

1 **Multiphase segmentation based on new signed pressure force functions and one**
2 **level set function**

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8 **Abstract:** In this paper we propose a new model to detect multiple objects of various
9 intensities in images having maximum, minimum or mid-intensity background by
10 evolving only one level set function. In this model, a new signed pressure force function
11 based on novel generalized averages is used for segmentation of images with maximum
12 or minimum intensity background. Although for images with mid-intensity background
13 which are indeed challenging for 2-phase models, we propose a new product generalized
14 signed pressure force function. Finally, to give experimental and qualitative evidence, our
15 model is tested on both synthetic and real images with Jaccard Similarity Index. The
16 experimental and qualitative results reveal that the proposed method is efficient in both
17 global and selective segmentation. Our new model is also tested on colour images and the
18 results are compared with the state of the art models.

19 **Key words:** Segmentation, Level set, Geodesic active contours, spf function

20 **1. Introduction**

21 Image segmentation aims to extract or distinguish objects from each other in
22 images. In simple words, image segmentation extracts objects by distinguishing
23 foreground and background in images [1-6]. Variational image segmentation models are
24 categorized into two classes namely: edge based models and region based models. In edge

1 based models objects are extracted by capturing their boundaries while in region based
 2 models objects are detected by detecting their occupied optimal regions. For edge
 3 detection most of the models use an edge detector function which mainly depends on the
 4 gradient of a given image [4, 7, 8]. On the other hand, region based segmentation models
 5 use region detectors, named as fidelity terms, which use image statistical information to
 6 capture objects/regions [9-11]. In region based models one of the famous models is
 7 Mumford-Shah (MS) [10] which is given by:

$$8 \quad F(u, K)^{MS} = \int_{\Omega} (u - u_0)^2 dx dy + \alpha^* \int_{\Omega - K} |\nabla u|^2 dx dy + \beta \int_K ds \quad (1)$$

9 where u_0 is the given image, u is the required solution and is a smooth approximation
 10 of u_0 , $\alpha^*, \beta > 0$, Ω is the image domain and K is the set of edges (discontinuities) in the
 11 image. This model is theoretically very strong but computationally very complex. First
 12 variation of this model is piecewise constant MS model which consider u to be piecewise
 13 constant in each region. The level set methods [12] provided a good numerical
 14 implementation of piecewise constant (PC) MS models [5, 13] but unfortunately, for
 15 multi-region image segmentation usually multi-level set functions are evolved which are
 16 indeed time consuming approach in many situations. To enhance the performance of the
 17 PC model, Li et al. [9] proposed the local binary fitting (LBF) model by incorporating
 18 the image local statistical information in the PC model [5, 14,15]. This model may
 19 segment images with intensity in-homogeneity and gives competing results compare to
 20 the state of art models. In this model Gaussian kernel functions are used instead of
 21 average intensities. This model works well in images having intensity in-homogeneity,
 22 but computational cost is very high [16]. To overcome this drawback Wang et al.
 23 proposed local energy based model (Local Chan-Vese (LCV)) [15] which is given by:

$$F(\phi, c_1, c_2, d_1, d_2)^{LCV} = \mu \int_{\Omega} |\nabla H(\phi)| dx dy + \lambda_1 \int_{\Omega} ((u_0 - c_1)^2 + (u_0^* - u_0 - d_1)^2) H(\phi) dx dy$$

$$+ \lambda_2 \int_{\Omega} ((u_0 - c_2)^2 + (u_0^* - u_0 - d_2)^2) (1 - H(\phi)) dx dy, \quad (2)$$

where u_0^* denotes the smooth version of given image u_0 [15]. $\mu, \lambda_1, \lambda_2 > 0$ are trade off parameters and c_1, c_2, d_1, d_2 are average intensities of given image and difference image inside and outside level set function ϕ respectively. $H(\phi)$ is Heaviside function which is 0 if $\phi < 0$ and is 1 if $\phi > 0$. Similarly, Zhang et al. proposed an active contour model based on local image fitting (LIF) [16] for images with intensity in-homogeneity. Recently, Zhang et al. [16-18] combined the idea of geodesics [4] and Chan-Vese [5] model and proposed geodesic aided Chan-Vese (GCV) model as follows:

$$\frac{\partial \phi}{\partial t} = \alpha (spf) \cdot |\nabla \phi|, \text{ in } \Omega, \quad (3)$$

$$\phi(t, x, y) = \phi_0(x, y), \text{ in } \Omega,$$

where spf denotes the signed pressure force function [16-18] and $\alpha > 0$. In similar connection, Akram et al. [1] modified the model in eq. (3) for images with intensity in-homogeneity by replacing the spf function with their local- spf function [1]. Although these models have improved performance of the region based active contour models, they are not designed to handle images having multiple intensity objects [16-17]. For quantitative comparison we will use the Jaccard Similarity (JS) for different models. If we denote the segmented region by R_1 and the ground truth (GT) by R_2 , the JS is the ratio

of the areas of the intersection by the union of the regions, i.e. $JS(R_1, R_2) = \frac{|R_1 \cap R_2|}{|R_1 \cup R_2|}$. For

better results we want JS to be close to 1. The ground truth GT used in this paper is obtained manually in the following way: based on the maximum intensity we set a threshold value and then we choose GT as image \leq threshold value. In hardware image,

1 the intensity values of all objects are at the most 230, we choose threshold value 240 and
2 take GT as $\text{image} \leq 240$.

