Relations in the space of $(2, 0)$ heterotic string models

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by

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Abstract

Understanding better the landscape of string models and eventually finding, if possible, a dynamical way to select among them is one of the most interesting, open problems in string theory. In this thesis, we investigate aspects of the heterotic landscape and discuss relations among large classes of vacua.

The first part of the thesis is devoted to the equivalence between free fermionic models and orbifolds. Free fermionic models and symmetric heterotic toroidal orbifolds both constitute exact backgrounds that can be used effectively for phenomenological explorations within string theory. It is widely believed that for $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifolds the two descriptions should be equivalent, but a detailed dictionary between both formulations was lacking. A detailed account of how the input data of both descriptions can be related to each other can be found in this thesis. In particular, we show that the generalized GSO phases of the free fermionic model correspond to generalized torsion phases used in orbifold model building. We illustrate our translation methods by providing free fermionic realizations for all $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold geometries in six dimensions.

In the second half of the thesis, we turn our attention to a novel idea called spinor-vector duality. In its original form, spinor-vector duality was limited to $\mathbb{Z}_2$ structures. Here, we use the language of simple currents to generalize this idea to theories with arbitrary internal RCFTs. We also elucidate the underlying spectral flow structure. Even though the spectral flow has been traditionally used to relate states within a single model, we offer a new way to look at it, allowing relations between different models. The idea of grouping together models into families according to the spectral flow orbit is quite important: the spectra of the models, though not identical, are related and we can make statements about models in the entire family by examining one representative. The grouping also offers a conceptual handle, acting as an organization principle in a vast landscape of models.

Publication List

This thesis contains material that has appeared in the following publications by the author:

- P. Athanasopoulos, A. E. Faraggi, and D. Gepner "Spectral flow as a map between N = (2, 0)-models" *Phys. Lett.* **B735** (2014) 357–363 [arXiv:1403.3404].
- P. Athanasopoulos "Discrete symmetries in the heterotic-string landscape" *J. Phys. Conf. Ser.* **631** (2015) no. 1, 012083 [arXiv:1502.04986].
- P. Athanasopoulos, A. E. Faraggi, S. G. Nibbelink, and V. M. Mehta "Heterotic free fermionic and symmetric toroidal orbifold models" *JHEP* **04** (2016) 038 [arXiv:1602.03082].

The following publications by the author are not presented in this thesis:

- P. Athanasopoulos, A. E. Faraggi, and V. M. Mehta "Light Z' in heterotic string standardlike models" *Phys. Rev.* **D89** (2014) no. 10, 105023 [arXiv:1401.7153].
- J. M. Ashfaque, P. Athanasopoulos, A. E. Faraggi, and H. Sonmez "Non-Tachyonic Semi-Realistic Non-Supersymmetric Heterotic String Vacua" *Eur. Phys. J.* **C76** (2016) no. 4, 208 [arXiv:1506.03114].

Declaration

I hereby declare that all work described in this thesis is the result of my own research unless reference to others is given. None of this material has previously been submitted to this or any other university. All work was carried out in the Theoretical Physics Division of the Department of Mathematical Sciences, University of Liverpool, UK during the period of September 2012 until August 2016.

Acknowledgments

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Contents

Chapter 1

Introduction

The desire to understand our universe has been around for as long as humans existed and the quest for deeper knowledge is a strong driving force for many of us today. Physics has been at the forefront of this endeavor, offering great opportunities to combine physical observations with elegant mathematics.

The main topic of this thesis is string theory. String theory emerged as the leading framework in the effort to unify the gravitational interactions with the gauge interactions described by the Standard Model. It is a beautiful and inspiring theory but, despite being around for decades, it is in many ways still at its infancy, readily offering a variety of open problems and challenges to the ambitious practitioner.

One of the famous problems in the field is the landscape problem. String theory admits a multitude of solutions that are consistent and could *a priori* describe our world. The vast number of possible models impedes, rather than assists, the progress towards the construction of a standard string model. In a theory that allows the construction of such a great number of models, studying how different models relate to each other becomes essential. Grouping models together according to equivalence or shared properties might allow us to better conceptualize the vast landscape and might also help us understand why we find ourselves at this very special point in the space of possible models. Understanding this means understanding why the universe is the way it is.

In this thesis we make a contribution towards this noble goal by discussing certain relationships in the space of heterotic $(2,0)$ string models. We chose to focus on this particular type of models because of their interesting phenomenological potential and applications. Physics is an experimental science and we should always try and make contact with observations. One might randomly wander around the space of ideas, but as long as there is a constant driving force towards experimental verification then one knows that it is impossible to get lost.

We begin in chapter 2 by reviewing the necessary concepts of the theory that will be used through the thesis. The following chapters are devoted to discussing in detail the correspondence between free fermionic models and orbifolds. Chapter 3 introduces the former, chapter 4 the latter and chapter 5 gives the details of how to translate one to the other. In chapter 6 we investigate a different type of relationship in the space of heterotic vacua, generalizing an idea called the spinor-vector duality to a much bigger set of string models. Finally, we conclude with an appendix on lattices, which uses concepts familiar from linear algebra and computer science in a way that, as far as we know, cannot be found in the standard textbooks.

Chapter 2

Basic Concepts of String Theory

This chapter introduces the fundamental concepts of string theory and its main purpose is to establish notation and conventions. Expanded versions of most of this material can be found in standard textbooks [6–13]. For the more specialized subtopics, further references are given at the beginning of the corresponding sections.

2.1 The bosonic string

String theory is the study of one-dimensional objects (strings) propagating in a d-dimensional spacetime. In the same way that a propagating point particle creates a worldline, a propagating string creates a worldsheet (fig. 2.1). Since the action for a particle is proportional to the proper length of the worldline, a natural action for the string is one that is proportional to the worldsheet area.

Definition. The *Nambu-Goto action* is defined as

$$S_{NG} = -\frac{1}{2\pi\alpha'}\int_\Sigma dA \tag{2.1}$$

$$= -\frac{1}{2\pi\alpha'}\int d^2\sigma\left[-\det_{\alpha\beta}\frac{\partial X^\mu}{\partial\sigma^\alpha}\frac{\partial X^\nu}{\partial\sigma^\beta}\eta_{\mu\nu}\right]^{1/2} \tag{2.2}$$

$$= -\frac{1}{2\pi\alpha'}\int d^2\sigma\sqrt{-\Gamma}\ , \tag{2.3}$$

where $\sigma^\alpha = (\tau, \sigma)$ are the worldsheet coordinates, $\Gamma_{\alpha\beta} = \frac{\partial X^\mu}{\partial\sigma^\alpha}\frac{\partial X^\nu}{\partial\sigma^\beta}\eta_{\mu\nu}$ is the induced metric on the worldsheet and α' having dimensions of (length)^2 is the only dimensionful quantity in string theory. It is known as the *Regge slope.*

This action includes a square root which is cumbersome to deal with. A way to get around this is the following:

Definition. The *Polyakov action* is

$$S_P = -\frac{1}{4\pi\alpha'}\int d^2\sigma\sqrt{-h}h^{\alpha\beta}\frac{\partial X^\mu}{\partial\sigma^\alpha}\frac{\partial X^\nu}{\partial\sigma^\beta}\eta_{\mu\nu} \tag{2.4}$$

$$= -\frac{1}{4\pi\alpha'}\int d^2\sigma\sqrt{-h}h^{\alpha\beta}\Gamma_{\alpha\beta}\ . \tag{2.5}$$

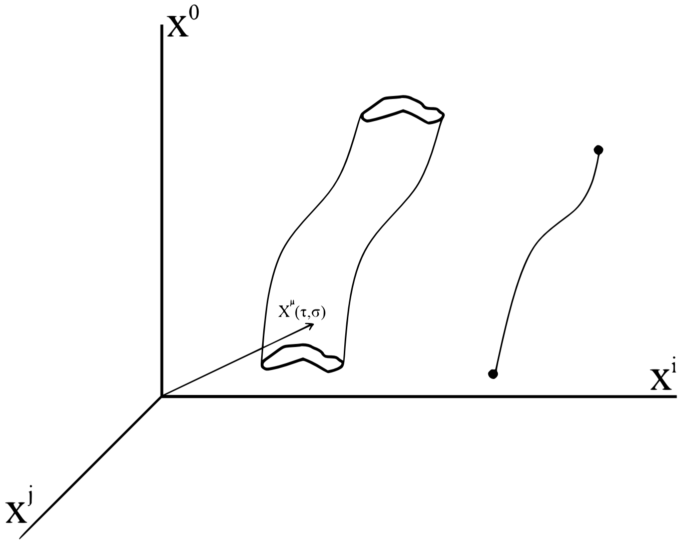

Figure 2.1: A string and a point particle propagating on a d-dimensional spacetime.

In this form the worldsheet metric $h_{\alpha\beta}$ is arbitrary, but not dynamical since no derivatives appear in the action. Note the appearance of the Minkowski metric $\eta_{\mu\nu}$ in the action, which means that the string is propagating in flat spacetime. We will study more general cases where $\eta_{\mu\nu}$ is replaced by a dynamical field $G_{\mu\nu}(X)$ in later chapters. When this happens, (2.4) is simply the zeroth order approximation of the exact result.

Definition. One of the most fundamental quantities for any model is the *stress-energy tensor*. It is the quantity that describes how small variations of the metric affect the action and it is defined as

$$T_{\alpha\beta} = \frac{4\pi}{\sqrt{-h}} \frac{\delta S_P}{\delta h^{\alpha\beta}} \ . \tag{2.6}$$

We also note that the Polyakov action is invariant under the following:

- Poincaré transformations: $X^\mu(\tau,\sigma) \to a^\mu{}_\nu X^\nu(\tau,\sigma) + b^\mu$,
- reparameterizations of the worldsheet: $(\tau,\sigma) \to (\tilde{\tau},\tilde{\sigma})$,
- Weyl rescaling: $h^{\alpha\beta}(\tau,\sigma) \to \Omega(\tau,\sigma) h^{\alpha\beta}(\tau,\sigma)$.

