Perpetually Dominating Large Grids

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Abstract. In the *Eternal Domination* game, a team of guard tokens initially occupies a dominating set on a graph G. A rioter then picks a node without a guard on it and attacks it. The guards defend against the attack: one of them has to move to the attacked node, while each remaining one can choose to move to one of his neighboring nodes. The new guards' placement must again be dominating. This attack-defend procedure continues perpetually. The guards win if they can eternally maintain a dominating set against any sequence of attacks, otherwise the rioter wins.

We study rectangular grids and provide the first known general upper bound for these graphs. Our novel strategy implements a square rotation principle and eternally dominates $m \times n$ grids with the guards' number converging to 20% of the nodes as m, n grow.

Keywords: Eternal Domination, Combinatorial Game, Two players, Graph Protection, Grid

1 Introduction

Protection and security needs have always remained topical throughout human history. Nowadays, patrolling a network of premises, forcefully defending against attacks and ensuring a continuum of safety are top-level affairs in any military strategy or homeland security agenda.

Going back in time, the *Roman Domination* problem was introduced in [23]: where should Emperor Constantine the Great have located his legions in order to optimally defend against attacks in unsecured locations without leaving another location unsecured? In computer science terms, the interest is in producing a placement of guards on a graph such that any node without a guard has at least one neighbor with two guards on it. In other words, we are looking for a *dominating set* of the graph (i.e. each node must have a guard on it or on at least one of its neighbors), but with some extra qualities. Some seminal work on this topic includes [15, 22].

The above modeling caters only for a single attack on an unsecured node. A natural question is to consider special domination strategies against a sequence of attacks on the same graph [5]. In this setting, (some of) the guards are allowed to move after each attack to defend against it and modify their overall placement. The difficulty here lies in establishing a robust guards' placement in order to retain domination after coping with each attack. Such a sequence of attacks can be of finite (i.e. a set of k consecutive attacks) or even *infinite* length.

In this paper, we focus on the latter. We wish to protect a graph against attacks happening indefinitely on its nodes. Initially, the guards are placed on some nodes of the graph such that they form a dominating set (a simple one; not a Roman one). Then, an attack occurs on an unguarded node. All the guards (may) now move in order to counter the attack: one of them moves to the attacked node, while each of the others moves to one of his neighboring nodes such that the new guards' placement forms again a dominating set. This scenario takes place ad infinitum. The attacker's objective is to devise a sequence of attacks which leads the guards to a non-dominating placement. On the other hand, the guards wish to maintain a sequence of dominating sets without any interruption. The *Eternal Domination* problem, studied in this paper, deals with determining the minimum number of guards such that they perpetually protect the graph in the above fashion. The focus is on rectangular grids, where we provide a first, up to our knowledge, upper bound.

1.1 Related Work

Infinite order domination was originally considered by Burger et al. [4] as an extension to finite order domination. Later on, Goddard et al. [12] proved some first bounds with respect to some other graph-theoretic notions (like independence and clique cover) for the one-guard-moves and all-guards-move cases. The relationship between eternal domination and clique cover is examined more carefully in [1]. There exists a series of other papers with several combinatorial bounds, e.g. see [13, 16, 17, 19, 21].

Regarding the special case of grid graphs, Chang [6] gave many strong upper and lower bounds for the domination number. Indeeed, Gonçalves et al. [14] proved Chang's construction optimal for rectangular grids where both dimensions are greater or equal to 16. Moving onward to eternal domination, bounds for $3 \times n$ [8], $4 \times n$ [2] and $5 \times n$ [24] grids have been examined, where for $3 \times n$ the bounds are almost tight and for $4 \times n$ exactly tight.

Due to the mobility of the guards in eternal domination and the breakdown into alternate turns (guards vs attacker), one can view this problem as a pursuit-evasion combinatorial game in the same context as *Cops & Robber* [3] and the *Surveillance Game* [10, 11]. In all three of them, there are two players who take turns alternately with one of them pursuing the other possibly indefinitely.

Besides, an analogous *Eternal Vertex Cover* problem has been considered [9, 18], where attacks occur on the edges of the graph. In that setting, the guards defend against an attack by traversing the attacked edge, while they move in order to preserve a vertex cover after each turn.

For an overall picture and further references on the topic, the reader is suggested to tend to a recent survey on graph protection [20].

1.2 Our Result

We make a first step towards answering an open question in [20] and show that, in order to ensure eternal domination in rectangular grids, only a linear number of extra guards is needed compared to the case of ensuring domination.

To obtain this result, we devise an elegantly unraveling strategy of successive (counter) clockwise rotations for the guards to perpetually dominate an infinite grid. This strategy is referred to as the *Rotate-Square* strategy. Then, we apply the same strategy to finite grids with some extra guards to ensure the boundary remains always guarded. Overall, we show $\lceil \frac{mn}{5} \rceil + \mathcal{O}(m+n)$ guards suffice to perpetually dominate a big enough $m \times n$ grid. This is the first general result for rectangular grids.

1.3 Outline

In Section 2, we define some basic graph-theoretic notions and *Eternal Domination* as a two-player combinatorial pursuit-evasion game. Forward, in Section 3, we describe the basic components of the Rotate-Square strategy and prove it can be used to dominate an infinite grid forever. Later, in Section 4 we show how the strategy can be adjusted to perpetually dominate finite grids by efficiently handling moving near the boundary and the corners. Finally, in Section 5, we shortly mention some concluding remarks and open questions.