3 In this paper, we will propose models to handle the problems discussed above.
4 Our novel model is using a single level set function for segmentation of multi-region
5 images having background of either maximum intensity, minimum intensity or average
6 intensity and the objects are of homogeneous intensities. The new model is tested on both
7 synthetic and real images. Moreover, some tests on colour images are also conducted and
8 results are compared with a standard model.

9 Furthermore, for quantitative comparison of different segmentation models, the
10 Jaccard Similarity (JS) Index [13] is presented. From experimental results, it can be seen
11 that our proposed model perform well than the state of the art models. It can also be seen
12 that our proposed model equally works in both global and selective segmentation.

13 The rest of paper is organised in the following way. In the next section some
14 important materials and methods are discussed and the proposed model is given. In
15 section 3 experimental results, discussions about the experiments are given. In section 4
16 conclusions are given.

17 **2. The Proposed Model for Multi-region Segmentation**

18 In 2-phase framework, the averages of the Chan-Vese (CV) model [5] consider
19 objects of high intensities as background and objects with low intensities as foreground,
20 so the CV and GCV models are only capable of detecting objects of either high or lower
21 intensities. To handle this type of issues we design a new model based on generalized
22 averages which is defined in next section. In images with maximum intensity background,
23 we need large values of generalized averages than the CV averages to detect all the

1 objects. Similarly, in images with minimum intensity background, we need small values
 2 of generalized averages than the CV averages to detect all the objects.

3 **2.1 Generalized averages in segmentation framework**

4 **Definition 1:** We define generalized averages in the following way:

$$5 \quad G_{c_1} = \frac{\int_{\Omega} u_0^{\beta} H(\phi) dx dy}{\int_{\Omega} u_0^{\beta-1} H(\phi) dx dy}, \quad G_{c_2} = \frac{\int_{\Omega} u_0^{\beta} (1-H(\phi)) dx dy}{\int_{\Omega} u_0^{\beta-1} (1-H(\phi)) dx dy}, \quad (4)$$

6 where β is any real number, u_0 is the given image and H is the Heaviside function whose
 7 value is 0 if $\phi < 0$ and 1 if $\phi > 0$. The above family of averages is named as Av –
 8 family.

9 **Remark 1:** The value of the parameter β should be chosen greater than 1 for images of
 10 maximum intensity background, like $\beta = 2$. For images of minimum intensity background
 11 the value of the parameter β should be chosen smaller than 1, such as $\beta = -2$. For images
 12 of mid-intensity background, it has been observed that the β and number of pixels
 13 carrying maximum intensity are proportional. Experimentally we have derived the
 14 following relation, where 0.03 is the average of ratios of β to number of pixels carrying
 15 maximum intensity:

$$16 \quad \beta = 0.03 \times \text{number of pixels carrying maximum intensity.}$$

17 Now we consider the generalized averages in discrete form as

$$18 \quad G_{average} = \frac{\sum_{(i,j)} u_0(i,j)^{\beta}}{\sum_{(i,j)} u_0(i,j)^{\beta-1}}. \quad (5)$$

19 Let's consider the following theorem for analysis.

20 **Theorem 1:** For any positive real numbers x_1, x_2, \dots, x_n then

$$21 \quad (I) \quad \lim_{\beta \rightarrow \infty} G_{average} = \text{Max}\{x_1, x_2, \dots, x_n\}, \quad (II) \quad \lim_{\beta \rightarrow -\infty} G_{average} = \text{Min}\{x_1, x_2, \dots, x_n\}.$$

1 **Proof:** Let us suppose that $x_1 \geq x_2 \geq \dots \geq x_n$ (which is possible by relabeling and
 2 combining terms together). Then

$$3 \quad \lim_{\beta \rightarrow \infty} G_{average}(x_1, x_2, \dots, x_n) = \lim_{\beta \rightarrow \infty} \frac{\sum_{i=1}^n x_i^\beta}{\sum_{i=1}^n x_i^{\beta-1}} = \lim_{\beta \rightarrow \infty} \frac{x_1^\beta \sum_{i=1}^n \left(\frac{x_i}{x_1}\right)^\beta}{x_1^{\beta-1} \sum_{i=1}^n \left(\frac{x_i}{x_1}\right)^{\beta-1}} = x_1 = \text{Max}\{x_1, x_2, \dots, x_n\}.$$

4 Next, to prove the second part we proceed as follows:

$$5 \quad \lim_{\beta \rightarrow -\infty} G_{average} = \lim_{\beta \rightarrow -\infty} \frac{\sum_{i=1}^n x_i^\beta}{\sum_{i=1}^n x_i^{\beta-1}} = \lim_{\beta \rightarrow -\infty} \frac{x_n^\beta \sum_{i=1}^n \left(\frac{x_i}{x_n}\right)^\beta}{x_n^{\beta-1} \sum_{i=1}^n \left(\frac{x_i}{x_n}\right)^{\beta-1}} = x_n = \text{Min}\{x_1, x_2, \dots, x_n\}.$$

6 This result validates that, the more β deviates from 1 in positive direction, i.e. $\beta = 2, 3,$
 7 $4, 5, \dots$, the averages deviate from the central value and tend towards maximum value of
 8 their respective data set. Similarly, the more β deviates from 1 in negative direction, i.e.
 9 $\beta = 0, -1, -2, -3, \dots$, the averages deviate from the central value and tend towards
 10 minimum value of their respective data set. Thus we have now formulae (2) which can
 11 provide averages according to the images.