In fact, we can use the last to set $h_{\alpha\beta} = \eta_{\alpha\beta} = \mathrm{diag}(-1,1)$ bringing the action to the form

$$S_P = -\frac{1}{2\pi\alpha'} \int d^2\sigma \ \eta^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X_\mu \ . \tag{2.7}$$

The equations of motion for X read

$$\partial_\alpha \partial^\alpha X^\mu = 0 \tag{2.8}$$

and when supplemented with the closed string boundary conditions

$$X^\mu(\tau, \sigma + 2\pi) = X^\mu(\tau,\sigma) \tag{2.9}$$

lead to a general wave solution of the form

$$X^\mu(\tau,\sigma) = X^\mu_R(\tau - \sigma) + X^\mu_L(\tau+\sigma) \ , \tag{2.10}$$

where

$$X^\mu_R(\tau-\sigma) = \frac{1}{2}x^\mu + \frac{\alpha'}{2}p^\mu(\tau-\sigma) + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\alpha^\mu_n e^{-in(\tau-\sigma)} , \tag{2.11}$$

$$X^\mu_L(\tau+\sigma) = \frac{1}{2}x^\mu + \frac{\alpha'}{2}p^\mu(\tau+\sigma) + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\bar{\alpha}^\mu_n e^{-in(\tau+\sigma)} . \tag{2.12}$$

When quantizing the theory, all the above quantities are promoted to operators and we impose the commutation relations

$$[x^\mu, p^\nu] = i\eta^{\mu\nu} , \tag{2.13}$$

$$[\bar{\alpha}^\mu_m, \alpha^\nu_n] = 0 , \tag{2.14}$$

$$[\alpha^\mu_m, \alpha^\nu_n] = [\bar{\alpha}^\mu_m, \bar{\alpha}^\nu_n] = m\delta_{m+n,0}\eta^{\mu\nu} . \tag{2.15}$$

Furthermore, the reality (hermiticity) of the field X leads to

$$(\alpha^\mu_m)^\dagger = \alpha^\mu_{-m} \quad \text{and} \quad (\bar{\alpha}^\mu_m)^\dagger = \bar{\alpha}^\mu_{-m} . \tag{2.16}$$

If we absorb the factor m in (2.15) in the oscillators by redefining $\alpha^\mu_m \to \frac{1}{\sqrt{|m|}}\alpha^\mu_m$, we obtain

$$[\alpha^\mu_m, (\alpha^\nu_n)^\dagger] = \delta_{m,n}\eta^{\mu\nu} , \tag{2.17}$$

which are the familiar from the harmonic oscillator commutation relations. They allow us to interpret modes of the form α^μ_{-m}, with $m > 0$, as creation operators and modes of the form α^μ_m, with $m > 0$, as annihilation operators. The number operator and the Hamiltonian operator will then be

$$N_m =: \alpha_m \cdot \alpha_{-m} := \alpha_{-m}\alpha_m , \tag{2.18}$$

$$H = \frac{1}{2}\sum_{n\in\mathbb{Z}} : \alpha_{-n}\cdot\alpha_n + \bar{\alpha}_{-n}\cdot\bar{\alpha}_n : , \tag{2.19}$$

where the normal ordering symbol means that annihilation operators should appear on the right of creation operators. If we also define the Virasoro operators as

$$L_n = \frac{1}{2}\sum_{m\in\mathbb{Z}} : \alpha_{n-m}\cdot\alpha_m : , \tag{2.20}$$

we note that

$$H = L_0 + \overline{L}_0 . \tag{2.21}$$

We will further discuss the Virasoro operators and the algebra they satisfy in section 2.5. For now, we conclude with two further remarks presented here without proof. The first is that there is a normal ordering ambiguity for L_0, implying that the general quantum version of L_0 will differ by the normal ordered one by a constant. This effectively means we should replace $L_0 \to L_0 + a$, where a is a constant, in every formula.

The second is that this formalism allows for states with negative norm. For example using $[\alpha^0_m, \alpha^0_{-m}] = -m$ we see that $\langle 0|\alpha^0_m\alpha^0_{-m}|0\rangle = -m\,\langle 0|0\rangle < 0$. The issue can be circumvented by making sure that physical states are not of this type. This can be guaranteed by imposing the physicality conditions

$$L_n\,|\text{phys}\rangle = 0 \quad \text{and} \quad \overline{L}_n\,|\text{phys}\rangle = 0, \quad n > 0\ , \tag{2.22}$$

$$(L_0 + a)\,|\text{phys}\rangle = 0 \quad \text{and} \quad (\overline{L}_0 + a)\,|\text{phys}\rangle = 0\ , \tag{2.23}$$

$$(L_0 - \overline{L}_0)\,|\text{phys}\rangle = 0\ . \tag{2.24}$$

It then remains to show that unphysical states, known as ghosts, completely decouple from the spectrum. The following "no-ghost theorem" is a famous result in string theory.

Theorem. *In 26 spacetime dimensions the physical spectrum defined by the above conditions contains only positive norm states if* $a = -1$.

2.2 The fermionic string

There are many reasons why one might want to further generalize the action (2.4). The mathematical reason is that there is no *a priori* justification from the two-dimensional point of view for restricting to the Polyakov action. Instead, it is preferable to fix the symmetries of the action and then ask what is the most general action that can be written. We will further elaborate on this point in section 2.6.

From a physical point of view, the spectrum of the bosonic string includes a tachyon which is highly undesirable. Including worldsheet fermions in the action is the first step towards further generalizations and it can solve the tachyon problem as well.

The action of a Majorana fermion in 2d Minkowski space with metric $h_{\alpha\beta} = \text{diag}(1, -1)$ is

$$S = \frac{1}{4\pi} \int dx^0 dx^1 \sqrt{|h|} (-i) \overline{\Psi} \gamma^\alpha \partial_\alpha \Psi\ , \tag{2.25}$$

where $\Psi = \binom{\psi}{\bar{\psi}}$ is a two (real) component spinor, $\overline{\Psi} = \Psi^\dagger \gamma^0$ and the γ's satisfy the *Clifford algebra*

$$\{\gamma^\alpha, \gamma^\beta\} = 2h^{\alpha\beta}\mathbb{1}\ . \tag{2.26}$$

Performing a Wick-rotation the metric becomes Euclidean and if we also complexify the coordinates as $z = x^0 + ix^1$ the action becomes

$$S = \frac{1}{4\pi} \int dz d\bar{z} \left(\psi \bar{\partial} \psi + \bar{\psi} \partial \bar{\psi} \right)\ , \tag{2.27}$$

which is usually taken as the starting point when discussing fermions from a conformal field theory point of view.

The equations of motion for ψ and $\bar{\psi}$ read

$$\partial \bar{\psi} = \bar{\partial} \psi = 0 \tag{2.28}$$

and when supplemented with the closed string boundary conditions

$$\psi(e^{2\pi i} z) = +\psi \quad \text{Neveu-Schwarz (NS)} \tag{2.29}$$

$$\text{or} \quad \psi(e^{2\pi i} z) = -\psi \quad \text{Ramond (R)}\ , \tag{2.30}$$

they lead to a general solution of the form

$$\psi(z) = \sum_r \psi_r \; z^{-r-\frac{1}{2}} \quad \text{with} \quad \begin{cases} \text{NS} & \leftrightarrow \quad r \in \mathbb{Z} + \frac{1}{2} \\ \text{R} & \leftrightarrow \quad r \in \mathbb{Z} \end{cases} \tag{2.31}$$

and similarly for $\bar{\psi}$. Quantizing fermionic fields is achieved through anti-commutation (rather than commutation) relations, so we impose

$$\{\psi_r, \psi_s\} = \delta_{r+s,0} \; . \tag{2.32}$$

The stress-energy tensor for this theory is

$$T(z) = \frac{1}{2} : \psi \partial \psi := \sum_{n \in \mathbb{Z}} L_n \; z^{-n-2} \; , \tag{2.33}$$

where in the second equation we have also introduced the Virasoro operators as the Laurent modes of T. The analogue of (2.20) in this case is

$$L_n = \frac{1}{2} \sum_r : \psi_{n-r} \cdot \psi_r : \quad \text{with} \quad \begin{cases} \text{NS} & \leftrightarrow \quad r \in \mathbb{Z} + \frac{1}{2} \\ \text{R} & \leftrightarrow \quad r \in \mathbb{Z} \end{cases} . \tag{2.34}$$

2.3 $N = 1$ worldsheet supersymmetry

By adding the action of the free fermion (2.27) to the action of the free boson (2.7) we obtain

$$S = \frac{1}{4\pi} \int dz d\bar{z} \; \left(\partial_\alpha X^\mu \partial^\alpha X_\mu + \psi \bar{\partial} \psi + \bar{\psi} \partial \bar{\psi} \right) \; , \tag{2.35}$$

using $\alpha' = 2$ in this section. Unsurprisingly, the stress-energy tensor in this case is

$$T(z) = \frac{1}{2} : \partial X \partial X : + \frac{1}{2} : \psi \partial \psi : \; . \tag{2.36}$$

However, this action now possesses more symmetry than before (worldsheet supersymmetry). Namely, it is invariant under the worldsheet supersymmetry transformations

$$\delta X \propto \epsilon \psi \; , \tag{2.37}$$

$$\delta \psi \propto \epsilon \gamma^\alpha \partial_\alpha X \; , \tag{2.38}$$

where ϵ is a constant infinitesimal spinor. This implies the existence of a superpartner for every field in the theory. The superpartner of the stress-energy tensor is

$$G(z) = : \partial X \psi : (z) \; . \tag{2.39}$$

It can be expanded in terms of its Laurent modes as

$$G(z) = \sum_{r \in \mathbb{Z} + \frac{1}{2}} G_r z^{-r-\frac{3}{2}} \quad \text{with} \quad G_r = \sum_{s \in \mathbb{Z} + \frac{1}{2}} j_{r-s} \psi_s \; . \tag{2.40}$$

In the previous equation j_m are the Laurent modes of $j(z) = \partial X(z)$. We state here without proof [14] that the fields above satisfy the super-Virasoro algebra, where:

Definition. The $N = 1$ *super-Virasoro algebra* is defined through the following commutation and anti-commutation relations:

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m,-n} \ , \tag{2.41}$$

$$[L_m, G_r] = \left(\frac{m}{2} - r\right) G_{m+r} \ , \tag{2.42}$$

$$\{G_r, G_s\} = 2L_{r+s} + \frac{c}{3}\left(r^2 - \frac{1}{4}\right)\delta_{r,-s} \ . \tag{2.43}$$

Theories that satisfy this algebra are called $N = 1$ superconformal field theories (SCFTs).

2.4 $N = 2$ worldsheet supersymmetry

We can further generalize the idea in the previous section to define an $N = 2$ SCFT, *i.e.* a theory with $N = 2$ worldsheet supersymmetry. By definition a CFT is said to have $N = 2$ worldsheet supersymmetry if it includes four fields:

$$T(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2} \ , \tag{2.44}$$

$$G^{\pm}(z) = \sum_{n\in\mathbb{Z}} G^{\pm}_{n\pm a} z^{-n-\frac{3}{2}\mp a} \ , \tag{2.45}$$

$$J(z) = \sum_{n\in\mathbb{Z}} J_n z^{-n-1} \ , \tag{2.46}$$

that satisfy the algebra:

$$\begin{aligned}
[L_m, L_n] &= (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0} \ , \\
\left[L_m, G^{\pm}_{n\pm a}\right] &= (\frac{m}{2} - n \mp a)G^{\pm}_{m+n\pm a} \ , \\
[L_m, J_n] &= -nJ_{m+n} \ , \\
[J_m, J_n] &= \frac{c}{3} m\delta_{m+n,0} \ , \\
\left[J_m, G^{\pm}_{n\pm a}\right] &= \pm G^{\pm}_{m+n\pm a} \ , \\
\{G^{+}_{m+a}, G^{-}_{n-a}\} &= 2L_{m+n} + (m-n+2a)J_{m+n} + \frac{c}{3}\Big((m+a)^2 - \frac{1}{4}\Big)\delta_{m+n,0} \ , \\
\{G^{+}_{m+a}, G^{+}_{n+a}\} &= \{G^{-}_{m-a}, G^{-}_{n-a}\} = 0 \ ,
\end{aligned} \tag{2.47}$$

where a is a real parameter that describes how the fermionic superpartners $G^{\pm}$ of T transform:

$$G^{\pm}(e^{2\pi i}z) = -e^{\mp 2\pi i a}G^{\pm}(z). \tag{2.48}$$

The algebras for a and $a+1$ are isomorphic. $a \in \mathbb{Z}$ corresponds to the R sector and $a \in \mathbb{Z} + \frac{1}{2}$ corresponds to the NS sector. The first three equations in (2.47) are simply the super-Virasoro algebra. The next two equations specify a $U(1)$ current algebra and that $G^{\pm}$ has $j(z)$ charge ± 1. The last two determine the anti-commutation relations between the fields $G^{\pm}(z)$.

