Matrix Semigroup Freeness Problems in $\mathbf{SL(2,\mathbb{Z})}$*

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Abstract. In this paper we study decidability and complexity of decision problems on matrices from the special linear group $\mathrm{SL}(2,\mathbb{Z})$. In particular, we study the freeness problem: given a finite set of matrices G generating a multiplicative semigroup S, decide whether each element of S has at most one factorization over G. In other words, is G a code? We show that the problem of deciding whether a matrix semigroup in $\mathrm{SL}(2,\mathbb{Z})$ is non-free is NP-hard. Then, we study questions about the number of factorizations of matrices in the matrix semigroup such as the finite freeness problem, the recurrent matrix problem, the unique factorizability problem, etc. Finally, we show that some factorization problems could be even harder in $\mathrm{SL}(2,\mathbb{Z})$, for example we show that to decide whether every prime matrix has at most k factorizations is PSPACE-hard.

Keywords: matrix semigroups, freeness, decision problems, decidability, computational complexity

1 Introduction

In general, many computational problems for matrix semigroups are proven to be undecidable starting from dimension three or four [3, 5, 8, 16, 25]. One of the central decision problems for matrix semigroups is the membership problem. Let $S = \langle G \rangle$ be a matrix semigroup generated by a generating set G. The *membership problem* is to decide whether or not a given matrix M belongs to the matrix semigroup S. In other words the question is whether a matrix M can be factorized over the generating set G or not.

Another fundamental problem for matrix semigroups is the *freeness problem*, where we want to know whether every matrix in the matrix semigroup has a unique factorization over G. Mandel and Simon [22] showed that the freeness problem is decidable in polynomial time for matrix semigroups with a single generator for any dimension over rational numbers. Indeed, the freeness problem for matrix semigroups with a single generator is the complementary problem of the *matrix torsion problem* which asks whether there exist two integers $p, q \geq 1$ such that $M^p = M^{q+p}$. Klarner et al. [17] proved that the freeness problem in dimension three over natural numbers is undecidable.

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Decidability of the freeness problem in dimension two has been already an open problem for a long time [7, 8]. However the solutions for some special cases exist. For example Charlier and Honkala [10] showed that the freeness problem is decidable for upper-triangular matrices in dimension two over rationals when the products are restricted to certain bounded languages. Bell and Potapov [4] showed that the freeness problem is undecidable in dimension two for matrices over quaternions.

The study in [8] revealed a class of matrix semigroups formed by two 2×2 matrices over natural numbers for which the freeness in unknown, highlighting a particular pair:

$$\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \text{ and } \begin{pmatrix} 3 & 5 \\ 0 & 5 \end{pmatrix}.$$

The above case was simultaneously solved in two papers [9] and [14], where authors were providing new algorithms for checking freeness at some subclasses.

However the status of the freeness problem for natural, integer and complex numbers is still unknown. The decidability of the freeness problem for $\mathrm{SL}(2,\mathbb{Z})$ was shown in [9] following the idea of solving the membership problem in $\mathrm{SL}(2,\mathbb{Z})$ shown in [11].

The effective symbolic representation of matrices in $\mathrm{SL}(2,\mathbb{Z})$ leads recently to several decidability and complexity results. The mortality, identity and vector reachability problems were shown to be NP-hard for $\mathrm{SL}(2,\mathbb{Z})$ in [1, 6]. For the modular group, the membership problem was shown to be decidable in polynomial time by Gurevich and Schupp [15]. Decidability of the membership problem in matrix semigroups in $\mathrm{SL}(2,\mathbb{Z})$ and the *identity problem* in $\mathbb{Z}^{2\times 2}$ was shown to be decidable in [11] in 2005. Later in 2016, Semukhin and Potapov showed that the *vector reachability problem* is also decidable in $\mathrm{SL}(2,\mathbb{Z})$ [27].

