Do the same for the other level L', attaching a new tree $T_{L'}$ of appropriate depth to x.

Note that this operation affects the edges incident with x only, and every other vertex of G', even those adjacent with x, retains its vertex degree. Thus we can perform this operation on every such vertex x simultaneously, with the understanding that if $e = xx'$ is an edge of G', and both x, x' are replaced by trees T, T' respectively by the above operation, then e becomes an edge joining a leaf of T to a leaf of T'; see Figure 6. There are many ways to match the leaves of the trees coming from vertices in one layer of D_n to the leaves of the trees coming from vertices in a subsequent layers, and so we have not uniquely identified the resulting graph, but what matters is that such a matching is possible because we have the same number of leaves on each side.
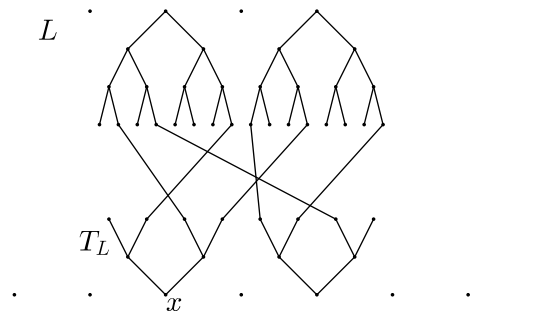

Figure 6: The tree T_L we replaced x with in order to turn G' into a bounded degree graph G, and a few similar trees for other vertices in the level of x and the level L above.

Let G denote a graph obtained by performing this operation to every vertex x as above. Note that G has maximum degree 6 (we did not need to modify the vertices at levels $n, -n$ in C, as they already had degree 6).

Now let's check that G is still transient, by considering the obvious flow to infinity: we start from the canonical flow f of strength 1 from o to infinity in G'. Recall that every edge $e = xx'$ of G' of resistance k was subdivided into a path P_e of length k consisting of edges of resistance 1, then x, x' were replaced by trees T, T', and now P_e joins a leaf of

T to a leaf of T' in G. Note that there is a unique path $Q_e \supset P_e$ in $T \cup P_e \cup T'$ from the root of T to the root of T'. For each edge e of G', we send a flow of intensity $f(e)$ along that path Q_e; easily, this induces a flow j on G from o to infinity.

We claim that the energy of j is finite, which means that G is transient by Lyons' criterion. Indeed, the contribution of the path P_e to the energy of j coincides with the contribution of e to the energy of f, and so their total contribution is finite. Let us now bound the contributions of the trees we introduced when defining G from G'. For this, we will use the following basic observation about flows on binary trees

> Let T be a binary tree of depth k, and let j be a flow from the root of T to its leaves such that every two edges at the same layer carry the same flow. Then the energy dissipated by j in all of T equals $(2^{k+1} - 1)$ times the energy dissipated by j in the last layer of T. (7.1)

Indeed, it is straightforward to check that the energy dissipated in each layer equals twice the energy dissipated in the next layer, and so the energy dissipated by j in all of T eqals $(1 + 2 + \ldots + 2^k)$ times the energy dissipated by j in the last layer.

Consider now two consecutive levels L, M in a copy of some gadget D_n in G', and suppose L has 2^k vertices and M has 2^{k+1} vertices. Recall that each L-M edge had resistance 2^k in G'. Furthermore, the f value is the same for all these edges; let b denote that common value. Thus, letting E denote the number of L-M edges, the total energy dissipated by f on L-M edges is $E2^k b^2$.

Note that for each tree T we introduced in the definition of G, each leaf of T was joined with exactly one edge of G'. It follows that for each such tree T between the layers L and M, the value of j at any edge in the last layer of T is b. Since each L-M edge of G' gave rise to exactly two such last-layer edges, namely one in the tree substituting each of its end-vertices, the total energy dissipated by j in all last-layer edges of G between the layers L and M is $2Eb^2$. By (7.1), the total energy dissipated by j in all layers of all trees we introduced between layers L and M, equals that amount multiplied by a constant smaller than 2^{k+1}. Recalling that the total energy dissipated by f on L-M edges was $E2^k b^2$, we see that the energy dissipated by j between layers L and M is less than 5 times that dissipated by f. Since this holds for each copy of each D_n, we deduce that j has finite energy since f does, proving that G is transient too.

