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How many wireless sensors are needed to guarantee connectivity of a one-dimensional network with random inter-node spacings?

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SUMMARY

An important problem in wireless sensor networks is to find an optimal number of randomly deployed sensors to guarantee connectivity of the resulting network with a given probability. The authors describe a general method to compute the probabilities of connectivity and coverage for one-dimensional networks with arbitrary densities of inter-node spacings. A closed formula for the probability of connectivity is derived when inter-node spacings have arbitrary different piecewise constant densities. Explicit estimates for a number of sensors to guarantee connectivity of the network are found for constant and normal densities.

Keywords and phrases: wireless sensor network, connectivity, probability, density

Mathematics Subject Classification: 90B18, 68M10, 46N30

1 Introduction

The problems of connectivity and coverage in wireless sensor networks have been extensively investigated [1]. Theoretical properties of random networks or graphs are summarized in [10]. One-dimensional networks are simple, but are widely used in practice for monitoring roads, rivers, coasts and boundaries of restricted areas. One-dimensional networks that are deployed along paths can often provide nearly the same information about moving objects as two-dimensional networks, but require fewer sensors and have a lower cost. Random networks are usually modelled by using a probability density of positions of sensors defined over an entire domain. For instance, all sensors are uniformly distributed in a segment [14].

Sensors of random one-dimensional networks are practially deployed one by one along a trajectory of a vehicle. Hence the authors consider probability densities of *inter-node spacings* (distances between successive sensors), not densities of positions of sensors. It is assumed that inter-node spacings are independent random variables, but not necessarily identical. So a network may contain some powerful sensors that can be at a larger distance apart. The proposed approach expresses the probability of connectivity as a function in the number of nodes for arbitrary densities, in a closed form for piecewise constant densities. Explicit estimates closed to optimal are found for a number of sensors to guarantee connectivity of a random network for classical probability densities (constant and normal).

The paper is organised as follows. Section 2 reviews previous results and states problems. Section 3 explores relations between connectivity of one-dimensional networks and two-dimensional networks. The method for computing the probabilities of connectivity and coverage is introduced in Section 4. Section 5 discusses explicit estimates for a number of sensors needed to guarantee connectivity for a constant density with two parameters and a truncated normal density. Conclusions and further problems are summarised in Section 6. Appendices A–B contain proofs of the main theorems and corollaries from Sections 4 and 5. Appendices C–D discuss more general results for arbitrary piecewise constant densities.

2 Previous results and formulation of problems

Many previous results on connectivity are *asymptotic* in the number of sensors, see [2, 4] for two-dimensional networks and [10] for random graphs. Rates of convergence in asymptotic formulae are often hard to analyse. For example, connectivity may be guaranteed for 10^6 sensors, but not for 1000 sensors. A standard assumption for finite networks is the uniform distribution of sensors. *Exponential* densities of positions of sensors were also considered in a segment [5] and in a square [7, 10, 11, 12, 13]. For n sensors distributed independently and uniformly in a segment $[0, L]$, the probability of connectivity was found in [3]:

$$P'_n = \sum_{i=0}^{i<L/R} (-1)^i \binom{n-1}{i} \Big(1 - i\frac{R}{L}\Big)^n, \text{ where } \binom{n-1}{i} = 0 \text{ for } i \geq n.$$

The upper bound $i < \dfrac{L}{R}$ in the sum implies that $1 - i\dfrac{R}{L} > 0$, but $P'_n \geq 0$ seems hard to prove by combinatorial methods. This method was generalised to the exponential density in [6] and we shall extend it to arbitrary densities of inter-node spacings in section 4.

Suppose that a sink node at the origin $x_0 = 0$ collects some information from other sensors. Let L be the *length* of a segment, where one deploys n sensors that have a *transmission radius* R. Number all sensors in the increasing order of their positions: $0 = x_0 \leq x_1 \leq \cdots \leq x_n \leq L$. Let $f_i(s)$ be the probability density function of the i-th distance $d_i = x_i - x_{i-1}$. Technically, the density $f_i(s)$ should be defined over the whole segment $[0, L]$. Our practical examples consider densities $f_i(s)$ that are concentrated around a transmission radius R. The resulting network is *connected* if the distance y_i between any

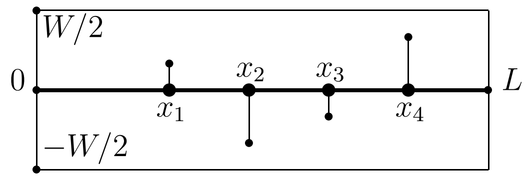

Figure 1: A 2-dimensional network along a narrow road

successive sensors, including the sink node, is not greater than R. The densities f_i depend on a practical way to deploy sensors. The transmission radius R is an input parameter, because the range of radii is often restrictive. The key practical problem is to find relative estimates for a minimum number of sensors to guarantee connectivity and coverage.

The Connectivity Problem is to find an optimal number of randomly deployed sensors in the segment $[0, L]$ such that the network is connected with a given probability p.

The Coverage Problem is to find an optimal number of randomly deployed sensors such that the network is connected and covers the segment $[0, L]$ with a given probability p.

3 A reduction of dimension: from two to one

This section shows how the connectivity problem in dimension 2 can be reduced to dimension 1. Let us deploy sensors from a vehicle moving along a river or a path in a forest. Then all sensors are located in a narrow road of some width W, as shown in Fig. 1.

Assume that the width W of the road is less than a transmission radius R. Denote the two-dimensional positions of the sensors by pairs $(x_1, y_1), \dots, (x_n, y_n)$. Order all sensors by their x-coordinates. The coordinate $y_i \in [-W/2, W/2]$ can be considered as a deviation of the i-th sensor from the central horizontal segment $[0, L]$.

The results below relate the connectivity of the two-dimensional network of sensors $(x_1, y_1), \dots, (x_n, y_n)$ and the one-dimensional network of the projections $x_1, \dots, x_n$.

Proposition 3.1 (The Dimension Down Proposition). *Consider a connected 2-dimensional network along a straight road of a width W. Each distance between successive sensors is less than a transmission radius R. Then the 1-dimensional network of the projected sensors with the positions $x_1, \dots, x_n$ is connected for the same transmission radius R. If a road is not straight, then connectivity is guaranteed for the larger radius $R + W$, see Fig. 2.*

Proposition 3.2 (The Dimension Up Proposition). *Consider a 2-dimensional network along a straight road of a width W. Suppose that the 1-dimensional network of the projected sensors $x_1, \dots, x_n$ is connected for a transmission radius R. Then the original 2-dimensional network is connected for the transmission radius $\sqrt{R^2 + W^2}$. If a road is not straight, then connectivity is guaranteed for the larger transmission radius $R + W$.*

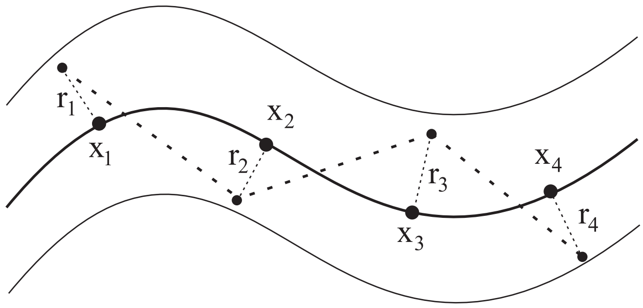

Figure 2: A 2-dimensional network along a non-straight road

Proof of the Dimension Down Proposition. Let $\tilde{d}_i$ be the Euclidean distance between successive sensors (x_{i-1}, y_{i-1}) and (x_i, y_i) of a 2-dimensional network that is connected for a transmission radius R. The Pythagoras theorem says that the distance between successive projected sensors is $|x_i - x_{i-1}| = \sqrt{\tilde{d}_i^2 - |y_i - y_{i-1}|^2} \le \sqrt{\tilde{d}_i^2} \le R$. Hence, the successive sensors at x_{i-1} and x_i are within the radius R as required.

For a non-straight road, let r_i be the shortest distance from the sensor (x_i, y_i) to the central curve, so $r_i \le W/2$. Two successive sensors (x_{i-1}, y_{i-1}) and (x_i, y_i) and their projections form a quadrilateral (possibly, self-intersecting), see Fig. 2. Estimate one side of the quadrilateral by using three others: $|x_i - x_{i-1}| \le \tilde{d}_i + r_{i-1} + r_i \le R + W$. Here $|x_i - x_{i-1}|$ denotes the Euclidean distance on the plane between the projected sensors. Hence the projected sensors x_{i-1}, x_i are within the radius $R + W$. $\square$

Proof of the Dimension Up Proposition. The projected sensors x_i form a connected network for a transmission radius R, so $|x_i - x_{i-1}| \le R$. The width of a straight road is W, hence $|y_i - y_{i-1}| \le W$. The Pythagoras theorem says that the Euclidean distance $\tilde{d}_i$ between successive sensors (x_{i-1}, y_{i-1}) and (x_i, y_i) of the original 2-dimensional network is $\tilde{d}_i = \sqrt{|x_i - x_{i-1}|^2 + |y_i - y_{i-1}|^2} \le \sqrt{|x_i - x_{i-1}|^2 + W^2} \le \sqrt{R^2 + W^2}$. So the sensors (x_{i-1}, y_{i-1}) and (x_i, y_i) are within the radius $\sqrt{R^2 + W^2}$ as required.