In this paper we study decidability and complexity questions related to freeness and various other factorization problems in $\mathrm{SL}(2,\mathbb{Z})$. The new hardness results are interesting in the context of understanding complexity in matrix semigroups in general and the decidability results on factorizations in $\mathrm{SL}(2,\mathbb{Z})$ can be important in other areas and fields. In particular, the special linear group $\mathrm{SL}(2,\mathbb{Z})$ has been extensively exploited in hyperbolic geometry [12, 32], dynamical systems [24], Lorenz/modular knots [21], braid groups [26], high energy physics [30], M/en theories [13], music theory [23], and so on.

In this paper, we show that the question about non-freeness for matrix semigroups in $\mathrm{SL}(2,\mathbb{Z})$ is NP-hard by finding a different reduction than the one used in [1, 6]. Then we show both decidability and hardness results for the *finite freeness problem*: decide whether or not every matrix in the matrix semigroup has a finite number of factorizations. Also we prove NP-hardness of the problem whether a given matrix has more than one factorization in $\mathrm{SL}(2,\mathbb{Z})$ and undecidability of this problem in $\mathbb{Z}^{4\times 4}$, or more specifically in $\mathrm{SL}(4,\mathbb{Z})$. Then it is shown that both problems whether a particular matrix has an infinite number factorizations or it has more than k factorizations are decidable and NP-hard in $\mathrm{SL}(2,\mathbb{Z})$ while they are undecidable in $\mathbb{Z}^{4\times 4}$. Finally we show that some of the factorizations problems could be even harder in $\mathrm{SL}(2,\mathbb{Z})$, for example we show that to decide whether every prime matrix has at most k-factorizations is PSPACE-hard.

2 Preliminaries

In this section we formulate several problems, provide important definitions and notation as well as several technical lemmas used throughout the paper.

Basic definitions. A *semigroup* is a set equipped with an associative binary operation. Let S be a semigroup and X be a subset of S. We say that a semigroup S is *generated* by a subset X of S if each element of S can be expressed as a composition of elements of X. Then, we call X the *generating set* of S. Then, X is a *code* if and only if every element of S has a unique factorization over X. A semigroup S is *free* if there exists a subset $X \subseteq S$ which is a code and $S = X^+$.

Given an alphabet $\Sigma = \{1, 2, \ldots, m\}$, a word w is an element of Σ^*. For a letter $a \in \Sigma$, we denote by $\overline{a}$ the inverse letter of a such that $a\overline{a} = \varepsilon$ where ε is the empty word.

A *nondeterministic finite automaton* (NFA) is a tuple $A = (\Sigma, Q, \delta, q_0, F)$ where Σ is the input alphabet, Q is the finite set of states, $\delta\colon Q \times \Sigma \to 2^Q$ is the multivalued transition function, $q_0 \in Q$ is the initial state and $F \subseteq Q$ is the set of final states. In the usual way δ is extended as a function $Q \times \Sigma^* \to 2^Q$ and the language accepted by A is $L(A) = \{w \in \Sigma^* \mid \delta(q_0, w) \cap F \neq \emptyset\}$. The automaton A is a deterministic finite automaton (DFA) if δ is a single valued function $Q \times \Sigma \to Q$. It is well known that the deterministic and nondeterministic finite automata recognize the class of *regular languages* [29].

Factorization and freeness problems. Let S be a matrix semigroup generated by a finite set G of matrices. Then we define a matrix M is *k-factorizable* for $k \in \mathbb{N}$ if there are at most k different factorizations of M over G. In the matrix semigroup freeness problem, we check whether every matrix in S is 1-factorizable.

Problem 1. Given a finite set G of $n \times n$ matrices generating a matrix semigroup S, is S free? (i.e., does every element $M \in S$ have a unique factorization over G?)

The above problem is well-known as the *freeness problem*. Clearly, the *non-freeness problem* is to decide whether the matrix semigroup S is not free.

For a matrix M, if there exists $k < \infty$ where M is k-factorizable, then we say that M is *finitely factorizable*. In other words, a finitely factorizable matrix M has finitely many different factorizations over G. We define a matrix semigroup S is *finitely free* if every matrix in S is finitely factorizable and define the *finite freeness problem* as follows:

Problem 2. Given a finite set G of $n \times n$ matrices generating a matrix semigroup S, does every element $M \in S$ have a finite number of factorizations over G?