Name	Transformation	Generator
translation	$x'^{\mu} = x^{\mu} + a^{\mu}$	$-i\partial_{\mu}$
dilation	$x'^{\mu} = \alpha x^{\mu}$	$-ix^{\mu}\partial_{\mu}$
rotation	$x'^{\mu} = M^{\mu}_{\nu} x^{\mu}$	$i(x_{\mu}\partial_{\nu} - x_{\nu}\partial_{\mu})$
special conformal transformation	$x'^{\mu} = \frac{x^{\mu} - (x \cdot x) a^{\mu}}{1 - 2a \cdot x + (a \cdot a)(x \cdot x)}$	$-i(2x_{\mu}x^{\nu}\partial_{\nu} - (x \cdot x)\partial_{\mu})$

Table 2.1: Summary of the finite conformal transformations and the corresponding generators in $d \geq 3$.

The Cartan subalgebra of the $N = 2$ super-Virasoro algebra is generated by L_0 and j_0. These operators can be diagonalized simultaneously and each state in the Hilbert space is specified by the corresponding eigenvalues acting as labels:

$$L_0 \left| h, Q \right\rangle = h \left| h, Q \right\rangle \quad \text{and} \quad j_0 \left| h, Q \right\rangle = Q \left| h, Q \right\rangle \ . \tag{2.49}$$

We will study the properties of this algebra in more depth in chapter 6.

2.5 Conformal Field Theory

2.5.1 Conformal transformations

We will now temporarily interrupt the discussion of strings to introduce the basic language of two-dimensional Conformal Field Theory (CFT) and the concepts that will be used later on. This section includes material from [14, 15] and proofs omitted in this short review can be found there.

Definition. A transformation $x \mapsto x'$ is called *conformal* if it preserves the metric up to an overall factor, *i.e.* if $g'(x') = \Lambda(x) g(x)$. $\Lambda(x)$ is called the *scale factor*.

The different types of conformal transformations in $d \geq 3$ are summarized in table 2.1. In two dimensions, we usually complexify the coordinates as

$$z = x^0 + ix^1 \qquad \bar{z} = x^0 - ix^1. \tag{2.50}$$

It is then true that

Proposition. *Any holomorphic function $f(z) = z + \epsilon(z)$ gives rise to an infinitesimal two-dimensional conformal transformation $z \mapsto f(z)$.*

Proof. When $z \mapsto f(z)$ the metric transforms as

$$ds^2 = dz d\bar{z} \ \mapsto \ \frac{\partial f}{\partial z} \frac{\partial \bar{f}}{\partial \bar{z}} dz d\bar{z}$$

so the transformation is conformal and the scale factor is $|\frac{\partial f}{\partial z}|^2$. □

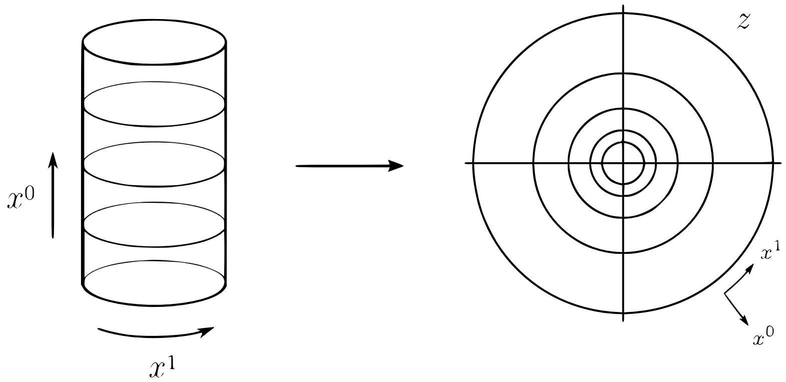

Figure 2.2: The map of the cylinder to the complex plane.

By performing the Laurent expansion of such a function around say $z = 0$ we obtain:

$$z' = z + \epsilon(z) = z + \sum_{n\in\mathbb{Z}} \epsilon_n(-z^{n+1}), \tag{2.51}$$

$$\bar{z}' = \bar{z} + \epsilon(\bar{z}) = \bar{z} + \sum_{n\in\mathbb{Z}} \epsilon_n(-\bar{z}^{n+1}), \tag{2.52}$$

which allows us to identify the generators for a particular n as

$$l_n = -z^{n+1}\partial_z \quad , \quad \bar{l}_n = -\bar{z}^{n+1}\partial_{\bar{z}} \ . \tag{2.53}$$

Definition. These generators furnish two copies of what is known as the *Witt algebra.* Namely, they satisfy the commutation relations:

$$\begin{aligned} [l_m, l_n] &= (m-n)l_{m+n}, \\ \left[\bar{l}_m, \bar{l}_n\right] &= (m-n)\bar{l}_{m-n}, \\ \left[l_m, \bar{l}_n\right] &= 0. \end{aligned} \tag{2.54}$$

The above relations demonstrate that the algebra of infinitesimal conformal transformations in $d = 2$ dimensions is infinite dimensional. It also turns out that the Witt algebra admits a central extension leading to the following definition.

Definition. The *Virasoro algebra* with *central charge* c is the extension of the Witt algebra and it is defined via the commutation relations:

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m,-n} \ . \tag{2.55}$$

2.5.2 Correlation functions, ordering and operator product expansion

In CFT the observables are correlation functions of the form

$$\langle A_a(z_1, \bar{z}_1) \cdots A_n(z_n, \bar{z}_n) \rangle \ . \tag{2.56}$$

Definition. The *expectation value* is defined as

$$\langle F[x(z,\bar{z})] \rangle = \frac{\int [dx] F[x(z,\bar{z})] e^{-S}}{\int [dx] e^{-S}} \tag{2.57}$$

where $x(z, \bar{z})$ collectively denotes all the fields and S is the action.

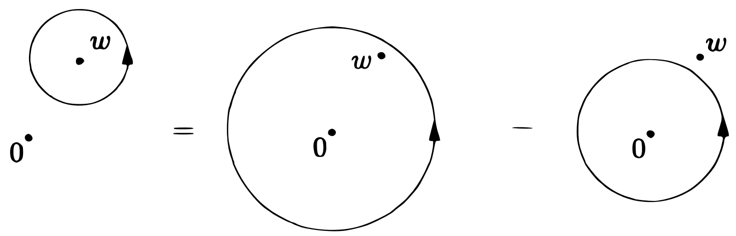

Figure 2.3: Contours of the integrals appearing in (2.61).

Everything inside the expectation value is implicitly assumed to be radially ordered. This is the analogue of time ordering in QFT. However, when discussing closed strings the worldsheet is a cylinder with coordinates

$$w = x^0 + ix^1 \quad \text{with} \quad w \sim w + 2\pi i \tag{2.58}$$

and it is usually convenient to map the cylinder to the plane via the transformation (see fig. 2.2)

$$z = e^w = e^{x^0} \cdot e^{ix^1} \ . \tag{2.59}$$

When mapped to the plane, an increase in the radial coordinate corresponds to a later time, leading to the following definition:

Definition. The *radial ordering* of two operators is defined as

$$R(A(z)B(w)) = \begin{cases} A(z)B(w) & \text{when } |z| > |w| \\ B(w)A(z) & \text{when } |w| > |z| \end{cases} \ . \tag{2.60}$$

The radial ordering is in a sense an equal time commutator. This can be seen via the following equalities, with the curves of the contour integrals defined in fig. 2.3.

$$\oint_{C(w)} dz\ R(A(z), B(w)) = \oint_{|z|>|w|} dz\ A(z)B(w) - \oint_{|z|<|w|} dz\ B(w)A(z) \tag{2.61}$$

$$= \oint dz\ [A(z), B(w)] \tag{2.62}$$

Definition. Let A_i be the set of all local operators of the CFT. The *operator product expansion (OPE)* of any two of them is

$$A_i(z,\bar{z})A_j(w,\bar{w}) = \sum_k C_{ij}^k(z-w,\bar{z}-\bar{w})A_k(w,\bar{w}) \ , \tag{2.63}$$

where such an expression is always understood to hold inside expectation values. For example,

$$\langle A_i(z,\bar{z})A_j(w,\bar{w})\cdots\rangle = \sum_k C_{ij}^k(z-w,\bar{z}-\bar{w})\,\langle A_k(w,\bar{w})\cdots\rangle \ , \tag{2.64}$$

where $\cdots$ denote other arbitrary operators inserted at a distance large compared to $|z-w|$.

2.5.3 The stress-energy tensor and primary fields

Conformal theories are invariant under the transformation $x^\mu \mapsto x^\mu + \epsilon^\mu(x)$. By Noether's theorem, this implies a conserved current which in this case can be written as

$$j_\mu = T_{\mu\nu}\epsilon^\nu \ . \tag{2.65}$$

Definition. The quantity $T_{\mu\nu}$ appearing above is symmetric and is called the *stress-energy tensor.*

The stress-energy tensor is one of the most fundamental objects in any CFT. It can be expanded in terms of its Laurent series modes L_n:

$$T(z) = \sum_{n\in\mathbb{Z}} z^{-n-2}L_n \quad \Leftrightarrow \quad L_n = \frac{1}{2\pi i}\oint dz\ z^{n+1}T(z) \ , \tag{2.66}$$

and it is these modes that (after quantization) will satisfy the Virasoro algebra (2.55). We now proceed with some further definitions.

Definition. If a field $\phi(z,\bar{z})$ transforms under scalings $z \mapsto \lambda z$ as

$$\phi(z,\bar{z}) \mapsto \phi'(z,\bar{z}) = \lambda^h \bar{\lambda}^{\bar{h}} \phi(\lambda z, \bar{\lambda}\bar{z}) \ , \tag{2.67}$$

it is said to have *conformal dimension* $(h,\bar{h})$.

Definition. If a field $\phi(z,\bar{z})$ transforms under conformal transformations $z \mapsto f(z)$ as

$$\phi(z,\bar{z}) \mapsto \phi'(\lambda z,\bar{\lambda}\bar{z}) = \left(\frac{\partial f}{\partial z}\right)^{-h}\left(\frac{\partial}{\partial \bar{z}}\right)^{-\bar{h}} \phi(z,\bar{z}) \ , \tag{2.68}$$

it is called a *primary field* with conformal dimensions $(h,\bar{h})$.

Proposition (Equivalent definition)**.** *A field $\phi(z,\bar{z})$ is a* primary field *with conformal dimensions $(h,\bar{h})$ if and only if its OPE with the stress-energy tensor takes the form*

$$T(z)\phi(w,\bar{w}) = \frac{h}{(z-w)^2}\phi(w,\bar{w}) + \frac{1}{z-w}\partial_w\phi(w,\bar{w}) + \cdots \tag{2.69}$$

$$\overline{T}(z)\phi(w,\bar{w}) = \frac{\bar{h}}{(\bar{z}-\bar{w})^2}\phi(w,\bar{w}) + \frac{1}{\bar{z}-\bar{w}}\partial_w\phi(w,\bar{w}) + \cdots \ , \tag{2.70}$$

where the ellipses denote non-singular terms.

2.5.4 Simple currents

Definition. The conformal fields ϕ_a of a theory satisfy an operator product algebra which we can then use to define the *fusion algebra* as:

$$\phi_a \times \phi_b = N_{ab}{}^c\ \phi_c \ , \tag{2.71}$$

where $N_{ab}{}^c$ are just the non-negative constants in the coefficients appearing in the OPE (2.63) of ϕ_a with ϕ_b.

