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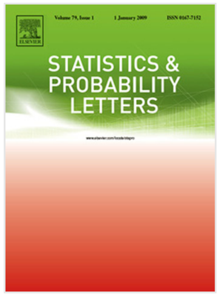

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Estimating Conditional Means with Heavy Tails

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Abstract

When a conditional distribution has an infinite variance, commonly employed kernel smoothing methods such as local polynomial estimators for the conditional mean admit non-normal limiting distributions (Hall, Peng and Yao 2002). This complicates the related inference as the conventional tests and confidence intervals based on asymptotic normality are no longer applicable, and the standard bootstrap method often fails. By utilizing the middle part of data nonparametrically and the tail parts parametrically based on extreme value theory, this paper proposes a new estimation method for conditional means, resulting in asymptotically normal estimators even when the conditional distribution has infinite variance. Consequently the standard bootstrap method could be employed to construct, for example, confidence intervals regardless of the tail heaviness. The same idea can be applied to estimating the difference between a conditional mean and a conditional median, which is a useful measure in data exploratory analysis.

Key words: Asymptotic normality; conditional mean; extreme value theory; heavy tail.

1 Introduction

Mean and median are two important location parameters in data exploratory analysis and the difference between them is indicative for the skewness of the underlying distribution. When the underlying distribution has a finite variance, the sample mean has a normal limit. However, when the underlying distribution has heavy tails with a finite mean but an infinite variance, the sample mean admits a stable law limit. Therefore, in order to construct a valid confidence interval for the mean, one has to know if the variance of the underlying distribution is finite or not. When the distribution is heavy tailed with infinite variance, the standard bootstrap method does not work, and a subsample bootstrap method should be employed to construct a valid confidence interval for the mean; see Hall and Jing (1998).

A different and unified approach has been proposed by Peng (2001) which constructs a mean estimator by using the middle part of data non-parametrically and the tail parts parametrically based on extreme value theory. The resulted estimator for the mean is alway asymptotically normal regardless of the tail heaviness of the underlying distribution. Hence one could simply employ the standard bootstrap method or empirical likelihood method to construct the confidence intervals for the mean even when the underlying distribution has an infinite variance; see Peng (2004). This idea has been taken further for estimating expected shortfall in risk management by Necir and Meraghni (2009). This paper aims to further extend this idea for estimating a conditional mean and the difference between a conditional mean and a conditional median, which are useful quantities in data exploratory analysis.

Suppose that $\{(X_i, Y_i)^T\}$ is a sequence of independent and identically distributed random vectors and the conditional distribution function $F(y|x) = P(Y_i \le y|X_i = x)$ satisfies

$$\begin{cases} \lim_{t\to\infty} \frac{1-F(ty|x)}{1-F(t|x)} = y^{-\alpha(x)}, \quad y > 0 \\ \lim_{t\to\infty} \frac{1-F(t|x)}{1-F(t|x)+F(-t|x)} = p(x) \in [0,1], \end{cases} \tag{1}$$

where $m(x)$ is an unknown smooth function and $\alpha(x) > 1$. Like mean and median, the conditional mean $E(Y_i|X_i = x)$ is of importance in many applications, which includes the random design regression model as a special case:

$$Y_i = m(X_i) + \epsilon_i, \tag{2}$$

where $\epsilon_i's$ are independent and identically distributed random variables with zero mean and satisfy

$$\begin{cases} \lim_{t\to\infty} \frac{P(\epsilon_i > ty)}{P(\epsilon_i > t)} = y^{-\beta}, \quad y > 0 \\ \lim_{t\to\infty} \frac{P(\epsilon_i > t)}{P(|\epsilon_i| > t)} = p \in [0,1], \end{cases} \tag{3}$$

for some $\beta > 1$.

Under model (2) and condition (3), model (1) holds with $\alpha(x) \equiv \beta$. Furthermore, the limiting distribution of a local smoothing estimator for $m(x)$ is normal or non-normal, respectively, when $\beta > 2$ or $\beta < 2$. This makes interval estimation nontrivial. However, when ϵ_i has a median zero, i.e., $m(x)$ is a conditional median, Hall, Peng and Yao (2002) showed that the least absolute deviations estimator has a normal limit for any $\beta > 1$. Consequently the standard bootstrap method can be employed to construct a confidence interval for the conditional median even when β is less than 2.

In this paper, we seek a new estimator for conditional mean $E(Y_i|X_i = x)$, and the difference between this conditional mean and the conditional median of Y_i given $X_i = x$ under the general setting (1). The new estimator is always asymptotically normal provided $\alpha(x) > 1$. Therefore the standard bootstrap method can be employed to construct confidence intervals for the conditional mean in a straightforward manner.

We organize the paper as follows. Section 2 presents the new method and the asymptotic results. A simulation study is given in Section 3. All proofs are put in Section 4.

2 Main Results

First we propose a new estimator for the conditional mean $E(Y_i|X_i = x)$, which admits a normal limiting distribution regardless of $\mathrm{Var}(Y_i|X_i = x)$ being finite or not.