Freeness and finite freeness problems are asking about factorization properties for all matrices in the semigroup. In case where a semigroup is not free or not finitely free, instead of asking whether the semigroup is free or finitely free, it is possible to define problems for a given particular matrix in the semigroup as follows:

Problem 3. Given a finite set G of $n \times n$ matrices generating a matrix semigroup S and a matrix M in S, does M have

a. a unique factorization over G? (matrix unique factorizability problem)
b. at most k factorizations over G? (matrix k-factorizability problem)
c. an infinite number of factorizations over G? (recurrent matrix problem)

Group alphabet encodings. Let us introduce several technical lemmas that will be used in encodings for NP-hardness and undecidability results. Our original encodings require the use of group alphabet and the following lemmas for showing the transformation from an arbitrary group alphabet into a binary group alphabet and later into matrix form that is computable in polynomial time.

Lemma 4. *Let $\Sigma = \{z_1, z_2, \ldots, z_l\}$ be a group alphabet and $\Sigma_2 = \{a, b, \overline{a}, \overline{b}\}$ be a binary group alphabet. Define the mapping $\alpha : \Sigma \to \Sigma_2^*$ by:*

$$\alpha(z_i) = a^i b \overline{a}^i, \quad \alpha(\overline{z_i}) = a^i \overline{b} \overline{a}^i,$$

where $1 \leq i \leq l$. Then α is a monomorphism. Note that α can be extended to domain Σ^* in the usual way.

Lemma 5 (Lyndon and Schupp [20]). *Let $\Sigma_2 = \{a, b, \overline{a}, \overline{b}\}$ be a binary group alphabet and define $f : \Sigma_2^* \to \mathbb{Z}^{2\times 2}$ by:*

$$f(a) = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, f(\overline{a}) = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}, f(b) = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}, f(\overline{b}) = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}.$$

Then f is a monomorphism.

The composition of Lemmas 4 and 5 gives us the following lemma that ensures that encoding the *subset sum problem* (SSP) and the *equal subset sum problem* (ESSP) instances into matrix semigroups can be completed in polynomial time.

Lemma 6 (Bell and Potapov [6]). *Let z_j be in Σ and α, f be mappings as defined in Lemmas 4 and 5, then, for any $i \in \mathbb{N}$,*

$$f(\alpha(z_j^i)) = f((a^j b \overline{a}^j)^i) = \begin{pmatrix} 1 + 4ij & -8ij^2 \\ 2i & 1 - 4ij \end{pmatrix}.$$

Symbolic representation of matrices from $\mathbf{SL}(2, \mathbb{Z})$. Here we provide another technical details about the representation of $\mathrm{SL}(2, \mathbb{Z})$ and their properties [2, 28]. It is known that $\mathrm{SL}(2, \mathbb{Z})$ is generated by two matrices

$$\mathbf{S} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \text{ and } \mathbf{R} = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix},$$

which have respective orders 4 and 6. This implies that every matrix in $\mathrm{SL}(2,\mathbb{Z})$ is a product of $\mathbf{S}$ and $\mathbf{R}$. Since $\mathbf{S}^2 = \mathbf{R}^3 = -\mathbf{I}$, every matrix in $\mathrm{SL}(2,\mathbb{Z})$ can be uniquely brought to the following form:

$$(-\mathbf{I})^{i_0}\mathbf{R}^{i_1}\mathbf{S}\mathbf{R}^{i_2}\mathbf{S}\cdots\mathbf{S}\mathbf{R}^{i_{n-1}}\mathbf{S}\mathbf{R}^{i_n}, \tag{1}$$

where $i_0 \in \{0,1\}$, $i_1, i_n \in \{0,1,2\}$, and $i_j \neq 0 \mod 3$ for $1 < j < n$.

