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Simply Connected Fast Escaping Fatou Components

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Abstract: We give an example of a transcendental entire function with a simply connected fast escaping Fatou component, but with no multiply connected Fatou components. We also give a new criterion for points to be in the fast escaping set.
Keywords: transcendental entire functions; wandering domain; fast escaping set.

1. Introduction

Suppose that $f : \mathbb{C} \to \mathbb{C}$ is a transcendental entire function. The *Fatou set* $F(f)$ is defined as the set of points $z \in \mathbb{C}$ such that $(f^n)_{n\in\mathbb{N}}$ is a normal family in a neighborhood of z. The *Julia set* $J(f)$ is the complement of $F(f)$. An introduction to the properties of these sets was given in [3].

The *fast escaping set* $A(f)$ was introduced in [5]. We use the definition

$$A(f) = \{z : \text{there exists } \ell \in \mathbb{N} \text{ such that } |f^{n+\ell}(z)| \geq M^n(R, f), \text{ for } n \in \mathbb{N}\},$$

given in [10]. Here, the *maximum modulus function* $M(r, f) = \max_{|z|=r} |f(z)|$, for $r > 0$, $M^n(r, f)$ denotes repeated iteration of $M(r, f)$ with respect to r, and $R > 0$ can be taken to be any value such that $M(r, f) > r$, for $r \geq R$. We write $M(r)$ when it is clear from the context which function is being considered.

Suppose that $U = U_0$ is a component of $F(f)$. If $U \cap A(f) \neq \emptyset$, then $\overline{U} \subset A(f)$ [10, Theorem 1.2]. We call a Fatou component in $A(f)$ *fast escaping*. Denote by U_n the component of $F(f)$ containing $f^n(U)$. We say that a component is

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wandering if $U_n = U_m$ implies that $n = m$. All fast escaping Fatou components are wandering; [5, Lemma 4] and [10, Corollary 4.2].

For a transcendental entire function, all multiply connected Fatou components are fast escaping; [8, Theorem 2] and [10, Theorem 4.4]. The first example of a transcendental entire function with a multiply connected Fatou component was constructed by Baker in [1]. Other examples are found in, for example, [2], [4], [6] and [7].

The only known example of a simply connected fast escaping Fatou component was given by Bergweiler [4], using a quasi-conformal surgery technique from [7]. This function also has multiply connected Fatou components. In fact, in [4], the properties of the multiply connected Fatou components are used to show that the simply connected Fatou components are fast escaping.

This prompts the question of whether a transcendental entire function can have simply connected fast escaping Fatou components without having multiply connected Fatou components. We answer this in the affirmative, using a direct construction and a recent result on the size of multiply connected Fatou components [6, Theorem 1.2] to prove the following.

Theorem 1. *There is a transcendental entire function with a simply connected fast escaping Fatou component, and no multiply connected Fatou components.*

To prove Theorem 1 we require a new sufficient condition for points to be in $A(f)$, which may be of independent interest.

Theorem 2. *Suppose that f is a transcendental entire function. Suppose also that, for $R_0 > 0$, $\epsilon : [R_0, \infty) \to (0, 1)$ is a nonincreasing function, such that*

$$\epsilon(M^n(r)) \geq \epsilon(r)^{n+1}, \quad \text{for } r \geq R_0 \text{ and } n \in \mathbb{N}. \tag{1.1}$$

Define $\eta(r) = \epsilon(r)M(r)$, for $r \geq R_0$. Then there exists $R_1 \geq R_0$ such that

$$A(f) = \{z : \text{ there exists } \ell \in \mathbb{N} \text{ such that } |f^{n+\ell}(z)| \geq \eta^n(R'), \text{ for } n \in \mathbb{N}\},$$

for $R' \geq R_1$.

Note that this is a generalisation of [10, Theorem 2.7], which is obtained from Theorem 2 when ϵ is constant.

2. The definition of the function

In this section we define a transcendental entire function, f, which has all the properties defined in Theorem 1. Since f is very complicated, we first outline informally the construction of f, starting with simpler functions which only have some of these properties. We then give the full construction. A detailed proof of Theorem 1 is given in subsequent sections.

Consider first a transcendental entire function defined by a power series;

$$g(z) = z \prod_{k=1}^{\infty} \left(1 + \frac{z}{a_k}\right)^2, \quad 0 < a_1 < a_2 < \cdots .$$

The sequence $(a_n)_{n\in\mathbb{N}}$ can be chosen so that the following holds: we can define another sequence, $(b_n)_{n\in\mathbb{N}}$, such that b_n is approximately equal to a_n, $-b_n$ is close to a critical point of g, and $g(-b_n)$ is close to $-b_{n+1}$. It can then be shown that a small disc centred at $-b_n$ is mapped by g into a small disc centred at $-b_{n+1}$. By Montel's theorem, these discs must be in the Fatou set of g. Moreover, these discs cannot be in multiply connected Fatou components of g since, by [6, Theorem 1.2], any open set contained in a multiply connected Fatou component of g must, after a finite number of iterations of g, cover an annulus surrounding the origin. Finally, it can be shown, by comparing $|g(-b_n)|$ to $M(b_n, g) = g(b_n)$, that these discs are contained in fast escaping Fatou components of g.

However, g does not have all the properties defined in Theorem 1. In particular, by considering the behaviour of g in large annuli which omit the zeros of g, it can be shown that g has multiply connected Fatou components. Thus g has very similar properties to the example in [4].

