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The Complexity of the Empire Colouring Problem

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Abstract We investigate the computational complexity of the empire colouring problem (as defined by Percy Heawood in 1890) for maps containing empires formed by exactly $r > 1$ countries each. We prove that the problem can be solved in polynomial time using s colours on maps whose underlying adjacency graph has no induced subgraph of average degree larger than s/r. However, if $s \geq 3$, the problem is NP-hard even if the graph is a for forests of paths of arbitrary lengths (for any $r \geq 2$, provided $s < 2r - \sqrt{2r + \frac{1}{4}} + \frac{3}{2}$). Furthermore we obtain a complete characterization of the problem's complexity for the case when the input graph is a tree, whereas our result for arbitrary planar graphs fall just short of a similar dichotomy. Specifically, we prove that the empire colouring problem is NP-hard for trees, for any $r \geq 2$, if $3 \leq s \leq 2r - 1$ (and polynomial time solvable otherwise). For arbitrary planar graphs we prove NP-hardness if $s < 7$ for $r = 2$, and $s < 6r - 3$, for $r \geq 3$. The result for planar graphs also proves the NP-hardness of colouring with less than 7 colours graphs of thickness two and less than $6r - 3$ colours graphs of thickness $r \geq 3$.

Insert your abstract here. Include keywords, PACS and mathematical subject classification numbers as needed.

Keywords colouring · planar graphs · NP-hardness · algorithms

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1 Introduction

Let r and s be fixed positive integers. Assume that a partition is defined on the n vertices of a planar graph G. In this paper we usually call the blocks of such partition the *empires* of G and we assume that each block contains exactly r vertices. The graph G along with a partition of this type will be referred to as an *r-empire graph*. The *(s,r)-colouring* problem (s-COL$_r$) asks for a colouring of the vertices of G that uses at most s colours, never assigns the same colour to adjacent vertices in different empires and, conversely, assigns the same colour to all vertices in the same empire, disregarding adjacencies.

For $r = 1$, the problem coincides with the classical vertex colouring problem on planar graphs. The generalization for $r \geq 2$ was defined by Heawood [11] in the same paper in which he refuted a previous "proof" of the famous Four Colour Theorem. It has since been shown that $6r$ colours are always sufficient and in some cases necessary to solve this problem [13].

In [19] (also see [18]), we proved that $2r$ colours suffice and are sometimes needed to colour a collection of empires defined in an arbitrary tree. We also looked at the proportion of (s,r)-colourable trees on n vertices. We showed that, as n tends to infinity, for each r there exists a value s_r such that almost no tree can be coloured with at most s_r colours and, conversely, for s sufficiently larger than s_r, s colours are sufficient with (at least) constant positive probability. Later on [6] we improved on this showing that, as n tends to infinity, the minimum value s for which a random tree is (s,r)-colourable is concentrated in a very short interval with high probability.

Although our investigation considerably expanded the state of knowledge on s-COL$_r$, it failed to shed light on its computational complexity. Heawood [11] was the first to argue that there is a simple algorithm that can find a $(6r,r)$-colouring in any planar graph G in polynomial time. The same process uses at most $2r$ colours if G is a tree. But what if we only have r available colours? How difficult is it to decide whether G has an (r,r)-colouring? In this paper we show that s-COL$_r$ can be solved in polynomial time on planar graphs containing no induced subgraph of average degree greater than s/r. This implies that, for instance, $(2r-1)$-COL$_r$ (resp. $(6r-1)$-COL$_r$) can be solved in polynomial time on forests consisting of paths of length at most $2r-1$ (resp. planar graphs with components of size at most $12r$). Unfortunately, the outcome of our investigation seems to indicate that such algorithmic results cannot be extended much further. If $r \geq 2$ and $s \geq 3$, we prove that s-COL$_r$ NP-hard on linear forests if $s < 2r - \sqrt{2r + \frac{1}{4}} + \frac{3}{2}$. Furthermore, the hardness extends to $s < 6r - 3$ (resp. $s < 7$) when $r \geq 3$ (resp. for $r = 2$) on arbitrary planar graphs. Finally, for trees, our argument entails a nice dichotomy: s-COL$_r$ is NP-hard for any fixed $r \geq 2$, if $s \in \{3, \ldots, 2r-1\}$ and solvable in polynomial time for any other positive value of s.

The hardness proofs mentioned above hinge on the fact that the connectivity within empires has no effect on the graph colourability. Essentially, to find an (s,r)-colouring in a planar graph G, it suffices to be able to colour with at most s distinct colours (in such a way that no two distinct vertices connected by an edge receive the same colour) its *reduced graph* $R_r(G)$. This is a (multi)graph obtained by contracting each empire to a distinct pseudo-vertex and adding an edge between a pair of pseudo-vertices u and v for each edge connecting two vertices in the original graph, one belonging to the empire represented by u, the other one to that represented by v. The algorithmic results are based on the use of simple minimum degree greedy colouring strategies [11] or more refined heuristics providing algorithmic proofs (see [10, Theorem 7.9] or [15, Exercises 9.12, 9.13]) of the well-known Brooks theorem [4] on such reduced graphs.

The reader at this point may question the reasons for studying this type of colourings. Our main interest in the problem comes from its relationship with other important colouring problems. Each instance of s-COL$_r$ can be translated to an instance of the classical colouring problem, but it is not clear to what extent the two problems are equivalent. The empire colouring problem is also related to the problem of colouring graphs of given thickness (a graph has *thickness* t [12,20,17], if t is the minimum integer such that its edges can be partitioned into at least t planar graphs). Bipartite graphs can have high thickness [3] but only need two colours, and on the other hand a graph of thickness t may have chromatic number as larger as $6t$. Theorem 12 in this paper implies that deciding whether a graph of thickness $t \geq 3$ can be coloured with $s < 6t - 3$ colours is NP-hard.

The rest of the paper is organized as follows. In Section 2 we present our positive results concerning sparse planar graphs. We then move on (Section 3) to describe a new reduction from the well-known satisfiability problem to the problem of colouring a particular type of graph. Hardness results for the colourability of these graphs will be instrumental to our main results. The next Section is devoted to the definition and analysis of a number of gadgets that will be used in the subsequent reductions. Section 5 deals with the hardness result for forests of paths. The last two sections deal with the hardness results for trees and arbitrary planar graphs.

Let k and s be positive integers greater than two. In what follows k-SAT (resp. s-COL) denotes the well known [9,14] NP-complete problem of checking the satisfiability of a k-CNF boolean formula (resp. deciding whether the vertices of a graph G can be coloured using at most s distinct colours in such a way that no edge of G connects two vertices of the same colour). Also, if Π is a decision problem and $\mathcal{I}$ is a particular set of instances for it, then $\Pi(\mathcal{I})$ will denote the restriction of Π to instances belonging to $\mathcal{I}$. If Π_1 and Π_2 are decision problems, then $\Pi_1 \leq_p \Pi_2$ will denote the fact that Π_1 is polynomial-time reducible to Π_2. Unless otherwise stated we follow [8] for all our graph-theoretic notations.

2 Algorithms

The main outcome of our work is that the empire colouring problem is much harder than the problem of colouring planar graphs in the classical sense. However there are cases where things are easy. Let σ be an arbitrary positive real number. In the following result SPARSE(σ) denotes the class of planar graphs G containing no induced subgraph of average degree larger than σ.

Theorem 1 *Let r be an arbitrary positive integer and σ be a positive real number such that $r\sigma$ is a whole number. The decision problem $r\sigma$-COL$_r$(SPARSE(σ)) can be solved in polynomial time.*

Proof Let r and σ be two positive numbers satisfying the assumptions above, and assume that $G \in$ SPARSE(σ), and its vertex set is partitioned into empires of size r.

If $R_r(G)$ contains a copy of $K_{r\sigma+1}$ then there can be no $(r\sigma, r)$-colouring of G. We now argue that if $R_r(G)$ does not contain a copy of $K_{r\sigma+1}$ then it is $r\sigma$-colourable (and therefore G admits an $(r\sigma, r)$-colouring).

Let S be a connected component of $R_r(G)$. In what follows we denote by G^S the subgraph of G such that $R_r(G^S) \equiv S$. Because all edges of S are edges in G^S, the average degree of this graph satisfies

$$|E(S)| = |E(G^S)| = \frac{d(G^S) \cdot |V(G^S)|}{2}.$$

From this, using the fact that $|V(S)| = |V(G^S)|/r$ and the definition of SPARSE(σ), we have

$$|E(S)| \leq \frac{r\sigma}{2} \cdot |V(S)|.$$

This implies that the average degree of S is at most $r\sigma$. It follows that S is either a regular graph of degree $r\sigma$ or it must contain at least a vertex of degree less than $r\sigma$. In the former case S can be coloured with $r\sigma$ colours using, say, the algorithm in the proof of Brooks' Theorem described in [10]. If S contains a vertex of degree less than $r\sigma$ we argue that, in fact, the assumptions about the average degree of all subgraphs of G imply that any induced subgraph of S is either $r\sigma$-regular or, in turn, contains a vertex of degree at most $r\sigma - 1$. Assume that some induced subgraph of S, S' is not $r\sigma$-regular and its minimum degree is at least $r\sigma$. This implies that in particular $d(S') \geq r\sigma$. But, by the assumptions on G the average degree of S' cannot exceed $r\sigma$. Therefore $d(S') = r\sigma$ and this implies S' must contain a vertex of degree less than $r\sigma$. $\square$

The result above has a number of interesting consequences. Let k be a positive integer. Any induced subgraph on n vertices of a forest of paths of length at most k cannot span more than $kn/(k+1)$ edges. Hence Theorem 1 implies, for instance, that $\left\lceil \frac{2kr}{k+1} \right\rceil$-COL$_r$ can be decided in polynomial time for forests of paths of length at most k. Similarly $(6r-1)$-COL$_r$ can be decided in polynomial time for graphs G formed by arbitrary planar components of size at most $12r$.