Suppose that our observations $\{(X_i, Y_i)^T\}_{i=1}^n$ are independent and identically distributed random vectors with distribution function $F(x, y)$ and the conditional distribution $F(y|x)$ of Y_i given $X_i = x$ satisfies (1). For a given $h = h(n) > 0$, define $N = \sum_{i=1}^n I(|X_i - x| \le h)$, let $\{(\bar{X}_j, \bar{Y}_j)\}_{j=1}^N$ denote those data pairs $\{(X_i, Y_i)\}_{i=1}^n$ such that $|X_i - x| \le h$, and let $\bar{Y}_{N,1} \le \cdots \le \bar{Y}_{N,N}$ denote the order statistics of $\bar{Y}_1, \cdots, \bar{Y}_N$. Obviously, when $h \to 0$ and $hn \to \infty$, we have $N/(nh) \xrightarrow{p} f_1(x)$, where f_1 denotes the density of X_i. Therefore we write $N_0 = nh$ and say $N_0 \to \infty$ instead of $N \xrightarrow{p} \infty$.

Similar to Peng (2001), we write

$$\begin{aligned} & E(Y_i|X_i = x) = \int_{-\infty}^{\infty} y\, dF(y|x) = \int_0^1 F^-(y|x)\, dy \\ = \; & \int_0^{k/N} F^-(y|x)\, dy + \int_{k/N}^{1-k/N} F^-(y|x)\, dy + \int_{1-k/N}^1 F^-(y|x)\, dy \\ := \; & m_1(x) + m_2(x) + m_3(x), \end{aligned} \tag{4}$$

where $F^-(y|x)$ denotes the generalized inverse of the conditional distribution $F(y|x)$, and $k = k(N_0) \to \infty$ and $k/N_0 \to 0$ as $N_0 \to \infty$. Based on (4) we propose to estimate the first and third terms by a parametric approximation for $F(y|x)$ via extreme value theory and to estimate the second term nonparametrically. More specifically, when $F^-(y|x) \sim c_1 y^{-1/\alpha_1}$ and $F^-(1 - y|x) \sim c_2 y^{-1/\alpha_2}$ for some constants c_1 and c_2 as $y \to 0$, the tail indices α_1 and α_2 can be estimated by the well-known Hill estimator (Hill (1975))

$$\hat{\alpha}_1 = \{\frac{1}{k} \sum_{i=1}^k \log^+(-\bar{Y}_{N,i}) - \log^+(-\bar{Y}_{N,k})\}^{-1}$$

and

$$\hat{\alpha}_2 = \{\frac{1}{k} \sum_{i=1}^k \log^+(\bar{Y}_{N,N-i+1}) - \log^+(\bar{Y}_{N,N-k+1})\}^{-1}$$

with $\log^+ x = \log(x \vee 1)$. In our simulation, we set $\hat{\alpha}_1 = 0$ when all $\bar{Y}_{N,i} > -1$ for $1 \le i \le k$, and $\hat{\alpha}_2 = 0$ when all $\bar{Y}_{N,i} < 1$ for $N - k + 1 \le i \le N$. Note that as $N_0 \to \infty$

$$\frac{m_1(x)}{\frac{k}{N} F^-(k/N|x)} \xrightarrow{p} \int_0^1 y^{-1/\alpha(x)}\, dx = \frac{\alpha(x)}{\alpha(x) - 1}$$

and

$$\frac{m_3(x)}{\frac{k}{N} F^-(1 - k/N|x)} \xrightarrow{p} \int_0^1 y^{-1/\alpha(x)}\, dx = \frac{\alpha(x)}{\alpha(x) - 1}.$$

Therefore the three terms in (4) can be estimated separately by

$$\hat{m}_1(x) = \frac{k}{N} \bar{Y}_{N,k} \frac{\hat{\alpha}_1}{\hat{\alpha}_1 - 1}, \quad \hat{m}_2(x) = \frac{1}{N} \sum_{i=k+1}^{N-k} \bar{Y}_{N,i}, \quad \hat{m}_3(x) = \frac{k}{N} \bar{Y}_{N,N-k+1} \frac{\hat{\alpha}_2}{\hat{\alpha}_2 - 1},$$

which leads to our new estimator for the conditional mean $m(x) = E(Y_i|X_i = x)$ as $\hat{m}(x) = \hat{m}_1(x) + \hat{m}_2(x) + \hat{m}_3(x)$. Note that one could also use other tail index estimators instead of the Hill's estimator such that the one in Dierckx, Goegebeur and Guillou (2014). Moreover one may employ a different k in $\hat{\alpha}_1$ and $\hat{\alpha}_2$.