We note that no zero of g can be in a multiply connected Fatou component, since 0 is a fixed point. In order to prevent the existence of multiply connected Fatou components, we add further zeros to the function, along the negative real axis. This requires some care. The addition of too many zeros – for example, spaced linearly along the negative real axis – leads to a breakdown of other parts of the construction. The addition of a zero with modulus insufficiently distant from a_n leads to a similar breakdown.

We use [6, Theorem 1.2] to show that only a relatively small number of additional zeros are required. In particular, suppose that h is a transcendental entire function with $h(0) = 0$ and with zeros of modulus $r_0 < r_1 < r_2 < \cdots$.

Then, by [6, Theorem 1.2], h has no multiply connected Fatou components if $\lim_{k\to\infty} \log r_{k+1}/\log r_k$ exists and is equal to 1.

To use this result, we need to understand the behaviour of $\log a_{n+1}/\log a_n$, for large n. From the recursive definition that we use to ensure that the sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ have the required properties, (see (3.15)), we find that, for large n, $\log a_{n+1}/\log a_n$ is close to n^3. See (3.9) for a precise statement of how the term n^3 arises here.

This suggests the following. Define $\mu_n = n^{3/n}$, for $n \in \mathbb{N}$. To simplify some displays we set $\mu_{n,m} = \mu_n^m$, and observe that $\mu_{n,0} = 1$ and $\mu_{n,n} = n^3$, for $n \in \mathbb{N}$. We now define a more complicated transcendental entire function;

$$h(z) = z \prod_{k=1}^{\infty} \prod_{l=0}^{k-1} \left(1 + \frac{z}{a_k^{\mu_{k,l}}}\right)^2, \quad 0 < a_1 < a_2 < \cdots .$$

The sequence $(a_n)_{n\in\mathbb{N}}$ in this definition is not the same as in the definition of g, but serves the same purpose, and is chosen similarly. This function has zeros of modulus

$$\cdots, a_n,\ a_n^{\mu_n},\ a_n^{\mu_{n,2}},\ \cdots,\ a_n^{\mu_{n,n-1}}, a_{n+1},\ a_{n+1}^{\mu_{n+1}} \cdots .$$

Since it is readily seen that $\mu_n \to 1$ as $n \to \infty$, this function does not have multiply connected Fatou components. However, two further adjustments are required. Firstly, the zero of modulus $a_n^{\mu_{n,n-1}}$ is sufficiently close to the zero of modulus a_{n+1} that the original construction breaks down. We resolve this by omitting this zero. Secondly, the value of $\log a_{n+1}/\log a_n$ is not close enough to n^3, for large n, to ensure that $\lim_{k\to\infty} \log a_{k+1}/\log a_k^{\mu_{k,k-2}} = 1$. We resolve this by adding one additional zero, which serves no other purpose in the construction. This zero is defined using two additional sequences, $(\alpha_n)_{n\in\mathbb{N}}$ and $(\beta_n)_{n\in\mathbb{N}}$, which we choose to keep $\log a_{n+1}/\log a_n$ sufficiently close to n^3.

Now we are able to indicate the form of the function f in Theorem 1. Let f be the transcendental entire function;

$$f(z) = z \prod_{k=3}^{\infty} \left\{ \left(1 + \frac{z}{a_k^{\beta_k}}\right)^{2\alpha_k} \prod_{l=0}^{k-2} \left(1 + \frac{z}{a_k^{\mu_{k,l}}}\right)^2 \right\}, \tag{2.1}$$

where $0 < a_3 < a_4 < \cdots$, $\alpha_n \in \{0, 1, 2, \ldots\}$, $\beta_n \in \mathbb{R}$, for $n \in \mathbb{N}$. Again, the sequence $(a_n)_{n\in\mathbb{N}}$ in this definition is not the same as that in the definition of g or h, but serves the same purpose, and is chosen similarly. The related sequence

$(b_n)_{n\in\mathbb{N}}$, discussed after the definition of g, is defined for f by (3.14). The sequences $(\alpha_n)_{n\in\mathbb{N}}$ and $(\beta_n)_{n\in\mathbb{N}}$ are the two sequences mentioned at the end of the previous paragraph.

The structure of the proof of Theorem 1 is as follows. First, in Section 3, we give the definition of the various sequences in (2.1), and we prove a number of estimates on the modulus of the zeros of f. Next, in Section 4, we show that there are no multiply connected Fatou components of f. In Section 5 we show that there are intervals on the negative real axis each contained in some Fatou component of f. Finally, in Section 6 we prove Theorem 2 and then use this to show that these Fatou components of f are fast escaping. It is clear that Theorem 1 follows from these results.

Remark: Rippon and Stallard asked [10, Question 1] if there can be unbounded fast escaping Fatou components of a transcendental entire function. It can be shown that the Fatou components of the function f are all bounded. Indeed, it is straightforward to prove that the number of zeros of f in the disc $\{z : |z| < r\}$ is $O(\log r)$, and hence that $\log M(r, f) = O((\log r)^2)$. It follows that f has no unbounded Fatou components [10, Theorem 1.9(b)], and, moreover, that the set $A(f)$ has a structure known as a spider's web. For more details we refer to [10].

3. Defining the sequences

In this section we first define the sequences $(\alpha_n)_{n\in\mathbb{N}}$ and $(\beta_n)_{n\in\mathbb{N}}$, and then define the sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$.