12 **Remark:** Although $G_{average}$ finds minimum and maximum value as $\beta \rightarrow -\infty$, and $\beta \rightarrow$
 13 ∞ respectively is shown mathematically in double precision but this is not the case for
 14 numerical overflows. This is given in Figure 1 (a) and (b) by taking a set of ten numbers
 15 $\{3, 5, 7, \dots, 21\}$ and $G_{average}$ is plotted against β .

16 2.2 Segmenting images with minimum or maximum intensity background.

17 In this subsection we design a new formulation based on spf function with generalized
 18 averages to segment images having minimum or maximum intensity background. For this
 19 we define novel generalized spf function based on generalized averages as:

1 **Definition 2:** The generalized spf function denoted by $Gspf$ is defined as follows:

$$2 \quad Gspf_{\beta}(u_0) = \frac{u_0(x, y) - \frac{G_{c_1} + G_{c_2}}{2}}{\max\left(u_0(x, y) - \frac{G_{c_1} + G_{c_2}}{2}\right)}, \text{ where } (x, y) \in \Omega. \quad (6)$$

3 In particular for $\beta = 1$ we have averages of the CV model [6] and consequently we have
4 spf function of [18]. Since the quality of detection in spf based models is totally dependent
5 on signs of a spf function and 2-phase homogeneous case is simple and can be seen in
6 [18]. For complete detection of foreground and background in an image, $Gspf$ function
7 must have opposite signs in foreground and background, otherwise result will be
8 incomplete. The signs of $Gspf$ function depend on values of averages which are clear
9 from equation (2) and we validate from the following tests that the types of averages
10 which we use, matter the quality of detection. Figure 1 (c), (d), (e) and (f) illustrate that
11 for different values of the parameter β we have different values of averages which help
12 in capturing all multi-intensity objects in an image. The given image in this case is a
13 maximum background image and the increment in β causes the detection of all the
14 objects. Now we give some theoretical results for justification of proposed model.

15 **Theorem 2:** Consider an image having q -objects with intensities I_1, I_2, \dots, I_q , let I_b be the
16 background intensity. Also let G_b and G_f be background and foreground averages. Then

17 (I) If $I_1 < I_2 < \dots < I_q < I_b$ (i.e., background intensity is maximum), then for
18 large value of β , $Gspf > 0$ in background region and $Gspf < 0$ in
19 foreground.

20 (II) If $I_b < I_1 < I_2 < \dots < I_q$ (i.e., background intensity is minimum), then for
21 large or small β , $Gspf > 0$ in foreground and $Gspf < 0$ in background
22 region.

1 **Proof:** (I) For investigating the sign of G_{spf} in background and foreground regions, we
 2 proceed as follows:

$$3 \quad \lim_{\beta \rightarrow \infty} G_{spf}_{\beta} = \lim_{\beta \rightarrow \infty} \frac{I_b - \frac{G_b + G_f}{2}}{\max\left(\left|I_b - \frac{G_b + G_f}{2}\right|\right)} = \frac{I_b - \frac{I_b + I_q}{2}}{\max\left(\left|I_b - \frac{I_b + I_q}{2}\right|\right)} > 0,$$

4 which means that the sign of G_{spf} functions positive in background region.

5 Next, we consider

$$6 \quad \lim_{\beta \rightarrow \infty} G_{spf}_{\beta} = \lim_{\beta \rightarrow \infty} \frac{I_q - \frac{G_b + G_f}{2}}{\max\left(\left|I_q - \frac{G_b + G_f}{2}\right|\right)} = \frac{I_q - \frac{I_q + I_b}{2}}{\max\left(\left|I_q - \frac{I_q + I_b}{2}\right|\right)} < 0,$$

7 which means that the sign of G_{spf} functions negative in foreground region.

8 Consequently, sign of G_{spf} is negative in objects $I_1 < I_2 < I_3 < \dots < I_{q-1}$. This means
 9 that the sign of G_{spf} is opposite in background and foreground. Thus by choosing a
 10 suitable large β all the objects in the image can be detected.

11 (II) When background has the minimum intensity and in foreground we have multi objects
 12 of different intensities. Let we have $I_b < I_1 < I_2 < I_3 < \dots < I_q$.

