Efficient Simulation Approaches for Reliability Analysis of Large Systems

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Abstract. Survival signature has been presented recently to quantify the system reliability. However, survival signature-based analytical methods are generally intractable for the analysis of realistic systems with multi-state components and imprecisions on the transition time. The availability of numerical simulation methods for the analysis of such systems is required. In this paper, novel simulation methods for computing system reliability are presented. These allow to estimate the reliability of realistic and large-scale systems based on survival signature including parameter uncertainties and imprecisions. The simulation approaches are generally applicable and efficient since only one estimation of the survival signature is needed while Monte Carlo simulation is used to generate component transition times. Numerical examples are presented to show the applicability of the proposed methods.

Keywords: Reliability Analysis; Survival Signature; Monte Carlo Simulation

1 Introduction

The structure of a complex system cannot be sequentially reduced considering alternative series and parallel sections. Consequentially, the study of the reliability of such systems is still a topical subject in the literature and it has obvious importance in many application areas [11]. Traditionally, the reliability analysis of systems is performed adopting different well-known tools such as reliability block diagram, fault tree and success tree methods, failure mode and effect analysis, and master logic diagram [12]. The main limitation of these traditional approaches is their lack of applicability for very large systems.

In recent years, the system signature [16] has been recognized as an important tool to quantify the reliability of systems consisting of independent and identically distributed (*iid*) or exchangeable components with respect to the random failure times. System signature separates the system structure from the component probabilistic failure distribution. However, when it is adopted to solve a complex systems with more than one component type, it requires the computation of the probabilities of all possible different ordering statistics of each component failure lifetime distributions, which is an intractable and tedious procedure. In order to overcome the limitations of the system signature, Coolen and Coolen-Maturi [7] proposed the use of survival signature for analysing complex systems consisting of more than one single component type increasing the applicability of such approach to characterise complex systems and networks. System survival signature can also be derived from the subsystems survival signature [8], which provides a theory for reliability analysis on real world systems of non-trivial size. Based on the above concepts, Feng et al. [10] developed an analytical method to calculate survival functions of systems with uncertain components parameters which belong to the exponential family. The analytical solutions are exact within the assumptions made, but they are sometimes hard or impossible to derive for large complex systems. Hence, general applicable numerical solutions are required to perform numerical experiments, analysing the effect of different distribution types and to overcome the limitation of analytical methods that are usually based on ad-hoc solutions and limited to some specific families of probability distribution functions. Simulation methods can be used for the sensitivity analysis of multi-criteria decision models [6], optimise models with rare events [15] and perform multi-attribute decision making [18].

In this paper, efficient simulation approaches are proposed to estimate the reliability of large systems based on survival signature without the calculation of all the cut-sets, which is a challenging and error prone task. The proposed simulation approaches are applicable to any system configuration and able to consider different representations of the uncertainties. The numerical implementation of the proposed approaches is based on two open source packages: the R package "ReliabilityTheory" [4, 3] adopted to calculate the survival signature and OpenCossan [14] a Matlab toolbox for uncertainty quantification and reliability analysis. The applicability and efficiency of the proposed approaches are demonstrated by solving numerical examples.

2 Background

2.1 Survival signature

Suppose there is one system formed by M components. Let the state vector of components be $\underline{x} = (x_1, x_2, ..., x_M) \in \{0, 1\}^M$ with $x_i = 1$ if the i-th component is in working state and $x_i = 0$ if not. $\Phi = \Phi(\underline{x}) : \{0, 1\}^m \rightarrow \{0, 1\}$ defines the system structure function, i.e., the system status based on all possible $\underline{x}$. Φ is 1 if the system functions for state vector $\underline{x}$ and 0 if not.

Now consider a system with $K \geq 2$ types of M components, with m_k indicating the number of components of each type and $\sum_{k=1}^{K} m_k = M$. It is assumed that the failure times of the same component type are independently and identically distributed (*iid*) or exchangeable. Coolen et al. [8] introduced the survival signature for such a system, denoted by $\phi(l_1, l_2, ..., l_K)$, with $l_k = 0, 1, ..., m_k$ for $k = 1, 2, ..., K$, which is defined to be the probability that the system functions given that l_k of its m_k components of type k work, for each $k \in \{1, 2, ..., K\}$.