Note that G' can be obtained from G by contracting edges. Thus any transient $H \subseteq G$ has a transient minor $H' \subseteq G'$, because contracting edges preserves transience by Lyons' criterion. As we have proved that H' has a K^r minor (Lemma 7.3), so does H as any minor of H' is a minor of H.

Despite Proposition 7.1, the following remains open

Question 7.4 (I. Benjamini (private communication))**.** Does every bounded-degree transient graph have a transient subgraph which is sphere-packable in R^3?

8 Problems

It is not hard to see that our hyperbolic souvlaki Ψ is *amenable*, that is, we have $\inf_{\emptyset \neq S \subset \Psi \text{ finite}} \frac{|\partial S|}{|S|} = 0$, where $\partial S = \{v \in V(\Psi) \setminus S \mid \text{there exists } w \in S \text{ adjacent to } v\}$. We do not know if this is an essential feature:

Problem 8.1. Is there a non-amenable counterexample to Conjecture 1.1?

Similarly, one can ask

Problem 8.2. Is there a non-amenable, hyperbolic graph with bounded-degrees, C-dense infinite geodesics, and the Liouville property, the hyperbolic boundary of which consists of a single point?

Here we did not ask for transience as it is implied by non-amenability [6].

We conclude with further questions asked by I. Benjamini (private communication)

Problem 8.3. Is there a uniformly transient counterexample to Conjecture 1.1? Is there an 1-ended counterexample?

Here *uniformly transient* means that there is an upper bound on the effective resistance between any vertex of the graph and infinity.

Our last problem, also by I. Benjamini (private communication), is motivated by our construction in Section 6 and the Appendix, where unimodular random graphs are defined.

Problem 8.4. Is there a bounded degree unimodular random graph that is non-Liouville but contains no transient subtree?

A   Appendix: a unimodular Liouville hyperbolic souvlaki

authored by

Gábor Pete[4]
Rényi Institute, Hungarian Academy of Sciences, Budapest, and
Institute of Mathematics, Budapest University of Technology and Economics
`http1://www.math.bme.hu/~gabor`

and
Gourab Ray [5]
Statistical Laboratory, University of Cambridge
`https://sites.google.com/site/gourabmathematics/`

Dedicated to Findus, who has always wanted to grow a tree out of meatballs [24]

A.1   Motivation and the result

Graphs with strange properties as constructed in the main paper cannot be transitive. On the one hand, a transitive transient graph has at least 3-dimensional volume growth (as follows from an extension of Gromov's polynomial growth theorem by Trofimov [28] and Losert [19]; see also [29, Theorem 5.11]), hence by the Coulhon-Saloff-Coste isoperimetric inequality (see [11] or [21, Theorem 6.29]) it has at least 3-dimensional isoperimetry, and hence by Thomassen's result [27] it contains a transient subtree. On the other hand, a transitive transient hyperbolic graph must be non-amenable [10], and non-amenable transitive graphs are non-Liouville [17] and contain non-amenable subtrees [6].

This raises the question whether such exceptional graphs can possess any sort of homogeneity. Beyond transitivity, a very natural class of graphs, especially when random walks are considered, is the class of unimodular random graphs, introduced in [8], studied in depth in [1] and in many works since; see [25, Chapter 14] for an overview. Here are the main definitions.

Let $\mathcal{G}_\star$ be the space of isomorphism classes of locally finite labeled rooted graphs, and let $\mathcal{G}_{\star\star}$ be the space of isomorphism classes of locally finite labeled graphs with an ordered pair of distinguished vertices, each equipped with the natural local topology: two (doubly) rooted graphs are "close" if they agree in "large" neighborhoods of the root(s).