The representation (1) for a given matrix in $\mathrm{SL}(2,\mathbb{Z})$ is unique, but it is very common to present this result ignoring the sign, i.e. considering the projective special linear group. Let $\Sigma_{SR} = \{s, r\}$ be a binary alphabet. We define a mapping $\varphi : \Sigma_{SR} \to \mathrm{SL}(2,\mathbb{Z})$ as follows: $\varphi(s) = \mathbf{S}$ and $\varphi(r) = \mathbf{R}$. Naturally, we can extend the mapping φ to the morphism $\varphi : \Sigma_{SR}^* \to \mathrm{SL}(2,\mathbb{Z})$. We call a word $w \in \Sigma_{SR}^*$ *reduced* if there is no occurrence of subwords ss or rrr in w. Then, we have the following fact.

Theorem 7 (Lyndon and Schupp [20]). *For every matrix $M \in \mathrm{SL}(2,\mathbb{Z})$, there exists a unique reduced word $w \in \Sigma_{SR}^*$ in form of (1) such that either $M = \varphi(w)$ or $M = -\varphi(w)$.*

Following Theorem 7, all word representations of a particular matrix M in $\mathrm{SL}(2,\mathbb{Z})$ over the alphabet Σ_{SR} can be expressed as a context-free language.

Lemma 8. *Let M be a matrix in $SL(2,\mathbb{Z})$. Then, there exists a context-free language over Σ_{SR} which contains all representations $w \in \Sigma_{SR}^*$ such that $\varphi(w) = M$.*

3 Matrix semigroup freeness

The matrix semigroup freeness problem is to determine whether every matrix in the semigroup has a unique factorization. Note that the decidability of the matrix semigroup freeness in $\mathrm{SL}(2,\mathbb{Z})$ has been shown by Cassaigne and Nicolas [9] but the complexity of the problem has not been resolved yet despite various NP-hardness results on other matrix problems [1, 6]. Here we show that the problem of deciding whether the matrix semigroup in $\mathrm{SL}(2,\mathbb{Z})$ is not free is NP-hard by encoding different NP-hard problem comparing to the one used in [1, 6].

Theorem 9. *Given a matrix semigroup S in $SL(2,\mathbb{Z})$ generated by the set G of matrices, the problem of deciding whether S is not free is NP-hard.*

Proof. We use an encoding of the *equal subset sum problem* (ESSP), which is proven to be NP-hard, into a set of two-dimensional integral matrices [31]. The ESSP is, given a set $U = \{s_1, s_2, \ldots, s_k\}$ of k integers, to decide whether or not there exist two disjoint nonempty subsets $U_1, U_2 \subseteq U$ whose elements sum up to the same value. Namely, $\sum_{s_1 \in U_1} s_1 = \sum_{s_2 \in U_2} s_2$.

Define an alphabet $\Sigma = \{0, 1, \ldots, k-1, \overline{1}, \overline{2}, \ldots, \overline{(k-2)}, \overline{(k-1)}, \overline{k}, a\}$. We define a set W of words which encodes the ESSP instance.

$$W = \{i \cdot a^{i+1} \cdot \overline{(i+1)},\ \ i \cdot \varepsilon \cdot \overline{(i+1)} \mid 0 \leq i \leq k-1\} \subseteq \Sigma^*.$$

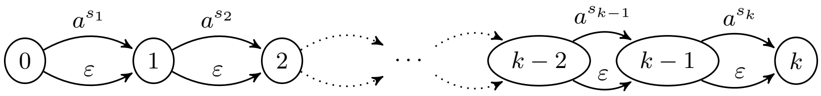

Fig. 1. Structure of the matrix semigroup encoded by the set W. Each matrix in the generating set of the matrix semigroup corresponds to each transition of the automaton structure.

We define 'border letters' as letters from $\Sigma \setminus \{a\}$ and the inner border letters of a word as all border letters excluding the first and last. We call a word a 'partial cycle' if all inner border letters in that word are inverse to a consecutive inner border letter. Moreover, we note that for any partial cycle $u \in W^+$ the first border letter of u is strictly smaller than the last border letter if we compare them as integers. Fig. 1 shows the structure of our encoding of the ESSP instance.