Theorem 1 also implies that the minimum s for which G admits an (s,r)-colouring can be determined in polynomial time for any $G \in \mathrm{SPARSE}(\sigma)$, with $r\sigma \leq 3$.

3 A Useful Reduction

Let s and k be positive integers with $s > \max(2,k)$. Also, let n and m be positive integers. An (s,k)-*formula graph* is an undirected graph Φ such that $V(\Phi) = \mathcal{T} \cup \mathcal{C} \cup \mathcal{A}$ where $\mathcal{T} = \{T, F, X^1, \ldots, X^{s-2}\}$, $\mathcal{C}$ contains m groups of vertices $\{c^{1,1}, \ldots, c^{1,s-1}\}$, $\{c^{2,1}, \ldots c^{2,s-1}\}, \ldots, \{c^{m,1}, \ldots, c^{m,s-1}\}$ and $\mathcal{A}$ is a set of $2n$ vertices paired up in some recognizable way. In particular, in what follows we will denote the elements of $\mathcal{A}$ by $a_1, \ldots, a_n, \overline{a_1}, \ldots, \overline{a_n}$, and we will say that for each $i \in \{1, \ldots, n\}$, a_i and $\overline{a_i}$ are a *pair of complementary vertices.* Set $\mathcal{T}$ spans a complete graph; for each pair of complementary vertices a and $\overline{a}$, $\{a, \overline{a}, X^j\}$ spans a complete graph for each $j \in \{1, \ldots, s-2\}$; for each $i \in \{1, \ldots, m\}$, $\{T, c^{i,1}, \ldots, c^{i,s-1}\}$ spans a complete graph and if $j \in \{1, \ldots, k\}$ then there is a single edge connecting $c^{i,j}$ to some vertex in $\mathcal{A}$, else if $j \geq k+1$ then $\{c^{i,j}, F\} \in E(\Phi)$. Figure 1 gives a simple example of a $(5,3)$-formula graph.

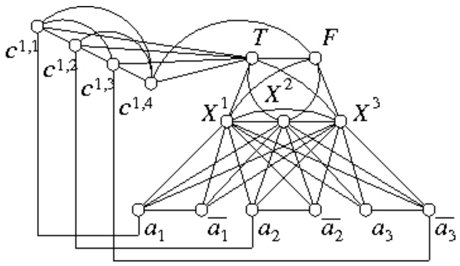

Fig. 1 A small formula graph

Let $\mathrm{FG}(s,k)$ denote the class of all (s,k)-formula graphs. We will now describe a reduction from k-SAT to the problem of colouring using at most s distinct colours the vertices of a given (s,k)-formula graph. The reduction shows the NP-hardness of s-COL(FG(s,k)) for any $k \geq 3$ and $s > k$. This in turn will be used repeatedly to prove our hardness results on s-COL$_r$.

Theorem 2 *Let s be an integer with $s \geq 3$. Then k-SAT $\leq_p$ s-COL(FG(s,k)) for any positive integer $k < s$.*

Proof Given a k-CNF formula $\phi \equiv C_1 \wedge \ldots \wedge C_m$ where C_i is the disjunction of k literals $\mathsf{c}^{i,1}, \ldots, \mathsf{c}^{i,k}$ for each $i \in \{1, \ldots, m\}$, we devise an (s,k)-formula graph Φ that admits an s-colouring if and only if ϕ is satisfiable. The graph

Φ will consist of one *truth gadget*, one *variable gadget* for each variable in ϕ, and one *clause gadget* for each clause in ϕ.

The truth gadget is a complete graph on s vertices labelled T, F, and $X^1, \ldots, X^{s-2}$. Note that every vertex in this gadget must be given a different colour in any s-colouring. Hence w.l.o.g. we call these colours "TRUE", "FALSE", "OTHER^1", ..., "OTHER^{s-2}" respectively. For each variable a of ϕ the variable gadget consists of two complementary vertices labelled a, and $\overline{a}$, connected by an edge and also adjacent to $X^1, \ldots, X^{s-2}$. There are therefore only two ways to colour a and $\overline{a}$: either a is TRUE and $\overline{a}$ is FALSE or a is FALSE and $\overline{a}$ is TRUE. Thus the two colourings of a and $\overline{a}$ encode the two truth-assignments of the variable a. Each clause $\mathsf{c}^{i,1} \vee \ldots \vee \mathsf{c}^{i,k}$ will be represented by $s+k+1$ vertices of Φ. Of these, k will correspond to the clause literals and will be labelled $c^{i,1}, \ldots, c^{i,k}$, $s-1-k$ will be labelled $c^{i,k+1}, \ldots, c^{i,s-1}$, and the remaining $k+2$ will be k vertices from variable gadgets and the vertices T and F from the truth gadget. Vertices $T, c^{i,1}, \ldots, c^{i,s-1}$ form a clique and, furthermore, for each $j \in \{1, \ldots, k\}$, the vertex $c^{i,j}$ is connected to the corresponding literal in a variable gadget. For $k \leq s-2$ vertices $c^{i,j}$, for $j \in \{k+1, \ldots, s-1\}$, are adjacent to F. Note that, in any colouring of a clause gadget, vertices $c^{i,j}$, for $j \leq k$, cannot have the same colour of vertex T, and vertices $c^{i,j}$ for $j \geq k$ cannot be coloured like F either. The reader can readily verify that $\Phi \in$FG(s,k). The graph in Figure 1 is the $(5,3)$-formula graph corresponding to the formula ϕ consisting of the single clause $\mathsf{a}_1 \vee \mathsf{a}_2 \vee \overline{\mathsf{a}_3}$.

If ϕ is satisfiable, the elements of $\mathcal{A}$ in Φ can be assigned a colour in $\{$TRUE, FALSE$\}$ so that, for each $i \in \{1, \ldots, m\}$ at least one of the $c^{i,j}$ (say for $j = j^*$) is adjacent to some literal coloured TRUE. This implies that c^{i,j^*} can be coloured FALSE, while all other $c^{i,j}$ for $j \in \{1, \ldots, s-1\} \setminus \{j^*\}$ can be assigned a distinct colour in $\{$OTHER^1, OTHER$^2, \ldots,$ OTHER$^{s-2}\}$. Conversely if there is no way to colour $\mathcal{A}$ so that for each $i \in \{1, \ldots, m\}$ at least one of the $c^{i,j}$ is adjacent to some literal coloured TRUE, then the clause gadget will need $s+1$ colours as the $s-1$ vertices $c^{i,j}$ only have $s-2$ colours available (as TRUE and FALSE are used up by T, F, and the corresponding literals). From this we can see that Φ admits an s-colouring if and only if there is some way to assign the variables of ϕ as TRUE or FALSE in such a way that every clause contains at least one TRUE literal. □

4 Gadgetry

Before moving to our hardness results it is convenient to introduce a number of gadgets.

Clique Gadgets. Let r and s be positive integers with $s < 2r$. In what follows the *clique gadget* $B_{r,s}$ is an r-empire graph satisfying the following properties.

B0 $B_{r,s}$ has $r(s+1)$ vertices partitioned into $s+1$ empires of size r.
B1 The graph $B_{r,s}$ is a forest consisting of r paths.
B2 No path in the graph $B_{r,s}$ contains two vertices from the same empire.

B3 The reduced graph of $B_{r,s}$ contains a copy of K_{s+1}. Hence $B_{r,s}$ admits an $(s+1,r)$-colouring and cannot be coloured with fewer colours.

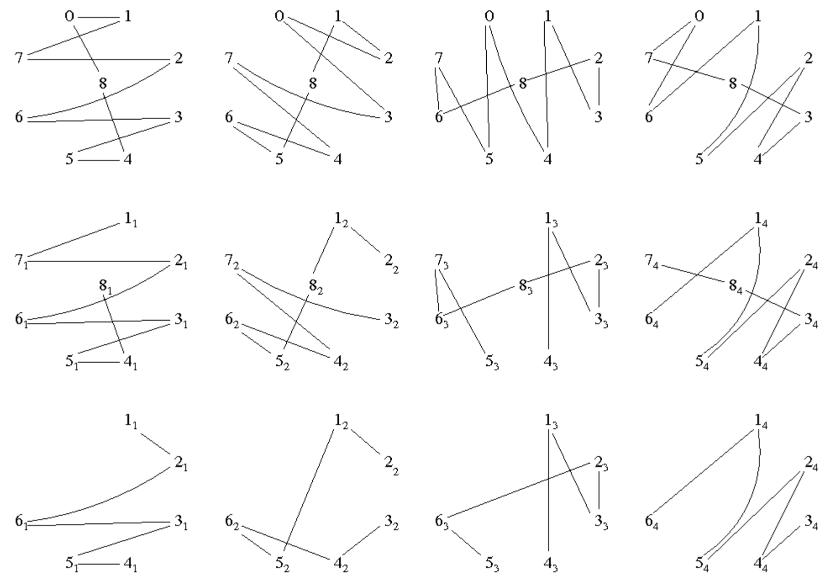

Fig. 2 Top row: Decomposition of K_9 into Hamiltonian cycles. Middle row: $B_{4,7}$. Bottom row: $B_{4,5}$.