Definition A.1. *We say that a Borel measure μ on $\mathcal{G}_\star$ is* unimodular *if it obeys the* Mass Transport Principle*:*

$$\int_{\mathcal{G}_\star} \sum_{x \in V(G)} f(G, o, x) \, d\mu(G, o) = \int_{\mathcal{G}_\star} \sum_{x \in V(G)} f(G, x, o) \, d\mu(G, o),$$

for any Borel function $f : \mathcal{G}_{\star\star} \to [0, \infty]$.

There are several other equivalent definitions; see [25, Definition 14.1]. Probably the nicest one, which works in most situations (e.g., bounded degree non-deterministic graphs), is that the Markov chain on $\mathcal{G}_\star$ generated by continuous time random walk on G with rate 1 exponential clocks on the edges is reversible.

An important class of unimodular graphs consists of Cayley graphs of finitely generated groups and of invariant random subgraphs of a Cayley graph. Another one is the class of sofic measures: the closure of the set of finite graphs with a uniform random root under local weak convergence, which is just weak convergence of measures in the space $\mathcal{G}_\star$.

Since many results on random walks and harmonic functions on transitive graphs generalize to unimodular or, more generally, stationary random graphs [5, 13, 3], it is natural to ask what the situation is in the present case. Here is our answer:

Theorem A.2. *There exists a bounded degree unimodular random graph that is a.s. transient and hyperbolic, but Liouville and has no transient subtree.*

Our construction will be a splice between the one in the main paper and the so-called d-regular canopy tree, which is the local weak limit of larger and larger balls in the d-regular tree. It is partly motivated by [9], where similar counterexamples for Bernoulli percolation are constructed based on the canopy tree. However, making the splice is not entirely straightforward here, since we have to put the meatballs on the canopy tree in a way that the graph remains unimodular. For this, the exponentially growing meatballs of the original construction would not work.

A.2 The construction

Take a d-ary tree T_n of height n (i.e., the root has d children, each of which has d children, and so on, stopping with the nth descendent generation). We are going to replace each edge of T_n by a modification of the graphs M_k in the Souvlaki construction of the main paper. Recall what W, H_2, H_3 are. Consider a subpath of the bottom double ray P_k of W of length $(k-1)^2 + k^2 + k^4$. Define M_k to be the subgraph of H_3 induced by vertices of the form (t, w) such that w lies at or above P_k and has height $h(w)$ at most k. We call M_k the meatballs.

We now "replace" each edge of T_n at height $n - k + 1$ (edges such that the vertex closer to the root has height $n - k$) by M_k for $k = 1, \ldots, n$. The word replace is within quotes because we have to specify the way we glue adjacent meatballs. We divide $P_k = L_k \cup R_k \cup A_k$, where L_k is the segment of the leftmost k^2 vertices, R_k is the segment of the rightmost $(k-1)^2$ vertices, and A_k is the middle k^4 vertices. Now let M_k^L denote the set of vertices (t, w) so that w lies on or above $L_k \cup A_k$ and has height at most k. Define M_k^R to be the graph induced by the rest of the vertices. Note that M_k^R and M_k^L are joined together by a set of edges. Let B_k denote the endpoint of these edges that lie in M_k^L.

Since we have the tree T_n instead of just a line, we need to modify M_k a bit, so that it branches into d copies of M_k^R for the identifications. That is, we take one copy of M_k^L and d copies of M_k^R, then glue each of the latter with M_k^L along B_k. Call this new gadget M_k'. The height function h extends to M_k', with values between 0 and k.

Now take an edge e at height $n - k + 1$, for $k \geq 1$. Remove it and replace it with M_k' so that e corresponds to the segment $L_k \cup A_k$, while its d children $e_1, \ldots e_d$ at height $n - k + 2$ correspond to the d copies of R_k for e. Since each R_k contains $(k - 1)^2$ vertices, we can identify them with the copies of L_{k-1} for the edges e_i. This completes the gluing procedure. For an edge e at height n, replace e just by M_1^L, without branching into copies of M_1^R. Call the new graph so obtained T_n'.