There are $\binom{m_k}{l_k}$ state vectors $\underline{x}^k$ with precisely l_k components x_i^k equal to 1, so with $\sum_{i=1}^{m_k} x_i^k = l_k$. x_i^k denotes the state of the i-th component of type k.

Let $S_{l_1,l_2,...,l_K}$ denote the set of all state vectors for the whole system for which $\sum_{i=1}^{m_k} x_i^k = l_k$, $k = 1, 2, ..., K$. Assume that the random failure times of components of the different types are fully independent, and in addition the components are exchangeable within the same component types, the survival signature can be rewritten as:

$$\phi(l_1, ..., l_K) = \left[\prod_{k=1}^{K} \binom{m_k}{l_k}^{-1}\right] \times \sum_{\underline{x} \in S_{l_1,l_2,...,l_K}} \phi(\underline{x}), \tag{1}$$

$C_k(t) \in \{0, 1, ..., m_k\}$ denotes the number of type k components working at time t. Assume that the components of the same type have a known CDF, $F_k(t)$ for type k. Moreover, the failure times of different component types are assumed independent, then:

$$P(\bigcap_{k=1}^{K}\{C_k(t) = l_k\}) = \prod_{k=1}^{K} P(C_k(t) = l_k) = \prod_{k=1}^{K} \binom{m_k}{l_k} [F_k(t)]^{m_k - l_k} [1 - F_k(t)]^{l_k} \tag{2}$$

Hence, the survival function of the system with K types of components becomes:

$$P(T_s > t) = \sum_{l_1=0}^{m_1} ... \sum_{l_K=0}^{m_K} \phi(l_1, ..., l_K) P(\bigcap_{k=1}^{K}\{C_k(t) = l_k\}) \tag{3}$$

Equation (3) separates the structure of the system from the failure time distribution of its components, which is the main advantage of the system signature. The survival signature only needs to be calculated once for any system, which is similar to the system signature for systems with only single type of components. For a special case of a system with only one type ($K = 1$) of components, the survival signature and the system signature [16] are directly linked to each other through a simple equation, however, the latter cannot be easily generalized for systems with multiple types ($K \geq 2$) of components [7]. This implies that all attractive properties of the system signature also hold for the method using the survival signature. The survival signature is easy to apply for systems with multiple types of components, and one could argue it is much easier to interpret than the system signature.

2.2 Modelling the uncertainties

Multiple mathematical concepts can be used to characterize variability and uncertainty. Often in practical situations very limited data are available, and to avoid the inclusion of subjective and often unjustified hypothesis, the imprecision and vagueness of the data can be treated combining probabilistic and set theoretical components in a unified construct allowing the identification of bounds on probabilities for the events of interest in order to give a different prospective to the resultss [5]. Random Set theory is a general framework suited to model uncertainty represented as cumulative distribution functions (CDFs), intervals, probability boxes, normalized fuzzy sets (also known as possibility distributions) and Dempster-Shafer [9, 17] structures without making any implicit or explicit assumption at all. Explanatory examples of such flexible frameworks are provided in [13, 2, 1].

Without entering in the mathematical formalism, the Random Sets can be understood as random variables that sample sets and not points as realizations. These realization are called focal elements. When all focal elements are singletons, then the Random Set becomes a random variable.

Focal elements propagated through a model produce a collection of sets and not points. These sets are generally identified by means of an opportune optimization strategy. The collection of these set produces the so called Dempster-Shafer structure [9]. The upper and lower bounds of these structures form the distribution bounds of model output.

3 Simulation approaches

The survival signature presented in the Section 2.1 can be adopted in a Monte Carlo based simulation method to estimate the system reliability in a simple and efficient way. A possible system evolution is simulated by generating random events (i.e. the random transition time such as failures of the system components) and then estimating the status of the system based on the survival signature (Eq. 1). Then, counting the occurrence number of a specific condition (i.e. number of system failures), it is possible to estimate the reliability of the system.