Theorem 3 *Let r and s be positive integers with $s < 2r$. Then there exists an r-empire graph $B_{r,s}$ satisfying properties* **B0**, **B1**, **B2**, **B3**. *Furthermore $B_{r,s}$ can be constructed in time polynomial in r.*

Proof For any positive integer r, the clique K_{2r+1} can be decomposed into r edge-disjoint Hamiltonian cycles. The result, reported in [5, p. 71], is attributed to Walecki (see [16]). A dummy ∞ is added to the vertex set of K_{2r+1}. The sequence

$$0, 1, 2r-1, 2, 2r-2, 3, 2r-3, \ldots, r-1, r+1, r, \infty$$

can be seen as a Hamiltonian cycle of K_{2r+1} after label "∞" is identified with vertex $2r$. The remaining cycles are obtained as cyclic rotations of the first one.

Given one such decomposition (see top row in Figure 2) we define $B_{r,2r-1}$ (see middle part of Figure 2) by copying cycle i from the decomposition onto vertices $0_i, \ldots, (2r)_i$, and then taking the induced graph formed by deleting the vertex 0_i from the cycle on $0_i, \ldots, (2r)_i$. Also, if $r > 1$, for any $s \in \{1, \ldots, 2r-2\}$ graph of $B_{r,s}$ is obtained from that of $B_{r,s+1}$, by removing all vertices in the empire labelled $\mathbf{s+2}$ and adding an edge $\{u, v\}$ whenever u and v are the only two neighbours of $(s+2)_i$. $\square$

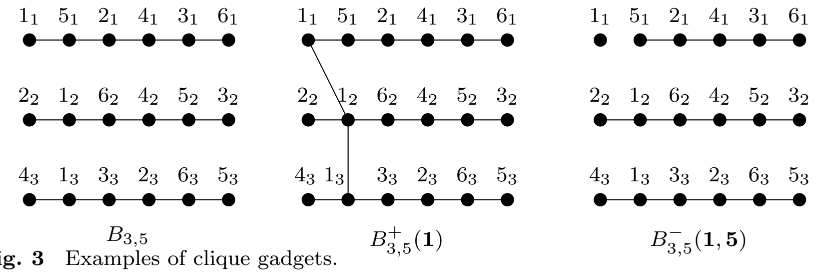

Fig. 3 Examples of clique gadgets.

Our results on trees will also need variants of these gadgets having particular connectivity features. Thus if $r > 1$ and $\mathbf{v} \equiv \{v_1, \ldots, v_r\}$ is some set of r vertices, the *connected clique gadget rooted at* $\mathbf{v}$, $B^+_{r,s}(\mathbf{v})$, is formed from $B_{r,s}$, as defined in Theorem 3, by adding edges $\{v_i, v_{i+1}\}$ for all i such that $1 \le i \le r-1$. Note that the graph of such gadget is a tree. Furthermore $B^+_{r,s}(\mathbf{v})$ still satisfies **B0**, and **B3**. Finally, if $\mathbf{u}$ and $\mathbf{v}$ are two empires, the $(\mathbf{u}, \mathbf{v})$-*colour constraining gadget* $B^-_{r,s}(\mathbf{u}, \mathbf{v})$ is an r-empire graph obtained from $B_{r,s}$, without loss of generality, by removing a single edge connecting the end-point u_1 of a path to its neighbour v_1. Thus u_1 becomes isolated in the graph of $B^-_{r,s}(\mathbf{u}, \mathbf{v})$. The graph $R_r(B^-_{r,s}(\mathbf{u}, \mathbf{v}))$ contains a copy of K_{s-1} in which every vertex is also adjacent to the vertices corresponding to $\mathbf{u}$ and $\mathbf{v}$. Thus any (s, r)-colouring of $B^-_{r,s}(\mathbf{u}, \mathbf{v})$ must give $\mathbf{u}$ and $\mathbf{v}$ the same colour. Figure 3 gives a few examples. In the remainder of the paper we will often need to describe schematically the colour constraining gadgets. Figure 4 gives an example of the graphical notation that will be used.

Fig. 4 A schematic representation of a $(\mathbf{u}, \mathbf{v})$-colour constraining gadget. the diagram shows the isolated vertex in empire $\mathbf{u}$. The two dashed blobs denote, respectively, the other vertices in $\mathbf{u}$ and the vertices in $\mathbf{v}$. The thick black line stands for the part of the gadget constraining the colour of $\mathbf{u}$ and $\mathbf{v}$: the two empires must be given the same colour in any s-colouring of $B^-_{r,s}(\mathbf{u}, \mathbf{v})$.

Connectivity Gadgets. For positive integers r, s and m with $r \ge 2$ and $s \ge 3$, the *connectivity gadget*, denoted by $A_{r,s,m}$, is an r-empire graph satisfying the following conditions:

A0 The graph $A_{r,s,m}$ contains $O(r^2 sm)$ vertices split into empires of size r.
A1 The graph $A_{r,s,m}$ is a linear forest.
A2 There is a set of at least m isolated vertices in the graph of $A_{r,s,m}$ and such vertices must be given the same colour in any (s, r)-colouring of $A_{r,s,m}$. These vertices define the so called *monochromatic set* of the gadget and

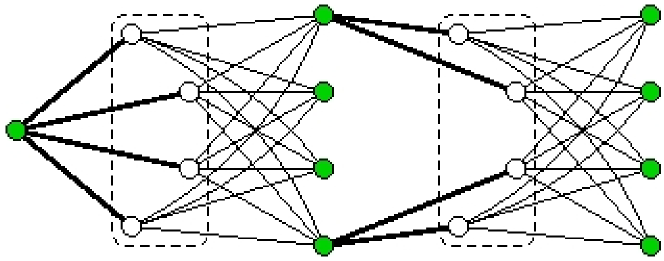

Fig. 5 The graph $E_{5,4,2}$. Vertices in the monochromatic set are greyed, the cliques connecting the colour constraining vertices are shown as dashed shapes, whereas the edges connecting a clique to the plug vertex or socket vertices used in its place are shown as thick lines.

will collectively be denoted by Z. The elements of such set will be generally denoted by z.

Let q and t be arbitrary positive integers. In what follows $E_{s,q,t}$ is a (non-empire) graph satisfying the following properties:

E0 $E_{s,q,t}$ contains $(s+q-1)t+1$ vertices.
E1 $E_{s,q,t}$ contains a set of $qt+1$ *monochromatic vertices*. Each of these must be given the same colour in any proper s-colouring of the graph. Among these we identify a *plug vertex* which we denote by u^0, and q *socket vertices* denoted by $u^1,\dots,u^q$, all of degree exactly $s-1$. The other $q(t-1)$ monochromatic vertices are termed *internal monochromatic vertices*. The remaining $(s-1)t$ vertices in $E_{s,q,t}$ are called *colour constraining vertices*, and usually denoted by the letter w, appropriately indexed.
E2 The maximum degree of $E_{s,q,t}$ is at most $s+q-1$.
E3 When $s-1$ and q are both even, every vertex in the graph has even degree.

Figure 5 shows the graph $E_{5,4,2}$. Graphs $E_{s,q,t}$ will "guide" the constrution of gadgets $A_{r,s,m}$ in the sense that for each r, s, and m there will be values of q and t such that $E_{s,q,t}$ will be the reduced graph of $A_{r,s,m}$.

Lemma 1 *Let s, q and t be positive integers such that $s \geq 3$, and $q \geq \sqrt{s-1}$. Then there exists a graph $E_{s,q,t}$ satisfying conditions* **E0**, **E1**, **E2**, *and* **E3**.

Proof The graph $E_{s,q,1}$ consists of a *plug vertex* u^0, $s-1$ *colour constraining vertices* $w^1,\dots,w^{s-1}$, and q *socket vertices* $u^1,\dots,u^q$. We can see immediately that condition **E0** is satisfied. The edges of $E_{s,q,1}$ are defined as follows: there is a clique on the $s-1$ vertices $w^1,\dots,w^{s-1}$, also for every $i \in \{0,\dots,q\}$ and $j \in \{1,\dots,s-1\}$ there is an edge $\{u^i,w^j\}$. In any proper s-colouring of $E_{s,q,1}$ the vertices $u^0,\dots,u^q$ must use a colour not used by the $s-1$ vertices in the clique, condition **E1** follows from this. The vertices $w^1,\dots,w^{s-1}$ all have degree $s+q-1$ while the vertices $u^0,\dots,u^q$ all have degree $s-1$, conditions **E2** and **E3** follow from this.

