The strange and charm quark contributions to the anomalous magnetic moment of the muon from lattice QCD

Jonna Koponen[a,*], Bipasha Chakraborty[a], Christine T. H. Davies[a], Gordon Donald[b], Rachel Dowdall[c], Pedro Gonçalves de Oliveira[a], G. Peter Lepage[d], Thomas Teubner[e]

[a]*SUPA, School of Physics and Astronomy, University of Glasgow, Glasgow, G12 8QQ, UK*
[b]*Institut für Theoretische Physik, Universität Regensburg, 93040 Regensburg, Germany*
[c]*DAMTP, University of Cambridge, Wilberforce Road, Cambridge, CB3 0WA, UK*
[d]*Laboratory for Elementary-Particle Physics, Cornell University, Ithaca, New York 14853, USA*
[e]*Department of Mathematical Sciences, University of Liverpool, Liverpool, L69 3BX, UK*

Abstract

We describe a new technique (published in [1]) to determine the contribution to the anomalous magnetic moment of the muon coming from the hadronic vacuum polarisation using lattice QCD. Our method uses Padé approximants to reconstruct the Adler function from its derivatives at $q^2 = 0$. These are obtained simply and accurately from time-moments of the vector current-current correlator at zero spatial momentum. We test the method using strange quark correlators calculated on MILC Collaboration's $n_f = 2 + 1 + 1$ HISQ ensembles at multiple values of the lattice spacing, multiple volumes and multiple light sea quark masses (including physical pion mass configurations). We find the (connected) contribution to the anomalous moment from the strange quark vacuum polarisation to be $a_\mu^s = 53.41(59) \times 10^{-10}$, and the contribution from charm quarks to be $a_\mu^c = 14.42(39) \times 10^{-10}$ - 1% accuracy is achieved for the strange quark contribution. The extension of our method to the light quark contribution and to that from the quark-line disconnected diagram is straightforward.

Keywords: muon anomalous magnetic moment, hadronic vacuum polarisation, Lattice QCD

1. Motivation

The magnetic moment of the muon can be determined extremely accurately in experiment and the anomaly, $a_\mu = (g_\mu - 2)/2$, is known to 0.5 ppm (Brookhaven E821, [2]). Theoretical calculation of a_μ in the Standard Model shows a discrepancy with the experimental result of about $25(9) \times 10^{-10}$, which could be an indication of new virtual particles. Improvements of a factor of 4 in the experimental uncertainty are expected in 2017-18 (Fermilab E989 experiment) — improvements in the theoretical determination would make the discrepancy (if it remains) really compelling.

The theoretical uncertainty is dominated by that from the hadronic vacuum polarisation (HVP) contribution
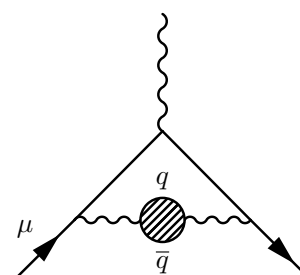

Figure 1: Diagrammatic representation of the hadronic vacuum polarisation contribution to the muon anomalous magnetic moment. The wavy lines are photons and the shaded blob is the hadronic vacuum polarisation contribution with all the quark and gluon interactions.

---

[*]Speaker

(Fig. 1). At the moment the most accurate determination of HVP contribution comes from experiment: dispersion relation + cross section for e^+e^- (and τ) → hadrons gives $\sim 700 \times 10^{-10}$ with a 1% error [3, 4]. Higher order contributions from QCD processes have larger percentage uncertainty but make an order of magnitude smaller contribution, so we set out to calculate the lowest order HVP contribution and aim at $1 - 2\%$ accuracy.

2. Method

We express the a_μ HVP contribution in terms of a small number of derivatives of the hadronic vacuum polarisation function Π evaluated at zero momentum and use lattice current-current correlators to calculate the derivatives [1].

The HVP contribution to the anomalous magnetic moment for quark flavour f is

$$a_{\mu,\mathrm{HVP}}^{(f)} = \frac{\alpha}{\pi} \int_0^\infty dq^2 f(q^2) (4\pi\alpha Q_f^2) \hat{\Pi}_f(q^2), \qquad (1)$$

where

$$f(q^2) \equiv \frac{m_\mu^2 q^2 A^3 (1 - q^2 A)}{1 + m_\mu^2 q^2 A^2},$$
$$A \equiv \frac{\sqrt{q^4 + 4m_\mu^2 q^2} - q^2}{2m_\mu^2 q^2} \qquad (2)$$

and $\alpha = \alpha_{\mathrm{QED}}$. The integrand peaks at $q^2 \sim O(m_\mu^2)$. Previous methods calculated $\hat{\Pi}(q^2)$ at larger q^2 and extrapolated to zero, which leads to model uncertainties, and calculating directly at small q^2 using "twisted boundary conditions" produces noisy results. We avoid this by working, in effect, from $q^2 = 0$ upwards.

