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Bayesian Combinatorial Auctions

George Christodoulou, Computer Science Department, University of Liverpool, Liverpool, UK. Email: gchristo@liv.ac.uk
Annamária Kovács, Informatics Institute, Goethe University, Frankfurt M., Germany. Email: panni@cs.uni-frankfurt.de
Michael Schapira, The School of Computer Science and Engineering, The Hebrew University of Jerusalem, Israel. Email:schapiram@cs.huji.ac.il

We study the following simple Bayesian auction setting: m items are sold to n selfish bidders in m independent second-price auctions. Each bidder has a *private* valuation function that specifies his complex preferences over *all* subsets of items. Bidders only have *beliefs* about the valuation functions of the other bidders, in the form of probability distributions. The objective is to allocate the items to the bidders in a way that provides a good approximation to the optimal social welfare value. We show that if bidders have submodular, or more generally XOS valuation functions, every Bayes-Nash equilibrium of the resulting game provides a 2-approximation to the optimal social welfare. Moreover, we show that in the full-information game a pure Nash always exists and can be found in time that is polynomial in both m and n.

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1. INTRODUCTION

1.1. (Bayesian) Combinatorial Auctions with Item Bidding

Combinatorial auctions are fundamental to economic theory and have also been the subject of much research in computer science. In a *combinatorial auction* m items $M = \{1, \ldots, m\}$ are sold to n bidders $N = \{1, \ldots, n\}$. Each bidder i has a *valuation function* (or valuation, in short) v_i that assigns a nonnegative real number to *every* subset of the items. v_i expresses i's preferences over *bundles* of items. The value $v_i(S)$ can be thought of as specifying i's maximum willingness to pay for S. Two standard assumptions are made regarding v_i: $v_i(\emptyset) = 0$ (normalization), and $v_i(S) \leq v_i(T)$ for every two bundles $S \subseteq T$ (monotonicity). The objective is to find a partition of the items among the bidders $S_1, \ldots, S_n$ (where $S_i \cap S_j = \emptyset$ for all $i \neq j$) such that the social welfare $\Sigma_i v_i(S_i)$ is maximized.

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The interplay between selfishness and computation in combinatorial auctions is well studied. In particular, much work has been devoted to the design of mechanisms that are both truthful, i.e., incentivize bidders to report their private information, and computationally efficient. Unfortunately, in general, truthfulness and computational efficiency are at odds. The celebrated VCG mechanisms [Vickrey 1961; Clarke 1971; Groves 1973] optimize the social welfare and motivate agents to truthfully report their private information. The caveat is that this may take exponential time [Nisan and Ronen 1999; 2000] (in the natural parameters of the problem m and n). If we disregard incentives, non-truthful algorithms obtain good approximations to the optimal social welfare in polynomial time for restricted, yet very expressive, special cases of combinatorial auctions (e.g., combinatorial auctions in which bidders have *submodular* valuations, also called "combinatorial auctions with *submodular bidders*" [Lehmann et al. 2001; Dobzinski et al. 2005; Dobzinski and Schapira 2006; Feige 2006; Feige and Vondrak 2006; Vondrak 2008]). Unfortunately, no truthful mechanism can approximate the optimum social welfare in combinatorial auctions with submodular bidders within any constant factor [Dobzinski 2011], so it is natural to relax the equilibrium condition. Also, practitioners typically resort to non-truthful auction mechanisms, e.g., the famous Generalized Second Price auction (GSP) for selling adwords.

We approach the central issue of reconciling selfishness and computation in auctions from an old-new perspective: Harsanyi [1968] introduced *Bayesian games* as an elegant way of modeling selfishness in partial-information settings. In a Bayesian game, players do not know exactly the private information of the other players, but only have *beliefs*, expressed by probability distributions over the different possible realizations of this private information. In combinatorial auctions, this translates to probability distributions over the possible valuation functions of the other bidders. We are interested in maximizing the social welfare in a way that is aligned with the interests of the different bidders. We ask the following question: Can we design an auction for which *any* Bayesian Nash equilibrium provides a good approximation to the optimal social welfare? We thus seek a Bayesian analogue of the *price of anarchy* [Koutsoupias and Papadimitriou 1999; Roughgarden and Tardos 2002] in this context[1].

Of much theoretical and practical interest are combinatorial auctions with item bidding, where the auctioneer sells the items by simultaneously running m independent single-item auctions. Inspired by auctions on eBay, we investigate the simple auction setting in which m items are sold in m independent second-price auctions, and each bidder can participate in any number of these single-item auctions. This auction setting induces a game in which a bidder's strategy is the m-dimensional vector of bids he submits in the different single-item auctions, and his payoff is his value for the set of items he is allocated minus his payments. Unfortunately, some unnatural complications arise: Consider the scenario that $m = 1$, $n = 2$, and the two bidders have complete information about each other. Let $v_1(1) = 1$ and $v_2(1) = 0$. Observe that the optimal social welfare is 1 (assign item 1 to bidder 1). Also observe that if bidder 1 bids 0 and bidder 2 bids 1 then this is a pure Nash equilibrium with a social welfare value of 0. Hence, the price of anarchy of this full-information game, that is, the ratio between the optimal solution and worst-case social welfare in equilibrium, is unbounded.

Observe that in the above scenario bidder 2 bid for (and got) an item he was not interested in possessing. We argue that such situations are unlikely to occur in practice, especially if bidders are only partially informed and are thus more inclined to avoid risks. We therefore make the assumption that bidders 'play it safe', in the sense that a bidder will not submit bids that might (in some scenario) result in getting a negative payoff. We call this the *no-overbidding* assumption. A strategy of bidder i is a *bid-*

[1] See [Garg and Narahari 2005; Gairing et al. 2005] for a similar approach to ours in selfish routing problems.

vector $b_i = (b_i(1), b_i(2), \ldots, b_i(m))$, where $b_i(j)$ represents i's bid for item j. A bid-vector is called *non-overbidding*, if the sum of bids over any set of items S does not exceed the value $v_i(S)$ of this set (see Definition 2.1). (We point out that the no-overbidding assumption can simply be viewed as the ex-post individual-rationality assumption from decision theory.)

1.2. Our Contributions

(Bayesian) Price of Anarchy in Auctions. We initiate the study of (Bayesian) price of anarchy in auctions and, in particular, of combinatiorial auctions with item bidding. Our main result establishes that when bidders valuations are *submodular*, i.e., exhibit decreasing marginal utilities, *any* (mixed) Bayes-Nash equilibrium of this auction game provides a good approximation to the optimal social welfare. A bidder i is said to have a *submodular* valuation function if for all $S, T \subseteq M$ $v_i(S \cup T) + v_i(S \cap T) \leq v_i(S) + v_i(T)$. Equivalently (see, e.g., [Lehmann et al. 2001]) v_i is submodular if for every two bundles $S \subseteq T$ that do not contain an item j it holds that

$$v_i(S \cup \{j\}) - v_i(S) \geq v_i(T \cup \{j\}) - v_i(T).$$

We present the following result:

Theorem: Under no-overbidding, and when all valuations are submodular, any (mixed) Bayes-Nash equilibrium of the auction game approximates the optimum social welfare within a factor of 2.

Our proof of the above result combines several new ideas and is interesting in its own right. In particular, the proof involves the first application of *smoothness* [Roughgarden 2012a] to games with incomplete information [Roughgarden 2012b].