2 Preliminaries

2.1 Graphs and Domination

Let $G = (V(G), E(G))$ be a simple connected graph. We denote an edge between two connected vertices, namely v and u, as $(u, v) \in E(G)$ (or equivalently (v, u)). The *open-neighborhood* of a subset of vertices $S \subseteq V(G)$ is defined as $N(S) = \{v \in V(G) \setminus S : \exists u \in S \text{ such that } (u, v) \in E(G)\}$ and the *closed-neighborhood* as $N[S] = S \cup N(S)$. A *path* of length $n \in \mathbb{N}$, namely P_n, is a graph where $V(P_n) = \{v_0, v_1, \ldots, v_{n-1}\}$ and $E(P_n) = \{(v_0, v_1), (v_1, v_2), \ldots (v_{n-2}, v_{n-1})\}$. The *Cartesian product* of two graphs G and H is another graph denoted $G \times H$ where $V(G \times H) = V(G) \times V(H)$ and two vertices (v, v') and (u, u') are adjacent if either $v = u$ and $(v', u') \in E(H)$ or $v' = u'$ and $(v, u) \in E(G)$. A *grid*, namely $P_m \times P_n$, is the Cartesian product of two paths of lengths $m, n \in \mathbb{N}$.

A set of vertices $S \subseteq V(G)$ is called a *dominating set* of G if $N[S] = V(G)$. That is, for each $v \in V(G)$ either $v \in S$ or there exists a node $u \in S$ $(u \neq v)$ such that $(u, v) \in E(G)$. A minimum-size such set, say S^*, is called a *minimum dominating set* of G and $\gamma(G) = |S^*|$ is defined as the *domination number* of G. For grids, we simplify the notation $\gamma(P_m \times P_n)$ to $\gamma_{m,n}$.

2.2 Eternal Domination

Eternal Domination can be regarded as a combinatorial pursuit-evasion game played on a graph G. There exist two players: one of them controls the *guards*, while the other controls the *rioter* (or *attacker*). The game takes place in *rounds*. Each round consists of two *turns*: one for the guards and one for the rioter.

Initially (round 0), the guard tokens are placed such that they form a dominating set on G. Then, without loss of generality, the rioter attacks a node without a guard on it. A guard, dominating the attacked node, must now move on it to counter the attack. Notice that at least one such guard exists because their initial placement is dominating. Moreover, the rest of the guards may move; a guard on node v can move to any node in $N[\{v\}]$. The guards wish to ensure that their modified placement is still a dominating set for G. The game proceeds in a similar fashion in any subsequent rounds. Guards win if they can counter any attack of the rioter and perpetually maintain a dominating set; that is, for an infinite number of attacks. Otherwise, the rioter wins if she manages to force the guards to reach a placement that is no longer dominating; then, an attack on an undominated node suffices to win.

Definition 1. $\gamma^\infty(G)$ *stands for the* eternal domination number *of a graph G, i.e. the minimum size of a guards' team that can eternally dominate G (when* all *guards can move at each turn).*

As above, we simplify $\gamma^\infty(P_m \times P_n)$ to $\gamma^\infty_{m,n}$. Since the initial guards' placement is dominating, we get $\gamma^\infty(G) \geq \gamma(G)$ for any graph G. By a simple rotation, we get $\gamma_{m,n} = \gamma_{n,m}$ and $\gamma^\infty_{m,n} = \gamma^\infty_{n,m}$. Finally, multiple guards are *not* allowed to lie on a single node, since this could provide an advantage for the guards, as proved in [7].

3 Eternally Dominating an Infinite Grid

In this section, we describe a strategy to eternally dominate an *infinite grid*. We denote an infinite grid as G_∞ and define it as a pair $(V(G_\infty), E(G_\infty))$, where $V(G_\infty) = \{(x, y) : x, y \in \mathbb{Z}\}$ and any node $(x, y) \in V(G_\infty)$ is connected to $(x, y-1)$, $(x, y+1)$, $(x-1, y)$ and $(x+1, y)$. In Figure 1 (and in the figures to follow), we depict the grid as a square mesh where each *cell* corresponds to a node of $V(G_\infty)$ and neighbors only four other cells: the one above, below, left and right of it. We assume row x is *above* row $x+1$ and column y is *left* of column $y+1$.

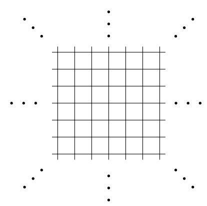


Fig. 1: The Infinite Grid G_∞

Initially, let us consider a family of dominating sets for G_∞. In the following, let $\mathbb{Z}^2 := \mathbb{Z} \times \mathbb{Z}$ and $\mathbb{Z}_5 := \{0, 1, 2, 3, 4\}$ stand for the group of integers modulo 5. We then define the function $f : \mathbb{Z}^2 \to \mathbb{Z}_5$ as $f(x, y) = x + 2y \pmod 5$ for any $(x, y) \in \mathbb{Z}^2$. This function appears in [6] and is central to providing an optimal dominating set for sufficiently large finite grids. Now, let $D_t = \{(x, y) \in$

$V(G_\infty) : f(x,y) = t\}$ for $t \in \mathbb{Z}_5$ and $\mathcal{D}(G_\infty) = \{D_t : t \in \mathbb{Z}_5\}$. For purposes of symmetry, let us define $f'(x,y) = f(y,x)$ and then $D'_t = \{(x,y) \in V(G_\infty) : f'(x,y) = t\}$ and $\mathcal{D}'(G_\infty) = \{D'_t : t \in \mathbb{Z}_5\}$.

Proposition 1. *Any $D_t \in \mathcal{D}(G_\infty)$ is a dominating set for G_∞.*

Proposition 2. *Any $D'_t \in \mathcal{D}'(G_\infty)$ is a dominating set for G_∞.*

Notice that the above constructions form *perfect* dominating sets, i.e. dominating sets where each node is dominated by *exactly* one other node, since for each node $v \in V(G_\infty)$ exactly one node from $N[\{v\}]$ lies in D_t (respectively D'_t) by the definition of D_t (respectively D'_t).