Definition. A *simple current* J is a primary field with the property that

$$J \times \phi_a = \phi_{J_a} \quad \text{for all primary fields } \phi_a \ . \tag{2.72}$$

In other words, when a simple current fuses with any other primary field then (by definition) only one field appears on the right-hand-side, as opposed to a linear combination of fields as stated in (2.71).

2.6 Minimal models and Gepner models

This section is mainly based on [16–20].

2.6.1 Introduction

As we have already briefly discussed, there are two different ways to think about string theory: the *spacetime view* and the *worldsheet view*. The spacetime view is the one presented in section 2.1 with the string propagating in a d-dimensional spacetime. The no-ghost requirement then imposes $d = 26$ and we are left with the non-trivial task of explaining why we do not observe the extra dimensions in the real world, the most common way out being to claim that they are compactified in some way.

These interpretation issues do not arise in the alternative worldsheet point of view. In this approach, the goal is to write down the most general action that can be defined on a two-dimensional surface that is invariant under Poincaré transformations, worldsheet reparameterizations and Weyl rescaling[1]. Imposing that the algebra of the CFT is the Virasoro algebra, the consistency of the theory requires that the central charge is $c = 26$. The main difference from the previous approach comes from splitting the CFT as:

$$\mathrm{CFT}_{c=26} = \mathrm{CFT}_{c=4} \oplus \mathrm{CFT}_{\mathrm{internal}} \ . \tag{2.73}$$

A realization of $\mathrm{CFT}_{c=4}$ must be given in terms of 4 bosons X^μ which will be interpreted as coordinates in a 4-dimensional spacetime. The realization of the internal CFT is in principle arbitrary.

Any concrete model about the real world will of course have to specify the internal CFT part, but the significant advantage of this approach is that we do not need to realize it in terms of bosons. There are many other ways it can be done. This also means that we do not have to propose elaborate ways of compactifying the extra dimensions, since there are no extra dimensions. The effects of different realizations of the internal part will present themselves in the physical spectrum of the theory.

The choice of realizing the internal CFT as bosons, identifying these bosons as extra dimensions and then explaining how they are compactified, results in the so called *geometric theories*. If we chose a different description for the internal CFT we get a *non-geometric theory*. It is not *a priori* obvious if geometric and non-geometric

[1] We do not add worldsheet supersymmetry just yet, in order to make a comparison with the bosonic string. We will add it in the next section.

theories are equivalent or not. The equivalence of the non-geometric free fermionic models with the geometric orbifold models is discussed in great detail in this thesis.

2.6.2 Minimal models

If we focus temporarily on the superstring, for which consistency requires $c = 15$, we implement the separation of spacetime and internal parts as:

$$\mathrm{CFT}_{c=15} = \mathrm{CFT}_{c=6} \oplus \mathrm{CFT}_{c=9} \ . \tag{2.74}$$

Note that in this case the four spacetime bosons (that have $c = 1$) come with their superpartners (that have $c = 1/2$) and $4 \cdot (1 + \frac{1}{2}) = 6$, so all of these are grouped into the first term. We would like to focus on the internal $c = 9$ CFT next.

One of the simplest options (beyond a bosonic or fermionic realization) for the internal CFT is to consider a tensor product of the so called *minimal models* $\mathcal{A}_k$. These are $N = 2$ SCFTs with $c = \frac{3k}{k+2}$ where k is an integer. The motivation for using $N = 2$ SCFTs as the starting point is spacetime supersymmetry. It is well known that to preserve $N = 1$ spacetime supersymmetry we must compactify on a Calabi-Yau manifold. The analogue of this statement from the worldsheet point of view is that in order to guarantee $N = 1$ spacetime supersymmetry we must start from an $N = 2$ SCFT.

The algebra of these theories is (2.47). In such theories, it is particularly convenient to describe a (left-moving) state in terms of three numbers l, q, s:

$$\phi = |l, q, s\rangle \ . \tag{2.75}$$

However, we know from (2.49) that the Cartan subalgebra has rank 2, so not all three variables are independent. The restrictions on them are [16, 17, 21]:

$$\begin{aligned} 0 \le l \le k, \quad 0 \le |q - s| \le l, &\qquad l + q + s = 0 \mod 2 \\ q = q \mod 2(k+2), &\qquad s = s \mod 4 \\ s = 0, 2 \to \mathrm{NS}, &\qquad s = \pm 1 \to \mathrm{R} \end{aligned} \tag{2.76}$$

and similarly for a right-moving state for which we use barred numbers $\bar{l}, \bar{q}, \bar{s}$. The conformal charge h and the superconformal $U(1)$ charge Q of such a state are given by:

$$h = \frac{1}{4(k+2)}[l(l+2) - q^2] + \frac{1}{8}s^2 \ , \tag{2.77}$$

$$Q = -\frac{q}{k+2} + \frac{1}{2}s \ . \tag{2.78}$$

Left and right minimal models can be tensored into $\mathcal{A}_k \otimes \bar{\mathcal{A}}_k$. In such products the following states are identified:

$$(l, q, s) \otimes (\bar{l}, \bar{q}, \bar{s}) \equiv (k - l, q + k + 2, s + 2) \otimes (k - \bar{l}, \bar{q} + k + 2, \bar{s} + 2) \ . \tag{2.79}$$

For future convenience let us also define:

$$\phi_L = \otimes_{i=1}^{r} (l_i, q_i, s_i) \tag{2.80}$$

and similarly for ϕ_R, where we have considered a tensor product of r minimal models which gives a total central charge of 9:

$$c_T = \sum_{i=1}^{r} c_i = \sum_{i=1}^{r} \frac{3k_i}{k_i+2} \stackrel{!}{=} 9 \tag{2.81}$$

Another common notation for the state $|\phi_L, \phi_R\rangle$ is :

$$|\phi_L, \phi_R\rangle = \phi^{l;\bar{l}}_{q;\bar{q};s;\bar{s}} \tag{2.82}$$

where the indices on the right hand side are understood to be vectors with r components. Sometimes we might suppress the s and $\bar{s}$ indices for convenience.

2.6.3 Gepner models

Gepner provided a procedure for constructing consistent CFTs with all the desired properties [16, 17]. We will use these models as a test ground for ideas that go beyond the bosonic and fermionic realizations of string theory. In the original construction, a type II theory was used to construct a heterotic theory. We will present here a similar construction, in which one starts from the bosonic string and then gets either a type II theory or a heterotic theory. We will restrict ourselves to the heterotic theory.

In a heterotic Gepner model, the left-moving sector (the superstring) has a CFT which is split into the following parts:

1. Two spacetime bosons/coordinates (in the lightcone gauge) with $c = 2 \cdot 1 = 2$,
2. an $SO(2)_1$ part associated with (the fermionic superpartners of) the spacetime-coordinates with $c = 2 \cdot \frac{1}{2} = 1$,
3. an internal CFT with $c = 9$ which is realized as a product of minimal models.

In the right-moving sector we split the CFT into three parts:

1. Two spacetime bosons/coordinates (in the lightcone gauge) with $c = 2 \cdot 1 = 2$. We will ignore these (as well as their left-moving counterparts) from now on.
2. A CFT with $c = 9$ which is realized as a product of minimal models to match the left-moving sector,
3. a CFT of $c = 13$ which we choose to also realize as bosons. These 13 bosons are taken to be compactified on the root lattice of $SO(10) \times E_8$. Since these bosons have no counterpart on the left-moving sector we cannot interpret them as extra dimensions and thus naively one might expect that there is no need to compactify them. The reason we actually have to do the compactification on what appears to be a random lattice is modular invariance. It turns out that constructing heterotic modular invariant partition functions is extremely difficult and using the root lattice of $SO(10) \times E_8$ is one of the few ways it can be done. Another alternative is the $SO(26)$ root lattice. Descriptions of all these lattices are given in appendix A.

Finally, let us establish some notation for future use. It is customary to represent states in a Gepner model as vectors:

$$V = V_L \otimes V_R = (w; q_1, \cdots, q_r; s_1, \cdots, s_r) \otimes (\bar{w}; \bar{q}_1, \cdots, \bar{q}_r; \bar{s}_1, \cdots, \bar{s}_r) \tag{2.83}$$

where w and $\bar{w}$ are $SO(2)$ and $SO(10)$ weights respectively. The dot product between two vectors is defined as:

$$V_L \cdot V_L' = w \cdot w' - \frac{1}{2}\sum_{i=1}^{r} \frac{q_i \cdot q_i'}{k_i + 2} + \frac{1}{4}\sum_{i=1}^{r} s_i \cdot {s'}_i \tag{2.84}$$

and similarly for the right-moving sector. The product between the w's is the usual Euclidean vector product. Some properties of the weight vectors of $SO(2n)$ can be found in appendix A.2.1.

Orbifolding Gepner models

Consider now the group $H = \mathbb{Z}_{m_1} \times \cdots \times \mathbb{Z}_{m_r}$ (as an additive group) and take an element $\gamma = (\gamma_1, \cdots, \gamma_r) \in H$. We also define:

$$\Gamma = (0; \gamma_1, \dots, \gamma_r; 0, \dots, 0) \tag{2.85}$$

to mimick the structure of $V_{L/R}$. With each γ we associate the operator $g(\gamma)$ acting on a state ϕ as:

$$g(\gamma)\phi^{l;\bar{l}}_{q;\bar{q}} = e^{2\pi i \sum_{j=1}^{r} \frac{\gamma_j (q_j + \bar{q}_j)}{2(k_j+2)}} \phi^{l;\bar{l}}_{q;\bar{q}} = e^{-2\pi i \Gamma \cdot (V_L + V_R)} \phi^{l;\bar{l}}_{q;\bar{q}} \ . \tag{2.86}$$

The quantity $z_j = \frac{q_j + \bar{q}_j}{2}$ is sometimes referred to as the $\mathbb{Z}_{k_j+2}$ charge. Note also that since q_j and $\bar{q}_j$ are only defined $\mathrm{mod} 2(k_j + 2)$ we need to have this number appearing in the denominator in the exponent if we want $g(\gamma)$ to be well defined. From this we can also see that the biggest possible symmetry group of the theory is:

$$H \subset G = \mathbb{Z}_{k_1+2} \times \cdots \times \mathbb{Z}_{k_r+2} \ . \tag{2.87}$$

If $k_i = k_j$ for some factors, then we also have the symmetry of exchanging these factors. We will ignore this permutation symmetry here.

The next step is to mod out by the discrete subgroup H (which we take to be generated by γ). This way we obtain a $\mathbb{Z}_M$ orbifold where $M = \mathrm{lcm}$[2]$(m_1, \dots, m_r)$. The boundary conditions for such an orbifold can be determined from modular invariance and they are:

$$V_L - V_R = n 2\Gamma + n_0 \beta_0 + n_i \beta_i \tag{2.88}$$

where $n \in \{0, \dots, M-1\}$ labels the twisted sectors and the vectors β_0 and β_i are defined as:

$$\beta_0 = (c; 1, \dots, 1; 1, \dots, 1) \ , \tag{2.89}$$

$$\beta_i = (v; 0, \dots, 0; 0, \dots, 2, \dots, 0) \ , \tag{2.90}$$

[2]least common multiple

where c and v are weights of the spinor and vector representation of SO(2) (or SO(10) for the right-movers) defined in appendix A.2.1 and 2 appears in the i^{th} position in the definition of β_i.