Moreover, this result holds for the strictly broader class "fractionally-subadditive valuations" [Feige 2006] (defined, and termed XOS, in [Nisan 2000]). Importantly, our result is independent of the bidders' beliefs, i.e., the 2-approximation ratio is guaranteed for *any* common probability distribution ("common prior") over the valuation functions (we do require the common prior to be the product of independent probability distributions). Our approach thus suggests a middle-ground between the classical economic and the standard computer science approaches: Works in economics typically assume that the "input" is drawn from some *specific* probability distribution, and prove results that apply to that specific distribution. In contrast, computer scientists typically prefer worst-case analysis.

Existence and computability of pure Nash equilibria. We study the scenario of interest in which the valuation function of each bidder is known to all other bidders (or, equivalently, the common prior selects a single valuation profile with probability 1). We show that such full-information games always possess a *pure* Nash equilibrium even if the no-overbidding assumption does not hold. In fact, a simple argument establishes that the socially-optimal allocation of items to bidders is achievable in pure equilibrium. Hence, while the price of anarchy in these full-information games is unbounded (without the no-overbidding assumption), the *price of stability* (the social welfare of the *best* Nash equilibrium relative to the optimum [Anshelevich et al. 2004]) is 1.

Optimizing the social welfare in combinatorial auctions with submodular bidders is NP-hard [Lehmann et al. 2001]. Can a pure Nash equilibrium that provides a good approximation to the optimal social welfare be computed in polynomial time? We give the following answer for submodular bidders:

Theorem: When bidders have submodular valuations, a pure Nash equilibrium of a full-information game that approximates the optimal social welfare within a ratio of 2 can be computed in polynomial time.

The proof of this theorem shows that the 2-approximation algorithm for maximizing social welfare in combinatorial auctions with submodular bidders, due to Lehmann et al. [Lehmann et al. 2001], can be used to compute the bids in a pure Nash equilibrium. We note that similar questions have been studied by Vetta in [2002][2].

We provide, for the wider class of fractionally-subadditive valuation functions, a constructive way of finding a pure Nash that yields a 2-approximation via a simple and natural myopic procedure (inspired by the greedy approximation-algorithm in [Dobzinski et al. 2005]). Unfortunately, while this myopic procedure does compute a pure Nash equilibrium in polynomial time for some interesting (non-submodular) valuations, this is not true in general. We show that the myopic procedure may take exponential time by exhibiting a non-trivial construction of an instance on which this can occur.

1.3. Follow-Up work

Since the publication of the conference version of this paper [Christodoulou et al. 2008] there has been a fair amount of follow-up work on studying the price of anarchy of Bayes-Nash equilibria, and on combinatorial auctions with item bidding. Below, we briefly discuss the most relevant references.

Bhawalkar and Roughgarden [2011] and Feldman et al. [2013] study the price of anarchy of *simultaneous second-price auctions* for bidders with subadditive valuation functions. In [Bhawalkar and Roughgarden 2011] it is shown that the price of anarchy for pure Nash equilibria is 2, but is strictly greater than 2 for Bayes-Nash equilibria. Fu, Kleinberg and Lavi [2012] extended this bound for general valuation functions. [Bhawalkar and Roughgarden 2011] coined the term "combinatorial auctions with item bidding" that refers to settings like ours in which multiple items are sold concurrently in single-item auctions. [Feldman et al. 2013] proves upper and lower bounds on the price of anarchy for Bayes-Nash equilibria and presents the best upper and lower bounds to date: 2.061 and 4, respectively.

The Bayesian price of anarchy of *simultaneous first-price auctions* was studied in [Hassidim et al. 2011; Syrgkanis and Tardos 2013; Feldman et al. 2013; Christodoulou et al. 2013]. Hassidim et al. [2011] investigated the existence of pure and mixed equilibria in such auctions and also studied the Bayesian price of anarchy in this context. They showed that pure Nash equilibria are always efficient (when they exist), and they also upper bounds of 4, $O(\log m)$, and $O(m)$, for the price of anarchy of Bayes-Nash equilibria with XOS, subadditive and general valuations respectively. They also presented superconstant lower bounds for auctions with superadditive valuations. Syrgkanis and Tardos [2013], and Feldman et al. [2013], established upper bounds of $e/(e-1)$ for XOS valuations and 2 for subadditive valuations, respectively. Recently Christodoulou et al. [2013] showed tight lower bounds for both cases.

Paes Leme and Tardos [2010] initiated the study of the price of anarchy of *generalized second price auctions*. This work was followed in [Caragiannis et al. 2011; Caragiannis et al. 2012] for the full information and for the Bayesian setting and even for correlated distributions [Lucier and Paes Leme 2011]. Borodin and Lucier [2010]

[2]Vetta [2002] considers a general setting in which decisions are made by non-cooperative agents, and the utility functions are submodular. He proves (among others), that in this setting the price of anarchy is at most 2. The framework discussed there assumes players whose pure strategies are modeled by certain *subsets* of a ground set (e.g., the items). Apparently, even in the full-information, and submodular case, this framework is not applicable to our auction, where the bids on the items play a crucial role.

study the Bayesian price of anarchy of greedy mechanisms and Roughgarden [2012b] explores methods to bound the price of anarchy in games with incomplete information.

Markakis and Telelis [2012] studied the inefficiency of uniform price *multi-unit* auctions. De Keijzer et al. [2013] showed bounds for the price of anarchy of Bayesian equilibria for several formats of multi-unit auctions with first or second pricing schemes. Bhawalkar and Roughgarden [2012] study the effect of the price rule of a single item in the equilibrium performance of simultaneous auctions. Alon et al. [2012] investigate the price of anarchy in network *cost sharing* games in a Bayesian context.

2. BAYESIAN PRICE OF ANARCHY

In Subsection 2.1 we present the setting explored in this paper. Section 2.2 presents our Bayesian price of anarchy result for pure and mixed Bayes-Nash equilibria. The general proof for mixed equilibria can be found in Appendix A.

2.1. Bayesian Combinatorial Auctions with Item Bidding

The auction. m items are sold to n bidders in m independent second-price auctions (with some tie-breaking rule). A bidder's strategy is a *bid-vector* $b_i \in \mathbb{R}^m_{\geq 0}$ ($b_i(j)$ represents i's bid for item j). A *(pure) strategy profile* of all players is an n-tuple $b = (b_1, \ldots, b_n)$. We will use the notation $b = (b_i, b_{-i})$, to denote a strategy profile in which bidder i bids b_i and other bidders bid as in $b_{-i} = (b_1, \ldots, b_{i-1}, b_{i+1}, \ldots, b_n)$. We use $[j]$ to denote the set of integers from 1 to j.

Given a strategy profile b, the items are allocated according to the second price rule, i.e., every object is sold to the highest bidder at a price equal to the second highest bid.

For any fixed b we denote by $X_i(b)$ the set of items obtained by player i in the auction. For a set $S \subseteq M$, let the sum of the highest bids be denoted by

$$Bids(S, b) = \sum_{j \in S} \max_k b_k(j),$$

$$Bids(S, b_{-i}) = \sum_{j \in S} \max_{k \neq i} b_k(j),$$

and

$$Bids(S, b_i) = \sum_{j \in S} b_i(j).$$

The *utility* (payoff) of player i is then given by

$$u_i(b) = v_i(X_i(b)) - Bids(X_i(b), b_{-i}).$$

We make two assumptions about the bidders: no-overbidding, and that the v_is are *fractionally-subadditive*. A valuation is fractionally-subadditive if it is the pointwise maximum of a set of additive valuations: A valuation a is additive if for every $S \subseteq M$ $a(S) = \Sigma_{j \in S} a(\{j\})$. A valuation v_i is fractionally-subadditive if there are additive valuations $A_i = \{a_1, \ldots, a_l\}$ such that for every $S \subseteq M$ $v_i(S) = \max_{a \in A} a(S)$. (We will call $a_k \in A$ a *maximizing additive valuation* for the set S if $v_i(S) = a_k(S)$.)[3]

The class of fractionally-subadditive valuations is known to be strictly contained in the class of subadditive valuations and to strictly contain the class of submodular valuations [Nisan 2000; Lehmann et al. 2001].