3.1 A First Eternal Domination Strategy

Let us now consider a *shifting-style* strategy as the simplest and most straightforward strategy to eternally dominate G_∞. The guards initially pick a placement D_t for some $t \in \mathbb{Z}_5$. Next, an attack occurs on some unguarded node. Since the D_t placement perfectly dominates G_∞, there exists exactly one guard adjacent to the attacked node. Thence, it is mandatory for him to move onto the attacked node. His move defines a direction in the grid: left, right, up or down. The rest of the strategy reduces to each guard moving according to the defined direction.

Altogether, the guards are all shifting toward the same direction. Therefore, they abandon their original D_t placement, but end up in a $D_{t'}$ placement where t' depends on t and the direction coerced by the attack. The last holds since moving toward the same direction has the same effect to the outcome of the $f(\cdot)$ function found in the definition of D_t. It is easy to see that the above strategy can be repeated after any attack of the rioter. Thus, the guards always occupy a placement in $\mathcal{D}(G_\infty)$ and, by Proposition 1, they dominate G_∞ perpetually.

The aforementioned strategy works fine for the infinite grid, as demonstrated above. Nonetheless, applying it (directly or modified) to a finite grid encounters many obstacles. Shifting the guards toward one course leaves some nodes in the very end of the opposite course (near the boundary) undominated, since there is no longer an unlimited supply of guards to ensure protection. To overcome this problem, we propose a different strategy whose main aim is to redistribute the guards without creating any bias to a specific direction.

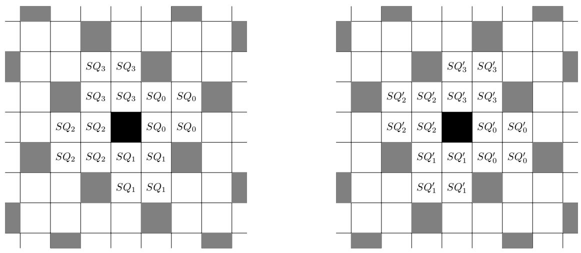
3.2 Empty Squares

The key idea toward another eternal domination strategy is to *rotate* the guards' placement around some *squares* (i.e. subgrids of size 2×2) such that, intuitively, the overall movement is zero and the guards always occupy a placement in $\mathcal{D}(G_\infty) \cup \mathcal{D}'(G_\infty)$ after an attack is defended.

Consider a node $(x,y) \in V(G_\infty)$, where $(x,y) \in D_t$ for some value t. Now, assume that the guards lie on the nodes dictated in D_t and thence form a dominating set. By looking around (x,y), we identify the existence of 4 *empty squares* (i.e. sets of 4 cells with no guard on them):

- $SQ_0 = \{(x-1, y+1), (x-1, y+2), (x, y+1), (x, y+2)\}$
- $SQ_1 = \{(x+1, y), (x+1, y+1), (x+2, y), (x+2, y+1)\}$
- $SQ_2 = \{(x, y-2), (x, y-1), (x+1, y-2), (x+1, y-1)\}$
- $SQ_3 = \{(x-2, y-1), (x-2, y), (x-1, y-1), (x-1, y)\}$

One can verify that, for every $(w, z) \in \bigcup_{i=0}^{3} SQ_i$, we get $f(w, z) \neq f(x, y)$ and thus $(w, z) \notin D_t$. Figure 2a demonstrates the above observation. Notice that (x, y) has exactly one neighbor in each of these squares and is the only guard who dominates these 4 neighbors, since the domination is perfect. Furthermore, an attack on the neighbor lying in SQ_i would mean the guard moves there and *slides* along an edge of $SQ_{(i+1) \bmod 4}$, i.e. both its current and previous position is neighboring to a node in $SQ_{(i+1) \bmod 4}$. For example, in Figure 2a, an attack on the bottom-right cell of SQ_3 would mean the guard slides along SQ_0. Finally, each square is protected by exactly 4 guards around it (one for each of its vertices) in a formation as seen in Figure 2a.



(a) The 4 empty squares around a node in D_t

(b) The 4 empty squares around a node in D'_t

Fig. 2: Empty Squares

The aforementioned observations also extend to a node (x, y) lying on a dominating set D'_t. We now define the 4 empty squares as follows (see Figure 2b):

- $SQ'_0 = \{(x, y+1), (x, y+2), (x+1, y+1), (x+1, y+2)\}$
- $SQ'_1 = \{(x+1, y-1), (x+1, y), (x+2, y-1), (x+2, y)\}$
- $SQ'_2 = \{(x-1, y-2), (x, y-1), (x, y-2), (x, y-1)\}$
- $SQ'_3 = \{(x-2, y), (x-2, y+1), (x-1, y), (x-1, y+1)\}$

Similarly to before, the squares are empty, since for every $(w, z) \in \bigcup_{i=0}^{3} SQ'_i$ we get $f'(w, z) \neq f'(x, y)$ and thus $(w, z) \notin D'_t$. The (x, y)-guard has exactly one neighbor in each of these squares and protecting an attack on SQ_i now means sliding along the edge of $SQ_{(i-1) \bmod 4}$. Finally, each square is protected by exactly 4 guards in a formation that looks like a clockwise step of the formation seen before for D_t.

3.3 The Rotate-Square Strategy

We hereby describe the *Rotate-Square* strategy and prove that it perpetually dominates G_∞. The strategy makes use of the empty squares idea and, once an attack occurs, the square along which the defence-responsible guard slides is identified as the *pattern square*. Then, the other 3 guards corresponding to the pattern square perform a (counter) clockwise step depending on the move of the defence-responsible guard. Let us break the guards' turn down into some distinct components to facilitate a formal explanation. Of course, the guards are always assumed to move concurrently during their turn. That is, they *centrally* compute the whole strategy move and then each one moves to the position dictated by the strategy at the same time.