The vacuum polarisation function is a Fourier transform of the lattice current-current correlator:

$$\Pi^{ii} = q^2 \Pi(q^2) = a^4 \sum_t e^{iqt} \sum_{\vec{x}} \langle j^i(\vec{x}, t) j^i(0) \rangle. \qquad (3)$$

We calculate the derivatives of the renormalised vacuum polarisation function $\hat{\Pi}(q^2) \equiv \Pi(q^2) - \Pi(0)$ from the time moments of the correlator:

$$G_{2n} \equiv a^4 \sum_t \sum_{\vec{x}} t^{2n} Z_V^2 \langle j^i(\vec{x}, t) j^i(0) \rangle$$
$$= (-1)^n \frac{\partial^{2n}}{\partial q^{2n}} q^2 \hat{\Pi}(q^2) \Big|_{q^2=0}. \qquad (4)$$

Here the correlator $\langle j^i(\vec{x}, t) j^i(0) \rangle$ is a local spatial vector current at zero spatial momentum.

We define $\hat{\Pi}(q^2)$ through the series expansion

$$\hat{\Pi}(q^2) = \sum_{J=1}^\infty q^2 \Pi_j, \quad \Pi_j = (-1)^{j+1} \frac{G_{2j+2}}{(2j+2)!} \qquad (5)$$

and use Padé approximants for $\hat{\Pi}(q^2)$ to control the high-q^2 region. High-order Padé approximants converge to the exact result [1]. We use 4th, 6th, 8th and 10th time moments (i.e. $j = 1, 2, 3$ and 4). Only quark-line connected contributions to the lowest order HPV are considered here — disconnected contributions will need to be addressed separately.

3. Lattice configurations

We use lattice ensembles made by MILC collaboration [5, 6] that have u/d, s and c quarks in the sea (i.e. $n_f = 2 + 1 + 1$). The lattice spacings are $a \approx 0.15$ fm (very coarse), 0.12 fm (coarse) and 0.09 fm (fine), determined using the Wilson flow parameter w_0 [7]. We use Highly Improved Staggered Quark (HISQ) action, which is known for very small discretisation errors. We use ensembles that have different light sea quark masses, including ensembles with physical light quark mass. The strange valence quark is tuned using the η_s meson mass $m_{\eta_s} = 688.5$ MeV [7]. We also test tuning effects by deliberately mistuning the strange quark by 5% (set 6). We use large volumes, $(5.6 \text{ fm})^3$ on the finest lattices, and have ensembles with different volumes for testing finite volume effects. Details of the lattice ensembles are listed in Table 1.

The local current used here is not the conserved vector current for this quark action and must be normalised. Renormalisation constant $Z_{V,\bar{s}s}$ is calculated completely nonperturbatively by demanding that the vector form factor for this current be 1 between two equal mass mesons at rest ($q^2 = 0$) [8]. Accurate normalisation is crucial if the target is total uncertainty at the 1% level.

4. Meson correlators

The 2-point correlators used in this study are the ϕ meson correlators made using a local spatial vector operator. As a cross-check and illustration of the accuracy of the correlators we plot the mass difference of the $s\bar{s}$ vector and pseudoscalar mesons, $m_\phi - m_{\eta_s}$, in Fig. 2, and also plot the ϕ meson decay constant in Fig. 3. Both figures show that the discretisation errors are indeed small and also emphasize the importance of working at the physical light quark masses (here sea quarks, but even more important in the case of light valence quarks).
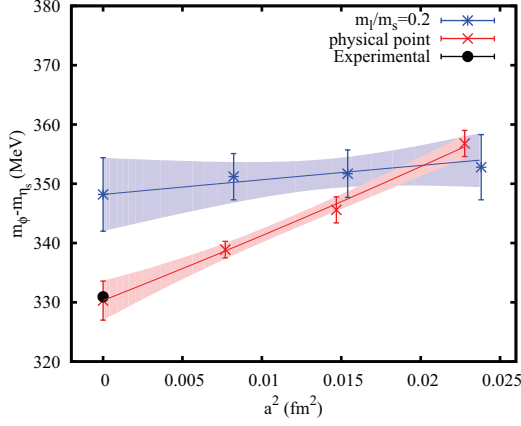
Figure 2: Mass difference $m_\phi - m_{\eta_s}$ as a function of a^2. Comparing results on ensembles that have different light sea quark masses shows the advantage of working at the physical point.
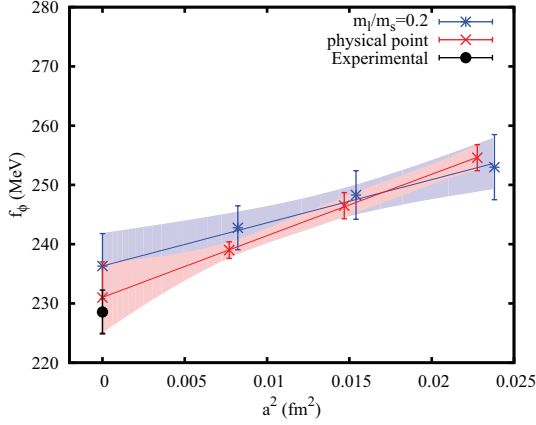