[3]In order to have a finite (discrete) model, we can restrict acceptable bids to multiples of an arbitrary fixed ϵ, and not to exceed some maximum value B_{max}. In this case either we have to make the additional assumption that all $a(j)$ in the additive valuations of the sets A_i adhere to the same restrictions, or admit implicit ϵ errors in the derived bounds on the price of anarchy.

Definition 2.1. A bid vector b_i is said to be *non-overbidding* given a valuation v_i, if for all $S \subseteq M$ $v_i(S) \geq \Sigma_{j \in S} b_i(j)$.

Bayes-Nash Equilibria. For all i, let V_i denote the finite set of possible valuations of player i. The set of possible valuation profiles of the players is then $V = V_1 \times \ldots \times V_n$. There is a *known* probability distribution D over the valuations V (a *common prior*). D can be regarded as some market statistics that is known to all bidders (and to the auctioneer), and specifies their beliefs. We assume that $D = D_1 \times \ldots \times D_n$ is the cartesian product of independent probability distributions D_i : any valuation profile $v = (v_1, \ldots, v_n)$ occurs with probability $D(v) = \Pi_{i=1}^{n} D_i(v_i)$, where $D_i(v_i)$ is the probability that bidder i has the valuation function v_i. We will use the short notation $V_{-i} = \times_{k \neq i} V_k$, $D_{-i} = \times_{k \neq i} D_k$, and $v_{-i} = (v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n)$.

A *bidding-function* B_i for player i is a function that assigns a bid-vector $b_i = B_i(v_i)$ to every valuation function $v_i \in V_i$. The reader may find it helpful to think of B_i as a suggestion made to player i by the auctioneer as to which bid to submit. An n-tuple of bidding-functions $B = (B_1, \ldots, B_n)$ is a *Bayes-Nash equilibrium* if for every $i \in N$, and for every valuation function v_i, the bid $B_i(v_i)$ maximizes i's expected utility given that his valuation function is v_i, and that the bid of every other bidder j is $B_j(v_j)$, where v_j is drawn from D_j. That is, a Bayes-Nash maximizes i's expected payoff for any valuation function he may have, given his beliefs about the other bidders.

Bayesian price of anarchy. For a fixed valuation profile of the bidders $v = (v_1, \ldots, v_n)$, the optimal social welfare is $OPT(v) = max_{S_1, \ldots, S_n} \Sigma_i v_i(S_i)$, where the maximum is taken over all partitions of M into disjoint bundles $S_1, \ldots, S_n$. For given D, the (expected) optimal social welfare $SW(OPT)$ is the expectation $E[OPT(v)]$, where v is drawn from D. That is,

$$SW(OPT) = \sum_{v \in V} D(v) OPT(v)$$

Given a valuation profile v, every pure strategy profile b induces a social welfare value $SW(b) = \sum_{i \in N} v_i(X_i(b))$. For an n-tuple of bidding-functions $B = (B_1, \ldots, B_n)$, we denote by $SW(B)$ the expected social welfare $E[SW(B_1(v_1), \ldots, B_n(v_n))]$, where the $v = (v_1, \ldots, v_n)$ is drawn from D :

$$SW(B) = \sum_{v \in V} D(v) SW(B(v)).$$

We are interested in Bayes-Nash equilibria B for which the ratio $\frac{SW(OPT)}{SW(B)}$ is small. The *Bayesian price of anarchy* of a game is

$$PoA = \max_{D,\ B\ Bayes-Nash} \frac{SW(OPT)}{SW(B)},$$

that is the maximum of the expression $\frac{SW(OPT)}{SW(B)}$, taken over *all* probability distributions D, and *all* Bayes-Nash equilibria B (for these probability distributions). Intuitively, Bayesian price of anarchy of α means that no matter what the bidders' beliefs are, every Bayes-Nash equilibrium provides an α-approximation to the optimal social welfare.

2.2. Bayesian Price of Anarchy of 2

This subsection exhibits our main result. For the sake of readability we prove the theorem regarding the Bayesian price of anarchy for *pure* Bayes-Nash equilibria. The extension of the theorem to *mixed* Bayes-Nash equilibria is presented at the end of the section. The proof exploits the fractional subadditivity of the valuations via the following lemma:

LEMMA 2.2. *Let S be a set of items, and a be a maximizing additive valuation of player i for this set. If i bids according to a for the elements of S (and 0 for other elements), while all the others bid according to any pure profile b_{-i}, then*

$$u_i(a, b_{-i}) \geq v_i(S) - Bids(S, b_{-i}).$$

PROOF. Let $X_i := X_i(a, b_{-i})$ be the set of items that player i is going to get. Note that if i wins any item $j \notin S$ then the maximum bid on this j was 0. Thus we can assume w.l.o.g. that $X_i \subseteq S$ (otherwise the proof holds with $X_i \cap S$ instead of X_i). Moreover, $a(j) - Bids(\{j\}, b_{-i}) \leq 0$ holds for every non-obtained item $j \in S - X_i$. Therefore, we have

$$\begin{aligned} u_i(a, b_{-i}) &= v_i(X_i) - Bids(X_i, b_{-i}) \\ &\geq \sum_{j \in X_i} a(j) - Bids(X_i, b_{-i}) \\ &\geq \sum_{j \in S} a(j) - Bids(S, b_{-i}) \\ &= v_i(S) - Bids(S, b_{-i}). \end{aligned}$$

The first inequality uses the definition of $v_i(X_i)$ as the maximizing additive valuation for X_i whereas a is one of the additive valuations. The second inequality follows from the inequality for non-obtained items. □

THEOREM 2.3. *Let D be a distribution over fractionally-subadditive valuations of the bidders. If $B = (B_1, \ldots, B_n)$ is a Bayes-Nash, such that each B_i maps every valuation function v_i to a non-overbidding bid (given v_i) then $\frac{SW(OPT)}{SW(B)} \leq 2$.*

PROOF. Let $v = (v_1, \ldots, v_n)$ be a fixed valuation profile. We denote by $O^v = (O^v_1, \ldots, O^v_n)$ the optimum allocation with respect to profile v.

Now for every player i, let a_i denote the maximizing additive valuation for the set O^v_i, (we set $a_i(j) = 0$ if $j \notin O^v_i$). For all i, we consider a_i as an alternative strategy to $B_i(v_i)$.

Let us fix a bidder i. Let w_{-i} be an arbitrary valuation profile of all bidders except for i. We introduce the short notation

$$X_i^{w_{-i}} \stackrel{\text{def}}{=} X_i(B_i(v_i), B_{-i}(w_{-i})).$$

Furthermore, for any $S \subseteq M$ we will use

$$Bids^{w_{-i}}(S) \stackrel{\text{def}}{=} Bids(S, B_{-i}(w_{-i})),$$

resp.

$$Bids^{w}(S) \stackrel{\text{def}}{=} Bids(S, B(w)),$$

where $w = (w_i, w_{-i})$ is a complete valuation profile.