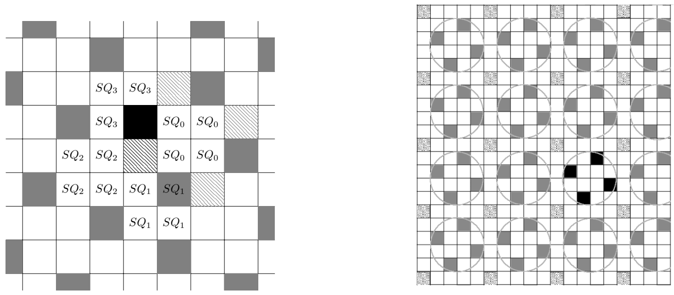
Initially, the guards are assumed to occupy a dominating set D in $\mathcal{D}(G_\infty) \cup \mathcal{D}'(G_\infty)$. Then, an attack occurs on a node in $V(G_\infty) \setminus D$. To defend against it, the guards apply *Rotate-Square*:

(1) Identify the defence-responsible guard; there is exactly one since the domination is perfect.
(2) Identify the pattern square SQ_j from the 4 empty squares around this guard.
(3) Rotate around SQ_j according to the defence-responsible guard's move.
(4) Repeat the rotation pattern in horizontal and vertical lanes in hops of distance 5.

Let us examine each of these strategy components more carefully. Step (1) requires looking at the grid and spotting the guard who lies on a neighboring node of the attack. In step (2), the pattern square is identified as described in the previous subsection following the $(i \pm 1) \bmod 4$ rule depending on the current dominating set (Figures 2a and 2b). In step (3), the 4 guards around the pattern square (including the defence-responsible guard) take a (counter) clockwise step based on the node to be defended. For an example, see Figure 3a: the defence-responsible guard (in black) defends against an attack on the bottom-right cell of SQ_3 by sliding along SQ_0 in clockwise fashion. Then, the other 3 guards around SQ_0 (in gray) take a clockwise step sliding along an SQ_0-edge as well. The latter happens in order to preserve that SQ_0 remains empty. Eventually, in step (4), the pattern square (SQ_0) is used as a guide for the move of the rest of the guards. Consider an SQ_0-guard initially lying on node (w, z). By construction of D_t, guards lie on all nodes $(w \pm 5\alpha, z \pm 5\beta)$ for $\alpha, \beta \in \mathbb{N}$, since adding multiples of 5 in both dimensions does not affect the outcome of $f(\cdot)$. In the end, all these corresponding guards mimic the move of (w, z), i.e. they move toward the same direction. This procedure is executed for all the guards of SQ_0. The rest of the guards, i.e. guards that do not correspond to any SQ_0-guard, remain still during this turn. We vizualise such an example in Figure 3b. The circles enclose the repetitions of the pattern square, where the original pattern square is given in black. The dotted nodes remain still during this turn.

Lemma 1. *Assume the guards occupy a dominating placement $D \subseteq V(G_\infty)$ in $\mathcal{D}(G_\infty) \cup \mathcal{D}'(G_\infty)$ and an attack occurs on a node in $V(G_\infty) \setminus D$. After applying the Rotate-Square strategy, the guards successfully defend against the attack and again form a dominating set in $\mathcal{D}(G_\infty) \cup \mathcal{D}'(G_\infty)$.*

Proof. In this proof, we are going to demonstrate that any of the 4 possible attacks (one per empty square) around a node in a D_t (or D'_t) placement can



(a) A clockwise step around SQ_0 in D_t

(b) A visualization of step (4) of Rotate-Square

Fig. 3: Steps (3) and (4) of Rotate-Square

be defended by Rotate-Square and, most importantly, the guards still occupy a placement in $\mathcal{D}(G_\infty) \cup \mathcal{D}'(G_\infty)$ after their turn. Below, in Figure 4, we provide pictorial details for 1 out of 8 cases (4 for D_t and 4 for D'_t); we need not care about the value of t, since all D_t (respectively D'_t) placements are mere shifts to each other. The defence-responsible guard is given in black, while the rest in gray. Their previous positions are observable by a slight shade. The guards with no shade around them are exactly the ones who do not move during their turn. Also, notice that the guards who are mimicking the strategy of the pattern square occupy positions $(w \pm 5\alpha, z \pm 5\beta)$ for $\alpha, \beta \in \mathbb{N}$, where (w, z) is the new position of a pattern square guard. Then, $f(w, z) = f(w \pm 5\alpha, z \pm 5\beta)$ and $f'(w, z) = f'(w \pm 5\beta, z \pm 5\alpha)$ since the modulo 5 operation cancels out the addition (subtraction) of 5α and 5β. A similar observation holds for the set of guards that stand still during their turn. We identify a model guard, say on position (a, b), and then the rest of such guards are given by $(a \pm 5\alpha, b \pm 5\beta)$. Again, the $f(\cdot)$ (respectively $f'(\cdot)$) values of all these nodes remain equal. For this reason, we focus below only on the pattern square and the model guards and demonstrate that they share the same value of $f(\cdot)$ (respectively $f'(\cdot)$).

We hereby consider a potential attack around a node $(x, y) \in D_t$.