For a non-straight road, let r_i be the shortest distance from the sensor (x_i, y_i) to the road, so $r_i \le W/2$. Two successive sensors (x_{i-1}, y_{i-1}) and (x_i, y_i) and their projections form a quadrilateral, see Fig. 2. Estimate one side of the quadrilateral by using others: $\tilde{d}_i \le |x_i - x_{i-1}| + r_{i-1} + r_i \le R + W$. Here $|x_i - x_{i-1}|$ denotes the Euclidean distance on the plane between the projected sensors. Hence the original sensors at (x_{i-1}, y_{i-1}) and (x_i, y_i) are within the radius $R + W$ of each other as required. $\square$

4 General expression for the probability of connectivity

Recall that n sensors with a transmission radius R are deployed in a segment $[0, L]$ so that the i-th distance $d_i = x_i - x_{i-1}$ between successive sensors has a probability density $f_i(s)$

for $i = 1, \ldots, n$. Assume that the densities $f_1, \ldots, f_n$ are integrable and $\int\limits_0^L f_i(s)ds = 1$, $i = 1, \ldots, n$. Hence the i-th distance can take values from 0 to L. So the n-th sensor may not be within $[0, L]$ and only $x_n \leq nL$ holds. A network is *proper* if all sensors are within $[0, L]$. A proper network is *connected* if the distance $d_i = x_i - x_{i-1}$ between any successive sensors, including the sink node $x_0 = 0$, is less than R.

Our aim is to compute the *conditional* probability that a proper network is connected, namely the probability that the network is connected assuming that the network is proper. By Bayes' formula the answer will be a quotient, the probability that the network is proper and connected over the probability that the network is proper.

In practice all networks are proper, because sensors are always deployed along a fixed trajectory of a vehicle. Hence the probability that a network is proper is close to 1 for all practical densities f_i concentrated around a transmission radius R. The key ingredients are evaluations of the function $v_n(r, l)$ that is recursively defined for $n \geq 0$ as follows:

$$
\begin{aligned}
&v_0(r,l) = 1 && \text{if } r, l > 0;\\
&v_n(r,l) = 0 && \text{if } r \leq 0 \text{ or } l \leq 0;\\
&v_n(r,l) = 1 && \text{if } r \geq l > 0,\ n > 0;\\
&v_n(r,l) = \int\limits_0^r f_n(s) v_{n-1}(r, l-s)ds && \text{if } r < l,\ n > 0.
\end{aligned}
$$

Proposition 4.1 (The Probability Proposition)**.** *For $0 < r \leq l$ in the above notations, $v_n(r,l)$ is the probability that an array of random distances $(d_1, \ldots, d_n)$ with probability densities $f_1, \ldots, f_n$, respectively, satisfies $\sum\limits_{i=1}^n d_i \leq l$ and $0 \leq d_i \leq r$ for $i = 1, \ldots, n$.*

The variables r and l play the roles of the upper bounds for the distance between successive sensors and the sum of distances, respectively. Namely, $v_n(L, L)$ is the probability that a 1-dimensional network is *proper*, i.e. all sensors are within $[0, L]$, and $v_n(R, L)$ is the probability that a network is proper and connected.

Theorem 1 (The Connectivity Theorem)**.** *Let n sensors $x_1, \ldots, x_n$ having a transmission radius R be deployed in $[0, L]$ so that a sink node is fixed at $x_0 = 0$ and the distances $d_i = x_i - x_{i-1}$, $i = 1, \ldots, n$, have given probability density functions $f_1, \ldots, f_n$. Then the probability of connectivity of the resulting network is $P_n = \dfrac{v_n(R,L)}{v_n(L,L)}$, where $v_n(r,l)$ was recursively defined above. So the probability P_n is independent of the order of sensors.*

Given a required probability p of connectivity, the answer to the Connectivity Problem from section 2 is a minimum number n of sensors such that $P_n \geq p$. For $n = 1$, Connectivity Theorem 1 above gives the probability of connectivity

$$P_1 = P(0 \leq d_1 \leq R) = v_1(R, L) = \int\limits_0^R f_1(l)dl \quad \text{since} \quad v_1(L, L) = \int\limits_0^L f_1(l)dl = 1.$$

Theorem 2 (The Coverage Theorem)**.** *Under the conditions of Connectivity Theorem 1, the probability that the network is connected and covers* $[0, L]$ *is* $\dfrac{v_n(R, L) - v_n(R, L - R)}{v_n(L, L)}$.

Connectivity Theorem 1 leads to explicit estimates for a number of sensors to guarantee connectivity with a given probability for classical densities in section 5. The Connectivity and Coverage Theorems will be proved in Appendix A by generalising a method of [3]. More practical densities depend on a radius R and will be discussed in appendices C–D.

5 Corollaries for constant and normal densities

In this section one derives explicit estimates for a number of sensors to guarantee connectivity of networks for two classical probability densities. Start from the constant density $f(l) = \dfrac{1}{b-a}$ over any subsegment $[a, b] \subset [0, L]$. It means that each sensor is thrown from a moving vehicle at a random uniform distance between a and b away from the previous sensor. This case includes the simplest uniform density $f(l) = \dfrac{1}{L}$ for $a = 0$, $b = L$.

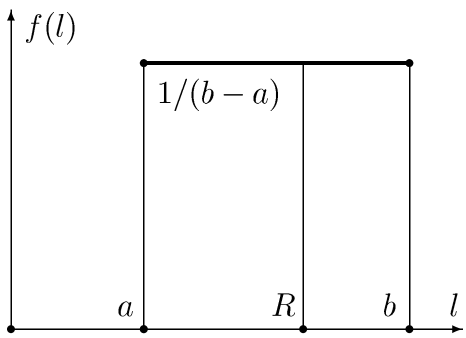

Figure 3: The constant density over the segment $[a, b]$

The left endpoint a should be less than the transmission radius R, otherwise no sensor communicates with its neighbours. The mathematical *expectation* of the distance between successive sensors is $\dfrac{a+b}{2}$. This average should be lower than the transmission radius R in practice to increase the probability of connectivity. For example, for a network of a sink node at 0 and one sensor at d_1, the probability of connectivity is $P(0 \le d_1 \le R) = \dfrac{R-a}{b-a}$. It is the area under the density $f(l) = \dfrac{1}{b-a}$ over the interval $a \le l \le R$, see Fig. 3.

Corollary 5.1 (The Constant Corollary)*.* If in Connectivity Theorem 1 the distances between successive sensors have the constant density $f(l) = \dfrac{1}{b-a}$ on $[a,b]$, then

$$\text{the probability of connectivity is} \qquad P_n^c = \frac{\sum\limits_{k=0}^{n} (-1)^k \binom{n}{k} (L - a(n-k) - Rk)^n}{\sum\limits_{k=0}^{n} (-1)^k \binom{n}{k} (L - a(n-k) - bk)^n}.$$

Assume that $\dfrac{a+b}{2} \le R \le b$. The resulting network is connected with a probability p, where $0.17 \approx 1 - \sqrt{\ln 2} \le p < 1$, if the number of sensors satisfies $n \ge \max\left\{1 + \dfrac{L-b}{a},\ 3\log_2 \dfrac{1}{1-p}\right\}$.

The sums in P_n^c above include all expressions taken to the power n if they are positive. The computational *complexity* is the number of standard operations like multiplications and evaluating simple functions like $\ln x$. The computational complexity of P_n^c is linear in the number n of sensors. A linear algorithm computing P_n^c above initialises the array consisting of $n+1$ elements $L - iR$, $i = 0, \dots, n$, then finds $n \ln(L - a(n-k) - Rk)$ and $\exp(n \ln(L - a(n-k) - Rk)) = (L - a(n-k) - Rk)^n$. All $n+1$ binomial coefficients $\dbinom{n}{k}$ can be computed in advance. So the total complexity of computing P_n^c is $O(n)$.