Since B is a Bayes-Nash, the strategy $B_i(v_i)$ provides higher expected utility to player i than the strategy a_i :

$$\sum_{w_{-i} \in V_{-i}} D(w_{-i}) u_i(B_i(v_i), B_{-i}(w_{-i})) \geq \sum_{w_{-i} \in V_{-i}} D(w_{-i}) u_i(a_i, B_{-i}(w_{-i})).$$

The utility values on the left-hand side are

$$u_i(B_i(v_i), B_{-i}(w_{-i})) = v_i(X_i^{w_{-i}}) - Bids^{w_{-i}}(X_i^{w_{-i}}) \leq v_i(X_i^{w_{-i}}).$$

On the right-hand side, applying Lemma 2.2 yields

$$u_i(a_i, B_{-i}(w_{-i})) \geq v_i(O_i^v) - Bids^{w_{-i}}(O_i^v).$$

By merging the inequalities above, we get

$$\begin{aligned}
\sum_{w_{-i}\in V_{-i}} D(w_{-i})v_i(X_i^{w_{-i}}) \;&\geq\; \sum_{w_{-i}\in V_{-i}} D(w_{-i})[v_i(O_i^v) - Bids^{w_{-i}}(O_i^v)] \\
&=\; v_i(O_i^v) \sum_{w_{-i}\in V_{-i}} D(w_{-i}) - \sum_{w_{-i}\in V_{-i}} D(w_{-i})Bids^{w_{-i}}(O_i^v) \\
&=\; v_i(O_i^v)\cdot 1 - \sum_{w\in V} D(w)Bids^{w_{-i}}(O_i^v) \\
&\geq\; v_i(O_i^v) - \sum_{w\in V} D(w)Bids^{w}(O_i^v).
\end{aligned}$$

Here the expected highest bids $\sum_{w_{-i}\in V_{-i}} D(w_{-i})Bids^{w_{-i}}(O_i^v)$, and $\sum_{w\in V} D(w)Bids^{w_{-i}}(O_i^v)$ are equal, because D is independent for all bidders. Finally, $Bids^{w_{-i}}(O_i^v) \leq Bids^{w}(O_i^v)$ obviously holds, since in the latter case we consider maximum bids over a larger set of players. We obtained

$$v_i(O_i^v) \leq \sum_{w_{-i}\in V_{-i}} D(w_{-i})v_i(X_i^{w_{-i}}) + \sum_{w\in V} D(w)Bids^{w}(O_i^v).$$

We sum over all i, and then take the expectation over all valuations $v = (v_1, \dots, v_n)$ on both sides:

$$\begin{aligned}
\sum_{v\in V} D(v) \sum_{i\in N} v_i(O_i^v) \;&\leq\; \sum_{v\in V} D(v) \sum_{i\in N} \sum_{w_{-i}\in V_{-i}} D(w_{-i})v_i(X_i^{w_{-i}}) \\
&\quad+\; \sum_{v\in V} D(v) \sum_{i\in N} \sum_{w\in V} D(w)Bids^{w}(O_i^v).
\end{aligned}$$

Note that $\sum_{v\in V} D(v) \sum_{i\in N} v_i(O_i^v) = SW(OPT)$. Furthermore, we claim that both summands on the right-hand side are at most $SW(B)$, so that $SW(OPT) \leq 2SW(B)$, which will conclude the proof. The first summand solves to

$$\begin{aligned}
&\sum_{i\in N} \sum_{v_i\in V_i} D(v_i) \sum_{v_{-i}\in V_{-i}} D(v_{-i}) \sum_{w_{-i}\in V_{-i}} D(w_{-i})v_i(X_i^{w_{-i}}) \\
=\;& \sum_{i\in N} \sum_{v_i\in V_i} D(v_i) \sum_{w_{-i}\in V_{-i}} D(w_{-i})v_i(X_i^{w_{-i}}) \sum_{v_{-i}\in V_{-i}} D(v_{-i}) \\
=\;& \sum_{i\in N} \sum_{v_i\in V_i} \sum_{w_{-i}\in V_{-i}} D(v_i)D(w_{-i})v_i(X_i^{w_{-i}})\cdot 1 \\
=\;& \sum_{i\in N} \sum_{v\in V} D(v)v_i(X_i^{v_{-i}}) \\
=\;& \sum_{v\in V} D(v) \sum_{i\in N} v_i(X_i(B(v))) = SW(B).
\end{aligned}$$

Finally, the second summand is

$$\begin{aligned}\sum_{v\in V} D(v) \sum_{w\in V} D(w) \sum_{i\in N} Bids^w(O_i^v) &= \sum_{v\in V} D(v) \sum_{w\in V} D(w) Bids^w(M) \\ &= \sum_{w\in V} D(w) Bids^w(M) \sum_{v\in V} D(v) \\ &= \sum_{w\in V} D(w) Bids^w(M)\cdot 1 \\ &= \sum_{w\in V} D(w) \sum_{i\in N} Bids(X_i(B(w)), B_i(w_i)) \\ &\le \sum_{w\in V} D(w) \sum_{i\in N} w_i(X_i(B(w))) = SW(B).\end{aligned}$$

The last inequality holds since for all i, the $B_i(w_i)$ contains non-overbidding bids for any set of items including the obtained set $X_i(B(w))$. $\square$

The bound on the price of anarchy holds in general for *mixed* Bayes-Nash equilibria:

THEOREM 2.4. *Let D be a distribution on the valuations of the bidders. If $p = (p_1, ..., p_n)$ is a mixed Bayesian Nash, such that for every i and every valuation v_i, the $p^{v_i}(= p_i^{v_i})$ is a probability distribution over* non-overbidding *bids w.r.t. v_i,*[4] *then* $\frac{SW(OPT)}{SW(p)} \le 2$.

The proof of the general case consists in a straightforward extension of all (in)equalities to expectations over the mixed bidding strategies. Appendix A contains the proof.

A simple example shows that even in the full-information setting, this Bayesian price of anarchy result is tight:

Example 2.5. Consider the following example, with 2 items and 2 players. The first player values the objects $v_1(1) = v_1(\{1,2\}) = 2$, and $v_1(2) = 1$, and symmetrically for the second player $v_2(2) = v_2(\{1,2\}) = 2$, and $v_2(1) = 1$. In the optimum partition, the first player gets the first object and the second player gets the second object. This results to a social welfare of 4.

If the first player bids $b_1(2) = 1, b_1(1) = 0$, and the second $b_2(1) = 1, b_2(2) = 0$, then the first player will get the second object, while the second player will get the first object. This results to a social welfare of 2. In addition, b is a pure Nash equilibrium. Consequently, the price of anarchy is at least $4/2 = 2$.

3. COMPUTING PURE NASH EQUILIBRIA

In this section we consider the following full-information game: The m items are sold to n bidders with fractionally-subadditive valuation functions in m independent second-price auctions. The players' valuation functions are assumed to be common knowledge.

In Subsection 3.1, we show that a pure Nash (with non-overbidding bids) that provides a good approximation to the social welfare, always exists in such games and provide a constructive way of finding one. We also prove that the price of stability [Anshelevich et al. 2004] is 1, i.e. the optimum can always be achieved in a Nash equilibrium. Finally, we describe an example to demonstrate that with this procedure it might take exponential time to find an equilibrium.

[4]Technically this condition can be realized so that $p^{v_i}(b_i) = 0$ whenever b_i is an *overbidding* bid.

In Subsection 3.2 we show that if bidders have submodular valuation functions then a good pure Nash can be reached in polynomial time.

3.1. Fractionally-Subadditive Valuation Functions

Despite the fact that (as shown in the Introduction) some Nash equilibria may fail to provide good approximation to the social welfare, we present a constructive way for finding a pure Nash that yields a 2-approximation. We introduce a natural procedure we call POTENTIAL PROCEDURE, which always reaches such an equilibrium. The POTENTIAL PROCEDURE is a simple myopic procedure for fractionally-subadditive bidders.[5]

For every i let $A_i = \{a^i_1, ..., a^i_{l_i}\}$ be a set of additive valuations such that for every $S \subseteq M$, $v_i(S) = \max_{a \in A_i} a(S)$. Since v_i is fractionally subadditive, such A_i must exist. The procedure simply starts with zero bids $b^*_i(j)$, zero per-item prices r_j, and an empty set of items $S_i = \emptyset$ allocated to each player i. Then it lets players best-reply one by one, to the bids of other players by switching to new non-overbidding bids. After every round, the sets S_i of all agents form a partition (w.l.o.g.) of the item-set, and for every player i the bids $b^*_i(j)$, on his 'own' items from S_i are determined by a maximizing additive valuation from A_i for S_i, and are zero for items not in S_i. The bids of *other* players we regard as current prices r_j on each item j for choosing a next best response.