Like the study of extreme value statistics, in order to derive the asymptotic limits for $\hat{m}_1(x)$ and $\hat{m}_3(x)$, one needs to specify an approximate rate in (1), which is generally called a second order condition in extreme value theory; see De Haan and Ferreira (2006). Here we simply assume that there exist positive smoothing functions $d(x), c_1(x), c_2(x), \alpha(x) > 1, \beta(x)$ such that for y large enough

$$|1 - F(y|x) - c_1(x) y^{-\alpha(x)}| + |F(-y|x) - c_2(x) y^{-\alpha(x)}| \le d(x) y^{-\alpha(x) - \beta(x)} \quad (5)$$

uniformly in $|x - x_0| \le h$. Note that $\beta(x)$ is slightly smaller than the so-called second order parameter in extreme value theory, which can be seen from the inequality for a second order regular variation in De Haan and Ferreira (2006). Furthermore we assume the following regularity conditions:

A1) the marginal density f_1 of X_i is positive and continuous at x_0;

A2) functions $c_1(x), c_2(x)$ and $\alpha(x)$ have a continuous second order derivative at x_0, and functions $d(x)$ and $\beta(x)$ have a continuous first order derivative at x_0;

A3) the conditional mean function $m(x) = \int_{-\infty}^0 F(y|x)\, dy + \int_0^\infty (1 - F(y|x))\, dy$ has a continuous second order derivative at x_0.

To show that the new estimator always has a normal limit, we rely on the following approximations.

Let $H(y)$ denote the distribution function $\bar{Y}_i$ with $x = x_0$, i.e., the conditional distribution of Y_i given $|X_i - x_0| \le h$. Put $U_i = H(\bar{Y}_i)$ for $i = 1, \cdots, N$, and so $U_1, \cdots, U_N$ are i.i.d. random variables with uniform distribution on $(0,1)$. Let $U_{N,1} \le \cdots \le U_{N,N}$ denote the order statistics of $U_1, \cdots, U_N$. Define $G_N(v) = \frac{1}{N}\sum_{i=1}^N \mathbf{1}(U_i \le v)$, $\alpha_N(v) = \sqrt{N}\{G_N(v) - v\}$, $Q_N(0) = U_{N,1}$, $Q_N(s) = U_{N,i}$ if $\frac{i-1}{N} < s \le \frac{i}{N}$, and $\beta_N(s) = \sqrt{N}\{Q_N(s) - s\}$. Then it follows from Csörgő, Csörgő, Horváth and Mason (1986) that there exists a sequence of Brownian bridges $\{B_N(u)\}$ such that for any $\nu \in [0, 1/4)$ and $\lambda > 0$

$$\begin{cases} \sup_{U_{N,1} \le u \le N_{N,N}} \frac{u^\nu |\alpha_N(u) - B_N(u)|}{u^{1/2-\nu}(1-u)^{1/2-\nu}} = O_p(1) \\ \sup_{\lambda/N \le s \le 1-\lambda/N} \frac{N^\nu |\beta_N(s) + B_N(s)|}{s^{1/2-\nu}(1-s)^{1/2-\nu}} = O_p(1). \end{cases} \tag{6}$$

Theorem 1. *Suppose (5) and Conditions A1)–A3) hold. Put $N_0 = nh$, $\alpha_0 = \alpha(x_0)$, $\beta_0 = \beta(x_0)$, and further assume that as $n \to \infty$*

$$\begin{cases} k \to \infty, \quad \sqrt{k} h^2 (\log N_0)^2 = o(1), \\ k = o(N_0^{\frac{2\beta_0}{\alpha_0 + 2\beta_0}}), \quad \frac{\sqrt{N_0}}{\sigma(k/N_0)} h^2 = o(1), \end{cases} \tag{7}$$

where

$$\sigma^2(s) = \int_s^{1-s}\int_s^{1-s} (u \wedge v - uv)\, dH^-(u) dH^-(v).$$

Then as $n \to \infty$,

$$\begin{aligned} & \frac{\sqrt{N}}{\sigma(k/N)}\{\hat{m}(x_0) - m(x_0)\} \\ = & -\frac{\Delta_2 \alpha_0}{(\alpha_0 - 1)^2}\int_0^1 \sqrt{\frac{N}{k}}\Big(\frac{B_N(\frac{k}{N}s)}{s} - B_N(\frac{k}{N})\Big)\, ds - \frac{\Delta_2}{\alpha_0 - 1}\sqrt{\frac{N}{k}} B_n(\frac{k}{N}) \\ & -\frac{\Delta_1 \alpha_0}{(\alpha_0 - 1)^2}\int_0^1 \sqrt{\frac{N}{k}}\Big(\frac{B_N(1-\frac{k}{N}s)}{s} - B_N(1 - \frac{k}{N})\Big)\, ds - \frac{\Delta_1}{\alpha_0 - 1}\sqrt{\frac{N}{k}} B_N(1 - \frac{k}{N}) \\ & -\frac{\int_{k/N}^{1-k/N} B_N(s)\, dH^-(s)}{\sigma(k/N)} + o_p(1) \\ \stackrel{d}{\to} & \; N(0, 1 + \{\frac{(2-\alpha_0)(2\alpha_0^2 - 2\alpha_0 + 1)}{2(\alpha_0 - 1)^4} + \frac{2 - \alpha_0}{\alpha_0 - 1}\} I(\alpha_0 < 2)), \end{aligned}$$

where $m(x_0) = E(Y_i | X_i = x_0)$,

$$\Delta_1 = \{\frac{2 - \alpha_0}{2(c_1^{2/\alpha_0}(x_0) + c_2^{2/\alpha_0}(x_0))}\}^{1/2} c_1^{1/\alpha_0}(x_0) I(\alpha_0 < 2)$$

and

$$\Delta_2 = \{\frac{2 - \alpha_0}{2(c_1^{2/\alpha_0}(x_0) + c_2^{2/\alpha_0}(x_0))}\}^{1/2} c_2^{1/\alpha_0}(x_0) I(\alpha_0 < 2).$$