Now pick a uniform random vertex ρ_n from T_n' and take a weak limit. Call the limit (T, ρ). Clearly this graph is unimodular, Gromov hyperbolic and bounded degree, from arguments in the main paper.

A.3 Proofs

Root height Pick any integer $d > 6$. At height i of M_k^L, the number of vertices is $3^i 2^i (k^2 + k^4)$, with the factor 3^i coming from H_2, and the factor 2^i coming from W. Thus, the volume of M_k^L is

$$v_k := \frac{6^{k+1} - 1}{5}(k^4 + k^2),$$

and the probability that the uniform root in T_n is a vertex in one of the M_k^Ls is

$$p_{k,n} := \frac{v_k d^{n-k+1}}{\sum_{j=1}^{n} v_j d^{n-j+1}} = \frac{v_k d^{-k}}{\sum_{j=1}^{n} v_j d^{-j}}\,.$$

Since $d > 6$ implies that $v_k d^{-k}$ is summable, the limit $p_k := \lim_{n \to \infty} p_{k,n}$ is a proper probability distribution for $k = 1, 2, \ldots$. Therefore, the root in the limit (T, ρ) is almost surely at a level corresponding to a finite k, at a finite distance from the leaves (of the underlying canopy tree that is the local weak limit of the original trees T_n). In other words, we can think of the limit (T, ρ) as a souvlaki with a canopy tree skewer, with a random root somewhere.

Constructing a good flow One can think of the canopy tree as an infinite spine with finite bushes hanging off of it. Similarly, our canopy tree souvlaki has an infinite spine, a "traditional" infinite souvlaki. It is of course enough to show that this infinite spine is transient. We will construct for each $k \geq 1$ a unit flow g from R_{k+1} to L_{k+1}, with an energy that is summable in k. Concatenating these flows yields a flow along the spine to infinity, with finite energy, hence the spine turns out to be transient by Terry Lyons' criterion. (Unfortunately, the roles of Rs and Ls are now swapped compared to the main paper, due to the way that the infinite limit is constructed.)

Note that R_{k+1} has k^2 vertices and L_{k+1} has $(k + 1)^2$ vertices. We name the vertices in R_{k+1} as $r_1, r_2, \ldots, r_{k^2}$ and the vertices in L_{k+1} as $l_1, \ldots, l_{(k+1)^2}$. We will construct a flow g_j from r_j to L_{k+1} and g will be the sum the flows g_j. The flow g will have outflow $1/k^2$ from each vertex in R_{k+1} and inflow $1/(k + 1)^2$ for each vertex in L_{k+1}.

In the main paper, since the meatballs had exponentially growing lengths, there was a straightforward division of the total flow from each vertex into two vertices. Since here everything is polynomial, the division is slightly more complicated, but this is just a technicality. We deal with this as follows.

Since H_2 is transient, there exists a natural unit flow t (equally branching off at each vertex) with finite energy. Then, the flow g_j is just $\frac{1}{k^2} t$ up to height k, for each $j = 1, \ldots, k^2$. After this, g_j takes $k^2/(k + 1)^2$ fraction of the total incoming flow at height k, flows along the horizontal edges at height k until reaching above l_j, then flows along the tree proportionally with t (in reverse direction) to reach l_j. So, for $j = 1, \ldots, k^2$, the total flow into l_j is already $1/(k + 1)^2$.

There is still $(1 - k^2/(k+1)^2)$ fraction of the outflow from R_{k+1} left at level k. In fact, there is $(k^{-2} - (k+1)^{-2})3^{-k}$ amount left in each vertex at height k, in each g_j, $1 \le j \le k^2$. Flow this amount horizontally along level k until we reach above the vertex l_{k^2+1}. The total flow here (summed over j) is now $((k+1)^2 - k^2)/(k+1)^2$. Now we take $1/(k+1)^2$ out of this, and flow it along the tree above l_{k^2+1} proportionally with t, in reverse direction. We flow the remaining amount of flow at level k horizontally to reach above l_{k^2+2}, and again drop $1/(k+1)^2$ amount along the tree with t. We continue like this until all the flow is exhausted. Note that we input a flow $1/(k+1)^2$ for each l_j, $k^2 < j \le (k+1)^2$. Thus all in all, this is a unit flow from R_{k+1} to L_{k+1}.