In each round, an arbitrary dissatisfied player k selects a new set T that maximizes his utility given the current prices. Since prices are 0 over his own set S_k, it can be assumed w.l.o.g., that $T \supseteq S_k$ (otherwise $T \cup S_k$ could be chosen), and also that the new bids of player k are never lower than the current price (otherwise it would be better for him to choose a subset of T). This T becomes his new set, i.e., he robs items from other players, which results in a new item-partition; subsequently the bids and prices get updated as described above. We will prove that if after such a round all players are satisfied with their current sets, then the bids $(b^*_i)_{i \in N}$ correspond to a Nash equilibrium, (even in the case when overbidding is allowed).

Note that after every round the bids of *all* players need to be adjusted to sum up to the exact value of the current set of the player, and be zero elsewhere. In this sense, the procedure slightly differs from a pure best-response procedure, but can nevertheless be understood as a natural and realistic distributed process that non-colluding agents could use to converge to an equilibrium.

Remark 3.1. This procedure requires bidder i to be able to determine which bundle he would prefer most, given a vector of per-item prices $r = (r_1, ..., r_m)$. That is, to declare a bundle S for which $v_i(S) - \Sigma_{j \in S} r_j$ is maximized. This type of query is called a *demand query* and is very common in the combinatorial auctions literature.[6] In addition, the agents must be capable of responding XOS queries, in that for given S they can choose their maximizing additive valuation over S.

<u>POTENTIAL PROCEDURE:</u>

(1) Initialize $b^*_i(j) \leftarrow 0, S_i \leftarrow \emptyset, r_j \leftarrow 0$, for $i = 1, \ldots, n$ and $j = 1, \ldots, m$.
(2) While there is a bidder k such that $S_k \neq \arg\max_{S \subseteq M}(v_k(S) - \Sigma_{j \in (S \setminus S_k)} r_j)$:
  (a) Let $T = \arg\max_{S \subseteq M}(v_k(S) - \Sigma_{j \in (S \setminus S_k)} r_j)$; $\{$*w.l.o.g.* $S_k \subseteq T\}$
  (b) Set $S_k \leftarrow T$
  (c) For all $i \neq k$ let $S_i \leftarrow S_i \setminus S_k$
  (d) For all $i \in N$ do

[5] It can also be understood as a centralized local search procedure.
[6] See, e.g., [Blumrosen and Nisan 2005; Dobzinski et al. 2005; Feige 2006; Feige and Vondrak 2006].

i. Let $a^i \in A_i$ be such that $v_i(S_i) = a^i(S_i)$.
ii. Set $b_i^*(j) \leftarrow a^i(j)$ and $r_j \leftarrow a^i(j)$ for all $j \in S_i$ and $b_i^*(j) \leftarrow 0$ for all $j \notin S_i$

(3) Output $b^* = (b_1^*, \ldots, b_n^*)$.

We use a potential-function argument and the fractional-subadditivity of the bidders to show that the POTENTIAL PROCEDURE eventually converges to a “good” pure Nash.

THEOREM 3.2. *If the valuation functions are fractionally subadditive, then the* POTENTIAL PROCEDURE *ends in a pure Nash equilibrium that provides a* 2-*approximation to the optimal social welfare.*

PROOF. We divide the proof into two claims: first, we show that the procedure terminates, and second, that the last bids b^* correspond to a Nash equilibrium.

CLAIM 1. *If the valuation functions are fractionally subadditive, then the* POTENTIAL PROCEDURE *terminates after a finite number of rounds.*

PROOF. We use a potential function argument. For this purpose, we use as a potential function the function that assigns to a partition of items $X = (X_1, \ldots, X_n)$, the social welfare

$$SW(X) = \sum_{i \in N} v_i(X_i).$$

In particular, we are going to show that whenever there is a strict improvement to a player's utility, this translates into a strict improvement of the social welfare. This will prove that this procedure will terminate, as it converges to a local maximum of the social welfare. Later we will prove that any local maximum of the social welfare, corresponds to a pure Nash equilibrium.

Let $S^t = (S_1^t, \ldots, S_n^t)$ be the partition of the items and r_j^t the maximum bid for item j, after round $t = 1, 2, \ldots$ of the POTENTIAL PROCEDURE. W.l.o.g. assume that at time step t, player k is chosen by the procedure to improve his utility, i.e.

$$v_k(S_k^t) - \sum_{j \in S_k^t \setminus S_k^{t-1}} r_j^{t-1} > v_k(S_k^{t-1}),$$

where S_k^t and S_k^{t-1} play the roles of T and S_k, respectively, in step (2) of the procedure. For $i \neq k$, the prices r_j^{t-1} for items in S_i^{t-1} are determined by a maximizing additive valuation $a \in A_i$ for S_i^{t-1}. Therefore

$$v_i(S_i^t) = v_i(S_i^{t-1} \setminus S_k^t) \geq \sum_{j \in S_i^{t-1} \setminus S_k^t} r_j^{t-1} = v_i(S_i^{t-1}) - \sum_{j \in S_k^t \cap S_i^{t-1}} r_j^{t-1}.$$

The inequality holds by fractional subadditivity, since the r_j^{t-1} are defined by some additive valuation from A_i, whereas $v_i(S_i^{t-1} \setminus S_k^t)$ is determined by the *maximizing* additive valuation for this set. Since the $(S_i^{t-1})_{i \in N}$ form a partition, clearly $S_k^t \setminus S_k^{t-1} = \bigcup_{i \neq k} S_k^t \cap S_i^{t-1}$. Therefore, after summing up the above inequalities over all players i (also for $i = k$), the prices cancel out and we obtain

$$\sum_{i \in N} v_i(S_i^t) \; > \; v_k(S_k^{t-1}) + \sum_{j \in S_k^t \setminus S_k^{t-1}} r_j^{t-1}$$

$$
\begin{aligned}
&+ \sum_{i \neq k} \left(v_i(S_i^{t-1}) - \sum_{j \in S_k^t \cap S_i^{t-1}} r_j^{t-1} \right) \\
&= \sum_{i \in N} v_i(S_i^{t-1}) + \sum_{j \in S_k^t \setminus S_k^{t-1}} r_j^{t-1} - \sum_{i \neq k} \sum_{j \in S_k^t \cap S_i^{t-1}} r_j^{t-1} \\
&= \sum_{k \in N} v_k(S_k^{t-1}).
\end{aligned}
$$

Hence, the procedure will terminate, giving a partition that corresponds to a (local) maximum of the social welfare function. □

Now we will prove that the strategy profile b^* resulting the last partition of the POTENTIAL PROCEDURE, is a Nash equilibrium.