Attack on $(x-1, y)$ (i.e. on SQ_3). We apply Rotate-Square around SQ_0. The four guards around SQ_0 and the model guard standing still move as follows (Figure 4): Let P stand for the set of new positions given in Table 1. The

Table 1: Attack on $(x-1, y)$ (rotate around SQ_0); Figure 4

Old Position (w, z)	New Position (w', z')	$f'(w', z')$ (mod 5)
(x, y)	$(x-1, y)$	$2x + y - 2$
$(x-2, y+1)$	$(x-2, y+2)$	$2x + y - 2$
$(x-1, y+3)$	$(x, y+3)$	$2x + y + 3$
$(x+1, y+2)$	$(x+1, y+1)$	$2x + y + 3$
$(x-3, y-1)$	$(x-3, y-1)$	$2x + y - 2$

guards now occupy positions $(w, z) \in P$ where $f'(w, z) = 2x + y - 2 \pmod 5 = 2x + y + 3 \pmod 5 = t'$. By this fact, we get $P \subseteq D'_{t'}$. Now, assume there exists a node $(w, z) \notin P$, but $(w, z) \in D'_{t'}$. Without loss of generality, we assume $w \in [x-3, x+1]$ and $z \in [y-1, y+3]$, since the configuration of the guards in this window is copied all over the grid by the symmetry of D_t or D'_t placements. Since $(w, z) \notin P$, this is a node with no guard on it. However, by construction, any such node is dominated by a *neighboring* node (w_1, z_1) with $f'(w_1, z_1) = t'$. Then, by assumption, $f'(w, z) = f'(w_1, z_1) = t'$, which is a contradiction because, by definition of $f'(\cdot)$, two neighboring nodes never have equal values.

All other cases can be proved in a similar fashion. Notice that an attack against a D_t placement leads to a $D'_{t'}$ placement for some t' and vice versa. □

An induction on the application of Lemma 1 provides the following Theorem.

Theorem 1. *The guards perpetually dominate G_∞ by following the Rotate-Square strategy starting from an initial dominating set in $\mathcal{D}(G_\infty) \cup \mathcal{D}'(G_\infty)$.*

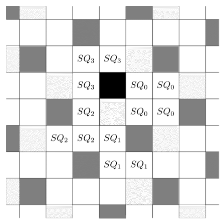


Fig. 4: Attack on SQ_3

4 Eternally Dominating Finite Grids

We now apply the Rotate-Square strategy to finite grids, i.e. graphs of the form $P_m \times P_n$. The idea is to follow the rules of the strategy, but to never leave any boundary or corner node without a guard on it. A finite $m \times n$ grid consists of nodes (i, j) where $i \in \{0, 1, 2, \ldots, m-1\}$ and $j \in \{0, 1, 2, \ldots, n-1\}$. Nodes $(0, x), (m-1, x), (y, 0), (y, n-1)$ for $x \in \{1, 2, \ldots, n-2\}$ and $y \in \{1, 2, \ldots, m-2\}$ are called *boundary nodes*, while nodes $(0, 0), (0, n-1), (m-1, 0), (m-1, n-1)$ are called *corner nodes*. Connectivity is similar to the infinite grid. However, boundary nodes only have three neighbors, while corner nodes only have two.

Let us consider the intersection of D_t and D'_t with $P_m \times P_n$, namely $V(t) = D_t \cap (P_m \times P_n)$ and $V'(t) = D'_t \cap (P_m \times P_n)$, respectively. We cite the following counting lemma from [6].

Lemma 2 (Lemma 2.2 [6]). $\lfloor \frac{mn}{5} \rfloor \leq |V(t)| \leq \lceil \frac{mn}{5} \rceil$ *holds for all t, and there exist t_0, t_1, such that $|V(t_0)| = \lfloor \frac{mn}{5} \rfloor$ and $|V(t_1)| = \lceil \frac{mn}{5} \rceil$ hold.*

The main observation in the proof of the above lemma is that there exist either $\lfloor \frac{m}{5} \rfloor$ or $\lfloor \frac{m}{5} \rfloor + 1$ D_t-nodes in one column of a $P_m \times P_n$ grid. Then, a case-analysis counting provides the above bounds. The same observation holds for D'_t, since f' is defined based on the same function $f : \mathbb{Z}^2 \to \mathbb{Z}_5$. Thence, we can extend the above lemma for D'_t cases with the proof being identical.

Lemma 3. $\lfloor \frac{mn}{5} \rfloor \leq |V'(t)| \leq \lceil \frac{mn}{5} \rceil$ *holds for all t, and there exist t_0, t_1, such that $|V'(t_0)| = \lfloor \frac{mn}{5} \rfloor$ and $|V'(t_1)| = \lceil \frac{mn}{5} \rceil$ hold.*

In order to study the domination number of $P_m \times P_n$, the analysis is based on examining $V(t)$, but for an extended $P_{m+2} \times P_{n+2}$ mesh. Indeed, Chang [6] showed the following:

Lemma 4 (Theorem 2.2 [6]). *For any* $m, n \geq 8$, $\gamma_{m,n} \leq \lfloor \frac{(m+2)(n+2)}{5} \rfloor - 4$.

The result follows by picking an appropriate D_t placement and forcing into the boundary of $P_m \times P_n$ the guards on the boundary of $P_{m+2} \times P_{n+2}$. Moreover, Chang showed how to eliminate another 4 guards; one near each corner.

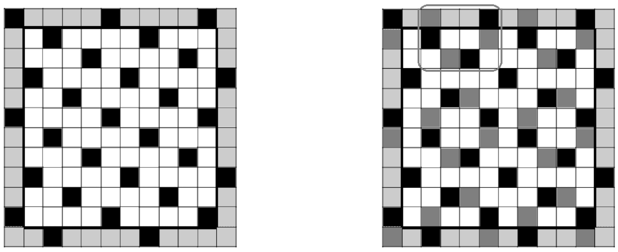
Below, to facilitate the readability of our analysis, we focus on a specific subcase of finite grids. We demonstrate an eternal dominating strategy for $m \times n$ finite grids where $m \bmod 5 = n \bmod 5 = 2$. Later, we discuss a simple extension to the general case.