Recall from Section 2 that $\mu_n = n^{3/n}$ and $\mu_{n,m} = \mu_n^m$; we also define, for $n \geq 3$,

$$\sigma_n = \sum_{l=1}^{n-2} \mu_{n,l} = \mu_n \frac{\mu_{n,n-2} - 1}{\mu_n - 1}. \tag{3.1}$$

We define $(\alpha_n)_{n\in\mathbb{N}}$ to be a sequence of integers and $(\beta_n)_{n\in\mathbb{N}}$ to be a sequence of real numbers. Assume that N_0 is even and chosen sufficiently large for subsequent

estimates to hold. Set

$$2\alpha_n = \begin{cases} 0, & \text{for } n < N_0, \\ N_0^3 + 2N_0^2 + 6N_0 + 2, & \text{for } n = N_0, \\ 3n^2 + n + 6, & \text{for } n > N_0,\ n \text{ even}, \\ 3n^2 + n + 4, & \text{for } n > N_0,\ n \text{ odd}. \end{cases} \tag{3.2}$$

Note that α_n is an integer, for $n \in \mathbb{N}$. Set

$$\beta_n = \begin{cases} 0, & \text{for } n < N_0, \\ \frac{1}{\alpha_n}(n^4 - \sigma_n), & \text{for } n \geq N_0,\ n \text{ even}, \\ \frac{1}{\alpha_n}(\frac{n^3(2n-1)}{2} - \sigma_n), & \text{for } n \geq N_0,\ n \text{ odd}. \end{cases}$$

We observe that these choices imply that

$$\tau_n = \frac{2}{n^3}\left(\alpha_n\beta_n + \sigma_n\right) \tag{3.3}$$

satisfies

$$\tau_n = \begin{cases} 2n, & \text{for } n \geq N_0,\ n \text{ even}, \\ 2n-1, & \text{for } n \geq N_0,\ n \text{ odd}, \end{cases} \tag{3.4}$$

and

$$2\alpha_n = 3n^2 + n + 3 + \tau_n - \tau_{n-1}, \quad \text{for } n > N_0. \tag{3.5}$$

We also define a sequence of integers $(T_n)_{n\in\mathbb{N}}$ by

$$T_n = \begin{cases} n^3 + 2n - 3, & \text{for } n \text{ even}, \\ n^3 + 2n - 2, & \text{for } n \text{ odd}. \end{cases} \tag{3.6}$$

Next we prove a result which gives various relationships between these sequences.

Lemma 3.1. *The following all hold for the choice of sequences above:*

$$\alpha_n \sim \frac{3}{2}n^2, \quad \beta_n \sim \frac{2}{3}n^2, \quad \textit{as } n \to \infty, \tag{3.7}$$

$$2\sum_{k=3}^{n}\alpha_k = n^3 + 2n^2 + 4n + 2 + \tau_n, \quad \textit{for } n \geq N_0, \tag{3.8}$$

$$1 + 2\sum_{k=3}^{n-1}\alpha_k + \sum_{k=3}^{n-1}\sum_{l=0}^{k-2} 2 = n^3 + \tau_{n-1} = T_n, \quad \textit{for } n \geq N_0, \tag{3.9}$$

and

$$\mu_{n,2} < \beta_n < \mu_{n,n-3}, \quad \textit{for large } n. \tag{3.10}$$

Proof. The first half of (3.7) is immediate from (3.2). Now, by (3.2), (3.3) and (3.4),

$$\beta_n \sim \frac{2n}{3}\left(n - \frac{\sigma_n}{n^3}\right), \quad \text{as } n \to \infty. \tag{3.11}$$

We have that

$$\frac{x}{2} \leq \log(1+x) \leq x, \quad \text{for } 0 < x < \frac{1}{2}. \tag{3.12}$$

Putting $x = \mu_n - 1$, we obtain

$$\frac{3}{n}\log n \leq \mu_n - 1 \leq \frac{6}{n}\log n, \quad \text{for large } n. \tag{3.13}$$

Hence, by (3.1),

$$\frac{\sigma_n}{n^3} = \frac{\mu_n}{\mu_{n,n}}\frac{\mu_{n,n-2}-1}{\mu_n - 1} \sim \frac{1}{\mu_n(\mu_n - 1)} = O\left(\frac{n}{\log n}\right) \quad \text{as } n \to \infty,$$

and the second half of (3.7) follows by (3.11).

We can see that (3.8) holds by induction. For, it is immediately satisfied when $n = N_0$. When $n = m > N_0$ we have, by (3.5) and (3.8) with $n = m-1$,

$$2\sum_{k=3}^{m}\alpha_k = 2\alpha_m + 2\sum_{k=3}^{m-1}\alpha_k = m^3 + 2m^2 + 4m + 2 + \tau_m.$$

Finally (3.9) follows from (3.4), (3.6), and (3.8), and (3.10) follows from (3.7). □

Next we define the sequence $(a_n)_{n\in\mathbb{N}}$ recursively, and for each $n \in \mathbb{N}$ put

$$b_n = a_n - \frac{2}{T_n + 2}a_n = \frac{T_n}{T_n + 2}a_n. \tag{3.14}$$

Choose a_3 and N_1 large, and set $a_{n+1} = a_n^{n^3}$, for $3 \leq n < N_1$. We assume that a_3 and N_1 are chosen sufficiently large for various estimates in the sequel to hold. For $n \geq N_1$, we define

(3.15)

$$a_{n+1} = \frac{(T_{n+1}+2)}{T_{n+1}}\, b_n \left(1 - \frac{b_n}{a_n}\right)^2 \prod_{k=3}^{n-1}\left\{\left(1 - \frac{b_n}{a_k^{\beta_k}}\right)^{2\alpha_k} \prod_{l=0}^{k-2}\left(1 - \frac{b_n}{a_k^{\mu_{k,l}}}\right)^2\right\}.$$

Finally in this section we prove a set of inequalities which concern the growth of the sequence (a_n), and the ratios of these numbers to the modulus of the other zeros of f. Note that (3.7) and (3.17) imply that the product in (2.1) is locally uniformly convergent in $\mathbb{C}$.