Energy computation The total energy cost for going up the tree to height k for each r_j is $E(t)/k^4$. The flow received at each vertex is $3^{-k}/k^2$. On each horizontal edge at height k, there are at most k^2 flows that we are summing up, hence total flow through is at most 3^{-k}. The total length of horizontal paths is $O(k^4)3^k 2^k$, hence the total energy along the horizontal edges is $O(k^4)6^k 3^{-2k}$. The energy for going down each tree above l_j is again at most $E(t)/k^4$. Therefore, the total energy dissipation is

$$O(k^4)6^k 3^{-2k} + O(k^2)E(t)/k^4 = O(k^4)(2/3)^k + O(1/k^2) = O(1/k^2),$$

which is summable in k. This concludes the proof of transience.

Liouville property Removing the infinite spine, the canopy tree souvlaki falls apart into finite pieces. Thus, random walk started anywhere in the graph will almost surely hit the spine. This and the Optional Stopping Theorem for bounded martingales imply that any bounded harmonic function is determined by its restriction to the spine. Also, the radial symmetry of the harmonic function in the meatballs along the spine is preserved, as in the main paper. Thus the graph is Liouville by the same argument as in the main paper.

Transient subtree We first claim that any subtree inside the infinite spine must be recurrent. This can be shown by the same argument as in the main paper. Namely, in the proof of Theorem 6.1, the degrees $d(v_k)$ and $d(v_{k+1})$ are $O(k^2)$, hence following the proof of Lemma 6.2 shows that taking $s = k^4$ is enough for the effective resistance between v_k and v_{k+1} to be uniformly positive. This is exactly the choice we made in defining A_k, thus the effective resistance of any subtree to infinity is infinite, hence it is recurrent.

Now, for a general subtree, when we do the contraction into the vertices v_k, then the portions of the subtree inside the finite bushes off the spine get contracted into finite pieces, each attached to the contracted graph from the spine at a single vertex. These finite pieces do not influence transience of the contracted graph, hence the original subtree is also recurrent.

References

[1] D. Aldous and R. Lyons. Processes on unimodular random networks. *Electron. J. Probab.* **12** (2007), Paper 54, 1454–1508. http://128.208.128.142/~ejpecp/viewarticle.php?id=1754 MR-2354165

[2] A. Ancona. Negatively Curved Manifolds, Elliptic Operators, and the Martin Boundary. *The Annals of Mathematics*, 125(3):495, 1987. MR-0890161

[3] O. Angel, T. Hutchcroft, A. Nachmias, and G. Ray. Hyperbolic and parabolic unimodular random maps. arXiv:1612.08693

[4] I. Benjamini. Coarse Geometry and Randomness. Lecture Notes in Mathematics. Springer, 2013.

[5] I. Benjamini and N. Curien. Ergodic theory on stationary random graphs. *Electron. J. Probab.* **17** (2012), paper no. 93, 1–20. MR-2994841

[6] I. Benjamini and O. Schramm. Every graph with a positive Cheeger constant contains a tree with a positive Cheeger constant. *GAFA* **7** (1997), 403–419. MR-1466332

[7] I. Benjamini and O. Schramm. Harmonic functions on planar and almost planar graphs and manifolds, via circle packings. *Invent. math.*, 126:565–587, 1996.

[8] I. Benjamini and O. Schramm. Recurrence of distributional limits of finite planar graphs. *Electron. J. Probab.* **6** (2001), no. 23, 13 pp.

[9] D. Beringer, G. Pete, and Á. Timár. On percolation critical probabilities and unimodular random graphs. *Preprint*, arXiv:1609.07043

[10] M. Bridson and A. Haefliger. *Metric spaces of non-positive curvature.* Springer-Verlag, Berlin, 1999.