Remark 1. *If $\alpha(x_0) > 2$, then as $N_0 \to \infty$*

$$\sigma^2(k/N) \xrightarrow{p} E(Y_i^2|X_i = x_0) - (E(Y_i|X_i = x_0))^2 < \infty.$$

In this case, we require $\sqrt{nh}h^2 \to 0$, which gives the same rate of convergence as the local smoothing estimator of a conditional mean without asymptotic bias. It also follows from the proof of the above theorem that the above $H(y)$ can be replaced by $F(y|x_0)$.

Remark 2. *It follows from the above theorem that a naive bootstrap method can be employed to construct a confidence interval for the conditional mean regardless of tail heaviness. We refer to Hall (1992) for an overview on bootstrap method. A review paper on applying bootstrap methods to extreme value statistics is Qi (2008).*

Remark 3. *When $\alpha_0 > 2$, the terms $m_1(x)$ and $m_3(x)$ in (4) become a smaller order than the term $m_2(x)$. Therefore the asymptotic limit of the new estimator is independent of the tail index α_0.*

Next we consider estimating the difference between conditional mean and conditional median, i.e., $\theta(x) = E(Y_i|X_i = x) - F^-(1/2|x)$. Based on the above estimator for $m(x)$, the proposed estimator for θ is $\hat{\theta}(x) = \hat{m}(x) - \bar{Y}_{N,[N/2]}$, and its asymptotic limit is given in the theorem below.

Theorem 2. *Under conditions of Theorem 1 and that the conditional density function $g(y|x) = \frac{dF(y|x)}{dy}$ is positive and continuous at $y = F^-(\frac{1}{2}|x_0)$ and $x = x_0$, we have, as $n \to \infty$,*

$$\begin{aligned}
&\frac{\sqrt{N}}{\sigma(k/N)}\{\hat{\theta}(x_0) - \theta(x_0)\} \\
= \; & -\frac{\Delta_2\alpha_0}{(\alpha_0-1)^2}\int_0^1 \frac{N}{k}\Big(\frac{B_N(\frac{k}{N}s)}{s} - B_N(\tfrac{k}{N})\Big)\,ds - \frac{\Delta_2}{\alpha_0-1}\sqrt{\frac{N}{k}}B_n(\tfrac{k}{N}) \\
& -\frac{\Delta_1\alpha_0}{(\alpha_0-1)^2}\int_0^1 \sqrt{\frac{N}{k}}\Big(\frac{B_N(1-\frac{k}{N}s)}{s} - B_N(1-\tfrac{k}{N})\Big)\,ds - \frac{\Delta_1}{\alpha_0-1}\sqrt{\frac{N}{k}}B_N(1-\tfrac{k}{N}) \\
& -\frac{\int_{k/N}^{1-k/N} B_N(s)\,dH^-(s)}{\sigma(k/N)} - \frac{B_N(1/2)}{\sigma(k/N)g(F^-(\frac{1}{2}|x_0)|x_0)} + o_p(1) \\
\xrightarrow{d} \; & N(0, \sigma_\theta^2),
\end{aligned}$$

where σ_θ^2 equals to the variance in Theorem 1 when $\alpha_0 \le 2$, and is

$$\begin{aligned}
&1 + \frac{1}{4g^2(F^-(1/2|x_0)|x_0)\int_0^1\int_0^1(u\wedge v - uv)\,dF^-(u|x_0)dF^-(v|x_0)} \\
&+ \frac{\int_0^{1/2} u\,dF^-(u|x_0) + \int_{1/2}^1(1-u)\,dF^-(u|x_0)}{g(F^-(1/2|x_0)|x_0)\int_0^1\int_0^1(u\wedge v - uv)\,dF^-(u|x_0)dF^-(v|x_0)}
\end{aligned}$$

when $\alpha_0 > 2$.

3 Simulation

We conduct a small scale simulation to illustrate the proposed method. To this end, we let $X_i's$ in (2) be independent $U(-1,1)$ random variables, and consider

$$m(x) = x + 4\exp(-4x^2).$$

Furthermore in (2) we let ϵ_i be independent scaled t-distribution with d degrees of freedom for $d = 1.5$ and 3. Then $\alpha(x) = d$ in (1). We re-scale ϵ_i such that its standard deviation is 0.5. We set sample size $n = 1000$ or 3000, and choose $k = 5, 10, 20, 30, 40$ and 50. We use bandwidth $h = 0.2$ when $n = 1000$, and $h = 0.1$ when $n = 3000$. This effectively sets the sample sizes 200 and 300, respectively, in the local estimation for $m(x)$ for each given x.