For $n = 1$, one gets $P_1^c = \dfrac{(L-a) - (L-R)}{(L-a) - (L-b)} = \dfrac{R-a}{b-a}$ as expected above. If a given probability p is too close to 1, then the estimate from Constant Corollary 5.1 depends on p, e.g. $n \ge 29$ for $p = 0.9999$. However, in all reasonable cases, the maximum is $1 + \dfrac{L-b}{a}$, which is independent of p. The restrictions $\dfrac{a+b}{2} \le R \le b$ say that the distance between successive sensors is likely to be less than R since $[0,R]$ covers more than a half of $[a,b]$.

Each distance $d_i = x_i - x_{i-1}$ between successive sensors belongs to the interval $[a,b]$. Such a network lies within the given segment $[0,L]$ only if $an \le L$. Hence the number n of sensors should satisfy $n < \dfrac{L}{a}$. In the boundary case $an = L$, all sensors should be located at the exact positions $x_i = \dfrac{ia}{n}$, $i = 1, \dots, n$. This event clearly happens with the probability 0, so the numerator of P_n^c vanishes for $L = an$ in Constant Corollary 5.1.

Table 1: Number of sensors to guarantee connectivity for $f(l) = \dfrac{1}{b-a}$, $a = 0.2R$, $b = 1.6R$

Transmission radius R, m.	200	150	100	50	25
Minimum number of sensors	14	19	30	63	132
Our estimate of min number	18	27	43	93	193
Max number of sensors L/a	25	34	50	100	200

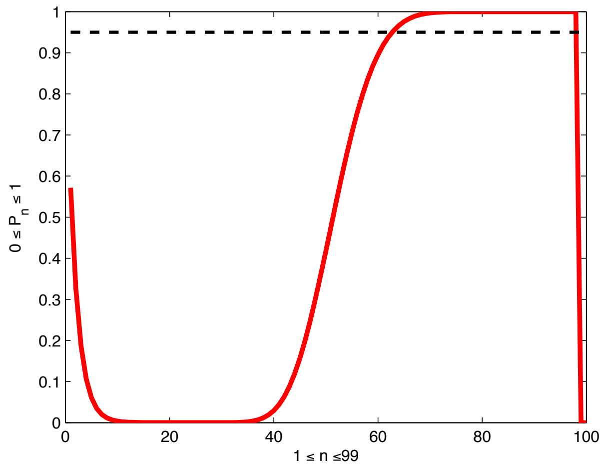

Figure 4: The probability of connectivity for $f(l) = \dfrac{1}{b-a}$, $a = 0.2R$, $b = 1.6R$, $R = 50$m

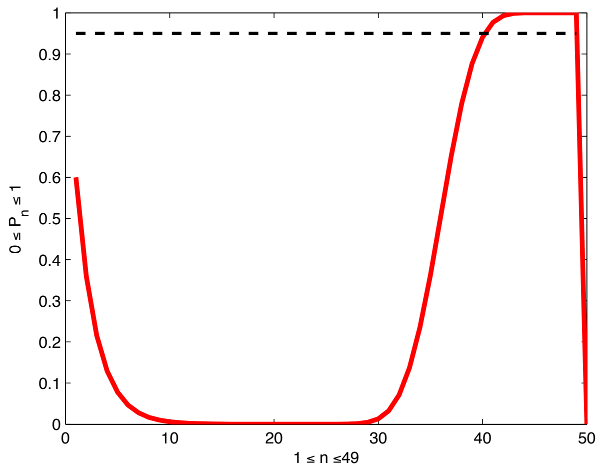

Figure 5: The probability of connectivity for $f(l) = \dfrac{1}{b-a}$, $a = 0.4R$, $b = 1.4R$, $R = 50$m

Figs. 4, 5, 6 show the probability of connectivity for different segments $[a, b]$ and $R =$ 50m. These probabilities were computed by randomly generating many networks according to a given density $f(l)$ and checking if the generated network is connected. Fig. 4 shows the probability P_n^c of connectivity for $1 \leq n \leq 100$, $L = 1$ km, $R = 50$m, $a = 0.2R$, $b = 1.6R$. This graph implies that if the required probability of connectivity is $p = 0.95$ and $R = 50$m,

Table 2: Number of sensors to guarantee connectivity for $f(l) = \dfrac{1}{b-a}$, $a = 0.4R$, $b = 1.4R$

Transmission radius R, m.	200	150	100	50	25
Minimum number of sensors	10	13	20	41	83
Our estimate of min number	10	14	22	47	97
Max number of sensors L/a	13	17	25	50	100

then the minimum number of sensors is 63, see the fourth number in the second row of Table 1. The estimates in the third row are from Constant Corollary 5.1.

The maximum possible number of sensors is $\dfrac{L}{a} = 100$. Then $P^c_{100} = 0$ since the sensors should be fixed at exact positions in $[0, L]$, which explains the drop to 0 in Fig. 6. The maximum number of sensors in Tables 1, 2, 3 is $\dfrac{L}{a}$, which gives probability 0 in this extreme case. All numbers slightly less than the maximum give a probability close to 1. Namely, subtracting $\dfrac{b}{a} - 1 = 7$ from the last row in Table 1 gives the previous row. This follows from the restriction $n \geq 1 + \dfrac{L-b}{a} = \dfrac{L}{a} - \left(\dfrac{b}{a} - 1\right)$.

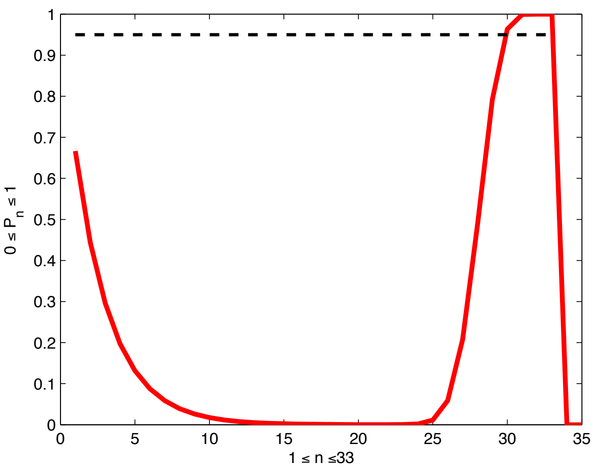

Figure 6: The probability of connectivity for $f(l) = \dfrac{1}{b-a}$, $a = 0.6R$, $b = 1.2R$, $R = 50$m

Tables 1, 2, 3 imply that the minimum number of sensors (to guarantee connectivity) decreases if $\dfrac{b-a}{R}$ decreases. For $b - a \leq R$, the estimate from Constant Corollary 5.1 is very

Table 3: Number of sensors to guarantee connectivity for $f(l) = \dfrac{1}{b-a}$, $a = 0.6R$, $b = 1.2R$

Transmission radius R, m.	200	150	100	50	25
Minimum number of sensors	8	10	15	31	61
Our estimate of min number	8	11	15	32	65
Max number of sensors L/a	10	13	17	34	67

close to the exact minimum number of sensors when sensors are deployed non-randomly at a distance slightly less than R. The found estimate for the minimum number of sensors requires few computations, which is a practical advantage.

Now consider the truncated normal density over $[0, L]$, i.e. $f(s) = \dfrac{c}{\sigma\sqrt{2\pi}} e^{-(s-\mu)^2/2\sigma^2}$, where the constant c guarantees that $\int\limits_0^L f(s)ds = 1$. The normal density has exponentially decreasing tails, so distances between successive sensors are likely to be close to the mean distance μ. Hence the mean μ should be less than the transmission radius R and the number n of sensors can not be greater than $\dfrac{L}{\mu}$, otherwise last sensors are likely to be outside $[0, L]$. That is why the Normal Corollary below gives an upper bound for the number of sensors that make a network connected, not a lower bound as in previous corollaries.

Corollary 5.2 (The Normal Corollary). If in Connectivity Theorem 1 the distances between successive sensors have the truncated normal density on $[0, L]$ with a mean μ and a standard deviation σ, then the network is connected with a given probability p for

$$n \le \min\left\{ \frac{p(1-p)}{\varepsilon},\ \frac{1}{4\mu^2}\left(\sqrt{4\mu L + \sigma^2\Phi^{-2}(p)} - \sigma\Phi^{-1}(p)\right)^2 \right\},\ \text{where}$$

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{x} e^{-s^2/2} ds \quad \text{and} \quad \varepsilon = \Phi\left(-\frac{\mu}{\sigma}\right) + 1 - \Phi\left(\frac{R-\mu}{\sigma}\right).$$

Values of the normal distribution $\Phi(x)$ are in standard tables. Table 4 shows estimates for the maximum number of sensors in $[0, L]$ with $L = 1$ km, $\mu = 0.6R$, $\sigma = 0.1R$ in such a way that the resulting network is connected with probability $p = 0.9975$. Then $\Phi^{-1}(p) \approx 2.8$, $\varepsilon \approx 0.000063$ and the first upper bound in Normal Corollary 5.2 gives $n \le \dfrac{p(1-p)}{\varepsilon} \approx 40$, which is the overall upper bound for $R = 25$m. For radii $R \ge 50$m, the second upper bound is smaller than the first one and is close to $\dfrac{L}{\mu}$. This is the exact number of sensors when all distances are not random and equal μ, because $\Phi^{-1}(p)\dfrac{\sigma}{R} \approx 0.28$ is small.