For $t>1$, assume that we already have a graph $E_{s,q,t-1}$ satisfying all the required conditions. To create the graph $E_{s,q,t}$, we add $E_{s,q,1}$, with its plug vertex removed, to $E_{s,q,t-1}$, and we use the socket vertices of $E_{s,q,t-1}$ to connect the two graphs. More precisely, the vertices of $E_{s,q,t}$ are

$$V(E_{s,q,t-1}) \cup \left(V(E_{s,q,1}) \backslash \{u^0\}\right).$$

Note that **E0** is satisfied and $E_{s,q,t}$ contains a single plug vertex and $s-1$ socket vertices. In what follows $w^1, \ldots, w^{s-1}$ are the $s-1$ colour constraining vertices belonging to the copy of $E_{s,q,1}$ used to define $E_{s,q,t}$. The edge set of $E_{s,q,t}$ contains all the edges of $E_{s,q,t-1}$ and $E_{s,q,1} - u^0$ plus $s-1$ additional edges to connect the socket vertices of $E_{s,q,t-1}$ to the colour constraining vertices of $E_{s,q,1}$. Each of the colour constraining vertices in $E_{s,q,1}$ is connected to a socket vertex of $E_{s,q,t-1}$, in such a way that, after this, the total degree of the socket vertices is $(q+1)(s-1)$. The assumption $q \geq \sqrt{s-1}$ is needed at this point, for otherwise the average degree of the socket vertices would be

$$\frac{(q+1)(s-1)}{q} = s-1+(s-1)/q > s-1+\sqrt{s-1} > s-1+q$$

where the expression on the right-hand side is the claimed bound on the maximum degree of $E_{s,q,t}$. Thus, if $q < \sqrt{s-1}$ at least one of the sockets would have degree larger than $s-1+q$ (thus contradicting **E2**).

In details, for s odd, we add edges $\{u^{i \bmod q}, w^{2i-1}\}$, and $\{u^{i \bmod q}, w^{2i}\}$ for $i \in \{1, \ldots, (s-1)/2\}$. Note that we connect an even number of vertices to each socket vertex thus preserving condition **E3**. For s even, we first add the edge $\{u^i, w^i\}$ for $i \in \{1, \ldots, \min(s-1, q)\}$. If $s-1 < q$ some sockets are not used by any of these edges and this completes the construction of $E_{s,q,t}$. Otherwise for $1 \leq i \leq (s-1-q)/2$ we also add edges $\{u^{i \bmod q}, w^{q+2i-1}\}$, $\{u^{i \bmod q}, w^{q+2i}\}$. Finally, if q is even, we add $\{u^q, w^{s-1}\}$.

As each of the colour constraining vertices of $E_{s,q,1}$ is adjacent to a socket vertex of $E_{s,q,t-1}$, the clique on these vertices must use all of the $s-1$ other colours in any proper s-colouring. The socket vertices of $E_{s,q,1}$ must therefore use the one remaining colour and hence are in the monochromatic set, condition **E1** follows. □

Theorem 4 *Let m, r, and s be positive integers, with $r \geq 2$, and s satisfying*

$$3 \leq s < 2r - \sqrt{2r + \frac{1}{4}} + \frac{3}{2}.$$

Then there exists a graph $A_{r,s,m}$ satisfying conditions **A0**, **A1**, *and* **A2**. *Furthermore $A_{r,s,m}$ can be constructed in time polynomial in r, s and m.*

Proof For $m \leq r$ a single empire of size r with no edges satisfies all conditions defining $A_{r,s,m}$. If $m > r$, we define $A_{r,s,m}$ in such a way that $R_r(A_{r,s,m})$ coincides with $E_{s,q,t}$, where $q = 2r - (s-1)$ and t is the smallest positive integer such that

$$r - 1 + t\left(qr - \frac{(q+1)(s-1)}{2}\right) \geq m.$$

Note that the stated bounds on s imply that q satisfies the conditions of Lemma 1.

In what follows the empires of $A_{r,s,m}$ will be denoted by bold type-face letters corresponding to the labels used to denote the vertices of $E_{s,q,t}$.

When s is odd, q is even and hence by condition **E3** every vertex in $E_{s,q,t}$ has even degree. By a well-known result of Euler the graph contains an Euler tour, and one such tour can be found in time polynomial in the size of the graph (see for instance [10, Chapter 6]). Given one such tour $\mathcal{T}$ we can construct the graph $A_{r,s,m}$ as follows. Let $A_{r,s,m}$ be the edgeless graph on $(s+q-1)t+1$ empires of r vertices, we visit the edges of $\mathcal{T}$ and add corresponding edges to $A_{r,s,m}$ keeping the invariant that one of the two end-points of the latest added edge has degree one in $A_{r,s,m}$. Without loss of generality we first add the edge $\{u^0{}_1, w^1{}_1\}$. Then, assuming we have visited the first $i-1$ edges of $\mathcal{T}$ and v_k is the vertex of degree one incident to the latest added edge, we connect v_k to an isolated vertex of empire $\mathbf{u}$, if $\{v,u\}$ is the next edge we visit in $\mathcal{T}$.

The edge set of graph $A_{r,s,m}$ consists of a single long path, and hence condition **A1** is satisfied. The degree distribution of $A_{r,s,m}$ is described in the following table.

vertex set	degree two	degree one	degree zero
$\mathbf{u}^0$	$\frac{s-1}{2}-1$	two	$r-\frac{s-1}{2}-1$
an empire corresponding to a colour constraining vertex	r		
one of the $t-1$ groups of q empires corresponding to internal monochromatic vertices	$(q+1)\frac{s-1}{2}$		$qr-(q+1)\frac{s-1}{2}$
$\mathbf{u}^i$ for $i>0$	$\frac{s-1}{2}$		$r-\frac{s-1}{2}$

Thus $A_{r,s,m}$ has

$$r-1+t\left(qr-\frac{(q+1)(s-1)}{2}\right)$$

isolated vertices within the monochromatic set. Increasing the value of t will increase this number provided that

$$qr>\frac{(q+1)(s-1)}{2}. \tag{1}$$

When s is even, $q=2r-(s-1)$ is odd. As before, we define $A_{r,s,m}$ using the graph $E_{s,q,t}$. However this time $E_{s,q,t}$ is not Eulerian. In particular, all colour constraining vertices have even degree $s+q-1$. However, by the construction used in Lemma 1, in each set of internal monochromatic vertices there are $\min\{s-1,q\}$ of even degree. Denote by $u^1,...,u^{q-s-1}$ the odd degree vertices in that set. Furthermore, the plug vertex u^0, and the final set of q socket vertices are all of odd degree. To understand the definition of $A_{r,s,m}$ we define a subgraph H of $E_{s,q,t}$. The edge set of H are defined as follows.

1. H contains a long path P_0 starting at u^0 and passing through w^{s-1} and u^q of each set of colour constraining and internal monochromatic vertices.
2. When $s-1<q$, for each set of internal monochromatic vertices and for all $i \in \{1,\ldots,(q-s-1)/2\}$, H contains a path $\{u^{2i-1}, w^{i \bmod (s-1)}, u^{2i}\}$.
3. Finally, for all i such that $i \le (q-1)/2$, there is a path $\{u^{2i-1}, w^i, u^{2i}\}$, where $u^1,\ldots,u^q$ are the socket vertices of $E_{s,q,t}$.

Note that $E_{s,q,t}-H$ is Eulerian. We construct $A_{r,s,m}$ in two stages. We first use the edges of an Euler tour of $E_{s,q,t}-H$ as we did in the case s odd. Then we define edges corresponding to the edges of H. This second type of edges involve different vertices from those used to deal with the Euler tour. Finally, if ${u^0}_1$ and ${u^0}_{s/2}$ are the start and the end point of the long path in $A_{r,s,m}$ corresponding to $E_{s,q,t}-H$ Euler tour, and ${u^0}_{s/2+1}$ is the starting point of the path P_0 in H, then we can actually attach the edge from ${u^0}_{s/2+1}$ to ${u^0}_{s/2}$. By doing this vertex ${u^0}_{s/2+1}$ becomes isolated, we lose a vertex of degree one, and gain a vertex of degree two.