First we prove that if there is a solution to the ESSP instance, then the matrix semigroup generated by matrices encoded from the set W is not free. Let us assume that there exists a solution to the ESSP instance, which is two sequences of integers where each of two sequences sums up to the same integer x. Then, the solution can be represented by the following pair of sequences:

$$Y = (y_1, y_2, \dots, y_{k-1}, y_k) \text{ and } Z = (z_1, z_2, \dots, z_{k-1}, z_k),$$

where $y_i, z_i \in \{0, s_i\}, 1 \leq i \leq k$ and $\sum_{i=1}^{k} y_i = \sum_{i=1}^{k} z_i = x$. Note that $y_i \neq z_i$ in at least one index i for $1 \leq i \leq k$.

For a sequence Y, there exists a word $w_Y = w_1 w_2 \cdots w_k \in W^+$ such that $w_i = (i-1) \cdot a^{y_i} \cdot \bar{i}$. Since $\sum_{i=1}^{k} y_i = x$, the reduced representation of w_Y is $r(w_Y) = 0 \cdot s^x \cdot \bar{k}$ as all inner border letters are cancelled. Analogously, we have a word w_Z for a sequence Z and its reduced representation $r(w_Z)$ is equal to $r(w_Y)$ as the sum of integers in the sequence Z is equal to the sum of integers in Y. As we have two words in W^+ whose reduced representations are equal, the semigroup generated by matrices encoded from the set W is not free.

Now we prove the opposite direction: if there is no solution to the ESSP instance, then the matrix semigroup is free. Assume that there is no solution to the ESSP instance and the matrix semigroup is not free. Since the matrix semigroup is not free, we have two different words $w, w' \in W^+$ whose reduced representations are equal, namely, $r(w) = r(w')$.

For a word w, we decompose w into subwords $w = u_1 u_2 \cdots u_m$ such that each $u_i \in W^+, 1 \leq i \leq m$ is a partial cycle of maximal size. Similarly, we decompose w' into subwords of maximal partial cycles as follows: $w' = u'_1 u'_2 \cdots u'_n$. Since $r(w) = r(w')$, it follows that $r(u_i) = r(u'_i)$ should hold for $1 \leq i \leq m$ and $m = n$. On the other hand, since $w \neq w'$, there exists $i, 1 \leq i \leq m$ where $u_i \neq u'_i$. Note that the maximal partial cycles u_i and u'_i should have the same number of a's since $r(u_i) = r(u'_i)$ and the letter a cannot be cancelled by the reduction of words. As we mentioned earlier, the first border letter and last border letter of a partial cycle are integers where the first border letter is strictly smaller than the last border letter. Let us say that i_1 is the first border letter and i_2 is the last border letter of u_i and u'_i. Then, the number of a's in u_i and u'_i is the sum of subset of integers from the set $\{s_{i_1+1}, s_{i_1+2}, \ldots, s_{i_2}\}$. It follows from the fact that $u_i \neq u'_i$ that we have two distinct subsets of the set $\{s_{i_1+1}, s_{i_1+2}, \ldots, s_{i_2}\}$ whose sums are the same. This contradicts our assumption since we have two disjoint subsets of equal subset sum. $\square$

Recently, Bell et al. proved that the problem of deciding whether the identity matrix is in S, where S is an arbitrary regular subset of $\mathrm{SL}(2,\mathbb{Z})$, is in NP [2]. Since we can show that the matrix semigroup S is not free by showing that the equation $M_1MM_2 = M_3M'M_4$ is satisfied where $M_1 \neq M_3$, $M_2 \neq M_4$, and $M_i, M, M' \in S$ for $1 \leq i \leq 4$. We can show that S is not free by showing that the matrix $M_1MM_2M_4^{-1}M'^{-1}M_3^{-1}$ is the identity matrix.