The Strategy. Initially, we place our guards on nodes belonging to $V(t) = D_t \cap (P_m \times P_n)$ for some value of t. Unlike the approach in [6], we do not force inside any guards lying outside the boundary of $P_m \times P_n$. Since a sequence of attacks may force the guards to any $V(t)$ or $V'(t)$ placement (i.e. for any value of t), we pick an initial placement (say $V(t_1)$) for which $|V(t_1)| = \lceil \frac{mn}{5} \rceil$ to make sure there are enough guards to maintain domination while transitioning from one placement to the other. By Lemma 2, there exists such a placement. Moreover, we cover the whole boundary by placing a guard on each boundary or corner node with no guard on it (see Figure 5a; the gray nodes denote the places where the extra guards are placed). We refer to any of these added guards as a *boundary guard*. This concludes the initial placement of the guards.

The guards now follow Rotate-Square limited within the grid boundaries. For grid regions lying far from the boundary, Rotate-Square is applied in the same way as in the infinite grid case. For pattern square repetitions happening near the boundary or the corners, Rotate-Square's new placement demands can be satisfied by performing shifts of boundary guards. In other words, when a guard needs to step out of the boundary, another guard steps inside to replace him, while the boundary guards between them shift one step on the boundary. An example can be found in Figure 5b depicting a step of our strategy (from the black to the dark gray placement). Let us examine the designated window at the top of the boundary. Non-boundary guards move from the black to the dark gray positions, while boundary guards (in light gray) take a step rightward to make room for the dark gray guard moving in at the left and cover the black guard leaving the boundary at the right. Finally, black to dark gray transitions, where both nodes are on the boundary, mean the corresponding guards there simply do not move; there is no need to swap them. Overall, we refer to this slightly modified version of Rotate-Square as *Finite Rotate-Square*.

Lemma 5. *Assume* $m \bmod 5 = n \bmod 5 = 2$ *and that the guards follow Finite Rotate-Square, for an Eternal Domination game in* $P_m \times P_n$. *Then, after every turn, their new placement* P *is dominating, all boundary and corner nodes have a guard on them and, for some* t, *there exists a set* $V(t)$ *(or* $V'(t)$*) such that* $V(t) \subseteq P$ *(or* $V'(t) \subseteq P$*).*

Proof. Consider the $(m-2) \times (n-2)$ subgrid remaining if we remove the boundary. Since $m \bmod 5 = n \bmod 5 = 2$, $(m-2)$ and $(n-2)$ perfectly divide 5. The latter means that each row (respectively column) of the subgrid has exactly $\frac{n-2}{5}$



(a) An initial placement for the guards (b) Boundary guards' shifting

Fig. 5: Finite Rotate-Square

(respectively $\frac{m-2}{5}$) guards on it. Now, without loss of generality, consider row one neighboring the upper boundary row, which is row zero. Let us assume that a pattern square propagation obligates a row-one guard to move to the boundary. Then, by symmetry of the pattern square, there exists another guard on the boundary who needs to move downward to row one. Notice that the same holds for each of the $\frac{n-2}{5}$ guards lying on row one, since the pattern square move propagates in hops of distance 5. Movements in and out of the boundary alternate due to the shape of the pattern square. Moreover, we need not care about where the pattern square is "cut" by the left/right boundary since, due to $n-2$ perfectly diving 5, there are exactly $\frac{n-2}{5}$ full pattern squares occuring subject to shifting. Thence, we can apply the shifting procedure demonstrated in Figure 5b to apply the moves and maintain a full boundary, while preserving the number of guards on row one. For some case visualizations of the proof, check Figure 6: it suffices to look at 12×12 grids since for larger $m \times n$ grids with this property the patterns evolve similarly and so we can omit grid regions in the middle.

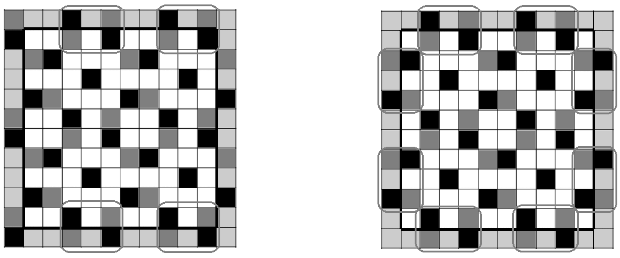


Fig. 6: Other examples of boundary shifting for Finite Rotate-Square

The new placement P is dominating, since the $(m-2) \times (n-2)$ subgrid is dominated by any $V(t)$ or $V'(t)$ placement and the boundary is always full of guards. Moreover, since we follow a modified Rotate-Square, P contains as a subset a node set $V(t)$ or $V'(t)$ after each guards' turn. $\square$

Lemma 6. *For $m, n \geq 7$ such that $m \bmod 5 = n \bmod 5 = 2$, $\gamma^{\infty}_{m,n} \leq \frac{mn}{5} + \frac{8}{5}(m+n) - \frac{24}{5}$ holds.*

Proof. By inductively applying Lemma 5, Finite Rotate-Square eternally dominates $P_m \times P_n$.

From the initial $V(t)$ placement, we get exactly $\frac{(m-2)(n-2)}{5}$ guards within $P_{m-2} \times P_{n-2}$, since $(m-2)$ and $(n-2)$ perfectly divide 5. Then, we need another $2(m+n)-4$ guards to cover the whole boundary. Overall, the guards sum to $\frac{(m-2)(n-2)}{5} + 2(m+n) - 4 = \frac{mn}{5} + \frac{8}{5}(m+n) - \frac{24}{5}$. □

So far, we focused on the special case where $m \bmod 5 = n \bmod 5 = 2$ and provided an upper bound for the eternal domination number. It is easy to generalize this bound for arbitrary values of m and n.