The degree distribution of $A_{r,s,m}$ is given in the following table.

vertex set	degree two	degree one	degree zero
$\mathbf{u}^0$	$\frac{s}{2}-1$	one	$r-\frac{s}{2}$
an empire corresponding to a colour constraining vertex	r		
one of the $t-1$ groups of q empires corresponding to internal monochromatic vertices	$(q+1)\frac{s-1}{2}-\max(q-s-1,0)$	$\max(q-s-1,0)$	$qr-(q+1)\frac{s-1}{2}-\max(q-s-1,0)$
$\mathbf{u}^i$ for $i>0$	$\frac{s}{2}-1$	one	$r-\frac{s}{2}$

In total this gives us

$$(q+1)\left(r-\frac{s}{2}\right)+(t-1)\left(qr-\frac{(q+1)(s-1)}{2}-\max(q-s-1,0)\right)$$

isolated vertices within the monochromatic set. Increasing t will increase this number provided that

$$qr > \frac{(q+1)(s-1)}{2}+\max(q-s-1,0) > 0. \tag{2}$$

When $s<r+1$ and hence $\max(q-s-1,0)=r-s+1$, the above inequality is always true. We therefore need only consider the case for larger r, in this

case the bound (1) on graphs where s is even is the same as the bound when s is odd. The bound can be rewritten as

$$\frac{(s-1)^2}{2} - \left(2r + \frac{1}{2}\right)(s-1) + 2r^2 > 0.$$

This inequality is satisfied for

$$s < 2r - \sqrt{2r + \frac{1}{4}} + \frac{3}{2},$$

and hence for any m and any s and r satisfying the above inequality, there exists some $A_{r,s,m}$ satisfying conditions **A0**, **A1**, and **A2**. □

Let r be a positive integer. Given an r-empire graph G, and an empire $\mathbf{v}$ in G, the *r-degree of* $\mathbf{v}$ is simply the degree of vertex $\mathbf{v}$ in the reduced graph of G (of course the 1-degree of a vertex in a graph is just its (ordinary) degree). Let r', s, and m be positive integers as specified at the beginning of this section. Gadgets $A_{r',s,m}$ will be used in the forthcoming reductions to replace particular empires with high r-degree by an array of vertices of degree one or two, chosen among the monochromatic vertices of the gadget. Let m be an integer at least as large as the r-degree of $\mathbf{v}$. The *linearization of* $\mathbf{v}$ *in* G is the process of replacing $\mathbf{v}$ in G with a copy of $A_{r',s,m}$ attaching each edge incident with some element of v to a distinct element of Z in $A_{r',s,m}$. We will say that these chosen elements of Z *simulate* the empire $\mathbf{v}$. Note that, in general, r' may be different from r. Thus repeated linearizations may be used to introduce larger empires in a given r-empire graph or even transform a standard graph into an r'-empire graph, for some fixed $r' > 1$.

Planar Gadgets. Let $\mathbf{u}$ and $\mathbf{v}$ be given set of r vertices and denote by $\delta_{x,y}$ the Kroeneker delta function. For positive integers r, and s with $r \geq 2$ and $s < 6r - 3 - 2\delta_{r,2}$, it is possible to define a family of r-empire graphs $D_{r,s}(\mathbf{u}, \mathbf{v})$ satisfying the following properties:

D0 The graph $D_{r,s}(\mathbf{u}, \mathbf{v})$ has $r(s+1)$ vertices partitioned into $s+1$ empires all of size r.
D1 The graph $D_{r,s}(\mathbf{u}, \mathbf{v})$ is planar and it contains an isolated vertex v_1.
D2 No connected component of the graph $D_{r,s}(\mathbf{u}, \mathbf{v})$ contains two vertices from the same empire.
D3 The graph K_{s+1} minus the edge $\{\mathbf{u}, \mathbf{v}\}$ is a subgraph of $R_r(D_{r,s}(\mathbf{u}, \mathbf{v}))$.

$D_{r,s}(\mathbf{u}, \mathbf{v})$ will serve a similar purpose in Theorem 13 to that of $B^-_{r,s}(\mathbf{u}, \mathbf{v})$ in Theorem 11.

Theorem 5 *Let r and s be positive integers with $r \geq 2$ and $s < 6r - 3 - 2\delta_{r,2}$. Let $\mathbf{u}$ and $\mathbf{v}$ be two disjoint sets of r vertices. There exists an r-empire graph $D_{r,s}(\mathbf{u}, \mathbf{v})$ satisfying conditions* **D0***,* **D1***,* **D2***, and* **D3***. Furthermore $D_{r,s}(\mathbf{u}, \mathbf{v})$ can be constructed in time polynomial in r.*

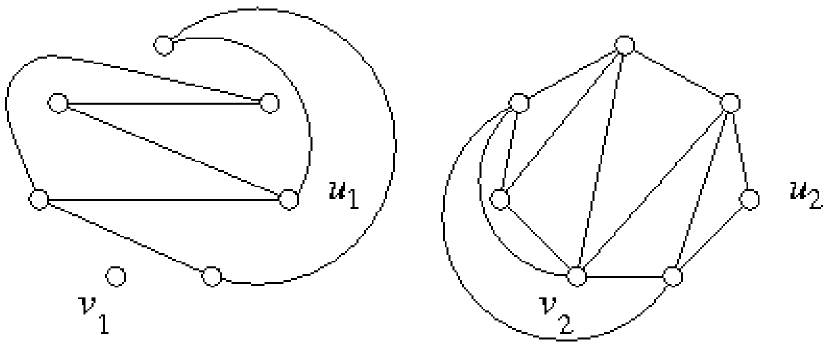

Fig. 6 The graph $D_{2,6}(\mathbf{u},\mathbf{v})$

Proof For $r = 2$, $s = 6$ a suitable graph is shown in Figure 6. For $r \geq 3$, we can derive $D_{r,6r-4}(\mathbf{u},\mathbf{v})$ from the proof in [2] that the thickness of K_{6r-3} is equal to r. In what follows we describe Beineke's construction highlighting a few points that are important to prove properties **D0**, **D1**, **D2**, and **D3**.

Beineke's construction starts by showing that there is a graph of thickness $r-1$ on $6(r-1)$ vertices labelled $u(i)$, $v(i)$, $w(i)$, $u'(i)$, $v'(i)$, $w'(i)$ for all $i \in \{1,\ldots,r-1\}$ in which there are edges connecting every pair of vertices except $\{u(i),u'(i)\}$, $\{v(i),v'(i)\}$ and $\{w(i),w'(i)\}$ for each $i \in \{1,\ldots,r-1\}$.

To do this, he defines D'_r to be a graph consisting of $r-1$ connected components $G_1,\ldots,G_{r-1}$ (see Figure 7), such that for each $i \in \{1\ldots r-1\}$ G_i consists of $6(r-1)$ vertices labelled $u(j)_i$, $v(j)_i$, $w(j)_i$, $u'(j)_i$, $v'(j)_i$, $w'(j)_i$ for all $j \in \{1,\ldots,r-1\}$. Vertices $u(i)_i$, $v(i)_i$, $w(i)_i$, $u'(i)_i$, $v'(i)_i$, $w'(i)_i$ will be called *external*, all others *internal* (as they are part of a copy of graph H). This satisfies property **D2**.

If corresponding vertices in distinct copies of G_i are grouped into empires of size $r-1$, the reduced graph of D'_r is a graph meeting Beineke's initial claim. It has $6(r-1)$ vertices labelled $u(i)$, $v(i)$, $w(i)$, $u'(i)$, $v'(i)$, $w'(i)$ for all $i \in \{1,\ldots,r-1\}$ in which there are edges connecting every pair of vertices except $\{u(i),u'(i)\}$, $\{v(i),v'(i)\}$ and $\{w(i),w'(i)\}$ for each $i \in \{1,\ldots,r-1\}$.

Three more empires $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, each of size $r-1$ are added to D'_r and connected to it in the following way:

$$\begin{array}{cl}
v\left(\lfloor\frac{r-1}{2}\rfloor\right)_1, v\left(\lfloor\frac{r-1}{2}\rfloor+1\right)_1 & \text{are adjacent to } a_1 \\
u(i)_i, u'(i+1)_i, v(i)_i & \text{are adjacent to } a_i (i>1),
\end{array}$$

$$\begin{array}{cl}
v(1)_1, v(2)_1, u'(1)_1 & \text{are adjacent to } b_1 \\
u(1)_{\lceil\frac{r-1}{2}\rceil+1}, u'(2)_{\lceil\frac{r-1}{2}\rceil+1} & \text{are adjacent to } b_{\lceil\frac{r-1}{2}\rceil+1} \\
v'(i)_i, v(i+1)_i, u(i)_i & \text{are adjacent to } b_i (i \in \{1,\ldots,\lceil\dfrac{r-1}{2}\rceil\}) \\
v(i)_i, v'(i+1)_i, u'(i)_i & \text{are adjacent to } b_i (i \in \{\lceil\dfrac{r-1}{2}\rceil+2,\ldots,r-1\}),
\end{array}$$

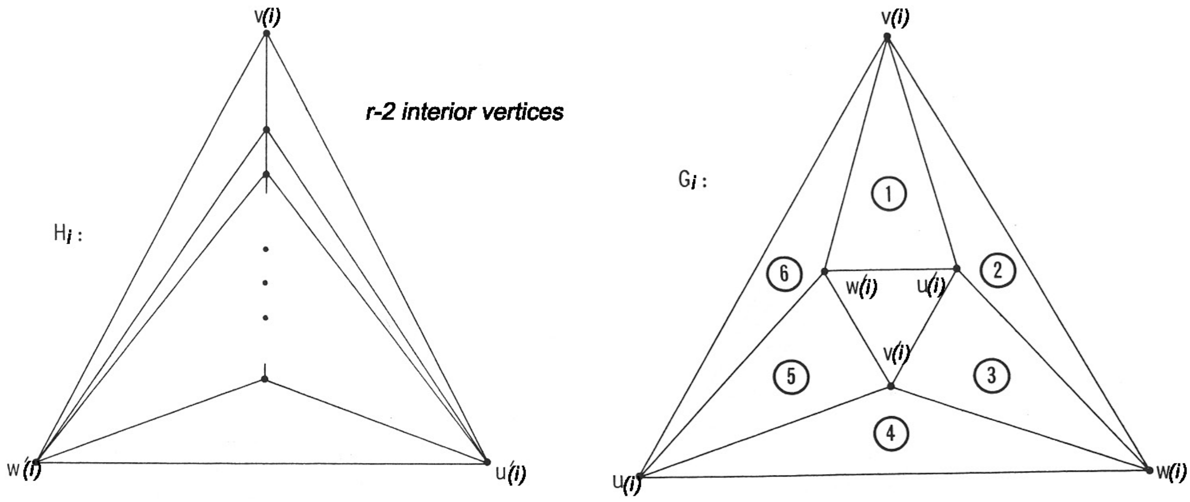

Fig. 7 The graphs H_i and G_i, the triangles labelled $1, \ldots 6$ in G_i contain copies of H_i in which the vertex v_i corresponds to $v(i)_i$, $u'(i)_i$, $w(i)_i$, $v'(i)_i$, $u(i)_i$, $w'(i)_i$ respectively. The labelling of the interior vertices of the H_i subgraphs is described in [2].