Lemma 3.2. *The following inequalities hold for the sequence (a_n) defined above. For $n \geq 3$,*

$$a_n^{n^3-2/n} \leq a_{n+1} \leq a_n^{n^3+2/n}, \tag{3.16}$$

$$a_n > \exp(e^n), \tag{3.17}$$

and, for large n,

$$\frac{a_n}{a_n^{\mu_n}} \leq \exp(-e^{n/2}), \qquad \frac{\alpha_n a_n}{a_n^{\beta_n}} \leq \exp(-e^{n/2}), \tag{3.18}$$

$$\frac{a_n^{\mu_{n,n-2}}}{a_{n+1}} \leq \exp(-e^{n/2}), \qquad \frac{\alpha_n a_n^{\beta_n}}{a_{n+1}} \leq \exp(-e^{n/2}). \tag{3.19}$$

Proof. First, assume that (3.16) holds for $3 \leq n \leq m$. Equation (3.17) follows for $3 \leq n \leq m$ by a simple induction. Hence, for sufficiently large m, by (3.7), (3.13), (3.16) and (3.17):

$$\begin{aligned}
&\frac{a_m}{a_m^{\mu_m}} = a_m^{1-\mu_m} \leq \exp(e^m(1-\mu_m)) \leq \exp(-e^{m/2});\\
&\frac{\alpha_m a_m}{a_m^{\beta_m}} \leq 3m^2 a_m^{1-m^2/2} \leq \exp(-e^{m/2});\\
&\frac{a_m^{\mu_{m,m-2}}}{a_{m+1}} \leq (a_m^{\mu_{m,m-2}})^{1-\mu_{m,2}+\frac{2}{m}} \leq \exp(e^m(1-\mu_{m,2}+\frac{2}{m})) \leq \exp(-e^{m/2});\\
&\frac{\alpha_m a_m^{\beta_m}}{a_{m+1}} \leq 3m^2 a_m^{m^2-m^3+2/m} \leq \exp(-e^{m/2}).
\end{aligned}$$

It remains to prove (3.16). We can assume, by taking N_1 sufficiently large, that (3.16) holds for $3 \leq n \leq m-1$, for some large m. We can assume also that m is sufficiently large that (3.9), (3.10) and various other estimates used in the following hold. We need to prove that (3.16) holds for $n = m$. Now, by (3.15)

and (3.3),

$$\begin{aligned} a_{m+1} &= \frac{(T_{m+1}+2)}{T_{m+1}}\, b_m^{\kappa_m} \left(1-\frac{b_m}{a_m}\right)^2 \frac{L_1}{\prod_{k=3}^{m-1}\left\{a_k^{2\alpha_k\beta_k}\prod_{l=0}^{k-2} a_k^{2\mu_{k,l}}\right\}} \\ &= k_m \frac{a_m^{\kappa_m}}{a_{m-1}^{2\alpha_{m-1}\beta_{m-1}}\prod_{l=1}^{m-3} a_{m-1}^{2\mu_{m-1,l}}}\frac{L_1}{L_2} \\ &= k_m \frac{a_m^{\kappa_m}}{a_{m-1}^{(m-1)^3\tau_{m-1}}}\frac{L_1}{L_2}, \end{aligned} \tag{3.20}$$

where, by (3.14),

$$k_m = \frac{(T_{m+1}+2)}{T_{m+1}}\left(\frac{T_m}{T_m+2}\right)^{\kappa_m}\left(\frac{2}{T_m+2}\right)^2,$$

by (3.9),

$$\kappa_m = 1 + 2\sum_{k=3}^{m-1}\alpha_k + \sum_{k=3}^{m-1}\sum_{l=0}^{k-2} 2 = m^3 + \tau_{m-1} = T_m,$$

$$L_1 = \prod_{k=3}^{m-1}\left\{\left(1-\frac{a_k^{\beta_k}}{b_m}\right)^{2\alpha_k}\prod_{l=0}^{k-2}\left(1-\frac{a_k^{\mu_{k,l}}}{b_m}\right)^2\right\},$$

and, by (3.3) again,

$$\begin{aligned} L_2 &= a_{m-1}^2\prod_{k=3}^{m-2}\left\{a_k^{2\alpha_k\beta_k}\prod_{l=0}^{k-2} a_k^{2\mu_{k,l}}\right\} \\ &= a_{m-1}^2\prod_{k=3}^{m-2} a_k^{2(1+\alpha_k\beta_k+\sum_{l=1}^{k-2}\mu_{k,l})} \\ &= a_{m-1}^2\prod_{k=3}^{m-2} a_k^{2+k^3\tau_k}. \end{aligned}$$

We now estimate the terms in this equality. Firstly, by (3.6), and noting that $(T_m/(T_m+2))^{T_m} > 1/e^2$, we obtain

$$\frac{1}{8}m^{-6} < k_m < 8m^{-6}.$$

Secondly, by (3.16), with $n = m-1$,

$$a_{m-1}^{(m-1)^3} \le a_m^{(1-\frac{2}{(m-1)^4})^{-1}} \le a_m^{1+\frac{4}{(m-1)^4}},$$

and so, by (3.4),

$$\frac{a_m^{\kappa_m}}{a_{m-1}^{(m-1)^3\tau_{m-1}}} \geq a_m^{\kappa_m-\tau_{m-1}-\frac{4\tau_{m-1}}{(m-1)^4}} = a_m^{m^3-\frac{4\tau_{m-1}}{(m-1)^4}} \geq a_m^{m^3-\frac{1}{m}}.$$