3.1 Reliability analysis of system without Imprecision

The simulation approach is based on the realizations of failure events of the system's components. Then, for each failure event the status of the system is generated based on the probability that the system is working knowing that a specific number of components are working. Such probability is given by the survival signature as defined in Eq. 1. Suppose there is a system with M components, K component types and m_k components of type k. Hence, $M = \sum_{k=1}^{K} m_k$. The survival signature is computed only once before starting the Monte Carlo simulation. Without loss of generality, the lifetime distributions of components are irrespective of the time they enter into service and once failed they can not be repaired. The reliability of the system can be estimated adopting the following procedure:

1 Sample the failure times for each component.The failure time of component type k is obtained sampling from the CDF F_k corresponding to it.
2 Order the failure times $t_i \leq t_{i+1}$ for $i = 1, 2, \ldots, M$. Hence, t_1 represents the first failure of a system component, t_2 the second failure and so on.
3 At each failure time, it is easy to calculate the number of components working for each component type: $C_k(t_i)$.

4 Evaluate the survival signature which applies immediately after the corresponding failure indicated as $\phi_{t_i} \equiv \phi(C_1(t_i), C_2(t_i), \ldots, C_K(t_i))$.
5 Set $i = 1$ and draw from a Bernoulli distribution with probability $1 - \phi_{t_i}$ the system status X_1 at time t_i, if $X_i = 1$ the system fails.
6 If the system does not fail at t_i, then consider t_{i+1}. The probability that the system functions at time t_{i+1} is $\phi_{t_{i+1}}/\phi_{t_i} = q_{i+1}$, given that it has survived at time t_i. So the system failure at time t_{i+1} X_{i+1} is drawn from a Bernoulli distribution with the probability $1 - q_{i+1}$.
7 Repeat Step 6 to process other failure times ($i \leftarrow i + 1$).

The above procedure is repeated for N samples and the status of the system over the time is collected in appropriated counters. It should be noted that with the assumption that the system fails if no component functions, this implies that $q_{i^*} = 0$ for $i^* \leq M$. Hence the system fails certainly at this t_{i^*} if it has not failed before. The proposed algorithm is applicable to any system and requires the calculation of the survival signature only once.

It is also possible to estimate the system reliability without the necessity to sample the system status at each component failure time. The idea is to interpret the survival signature as a normalised "production capability" of the system defined by the Equation 1. For instance, if all the components are working, the system output is 1. If all components are in failure status, the system output is 0. Hence, instead of sampling the system state at each failure time, the survival signature is evaluated immediately after each sampled component failure time and collected in proper counters. In other words, for each Monte Carlo simulation this method generates a random grid of time points at which to evaluate the probability of survival to those times that represents the "production level of the system". Finally, the survival function is obtained by directly averaging of the survival signature over the time, i.e. computing the expected production level of the system adopting an algorithm derived from the approach proposed in [19]. Hence, the reliability of the system can be estimated modifying the steps 5-7 of the proposed approach as follows:

5' Compute the probability of survival (production level) of the system by evaluating the survival signature ϕ at each sampled component failure time t_i. For instance, the probability that the system survivals at time t_1 is ϕ_{t_1}.
6' Collect the value of the survival signature in appropriate counters (i.e. $Y(j) = Y(j) + \phi_{t_i}$ for $j : t_{i-1} \leq j * dt < t_i$) where Y represents the counter and dt the discretisation time used to store the results.

The above procedure is repeated for N samples and the reliability of the system is computed averaging the values of the survival signatures ($P(T_s > t) \approx \frac{1}{N} Y(t)$). The uses of the survival signature makes this approach extremely efficient since it does not require to sample the system output at each component transition time (i.e. component failures). The flow chart of the simulation methods proposed for estimating the reliability of non-repairable systems is shown in Figure 1.