$w'(2)_1$	is adjacent to c_1
$w(i)_i, w'(i+1)_i$	are adjacent to $c_i (i > 1)$.

As each vertex from empires **a**, **b** and **c** was added to a single component of D'_r, property **D2** is still satisfied. Let G_r be the complement of $R_{r-1}(D'_r + \{\mathbf{a}, \mathbf{b}, \mathbf{c}\})$. It is not difficult to see that G_r is planar. Therefore by adding the vertices $u(j)_r$, $v(j)_r$, $w(j)_r$, $u'(j)_r$, $v'(j)_r$, $w'(j)_r$ for all $j \in \{1, \ldots, r-1\}$ with the same edge set as G_r we have a graph consisting of r planar components (that's G_r along with the components of the augmented graph $D'_r + \{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$) that reduces to K_{6r-3}.

G_1 contains a vertex c_1 of degree one which is adjacent to $w'(2)_1$. We can now form the graph $D_{r,6r-4}(\mathbf{u}, \mathbf{v})$ from G_r along with the components of the augmented graph $D'_r + \{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ by renaming empires c and $w'(2)$ as $\mathbf{v}$ and $\mathbf{u}$ respectively and removing all edges between $\mathbf{u}$ and $\mathbf{v}$. $D_{r,6r-4}(\mathbf{u}, \mathbf{v})$ satisfies property **D0**, **D1** (as the only edge incident to $\mathbf{v}$ has been deleted), **D2** and **D3** as the graph reduces to K_{6r-3} minus the edge $\{u, v\}$. For $s < 6r-4$, note that the induced graph formed by removing any empire other than $\mathbf{u}$ or $\mathbf{v}$ from $D_{r,s+1}(\mathbf{u}, \mathbf{v})$ is an example of $D_{r,s}(\mathbf{u}, \mathbf{v})$. As the size of the graph $D_{r,s}(\mathbf{u}, \mathbf{v})$ depends only on r and s, the graph can be constructed in polynomial time. □

5 Linear Forests

In Section 2 we showed (amongst other things) that there are specific values for s such that s-COL$_r$ becomes easy if the input graph is a collection short paths. Here we argue that if the paths are allowed to have arbitrary length (let LFOREST denote the set of all forests of this form) then the problem becomes NP-hard. We will prove the following result.

Theorem 6 *Let r and s be positive integers with $r \geq 2$ and $3 \leq s < 2r - \sqrt{2r + \frac{1}{4}} + \frac{3}{2}$. Then the s-COL$_r$(LFOREST) problem is NP-hard.*

Note that it follows from results in [18] that any r-empire graph defined on a linear forest can be coloured in polynomial time using $2r$ colours. Thus Theorem 6 is, at least for large values of r, close to best possible, in the sense that the largest values of s for which it holds are $2r - 1 + o(r)$.

The proof is split into two parts. The argument for $s = 3$ is based on a direct construction which is reminiscent of a well-known hardness proof for 3-COL [7, p.1103]. For $s > 3$, the hardness of s-COL$_r$(LFOREST) will then follow from that of s-COL(FG$(s, s-1)$).

We start from the case $s = 3$.

Theorem 7 *Let r be an integer with $r \geq 2$. Then* 3-SAT $\leq_p$ 3-COL$_r$(LFOREST).

Proof The proof construction is reminiscent of that used to show that 3-COL is NP-hard [7, p.1103].

Given an instance ϕ of 3-SAT we can produce a linear forest $P(\phi)$ and a partition of $V(P(\phi))$ into empires of size r such that $P(\phi)$ admits a $(3, r)$-colouring if and only if ϕ is satisfiable. $P(\phi)$ consists of one *truth gadget*, one *variable gadget* for each variable used in ϕ, and one *clause gadget* for each clause in ϕ.

To define the truth gadget, we start by adding $r-2$ distinct isolated vertices to each empire in $B_{2,2}$. The empires in the resulting graph (which we denote by $B_{2,2}^{+r}$) will be labelled $\mathbf{T}$, $\mathbf{F}$ and $\mathbf{X}$. Then, if ϕ uses n different variables and m clauses, we linearize $\mathbf{T}$ and $\mathbf{X}$ in $B_{2,2}^{+r}$, using one copy of $A_{r,3,\deg(\mathbf{T})+2m}$, and one copy of $A_{r,3,\deg(\mathbf{X})+n}$ (here $\deg(\mathbf{v})$ is the degree of empire $\mathbf{v}$ in $B_{2,2}^{+r}$),

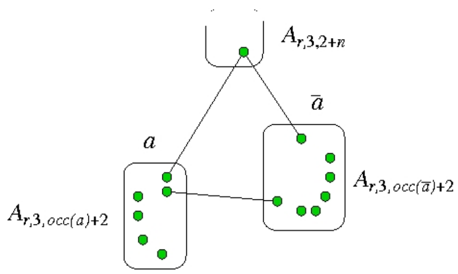

Fig. 8 The shape of a variable gadget for $s = 3$.

respectively. We denote such gadgets by $A(\mathbf{T})$ and $A(\mathbf{X})$ respectively. This completes the definition of the truth gadget. Since $\mathbf{T}$, $\mathbf{F}$ and $\mathbf{X}$ are all adjacent (in $B^{+r}_{2,2}$) and the linearization preserves colour constraints (because of property **A2**), the vertices of the truth gadget simulating the three empires of $B^{+r}_{2,2}$ must have different colours in any 3-colouring of the truth gadget. Without loss of generality we call TRUE, FALSE and OTHER respectively such colours.

For each variable a in ϕ, $P(\phi)$ contains a variable gadget. Let $\mathrm{occ}(\cdot)$ be a function taking as input a literal of ϕ and returning the number of occurrences of its argument in the given formula. The variable gadget for a is defined as the graph formed by the two connectivity gadgets $A_{r,3,\mathrm{occ}(\mathsf{a})+2}$ and $A_{r,3,\mathrm{occ}(\bar{\mathsf{a}})+2}$, along with a single monochromatic vertex z in $A(\mathbf{X})$ (a distinct monochromatic vertex is used for each variable of ϕ). The edges in the variable gadgets will be those of $A_{r,3,\mathrm{occ}(\mathsf{a})+2}$ and $A_{r,3,\mathrm{occ}(\bar{\mathsf{a}})+2}$ plus three further edges: $\{z, z_{\mathbf{a}}\}$, $\{z, z_{\bar{\mathbf{a}}}\}$, and $\{z'_{\mathbf{a}}, z'_{\bar{\mathbf{a}}}\}$. Here $z_{\mathbf{a}}$ and $z'_{\mathbf{a}}$ (resp. $z_{\bar{\mathbf{a}}}$ and $z'_{\bar{\mathbf{a}}}$) are monochromatic vertices in $A_{r,3,\mathrm{occ}(\mathsf{a})+2}$ and $A_{r,3,\mathrm{occ}(\bar{\mathsf{a}})+2}$. Figure 8 gives a schematic view of the variable gadget for an arbitrary variable a. Since $\mathbf{X}$ has colour OTHER, there are only two possible colourings for the vertices corresponding to $\mathbf{a}$ and $\bar{\mathbf{a}}$ — either all vertices for $\mathbf{a}$ are coloured TRUE and those for $\bar{\mathbf{a}}$ are coloured FALSE, or the vertices for $\mathbf{a}$ are coloured FALSE and those for $\bar{\mathbf{a}}$ TRUE.

Finally, for each clause in ϕ, $P(\phi)$ contains a gadget like the one depicted in Figure 9. This is connected to the rest of the graph via four connectivity gadgets. More specifically, the two vertices labelled T_1 and T_2 (in the Figure) are two monochromatic vertices in $A(\mathbf{T})$ (a distinct pair of such monochromatic vertices for each case clause gadget). Also, vertices labelled a, b and $\bar{c}$ in the Figure belong to the monochromatic set of three connectivity gadgets of the form $A_{r,3,\mathrm{occ}(\ell)+2}$ where ℓ is a literal ($\ell = \mathsf{a}$, b, and $\bar{\mathsf{c}}$ in the given example). Since the vertices of $A(\mathbf{T})$ corresponding to $\mathbf{T}$ will always be coloured TRUE, it can be shown that each clause gadget admits a proper $(3,r)$-colouring if and only if at least one of the empires corresponding to a literal in the clause is coloured TRUE.