Let $M_1M^*M_2M_4^{-1}(M^{-1})^*M_3^{-1}$ be a regular subset of $\mathrm{SL}(2,\mathbb{Z})$ subject to $M_1 \neq M_3$, $M_2 \neq M_4$ and $M \in S$. Then, we can decide whether or not S is free by deciding whether or not a regular subset of $\mathrm{SL}(2,\mathbb{Z})$ contains the identity matrix. Therefore, we can conclude as follows:

Corollary 10. *Given a matrix semigroup S in $SL(2,\mathbb{Z})$ generated by the set G of matrices, the problem of deciding whether S is not free is NP-complete.*

If the matrix semigroup is not free (not every matrix have unique factorization) we still have a question whether each matrix in a given semigroup has only a finite number of factorizations. Next we show that the problem of checking whether there exists a matrix in the semigroup which has an infinite number of factorizations is decidable and NP-hard in $\mathrm{SL}(2,\mathbb{Z})$.

Theorem 11. *Given a matrix semigroup S in $SL(2,\mathbb{Z})$ generated by the set G of matrices, the problem of deciding whether S contains a matrix with an infinite number of factorizations is decidable and NP-hard.*

Proof. Let us consider a matrix semigroup S which is generated by the set $G = \{M_1, M_2, \ldots, M_n\}$ of matrices. Let $w_1, w_2, \ldots, w_n \in \Sigma_{SR}^*$ be words encoding the generators, such that $\varphi(w_i) = M_i$ for $1 \leq i \leq n$. Then, we can define a regular language L_S corresponding to S as $L_S = \{w_1, w_2, \ldots, w_n\}^+$. Let $A = (Q, \Sigma, \delta, Q_0, F)$ be an NFA accepting L_S constructed based on S. For states q and p, where the state p is reachable from q by reading ss or rrr, we add an ε-transition from q to p. We repeat this process until there is no such pair of states following to the procedure proposed in [11].

If there exists a matrix M which can be represented by infinitely many factorizations over G, then there is an infinite number of accepting runs for the matrix M in A. It is easy to see that we have an infinite number of accepting runs for some matrix $M \in S$ if and only if there is a cycle only consisting of ε-transitions. As we can compute the ε-closure of states in A, the problem of deciding whether there exists a matrix with an infinite number of factorizations is decidable.

For the NP-hardness of the problem, we modify and adapt the NP-hardness proof of the identity problem in $\mathrm{SL}(2,\mathbb{Z})$ [6]. We use an encoding of the *subset*

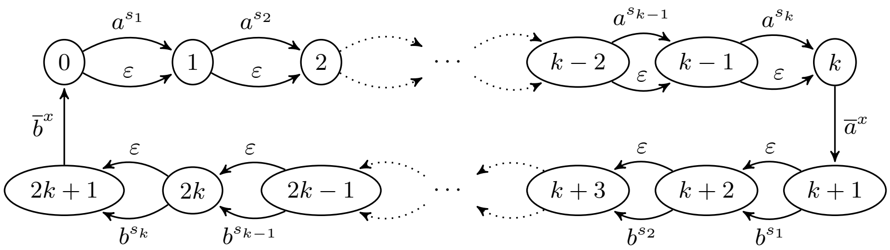

Fig. 2. Structure of the matrix semigroup encoded by the set W.

sum problem (SSP), which is, given a set $U = \{s_1, s_2, \ldots, s_k\}$ of k integers, to decide whether or not there exists a subset $U' \subseteq U$ whose elements sum up to the given integer x. Namely, $\sum_{s \in U'} s = x$.

Define an alphabet $\Sigma = \{0, 1, \ldots, 2k+1, \overline{1}, \overline{2}, \ldots, \overline{(2k+1)}, a, b, \overline{a}, \overline{b}\}$. We define a set W of words which encodes the SSP instance.

$$\begin{aligned} W = \; & \{i \cdot a^{i+1} \cdot \overline{(i+1)},\;\; i \cdot \varepsilon \cdot \overline{(i+1)} \mid 0 \leq i \leq k-1\} \;\cup \\ & \{i \cdot b^{i+1} \cdot \overline{(i+1)},\;\; i \cdot \varepsilon \cdot \overline{(i+1)} \mid k+1 \leq i \leq 2k\} \;\cup \\ & \{k \cdot \overline{a}^x \cdot \overline{(k+1)}\} \cup \{(2k+1) \cdot \overline{b}^x \cdot \overline{0}\} \subseteq \Sigma^*. \end{aligned} \tag{2}$$

Fig. 2 shows the structure of the word encoding of the SSP instance. The full proof for showing that the matrix semigroup S corresponding to W^+ has a matrix with an infinite number of factorizations if and only if the SSP instance has a solution can be found in the archive version [18] of the paper. $\square$

4 Matrix factorizability problems

In the matrix semigroup freeness problem, we ask whether every matrix in the semigroup has a unique factorization. Instead of considering a question about every matrix in the semigroup, we restrict our question to a given particular matrix, which may have a unique factorization, a finite number of unique factorizations or even an infinite number of unique factorizations.