Table 4: Number of sensors to guarantee connectivity, the normal case $\mu = 0.6R$, $\sigma = 0.1R$

Transmission radius R, metres	200	150	100	50	25
Our estimate of maximum number	7	11	16	33	65
Average number of sensors L/μ	8	11	17	33	67
Number of non-random sensors L/R	5	6	10	20	40

The estimates from Table 4 are close to optimal, e.g. for the radius $R = 150$m the non-random distribution of sensors at distance 149m apart requires 6 sensors plus a sink node, while the estimate above gives 11. The ratio 6/11 is close to the mean $\mu/R = 0.6$ since distances between successive sensors should be around $\mu = 0.6R$.

6 Conclusions and Further Problems

The main results of the paper consist of the following.

i) Derivation of general formulae for the probability of connectivity and coverage of random one-dimensional networks when inter-node spacings have arbitrary probability densities. Explicit expressions for piecewise constant densities are derived in Appendices C-D.

ii) Estimates for a number of sensors to guarantee connectivity of a random one-dimensional network in the classical case when inter-node spacings have a constant or normal density. Examples are presented and it is shown that these estimates are closed to optimal for constant densities and for normal densities as it can be seen from Tables 2, 3, 4.

Problems that are of theoretical and practical importance and we will consider in our future research are the following.

• For a given number of sensors, find an optimal probability density of inter-node spacings to maximise the probabilities of connectivity and coverage of the resulting network.

• extend the analytical approach from Appendix A to two-dimensional networks, when sensors are randomly deployed along non-straight paths that fill a two-dimensional area.

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Appendix A. Proofs of the theorems from section 4

Let us recall the convolution and Laplace transform, which will be used in the proof of Connectivity Theorem 1, Coverage Theorem 2, Constant Corollary 5.1 and Normal Corollary 5.2. The *convolution* of functions f and g is $f*g(s) = \int\limits_{-\infty}^{+\infty} f(l)g(s-l)dl$. The convolution is commutative, associative, distributive and respects constants c, namely

$$(cf)*g = c(f*g),\ f*g = g*f,\ (f*g)*h = f*(g*h)\ f*(g+h) = f*g+f*h.$$

The convolution is very important in probability theory, because the density of the sum of two random variables is the convolution of the densities of the variables.

For a function $f(l)$ and $r > 0$, introduce the *truncated* function: $f^{[r]}(l) = f(l)$ for $l \in [0,r]$ and $f^{[r]}(l) = 0$ otherwise. Let $u(l)$ be the *unit step* function equal to 1 for $l \geq 0$ and equal to 0 for $l < 0$. Then $f^{[r]}(l) = f(l)(u(l) - u(l-r))$. We use the *partial* convolution $f(r,l) * g(r,l)$ only for the argument l, while r remains constant. The following lemma rephrases the definition of $v_n(r,l)$ in terms of convolutions.

Lemma 6.1. *For probability densities $f_1, \ldots, f_n$, the function $v_n(r,l)$ from Section 3 equals the iterated convolution $f_n^{[r]} * \cdots * f_1^{[r]} * u(l)$, where $r < l$ and $n > 0$.*

Proof is by induction on n. The base $n = 1$ is trivial: $f_1^{[r]} * u(l) = \int\limits_0^r f_1(s)u(s-l)ds = \int\limits_0^r f_1(s)ds = v_1(r,l)$ since $s \leq r < l$. The inductive step from $n-1$ to n follows by the recursive definition of v_n in section 4, namely $v_n(r,l) = f_n^{[r]} * v_{n-1}(r,l)$. □

The *Laplace* transform of a function $f(l)$ is the function $\mathrm{LT}\{f(l)\}(s) = \int\limits_0^{+\infty} e^{-sl} f(l)dl$. The Laplace transform is a linear operator that converts a convolution into a product, i.e. $\mathrm{LT}\{f*g\} = \mathrm{LT}\{f\}\mathrm{LT}\{g\}$ and also $\mathrm{LT}\{af+bg\} = a\mathrm{LT}\{f\} + b\mathrm{LT}\{g\}$. The inverse Laplace transform LT^{-1} is also a linear operator. The following well-known properties of the Laplace transform can be easily checked by integration.

Lemma 6.2. *For any α, β and integer $m \geq 0$, one has*
(a) $\mathrm{LT}\{l^m u(l)\} = \dfrac{m!}{s^{m+1}}$,
(b) $\mathrm{LT}\{e^{-\alpha l} l^m u(l)\} = \dfrac{m!}{(s+\alpha)^{m+1}}$,
(c) $\mathrm{LT}\{(l-\beta)^m u(l-\beta)\} = \dfrac{m! e^{-\beta s}}{s^{m+1}}$,
(d) $\mathrm{LT}\{e^{-\alpha(l-\beta)}(l-\beta)^m u(l-\beta)\} = \dfrac{m! e^{-\beta s}}{(s+\alpha)^{m+1}}$. □

Lemma 6.2 allows one to compute the inverse Laplace transform, e.g. $\mathrm{LT}^{-1}\{1/s\} = u(l)$ by Lemma 6.2(a). Lemma 6.3 below gives a powerful method for computing the function $v_n(r,l)$, which is used in Connectivity Theorem 1.

Lemma 6.3. *For densities* $f_1, \dots, f_n$ *on* $[0, L]$*, set* $g(s) = \frac{1}{s}\mathrm{LT}\{f_n^{[r]}(l)\} \cdot \ldots \cdot \mathrm{LT}\{f_1^{[r]}(l)\}$*. Then the function* $v_n(r, l)$ *equals the inverse Laplace transform* $\mathrm{LT}^{-1}\{g(s)\}(l)$*.*

Proof of Lemma 6.3. One has $v_n(r, l) = f_n^{[r]} * \dots * f_1^{[r]} * u(l)$ by Lemma 6.1. Set $g(s) = \mathrm{LT}\{v_n(r, l)\}$ and $g_i(s) = \mathrm{LT}\{f_i^{[r]}(l)\}$, $i = 1, \dots, n$. The Laplace transform is considered with respect to l, the variable r is a fixed parameter. The Laplace transform converts the convolution into the product, hence $g(s) = g_1(s) \dots g_n(s)/s$ as expected since $\mathrm{LT}\{u(l)\} = 1/s$ by Lemma 6.2(a). The order in the product $g(s)$ does not matter as the convolution is commutative. So any reordering of the densities gives the same result and the probability of connectivity does not depend on this order. □

Proof of Connectivity Theorem 1. Let $0 = x_0 \le x_1 \le \dots \le x_n \le L$ be the positions of a sink node and n sensors. Let the distances $d_i = x_i - x_{i-1}$, $i = 1, \dots, n$, be independent and have probability densities $f_i(s)$. Any network can be represented by ordered sensors $(x_1, \dots, x_n)$ or, equivalently, by the distances $(d_1, \dots, d_n)$ between successive sensors. Then the conditional probability of connectivity is the probability that the network is proper and connected, i.e. $\sum\limits_{i=1}^{n} d_i \le L$ and $0 \le d_i \le R$, divided by the probability that the network is proper, i.e. $\sum\limits_{i=1}^{n} d_i \le L$ and $0 \le d_i \le L$. The required formula $P_n = \dfrac{v_n(R, L)}{v_n(L, L)}$ for the conditional probability of connectivity follows from Probability Proposition 4.1 stated in section 4. Permuting densities leads to the same probability P_n due to the commutativity property of the convolution from Lemma 6.1. □

Proof of Probability Proposition 4.1.
Let us illustrate the proof first in the partial cases $n = 1, 2$. For $n = 1$ and $L > r$,

$$P(0 \le d_1 \le r) = \int_0^r f_1(s)ds = v_1(r, l) \text{ and } P(0 \le d_1 \le l) = \int_0^l f_1(s)ds = v_1(l, l).$$

For $n = 2$, let the number d_2 belong to an interval $[s, s+\Delta] \subset [0, r]$ for some small $\Delta > 0$. The probability of this event E is $P(E) = P(s \le y_2 \le s + \Delta) \approx f_2(s)\Delta$, the area of the narrow rectangle below the graph of f_2 over $[s, s + \Delta]$. The random variables $d_1 = x_1 - x_0$ and $d_2 = x_2 - x_1$ are independent. The probability of connectivity is

$$P(E)P(0 \le y_1 \le l - s) \approx f_2(s)\Delta \cdot v_1(r, l - s).$$

The total probability is the limit sum of these quantities over all intervals $[s, s+\Delta]$ covering $[0, R]$ when $\Delta \to 0$. Hence the probability is $\int\limits_0^r f_2(s)v_1(l - s)ds = v_2(r, l)$.