Similarly, by (3.4) and (3.16),

$$\frac{a_m^{\kappa_m}}{a_{m-1}^{(m-1)^3\tau_{m-1}}} \leq a_m^{m^3+\frac{1}{m}}.$$

Thirdly, we consider L_1. Noting (3.10), and by (3.14) and (3.19), we see that the smallest term in this product is

$$1 > \left(1 - \frac{a_{m-1}^{\mu_{m-1,m-3}}}{b_m}\right)^2 > \left(1 - \frac{2a_{m-1}^{\mu_{m-1,m-3}}}{a_m}\right)^2 > 1 - \exp(-e^{\frac{m}{4}}).$$

There are fewer than m terms in L_1 of the form $(1 - a_{m-1}^p/b_m)^q,\ p \in \mathbb{R},\ q \in \mathbb{N}$, and so of comparable size to this term. The other terms in the product for L_1 tend to 1 sufficiently quickly, by (3.16), that $1/2 < L_1 < 1$.

Finally, we consider L_2. Observe that, by (3.4) and (3.16), the largest term in this product is

$$a_{m-1}^2 a_{m-2}^{2+(m-2)^3\tau_{m-2}} < a_{m-1}^2 a_{m-2}^{4(m-2)^4} < a_{m-1}^2 a_{m-1}^{8(m-1)} < a_m^{16/m^2}.$$

By (3.16), all other terms in the product for L_2 decrease sufficiently quickly that $1 < L_2 < a_m^{32/m^2}$.

Thus, by (3.20), for sufficiently large m,

$$a_{m+1} \geq \frac{1}{16} m^{-6} a_m^{m^3-\frac{1}{m}-\frac{32}{m^2}} \geq a_m^{m^3-\frac{2}{m}},$$

and similarly,

$$a_{m+1} \leq 8m^{-6} a_m^{m^3+\frac{1}{m}} \leq a_m^{m^3+\frac{2}{m}}.$$

This completes the proof of Lemma 3.2. □

4. There are no multiply connected Fatou components

In this section we prove the following result.

Lemma 4.1. *The transcendental entire function f does not have multiply connected Fatou components.*

We use the definition of an annulus

$$A(r_1, r_2) = \{z : r_1 < |z| < r_2\}, \quad \text{for } 0 < r_1 < r_2.$$

We need the following, which is part of [6, Theorem 1.2].

Theorem A. *Suppose that g is a transcendental entire function with a multiply connected Fatou component U. For each $z_0 \in U$ there exists $\alpha > 0$ such that, for sufficiently large $n \in \mathbb{N}$,*

$$g^n(U) \supset A(|g^n(z_0)|^{1-\alpha}, |g^n(z_0)|^{1+\alpha}).$$

Proof of Lemma 4.1. Observe that, for large n, in the closed annulus $\overline{A(a_n, a_{n+1})}$ there are zeros of f on the negative real axis of modulus $a_n, a_n^{\mu_n}, a_n^{\mu_{n,2}}, \ldots, a_n^{\mu_{n,n-2}}$ and a_{n+1}. Note also that 0 is a fixed point of f, and so no zero of f can be in a multiply connected Fatou component of f. Now, by (3.16),

$$a_{n+1} \leq a_n^{n^3+2/n} < (a_n^{\mu_{n,n-2}})^{\mu_{n,2}+2/n}.$$

Hence, for large n, there is at least one zero of f in any annulus $A(r, r^{\mu_{n,2}+2/n})$, for $a_n \leq r \leq a_{n+1}$. Note that $\mu_{n,2} + 2/n \to 1$ as $n \to \infty$.

Now, by Theorem A, if f has a multiply connected Fatou component, then there is a $c > 1$, and a sequence $(r_i)_{i\in\mathbb{N}}$, tending to infinity, such that the annuli $A(r_i, r_i^c)$ are contained in multiply connected Fatou components of f. This is in contradiction to the observations above regarding the distribution of zeros of f and the fact that these zeros do not lie in multiply connected Fatou components. Hence there can be no multiply connected Fatou components of f. □

5. There are simply connected Fatou components

Next we show that f has simply connected Fatou components.

Lemma 5.1. *Define $B_n = \{z : |z + b_n| < \delta_n b_n\}$, where $\delta_n = n^{-15}$. Then, for large n, we have $f(B_n) \subset B_{n+1}$, and B_n is contained in a simply connected Fatou component of f.*

Proof. Suppose that $z \in B_n$, in which case $z = -b_n + wb_n$ where $|w| < \delta_n$. We assume throughout this section that n is sufficiently large for various estimates

to hold. By (2.1), (3.14) and (3.15),

$$\frac{f(z)}{-b_{n+1}} = I_1 I_2,$$

where

(5.1)

$$I_1 = (1-w)\left(1+w\frac{b_n}{a_n-b_n}\right)^2 \prod_{k=3}^{n-1}\left\{\left(1+w\frac{b_n}{a_k^{\beta_k}-b_n}\right)^{2\alpha_k}\prod_{l=0}^{k-2}\left(1+w\frac{b_n}{a_k^{\mu_{k,l}}-b_n}\right)^2\right\},$$

and

$$I_2 = \prod_{l=1}^{n-2}\left(1+\frac{z}{a_n^{\mu_{n,l}}}\right)^2 \prod_{k=n}^{\infty}\left(1+\frac{z}{a_k^{\beta_k}}\right)^{2\alpha_k}\prod_{k=n+1}^{\infty}\prod_{l=0}^{k-2}\left(1+\frac{z}{a_k^{\mu_{k,l}}}\right)^2.$$