4.1 Unique factorizability problem

In the *matrix unique factorizability problem*, we consider the problem of deciding whether or not a particular matrix M in S has a unique factorization over G. We first establish the decidability and NP-hardness of the problem.

Theorem 12. *Given a matrix semigroup S in $SL(2, \mathbb{Z})$ generated by the set G of matrices and a particular matrix M in S, the problem of deciding whether the matrix M has more than one factorization over G is decidable and NP-hard.*

Proof. From Lemma 8, we can represent a set of all unreduced representations for M over $\Sigma_{SR} = \{s, r\}$ as a context-free language L_M.

We can also obtain a regular language that corresponds to the matrix semigroup S. Let $G = \{M_1, M_2, \dots, M_n\}$ be the generating set of S. Namely, $S = \langle M_1, M_2, \dots, M_n \rangle$. Let $w_1, w_2, \dots, w_n \in \Sigma_{SR}^*$ be words encoding the generators, such that $\varphi(w_i) = M_i$ for $1 \le i \le n$. Then, we can define a regular language L_S corresponding to S as $L_S = \{w_1, w_2, \dots, w_n\}^+$. Then, the intersection of $L_M \cap L_S$ contains all words that correspond to the matrix M in the semigroup S. If the cardinality of $L_M \cap L_S$ is larger than one, we immediately have two different factorizations for the matrix M over G. Therefore, let us assume that $|L_M \cap L_S| = 1$ and w be the only word in $L_M \cap L_S$. Clearly, $\varphi(w) = M$ and M can be generated by the set G. Note that each accepting path of w in L_S corresponds to a unique factorization of M over G. Now we can decide whether or not M has a unique factorization over G by counting the number of accepting paths of words in $L_M \cap L_S$ from an NFA accepting L_S.

The NP-hardness can be proven by the reduction from the SSP in a similar manner to the proof of Theorem 11. See Equation (2) for the word encoding of the SSP instance. Let us pick the word $w = 0 \cdot \varepsilon \cdot \overline{1}$ in W and notice that the matrix $M = f(\alpha(w))$ which is encoded from w is in the matrix semigroup S. We will show that the matrix M in S has at least two factorizations over the generating set $\{f(\alpha(w)) \mid w \in W\}$ of S if and only if the SSP instance has a solution. The full proof can be found in the archive version [18]. □

We reduce the *fixed element PCP* (FEPCP) [3] which is proven to be undecidable to the unique factorizability problem over $\mathbb{Z}^{4\times 4}$ for the following undecidability result.

Theorem 13. *Given a matrix semigroup S over $\mathbb{Z}^{4\times 4}$ generated by the set G of matrices and a particular matrix M in S, the problem of deciding whether the matrix M has more than one factorization over G is undecidable.*

4.2 Recurrent matrix problem

We first tackle the problem of deciding whether or not a particular matrix in the semigroup has an infinite number of factorizations. Note that we call this decision problem the *recurrent matrix problem* instead of the *matrix finite factorizability problem* as we named for the other variants. The recurrent matrix problem has been introduced by Bell and Potapov [3] and proven to be undecidable for matrices over $\mathbb{Z}^{4\times 4}$ based on the reduction from FEPCP.