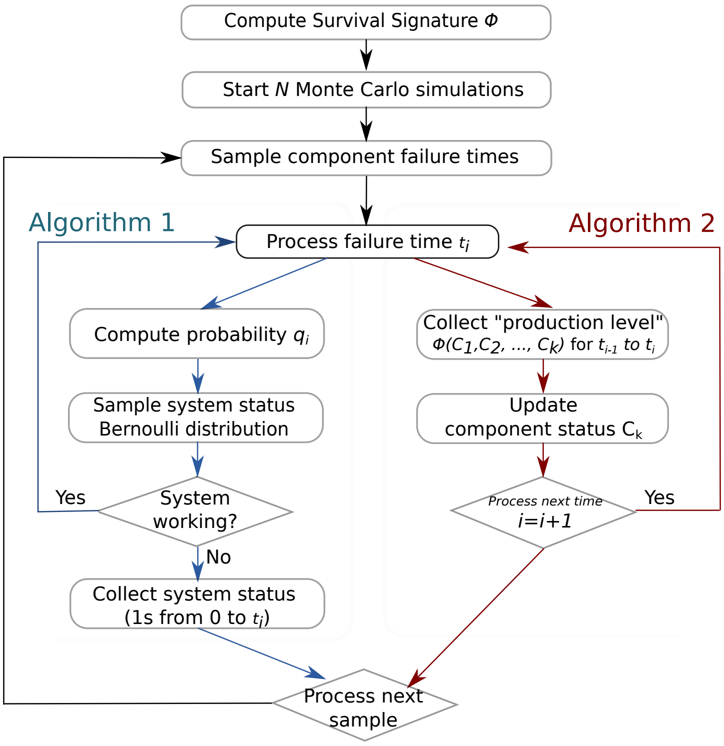

Fig. 1. Flow chart of the proposed numerical approaches.

3.2 Reliability Analysis of Systems with Imprecision

Reliability analysis of complex systems requires the probabilistic characterization of all the possible component transitions. This usually requires a large data-set that is not always available. In fact, it might not be possible to unequivocally characterize some component transitions due to lack of data or ambiguity. As mentioned in Section 2.2, to avoid the inclusion of subjective and often unjustified hypotheses, the imprecision and vagueness of the data can be treated by using concepts of imprecise probabilities. Randomness and imprecision are considered simultaneously but viewed separately at any time during the analysis and in the results. The probabilistic analysis is carried out conditional on the elements from the sets, which leads eventually to sets of probabilistic results.

Considering the imprecisions in the component parameters will lead to bounds of the survival function of the systems and it can therefore be seen as a conservative analysis, in the sense that it does not make any additional hypothesis with regard to the available information. In some instances, analytical methods will not be appropriate means to analyse a system. Again, simulation methods based on survival signature can be adopted to study systems considering parameter imprecision. A naive approach consists in adopting a double loop sampling where the outer loop is used to sample realizations in the epistemic space. In other words, each realization in the epistemic space defines a new probabilistic model that needs to be solved adopting the simulation methods proposed above. Then the envelop of the system reliability is identified. However, since almost all the systems are coherent (a system is coherent if each component is relevant, and the structure function is non decreasing) it is only necessary to compute the analysis reliability twice using the lower and upper bounds for all the parameters, respectively.

4 NUMERICAL EXAMPLES

4.1 Bridge System

The purpose of this numerical example is to verify the proposed algorithms since analytical solutions are available. The bridge system comprises six components, which belong to two types. It has no series section or parallel section which can enable simplification (see Figure 2). The survival signature can easily be computed either manually or using the R-package *ReliabilityTheory* [4]. The values of the survival signature are reported in Table 1. In this example the

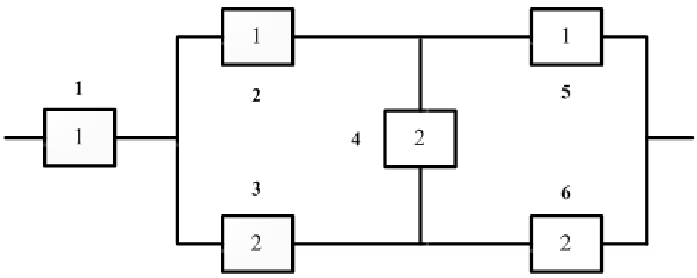

Fig. 2. Block diagramme of the bridge system with two types of components. The number inside the boxes represent the component type.

failure times of component type 1 and 2 both obey exponential distributions with parameters shown in Table 2. It is also assumed that the component once failed can not be repaired. The survival function of the bridge system is then calculated by means of the two proposed methods and compared with the analytical solution as shown in Figure 3. The simulations have been performed using $N = 5000$ samples and collecting the results in 2000 counters. The discretisation time is only required to collect the numerical results (i.e. survival function) although the simulation of the system is continuous with respect to the time. The variance of the estimators evaluated at time $t = 0.8$ and calculated by repeating the Monte