Note that $P(\phi)$ is $(3,r)$-colourable if and only if ϕ is satisfiable. This follows from the properties of the well known reduction 3-SAT $\leq_p$ 3-COL, as the graph

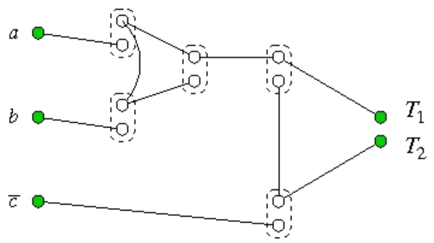

Fig. 9 The clause gadget for the clause $(\mathsf{a} \vee \mathsf{b} \vee \bar{\mathsf{c}})$, for $r = 2$. Each dashed curve encloses a pair of vertices belonging to the same empire. The vertices labelled T_1 and T_2 are in $Z(\mathbf{T})$, while vertices labelled a, b and $\bar{c}$ are in $Z(\mathbf{a})$, $Z(\mathbf{b})$ and $Z(\bar{\mathbf{c}})$ respectively.

obtained from $P(\phi)$ by shrinking each connectivity gadget first and then each remaining empire in $P(\phi)$ to a distinct (pseudo-)vertex coincides with that created from ϕ using the classical 3-COL reduction. $\square$

For $s > 3$ the NP-hardness of s-COL$_r$(LFOREST) follows from that of s-COL(FG$(s, s-1)$). The argument is much simpler than in the case described above. Given an $(s, s-1)$-formula graph Φ, the r-empire graph obtained by linearizing all vertices of Φ is an instance of s-COL$_r$(LFOREST). This immediately gives the following result.

Theorem 8 *Let r and s be fixed positive integers with $r \geq 3$, and $3 < s < 2r - \sqrt{2r + \frac{1}{4}} + \frac{3}{2}$. Then* s-COL(FG$(s, s-1)$) $\leq_p$ s-COL$_r$(LFOREST).

6 Trees

The result on linear forests of Section 5 already proves that s-COL$_r$ is NP-hard on planar graphs if $s \geq 3$ is sufficiently small. In this section we investigate the effect of connectedness on the computational complexity of the s-COL$_r$ problem. The outcome of our investigation is the following dichotomy result (in the next theorem TREE is the class of all trees).

Theorem 9 *Let r and s be fixed positive integers with $r \geq 2$, then the* s-COL$_r$(TREE) *problem is* NP*-hard if $2 < s < 2r$, and polynomial time solvable otherwise.*

The proof of Theorem 9 is split into two parts. The argument for $s = 3$ is very similar to the one we used for forests of paths, but simpler, as trees are allowed to have vertices of arbitrary large degree. We present the proof in some details only for the case $r = 2$ (see Theorem 10 below). For $r > 2$ note that a tree T_1 with empires of size r_1 can be translated into a tree T_2 with empires of size $r_2 > r_1$ by simply attaching $r_2 - r_1$ new leaves to a fixed element in each empire of T_1. For $s > 3$ we argue as in Section 5, translating formula graphs into pairs formed by a tree and a partition of its vertices into empires. The hardness of s-COL$_r$(TREE) follows from Theorem 2. Details in Theorem 11 below.

Theorem 10 3-SAT $\leq_p$ 3-COL$_2$(TREE).

Proof (Sketch) Given an instance ϕ of 3-SAT we define a tree $T(\phi)$ and a partition of its vertices into empires such that $T(\phi)$ admits a (3,2)-colouring if and only if ϕ is satisfiable. $T(\phi)$ will consist of one *truth gadget*, one *variable gadget* for each variable used in ϕ, and one *clause gadget* for each clause in ϕ.

The truth gadget is a copy of $B^+_{2,2}(\mathbf{T})$. Since empires $\mathbf{T}$, $\mathbf{F}$ and $\mathbf{X}$ are adjacent to each other (in the gadget's reduced graph) w.l.o.g. we assume they are coloured TRUE, FALSE and OTHER respectively. For each variable a in ϕ, $T(\phi)$ contains a copy of $B_{2,2}$ spanned by empires labelled $\mathbf{a}$, $\overline{\mathbf{a}}$, and $\mathbf{X}$.

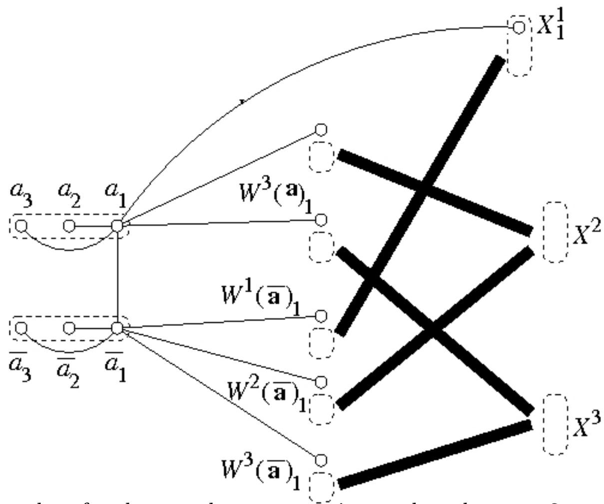

Fig. 10 The gadget for the complementary pair a and $\overline{a}$ when $r = 3$, $s = 5$. The dashed blobs represent either empires, or part of them. The diagram clearly shows all copies of $B^-_{3,5}(\mathbf{W}^i(\ell), \mathbf{X}^i)$, following the graphical notation introduced in Figure 4.

The construction forces empires $\mathbf{a}$, $\overline{\mathbf{a}}$ to be coloured differently from $\mathbf{X}$ (and each other). Finally, for each clause in ϕ, we use a clause gadget like the one in Figure 9.

Arguing like in the proof of Theorem 7 it is easy to see that $T(\phi)$ is $(3, 2)$-colourable if and only if there is some way to assign the variables of ϕ as TRUE or FALSE so that every clause contains at least one TRUE literal. □

Theorem 11 *s-COL(FG$(s, s-1)$) $\leq_p$ s-COL$_r$(TREE), for any $r \geq 3$ and $3 < s < 2r$.*

Proof As in the proof of Theorem 8 we give a set of replacement rules that translate an $(s, s-1)$-formula graph Φ into a tree $T(\Phi)$ and a partition of $V(T(\Phi))$ into empires of size r such that $T(\Phi)$ is (s, r)-colourable if and only if the formula graph is s-colourable. This time there is no need to use the connectivity gadgets $A_{r,s,m}$ as the vertices of $T(\Phi)$ can have arbitrarily large degrees. However some care is needed to make sure that the resulting graph is in fact a tree.

In details, the complete graph on $\{T, F, X^1, \ldots, X^{s-2}\}$ is replaced by a copy of $B^+_{r,s-1}(\mathbf{T})$ with empires labelled $\mathbf{T}$, $\mathbf{F}$, and $\mathbf{X}^1, \ldots, \mathbf{X}^{s-2}$. Note that, as discussed in Section 4, this graph is in fact a tree (Figure 3 displays the connected clique gadget for $r = 3$ and $s = 5$). Also, because of constraint **B3** in the definition of $B_{r,s}$, w.l.o.g. we may assume that colours "TRUE", "FALSE", "OTHER^1", …, "OTHER^{s-2}" are assigned to empires $\mathbf{T}$, $\mathbf{F}$, $\mathbf{X}^1 \ldots, \mathbf{X}^{s-2}$ respectively.

For each complementary pair $a, \overline{a}$ of $V(\Phi)$ we create $2s-5$ empires $\mathbf{W}^2(a), \ldots, \mathbf{W}^{s-2}(a)$ and $\mathbf{W}^1(\overline{a}), \ldots, \mathbf{W}^{s-2}(\overline{a})$. These are then connected to $B^+_{r,s-1}(\mathbf{T})$ using the graphs $B^-_{r,s}(\mathbf{W}^i(a), \mathbf{X}^i)$, and $B^-_{r,s}(\mathbf{W}^i(\overline{a}), \mathbf{X}^i)$ for all $i \in \{1, \ldots, s-2\}$. For each $a \in \mathcal{A}$ the subgraph of Φ spanned by $\bigcup_i \{a, \overline{a}, X^i\}$ is represented by a graph like the one sketched in Figure 10 for $r = 3$ and $s = 5$. This graph involves empires $\mathbf{a}$, $\overline{\mathbf{a}}$, $\mathbf{X}^1, \ldots, \mathbf{X}^{s-2}$, $\mathbf{W}^2(a), \ldots, \mathbf{W}^{s-2}(a)$ and $\mathbf{W}^1(\overline{a}), \ldots, \mathbf{W}^{s-2}(\overline{a})$. Empires $\mathbf{a}$, and $\overline{\mathbf{a}}$, each span a tree with one vertex, w.l.o.g. a_1 (resp. $\overline{a}_1$) of degree $r-1$ and $r-1$ vertices of degree one, all adjacent to it. These two trees are connected by the edge $\{a_1, \overline{a}_1\}$. Vertex a_1 (resp. $\overline{a}_1$) is also connected to the vertex in $\mathbf{W}^2(a), \ldots, \mathbf{W}^{s-2}(a)$ left isolated in the graph $B^-_{r,s}(\mathbf{W}^i(a), \mathbf{X}^i)$ (resp. to the isolated vertex in $\mathbf{W}^1(\overline{a})_1, \ldots, \mathbf{W}^{s-2}(\overline{a})_1$ belonging to $B^-_{r,s}(\mathbf{W}^i(\overline{a}), \mathbf{X}^i)$). Finally a_1 is connected to X^1_1. The edge $\{a_1, X^1_1\}$ ensures that the union of $B^+_{r,s-1}(\mathbf{T})$ and the graph spanned by empires $\mathbf{a}$, $\overline{\mathbf{a}}$, $\mathbf{X}^1$, $\mathbf{W}^2(a), \ldots, \mathbf{W}^{s-2}(a)$ and $\mathbf{W}^1(\overline{a}), \ldots, \mathbf{W}^{s-2}(\overline{a})$ is just a single tree. The edges connecting empires $\mathbf{a}$, $\overline{\mathbf{a}}$, with $\mathbf{W}^2(a), \ldots, \mathbf{W}^{s-2}(a)$ and $\mathbf{W}^1(\overline{a}), \ldots, \mathbf{W}^{s-2}(\overline{a})$, because of the properties of the $(\mathbf{W}^i(\ell), \mathbf{X}^i)$-colour constraining gadgets, prevent $\mathbf{a}$ and $\overline{\mathbf{a}}$ from being able to use the colours of the $\mathbf{X}^i$ in any colouring of $T(\Phi)$.