We estimate $m(\cdot)$ on a regular grid of the 19 points between -0.9 and 0.9, and calculate the root mean square error:

$$\text{rMSE} = \Big\{\frac{1}{19}\sum_{j=-9}^{9}\{\hat{m}(0.1j) - m(0.1j)\}^2\Big\}^{1/2}. \tag{8}$$

For each setting, we replicate the exercise 500 times. To compare the performance with conventional nonparametric regression, we also calculate three nearest neighbor estimates, namely estimate $m(x)$ by the mean of Y_i's corresponding to those X_i's within, respectively, h-, $h/2$- and $h/4$-distance from x. Table 1 reports the mean and the standard deviation of rMSE for different settings over 500 replications. As we expected, the estimation error decreases when sample size n increases from 1000 to 3000, and the error also decreases when the tail index, reflected by the degrees of freedom (df), increases. With $t_{1.5}$-distributed errors, $k = 30$ gives a smallest standard deviation, and both $k = 20$ and $k = 30$ perform well. But with t_3-distributed errors, $k = 5$ leads to the most accurate estimates, which is in line with the theorem that tail parts do not play a role asymptotically in case of finite variance and so a smaller k is preferred. For the model with $t_{1.5}$-distributed errors, the nearest neighbor estimator is no longer asymptotically normal. Indeed our newly proposed estimator with either $k = 20$ or $k = 30$ performs better than the nearest neighbor estimator. However for the model with t_3-distributed errors, the nearest neighbor estimator is asymptotically normal and is indeed performs better than the new method.

(n, h, df)		New Estimator						NN Estimator		
		$k=5$	$k=10$	$k=20$	$k=30$	$k=40$	$k=50$	h	$h/2$	$h/4$
$(1000, 0.2, 1.5)$	Mean	3.674	0.381	0.188	0.200	0.236	0.279	0.218	0.250	0.340
	STD	47.66	1.836	0.291	0.053	0.071	0.090	0.447	0.621	0.701
$(3000, 0.1, 1.5)$	Mean	1.546	4.333	0.162	0.138	0.155	0.173	0.201	0.280	0.354
	STD	12.92	74.57	0.283	0.022	0.025	0.033	0.345	0.482	0.769
$(1000, 0.2, 3)$	Mean	0.134	0.154	0.208	0.274	0.348	0.428	0.122	0.080	0.103
	STD	0.021	0.023	0.033	0.045	0.059	0.072	0.021	0.018	0.024
$(3000, 0.1, 3)$	Mean	0.059	0.067	0.105	0.151	0.202	0.255	0.050	0.058	0.079
	STD	0.040	0.011	0.016	0.022	0.028	0.036	0.010	0.011	0.015

Table 1: Mean and standard deviation (STD) of rMSE defined in (8) for the proposed new estimator and the nearest neighbor (NN) estimator in simulation with 500 replications.

4 Proofs

Proof of Theorem 1. Write

$$\begin{aligned}
& \hat{m}_1(x_0) - \int_0^{k/N} H^-(v)\,dv \\
= \;& \tfrac{k}{N} H^-(U_{N,k})\left(\tfrac{\hat{\alpha}_1}{\hat{\alpha}_1-1} - \tfrac{\alpha_0}{\alpha_0-1}\right) \\
& + \tfrac{\alpha_0}{\alpha_0-1}\left(\tfrac{k}{N} H^-(U_{N,k}) - \tfrac{k}{N} H^-(k/N)\right) \\
& + \left(\tfrac{k}{N} H^-(k/N)\tfrac{\alpha_0}{\alpha_0-1} - \int_0^{k/N} H^-(v)\,dv\right) \\
= \;& \tfrac{k}{N} H^-(U_{N,k})\tfrac{\hat{\alpha}_1\alpha_0}{(\hat{\alpha}_1-1)(\alpha_0-1)}\tfrac{1}{k}\textstyle\sum_{i=1}^k \left\{\log\tfrac{H^-(U_{N,i})}{H^-(U_{N,k})} - \log(U_{N,i}/U_{N,k})^{-1/\alpha_0}\right\} \\
& + \tfrac{k}{N} H^-(U_{N,k})\tfrac{\hat{\alpha}_1\alpha_0}{(\hat{\alpha}_1-1)(\alpha_0-1)}\left\{\tfrac{1}{k}\textstyle\sum_{i=1}^k \log(U_{N,i}/U_{N,k})^{-1/\alpha_0} - 1/\alpha_0\right\} \\
& + \tfrac{k}{N} H^-(k/N)\tfrac{\alpha_0}{\alpha_0-1}\left\{\tfrac{H^-(U_{N,k})}{H^-(k/N)} - (\tfrac{N}{k}U_{N,k})^{-1/\alpha_0}\right\} \\
& + \tfrac{k}{N} H^-(k/N)\tfrac{\alpha_0}{\alpha_0-1}\left\{(\tfrac{N}{k}U_{N,k})^{-1/\alpha_0} - 1\right\} \\
& + \left\{\tfrac{k}{N} H^-(k/N)\tfrac{\alpha_0}{\alpha_0-1} - \int_0^{k/N} H^-(v)\,dv\right\} \\
:= \;& I_1 + I_2 + I_3 + I_4 + I_5,
\end{aligned}$$