13 Adopting the same approach we have

$$14 \quad \lim_{\beta \rightarrow \infty} G_{spf}_{\beta} = \lim_{\beta \rightarrow \infty} \frac{I_b - \frac{G_b + G_f}{2}}{\max\left(\left|I_b - \frac{G_b + G_f}{2}\right|\right)} = \frac{I_b - \frac{I_b + I_q}{2}}{\max\left(\left|I_b - \frac{I_b + I_q}{2}\right|\right)} < 0,$$

15 which means that the sign of G_{spf} function is negative in background region.

16 Next, we consider

$$17 \quad \lim_{\beta \rightarrow \infty} G_{spf}_{\beta} = \lim_{\beta \rightarrow \infty} \frac{I_1 - \frac{G_b + G_f}{2}}{\max\left(\left|I_1 - \frac{G_b + G_f}{2}\right|\right)} = \frac{I_1 - \frac{I_q + I_b}{2}}{\max\left(\left|I_1 - \frac{I_q + I_b}{2}\right|\right)} > 0,$$

1 if $I_1 > \frac{I_q + I_b}{2}$. This means that the sign of Gspf function is positive in region with
2 intensity I_1 . Thus in this case sign of Gspf will be positive in the remaining objects $I_2 <$
3 $I_3 <, \dots, < I_q$ and consequently all the objects in foreground can be detected. But if
4 $I_1 < \frac{I_q + I_b}{2}$ then we have an option to choose other suitable value of β for all objects
5 detection in foreground because

$$6 \quad \lim_{\beta \rightarrow -\infty} Gspf_{\beta} = \lim_{\beta \rightarrow -\infty} \frac{I_1 - \frac{G_b + G_f}{2}}{\max\left(I_1 - \frac{G_b + G_f}{2}\right)} = \frac{I_1 - \frac{I_b + I_1}{2}}{\max\left(I_1 - \frac{I_b + I_1}{2}\right)} > 0.$$

7 From above theorem 2 it can be seen that images with maximum or minimum intensity
8 background can be segmented with this formulation.

9 Thus we utilize Gspf function defined in equation (6) in the following manner [10]:

$$10 \quad \frac{\partial \phi}{\partial t} = \alpha Gspf \cdot |\nabla \phi|, \text{ in } \Omega, \text{ and } \phi(t, x, y) = \phi_0(x, y), \text{ in } \Omega. \quad (7)$$

11 The above proposed partial differential equation (PDE) contains the generalized statistical
12 image intensity information which derives the contour to the edges which truly represent
13 the objects boundaries. For numerical solution of equation (7) we used explicit scheme in
14 similar steps as done in [18]. Now we turn our attention towards images having
15 background of average intensity which is indeed a challenging case for 2-phase image
16 segmentation models. In the following section we propose a new model to tackle this
17 serious issue.

18 **2.3 Segmenting Images of Mid Intensity Background**

1 Here we propose a new model for segmentation of images having mid-intensity
 2 background. For this, let we have an image containing q-objects with the following
 3 intensities: $I_1 < I_2 < I_3 < \dots < I_{k-1} < I_k = I_b < I_{k+1} < \dots < I_q$.

4 Using (6) we define two functions denoted by G_B and G_F and are given by:

$$5 \quad G_B(u_0) = \frac{u_0(x, y) - \frac{G_{c_1}}{\theta}}{\max\left(u_0(x, y) - \frac{G_{c_1}}{\theta}\right)}, \quad G_F(u) = \frac{u_0(x, y) - \frac{G_{c_2}}{\theta}}{\max\left(u_0(x, y) - \frac{G_{c_2}}{\theta}\right)}, \quad \theta > 0. \quad (8)$$

6 The function G_B is obtained by putting G_{c_2} equals to zero and similarly the function
 7 G_F is obtained by putting G_{c_1} equals to zero in equation (4).

8 For images whose background is neither of maximum nor of minimum intensity we define
 9 a new product spf function given as:

$$10 \quad PGspf_{\beta, \theta} = G_B G_F. \quad (9)$$

11 For segmentation of images we need to have positive sign of G_B in the background and
 12 negative sign in foreground objects having intensities smaller than I_b i.e. objects having
 13 intensities I_1, I_2, \dots, I_{k-1} . Similarly, we need to have negative sign of G_F in the
 14 background and positive sign in foreground objects having intensities smaller than I_b i.e.
 15 objects having intensities I_1, I_2, \dots, I_{k-1} . This turns the product function $PGspf_{\beta, \theta}$ to
 16 have negative sign in background and positive sign in complete foreground in all the
 17 objects. Figure 2 exhibits a mid-intensity background image and explains the idea behind
 18 the design of the PGspf function where the aim is to get the image (PGspf function) where
 19 the background and foreground are having opposite signs. The given image u as shown
 20 in Figure 2 is the image to be segmented. The image u_1 as shown in Figure 2 is the first
 21 image which is obtained by subtracting $\frac{I_q}{\theta}$ from the given image, where I_q denotes the

22 maximum intensity and for this particular image $\theta \in (1, 2.5)$ is a parameter and in this

1 case we can take 2.1. Next, the image u_2 is the image shown in Figure 2 and it can be
2 easily seen that the sign of the background and foreground are opposite.