Each group $\{c^1, \ldots, c^{s-1}\}$ in $\mathcal{C}$ is replaced by empires $\mathbf{c}^1, \ldots, \mathbf{c}^{s-1}$ (different groups replaced by different sets of empires). The complete graph on $\{T, c^1, \ldots, c^{s-1}\}$ is replaced by a copy of $B_{r,s-1}$ on the corresponding empires (this ensures that the union of $B^+_{r,s-1}(\mathbf{T})$ and such $B_{r,s-1}$ form a single tree). We then attach to this graph $s-1$ graphs $B^-_{r,s}(\mathbf{b}^j, \mathbf{c}^j)$, for $j \in \{1, \ldots, s-1\}$. Empire $\mathbf{b}^j$ must have the same colour as $\mathbf{c}^j$ and it has, in $B^-_{r,s}(\mathbf{b}^j, \mathbf{c}^j)$, an isolated vertex, b^j_1. If ℓ is the unique element of $\mathcal{A}$ adjacent to c^j in the formula graph then $\{b^j_1, \ell_1\}$ is an edge of $T(\Phi)$. A schematic representation of the subgraph induced by $\mathbf{T}$, empires $\mathbf{c}^1, \ldots, \mathbf{c}^{s-1}$, along with the copies of $B^-_{r,s}(\mathbf{b}^j, \mathbf{c}^j)$ is given in Figure 11.

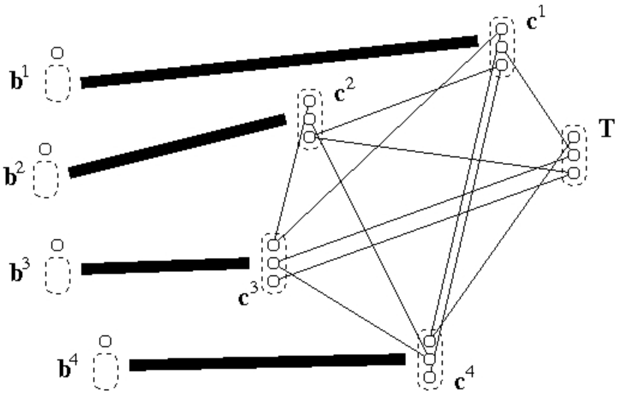

Fig. 11 A schematic representation of $\mathbf{T}$, empires $\mathbf{c}^1, \ldots, \mathbf{c}^{s-1}$, all edges among these along with the copies of $B^-_{r,s}(\mathbf{b}^j, \mathbf{c}^j)$.

The overall construction is such that for each vertex in $V(\Phi)$ there is an equivalent empire in $V(T(\Phi))$, and for each edge in $E(\Phi)$ there is an edge

$\{u, v\} \in E(T(\Phi))$ that either connects the corresponding empires $\mathbf{u}$ and $\mathbf{v}$ or connects $\mathbf{u}$ to an empire that must be given the same colour as $\mathbf{v}$ in any (s, r)-colouring of $T(\Phi)$. From this we can see that $T(\Phi)$ admits an (s, r)-colouring if and only if Φ admits an s-colouring. □

7 General Planar Graphs

Theorem 9 of last section does not exclude the possibility that s-COL$_r$ be solvable in polynomial time for arbitrary planar graphs provided $s \geq 2r$. Here we show that in fact this is not the case. The main result of this section is the following:

Theorem 12 *Let r and s be fixed positive integers with $r \geq 2$, then the s-COL$_r$ problem is* NP*-hard if $3 \leq s < 6r - 3 - 2\delta_{r,2}$, and solvable in polynomial time if $s = 2$ or $s \geq 6r$.*

Note that s-COL$_r$ can be solved in polynomial time for $s = 2$ (as checking if the reduced graph of a planar graph is bipartite is easy) and for $s \geq 6r$ (because of Heawood's result). Also, Theorem 9 proves the case $s < 2r$. Therefore only the case $s \geq 2r$ needs further discussion. The bulk of the argument is similar to that of Theorem 8 and 11 with a couple of differences. First, this time we only need the graph resulting from the transformation of the initial formula graph to be planar (note that the formula graph in general is NOT planar). On the other hand, we want the transformation to work for much larger values of s. Our solution hinges on proving that all complete subgraphs of the starting formula graph and a number of other gadgets attached to them have sufficiently large thickness. For the complete graphs we may use well-known results [1], whereas for the specific gadgets we need a bespoke construction.

Using the gadgets described above we can prove the following result, which completes the proof of Theorem 12.

Theorem 13 s-COL(FG$(s, s-1)$) $\leq_p$ s-COL$_r$*, for any $r \geq 2$ and $2r \leq s < 6r - 3 - 2\delta_{r,2}$.*

Proof The proof mirrors that of Theorem 11. We once again give a set of replacement rules to convert a $(s, s-1)$-formula graph Φ into a planar graph $G(\Phi)$ that is (s, r)-colourable if and only if Φ is s-colourable.

The copy of K_s induced by the vertex set $\mathcal{T}$ in Φ is replaced by r edge disjoint subgraphs of K_s. For $s \leq 6r-4$ the existence of such graphs is granted by known results on the thickness of K_s [1]. W.l.o.g. we may assume that the empires of the resulting graph (which, as usual, we label $\mathbf{T}$, $\mathbf{F}$, and $\mathbf{X}^1, \mathbf{X}^2, \ldots$) are coloured "TRUE", "FALSE", "OTHER^1", ..., "OTHER^{s-2}" respectively. The graph is then expanded, for each $a, \overline{a} \in \mathcal{A}$ using empires $\mathbf{W}^2(a), \ldots, \mathbf{W}^{s-2}(a)$ and $\mathbf{W}^1(\overline{a}), \ldots, \mathbf{W}^{s-2}(\overline{a})$ and the graphs $D_{r,s}(\mathbf{W}^i(a), \mathbf{X}^i)$ for all i such that $2 \leq i \leq s-2$, and $D_{r,s}(\mathbf{W}^i(\overline{a}), \mathbf{X}^i)$ for all i such that $1 \leq i \leq s-2$. The graphs $\Phi[\bigcup\{a, \overline{a}, X^j\}]$ and $\Phi[\{T\} \cup \mathcal{C}]$ are subject to transformations similar to those in Theorem 11 but using the planar decomposition of the complete graph instead of copies of $B_{r,s}$ and graphs $D_{r,s}(\mathbf{u},\mathbf{v})$ instead of $B^{-}_{r,s}(\mathbf{u},\mathbf{v})$.

As in Theorem 11, every vertex in $V(\Phi)$ has a corresponding empire in $V(G(\Phi))$, and every edge $\{u,v\} \in E(\Phi)$ has a corresponding edge in $E(G(\Phi))$ that connects either the empires $\mathbf{u}$ and $\mathbf{v}$ or empires that must be given the same colour as them in any proper (s,r)-colouring. It follows that $G(\Phi)$ admits a proper (s,r)-colouring if and only if Φ admits a proper s-colouring. $\square$

The reduction in the proof of Theorem 13 shows that for any given formula graph Φ one can define a planar graph $G(\Phi)$ which is formed by (at least) r connected components and reduces to Φ. Thus the proof is actually showing, for $s \geq 2r$, the NP-hardness of colouring, in the traditional sense, graphs of thickness r. The following result can be obtained extending the proof to any $s > 3$ and using a more direct reduction from 3-SAT for $s = 3$.

Theorem 14 *It is* NP*-hard to decide whether a graph of thickness* $r > 1$ *can be coloured with* $s < 6r - 3 - 2\delta_{r,2}$ *colours.*

An obvious way to improve Theorem 13 (and perhaps close the small gap between NP-hard and polynomially decidable cases) would be to use different gadgets to replace the complete subgraphs of Φ. However, it seems difficult to devise a graph with high thickness that shares the colour constraining properties of the complete graph. Perhaps, a more direct reduction from the satisfiability problem may provide a handle on the remaining open cases.

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