The generalized GSO projections associated with the modding of the theory by H are [22]:

$$\Gamma \cdot (V_L + V_R) \in \mathbb{Z} \ , \tag{2.91}$$

$$2\beta_0 \cdot \Gamma = -\sum_{i=1}^{r} \frac{\gamma_i}{k_i + 2} \in \mathbb{Z} \ . \tag{2.92}$$

The first equation above projects onto states invariant under the discrete symmetry and the second equation is needed to ensure spacetime supersymmetry. A powerful formalism that gives these results as well as many others is the *simple current construction*, which will be described in chapter 6. Moddings in which the gauge bosons which enlarge the $SO(10)$ gauge group to E_6 survive the projection are called $(2, 2)$ moddings. All other moddings are $(2, 0)$.

So far we have ignored the gauge degrees of freedom and focused solely on the $c = 9$ CFT. However, we can make an embedding of the $\mathbb{Z}_M$ symmetry in the gauge part of the (total) CFT. This embedding is represented as a shift of the $SO(10) \times E_8$ weight lattice by a vector A [23]. This mechanism is similar to the introduction of Wilson lines and indeed these shifts can often (but not always) be interpreted as Wilson lines. The main reason for implementing such an embedding is to break the gauge group to something smaller and phenomenologically more attractive. In chapter 6 we will examine the relationship between different models resulting from the breaking of the gauge group procedure described above. Since Gepner models will be used as examples there, we give here some more details about these models.

2.6.4 States of a heterotic Gepner model

Here we will give a brief outline of how we construct the states in a heterotic Gepner model. We start with a state of the form $V \otimes V$ of type II theory. The only restrictions that V must satisfy are

$$2\beta_0 \cdot V = \text{odd} \ , \tag{2.93}$$

$$2\beta_i \cdot V = \text{even} \ . \tag{2.94}$$

If the left-moving and right-moving sectors are different (*e.g.* in the heterotic string) we apply the above restrictions in each sector separately. The first condition is necessary for a supersymmetric theory[3] and the second ensures that the fermions in the various sectors have aligned boundary conditions (either NS or R).

We then apply the Gepner map by adding $(v; 0^r; 0^r)$ to the right to get $\tilde{V} = V + (v; 0^r; 0^r)$ which gives the state

$$V \otimes \tilde{V} \ . \tag{2.95}$$

[3] Using equation (2.78) we can see that this is equivalent to $Q_T =$odd, where $Q_T = 2c \cdot w + Q_{\text{int}}$.

This is our "starting state" in the heterotic string. We can then add multiples of β_0 on the left to get the superpartners of the state and on the right to get the E_6 partners:

$$\text{superpartners:} \qquad (V + m_0\beta_0) \otimes \tilde{V} \ , \tag{2.96}$$

$$E_6 \text{ partners:} \qquad V \otimes (\tilde{V} + m_0\beta_0) \ . \tag{2.97}$$

It is easy to verify that whenever V satisfies conditions (2.93) and (2.94) the above states satisfy the conditions as well[4] and are therefore allowed. For reasons of convenience we will drop the tilde from the right-moving sector whenever it is clear from the context that we are referring to the heterotic string.

Finally, let us give the mass formulae for the states:

$$\frac{\alpha' m_L^2}{2} = N_L + \frac{w^2}{2} + h_{\text{int}} - \frac{1}{2} \ , \tag{2.98}$$

$$\frac{\alpha' m_R^2}{2} = N_R + \frac{\bar{w}^2}{2} + \frac{p^2}{2} + h_{\text{int}} - 1 \ . \tag{2.99}$$

On the right-hand-side we recognize the contribution from the number operators, the internal CFT conformal dimension and the normal ordering constants. w and $\bar{w}$ are the contributions from the $SO(2)$ and $SO(10)$ weights respectively (we use the same symbol to remind us that the bosonic string map can be used to connect the two) and p is an E_8 weight.

2.6.5 The vectorial and spinorial states

We are now ready to identify the states in the spectrum that transform in the vectorial and spinorial representations of $SO(10)$. To this end, we focus on the right-moving sector for now. Any vectorial representation will have $\bar{w} = v \Rightarrow \bar{w}^2 = v^2 = 1, 3, 5, \ldots$ and it is easy to see that only $v^2 = 1$, *i.e.* the **10**, can give massless states. (2.99) then gives $h_{\text{int}} = \frac{1}{2}$, where we have used the fact that since $p \in \Gamma_{E_8}$ then $p^2 = 2k$ and the $p^2 \geq 2$ case does not give massless states when $\bar{w} = v$. Sometimes we group $\bar{w}$ and p together as $P = (\bar{w}, p) \in \Gamma_{SO(10)} \times \Gamma_{E_8}$.

From unitarity we also have the constraint

$$|Q| \leq 2h \quad \Rightarrow |Q_{\text{int}}| \leq 1$$

Also, Q_{int} must be an odd integer (this requirement comes from spacetime supersymmetry[5]) so $Q_{\text{int}} = \pm 1$. We chose (by convention) $Q_{\text{int}} = +1$ for the **10**. Then $Q_{\text{int}} = -1$ describes the $\overline{\mathbf{10}}$ (antimatter).

So the right-moving part for states in the **10** is:

$$(\bar{w})(h, Q) = (v_0)(\frac{1}{2}, 1)(p^2 = 0) \tag{2.100}$$

[4]Note however that $2\beta_0 \cdot V = \text{odd} \ \Leftrightarrow \ 2\beta_0 \cdot \tilde{V} = \text{even}$, so we must be careful to use the correct expression depending on if we have already applied the Gepner map or not. This issue does not appear for the other condition since $2\beta_i \cdot V = 2\beta_i \cdot \tilde{V}$.

[5]$2\beta_0 V = \text{even}$ (for heterotic) gives $Q_{\text{int}} + 2c \cdot w = \text{even}$. Here $w = v_0$ and $2c \cdot w = \pm 1$ so $Q_{\text{int}} = \text{odd}$.

with $v_0 = \underline{(\pm 1, 0, \ldots, 0)}$. The heterotic string can be thought of as originating from the type II string after applying the Gepner map on the right-moving sector :

$$(o, v, s, c)_{SO(2)} \quad \rightarrow \quad (v, o, -c, -s)_{SO(10)} \tag{2.101}$$

$$\text{or} \quad (o, v, s, c)_{SO(2)} \quad \rightarrow \quad (v, o, -s, -c)_{SO(10)}. \tag{2.102}$$

We will be using the first version in this work. If we ignore the signs (it is only $\bar{w}^2$ that appears in the mass formula anyway), the map can be succinctly described as

$$w_{SO(2)} \quad \rightarrow \quad w_{SO(10)} + v_{SO(10)}. \tag{2.103}$$

Therefore the state $(v_0)(\frac{1}{2}, 1)$ in the right-moving sector of the heterotic came from a $(0)(\frac{1}{2}, 1)$ state in type II. The full state in type II (left and right parts are identical) is

$$(0)(\frac{1}{2}, 1) \otimes (0)(\frac{1}{2}, 1),$$

which means that the full state in the heterotic is

$$(0)(\frac{1}{2}, 1) \otimes (v_0)(\frac{1}{2}, 1)(p^2 = 0) \tag{2.104}$$

with $v_0 = \underline{(\pm 1, 0, \ldots, 0)}$.

To find the superpartner of a state we add (multiples of) β_0 on the left (see (2.96)). Adding β_0 on a state with (h,Q) gives a state with $(h', Q') = (h + \frac{Q}{2} + \frac{3}{8})(Q + \frac{3}{2})$. The superpartner of equation (2.104) is

$$(s_0)(\frac{3}{8}, -\frac{1}{2}) \otimes (v_0)(\frac{1}{2}, 1)(p^2 = 0) \tag{2.105}$$

with $s_0 = \frac{1}{2}$. It is found by adding $-\beta_0$ on the left (no other multiple of β_0 gives massless states). (2.104) and (2.105) give the states transforming in the **10** of $SO(10)$.

To find the states in the **16** we just add $-\beta_0$ on the right side and we get:

$$(0)(\frac{1}{2}, 1) \quad \otimes \quad (c_0)(\frac{3}{8}, -\frac{1}{2})(p^2 = 0) \ , \tag{2.106}$$

$$(s_o)(\frac{3}{8}, -\frac{1}{2}) \quad \otimes \quad (c_0)(\frac{3}{8}, -\frac{1}{2})(p^2 = 0) \ . \tag{2.107}$$

If we add $-\beta_0$ on the right side one more time we get the singlets:

$$(0)(\frac{1}{2}, 1) \quad \otimes \quad (0)(1, -2)(p^2 = 0) \ , \tag{2.108}$$

$$(s_0)(\frac{3}{8}, -\frac{1}{2}) \quad \otimes \quad (0)(1, -2)(p^2 = 0) \ . \tag{2.109}$$

Note that everything up to this point has been general and applies to every Gepner model (as well as any other model with $N = 2$ superconformal symmetry for the internal CFT).

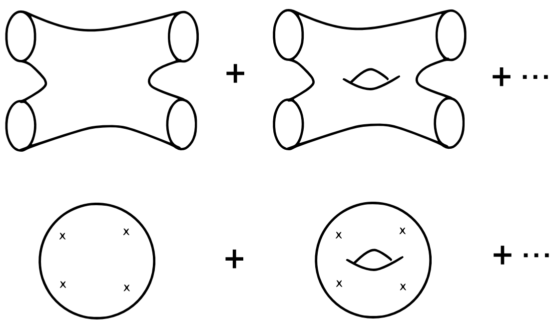

Figure 2.4: Schematic representation of a string amplitude as a series in the genus g parameter and the corresponding picture with the asymptotic states replaced by vertex operator insertions.

2.7 One loop amplitudes and modular invariance

Having introduced all the different types of models that will appear in this thesis, we now turn our attention again to fundamental concepts. Perturbative string theory means that amplitudes are expanded in terms of the genus g of the worldsheet. In other words, one is interested in diagrams like those appearing in fig. 2.4. Even though we will not go into the details of string loop calculations in this thesis, it is important to present the main idea. Schematically, the steps of the calculation involve replacing the asymptotic states with vertex operators, using what is known as the state/operator correspondence and then performing the integral over all inequivalent tori.

The case with no external legs and therefore no vertex operators is of particular interest. It describes the vacuum-to-vacuum amplitude and the amplitude is of the form

$$\mathcal{A} = \int_{\mathcal{F}} \frac{d^2\tau_2}{\tau_2^2} Z(\tau,\bar{\tau}) \ . \tag{2.110}$$

$\mathcal{F}$ is the fundamental domain of all inequivalent tori and $\frac{d^2\tau_2}{\tau_2^2}$ is the invariant measure. We will come back to these shortly. $Z(\tau,\bar{\tau})$ describes all the states that can run in the loop and therefore carries information about the entire spectrum of the theory. It is called the partition function.

Definition. The *partition function* is defined as

$$Z(\tau,\bar{\tau}) = \mathrm{Tr}_{\mathcal{H}}\, q^{L_0-\frac{c}{24}}\bar{q}^{\overline{L}_0-\frac{c}{24}}, \quad \text{where} \quad q = e^{2\pi i\tau} \ . \tag{2.111}$$

$\mathcal{H}$ is the Hilbert space of all states after the GSO projections have acted and, as we saw in section 2.5, the operators L_0 and $\overline{L}_0$ are related to the energy of the state. The name partition function is appropriate because once $Z(\tau,\bar{\tau})$ is calculated, it is of the form

$$Z(\tau,\bar{\tau}) = \sum d_{m,n} q^m \bar{q}^n \tag{2.112}$$

where the coefficient $d_{m,n}$ tells us how many states with fixed energy (mass level) m on the holomorphic sector and n on the anti-holomorphic sector exist in the spectrum.