The case $n > 1$ is by induction on n. If the network is proper and connected, the n-th distance $d_n = x_n - x_{n-1} \ge 0$ is not greater than r and not greater than $l - \sum\limits_{i=1}^{n-1} d_i$. The former condition means that the last sensor is close enough to the previous one. The latter condition guarantees that all the sensors are within $[0,l]$.

Split $[0,r]$ into equal segments of a small length $\Delta > 0$. Suppose for a moment that $f_n(s)$ is constant on each segment $[s, s+\Delta]$, where $s = j\Delta$, $j = 0, \ldots, [r/\Delta]-1$. The general case will be obtained by taking the limit when $\Delta \to 0$.

The probability $P(d_n \in [s, s+\Delta])$ approximately equals $f_n(s)\Delta$, the area below the graph of $f_n(s)$ which is assumed to be constant over a short segment $[s, s+\Delta]$. The probability that the $n-1$ sensors form a connected network in $[0, l-d_n]$ is approximately $v_{n-1}(r, l-s)$ by the induction hypothesis.

Since the distances are independent, the joint probability is $f_n(s)\Delta \cdot v_{n-1}(r, l-s)$. The total probability is $v_n(r,l)$, the limit sum over all these events as $\Delta \to 0$:

$$\sum_{j=1}^{[r/\Delta]} f_n(j\Delta)\Delta \cdot v_{n-1}(r, l-j\Delta) \to \int_0^r f_n(s) v_{n-1}(r, l-s) ds.$$

The final expression above is the standard definition of the Riemannian integral of the function $f_n(s)v_{n-1}(r, l-s)$ over $[0,r]$ as a limit sum. □

Proof of Coverage Theorem 2. By Probability Proposition 4.1, the function $v_n(R,L)$ equals the probability of the event $E(L)$ that n sensors are deployed within $[0,L]$ and form a connected network. The network covers $[0,L]$ if also at least one sensor is in $[L-R, L]$, i.e. the event $E(L-R)$ does not happen. Hence the probability that the network is proper, connected and covers $[0,L]$ is $v_n(R,L) - v_n(R, L-R)$. The required conditional probability that all sensors are within $[0,L]$ is $\dfrac{v_n(R,L) - v_n(R,L-R)}{v_n(L,L)}$. □

Appendix B. Proofs of the corollaries from section 5

Iterated convolutions respect constant factors, namely

$$(c_n f_n)^{[r]} * \cdots * (c_1 f_1)^{[r]} * u = c_n \ldots c_1 f_n^{[r]} * \cdots * f_1^{[r]} * u.$$

Hence we may consider probability densities without extra factors if we are interested only in the conditional probability P_n from Connectivity Theorem 1. Indeed, the product of these factors will cancel after dividing $v_n(R,L)$ by $v_n(L,L)$.

Proof of Constant Corollary 5.1. The truncated constant density over the segment $[a,b]$ without extra factors is $f^{[r]}(l) = u(l-a(r)) - u(l-b(r))$, where $[a(r), b(r)] = [a,b] \cap [0,r]$ is the domain, where the probability density is defined and restricted to $[0,r]$. For instance, if $0 < a < R < b < L$ then $a(L) = a(R) = a$ and $b(L) = b$, $b(R) = R$.

By Connectivity Theorem 1 and Lemma 6.1, P_n^c is expressed in terms of $v_n(r,l) = (f^{[r]})^{(n*)} * u(l)$. Lemma 6.2(c) for $m=0$, $\beta = a(r)$, $\beta = b(r)$ implies that

$$\mathrm{LT}\{u(l-a(r)) - u(l-b(r))\} = \frac{e^{-a(r)s} - e^{-b(r)s}}{s}.$$

Apply Lemma 6.3 multiplying n factors and dividing by s:

$$g(s) = \frac{(e^{-a(r)s} - e^{-b(r)s})^n}{s^{n+1}} = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{e^{-a(r)s(n-k)-b(r)sk}}{s^{n+1}}.$$

After expanding the binom, compute the inverse Laplace transform of each term above by Lemma 6.2(d) for the parameters $\alpha = 0$, $\beta = a(r)(n-k) + b(r)k$, $m = n$ as follows:

$$v_n(r,l) = \mathrm{LT}^{-1}\{g(s)\} = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{(l - a(r)(n-k) - b(r)n)^n}{n!}.$$

To get the final formula for the conditional probability $P_n^c = \dfrac{v_n(R,L)}{v_n(L,L)}$ of connectivity, substitute $a(L) = a(R) = a$, $b(L) = b$, $b(R) = R$ and cancel $n!$

We estimate a minimum number of sensors to guarantee connectivity with a probability p. The condition $n \geq 1 + \dfrac{L-b}{a}$, i.e. $L - a(n-1) - b \leq 0$, implies that the denominator of P_n^c from the Constant Corollary is equal to $(L - an)^n$ corresponding to $m = 0$. Another assumption $\dfrac{a+b}{2} \leq R$ means that $b \leq 2R$, hence the numerator of P_n^c contains only two terms: $(L - an)^n - n(L - a(n-1) - R)^n$. The equivalent inequalities

$$P_n^c = 1 - n\left(\frac{L - a(n-1) - R}{L - an}\right)^n \geq p, \quad \left(\frac{L - an}{L - a(n-1) - R}\right)^n \geq \frac{n}{1-p} \text{ and}$$

$\left(1 + \dfrac{R-a}{L - a(n-1) - R}\right)^n \geq \dfrac{n}{1-p}$ are weaker than $\left(1 + \dfrac{R-a}{b-R}\right)^n \geq \dfrac{n}{1-p}$, because $L - a(n-1) \leq b$. Since $\dfrac{a+b}{2} \leq R$, then $r = \dfrac{R-a}{b-R} \geq 1$. By Lemma 6.4 the required inequality $(1+r)^n \geq 2^n \geq \dfrac{n}{1-p}$ follows from the bound $n \geq 3\log_2 \dfrac{1}{1-p}$ in Constant Corollary 5.1. $\square$

Lemma 6.4. *For any $0.17 \approx 1 - \sqrt{\ln 2} \leq p < 1$, if $t \geq 3\log_2 \dfrac{1}{1-p}$, then $2^t \geq \dfrac{t}{1-p}$ holds.*

Proof. We shall prove that the function $f(t) = (1-p)2^t - t \geq 0$ for $t \geq -3\log_2(1-p)$ and $1 - \sqrt{\ln 2} \leq p < 1$. Under these restrictions, the function $f(t)$ is increasing. Indeed, $f'(t) = (1-p)2^t \ln 2 - 1 \geq 0$, because $2^t \geq 2^{-3\log_2(1-p)} = \dfrac{1}{(1-p)^3} \geq \dfrac{1}{(1-p)\ln 2}$ since the equivalent condition $\sqrt{\ln 2} \geq 1 - p$ is given. It remains to prove that $2^t \geq \dfrac{t}{1-p}$ for $t = -3\log_2(1-p)$. Substitute: $2^{-3\log_2(1-p)} = \dfrac{1}{(1-p)^3} \geq \dfrac{t}{1-p}$. So we need to deduce the inequality $t = 3\log_2 \dfrac{1}{1-p} \leq \dfrac{1}{(1-p)^2}$ or $x^2 \geq 3\log_2 x$, where $x = \dfrac{1}{1-p}$. The function $g(x) = x^2 - 3\log_2 x$ has $g'(x) = 2x - \dfrac{3}{x\ln 2}$, the minimum point $x = \sqrt{\dfrac{3}{2\ln 2}} \approx 1.47$ and the approximate minimum value $g(1.47) \approx 0.49$. Hence $g(x) = x^2 - 3\log_2 x > 0$ for $x > 0$. $\square$

In the proof of Normal Corollary 5.2 apply the following estimate for iterated convolutions of truncated densities by using tails of the normal density $f(s)$.