We show that the recurrent matrix problem is decidable and NP-hard for matrix semigroups in $\mathrm{SL}(2,\mathbb{Z})$. We first mention that the recurrent matrix problem is different with the identity problem. One may think that the recurrent matrix problem is equivalent to the identity problem since it is obvious that if the identity matrix exists then every matrix in the semigroup has an infinite number of factorizations. However, the opposite does not hold as follows:

Proposition 14. *Let S be a matrix semigroup generated by the generating set G and M be a matrix in S. Then, the matrix M has an infinite number of factorizations over G if the identity matrix exists in S. However, the opposite does not hold in general.*

Now we establish the results for the recurrent matrix problem in $\mathrm{SL}(2, \mathbb{Z})$.

Theorem 15. *The recurrent matrix problem in $SL(2, \mathbb{Z})$ is decidable and in fact, NP-hard.*

We also consider the matrix k-factorizability problem which is to decide whether a particular matrix M in the semigroup has at most k factorizations over the generating set G.

Lemma 16. *Given a matrix semigroup S in $SL(2, \mathbb{Z})$ generated by the set G of matrices, a particular matrix $M \in S$, and a positive integer $k \in \mathbb{N}$, the problem of deciding whether the matrix M has more than k factorizations over G is decidable and NP-hard.*

We mention that the matrix k-factorizability problem is also undecidable over $\mathbb{Z}^{4\times 4}$ following Theorem 13.

5 On the finite number of factorizations

Recall that the matrix semigroup freeness problem examines whether or not there exists a matrix in the semigroup has more than one factorization. The finite freeness problem asks whether there exists a matrix in the semigroup which has an infinite number of factorizations. In that sense, we may interpret these problems as the problems asking whether the number of factorizations in the semigroup is bounded by one (the freeness problem) or unbounded (the finite freeness problem).

In this section, we are interested in finding a number $k \in \mathbb{N}$ by which the number of factorizations of matrices in the matrix semigroup is bounded. In other words, we check whether every matrix in the semigroup is k-factorizable. However, it is not easy to define the *k-freeness problem* as we define the general freeness problem by the following observation.

Let S be a matrix semigroup generated by the set G of matrices and M be a k-factorizable matrix over G. Let us denote the number of factorizations of M by $\mathrm{dec}(M)$. Thus, we can write $\mathrm{dec}(M) = k$. It is easy to see that S is free if for every matrix M in S, $\mathrm{dec}(M) = 1$. Let us assume that $\mathrm{dec}(M_1) = m$ and $\mathrm{dec}(M_2) = n$ for $m, n \in \mathbb{N}$. Then, $\mathrm{dec}(M_1 M_2) = k$ where $k \geq mn$. This means that if S is not free, then there is no finite value k such that every matrix in S is k-factorizable.

In that reason, we define the following notion which prevents the multiplicative property of the number of factorizations. We say that a matrix M is *prime* if it is impossible to decompose M into $M = M_1 M_2$ such that $\mathrm{dec}(M) =$

$\mathrm{dec}(M_1) \times \mathrm{dec}(M_2)$, $\mathrm{dec}(M_1) \neq 1$, and $\mathrm{dec}(M_2) \neq 1$. We define a matrix semigroup S to be *k-free* if every prime matrix M in S has at most k different factorizations over G. Formally, a matrix semigroup S is k-free if and only if $\max\{\mathrm{dec}(M) \mid M \in S, \ \ M \text{ is prime}\} \leq k$.

This definition gives rise to the following problem which is a generalized version of the matrix semigroup freeness problem.

Problem 17. Given a finite set G of $n \times n$ matrices generating a matrix semigroup S, does every prime element $M \in S$ have at most k factorizations over G?

In this paper, we leave the decidability of the k-freeness problem open but establish the PSPACE-hardness result as a lower bound of the problem, which is interesting compared to the NP-hardness of the other freeness problems.

Theorem 18. *Given a matrix semigroup S in $SL(2,\mathbb{Z})$ generated by the set G of matrices and a positive integer $k \in \mathbb{N}$, the problem of deciding whether or not every prime matrix in S has at most k factorizations is PSPACE-hard.*

Proof. For the PSPACE-hardness of the problem, we reduce the DFA intersection emptiness problem [19] to the k-freeness problem. Note that given k DFAs, the DFA intersection emptiness problem asks whether the intersection of k DFAs is empty. The full proof can be found in the archive version [18]. □

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