First consider I_1. This is a polynomial in w of degree T_n+2, by (3.9). Write

$$I_1 = 1+\sum_{j=1}^{T_n+2}\eta_j w^j, \quad \text{for } \eta_j \in \mathbb{C}.$$

Then

$$|I_1-1| \leq |\eta_1 w| + |\eta_2 w^2| + \cdots + |\eta_{T_n+2} w^{T_n+2}|. \tag{5.2}$$

We consider η_1. We have, by (3.2), (3.9), (3.14), (3.16), (3.17) and (3.19),

$$\begin{aligned}
|\eta_1| &= \left|1+2\frac{b_n}{b_n-a_n}+2\sum_{k=3}^{n-1}\left\{\alpha_k\frac{b_n}{b_n-a_k^{\beta_k}}+\sum_{l=0}^{k-2}\frac{b_n}{b_n-a_k^{\mu_{k,l}}}\right\}\right| \\
&= \left|1-T_n+2\sum_{k=3}^{n-1}\left\{\alpha_k\left(1+\frac{a_k^{\beta_k}}{b_n-a_k^{\beta_k}}\right)+\sum_{l=0}^{k-2}\left(1+\frac{a_k^{\mu_{k,l}}}{b_n-a_k^{\mu_{k,l}}}\right)\right\}\right| \qquad (5.3)\\
&= 2\left|\sum_{k=3}^{n-1}\left\{\alpha_k\frac{a_k^{\beta_k}}{b_n-a_k^{\beta_k}}+\sum_{l=0}^{k-2}\frac{a_k^{\mu_{k,l}}}{b_n-a_k^{\mu_{k,l}}}\right\}\right| \\
&\leq 4\left|n\alpha_{n-1}\frac{a_{n-1}^{\beta_{n-1}}}{a_n}+n^2\frac{a_{n-1}^{\mu_{n-1,n-3}}}{a_n}\right| \\
&\leq \exp(-e^{n/4}).
\end{aligned}$$

Note that the cancellation in (5.3) occurs because, due to the choice of b_n and T_n, $-b_n$ is very close to a critical point of f.

Next consider η_k, for $k>1$. Note that the coefficients of w in the product for I_1 have modulus at most $b_n/(a_n-b_n) = T_n/2 < n^3$. Moreover, the degree of I_1

is less than $2n^3$, and so the expansion of (5.1) contains less than $(2n^3)^k$ terms in w^k. Hence

$$|\eta_k| < (n^3)^k.(2n^3)^k < n^{7k}, \quad \text{for large } n.$$

Hence $|\eta_2 w^2| < n^{-16}$. The other terms in (5.2) decrease sufficiently quickly that

$$|I_1 - 1| < 2n^{-16}, \quad \text{for large } n. \tag{5.4}$$

Now we consider I_2. Observe that each term in the product for I_2 has modulus less than 1. Hence, since $-\log(1-x) \leq \log(1+2x)$, for $0 < x < 1/2$, we have, by (3.12), (3.16), (3.17) and (3.18),

$$\begin{aligned}
0 &< -\log|I_2| \\
&\leq 2\left(\sum_{l=1}^{n-2}\log\left(1+\frac{2|z|}{a_n^{\mu_{n,l}}}\right)+\sum_{k=n}^{\infty}\alpha_k\log\left(1+\frac{2|z|}{a_k^{\beta_k}}\right)+\sum_{k=n+1}^{\infty}\sum_{l=0}^{k-2}\log\left(1+\frac{2|z|}{a_k^{\mu_{k,l}}}\right)\right) \\
&\leq 8\left(\sum_{l=1}^{n-2}\frac{a_n}{a_n^{\mu_{n,l}}}+\sum_{k=n}^{\infty}\alpha_k\frac{a_n}{a_k^{\beta_k}}+\sum_{k=n+1}^{\infty}\sum_{l=0}^{k-2}\frac{a_n}{a_k^{\mu_{k,l}}}\right) \\
&\leq 16\left(\exp(-e^{n/2})+\exp(-e^{n/2})+\exp(-e^{n/2})\right) \\
&\leq \exp(-e^{n/4}).
\end{aligned}$$

Thus, $1 - 2\exp(-e^{n/4}) \leq |I_2| \leq 1$. This, together with (5.4), establishes the first part of the lemma. It follows from Montel's theorem that, for large n, B_n is contained in a Fatou component, which must be simply connected by Lemma 4.1. □

Remark. Let V_n be the Fatou component containing B_n. These Fatou components are distinct. For, suppose that $V_m = V_n$ with $m \neq n$. Because all the coefficients of z in (2.1) are real, the Fatou set $F(f)$ must be invariant under reflection in the real axis. Hence, all points on the negative real axis between B_n and B_m must be in V_m, as otherwise V_m would be multiply connected. This is a contradiction since these points include the zeros of f.

6. The simply connected Fatou components are fast escaping

In this section we first prove Theorem 2, and then we use this result to prove the following.