Lemma 7. *For $m, n \geq 7$, $\gamma^{\infty}_{m,n} \leq \frac{mn}{5} + \mathcal{O}(m+n)$ holds.*

Proof. The idea behind this bound is to thicken the boundary in the cases when $m \bmod 5 = n \bmod 5 = 2$ does not hold and then apply Finite-Rotate-Square as above. More formally, one can identify an $(m-i) \times (n-j)$ subgrid where $i, j \leq 5$ such that $(m-j) \bmod 5 = (n-i) \bmod 5 = 2$ and run the strategy there. For the rest of the rows and columns, they can be perpetually secured with $\mathcal{O}(m+n)$ extra guards. □

Gonçalves et al. [14] showed $\gamma_{m,n} \geq \lfloor \frac{(m+2)(n+2)}{5} \rfloor - 4$ for any $m, n \geq 16$. By combining this with Lemma 4, we get the exact domination number $\gamma_{m,n} = \lfloor \frac{(m+2)(n+2)}{5} \rfloor - 4$ for $m, n \geq 16$. Then, by using Lemma 7, our main result follows.

Theorem 2. *For any $m, n \geq 16$, $\gamma^{\infty}_{m,n} \leq \gamma_{m,n} + \mathcal{O}(m+n)$ holds.*

5 Conclusions

We demonstrated a first strategy to eternally dominate general rectangular grids based on the repetition of a rotation pattern.

Regarding further work, a more careful case-analysis of the boundary may lead to improvements regarding the coefficient of the linear term. On the bigger picture, it remains open whether this strategy can be used to obtain a constant additive gap between domination and eternal domination in large grids. Furthermore, the existence of a stronger lower bound than the trivial $\gamma^{\infty}_{m,n} \geq \gamma_{m,n}$ one also remains open.

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A Proofs of Section 3

Proof of Proposition 1. Let $(x, y) \in V(G_\infty)$ and $f(x, y) = t \in \{0, 1, 2, 3, 4\}$. We consider all possible cases for another node $(w, z) \in V(G_\infty)$:

- If $f(w, z) = t$, then $(w, z) \in D_t$.
- If $f(w, z) = t + 1 \pmod 5$, then $f(w - 1, z) = t$ and so $(w - 1, z) \in D_t$ dominates (w, z).
- If $f(w, z) = t - 1 \pmod 5$, then $f(w + 1, z) = t$ and so $(w + 1, z) \in D_t$ dominates (w, z).
- If $f(w, z) = t + 2 \pmod 5$, then $f(w, z - 1) = t$ and so $(w, z - 1) \in D_t$ dominates (w, z).
- If $f(w, z) = t - 2 \pmod 5$, then $f(w, z + 1) = t$ and so $(w, z + 1) \in D_t$ dominates (w, z).

□

Proof of Proposition 2. Let $(x, y) \in V(G_\infty)$ and $f'(x, y) = t \in \{0, 1, 2, 3, 4\}$. We consider all possible cases for another node $(w, z) \in V(G_\infty)$:

- If $f'(w, z) = t$, then $(w, z) \in D'_t$.
- If $f'(w, z) = t + 1 \pmod 5$, then $f'(w, z - 1) = t$ and so $(w, z - 1) \in D'_t$ dominates (w, z).
- If $f'(w, z) = t - 1 \pmod 5$, then $f'(w, z + 1) = t$ and so $(w, z + 1) \in D'_t$ dominates (w, z).
- If $f'(w, z) = t + 2 \pmod 5$, then $f'(w - 1, z) = t$ and so $(w - 1, z) \in D'_t$ dominates (w, z).
- If $f'(w, z) = t - 2 \pmod 5$, then $f'(w + 1, z) = t$ and so $(w + 1, z) \in D'_t$ dominates (w, z).

□

Proof addendum for Lemma 1. We hereby provide the tables for the other 7 move cases regarding potential attacks on pattern squares. We assume the defence responsible guard lies on node (x, y). In all cases, the new placement is in $\mathcal{D}'(G_\infty)$ if the previous one is in $\mathcal{D}(G_\infty)$ and vice versa, as demonstrated by the value of $f(\cdot)$ (respectively $f'(\cdot)$) in the following tables. □

Table 2: Attack on $(x, y - 1)$ (rotate around SQ_3); Figure 7a

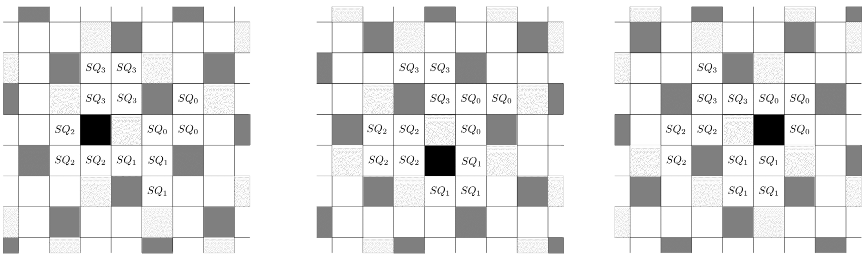
Old Position (w, z)	New Position (w', z')	$f'(w', z') \pmod 5$
(x, y)	$(x, y - 1)$	$2x + y - 1$
$(x - 1, y - 2)$	$(x - 2, y - 2)$	$2x + y - 1$
$(x - 3, y - 1)$	$(x - 3, y)$	$2x + y - 1$
$(x - 2, y + 1)$	$(x - 1, y + 1)$	$2x + y - 1$
$(x + 1, y + 2)$	$(x + 1, y + 2)$	$2x + y + 4$