3 Now to obtain the desired results from G_B and G_F we establish bounds on parameter θ .

4 Mathematically, we have the following constraints on θ :

$$5 \quad I_b - \frac{I_b}{\theta} > 0, \quad I_q - \frac{I_q}{\theta} > 0, \quad I_b - \frac{I_q}{\theta} > 0.$$

6 From first two inequalities we get the lower bound $1 < \theta$ and from the third, the upper

7 bound $\theta < \frac{I_q}{I_b}$. Thus the parameter θ exists and its value lies in the interval $1 < \theta < \frac{I_q}{I_b}$.

8 Thus, base on $PGspf_{\beta,\theta}$ we consider the following evolution problem:

$$9 \quad \frac{\partial \phi}{\partial t} = \alpha PGspf_{\beta,\theta} \cdot |\nabla \phi|, \text{ in } \Omega, \quad \text{where } \phi(t, x, y) = \phi_0(x, y), \text{ in } \Omega. \quad (10)$$

10 For numerical solution we use explicit scheme in similar steps as done in [18].

11 The overall algorithm can be described in the following steps:

- 12 1. Initialize the level set function ϕ as binary function.
- 13 2. Evolve the level set function ϕ according to equation (7) if an image is either of
14 maximum or minimum intensity background and evolve it according to (10) if
15 image is of mid-intensity background (we solve equation (7) and (10) explicitly).
- 16 3. Smooth the function ϕ by Gaussian kernel i.e. $\phi = G_\sigma * \phi$, where σ is the standard
17 deviation.
- 18 4. Check whether the evolution is stationary. If not, return to step 2.

19 For segmentation of **colour images** the above algorithm is applied on each channel (R, G
20 and B) of the image separately.

21 In next section we give some experimental results of the proposed model and some state
22 of the art models.

1 **3. Numerical Experiments**

2 In this section, we give experimental results of the proposed generalized signed pressure
3 function based (GSF) model and compare its results with the state of art models like GCV
4 [18], local Chan-Vese (LCV) model [15] and locally computed spf (LCS) model [1].
5 From the experimental results it can be seen that our proposed method perform efficiently
6 in images having maximum, minimum or mid-intensity background.

7 **Performance of the proposed model in noisy images**

8 In Figure 3, proposed method is tested on a noisy image. In Figure 3 (a) Original noisy
9 satellite image is given and in Figure 3(b) final result of the proposed GSF model is given.
10 Figure 3(c) displays a multi-region noisy synthetic image and Figure 3(d) shows the
11 successful detection by the GSF model. So we conclude that our proposed model works
12 very well in noisy images. In Figure 4, experimental results of all four models are given
13 on an aeroplane image.

14 **Comparison of the four models in images of minimum intensity background**

15 In Figure 5 proposed model performance on saw image having multi intensity region and
16 minimum intensity background is compared with LCV, GCV and LCS. Better
17 performance of our proposed model can be seen in the experimental results and also
18 verified by Jaccard similarity values.

19 **Comparison of all models on image with maximum intensity background**

20 In Figure 6, all the four models are tested on a real hardware image, which has maximum
21 intensity background and in-homogeneity in its foreground. Our proposed method has
22 performed very well in this type of images, while the other methods fail to segment this
23 image. For comparison JS values for our proposed model is 1.

24 **Comparison of all models on image with mid-intensity background**

1 The four models are tested on a real multi-objects image containing a bird and moon and
2 having mid-intensity background which is given in Figure 7. Our proposed model
3 performed has detected both objects in the image while the other models GCV, LCV and
4 LCS models could detect both objects. JS value for our model is 0.9.

5 **Selective segmentation results of the proposed model**

6 For selective image segmentation GCV and GSF are tested on the image given in Figure
7 8 where the initial contour encloses the kidney to be captured. The GCV model detected
8 the kidney with undesirable region whereas the GSF model captured purely the kidney of
9 interest without any undesirable result. Next, it can be easily observed that the proposed
10 GSF model captured the true boundaries of internal structure of given brain CT image
11 whereas the result by the GCV model can also be seen. JS values are given for
12 comparison.

13 **Performance of the GSF model in colour images**

14 To exhibit the performance of the proposed model on colour images, firstly, the proposed
15 GSF model and the vector valued Chan-Vese (VVCV) model [5, 12] are tested on colour
16 images having multi intensity objects as in Figure 9. It can be very easily observed from
17 these figures that the GSF model captured successfully all the different intensity objects
18 completely in contrast with the standard VVCV model. Similarly, Figure 9 displays an
19 aeroplane in inhomogeneous intensity sky. The proposed GSF model successfully
20 detected the aeroplane in contrast with the VVCV model.

21 **Speed comparison of all four models**

22 Next, using image in Figure 3(a), the time comparisons of all four models is given in
23 Table 1. These experiments are done by using MATLAB 7.11.0, with Windows 7,
24 2.53GHz Intel Core i3 personal computer with 2GB RAM. The following notations are

1 used in Table1; Size: the size of given image $m \times n$, Iter: total number of iterations, CPU:
2 the CPU time in seconds. From the table it can be seen that our model is as fast as GCV,
3 because in both models we are solving same partial differential equation, but we are using
4 different spf function. LCV is fast in time but its performance is limited in the above
5 discussed problems. In Table 2, quantitative comparison of different models in terms of
6 (mean \pm SD) of JS values is given, where SD is the standard deviation. For JS values
7 comparison we have used synthetic images with maximum, minimum and mid intensity
8 backgrounds like images given in Figures 1, 2, 5, 6, 7. It can be observed that our proposed
9 model gives better JS values in different problems.