Lemma 6.1. *Let V_n, $n \in \mathbb{N}$, be the simply connected Fatou components defined at the end of Section 5. Then $V_n \subset A(f)$, for large n.*

Proof of Theorem 2. We need two facts about the maximum modulus function, $M(\rho)$, of a transcendental entire function. The first is well known, and the second is from [9, Lemma 2.2]:

$$\frac{\log M(\rho)}{\log \rho} \to \infty \text{ as } \rho \to \infty, \tag{6.1}$$

and there exists $R > 0$ such that

$$M(\rho^c) \geq M(\rho)^c, \quad \text{for } \rho \geq R,\ c > 1. \tag{6.2}$$

Fix $r_0 \geq R_0$ such that $M(r) > r$, for $r \geq r_0$. Whenever $r \geq r_0$ there is a unique $n \in \mathbb{N}$ such that $M^{n-1}(r_0) \leq r < M^n(r_0)$. Hence, since ϵ is nonincreasing, by (1.1) and (6.1)

$$\epsilon(r)r \geq \epsilon(M^n(r_0))M^{n-1}(r_0) \geq \epsilon(r_0)^{n+1}M^{n-1}(r_0) \to \infty \text{ as } n \to \infty.$$

Hence

$$\epsilon(r)r \to \infty \text{ as } r \to \infty. \tag{6.3}$$

By (6.1) and (6.3) we see that, given $k > 0$, we can ensure that

$$\frac{\log M(\epsilon(r)R)}{\log(\epsilon(r)R)} > k, \quad \text{for large } r,\ R \geq r. \tag{6.4}$$

A little algebra shows that this is equivalent to

$$M(\epsilon(r)R)^{\frac{-\log \epsilon(r)}{\log(\epsilon(r)R)}} > \epsilon(r)^{-k}, \quad \text{for large } r,\ R \geq r. \tag{6.5}$$

In (6.2) we replace c with $\log R/\log(\epsilon(r)R)$, and replace ρ with $\epsilon(r)R$. We obtain, using (6.5) with $k = 3$, that there exists $R_1 \geq R_0$ such that

$$M(R) \geq \epsilon(r)^{-3}M(\epsilon(r)R), \quad \text{for } R \geq r \geq R_1. \tag{6.6}$$

Note that we can assume that R_1 is sufficiently large that both $M(r) > r$, for $r \geq R_1/\epsilon(r)$, and also, by (6.3), that $\eta(r) > r$, for $r \geq R_1$.

We claim next that we have

$$\eta^k(r) \geq \epsilon(r)^{-k-1}M^k(\epsilon(r)r) > M^k(\epsilon(r)r), \quad \text{for } r \geq R_1,\ k \in \mathbb{N}. \tag{6.7}$$

This can be seen by induction. When $k = 1$ we have, by (6.6),

$$\eta(r) = \epsilon(r)M(r) \geq \epsilon(r)^{-2}M(\epsilon(r)r), \quad \text{for } r \geq R_1.$$

Hence, by induction, for $r \geq R_1$,

$$\begin{aligned}
\eta^{k+1}(r) &= \epsilon(\eta^k(r))M(\eta^k(r)) \\
&\geq \epsilon(M^k(r))M(\eta^k(r)) && \text{as } \epsilon \text{ is nonincreasing} \\
&\geq \epsilon(r)^{k+1}M(\eta^k(r)) && \text{by (1.1)} \\
&\geq \epsilon(r)^{k+1}M(\epsilon(r)^{-k-1}M^k(\epsilon(r)r)) && \text{by (6.7)} \\
&\geq \epsilon(r)^{k+1}\epsilon(r)^{-3k-3}M^{k+1}(\epsilon(r)r) && \text{by repeated use of (6.6)} \\
&\geq \epsilon(r)^{-(k+1)-1}M^{k+1}(\epsilon(r)r) && \text{as required.}
\end{aligned}$$

(Note that in the penultimate step we have used the fact that $\epsilon(r)r \leq M(\epsilon(r)r)$, for $r \geq R_1$.)

It follows from (6.7) that, for $r \geq R_1$, $\eta^n(r) \to \infty$ as $n \to \infty$. Define

$$A'(f) = \{z : \text{ there exists } \ell \in \mathbb{N} \text{ such that } |f^{n+\ell}(z)| \geq \eta^n(R') \text{ for } n \in \mathbb{N}\},$$

for $R' \geq R_1$. We complete the proof by showing that $A'(f) = A(f)$.

First, suppose that $z \in A(f)$, in which case for some $\ell \in \mathbb{N}$ we have

$$|f^{n+\ell}(z)| \geq M^n(R), \quad \text{for } n \in \mathbb{N},$$

and some R with $M(r) > r$, for $r \geq R$.

Choose $K \in \mathbb{N}$ such that $M^K(R) = R' \geq R_1$. Then

$$|f^{n+\ell+K}(z)| \geq M^{n+K}(R) = M^n(R') \geq \eta^n(R'), \quad \text{for } n \in \mathbb{N}.$$

Hence $z \in A'(f)$.

Conversely, suppose that $z \in A'(f)$, in which case for some $\ell \in \mathbb{N}$ and $R' \geq R_1$ we have

$$|f^{n+\ell}(z)| \geq \eta^n(R'), \quad \text{for } n \in \mathbb{N}.$$

Choose $K \in \mathbb{N}$ so that $M^K(\epsilon(R')R') = R \geq R_1$. Then, by (6.7).

$$|f^{n+\ell+K}(z)| \geq \eta^{n+K}(R') \geq M^{n+K}(\epsilon(R')R') \geq M^n(R), \quad \text{for } n \in \mathbb{N}.$$

Hence $z \in A(f)$. □

Finally, we give the

Proof of Lemma 6.1. For some large R_0 define, for $r > R_0$,

$$\epsilon(r) = \frac{1}{16n^6}, \quad \text{for } a_{n-1}(1+\delta_{n-1}) < r \leq a_n(1+\delta_n), \tag{6.8}$$

where $\delta_n = n^{-15}$ as in Lemma 5.1. Define also $\eta(r) = \epsilon(r)M(r)$, for $r \geq R_0$.