$$\begin{aligned}
& \hat{m}_3(x_0) - \int_{1-k/N}^1 H^-(v)\,dv \\
= \;& \tfrac{k}{N} H^-(U_{N,N-k+1})\tfrac{\hat{\alpha}_2\alpha_0}{(\hat{\alpha}_2-1)(\alpha_0-1)}\tfrac{1}{k}\textstyle\sum_{i=1}^k \left\{\log\tfrac{H^-(U_{N,N-i+1})}{H^-(U_{N,N-k+1})} - \log(\tfrac{1-U_{N,N-i+1}}{1-U_{N,N-k+1}})^{-1/\alpha_0}\right\} \\
& + \tfrac{k}{N} H^-(U_{N,N-k+1})\tfrac{\hat{\alpha}_2\alpha_0}{(\hat{\alpha}_2-1)(\alpha_0-1)}\left\{\tfrac{1}{k}\textstyle\sum_{i=1}^k \log(\tfrac{1-U_{N,N-i+1}}{1-U_{N,N-k+1}})^{-1/\alpha_0} - 1/\alpha_0\right\} \\
& + \tfrac{k}{N} H^-(1-k/N)\tfrac{\alpha_0}{\alpha_0-1}\left\{\tfrac{H^-(U_{N,N-k+1})}{H^-(1-k/N)} - \left(\tfrac{N}{k}(1-U_{N,N-k+1})\right)^{-1/\alpha_0}\right\} \\
& + \tfrac{k}{N} H^-(1-k/N)\tfrac{\alpha_0}{\alpha_0-1}\left\{\left(\tfrac{N}{k}(1-U_{N,N-k+1})\right)^{-1/\alpha_0} - 1\right\} \\
& + \left\{\tfrac{k}{N} H^-(1-k/N)\tfrac{\alpha_0}{\alpha_0-1} - \int_{1-k/N}^1 H^-(v)\,dv\right\} \\
:= \;& III_1 + III_2 + III_3 + III_4 + III_5
\end{aligned}$$

and

$$\begin{array}{rl} & \hat{m}_2(x_0) - \int_{k/N}^{1-k/N} H^-(v)\,dv \\ = & \int_{U_{N,k}}^{k/N} H^-(v)\,dG_N(v) + \int_{1-k/N}^{U_{N,N-k}} H^-(v)\,dG_N(v) \\ & +H^-(1-k/N)\{G_N(1-k/N)-1+k/N\} - H^-(k/N)\{G_N(k/N)-k/N\} \\ & -\int_{k/N}^{1-k/N}\{G_N(v)-v\}\,dH^-(v) \\ := & II_1 + II_2 + II_3 + II_4 + II_5. \end{array}$$

Using Conditions A1)–A2), (5) and the fact that $|y^{\delta_3 h} - 1| \leq Mh\log y$ uniformly in $y \in [n^{\delta_1}, n^{\delta_2}]$ for any given $0 < \delta_1 < \delta_2 < 1$ and $\delta_3 > 0$, where $M > 0$ only depends on $\delta_1, \delta_2, \delta_3$, since $h \log n \to 0$, we have

$$\begin{array}{rl} & |1 - H(y) - c_1(x_0)y^{-\alpha_0}| \\ = & |\frac{\int_{x_0-h}^{x_0+h}\{1-F(y|z)\}f_1(z)\,dz}{P(|X_1-x_0|\leq h)} - c_1(x_0)y^{-\alpha(x_0)}| \\ \leq & |\frac{\int_{x_0-h}^{x_0+h}\{1-F(y|z)-c_1(z)y^{-\alpha(z)}\}f_1(z)\,dz}{P(|X_1-x_0|\leq h)}| \\ & +|\frac{\int_{x_0-h}^{x_0+h}\{c_1(z)y^{-\alpha(z)}-c_1(x_0)y^{-\alpha_0}\}f_1(z)\,dz}{P(|X_1-x_0|\leq h)}| \\ \leq & M_1 y^{-\alpha_0}\{y^{-\beta_0} + h^2(\log y)^2\} \end{array} \tag{9}$$

uniformly in $y \in [n^{\delta_1}, n^{\delta_2}]$ for any given $0 < \delta_1 < \delta_2 < 1$, where $M_1 > 0$ is independent of y. Similarly

$$|H(-y) - c_2(x_0)y^{-\alpha_0}| \leq M_2 y^{-\alpha_0}\{h^2(\log y)^2 + y^{-\beta_0}\} \tag{10}$$

uniformly in $y \in [n^{\delta_1}, n^{\delta_2}]$ for any given $0 < \delta_1 < \delta_2 < 1$, where $M_2 > 0$ is independent of y. Therefore

$$|H^-(1-t) - c_1^{1/\alpha_0}(x_0)t^{-1/\alpha_0}| \leq M_3 t^{-1/\alpha_0}\{h^2(\log t)^2 + t^{\beta_0/\alpha_0}\} \tag{11}$$

and

$$|H^-(t) + c_2^{1/\alpha_0}(x_0)t^{-1/\alpha_0}| \leq M_4 t^{-1/\alpha_0}\{h^2(\log t)^2 + t^{\beta_0/\alpha_0}\} \tag{12}$$

uniformly in $t \in [n^{-\delta_1}, n^{-\delta_2}]$ for any given $0 < \delta_2 < \delta_1 < 1$, where $M_3 > 0$ and $M_4 > 0$ are independent of t.