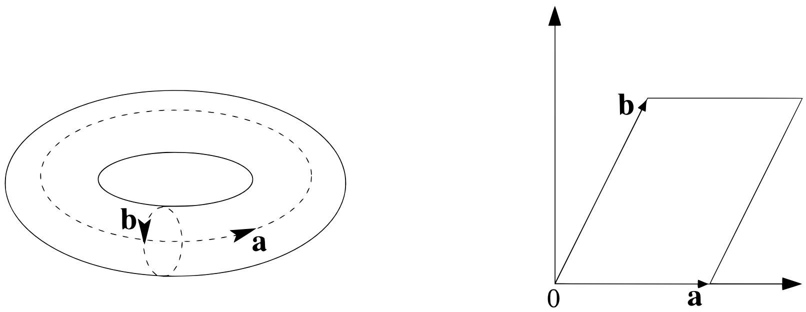

Figure 2.5: A torus arises from the periodic identification of points in the plane in two different directions specified by **a** and **b**. These two vectors also define a lattice on the plane.

2.7.1 Inequivalent tori

We now turn our attention on how to represent inequivalent tori. From fig. 2.5 we see that a torus is defined by identifying the following vectors on the plane

$$\mathbf{x} \sim m\mathbf{a} + n\mathbf{b}, \quad \text{with} \quad m, n \in \mathbb{Z}. \tag{2.113}$$

Definition. If a and b are the complex numbers corresponding to the vectors **a** and **b**, then the quantity describing the shape of the torus is called the *complex structure* or *Teichmüller parameter* or *modular parameter* τ and is defined as

$$\tau = \frac{b}{a} = \tau_1 + i\tau_2 \ . \tag{2.114}$$

We could of course choose a different set of basis vectors $\mathbf{a}'$ and $\mathbf{b}'$ to describe the torus. As long as the new basis is related to the old one via

$$\begin{pmatrix} \mathbf{a}' \\ \mathbf{b}' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \mathbf{a} \\ \mathbf{b} \end{pmatrix} \tag{2.115}$$

with $a, b, c, d \in \mathbb{Z}$ and $ad - bc = \pm 1$, then the torus is the same. Furthermore, $(-\mathbf{a}, -\mathbf{b})$ also describe the same torus as $(\mathbf{a}, \mathbf{b})$. We collect all this information in the following:

Proposition. *The transformations that leave a torus invariant belong to* $SL(2,\mathbb{Z})/\mathbb{Z}_2$ *and therefore*

$$\tau \quad \textit{and} \quad \frac{a\tau + b}{c\tau + d} \quad \textit{with} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})/\mathbb{Z}_2$$

describe the same torus. Such transformations are called modular transformations.

The physical meaning of the above proposition is that all observables must be invariant under modular transformations. We will see that this has strong implications about the form that partition functions are allowed to have. Before continuing, let us also state that:

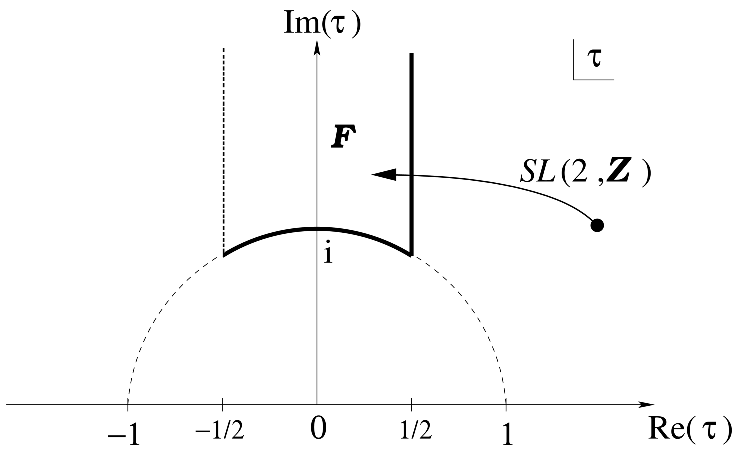

Figure 2.6: The fundamental domain $\mathcal{F}$ of $SL(2,\mathbb{Z})/\mathbb{Z}_2$.

Proposition. *The group of modular transformations is generated by*

$$T: \quad \tau \mapsto \tau + 1 \tag{2.116}$$

$$S: \quad \tau \mapsto -\frac{1}{\tau} \tag{2.117}$$

and therefore it is sufficient to check invariance only under T and S.

It is now easy to see that the tori that are not equivalent under the above transformations are those with a modular parameter in the set $\mathcal{F}$ given in fig. 2.6, *i.e.*

$$\mathcal{F} = \left\{ \tau \in \mathbb{C} : \ |\tau| \geq 1, \ -\frac{1}{2} < \tau_1 \leq \frac{1}{2}, \ \tau_2 > 0 \right\} \ . \tag{2.118}$$

2.7.2 Characters and modular invariants

To understand the partition function (2.111) better it is useful to disentangle the q and $\bar{q}$ parts further. In particular, looking at the form (2.112) we see that for Rational Conformal Field Theories (RCFTs) it is always possible[6] to write the partition function in the form

$$Z(\tau, \bar{\tau}) = \vec{\chi}(\tau)^{\mathrm{T}} \ M \ \vec{\bar{\chi}}(\bar{\tau}) \ . \tag{2.119}$$

Definition. The matrix M appearing in this form is called a *modular invariant.* The vector $\vec{\chi}(\tau)$ has components

$$\chi_i(\tau) = \mathrm{Tr}_{\mathcal{H}_i} \ q^{L_0 - \frac{c}{24}} \ , \tag{2.120}$$

and similarly for $\vec{\bar{\chi}}(\bar{\tau})$. For convenience, we will be omitting the arrows and the argument dependence of these vectors from now on. The index i runs over all irreducible representations and $\mathcal{H}_i$ is the Hilbert space built upon the highest weight state $|h_i\rangle$. For that reason χ_i is called the *character of the irreducible representation* $|h_i\rangle$.

[6] We could also write (2.119) for non-RCFTs with the understanding that the matrix M is infinite-dimensional. In this thesis we restrict ourselves to RCFTs.

What are the conditions on the matrix M such that the partition function (2.119) is modular invariant? By inspection of (2.120) we see that under T transformations χ picks up a phase which will then cancel the phase from $\bar{\chi}$ by virtue of the level-matching condition (2.24). On the other hand S transformations do impose certain restrictions. In particular, let S be the (symmetric) matrix implementing the transformations at the level of the characters, *i.e.*

$$\chi(-\frac{1}{\tau}) = S\chi(\tau) \ , \tag{2.121}$$

then M must satisfy

$$M = SMS^{\dagger} \ . \tag{2.122}$$

A very interesting question is if it is possible to identify all modular invariants. A classification of all modular invariants is only known for a few cases with certain symmetries [14,15]. The next best thing one can imagine is to be able to construct new modular invariants from a given one. The *simple currents* formalism [24,25] achieves this goal. We will introduce it and use it heavily in chapter 6.

Chapter 3

Free fermionic models

Next we review the free fermionic formulation as first outlined in [26, 27][1]. In this formalism, the internal sectors of the string are described by fermionic degrees of freedom. In general, there are n_f left-moving (or holomorphic) fermions f and $n_{\overline{f}}$ right-moving (or anti-holomorphic) fermions $\overline{f}$. In the case of heterotic string theories with four non-compact target space dimensions, again described by light-cone coordinates x^μ with superpartner ψ^μ, conformal invariance requires that we have

$$n_f = 18 \ , \qquad n_{\overline{f}} = 44 \ . \tag{3.1}$$

The holomorphic sector has worldsheet supersymmetry, which is non-linearly realized by the stress-energy tensor

$$T_F = \psi_\mu \, \partial x^\mu - \chi^i y^i w^i \ , \tag{3.2}$$

on the internal fermions χ^i, y^i, w^i, $i = 1, \ldots, 6$. The 44 real anti-holomorphic fermions are conventionally separated into two sets of real fermions $\overline{y}^i, \overline{w}^i$ and sixteen complex fermions $\overline{\lambda}^I$, $I = 1, \ldots, 16$. Often these fermions are further divided into three classes as indicated in table 3.1.

3.1 Basis vectors and the additive group

A 48-component vector $\boldsymbol{\alpha} = \big(\alpha(\psi), \alpha(\chi), \alpha(y), \alpha(w) \,\big|\, \alpha(\overline{y}), \alpha(\overline{w}); \alpha(\overline{\lambda})\big)$ characterizes a sector in a free fermionic model by defining a set of boundary conditions

$$f \mapsto -e^{i\pi\alpha(f)} \, f \ , \qquad \overline{f} \mapsto -e^{-i\pi\,\alpha(\overline{f})} \, \overline{f} \ , \tag{3.3}$$

for all the fermions. The line $|$ between the components of the vector $\boldsymbol{\alpha}$ separates the boundary conditions for holomorphic and anti-holomorphic fermions, $f = \psi^\mu, \chi^i, y^i, w^i$ and $\overline{f} = \overline{y}^i, \overline{w}^i; \overline{\lambda}^I$, and the semi-colon distinguishes the latter between real fermions, $\overline{y}^i, \overline{w}^i$, and complex fermions $\overline{\lambda}^I$. This convention means that when an entry $\alpha(f) = 0$, the fermion is anti-periodic, *i.e.* with NS boundary conditions.

[1] There exists an alternative fermionic description [28, 29]; a mapping between these formalisms may be found in Appendix A of [30].

Sector	Label	Description
SUSY	ψ^μ, χ^i	Real superpartners of the bosonic coordinate x^μ and of the six compactified coordinates
(holomorphic)	y^i, w^i	Real fermions that correspond to the bosons describing the six compactified directions
Non-SUSY	$\overline{y}^i, \overline{w}^i$	Real fermions that correspond to the bosons describing the six compactified dimensions
(anti-holomorphic)	$\overline{\lambda}^I = \begin{cases} \overline{\psi}^{1,\dots,5} \\ \overline{\eta}^{1,2,3} \\ \overline{\phi}^{1,\dots,8} \end{cases}$	Complex fermions that describe the visible gauge sector, corresponding to eight of the internal directions in T^{16} Complex fermions that describe the hidden gauge sector, corresponding to the remaining eight internal directions in T^{16}

Table 3.1: This table gives the fermionic states that freely propagate on the string worldsheet: $\mu = 1, 2$, $i = 1, \dots, 6$ and $I = 1, \dots, 16$, are four-dimensional light-cone, six real internal and sixteen complex indices, respectively. The left-moving sector is supersymmetric, while the right-moving sector is not.