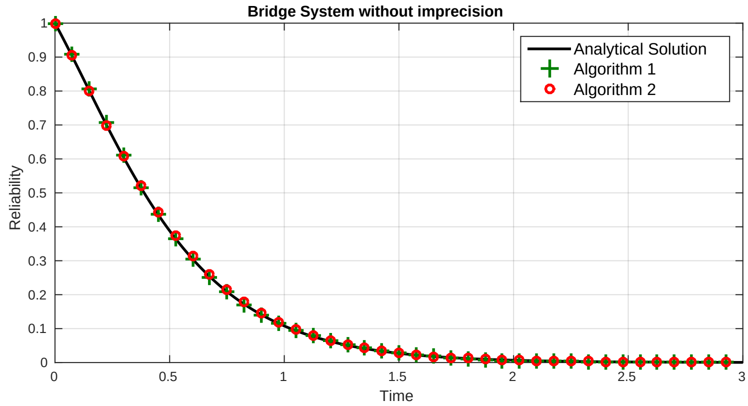

Fig. 3. Survival function of the bridge system calculated by means of the proposed simulation methods and analytical method, respectively.

Carlo simulation 20 times, is $3.0 \cdot 10^{-5}$ using the Algorithm 1 and $1.4 \cdot 10^{-5}$ with the Algorithm 2, respectively.

The system is also analysed in presence of imprecision on the parameter value. In particular, the only bounds of the exponential parameter is known as shown in Table 2. To estimate the bounds of the survival function, the analysis have been performed twice, using the lower and upper bound for all the parameters as explain in Section 3.2. The results are shown in Figure 4 and compared with analytical solutions estimated adopting the method presented in [10] showing a perfect agreement.

4.2 Imprecise System

In order to illustrate the efficiency and the applicability of the proposed simulation approaches a complex system composed by 8 components of 5 types is analysed. The component failure types and distribution parameters are shown in Table 4. In addition, it is assumed that the exact configuration of part of the system is unknown as shown in Figure 5. However, the system can still be described using the survival signature although affected by imprecision as shown in Table 3. Algorithm 2 is used to estimated the bounds of the survival function of the system by collecting the values (i.e. intervals) of the survival signature during the Monte Carlo simulation. The collected bounds are then used to estimate the bounds of the survival function as details in Ref [13]. For coherent systems as explain in Section 3.2. In principle, Algorithm 1 can also be used for the estimation of the reliability bounds although it requires some modifications in the sampling of the system status. The upper and lower bounds of survival function for the system with imprecision both in the survival signature and on the component distribution parameters are shown in Figure 6. The simulation have been performed using 5000 samples. This example shows the flexibility and

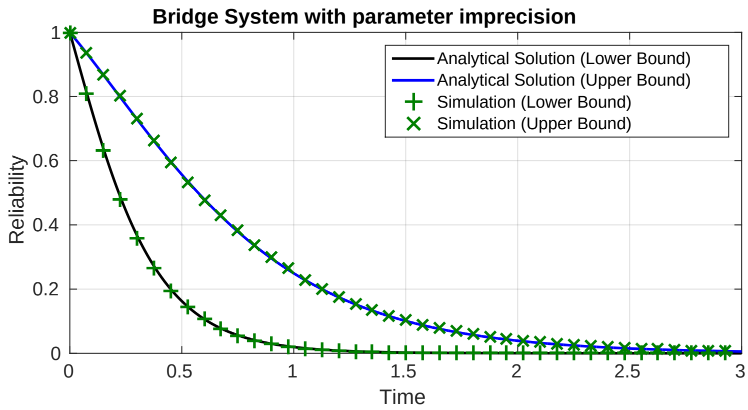

Fig. 4. Bounds of the survival function of the bridge system calculated by means of the simulation methods and compared with analytical solution.

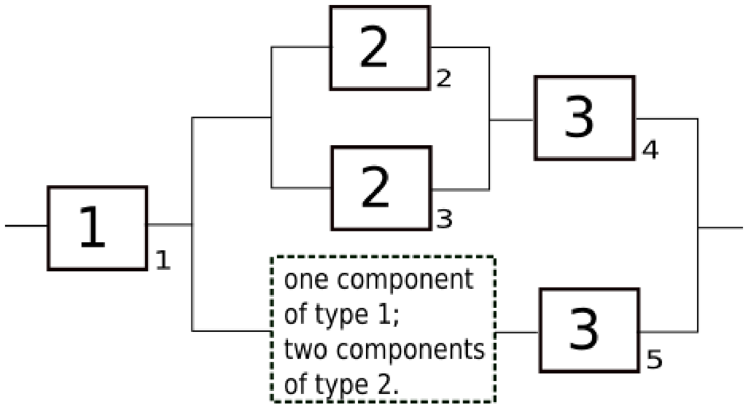

Fig. 5. An eight components system without knowing the exact configuration.

the applicability of the simulation approaches proposed for analysing complex systems affected by imprecision where no analytical solutions are available.