Lemma 6.5. *Let distances d_i have the normal density $f(s) = \dfrac{1}{2\pi\sqrt{\sigma}} \exp\Big(-\dfrac{(s-\mu)^2}{2\sigma^2}\Big)$ over $\mathbb{R}$ with a mean μ and a deviation σ. Then*

$$v_n(r,l) = (f^{[r]})^{(n*)} * u(l) \geq P\Big(\sum_{i=1}^{n} d_i \leq l\Big) - n\varepsilon, \text{ where } \varepsilon = 1 - \int_0^r f(s)ds.$$

Proof of Lemma 6.5 is by induction on n. The base $n=1$ is easy: $v_1(r,l) =$

$$\int_{-\infty}^{+\infty} f^{[r]}(s)u(l-s)ds = \int_{-\infty}^{+\infty} f(s)u(l-s)ds - \int_{\mathbb{R}-[0,r]} f(s)u(l-s)ds \geq \int_{-\infty}^{l} f(s)ds - \varepsilon$$

$$= P(d_1 \leq l) - \varepsilon \text{ since } u(l-s) \leq 1 \text{ and } \int_{\mathbb{R}-[0,r]} f(s)u(l-s)ds \leq \int_{\mathbb{R}-[0,r]} f(s)ds = \varepsilon.$$

The induction step from $n-1$ to n is similar: $v_n(r,l) = \int_{-\infty}^{+\infty} f^{[r]}(s)v_{n-1}(r,l-s)ds =$

$$\int_{-\infty}^{+\infty} f^{[r]}(s)P\Big(\sum_{i=1}^{n-1} d_i \leq l-s\Big)ds - (n-1)\varepsilon \int_0^r f(s)ds \geq$$

$$\geq \int_{-\infty}^{+\infty} f(s)P\Big(\sum_{i=1}^{n-1} d_i \leq l-s\Big)ds - \int_{\mathbb{R}-[0,r]} f(s)ds - (n-1)\varepsilon$$

$$\geq P\Big(\sum_{i=1}^{n} d_i \leq l\Big) - n\varepsilon \text{ using } P\Big(\sum_{i=1}^{n-1} d_i \leq l-s\Big) \leq 1. \qquad \square$$

Proof of Normal Corollary 5.2. By Connectivity Theorem 1 the probability of connectivity is $P_n = \dfrac{v_n(R,L)}{v_n(L,L)}$, where the denominator $v_n(L,L) = (f^{[L]})^{(n*)} * u(L)$ is computed using the truncated normal density over $[0,L]$, while in $v_n(R,L) = (f^{[R]})^{(n*)} * u(L)$ the same density is truncated over the shorter interval $[0,R]$.

As usual, forget about extra constants in front of $f(s) = \dfrac{1}{\sigma\sqrt{2\pi}} \exp\Big(-\dfrac{(s-\mu)^2}{2\sigma^2}\Big)$. For a given probability p we will find a condition on n such that $P_n \geq p$. We will make the inequality $P_n \geq p$ simpler and stronger after replacing the complicated values $v_n(L,L)$ and $v_n(R,L)$ by their upper and lower bounds, respectively.

The denominator $v_n(L,L)$ is the iterated convolution of normal densities truncated over $[0,L]$. This convolution of positive functions becomes greater if we integrate the same functions over $\mathbb{R}$. Then $v_n(L,L) \le P(\sum\limits_{i=1}^{n} d_i \le L)$, the probability that the sum of n normal variables with the mean μ and deviation σ is not greater than L. The sum $\sum\limits_{i=1}^{n} d_i$ is the normal variable with the mean $n\mu$ and deviation $\sigma\sqrt{n}$.

Then $P(\sum\limits_{i=1}^{n} d_i \le L) = \Phi\left(\dfrac{L-n\mu}{\sigma\sqrt{n}}\right)$, where the standard normal distribution is $\Phi(x) = \dfrac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{x} e^{-s^2/2} ds$. Taking into account the lower estimate of $v_n(R,L)$ from Lemma 6, replace the inequality $P_n \ge p$ by the following stronger one:

$$1 - n\varepsilon / P\Big(\sum_{i=1}^{n} y_i \le L\Big) \ge p \quad \text{or} \quad \Phi\left(\frac{L-n\mu}{\sigma\sqrt{n}}\right) \ge \frac{n\varepsilon}{1-p}.$$

Split the last inequality into two simpler ones:

$$\Phi\left(\frac{L-n\mu}{\sigma\sqrt{n}}\right) \ge p \text{ and } p \ge \frac{n\varepsilon}{1-p}.$$

The latter inequality gives $n \le \dfrac{p(1-p)}{\varepsilon}$ as expected, where

$$\varepsilon = \frac{1}{\sigma\sqrt{2\pi}} \int\limits_{\mathbb{R}-[0,R]} f(s)ds = \Phi\left(-\frac{\mu}{\sigma}\right) + 1 - \Phi\left(\frac{R-\mu}{\sigma}\right).$$

The former inequality above becomes quadratic for the variable $\sqrt{n}$, i.e.

$$L - n\mu \ge \sigma\Phi^{-1}(p)\sqrt{n}, \quad \mu n + \sigma\Phi^{-1}(p)\sqrt{n} - L \le 0.$$

The final quadratic inequality implies that $\sqrt{n}$ is not greater than the second root $\dfrac{1}{2\mu}\Big(\sqrt{4\mu L + \sigma^2\Phi^{-2}(p)} - \sigma\Phi^{-1}(p)\Big)$ of the quadratic polynomial. So $n \le \dfrac{1}{4\mu^2}\Big(\sqrt{4\mu L + \sigma^2\Phi^{-2}(p)} - \sigma\Phi^{-1}(p)\Big)^2$ as required. □

Appendix C. Networks with sensors of different types

We derive an explicit formula and an algorithm for computing the probability of connectivity when inter-node spacings have different constant densities. These general settings will help study *heterogeneous* networks containing sensors of different types, e.g. of different transmission radii. Assume that each distance between successive sensors has one of k independent constant densities $f_j(l) = c_j$ on $[a_j, b_j] \subset [0,L]$ and $f_j(l) = 0$ otherwise, $j = 1, \ldots, k$. The condition $\int\limits_0^L f_j(l)dl = 1$ implies that $\dfrac{1}{c_j} = b_j - a_j$ for each $j = 1, \ldots, k$.

Note that the types of densities may not respect the order of sensors in $[0, L]$, e.g. the 1st and 3rd distances can be from the 2nd group of densities equal to $f_2(l)$, while the 2nd distance can be from the 1st group. In this case we say that index 1 belongs to group 2, symbolically $(1) = 2$. Here the brackets $(\cdot)$ denote the operator transforming an index $i = 1, \ldots, n$ of a distance into its group number (i) that varies from 1 to k. For a heterogeneous network, the function $v_n(r, l)$ from section 4 is a sum over arrays of signs $Q = (q_1, \ldots, q_n)$ depending on the densities $\{f_1, \ldots, f_k\}$. Let $[a_j(r), b_j(r)]$ be the intersection of the intervals $[0, r]$ and $[a_j, b_j]$, where $f_j \neq 0$. Set $q_i^{\pm} = \dfrac{1 \pm q_i}{2}$, e.g. $1^+ = 1$, $1^- = 0$.

Corollary 6.1 (The Heterogeneous Corollary). In the above notations and under the conditions of the Connectivity Theorem assume that the i-th distance between successive sensors has the probability density $f_{(i)}(l) = c_{(i)}$ on $[a_{(i)}, b_{(i)}]$, $i = 1, \ldots, n$. Then the probability of connectivity of the resulting heterogeneous network equals $P_n^h = \dfrac{v_n(R, L)}{v_n(L, L)}$, where

$$v_n(r, l) = \sum_{Q=(q_1,\ldots,q_n)}^{\langle Q \rangle < l} \frac{C_Q}{n!} \Big(l - \langle Q \rangle\Big)^n, \; C_Q = \prod_{i=1}^{n} (-1)^{q_i^+} c_{(i)}, \; \langle Q \rangle = \sum_{i=1}^{n} \Big(a_{(i)}(r) q_i^- + b_{(i)}(r) q_i^+\Big).$$

The indices in the brackets (i) from the last formula above take values $1, \ldots, k$ for each $i = 1, \ldots, n$, i.e. $[a_{(i)}(r), b_{(i)}(r)]$ is the segment where the (i)-th density $f_{(i)}$ is defined after restricting to $[0, r]$. In particular, if each i-th distance has its own density, then $(i) = i$ and the indices $i, j = 1, \ldots, n$ coincide.

We show that Constant Corollary 5.1 follows from Heterogeneous Corollary 6.1 for one constant density $f_1 = \dfrac{1}{b - a}$ on $[a, b]$, i.e. $k = 1$. To compute $v_n(L, L)$, note that $[a_1(L), b_1(L)] = [a, b]$. Let k be the number of pluses in an array Q. Then $C_Q = \dfrac{(-1)^k}{b - a}$ and $\langle Q \rangle = a(n - k) + bk$. So the sum over Q can be rewritten as a sum over $0 \leq k \leq n$. For any fixed k, there are $\dbinom{n}{k}$ different arrays Q containing exactly k pluses. By the Heterogeneous Corollary, the common term in the sum $v_n(L, L)$ over k is $(-1)^k \dbinom{n}{k} (L - a(n - k) - bk)^n$. The only difference in computing $v_n(R, L)$ is that $b_1(R) = R$, which leads to the formula from Constant Corollary 5.1.