The reduced version $[\boldsymbol{\alpha}]$ of a vector $\boldsymbol{\alpha}$ has entries equal to those of $\boldsymbol{\alpha}$ up to even integers such that all entries of $[\boldsymbol{\alpha}]$ lie within the range

$$\big(-1, +1 \big] \ . \tag{3.4}$$

In particular, $[\boldsymbol{\alpha}](f)$ is the entry of α for the fermion f, restricted to the above range for complex fermions, and it is simply 0 or 1 for real fermions. Often the basis vectors are chosen to lie within this restricted range. The difference between a vector and its reduced representation is denoted by

$$2r(\boldsymbol{\alpha}) \equiv \boldsymbol{\alpha} - [\boldsymbol{\alpha}] \, . \tag{3.5}$$

Moreover, it is conventional to only indicate the fermions with non-vanishing entries: For illustration, in table 3.2 we have given a number of basis vectors that appear in many free fermionic models. They are described either by the names of the fermions that appear in them or equivalently by the values of all of their 48 entries. We represent any such vector by $\boldsymbol{\alpha}_L$ and $\boldsymbol{\alpha}_R$ with components $\alpha_L(f)$ and $\alpha_R(\overline{f})$. The Lorentzian inner product between two vectors, $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ is defined as

$$\boldsymbol{\alpha} \cdot \boldsymbol{\beta} = \boldsymbol{\alpha}_L^T \boldsymbol{\beta}_L - \boldsymbol{\alpha}_R^T \boldsymbol{\beta}_R = \tfrac{1}{2}\, \alpha(f)^T \beta(f) - \tfrac{1}{2}\, \alpha(\overline{f})^T \beta(\overline{f}) - \alpha(\overline{\lambda})^T \beta(\overline{\lambda}) \ , \tag{3.6}$$

with half-weighting for the real fermionic components $f = \psi^\mu, \chi^i, y^i, w^i$ and $\overline{f} = \overline{y}^i, \overline{w}^i$.

The collection of all such vectors defines a finite additive group, $\boldsymbol{\Xi} \cong \mathbb{Z}_{N_1} \oplus \cdots \oplus \mathbb{Z}_{N_K}$. This group

$$\boldsymbol{\Xi} = \text{span}\, \{ \mathbf{B}_1, \dots, \mathbf{B}_K \} \tag{3.7}$$

is generated by the set $\mathsf{B} = \{\mathbf{B}_a\}$ of basis vectors, which are linearly independent and non-redundant, in the sense that each $\boldsymbol{\alpha} \in \Xi$ can be written as $\boldsymbol{\alpha} = \sum m_a \mathbf{B}_a$, $m_a \in \mathbb{R}$, such that

$$m_a \mathbf{B}_a = 0 \bmod 2 \quad \Leftrightarrow \quad m_a = 0 \bmod N_a \tag{3.8}$$

for all $a = 1, \ldots, K$, where the mod 2 for vectors is understood component wise. Here N_a is the smallest integer satisfying $N_a \mathbf{B}_a = 0 \bmod 2$ and is called the order of $\mathbf{B}_a$.

Furthermore, any set of boundary conditions, $\boldsymbol{\alpha}$, has to be compatible with the worldsheet supersymmetry current T_F, *i.e.* all terms in (3.2) need to transform with the same phase:

$$T_F \mapsto -\delta_{\boldsymbol{\alpha}} T_F \ , \qquad \delta_{\boldsymbol{\alpha}} = e^{i\pi\alpha(\psi^\mu)} \ . \tag{3.9}$$

This is determined by the ψ^μ component of $\boldsymbol{\alpha}$, as it has been assumed that the non-compact Minkowski coordinates, x^μ, do not transform under any element of Ξ. Consequently, all vectors in the additive group Ξ must satisfy:

$$\alpha(\chi^i) + \alpha(y^i) + \alpha(w^i) = \alpha(\psi^\mu) \bmod 2 \ , \tag{3.10}$$

for all $i = 1, \ldots, 6$. This implies that if $\alpha(\psi^\mu) = 0$ then, for each i, the fermions $\{\chi^i, y^i, w^i\}$ may only appear in pairs in $\boldsymbol{\alpha}$; when $\alpha(\psi^\mu) = 1$, then, for each i, either just one fermion or all three out of these sets have to be present in $\boldsymbol{\alpha}$.

In order to ensure that the resulting partition function for the fermions is modular invariant, yet non-vanishing, it is crucial that all fermions can have both R and NS sectors. This means that the collection of all vectors in the additive set Ξ should affect all fermions. This is automatically guaranteed because the unit element $\mathbf{1}$ of the boundary condition addition rule is part of the additive set [27].

3.2 The free fermionic partition function

The full partition function of a free fermionic model [27],

$$Z(\tau, \bar{\tau}) = \sum_{\boldsymbol{\alpha}', \boldsymbol{\alpha} \in \Xi} C[{}^{\boldsymbol{\alpha}}_{\boldsymbol{\alpha}'}]\, Z[{}^{\boldsymbol{\alpha}}_{\boldsymbol{\alpha}'}](\tau, \bar{\tau}) \ , \tag{3.11}$$

is given by a sum over the additive set Ξ of partition functions defined by the boundary conditions $\boldsymbol{\alpha}$ and $\boldsymbol{\alpha}'$ when parallel transported around the non-contractible loops of the torus amplitude,

$$Z[{}^{\boldsymbol{\alpha}}_{\boldsymbol{\alpha}'}](\tau, \bar{\tau}) = \\ Z_x(\tau, \bar{\tau}) \left[\frac{\Theta[{}^{\alpha(y)}_{\alpha'(y)}]\Theta[{}^{\alpha(w)}_{\alpha'(w)}]\Theta[{}^{\alpha(\psi)}_{\alpha'(\psi)}]\Theta[{}^{\alpha(\chi)}_{\alpha'(\chi)}]}{\eta^{20}}(\tau) \right]^{\frac{1}{2}} \left[\frac{\overline{\Theta}[{}^{\alpha(\bar{y})}_{\alpha'(\bar{y})}]\overline{\Theta}[{}^{\alpha(\bar{w})}_{\alpha'(\bar{w})}]}{\bar{\eta}^{12}}(\bar{\tau}) \right]^{\frac{1}{2}} \frac{\overline{\Theta}[{}^{\alpha(\bar{\lambda})}_{\alpha'(\bar{\lambda})}]}{\bar{\eta}^{16}}(\bar{\tau}) \ , \tag{3.12}$$

$\mathbf{B}_a$	**Fermions included in** $\mathbf{B}_a$
$\mathbf{1}$	$\{\psi^\mu, \chi^{1\ldots6}; y^{1\ldots6}, w^{1\ldots6} \,\vert$ $\overline{y}^{1\ldots6}, \overline{w}^{1\ldots6}, \overline{\psi}^{1\ldots5}, \overline{\eta}^{123}, \overline{\phi}^{1\ldots8}\}$
$\mathbf{S}$	$\{\psi^\mu, \chi^{1\ldots6}\}$
$\boldsymbol{\xi}_1$	$\{\overline{\psi}^{1\ldots5}, \overline{\eta}^{123}\}$
$\boldsymbol{\xi}_2$	$\{\overline{\phi}^{1\ldots8}\}$
$\boldsymbol{\xi}$	$\boldsymbol{\xi}_1 + \boldsymbol{\xi}_2 = \{\overline{\psi}^{1\ldots5}, \overline{\eta}^{123}, \overline{\phi}^{1\ldots8}\}$
$\mathbf{e}_i$	$\{y^i, w^i \,\vert\, \overline{y}^i, \overline{w}^i\}$
$\mathbf{b}_1$	$\{\chi^{3456}; y^{3456} \,\vert\, \overline{y}^{3456}; \overline{\eta}^{23}\}$
$\mathbf{b}_2$	$\{\chi^{1256}; y^{12}, w^{56} \vert \overline{y}^{12}, \overline{w}^{56}; \overline{\eta}^{13}\}$

$\mathbf{B}_b \cdot \mathbf{B}_a$	$\mathbf{1}$	$\mathbf{S}$	$\boldsymbol{\xi}_1$	$\boldsymbol{\xi}_2$	$\boldsymbol{\xi}$	$\mathbf{e}_i$	$\mathbf{b}_s$
$\mathbf{1}$	-12	4	-8	-8	-16	0	0
$\mathbf{S}$	4	4	0	0	0	0	2
$\boldsymbol{\xi}_1$	-8	0	-8	0	-8	0	-2
$\boldsymbol{\xi}_2$	-8	0	0	-8	- 8	0	0
$\boldsymbol{\xi}$	-16	0	-8	-8	-16	0	-2
$\mathbf{e}_j$	0	0	0	0	0	0	0
$\mathbf{b}_t$	0	2	-2	0	-2	0	0

Table 3.2: The left part of this table gives a number of important basis vectors that appear in many free fermionic models. The vector $\mathbf{1}$ is necesarily part of the additive set $\boldsymbol{\Xi}$. The vector $\mathbf{S}$ is associated with target space supersymmetry. The right part gives their multiplication table using the product defined in (3.6).

in terms of the theta functions $\Theta[^{\alpha}_{\alpha'}](\tau) = \Theta[^{\alpha}_{\alpha'}](0;\tau)$:

$$\Theta[^{\alpha}_{\alpha'}](z;\tau) = e^{-\pi i \frac{1}{2}\alpha^T\alpha'} \sum_{n\in\mathbb{Z}^d} q^{\frac{1}{2}(n+\frac{1}{2}\alpha)^2}\, e^{2\pi i\,(n+\frac{1}{2}\alpha)^T(z+\frac{1}{2}\alpha')} \ . \tag{3.13}$$

In the defining equation (3.13) the arguments of the the theta functions are numbers, whereas in (3.12) we have grouped together fermions with the same name and therefore the arguments are vectors (of unequal length). For example

$$\begin{aligned} \alpha(\psi) &= \big(\alpha(\psi^1), \alpha(\psi^2)\big) \ , \\ \alpha(y) &= \big(\alpha(y^1), \alpha(y^2), \alpha(y^3), \alpha(y^4), \alpha(y^5), \alpha(y^6)\big) \ . \end{aligned}$$

The $Z_x(\tau,\overline{\tau})$ factor corresponds to the non-compact bosons x^μ. It is given for $d = 2$ (lightcone gauge) by the formula

$$Z_x(\tau,\overline{\tau}) = \left(\frac{1}{\sqrt{\tau_2}}\frac{1}{|\eta(\tau)|^2}\right)^d \ . \tag{3.14}$$

Modular invariance of the full partition function restricts both the choice of basis vectors of the additive group, $\boldsymbol{\Xi}$, as well as the generalized GSO phases $C[^{\boldsymbol{\alpha}}_{\boldsymbol{\alpha}'}]$. All pairs of basis vectors $\mathbf{B}_a, \mathbf{B}_b$ need to satisfy the following conditions (no sums implied here and the dot product is defined in (3.6)):

$$\mathrm{lcm}(N_a, N_b)\, \mathbf{B}_a \cdot \mathbf{B}_b = 0 \bmod 4 \ , \tag{3.15a}$$

hence in particular $N_a\, \mathbf{B}_a^2 = 0 \bmod 4$. Moreover, when N_a is even, an even stronger condition has to be imposed, namely,

$$N_a\, \mathbf{B}_a^2 = 0 \bmod 8 \ . \tag{3.15b}$$

This means that for models with only basis elements of order 2, $\mathbf{B}_a^2 = 0 \bmod 4$. Finally, real fermions which are simultaneously periodic under any three boundary condition basis vectors must come in pairs [29].