Figure 3: ϕ meson decay constant as a function of a^2, comparing results on ensembles that have different light sea quark masses ($m_l^{\text{lat}} = m_s/5$ and $m_l^{\text{lat}} = m_l^{\text{phys}}$).
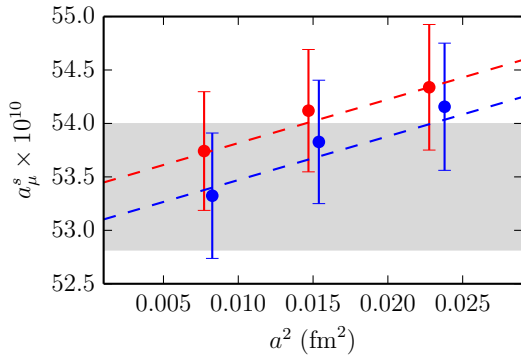

Figure 4: Continuum extrapolation fit. Here we compare results from ensembles with $m_l^{\text{lat}} = m_s/5$ (blue points) to results from ensembles with physical light quark mass (red points). The shaded error band is our final result.

5. Fitting the data

We use the [2,2] Padé approximant for each configuration set and then fit these results to a function of the form

$$a_{\mu,\text{lat}}^s = a_\mu^s \left(1 + c_{a^2} \left(\frac{a\Lambda_{\text{QCD}}}{\pi} \right)^2 + c_{\text{sea}}\delta_{\text{sea}} + c_{\text{val}}\delta_{\text{val}} \right), \quad (6)$$

where $\Lambda_{\text{QCD}} = 0.5$ GeV and

$$\delta_{\text{sea}} \equiv \sum_{q=u,d,s} \frac{m_q^{\text{sea}} - m_q^{\text{phys}}}{m_s^{\text{phys}}}, \quad \delta_{\text{val}} \equiv \frac{m_s^{\text{val}} - m_s^{\text{phys}}}{m_s^{\text{phys}}}. \quad (7)$$

Here the fit parameter c_{a^2} takes care of the (small) discretisation effects and c_{sea} and c_{val} take care of the dependence on the sea and valence quark masses. The fit results along with lattice data are shown in Fig. 4. More details of the fit can be found in [1].

6. Results

Our result for the leading order HVP contribution to the muon anomalous magnetic moment from the strange quarks is $a_\mu^s = 53.41(59) \times 10^{-10}$ (connected pieces only). We also used time moments we calculated in [9, 10] to get the charm quark contribution to a_μ: $a_\mu^c = 14.42(39) \times 10^{-10}$ (again, connected pieces only). A preliminary estimate of total light, strange and charm quark connected piece contributions (averaging a_μ^{light} on very coarse and coarse physical m_l ensembles) is $a_\mu^{\text{HVP,LO}} = a_\mu^{\text{light}} + a_\mu^s + a_\mu^c \sim 662(35) \times 10^{-10}$. This is to be compared with the dispersion relation + $e^+e^- \to$ hadrons cross section result of $\sim 700 \times 10^{-10}$ [3, 4] mentioned earlier. Note that our result does not include the disconnected diagrams.

The error budget for both strange and charm quark connected contributions is given in Table 2. The dominant error in a_μ^s is, by far, that coming from the uncertainty in the physical value of the Wilson flow parameter w_0, which we use to set the lattice spacings. The next largest contribution comes from the uncertainty in the renormalisation factor Z_V. For the charm quark contribution this is the dominant error, because a different method for calculating Z_V was used in that calculation. This could be improved by using the same method that was used here for the strange quark contribution. More details about the error budget are in [1].