The complexity to compute the function $v_n(r, l)$ from Heterogeneous Corollary 6.1 is $O(2^n)$, because $v_n(r, l)$ is a sum over 2^n arrays of signs and $\langle Q \rangle$ is a weighted sum of endpoints $a_i(r), b_i(r)$. In general, the expression $\langle Q \rangle$ can take 2^n different values. If there are only k different endpoints then the algorithm has the polynomial complexity $O(n^k)$, see 3-step Density Corollary 6.3 in Appendix D. If all $[a_j, b_j] \subset [0, R]$, then the network is connected and the formula above gives 1. Indeed, $v_n(R, L) = v_n(L, L)$ since $a_j(R) = a_j(L)$ and $b_j(R) = b_j(L)$, $j = 1, \ldots, k$.

Proof of Heterogeneous Corollary 6.1

extends the proof of Constant Corollary 5.1. Consider the truncated densities

$$f_i^{[r]}(l) = c_{(i)}\Big(u\big(l - a_{(i)}(r)\big) - u\big(l - b_{(i)}(r)\big)\Big),\ i = 1, \dots, n,\ \text{where}$$

(i) denotes the group containing the i-th distance. By Lemma 6.2(c) for $m = 0$ gives

$$\mathrm{LT}\{f_i^{[r]}\}(s) = c_{(i)} \frac{e^{-a_{(i)}(r)s} - e^{-b_{(i)}(r)s}}{s}.$$

Substitute each Laplace transform $\mathrm{LT}\{f_i^{[r]}\}(s)$ into the function $g(s)$ from Lemma 6.3 and expand the product $g(s)$, which gives the following sum of 2^n terms:

$$g(s) = \frac{1}{s}\prod_{i=1}^{n} c_{(i)} \frac{e^{-a_{(i)}(r)s} - e^{-b_{(i)}(r)s}}{s} = \sum_Q C_Q \frac{e^{-\langle Q\rangle s}}{s^{n+1}}.$$

The sum is taken over arrays $Q = (q_1, \dots, q_n)$ of signs. The sign $q_i = -1$ means that the term with $a_{(i)}(r)$ is taken from the i-th factor, the sign $q_i = +1$ encodes the second term with $b_{(i)}(r)$. The total power of the exponent in the resulting term corresponding to Q is $-\langle Q\rangle s$, where $\langle Q\rangle = \sum\limits_{i=1}^{n}(a_{(i)}(r)q_i^- + b_{(i)}(r)q_i^+)$. So each minus contributes $-a_{(i)}(r)s$ to the total power, while each plus contributes $-b_{(i)}(r)s$. Each plus contributes (-1) to the coefficient C_Q, i.e. $C_Q = \prod\limits_{i=1}^{n}(-1)^{q_i^+}c_{(i)}$ as required.

Compute the inverse Laplace transform by Lemma 6.2(d):

$$v_n(r,l) = \mathrm{LT}^{-1}\{g(s)\} = \sum_Q \frac{C_Q}{n!}\Big(l - \langle Q\rangle\Big)^n u(l - \langle Q\rangle),\ \text{where}$$

$u(l - \langle Q\rangle)$ can be replaced by the upper bound $l < \langle Q\rangle$ as in the final formula. □

Here is the algorithm to compute $v_n(r,l)$ in Heterogeneous Corollary 6.1:

- initialise two arrays $a_{(i)}(r)$ and $b_{(i)}(r)$, where $i = 1, \dots, n$;
- make a computational loop over 2^n arrays $Q = (q_1, \dots, q_n)$ of signs;
- for each array $Q = (q_1, \dots, q_n)$, compute $\langle Q\rangle$ and check the upper bound $l \le \langle Q\rangle$, then add $C_Q(l - \langle Q\rangle)^n$ to the current value of the function $v_n(r,l)$.

The algorithm to compute $v_n(L,L)$ is similar, simply replace R by L. If we need only P_n^h, forget about $n!$, which is cancelled after dividing $v_n(R,L)$ by $v_n(L,L)$.

Appendix D. Piecewise Constant Densities

This appendix shows how to compute the probability of connectivity by building any piecewise constant density from elementary blocks in Heterogeneous Corollary 6.1. The engine is Average Density Corollary 6.2 below dealing with the average $f(s) = \frac{1}{k}\sum\limits_{j=1}^{k} f_j(s)$ of constant densities $f_j(s) = c_j$ on $[a_j, b_j]$ and $f_j(s) = 0$ otherwise. The factor $\frac{1}{k}$ guarantees that $\int\limits_0^L f(s)ds = 1$, which follows from $\int_0^L f_j(s)ds = 1$.

For any ordered partition $n = n_1 + \cdots + n_k$ into k non-negative integers, denote by $(n_1, \ldots, n_k)$ the collection of densities, where the first n_1 densities equal f_1, the next n_2 densities equal f_2 etc. For example, for two constant densities f_1, f_2, number $n = 3$ can be split into two non-negative integers in one of the four ways, namely $3 = 0 + 3 = 1 + 2 = 2 + 1 = 3 + 0$. Then $(1, 2)$ denotes the collection (f_1, f_2, f_2), i.e. the 1st distance in such a network has the density f_1, while the remaining two distances have the density f_2. For each partition $(n_1, \ldots, n_k)$ or, equivalently, a collection of constant densities, let $v_n^{(n_1,\ldots,n_k)}(r, l)$ be the function defined by the formula from Heterogeneous Corollary 6.1 in Appendix C.

Corollary 6.2 (The Average Density Corollary). In the above notations and under the conditions of Connectivity Theorem 1, if distances between successive sensors have the probability density $f(l) = \frac{1}{k}\sum\limits_{j=1}^{k} f_j(l)$ on $[0, L]$, then the probability of connectivity is $P_n = \dfrac{\sum v_n^{(n_1,\ldots,n_k)}(R, L)/n_1!\ldots n_k!}{\sum v_n^{(n_1,\ldots,n_k)}(L, L)/n_1!\ldots n_k!}$. Both sums are taken over all collections of densities $(n_1, \ldots, n_k)$ corresponding to ordered partitions $n = n_1 + \cdots + n_k$.

The products $n_1!\ldots n_k!$ can not be cancelled in the formula above, because the numerator and denominator of P_n are sums of many terms involving different products $n_1!\ldots n_k!$ over all ordered partitions $n = n_1 + \cdots + n_k$. The complexity to compute P_n is $O(n2^n)$. Indeed, each $v_n^{(n_1,\ldots,n_k)}$ is computed by the algorithm described after Heterogeneous Corollary 6.1 using $O(2^n)$ operations. In partial cases the computational complexity can be reduced to polynomial, see comments after 3-step Density Corollary 6.3 below. The algorithm computing the probability from Average Density Corollary 6.2 applies the algorithm from Heterogeneous Corollary 6.1 to each function $v_n^{(n_1,\ldots,n_k)}(R, L)$ and $v_n^{(n_1,\ldots,n_k)}(L, L)$, then substitute results into the final formula.

Proof of Average Density Corollary 6.2. Forget about the factor $\frac{1}{k}$ as usual. Set $g_j(s) = \mathrm{LT}\{f_j^{[r]}\}$, $j = 1, \ldots, k$. Lemma 6.3 implies that $v_n(r, l) = \mathrm{LT}^{-1}\{g(s)\}$, where $g(s) = \frac{1}{s}\Big(\sum\limits_{j=1}^{k} g_j(s)\Big)^n$. Expand the brackets: $g(s) = \sum \dfrac{n!}{n_1!\ldots n_k!}\dfrac{g_1^{n_1}\ldots g_k^{n_k}}{s}$, where the sum is taken over all partitions $n = n_1 + \cdots + n_k$ into k non-negative integers.