10 **4. Conclusion**

11 In this paper we have proposed a novel active contour model based on generalized
12 averages and spf functions for segmenting images that have multi-objects, having either
13 maximum, minimum or mid-intensity backgrounds. For images with constant intensities,
14 all the four state of the art models work well but images having multi-regions can only be
15 segmented by our proposed GSF model which can be seen from the experiments.
16 However, images in which objects have intensity in-homogeneity may not be segmented
17 efficiently by proposed algorithm which is our future task.

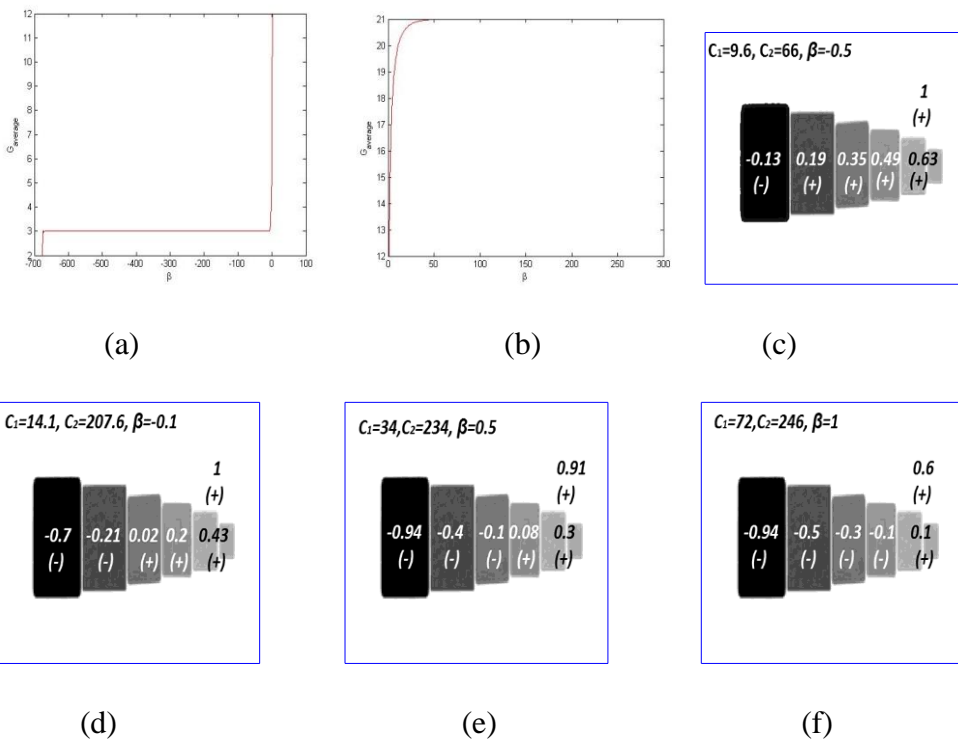
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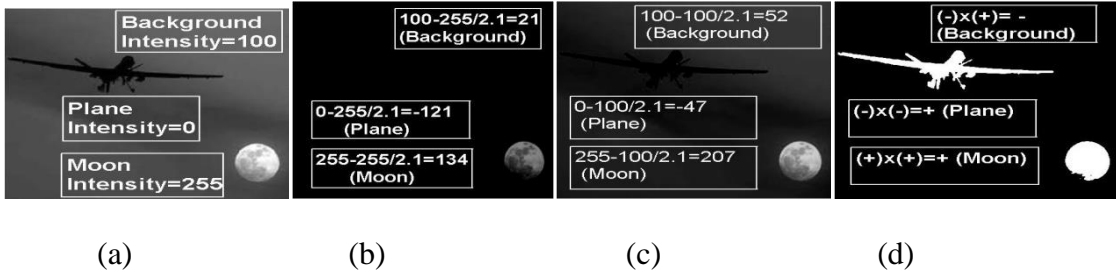
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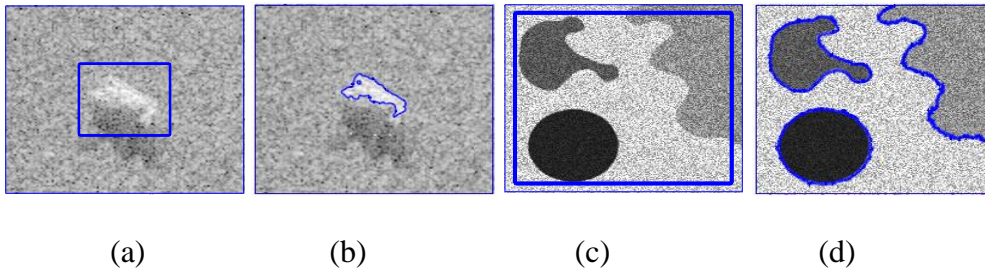
13 **Figures:**