Suppose that $x' \in B_n \cap \mathbb{R} \subset V_n$, for some n. We can assume that n is chosen sufficiently large for various estimates in this section to hold. We claim that $x' \in A(f)$. Thus, by [10, Theorem 1.2], $V_n \subset A(f)$.

Our approach to proving this claim is as follows. Set $x = -x'$, recalling that $x > 0$. We first show that

$$|f(x')| \geq \frac{1}{16n^6} M(x). \tag{6.9}$$

It follows from this, and since $f(x') \in B_{n+1} \cap \mathbb{R}$, that

$$|f^m(x')| \geq \eta^m(x), \quad \text{for } m \in \mathbb{N}. \tag{6.10}$$

Second, we show that ϵ satisfies (1.1). Thus, by (6.10) and Theorem 2, $x' \in A(f)$, as required.

First we need to establish (6.9). Since $M(x) = f(x)$, we have

$$\frac{|f(x')|}{M(x)} = J_1 J_2 J_3,$$

where

$$J_1 = \prod_{k=3}^{n-1} \left\{ \left(\frac{a_k^{\beta_k} - x}{a_k^{\beta_k} + x} \right)^{2\alpha_k} \sum_{l=0}^{k-2} \left(\frac{a_k^{\mu_{k,l}} - x}{a_k^{\mu_{k,l}} + x} \right)^2 \right\},$$

$$J_2 = \left(\frac{a_n - x}{a_n + x} \right)^2,$$

and

$$J_3 = \prod_{k=n+1}^{\infty} \left\{ \left(\frac{a_k^{\beta_k} - x}{a_k^{\beta_k} + x} \right)^{2\alpha_k} \sum_{l=0}^{k-2} \left(\frac{a_k^{\mu_{k,l}} - x}{a_k^{\mu_{k,l}} + x} \right)^2 \right\} \left(\frac{a_n^{\beta_n} - x}{a_n^{\beta_n} + x} \right)^{2\alpha_n} \sum_{l=1}^{n-2} \left(\frac{a_n^{\mu_{n,l}} - x}{a_n^{\mu_{n,l}} + x} \right)^2.$$

We consider these three terms separately. By (3.10) and (3.19), the smallest term in J_1 is

$$\left(\frac{a_{n-1}^{\mu_{n-1,n-3}}-x}{a_{n-1}^{\mu_{n-1,n-3}}+x}\right)^2=\left(\frac{1-\frac{a_{n-1}^{\mu_{n-1,n-3}}}{x}}{1+\frac{a_{n-1}^{\mu_{n-1,n-3}}}{x}}\right)^2\geq\left(1-\frac{8a_{n-1}^{\mu_{n-1,n-3}}}{a_n}\right)\geq\left(1-\exp(-e^{n/4})\right).$$

Thus, recalling the estimates of Lemma 3.2, $J_1\geq\frac{1}{2}$.

Secondly, recalling that $x=b_n+\omega b_n$, with $|\omega|<\delta_n=n^{-15}$, we have for large n,

$$J_2=\left(\frac{a_n-b_n-\omega b_n}{a_n+b_n+\omega b_n}\right)^2=\left(\frac{2-\omega T_n}{2T_n+2+\omega T_n}\right)^2\geq\frac{1}{4n^6}.$$

Thirdly, by (3.10) and (3.18), the smallest term in J_3 is

$$\left(\frac{a_n^{\mu_n}-x}{a_n^{\mu_n}+x}\right)^2\geq\left(\frac{1-\frac{x}{a_n^{\mu_n}}}{1+\frac{x}{a_n^{\mu_n}}}\right)^2\geq\left(1-\frac{8a_n}{a_n^{\mu_n}}\right)\geq\left(1-\exp(-e^{n/4})\right).$$

Thus, recalling again the estimates of Lemma 3.2, $J_3\geq\frac{1}{2}$. This establishes our first claim.

Finally, we need to show that ϵ satisfies (1.1). We claim that $M(a_n(1+\delta_n))\leq a_{n+2}$. In a very similar way to the proof of (6.9), we can split $f(2b_n)/a_{n+1}$ into three terms, and show that $f(2b_n)/a_{n+1}<2^{12n^3}$. Hence, by (3.14), (3.16) and (3.17),

$$M(a_n(1+\delta_n))\leq M(2b_n)=f(2b_n)<2^{12n^3}a_{n+1}\leq a_{n+1}^2\leq a_{n+2}, \tag{6.11}$$

as required.

Suppose then that r is such that $a_{n-1}(1+\delta_{n-1})<r\leq a_n(1+\delta_n)$. Since ϵ is nonincreasing, we deduce that, for $k\in\mathbb{N}$,

$$\begin{aligned}\epsilon(M^k(r))&\geq\epsilon(M^k(a_n(1+\delta_n))),\\&\geq\epsilon(a_{n+2k}),&&\text{by (6.11)}\\&=\frac{1}{16(n+2k)^6}\geq\frac{1}{(16n^6)^{k+1}}=\epsilon(r)^{k+1}.\end{aligned}$$

Thus ϵ satisfies (1.1). This completes the proof of Lemma 6.1 and hence the proof of Theorem 1. □

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