CLAIM 2. *If the valuation functions are fractionally subadditive, then the* POTENTIAL PROCEDURE *terminates with an n-tuple of bids that is a Nash equilibrium.*

PROOF. By contradiction, let's suppose that b^* is not a pure Nash equilibrium, i.e. there is a player i and a strategy b_i, such that $u_i(b_i, b^*_{-i}) > u_i(b^*) = v_i(S_i)$. Let $X_i = X_i(b_i, b^*_{-i})$ be the corresponding bundle of items that player i gets if he bids b_i. Based on b_i, we are going to construct a non-overbidding bid vector b'_i, that increases the utility of i at least as much as b_i does. We will show that b'_i corresponds to a set X'_i that would have been chosen by POTENTIAL PROCEDURE (X'_i or an even better set $T = \arg\max_{S \subseteq M}(v_i(S) - \Sigma_{j \in (S \setminus S_i)} r_j)$).

Let a be the maximizing additive valuation that corresponds to the set X_i. For every $j \notin X_i$, we assume that $b_i(j) = 0$ (we can decrease $b_i(j)$ to zero, without affecting player i's utility).

Let O be the set of all items $j \in X_i$ for which $a(j) < r_j$, and let a' be the maximizing additive utility of the set $X_i \setminus O$. The utility that player i would get if he bid 0 for all items in O, and a' on $X_i \setminus O$ would be

$$
\begin{aligned}
v_i(X_i \setminus O) - \sum_{j \in X_i \setminus O} r_j &= \sum_{j \in X_i \setminus O} (a'(j) - r_j) \\
&\geq \sum_{j \in X_i \setminus O} (a(j) - r_j) \\
&\geq \sum_{j \in X_i \setminus O} (a(j) - r_j) + \sum_{j \in O} (a(j) - r_j) \\
&= \sum_{j \in X_i} (a(j) - r_j) \\
&= v_i(X_i) - \sum_{j \in X_i} r_j \\
&= u_i(b_i, b^*_{-i}) \\
&> u_i(b^*) \\
&= v_i(S_i).
\end{aligned}
$$

Therefore, the POTENTIAL PROCEDURE could choose the set $X'_i = X_i \setminus O$, or a set that increases even more the utility of player i, i.e. $T = \arg\max_{S \subseteq M} v_i(S) - \sum_{j \in S \setminus S_i} r_j$.

This contradicts the fact that the stop condition (line 2) of the POTENTIAL PROCEDURE has been satisfied. □

The two claims and Theorem 2.3 imply that the POTENTIAL PROCEDURE results in a pure Nash equilibrium that is a 2-approximate solution.

Remark 3.3. Notice that by the second claim the global optimum of the social welfare potential function is also a pure Nash. This means that for any optimum partition $S^* = (S_1^*, \ldots, S_n^*)$ there always exist non-overbidding bids that form a Nash equlibrium (every player bids according to the maximizing additive valuation with respect to his set S_i^*), and the price of stability is 1.

Thus, we have a natural procedure, that is essentially a myopic best response sequence of the players, that leads to a pure Nash equilibrium. But, how long will it take the POTENTIAL PROCEDURE to converge? A non-trivial construction shows that unfortunately the worst case running time is exponential in n and m.

THEOREM 3.4. *There is an instance with* 2 *bidders, each with a fractionally-subadditive valuation function, on which the* POTENTIAL PROCEDURE *halts after* $\Omega(2^m)$ *steps.*

PROOF. Let $m = 2k$, and consider k pairs of items with the notation $M = \{1, 1', 2, 2', \ldots, k, k'\}$. We call $P \subset M$ a *proper* set, if ($|P| = k$ and) P contains exactly one item from each pair (e.g., the set $P = \{1, 2, \ldots, k\}$ is proper). The set of all proper sets is denoted by Π.

We define fractionally subadditive valuations v_1 and v_2 for the two bidders, so that the POTENTIAL PROCEDURE should proceed as follows. Assume first, that it starts from the partition $S_1 = \{1, 2, 3, \ldots, k\}$ and $S_2 = \{1', 2', 3', \ldots, k'\}$. In every two rounds the bidders exchange a pair of items: After any even round, S_1 and S_2 are complementary proper sets. In the next (odd) round, bidder 2 'robs' an item $y \in S_1$, by changing to $T = S_2 \cup y$. In return, in the coming even round bidder 1 takes the *pair* of the robbed item from bidder 2.

Bidder 1 (and also bidder 2) will possess each proper set exactly once. Since $|\Pi| = 2^k$, it takes $2 \cdot 2^k$ steps for the POTENTIAL PROCEDURE to converge. In order to go over every proper set systematically, the exchange of pairs follows the pattern of the Gray-code. This is a complete sequence of the k-long 0-1 vectors, such that every two consecutive vectors differ in exactly one bid. E.g., the Gray-code for $k = 3$ is (000, 001, 011, 010, 110, 111, 101, 100).

The jth bid being 0 in the vector means that $j \in S_1$ and $j' \in S_2$; the jth bid being 1 means the opposite. When the bid in the code is changed, the items j and j' should be exchanged in the corresponding two steps of the procedure (observe in the code that item j is exchanged 2^{j-1} times).

Next, we define fractionally subadditive valuations v_1 and v_2, for which the above sequence of exchanges is realized by the POTENTIAL PROCEDURE. The valuation v_1 consists of 2^k additive valuations $\{a_P\}_{P \in \Pi}$, one for each proper set. The rows in Table 1 show the additive valuations for $k = 3$, in the order in which these valuations and the respective proper sets appear as $a()$ and S_1 in even steps of the POTENTIAL PROCEDURE. Whenever an item x is in a proper set P, its value in the additive valuation a_P is $a_P(x) = 1 + s \cdot \epsilon$, where $\epsilon << 2^{-k}$, and the integer s indicates how many times this item was exchanged between the bidders so far.

Similarly, the valuation v_2 is the maximum of $2^k - 1$ additive valuations, as shown by Table 2. The rows in the table follow the order how bidder 2 responds in odd steps of the POTENTIAL PROCEDURE. Here the additive valuations take nonzero values on the current proper set $M \backslash S_1$ of bidder 2, and on an additional item y which is just being

1	1’	2	2’	3	3’
1		1		1	
1		1			$1+\epsilon$
1			$1+\epsilon$		$1+\epsilon$
1			$1+\epsilon$	$1+2\epsilon$	
	$1+\epsilon$		$1+\epsilon$	$1+2\epsilon$	
	$1+\epsilon$		$1+\epsilon$		$1+3\epsilon$
	$1+\epsilon$	$1+2\epsilon$			$1+3\epsilon$
	$1+\epsilon$	$1+2\epsilon$		$1+4\epsilon$	

Table I.

The additive valuations of bidder 1, in the order in which this bidder responds in even steps of the POTENTIAL PROCEDURE.

1	1’	2	2’	3	3’
	1		1	$1+\epsilon$	1
	1	$1+\epsilon$	1	$1+\epsilon$	
	1	$1+\epsilon$		$1+\epsilon$	$1+2\epsilon$
$1+\epsilon$	1	$1+\epsilon$			$1+2\epsilon$
$1+\epsilon$		$1+\epsilon$		$1+3\epsilon$	$1+2\epsilon$
$1+\epsilon$		$1+\epsilon$	$1+2\epsilon$	$1+3\epsilon$	
$1+\epsilon$			$1+2\epsilon$	$1+3\epsilon$	$1+4\epsilon$

Table II.

The additive valuations of bidder 2, in the order in which this bidder responds in odd steps of the POTENTIAL PROCEDURE.

’robbed’ from bidder 1 in the procedure. Again, the additive value of an item x is $1+s\cdot\epsilon$, where s shows how often x was exchanged so far.