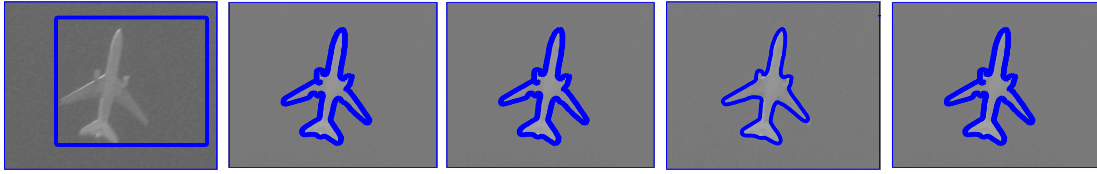
1 **Figure 1:** In (a) and (b) numerical overflow is given for $\beta < -679$ and $\beta > 233$ for a
 2 set of values $\{3, 5, \dots, 21\}$ and $G_{average} \rightarrow -\infty, \infty$ respectively. And in (c), (d), (e) and (f)
 3 signs of Gspf function inside and outside of the object for different values of averages.
 4 Clearly the above figures illustrate that a single pair of averages may not help in obtaining
 5 the desired result of segmentation. The suitable value of parameter β helps us in obtaining
 6 a suitable pair of averages from the A_v -family.



7
 8 (a) (b) (c) (d)
 9 **Figure 2:** Illustration of the $PGspf_{\beta, \theta}$ function in segmenting real image having mid-
 10 intensity background where maximum intensity (I_q) is 255, background intensity (I_b) is
 11 100 and choosing θ in interval $(I, \frac{I_q}{I_b}=2.5)$. (a) Given Image u ; (b) Image $u_1 = u - \frac{I_q}{\theta}$; (c)
 12 Image $u_2 = u - \frac{I_b}{\theta}$; (d) The product image $u_1 \times u_2$ where $\theta = 2.1$.



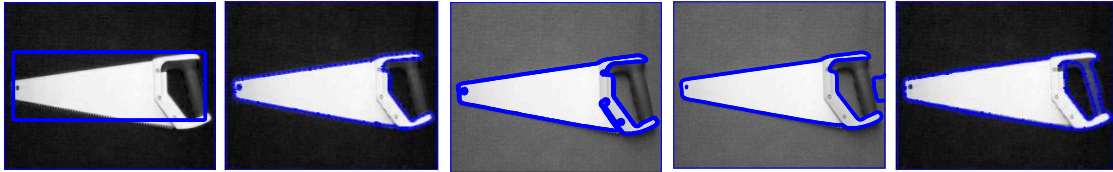
13
 14 (a) (b) (c) (d)
 15 **Figure 3:** Performance of the GSF model in segmenting real noisy satellite image and a
 16 multi region synthetic noisy image. (a) Real noisy satellite image; (b) Result by the GSF
 17 model, $\beta = 2, \mu = 10, iterations = 100$; (c) Multi region synthetic noisy image; (d)
 18 Result by the GSF model, $\beta = 1.5, \mu = 13, iterations = 250, size(u) = 200 \times 200$.



1

2 (a) Initial guess (b) GCV Result (c) LCV Result (d) LCS Result (e)GSF Result

3 **Figure 4:** Experimental test exhibiting that all the four models work well in simple image.



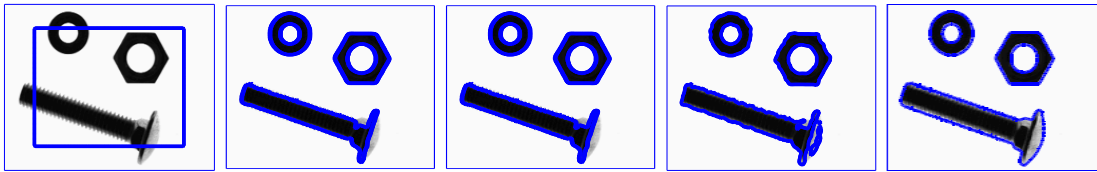
4

5 (a) Initial guess (b) GCV Result (c) LCV Result (d) LCS Result (e) GSF Result

6 **Figure 5:** Segmenting a real hardware image having minimum intensity background.

7 GCV (JS=0.78), LCV (JS=0.79), LCS (JS=0.7),GSF(JS=0.88). For proposed model

8 parameters used are: $\beta = -0.906, \mu = 77, iter = 65, size(u) = 200 \times 200$.