Table 3: Attack on $(x+1,y)$ (rotate around SQ_2); Figure 7b

Old Position (w,z)	New Position (w',z')	$f'(w',z')$ (mod 5)
(x,y)	$(x+1,y)$	$2x+y+2$
$(x+2,y-1)$	$(x+2,y-2)$	$2x+y+2$
$(x+1,y-3)$	$(x,y-3)$	$2x+y-3$
$(x-1,y-2)$	$(x-1,y-1)$	$2x+y-3$
$(x-2,y+1)$	$(x-2,y+1)$	$2x+y-3$

Table 4: Attack on $(x,y+1)$ (rotate around SQ_1); Figure 7c

Old Position (w,z)	New Position (w',z')	$f'(w',z')$ (mod 5)
(x,y)	$(x,y+1)$	$2x+y+1$
$(x+1,y+2)$	$(x+2,y+2)$	$2x+y+1$
$(x+3,y+1)$	$(x+3,y)$	$2x+y+1$
$(x+2,y-1)$	$(x+1,y-1)$	$2x+y+1$
$(x-1,y+3)$	$(x-1,y+3)$	$2x+y+1$



(a) Attack on SQ_2 (b) Attack on SQ_1 (c) Attack on SQ_0

Fig. 7: Rotate-Square against attacks on a D_t placement

Table 5: Attack on $(x-1,y)$ (rotate around SQ'_2); Figure 8a

Old Position (w,z)	New Position (w',z')	$f(w',z')$ (mod 5)
(x,y)	$(x-1,y)$	$x+2y-1$
$(x-2,y-1)$	$(x-2,y-2)$	$x+2y-1$
$(x-1,y-3)$	$(x,y-3)$	$x+2y-1$
$(x+1,y-2)$	$(x+1,y-1)$	$x+2y-1$
$(x-3,y+1)$	$(x-3,y+1)$	$x+2y-1$

Table 6: Attack on $(x,y-1)$ (rotate around SQ'_1); Figure 8b

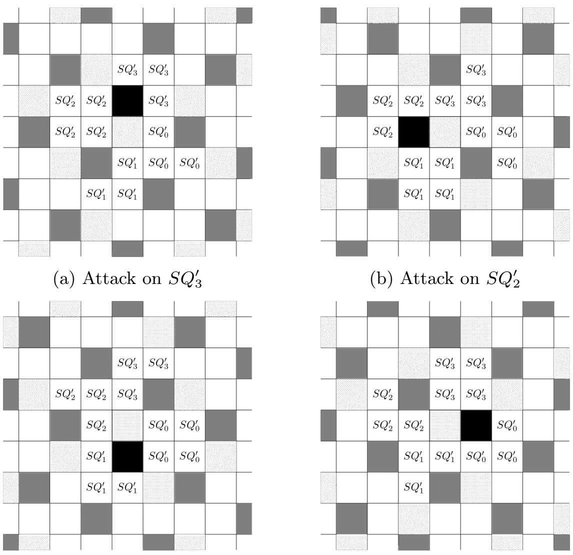
Old Position (w,z)	New Position (w',z')	$f(w',z')$ (mod 5)
(x,y)	$(x,y-1)$	$x+2y-2$
$(x+2,y+1)$	$(x+1,y+1)$	$x+2y+3$
$(x+3,y-1)$	$(x+3,y)$	$x+2y+3$
$(x+1,y-2)$	$(x+2,y-2)$	$x+2y-2$
$(x-1,y+2)$	$(x-1,y+2)$	$x+2y+3$

Table 7: Attack on $(x+1, y)$ (rotate around SQ'_0); Figure 8c

Old Position (w, z)	New Position (w', z')	$f(w', z')$ (mod 5)
(x, y)	$(x+1, y)$	$x+2y+1$
$(x+2, y+1)$	$(x+2, y+2)$	$x+2y+1$
$(x+1, y+3)$	$(x, y+3)$	$x+y+1$
$(x-1, y+2)$	$(x-1, y+1)$	$x+2y+1$
$(x-2, y-1)$	$(x-2, y-1)$	$x+2y-4$

Table 8: Attack on $(x, y+1)$ (rotate around SQ'_3); Figure 8d

Old Position (w, z)	New Position (w', z')	$f(w', z')$ (mod 5)
(x, y)	$(x, y+1)$	$x+2y+2$
$(x-1, y+2)$	$(x-2, y+2)$	$x+2y+2$
$(x-3, y+1)$	$(x-3, y)$	$x+2y-3$
$(x-2, y-1)$	$(x-1, y-1)$	$x+2y+-3$
$(x+1, y+3)$	$(x+1, y+3)$	$x+2y+2$



(a) Attack on SQ'_3

(b) Attack on SQ'_2

(c) Attack on SQ'_1

(d) Attack on SQ'_0

Fig. 8: Rotate-Square against attacks on a D'_t placement

Proof of Theorem 1. We prove by induction that the guards defend against any number of attacks and always maintain a placement in $\mathcal{D}(G_\infty) \cup \mathcal{D}'(G_\infty)$ after their turn.

In the first step, the guards apply Rotate-Square and by Lemma 1, they successfully defend against the first attack and now form another dominating set in $\mathcal{D}(G_\infty) \cup \mathcal{D}'(G_\infty)$.

Assume that i attacks have occured and the guards have successfully defended against all of them by following Rotate-Square. That is, they occupy a configuration in $\mathcal{D}(G_\infty) \cup \mathcal{D}'(G_\infty)$. The $(i+1)$-st attack now occurs and the guards again follow Rotate-Square and therefore defend against the attack and form another dominating set in $\mathcal{D}(G_\infty) \cup \mathcal{D}'(G_\infty)$ (by Lemma 1). $\square$