Comparing our results with other lattice QCD results shows good agreement: Fig. 5 shows our results (HPQCD) and European Twisted Mass Collaboration's results plotted against a^2. This again highlights the fact
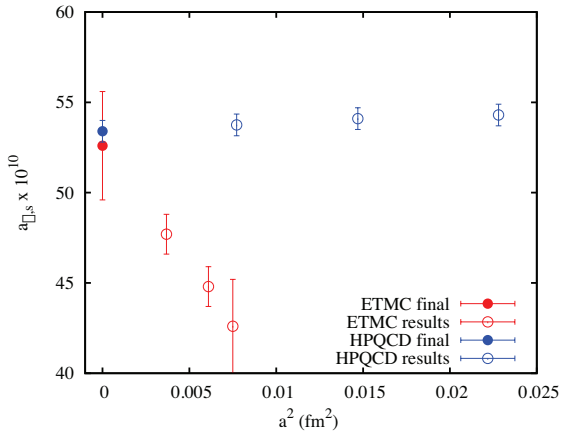
Figure 5: Comparison with other lattice results: a_μ^s as a function of a^2. ETMC results are from [11].

that HISQ action has very small discretisation errors compared to other lattice actions. RBC/UKQCD have also calculated a_μ^s and agree well with other determinations — see Table 3 for a list of the results.

7. Connected contribution to a_μ^{light}

The signal-to-noise ratio at large t is much worse for light quarks than for strange or charm quarks, which means we will need better statistics and improvements in the fitting method to get the error down to the 1% level. We can get better accuracy by calculating moments from best fit parameters instead of using raw lattice data. 5-6% precision is already achieved using 1000 configurations × 12 time sources (very coarse ensemble) and 400 configurations × 4 time sources (coarse ensemble) on physical light quark mass ensembles. 1% precision can be achieved by adding more time sources (need 4 times n_{src}) and up to 10 × configurations.

8. Conclusions

We have demonstrated in [1] that 1% precision can be achieved for the leading order HVP contribution to a_μ^s from the connected pieces. The error on a_μ^c could be pushed down to 1% by using the same method to calculate the renormalisation factor Z_V that was used here for the strange quark. However, the charm quark contribution to the total leading order HVP contribution $a_\mu^{\text{HVP,LO}}$ is small compared to contributions from strange and light quarks so this is not a top priority. The main task now is to push down the error coming from the light

quark contribution a_μ^l. We can get good enough statistics to achieve this: we can use more time sources, plus more very coarse and coarse configurations can be made relatively cheaply. Disconnected contributions need to be included in the future.

References

[1] B. Chakraborty, C. T. H. Davies, G. C. Donald, R. J. Dowdall, J. Koponen, G. P. Lepage, T. Teubner, Strange and charm quark contributions to the anomalous magnetic moment of the muon, Phys. Rev. D 89 (2014) 114501.

[2] G. W. Bennett, et al., Final report of the E821 muon anomalous magnetic moment measurement at BNL, Phys. Rev. D 73 (2006) 072003.

[3] K. Hagiwara, R. Liao, A. D. Martin, D. Nomura, T. Teubner, $(g2)_\mu$ and $\alpha(M_Z^2)$ re-evaluated using new precise data, Journal of Physics G: Nuclear and Particle Physics 38 (8) (2011) 085003.

[4] M. Davier, A. Hoecker, B. Malaescu, Z. Zhang, Reevaluation of the hadronic contributions to the muon g2 and to $\alpha(M_Z^2)$, The European Physical Journal C 71 (1).

[5] A. Bazavov, et al., Scaling studies of QCD with the dynamical highly improved staggered quark action, Phys. Rev. D 82 (2010) 074501.

[6] A. Bazavov, et al., Lattice QCD ensembles with four flavors of highly improved staggered quarks, Phys. Rev. D 87 (2013) 054505.

[7] R. J. Dowdall, C. T. H. Davies, G. P. Lepage, C. McNeile, V_{us} from π and K decay constants in full lattice QCD with physical u, d, s, and c quarks, Phys. Rev. D 88 (2013) 074504.

[8] B. Chakraborty, C. T. H. Davies, G. Donald, R. Dowdall, J. Koponen, G. P. Lepage, Nonperturbative tests of the renormalization of mixed clover-staggered currents in lattice QCD, PoS LATTICE2013 (2013) 309. arXiv:1401.0669.