It remains to prove that the procedure selects the given sets in the above order to be the set T (see the POTENTIAL PROCEDURE). Consider first bidder 2. After an even round he owns some proper set S_2. His payoff on every item of this S_2 is at least 1. On any new item the additional profit can be at most $s \cdot \epsilon << 1$. Therefore, the new set T will contain the current proper set S_2, and at most one more item, since every additive valuation $a()$ of bidder 2 takes nonzero values on a set of $k+1$ items. The proper set S_2 appears (with all members nonzero) in exactly two consecutive additive valuations, say in the sets $S_2 \cup \{y\}$ and $S_2 \cup \{z\}$. These correspond to the odd steps *before*, respectively *after* the current even step when bidder 1 took $S_1 = M \backslash S_2$. (E.g. the set $\{1, 2, 3\}$ appears in the sets $\{1, 2, 3, 3'\}$ and $\{1, 2, 2', 3\}$ in Table 2.) Setting $T = S_2 \cup \{y\}$ would be a step back, and would mean a negative profit of $-\epsilon$, since the previous response of bidder 1 increased r_y by ϵ (e.g., $r_{3'} = 1 + 3\epsilon$ in our example). On the other hand, setting $T = S_2 \cup \{z\}$ means an additional profit of ϵ to the value of S_2, so this is the best response for bidder 2.

The proof that the responses of bidder 1 follow the given order is analogous: at any step responding to bidder 2, bidder 1 has a value of $k-1$ from items, that currently only he bids for; giving up any of these would be a loss of about 1. Furthermore, he has the choice to bid for one of the two items in the remaining pair (one of which was just robbed by bidder 2), but he can only get the non-robbed item, because of the ϵ increase in the price for the robbed item.

Finally, we show how to modify this instance so that the procedure can start with the empty allocation $S_1 = S_2 = \emptyset$. We add two new items to the instance, named 0 and $0'$. For player one, item 0 appears with value 1 in the first row of the table, and with value 0 in every other row; item $0'$ has value $1+\epsilon$ in every row. Similarly, for player two, item $0'$ has value 1 in the first row, and value 0 in other rows; item 0 adds value $1+\epsilon$ to every row. Now, no matter which player responds first, the procedure follows the same order as shown above: Bidding the additive valuation of the first row offers a player an additional value of about 1 compared to any other row. In the next step he loses 1 by *any* response of the other player, and from then on all rows are equivalent concerning the items 0 and $0'$. □

Theorem 3.4 leaves us with two interesting open questions: First, will the POTENTIAL PROCEDURE converge in polynomial time for submodular valuations? Second, does the POTENTIAL PROCEDURE always run in time polynomial in the size of the sets A_i of additive valuations that underlie every fractionally-subadditive valuation (i.e., the number of the different additive valuations l_i constituting A_i)? An affirmative answer to this question would imply that the POTENTIAL PROCEDURE runs in polynomial time if the bidders have fractionally-subadditive valuations encoded in a bidding language (see [Nisan 2000; Lehmann et al. 2001; Dobzinski et al. 2005; Dobzinski and Schapira 2006]). Observe that in our example the size of these sets $|A_1|$ and $|A_2|$ is exponential in n and m.

3.2. Submodular Valuation Functions

In this subsection, we focus on submodular valuation functions. We present a polynomial time procedure that we call the MARGINAL-VALUE PROCEDURE, based on the algorithm due to Lehmann et al. [Lehmann et al. 2001]. We show that it results in a pure Nash equilibrium that also satisfies the premises of Theorem 2.3. Therefore, it provides a 2-approximation to the optimal social welfare.

<u>MARGINAL-VALUE PROCEDURE:</u>

(1) Fix an arbitrary order on the items. W.l.o.g. let this order be $1,\ldots,m$.
(2) Initialize $S_i \leftarrow \emptyset$, and $r_j \leftarrow 0$, for $i=1,\ldots,n$, and $j=1,\ldots,m$.
(3) For each item $j=1,\ldots,m$:
  (a) Let $i = \arg\max_{t\in N}\ v_t(S_t\cup\{j\}) - v_t(S_t)$. Set $S_i \leftarrow S_i \cup \{j\}$.
  (b) Set $r_j \leftarrow \max_{t\in N}\ v_t(S_t\cup\{j\}) - v_t(S_t)$.
(4) For every bidder i set $b_i^*(j) \leftarrow r_j$ for all $j\in S_i$ and $b_i^*(j)\leftarrow 0$ for all $j\notin S_i$.
(5) Output $b^* = (b_1^*, ..., b_n^*)$.

Observe, that the resulting n-tuple of bid-vectors b^* is such that for each item, only one bidder offers a non-zero bid for that item. Also notice that the MARGINAL-VALUE PROCEDURE only requires m rounds and so ends in polynomial time.

THEOREM 3.5. *If the valuation functions are submodular then a pure Nash equilibrium that provides a 2-approximation to the optimal social welfare can be computed in polynomial time.*

PROOF. We show that the execution of the MARGINAL-VALUE PROCEDURE, results in non-overbidding bids of the players that correspond to a pure Nash equilibrium.

For any $j\in[m]$ and an arbitrary set of items Y, let $Y^j \stackrel{\text{def}}{=} Y\cap[j]$.

CLAIM 3. *For every i, b_i^* is non-overbidding, given v_i.*

PROOF. Fix a player i. Clearly, it is enough to prove that $\sum_{j\in U} r_j \le v_i(U)$ for every $U\subseteq S_i$. For an arbitrary set of items T, the claim then follows from

$$\sum_{j\in T} b_i^*(j) = \sum_{j\in S_i\cap T} r_j \le v_i(S_i\cap T) \le v_i(T).$$

We fix a subset U. For each item $j \in U \subseteq S_i$, we define $a(j)$ to be the marginal value of the item, according to the given ordering, restricted to U (i.e., $a(j)$ depends also on the subset U). That is, for $j \in U$

$$a(j) \stackrel{\text{def}}{=} v_i(U^j) - v_i(U^{j-1}).$$

It is easy to see that $\sum_{j \in U} a(j) = v_i(U)$.

We claim that $r_j \leq a(j)$ for all $j \in U$. This will prove $\sum_{j \in U} r_j \leq v_i(U)$. The claim holds by submodularity, as

$$v_i(S_i^{j-1} \cup \{j\}) - v_i(S_i^{j-1}) \leq v_i(U^{j-1} \cup \{j\}) - v_i(U^{j-1}).$$

It follows from the MARGINAL-VALUE PROCEDURE, that on the left-hand side we have r_j. The right hand side is obviously $a(j)$. □

CLAIM 4. *The output b^* of the* POTENTIAL PROCEDURE *is a pure Nash equilibrium.*

PROOF. For the sake of contradiction, let's assume that b^* is not a Nash equilibrium. This means that player i instead of keeping his set $S_i = X_i(b^*)$, would rather bid for another set $T_i = X_i(b'_i, b^*_{-i})$, that he could get if he bid b'_i. Assume that T_i is maximal, that is, bidding for any superset of T_i would only decrease the utility of i. Player i is improving his utility by switching to b'_i, $u_i(b'_i, b^*_{-i}) > u_i(b^*)$. By construction of the MARGINAL-VALUE PROCEDURE, the items of S_i, are free for player i (since the other players bid 0 for those items), and therefore we obtain

$$v_i(S_i) < v_i(T_i) - \sum_{j \in T_i \setminus S_i} r_j. \tag{1}$$

Note that the maximality of T_i implies $S_i \subseteq T_i$, otherwise adding an item of $S_i \setminus T_i$ to T_i would not decrease $v_i(T_i)$ and would not increase the price, since the other players bid zero for this item.