5 Conclusions

Complex systems occur in various engineering applications, the survival signature has been shown to be a practical method for performing reliability analysis of complex systems with multiple component types. However, there always exist maintainability and imprecision within systems and the analytical methods are only applicable in a few cases (e.g., for components only with exponential distribution types).

In this paper, efficient simulation methods have been proposed for system reliability estimation. In principle, the simulation methods proposed in this paper

Table 1. Survival signature of the bridge system of Fig.2

l_1	l_2	$\phi(l_1, l_2)$
0	$[0, 1, 2, 3]$	0
$[1, 2]$	$[0, 1]$	0
1	2	1/9
1	3	1/3
2	2	4/9
2	3	2/3
3	$[0, 1, 2, 3]$	1

Table 2. Failure rates of the components in the bridge system.

Component type	Distribution type	λ	λ (with imprecision)
1	Exponential	0.8	$[\underline{0.4}, \overline{1.2}]$
2	Exponential	1.5	$[\underline{1.3}, \overline{2.1}]$

Table 3. Imprecise survival signature of the system of Fig.5, $\phi(l_1, l_2, l_3) = 0$ and $\phi(l_1, l_2, l_3) = 1$ for both lower and upper bounds are omitted.

l_1	l_2	l_3	$[\phi(l_1, l_2, l_3)]$
1	1	1	[1/8,1/8]
1	1	2	[1/4,1/4]
1	2	1	[1/5,1/4]
1	2	2	[3/7,1/2]
1	3	1	[1/4,3/8]
1	3	2	[1/2,1/2]
1	4	1	[1/4,1/2]
1	4	2	[1/2,1/2]
2	0	1	[0,1/2]
2	0	2	[0,1]
2	1	1	[1/4,3/4]
2	1	2	[1/2,1]
2	2	1	[1/2,1]
2	3	1	[3/4,1]

Table 4. Components failure types and distribution parameters for system of Fig.5

Component type	Distribution	Parameters
1	Weibull	$([1.6, 1.8], [3.3, 3.9])$
2	Exponential	$([2.1, 2.5])$
3	Weibull	$([3.1, 3.3], [2.3, 2.7])$

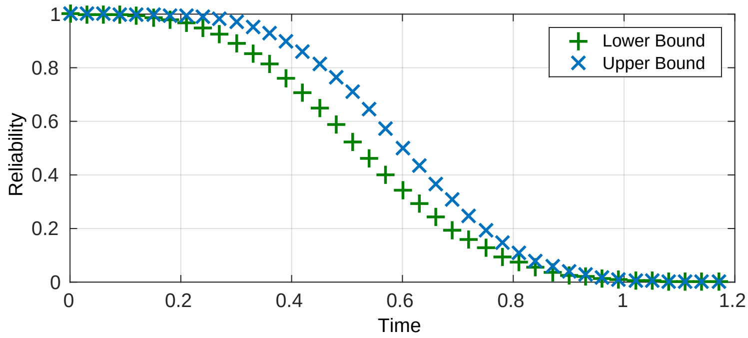

Fig. 6. Upper and lower bounds of survival function for the system in Fig 5.

have the ability to analyse system reliability by using only component failure time simulation and the survival signature, which is of great value for many systems in real world. The proposed simulation methods are generally applicable and able to deal with imprecision in the component distribution parameters as well in the system configuration (i.e. in the survival signature). Furthermore, they can easily be used to analyse non-repairable and repairable systems. The simulation methods are extremely fast and they can be adopted to analyse realistic and complex systems. The feasibility and effectiveness of the presented approaches have been illustrated with two numerical examples, the results indicate that simulation methods based on survival signature are efficient for analysing reliability on complex systems.

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