[9] C. Davies, G. Donald, R. Dowdall, J. Koponen, E. Follana, et al., Precision tests of the J/ψ from full lattice QCD: mass, leptonic width and radiative decay rate to η_c, PoS ConfinementX (2012) 288. arXiv:1301.7203.

[10] G. C. Donald, C. T. H. Davies, R. J. Dowdall, E. Follana, K. Hornbostel, J. Koponen, G. P. Lepage, C. McNeile, Precision tests of the J/ψ from full lattice QCD: Mass, leptonic width, and radiative decay rate to η_c, Phys. Rev. D 86 (2012) 094501.

[11] F. Burger, X. Feng, G. Hotzel, K. Jansen, M. Petschlies, D. B. Renner, Four-flavour leading-order hadronic contribution to the muon anomalous magnetic moment, JHEP 02 (2014) 099, (see also note in Phys. Rev. D89 (2014) 114501: private communication for the separate contributions a_μ^s and a_μ^c).

[12] S. Bodenstein, C. A. Dominguez, K. Schilcher, Hadronic contribution to the muon $g-2$ factor: A theoretical determination, Phys. Rev. D 85 (2012) 014029.

[13] K. Maltman, A new strategy for evaluating the LO HVP contribution to $(g-2)_\mu$ on the lattice, talk at the 32nd International Symposium on Lattice Field Theory (Lattice 2014) (June 2014).

Set	am_l^{sea}	am_s^{sea}	am_{η_s}	$Z_{V,\bar{s}s}$	$L/a \times T/a$	$n_{\text{cfg}} \times n_{\text{src}}$
1	0.01300	0.0650	0.54024(15)	0.9887(20)	16×48	1020×12
2	0.00235	0.0647	0.52680(8)	0.9887(20)	32×48	1000×12
3	0.01020	0.0509	0.43138(12)	0.9938(17)	24×64	526×16
4	0.00507	0.0507	0.42664(9)	0.9938(17)	24×64	1019×16
5	0.00507	0.0507	0.42637(6)	0.9938(17)	32×64	988×16
6	0.00507	0.0507	0.41572(14)	0.9938(17)	32×64	300×16
7	0.00507	0.0507	0.42617(9)	0.9938(17)	40×64	313×16
8	0.00184	0.0507	0.42310(3)	0.9938(17)	48×64	1000×16
9	0.00740	0.0370	0.31384(9)	0.9944(10)	32×48	504×16
10	0.00120	0.0363	0.30480(4)	0.9944(10)	64×96	621×16

Table 1: Lattice ensembles used in this study, made by MILC collaboration [5, 6]. The first two sets are "very coarse" (lattice spacing $a \sim 0.15$ fm), sets $3 - 8$ are "coarse" ($a \sim 0.12$ fm) and sets $9 - 10$ are "fine" ($a \sim 0.09$ fm) ensembles. am_l^{sea} and am_s^{sea} are the sea light and strange quark masses in lattice units and am_{η_s} is the η_s meson mass. $Z_{V,\bar{s}s}$ is the vector current renormalisation constant. L and T are the spatial and temporal extents of the lattice. n_{cfg} is the number of configurations and n_{src} is the number of time sources used in this study.

	a_μ^s	a_μ^c
Uncertainty in lattice spacing (w_0, r_1):	1.0%	0.6%
Uncertainty in Z_V:	0.4%	2.5%
Monte Carlo statistics:	0.1%	0.1%
$a^2 \to 0$ extrapolation:	0.1%	0.4%
QED corrections:	0.1%	0.3%
Quark mass tuning:	0.0%	0.4%
Finite lattice volume:	< 0.1%	0.0%
Padé approximants:	< 0.1%	0.0%
Total:	1.1%	2.7%

Table 2: Error budgets for connected contributions to the muon anomaly a_μ from vacuum polarization of s and c quarks. See [1] for more detailed discussion on the estimation of the errors.

$a_\mu^{s/c}$	dispersion + expt	HPQCD	ETMC (prelim.)	RBC/UKQCD (prelim.)
$a_\mu^s \times 10^{10}$	55.3(8)	53.4(6)	53(3)	52.4(2.1)
$a_\mu^c \times 10^{10}$	14.4(1)	14.4(4)	14.1(6)	–

Table 3: Comparison with other results. The dispersion relation + experiment results are from [3] and [12]; HPQCD results are from [1] (moments used for a_μ^c were calculated in [9, 10]); ETMC results are from [11]; RBC/UKQCD results are from [13].