Like in the proof of Claim 3, we define $a(j)$ on the items of the set $T_i :$ if $j \in T_i$ then $a(j) \stackrel{\text{def}}{=} v_i(T_i^j) - v_i(T_i^{j-1})$. We show that $a(j) \leq r_j$ for all $j \in T_i$. Using the definition of submodularity, $v_i(T_i^{j-1} \cup \{j\}) - v_i(T_i^{j-1}) \leq v_i(S_i^{j-1} \cup \{j\}) - v_i(S_i^{j-1})$. Note that on the left-hand side we have $a(j)$. Moreover, it follows from the definition of the MARGINAL-VALUE PROCEDURE that for $j \in S_i$ the right hand side equals r_j, whereas for $j \notin S_i$ it is at most r_j. Finally,

$$v_i(T_i) = \sum_{j \in T_i} a(j) \leq \sum_{j \in T_i} r_j = \sum_{j \in S_i} r_j + \sum_{j \in T_i \setminus S_i} r_j = v_i(S_i) + \sum_{j \in T_i \setminus S_i} r_j,$$

and this contradicts (1). □

The combination of the two claims above, together with Theorem 2.3 concludes the proof of Theorem 3.5.

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A. BAYESIAN PRICE OF ANARCHY OF 2 – MIXED NASH

Theorem. Let D be a distribution on the valuations of the bidders. If $p = (p_1, ..., p_n)$ is a mixed Bayesian Nash, such that for every i and every valuation v_i, the $p^{v_i}(= p_i^{v_i})$ is a probability distribution over non-overbidding bids w.r.t. v_i, then $\frac{SW(OPT)}{SW(p)} \leq 2$.

PROOF. Let $v = (v_1, ..., v_n)$ be a fixed valuation profile. We denote by $O^v = (O_1^v, \ldots, O_n^v)$ the optimum allocation with respect to profile v. Now for every player i, let a_i denote the maximizing additive valuation for the set O_i^v. For all i, we consider a_i as an alternative strategy to p^{v_i}.

Let us fix a bidder i. Let w_{-i} be an arbitrary valuation profile of all bidders except for i, respectively $w = (w_i, w_{-i})$ be a complete valuation profile.

Since p is a Bayes-Nash, the mixed strategy p^{v_i} provides better expected utility to player i than the (pure) strategy a_i :

$$\sum_{b_i} p^{v_i}(b_i) \sum_{w_{-i} \in V_{-i}} D(w_{-i}) \sum_{b_{-i}} p^{w_{-i}}(b_{-i}) u_i(b_i, b_{-i}) \geq$$

$$\geq \sum_{w_{-i} \in V_{-i}} D(w_{-i}) \sum_{b_{-i}} p^{w_{-i}}(b_{-i}) u_i(a_i, b_{-i}).$$

The utility values on the left-hand side are

$$u_i(b) = v_i(X_i(b)) - Bids(X_i(b), b_{-i}) \leq v_i(X_i(b)).$$

On the right-hand side, applying Lemma 2.2 yields

$$u_i(a_i, b_{-i}) \geq v_i(O_i^v) - Bids(O_i^v, b_{-i}).$$

By merging the inequalities above, we get

$$\sum_{b_i} p^{v_i}(b_i) \sum_{w_{-i} \in V_{-i}} D(w_{-i}) \sum_{b_{-i}} p^{w_{-i}}(b_{-i}) v_i(X_i(b)) \geq$$

$$\sum_{w_{-i} \in V_{-i}} D(w_{-i}) \sum_{b_{-i}} p^{w_{-i}}(b_{-i}) [v_i(O_i^v) - Bids(O_i^v, b_{-i})] =$$

$$v_i(O_i^v) \cdot 1 - \sum_{w_{-i} \in V_{-i}} D(w_{-i}) \sum_{b_{-i}} p^{w_{-i}}(b_{-i}) Bids(O_i^v, b_{-i}) =$$

$$v_i(O_i^v) - \sum_{w_{-i} \in V_{-i}} D(w_{-i}) \sum_{b_{-i}} p^{w_{-i}}(b_{-i}) Bids(O_i^v, b_{-i}) \sum_{w_i \in V_i} D(w_i) \sum_{b_i} p^{w_i}(b_i) =$$

$$v_i(O_i^v) - \sum_{w \in V} D(w) \sum_{b} p^{w}(b) Bids(O_i^v, b_{-i}) \geq$$

$$v_i(O_i^v) - \sum_{w \in V} D(w) \sum_{b} p^{w}(b) Bids(O_i^v, b).$$

Here we used that $\sum_{w_i \in V_i} D(w_i) \sum_{b_i} p^{w_i}(b_i) = 1$. Furthermore, note that $Bids(O_i^v, b_{-i}) \leq Bids(O_i^v, b)$ obviously holds, since in the latter case we consider maximum bids over a larger set of players. We obtained

$$v_i(O_i^v) \leq \sum_{b_i} p^{v_i}(b_i) \sum_{w_{-i} \in V_{-i}} D(w_{-i}) \sum_{b_{-i}} p^{w_{-i}}(b_{-i}) v_i(X_i(b)) +$$

$$+ \sum_{w \in V} D(w) \sum_{b} p^{w}(b) Bids(O_i^v, b).$$

We sum over all i, and then take the expectation over all valuations $v = (v_1, ..., v_n)$ on both sides:

$$\sum_{v \in V} D(v) \sum_{i \in N} v_i(O_i^v) \leq$$

$$\leq \sum_{v \in V} D(v) \sum_{i \in N} \sum_{b_i} p^{v_i}(b_i) \sum_{w_{-i} \in V_{-i}} D(w_{-i}) \sum_{b_{-i}} p^{w_{-i}}(b_{-i}) v_i(X_i(b)) +$$

$$+ \sum_{v \in V} D(v) \sum_{i \in N} \sum_{w \in V} D(w) \sum_{b} p^{w}(b) Bids(O_i^v, b).$$

On the left-hand side we have $\sum_{v \in V} D(v) \sum_{i \in N} v_i(O_i^v) = SW(OPT)$. Moreover, we show below that both summands on the right-hand side are at most $SW(p)$, so that $SW(OPT) \leq 2SW(p)$, which will conclude the proof. The first summand solves to

$$\sum_{i \in N} \sum_{v_i \in V_i} D(v_i) \sum_{v_{-i} \in V_{-i}} D(v_{-i}) \sum_{w_{-i} \in V_{-i}} D(w_{-i}) \sum_{b} p^{v_i, w_{-i}}(b) v_i(X_i(b)) =$$

$$\sum_{i \in N} \sum_{v_i \in V_i} D(v_i) \sum_{w_{-i} \in V_{-i}} D(w_{-i}) \sum_{b} p^{v_i, w_{-i}}(b) v_i(X_i(b)) =$$

$$\sum_{i \in N} \sum_{v \in V} D(v) \sum_{b} p^{v}(b) v_i(X_i(b)) =$$

$$\sum_{v \in V} D(v) \sum_{b} p^{v}(b) \sum_{i \in N} v_i(X_i(b)) = SW(p).$$

In the deduction above we obtained the third line from the second line by simply renaming w_{-i} by v_{-i}. Finally, the second summand is

$$\sum_{v \in V} D(v) \sum_{w \in V} D(w) \sum_{b} p^{w}(b) \sum_{i \in N} Bids(O_i^v, b) =$$

$$\sum_{v\in V} D(v) \sum_{w\in V} D(w) \sum_{b} p^w(b) Bids(M,b) =$$
$$\sum_{w\in V} D(w) \sum_{b} p^w(b) Bids(M,b) =$$
$$\sum_{w\in V} D(w) \sum_{b} p^w(b) \sum_{i\in N} Bids(X_i(b), b_i) \leq$$
$$\sum_{w\in V} D(w) \sum_{b} p^w(b) \sum_{i\in N} w_i(X_i(b)) = SW(p).$$

The last inequality holds since $p^w(b)$ can be nonzero only if for all i the bid b_i is non-overbidding w.r.t. w_i. $\square$