9

10 (a) Initial guess (b) GCV Result (c) LCV Result (d) LCS Result (e) GSF Result

11 **Figure 6:** Segmenting a real hardware image having maximum intensity

12 background.GCV (JS=0.8), LCV (JS=0.8), LCS (JS=0.85),GSF model (JS=1). For

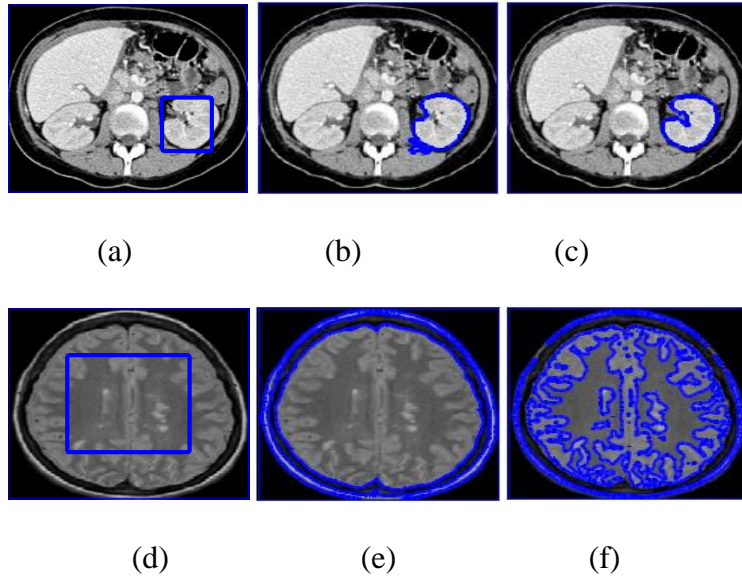
13 proposed model parameters used are: $\beta = 3, \mu = 15, iter = 100$.



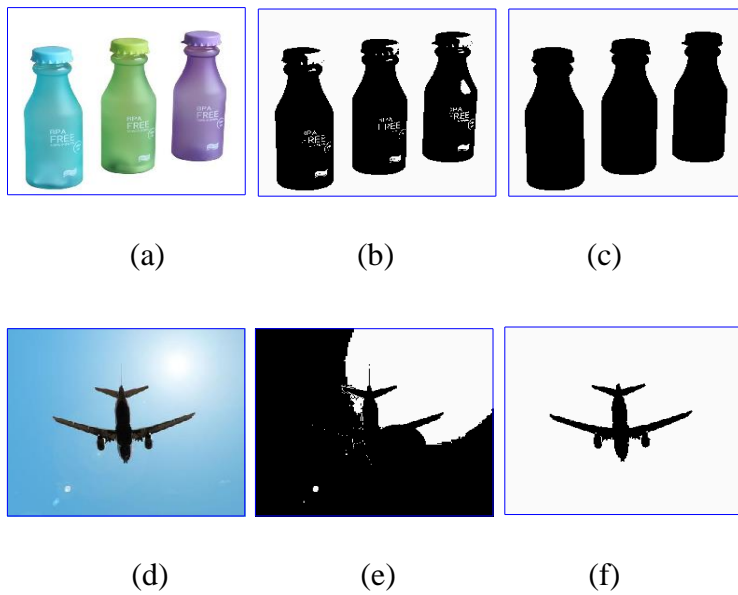
14

15 (a) Initial guess (b) GCV Result (c) LCV Result (d) LCS Result (e) GSF Result

1 **Figure 7:** Segmenting a real world image having background intensity neither maximum
 2 nor minimum. GCV (JS=0.48), LCV (JS=0.48), LCS (JS=0.1), GSF(JS=0.9). For
 3 proposed model parameters used are: $\beta = 3, \mu = 40, iter = 120, \theta = 4$.



8 **Figure 8:** Testing performance of the two models in single and multiple organ
 9 segmentation using real medical images. (a) Object of interest; (b) GCV Result
 10 (JS=0.992); (c) GSF (JS=0.996). (d) Object of interest;(e) GCV Result (JS=0.6); (f)
 11 GSF Result (JS=0.8). For GSF parameters are: $\beta = 2, \mu = 5$ (a), $\beta = 2, \mu = 15$.



1 **Figure 9:** Testing performance of the GSF and vector valued CV (VVCV) model. For
 2 (a) VVCV (JS=0.81), GSF (JS=0.996), for (d) VVCV (JS=0.5), GSF (JS=0.992). For
 3 GSF parameters used are: $\beta = 5, \mu = 20, iter = 100$ for (a); and $\beta = 0.5, \mu =$
 4 $10, iter = 100$ for (d).

5 **Table 1:**

Size	GSF		GCV Method		LCV Method		LCS Method	
	Iter	CPU	Iter	CPU	Iter	CPU	Iter	CPU
200× 200	17	0.4	17	0.4	2	0.3	62	3.4
400× 400	50	3	50	3	2	0.7	166	36
600× 600	80	17	80	17	2	2	290	208
800× 800	110	31	110	31	2	2.6	430	498

6
 7 **Table.1:** CPU time comparison of the LCV, GCV, LCS and our proposed GSF model.

Intensity	GCV Model	LCV Model	LCS Model	GSF Model
Maximum	0.68±0.1	0.6±0.1	0.88±0.03	0.98±0.02
Minimum	0.73±0.04	0.8±0.11	0.8±0.1	0.96±0.02
Mid	0.57±0.2	0.53±0.2	0.5±0.3	0.93±0.03

8
 9 **Table.2:** Mean and the SD of JS values for state of art models tested on images with
 10 different intensity backgrounds.