Note that

$$\frac{N}{nh} \xrightarrow{p} f_1(x_0), \quad P(\bar{Y}_{N,1} \geq -n^{-\delta}, \bar{Y}_{N,N} \leq n^{\delta}) \to 1 \tag{13}$$

for $\delta \in (0,1)$ large enough.

Write

$$\begin{aligned}
\sigma^2(s) &= \int_{H^-(s)}^0 \int_{H^-(s)}^0 \{H(u) \wedge H(v) - H(u)H(v)\}\, dudv \\
&\quad + \int_0^{H^-(1-s)} \int_0^{H^-(1-s)} \{H(u) \wedge H(v) - H(u)H(v)\}\, dudv \\
&= 2\int_{H^-(s)}^0 \int_v^0 H(v)\{1 - H(u)\}\, dudv \\
&\quad +2\int_0^{H^-(1-s)} \int_0^v H(u)\{1 - H(v)\}\, dudv \\
&= -2\int_{H^-(s)}^0 vH(v)\, dv - \{\int_{H^-(s)}^0 H(u)\, du\}^2 \\
&\quad +2\int_0^{H^-(1-s)} v\{1 - H(v)\}\, dv - \{\int_0^{H^-(1-s)} (1 - H(u))\, du\}^2 \\
&= IV_1(s) + IV_2(s) + IV_3(s) + IV_4(s).
\end{aligned}$$

Then it follows from (11)–(13) that

$$\begin{cases}
\frac{IV_1(k/N)}{(k/N)^{1-2/\alpha_0}} \xrightarrow{p} \frac{2c_2^{2/\alpha_0}(x_0)}{2-\alpha_0}, & \frac{IV_2(k/N)}{(k/N)^{1-2/\alpha_0}} \xrightarrow{p} 0, \\
\frac{IV_3(k/N)}{(k/N)^{1-2/\alpha_0}} \xrightarrow{p} \frac{2c_1^{2/\alpha_0}(x_0)}{2-\alpha_0}, & \frac{IV_4(k/N)}{(k/N)^{1-2/\alpha_0}} \xrightarrow{p} 0,
\end{cases}$$

when $\alpha_0 < 2$, and

$$\sigma^2(k/N) \xrightarrow{p} \begin{cases} \sigma_0^2 < \infty & \text{if} \quad \alpha_0 > 2 \\ \infty & \text{if} \quad \alpha_0 = 2, \end{cases}$$

where $\sigma_0^2 = \int_0^1 \int_0^1 (u \wedge v - uv)\, dF^-(u|x_0) dF^-(v|x_0)$. Therefore,

$$\frac{(k/N)^{1-2/\alpha_0}}{\sigma^2(k/N)} \xrightarrow{p} \frac{2-\alpha_0}{2(c_1^{2/\alpha_0}(x_0) + c_2^{2/\alpha_0}(x_0))} I(\alpha_0 < 2). \tag{14}$$

Now using (6), (9)–(14) and (7), we can show that

$$\frac{\sqrt{N}}{\sigma(k/N)}\{|I_1| + |I_3| + |I_5| + |III_1| + |III_3| + |III_5|\} = o_p(1),$$

$$\begin{aligned}
\frac{\sqrt{N}}{\sigma(k/N)} I_2 &= -\Delta_2 \frac{\alpha_0}{(\alpha_0-1)^2} \int_0^1 \sqrt{\frac{N}{k}} \{\frac{B_N(\frac{k}{N}s)}{s} - B_N(\frac{k}{N})\}\, ds + o_p(1), \\
\frac{\sqrt{N}}{\sigma(k/N)} I_4 &= -\Delta_2 \frac{1}{\alpha_0-1} \sqrt{\frac{N}{k}} B_N(\frac{k}{N}) + o_p(1), \\
\frac{\sqrt{N}}{\sigma(k/N)} III_2 &= -\Delta_1 \frac{\alpha_0}{(\alpha_0-1)^2} \int_0^1 \sqrt{\frac{N}{k}} \{\frac{B_N(1-\frac{k}{N}s)}{s} - B_N(1-\frac{k}{N})\}\, ds + o_p(1), \\
\frac{\sqrt{N}}{\sigma(k/N)} III_4 &= -\Delta_1 \frac{1}{\alpha_0-1} \sqrt{\frac{N}{k}} B_N(1-\frac{k}{N}) + o_p(1), \\
\frac{\sqrt{N}}{\sigma(k/N)} II_1 &= -\Delta_2 \sqrt{\frac{N}{k}} B_N(\frac{k}{N}) + o_p(1), \\
\frac{\sqrt{N}}{\sigma(k/N)} II_2 &= -\Delta_1 \sqrt{\frac{N}{k}} B_N(1-\frac{k}{N}) + o_p(1),
\end{aligned}$$

$$\frac{\sqrt{N}}{\sigma(k/N)} II_3 = \Delta_1 \sqrt{\frac{N}{k}} B_N(1-\frac{k}{N}) + o_p(1),$$