By Lemma 6.3 each term $g_1^{n_1}\ldots g_k^{n_k}/s$ is the inverse Laplace transform of the function $v_n^{(n_1,\ldots,n_k)}(r, l)$, where the first n_1 distributions equal f_1, the next n_2 distributions equal f_2 etc. It remains to cancel $n!$ in the final expression. $\square$

By taking sums of constant densities c_j on $[a_j, b_j]$, one can get any piecewise constant function on $[0, L]$. Any reasonable probability density can be approximated by sufficiently many piecewise constant functions. Hence Heterogeneous Corollary 6.1 and Average Density Corollary 6.2 are building blocks for computing the probability of connectivity for any real-life deployment of sensors. This universal approach is demonstrated for the sum of two constant densities over two different segments. So the density in question is a 3-step function depending on the transmission radius R and one more parameter C. The graph of the density is in Fig. 7. Let $f = \dfrac{f_1 + f_2}{2}$ be the density on $[0, L]$ such that

$$\frac{f_1(l)}{2} = \begin{cases} C & \text{if } l \in [0, R], \\ 0 & \text{otherwise;} \end{cases} \qquad \frac{f_2(l)}{2} = \begin{cases} \dfrac{1}{R} - C & \text{if } l \in [R/2, 3R/2], \\ 0 & \text{otherwise,} \end{cases}$$

where C, R are chosen so that $0 < C < \dfrac{1}{R}$, $\dfrac{3}{2}R \le L$ and $\int\limits_0^L f(l)dl = 1$, see Fig. 7.

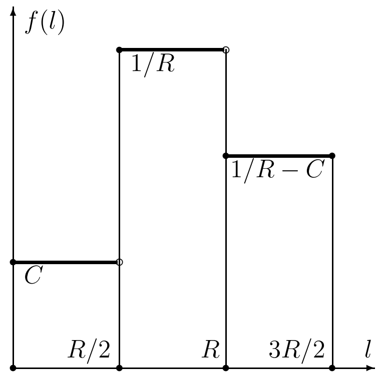

Figure 7: The piecewise constant distribution depending on R, C

From Fig. 7, for a network of a sink node at 0 and one sensor at d_1, the probability of connectivity is $P(0 \le d_1 \le R) = \dfrac{CR + 1}{2}$. This is the area of the first two rectangles below the graph of $f(l)$. For example, if $C = \dfrac{0.9}{R}$, then $P_1 = 0.95$ as shown in Fig. 8 below. Hence one sensor is likely to be close enough to the sink, although such a network can not cover the whole segment $[0, L]$. The 3-step Density Corollary shows how to compute the probability of connectivity explicitly for a piecewise constant density by using Heterogeneous Corollary 6.1 and Average Density Corollary 6.2.

Corollary 6.3 (The 3-step Density Corollary). Under the conditions of Connectivity Theorem 1 for the piecewise constant density $f(l)$ above, the probability of connectivity is

$$P_n^s = \frac{\sum\limits_{m=0}^{n} \sum\limits_{k_1=0}^{m} \sum\limits_{k_2=0}^{n-m} \dfrac{(-1)^{k_1+k_2}(L - (2k_1 + k_2 + n - m)R/2)^n}{D_m k_1!(m-k_1)!k_2!(n-m-k_2)!}}{\sum\limits_{m=0}^{n} \sum\limits_{k_1=0}^{m} \sum\limits_{k_2=0}^{n-m} \dfrac{(-1)^{k_1+k_2}(L - (2k_1 + 2k_2 + n - m)R/2)^n}{D_m k_1!(m-k_1)!k_2!(n-m-k_2)!}},$$

where $D_m = C^{-m}\Big(\frac{1}{R} - C\Big)^{m-n}$. The sums are over all possible values of m, k_1, k_2 such that the expressions in the brackets (taken to the power n above) are positive.

The complexity to compute the probability P_n^s above is $O(n^3)$, because the sums in the numerator and denominator are over three non-negative integers not greater than n and each term requires $O(1)$ operations. The result holds in the following extreme cases. If $C = \frac{1}{R}$, i.e all distances are in $[0, R]$, then set $d_m = 0$ for $m < n$. Hence $m = n$, $k_2 = 0$ and the sums over m, k_1, k_2 reduce to the same sum over $k_1 = 0, \dots, n$ in the numerator and denominator. So $P_n = 1$ as expected for $C = \frac{1}{R}$.

If $C = 0$, i.e. each distance is uniformly distributed on $[R/2, 3R/2]$, then set $D_m = 0$ for $m > 0$. So $m = 0$, $k_1 = 0$ and the result containing only sums over $k_2 = 0, \dots, n$ coincides with P_n^c from Constant Corollary 5.1 with $[a, b] = [R/2, 3R/2]$. Indeed, cancel D_0 and multiply the numerator and denominator by $k_2!$ to get $\binom{n}{k_2}$.

$$P_n^s = \frac{\sum\limits_{k_2=0}^{n} (-1)^{k_2} (L - (k_2 + n)R/2)^n / k_2!(n - k_2)!}{\sum\limits_{k_2=0}^{n} (-1)^{k_2} (L - (2k_2 + n)R/2)^n / k_2!(n - k_2)!}.$$

$$\text{or } P_n^s = \frac{\sum\limits_{k_2=0}^{n} (-1)^{k_2} \binom{n}{k_2} (L - (n - k_2)R/2 - k_2 R)^n}{\sum\limits_{k_2=0}^{n} (-1)^{k_2} \binom{n}{k_2} (L - (n - k_2)R/2 - 3k_2 R/2)^n}.$$

In 3-step Density Corollary 6.3 for $n = 1$, both sums contain only four non-zero terms corresponding to $(m, k_1, k_2) = (0, 0, 0);\ (0, 0, 1);\ (1, 0, 0);\ (0, 1, 0)$. Then all the factorials equal 1 and we get the probability expected from Fig. 7:

$$P_1^s = \frac{(\frac{1}{R} - C)(L - \frac{R}{2}) - (\frac{1}{R} - C)(L - R) + CL - C(L - R)}{(\frac{1}{R} - C)(L - \frac{R}{2}) - (\frac{1}{R} - C)(L - \frac{3R}{2}) + CL - C(L - R)} = \frac{CR + 1}{2}.$$

Proof of 3-step Density Corollary 6.3. In the notations of Heterogeneous Corollary 6.1 there are only $k = 2$ densities. Let the first m distances between successive sensors have the probability density f_1, while the last $n - m$ distances have the density f_2. An array Q splits into two parts of m signs and $n - m$ signs. Let k_1, k_2 be the number of pluses in each part.

To compute $v_n^{(m,n-m)}(L, L)$ from Average Density Corollary 6.2 for the partition $n = m + (n - m)$, note that $[a_1(L), b_1(L)] = [0, R]$, $[a_2(L), b_2(L)] = [R/2, 3R/2]$,

$$C_Q = \frac{(-1)^{k_1 + k_2}}{D_m}, \quad \langle Q \rangle = \Big(k_1 + k_2 + \frac{n - m}{2}\Big) R.$$

Table 5: Minimum number of sensors for the piecewise constant density, $C = \dfrac{0.9}{R}$.

Transmission radius, m.	250	200	150	100	50
Min number of sensors	12	17	25	44	105

So the sum over arrays Q can be rewritten as a sum over k_1, k_2. For fixed values of these parameters, there are $\binom{m}{k_1}\binom{n-m}{k_2}$ different arrays of signs. After cancelling the factorials $m!$ and $(n-m)!$ in Average Density Corollary 6.2, the sum $\sum\limits_{m=0}^{n} v_n^{(m,n-m)}(L, L)$ equals the denominator of the probability P_n^s. The only difference in computing $v_n(R, L)$ is that $b_2(R) = R$, not $3R/2$. Hence $2k_2$ is replaced by k_2. $\square$

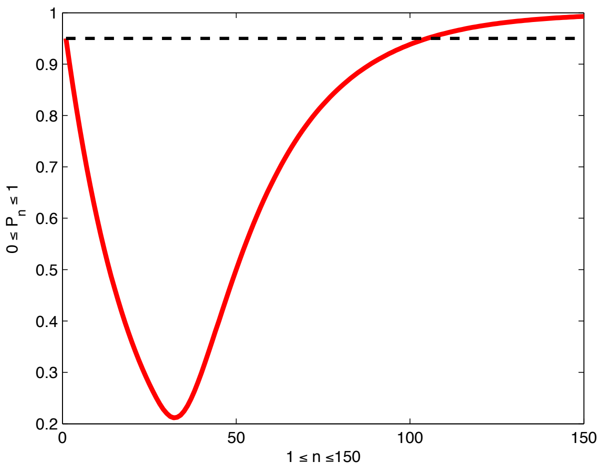

Figure 8: The probability of connectivity for the density $f(l)$ with $C = \dfrac{0.9}{R}$

Consider the piecewise constant density $f(l)$ for $C = \dfrac{0.9}{R}$ in Fig. 7. Table 5 shows the minimum number of sensors such that the network in $[0, L]$ is connected with probability 0.95, where $L = 1$km. Fig. 8 shows the probability $P_n^s \geq 0.2$, which were computed by generating many random networks. The threshold $p = 0.95$ is passed in Fig. 8 for the minimum number $n = 105$, see the last number in the second of row of Table 5.

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