[11] T. Coulhon and L. Saloff-Coste. Isopérimétrie pour les groupes et les variétés. *Rev. Mat. Iberoamericana* **9** (1993), 293–314. MR-1232845

[12] R. Diestel. *Graph Theory* (3rd edition). Springer-Verlag, 2005. Electronic edition available at: http://www.math.uni-hamburg.de/home/diestel/books/graph.theory.

[13] S. Ganguly, J. R. Lee, and Y. Peres. Diffusive estimates for random walks on stationary random graphs of polynomial growth. *Preprint*, arXiv:1609.04040

[14] A. Georgakopoulos. The boundary of a square tiling of a graph coincides with the poisson boundary. *Invent. math.*, 203(3):773–821, 2016.

[15] A. Georgakopoulos and J. Carmesin. Every planar graph with the Liouville property is amenable. Preprint 2015.

[16] M. Gromov. Hyperbolic Groups. In *Essays in group theory (ed. S. M. Gersten), MSRI series vol. 8*, pages 75–263. Springer, New York, 1987.

[17] V. A. Kaimanovich and A. M. Vershik. Random walks on discrete groups: boundary and entropy. *Ann. Probab.* **11** (1983), 457–490.

[18] A. V. Kostochka. Lower bound of the hadwiger number of graphs by their average degree. *Combinatorica*, 4(Issue 4):307–316, 1984. MR-0779891

[19] V. Losert. On the structure of groups with polynomial growth. *Math. Z.* **195** (1987), 109–117. MR-0888132

[20] T. J. Lyons. A simple criterion for transience of a reversible Markov chain. *Ann. Probab.*, 11:393–402, 1983.

[21] R. Lyons and Y. Peres. *Probability on Trees and Networks*. Cambridge University Press, New York. 2016. Electronic version available at http://mypage.iu.edu/~rdlyons/prbtree/prbtree.html.

[22] W. Mader. Homomorphieeigenschaften und mittlere Kantendichte von Graphen. *Mathematische Annalen*, 174(Issue 4):265–268, 1967. MR-0220616

[23] S. McGuinness. *Random Walks on Graphs and Digraphs*. PhD thesis, University of Waterloo, 1988.

[24] S. Nordqvist. *Findus plants meatballs.* Hawthorn Press, 2013.

[25] G. Pete. *Probability and geometry on groups.* Lecture notes for a graduate course, present version is at http://www.math.bme.hu/~gabor/PGG.pdf

[26] H. Short (editor). Notes on Hyperbolic Groups. http://www.cmi.univ-mrs.fr/~hamish/.

[27] C. Thomassen. Isoperimetric inequalities and transient random walks on graphs. *Ann. Probab.*, 20(3):1592–1600, 1992.

[28] V. I. Trofimov. Graphs with polynomial growth. *Math. USSR Sbornik* **51** (1985), 405–417.

[29] W. Woess. *Random walks on infinite graphs and groups*. Cambridge Tracts in Mathematics Vol. 138, Cambridge University Press, 2000.

Electronic Journal of Probability
Electronic Communications in Probability

Advantages of publishing in EJP-ECP

- Very high standards

- Free for authors, free for readers

- Quick publication (no backlog)

- Secure publication (LOCKSS[1])

- Easy interface (EJMS[2])

Economical model of EJP-ECP

- Non profit, sponsored by IMS[3], BS[4], ProjectEuclid[5]

- Purely electronic

Help keep the journal free and vigorous

- Donate to the IMS open access fund[6] (click here to donate!)

- Submit your best articles to EJP-ECP

- Choose EJP-ECP over for-profit journals

---

[1]LOCKSS: Lots of Copies Keep Stuff Safe http://www.lockss.org/
[2]EJMS: Electronic Journal Management System http://www.vtex.lt/en/ejms.html
[3]IMS: Institute of Mathematical Statistics http://www.imstat.org/
[4]BS: Bernoulli Society http://www.bernoulli-society.org/
[5]Project Euclid: https://projecteuclid.org/
[6]IMS Open Access Fund: http://www.imstat.org/publications/open.htm