$$\frac{\sqrt{N}}{\sigma(k/N)} II_4 = \Delta_2 \sqrt{\frac{N}{k}} B_N(\frac{k}{N}) + o_p(1),$$

$$\frac{\sqrt{N}}{\sigma(k/N)} II_5 = -\frac{\int_{k/N}^{1-k/N} B_N(s)\,dH^-(v)}{\sigma(k/N)} + o_p(1),$$

which implies that

$$\begin{array}{rl} & \frac{\sqrt{N}}{\sigma(k/N)}\{\hat{m}(x_0) - \int_0^1 H^-(v)\,dv\} \\ = & -\frac{\Delta_2\alpha_0}{(\alpha_0-1)^2}\int_0^1 \sqrt{\frac{N}{k}}(\frac{B_N(\frac{k}{N}s)}{s} - B_N(\frac{k}{N}))\,ds - \frac{\Delta_2}{\alpha_0-1}\sqrt{\frac{N}{k}} B_n(\frac{k}{N}) \\ & -\frac{\Delta_1\alpha_0}{(\alpha_0-1)^2}\int_0^1 \sqrt{\frac{N}{k}}(\frac{B_N(1-\frac{k}{N}s)}{s} - B_N(1-\frac{k}{N}))\,ds - \frac{\Delta_1}{\alpha_0-1}\sqrt{\frac{N}{k}} B_N(1-\frac{k}{N}) \\ & -\frac{\int_{k/N}^{1-k/N} B_N(s)\,dH^-(s)}{\sigma(k/N)} + o_p(1) \\ \xrightarrow{d} & N(0, 1 + \{\frac{(2-\alpha_0)(2\alpha_0^2-2\alpha_0+1)}{2(\alpha_0-1)^4} + \frac{2-\alpha_0}{\alpha_0-1}\} I(\alpha_0 < 2)) \end{array} \tag{15}$$

by noting that

$$\begin{array}{rl} & E\{\sqrt{\frac{N}{k}} B_N(\frac{k}{N}) \frac{\int_{k/N}^{1-k/N} B_N(s)\,dH^-(s)}{\sigma(k/N)} | N\} \\ = & \frac{\sqrt{\frac{N}{k}}\int_{k/N}^{1-k/N} \frac{k}{N}(1-s)\,dH^-(s)}{\sigma(k/N)} \\ = & \frac{\sqrt{k/N}}{\sigma(k/N)}\{\int_{H^-(k/N)}^0 (1-H(u))\,du + \int_0^{H^-(1-k/N)}(1-H(u))\,du\} \\ \xrightarrow{p} & \{\frac{2-\alpha_0}{2(c_1^{2/\alpha_0}(x_0)+c_2^{2/\alpha_0}(x_0))}\}^{1/2} c_2^{1/\alpha_0}(x_0) I(\alpha_0 < 2) \end{array}$$

and

$$\begin{array}{rl} & E\{\sqrt{\frac{N}{k}} B_N(1-\frac{k}{N}) \frac{\int_{k/N}^{1-k/N} B_N(s)\,dH^-(s)}{\sigma(k/N)} | N\} \\ = & \frac{\sqrt{\frac{N}{k}}\int_{k/N}^{1-k/N} \frac{k}{N}s\,dH^-(s)}{\sigma(k/N)} \\ = & \frac{\sqrt{k/N}}{\sigma(k/N)}\{\int_{H^-(k/N)}^0 H(u)\,du + \int_0^{H^-(1-k/N)} H(u)\,du\} \\ \xrightarrow{p} & \{\frac{2-\alpha_0}{2(c_1^{2/\alpha_0}(x_0)+c_2^{2/\alpha_0}(x_0))}\}^{1/2} c_1^{1/\alpha_0}(x_0) I(\alpha_0 < 2). \end{array}$$

It follows from A3) that

$$\begin{array}{rl} & \int_0^1 H^-(v)\,dv - \int_0^1 F^-(v|x_0)\,dv \\ = & \int_{-\infty}^0 H(v)\,dv + \int_0^\infty (1-H(v))\,dv - \int_{-\infty}^0 F(v|x_0)\,dv - \int_0^\infty (1-F(v|x_0))\,dv \\ = & \frac{\int_{x_0-h}^{x_0+h} f_1(z)\{\int_{-\infty}^0 F(y|z)\,dy + \int_0^\infty (1-F(y|z))\,dy - \int_{-\infty}^0 F(y|x_0)\,dy - \int_0^\infty (1-F(y|x_0))\,dy\}dz}{P(|X_1-x_0|\le h)} \\ = & O(h^2). \end{array} \tag{16}$$

Hence the theorem follows from (15), (16) and (7). □

Proof of Theorem 2. The theorem easily follows from the expansions in the proof of Theorem 1 and the fact that

$$\sqrt{N}\{\bar{Y}_{N,[N/2]} - H^{-}(\frac{1}{2})\} = \frac{\sqrt{N}(G_N(1/2) - 1/2)}{g(F^{-}(\frac{1}{2}|x_0)|x_0)} + o_p(1).$$

□

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