3.3 Conditions on generalized GSO phases

In addition, there are constraints on the generalized GSO phases coming from modular invariance [27]:

$$C\big[{}^{\boldsymbol{\alpha}}_{\boldsymbol{\alpha}'}\big] = C^*\big[{}^{-\boldsymbol{\alpha}}_{\boldsymbol{\alpha}'}\big] \,, \tag{3.16a}$$

$$C\big[{}^{\boldsymbol{\alpha}}_{\boldsymbol{\alpha}'}\big] = -e^{\frac{1}{4}i\pi\,\boldsymbol{\alpha}\cdot\boldsymbol{\alpha}}\, C\big[{}^{\boldsymbol{\alpha}}_{\boldsymbol{\alpha}'-\boldsymbol{\alpha}+\mathbf{1}}\big] \,, \tag{3.16b}$$

$$C\big[{}^{\boldsymbol{\alpha}}_{\boldsymbol{\alpha}'}\big] = e^{\frac{1}{2}i\pi\,\boldsymbol{\alpha}\cdot\boldsymbol{\alpha}'}\, C^*\big[{}^{\boldsymbol{\alpha}'}_{\boldsymbol{\alpha}}\big] \,, \tag{3.16c}$$

$$C\big[{}^{\boldsymbol{\alpha}}_{\boldsymbol{\beta}+\boldsymbol{\gamma}}\big] = \delta_{\boldsymbol{\alpha}}\, C\big[{}^{\boldsymbol{\alpha}}_{\boldsymbol{\beta}}\big]\, C\big[{}^{\boldsymbol{\alpha}}_{\boldsymbol{\gamma}}\big] \tag{3.16d}$$

$$C\big[{}^{\boldsymbol{\alpha}}_{\boldsymbol{\alpha}'}\big]\, C\big[{}^{\boldsymbol{\beta}}_{\boldsymbol{\beta}'}\big] = \delta_{\boldsymbol{\alpha}}\,\delta_{\boldsymbol{\beta}}\, e^{-\frac{1}{2}i\pi\,\boldsymbol{\alpha}\cdot\boldsymbol{\beta}}\, C\big[{}^{\boldsymbol{\alpha}}_{\boldsymbol{\alpha}'+\boldsymbol{\beta}}\big]\, C\big[{}^{\boldsymbol{\beta}}_{\boldsymbol{\beta}'+\boldsymbol{\alpha}}\big] \,, \tag{3.16e}$$

at the one- and two-loop level. The general solution to these conditions can be parameterized as follows [27]:

$$C\big[{}^{\boldsymbol{\alpha}}_{\boldsymbol{\alpha}'}\big] = \big(\delta_{\boldsymbol{\alpha}}\big)^{\sum_a n'_a - 1}\,\big(\delta_{\boldsymbol{\alpha}'}\big)^{\sum_a n_a - 1}\, e^{-\pi i\, r(\boldsymbol{\alpha})\cdot\boldsymbol{\alpha}'} \prod_{a,b} C\begin{bmatrix}\mathbf{B}_a\\ \mathbf{B}_b\end{bmatrix}^{n_a n'_b} \,, \tag{3.17}$$

for two arbitrary vectors $\boldsymbol{\alpha} = \sum n_a\,\mathbf{B}_a$, $\boldsymbol{\alpha}' = \sum n'_b\,\mathbf{B}_b \in \boldsymbol{\Xi}$, with $r(\boldsymbol{\alpha})$ defined in (3.5). It is important to note that (3.17) gives $C\big[{}^{\mathbf{0}}_{\mathbf{0}}\big] = 1$. This tells us that all generalized GSO phases are fixed in terms of the phases $C\big[{}^{\mathbf{B}_a}_{\mathbf{B}_b}\big]$ for all the basis vectors generating the additive group $\boldsymbol{\Xi}$. The phases that can be chosen freely are those of the upper triangular part of the GSO phase matrix C including the diagonal ($b \geq a$); the phases in the lower triangular part ($b < a$) are fixed by (3.16c).

It might sometimes happen that some vector $\boldsymbol{\alpha}$ does not lie in the reduced range defined in (3.4). One can bring it into this range by adding a vector $\boldsymbol{\delta}$ with only even entries. The generalized GSO phases are, in general, not invariant under such changes, but transform as

$$C\big[{}^{\boldsymbol{\alpha}+\boldsymbol{\delta}}_{\boldsymbol{\alpha}'+\boldsymbol{\delta}'}\big] = e^{\frac{1}{2}\pi i\,\boldsymbol{\delta}\cdot\boldsymbol{\alpha}'}\, C\big[{}^{\boldsymbol{\alpha}}_{\boldsymbol{\alpha}'}\big] \,, \tag{3.18}$$

as inferred from (3.11) and (3.13), provided that $\boldsymbol{\delta}, \boldsymbol{\delta}'$ have only even entries. This means that two sets of basis vectors, which only differ in vectors with only even entries, describe fully equivalent models as long as their generalized GSO phases are related via (3.18). It also shows that there is no loss of generality in enforcing the entries of all basis vectors to lie inside the range (3.4).

3.4 Massless spectrum

The spectrum in the $\boldsymbol{\alpha} \in \boldsymbol{\Xi}$ sector of a free fermionic model is built upon the left- and right-moving vacua, $|0\rangle_L^{\boldsymbol{\alpha}} \otimes |0\rangle_R^{\boldsymbol{\alpha}}$. When a fermion, f or $\overline{f}$, is strictly periodic, *i.e.* $\alpha(f) = 1$ or $\alpha(\overline{f}) = 1$, then this fermion has a zero mode. In all models, properties of the fermions are always defined pairwise, hence we can use complex fermions from which we can construct spin up/down generators. A single complex fermion zero mode leads to two degenerate vacua represented as $|\pm\rangle$; when we have a collection of fermionic zero modes we write $|\pm,\ldots,\pm\rangle$. Consequently, their vacua are associated with spinorial representations in target space. In particular, when the fermions ψ^μ have periodic boundary conditions, their zero modes form the light-cone version of the four-dimensional Clifford algebra and hence define target space fermions. Thus, whether the sector $\boldsymbol{\alpha}$ corresponds to bosons or fermions in target space is determined by the quantity $\delta_{\boldsymbol{\alpha}}$ defined in (3.9). Making use of (3.17) we then obtain

$$\delta_{\boldsymbol{\alpha}}^{-1} = C[{}^{\mathbf{0}}_{\boldsymbol{\alpha}}] = C[{}^{\boldsymbol{\alpha}}_{\mathbf{0}}] = \begin{cases} 1 & \text{spacetime bosons ,} \\ -1 & \text{spacetime fermions .} \end{cases} \tag{3.19}$$

Both bosonic and fermionic oscillator excitations may act on the vacuum of such sectors. The oscillator modes associated with the boson x^μ always have non-zero, integral frequencies. The smallest non-zero fermionic frequencies are

$$\nu(f) = \tfrac{1}{2}\left(1+\alpha_L(f)\right) , \qquad \nu(\overline{f}) = \tfrac{1}{2}\left(1+\alpha_R(\overline{f})\right) , \tag{3.20}$$

for real fermions, f and $\overline{f}$, while for the complex fermions, $\overline{\lambda}$, and their complex conjugates we have

$$\nu(\overline{\lambda}) = \tfrac{1}{2}\left(1+\alpha_R(\overline{\lambda})\right) , \qquad \nu(\overline{\lambda}^*) = \tfrac{1}{2}\left(1-\alpha_R(\overline{\lambda})\right) . \tag{3.21}$$

The left- and right-moving masses of such states are given by

$$M_L^2 = \tfrac{1}{8}\,\boldsymbol{\alpha}_L^2 - \tfrac{1}{2} + \sum_f \nu(f) + N_L , \qquad M_R^2 = \tfrac{1}{8}\,\boldsymbol{\alpha}_R^2 - 1 + \sum_{\overline{f}} \nu(\overline{f}) + N_R , \tag{3.22}$$

where $N_{R/L}$ are the number operators associated with bosonic oscillators on the right-/left-moving sides. Level-matching requires that these left- and right-moving masses are equal. Moreover, if we are only interested in massless states, both the left- and right-moving masses in (3.22) need to vanish. Hence, only for the values $\boldsymbol{\alpha}_L^2 \le 4$ and $\boldsymbol{\alpha}_R^2 \le 8$ are massless states possible.

On the states in each sector, $\boldsymbol{\alpha} \in \Xi$, the generalized GSO projections,

$$e^{i\pi\, \mathbf{B}_a\cdot F}\,|\text{state}\rangle_{\boldsymbol{\alpha}} = \delta_{\boldsymbol{\alpha}} C^*[{}^{\boldsymbol{\alpha}}_{\mathbf{B}_a}]\,|\text{state}\rangle_{\boldsymbol{\alpha}} , \tag{3.23}$$

are imposed for all basis elements $\mathbf{B}_a$, where

$$\mathbf{B}_a\cdot F = \sum_f \mathbf{B}_a\cdot F(f) - \sum_{\overline{f}} \mathbf{B}_a\cdot F(\overline{f}) . \tag{3.24}$$

Here we work in a complex basis for all fermions; the fermion number operator F is defined such that $F(f) = -F(f^*) = 1$. F vanishes on any NS-vacuum as well as on the highest weight R-vacuum $|+1^n\rangle$, which we define as $f_0^{i*}|+1^n\rangle = 0$ when it corresponds to n complexified fermions with periodic boundary conditions; $f_0^1|+1^n\rangle = |-1,1^{n-1}\rangle$, etc. (Note that $n = 10$ for the left-moving Ramond vacuum and $n = 28$ for the right-moving Ramond vacuum.) Only the states that survive the generalized GSO projections are physical, *i.e.* correspond to states in the four-dimensional target space.

3.5 Conditions for supersymmetry

The generator of spacetime supersymmetry is denoted by $\mathbf{S}$; its explicit form can be found in table 3.2. Different forms for $\mathbf{S}$ are, in principle, possible, but it was shown in [31] that they do not lead to models with less than $\mathcal{N}=2$ spacetime supersymmetry and will, therefore, not be considered further here. To preserve modular invariance, fermions with identical transformation properties always come in pairs, hence we can make use of a complex notation for the fermions as well.

Whenever $\mathbf{S}$ is part of the set of basis vectors $\{\mathbf{B}_a\}$, we know that associated with any sector $\boldsymbol{\alpha}$ there will be a sector $\boldsymbol{\alpha}+\mathbf{S}$. Since (3.19) decides whether a sector corresponds to target space bosons or fermions and $\mathbf{S}$ involves ψ^μ, it follows that if $\boldsymbol{\alpha}$ is bosonic then $\boldsymbol{\alpha}+\mathbf{S}$ is fermionic and vice versa. The supersymmetry element $\mathbf{S}$ then leads, via (3.23), to the projection, that imposes the following for the signs s:

$$\sum_\alpha s_\alpha = \begin{cases} \text{even} \\ \text{odd} \end{cases} \qquad \text{for} \qquad C[{}^{\mathbf{S}}_{\mathbf{S}}] = \mp 1 \ . \tag{3.25}$$

Either choice corresponds to $\mathcal{N}=4$ spacetime supersymmetry in four dimensions, but of opposite chirality in ten dimensions; conventionally one takes for positive chirality that the spinors' sums are even, so that $C[{}^{\mathbf{S}}_{\mathbf{S}}]=-1$.

In order to break $\mathcal{N}=4$ down to $\mathcal{N}=1$ supersymmetry in four dimensions, the set of basis vectors $\{\mathbf{B}_a\}$ must contain elements that overlap with the vector $\mathbf{S}=\{\psi^\mu,\chi^i\}$. In light of (3.15a), their overlaps always involve an even number of complexified combinations of the fermions in $\mathbf{S}$. To fix conventions, we choose the surviving four-dimensional gravitino,

$$|s\rangle_L^{\mathbf{S}} \otimes \overline{\partial} x^\mu_{-1}\,|0\rangle_R^{\mathbf{S}}\,, \tag{3.26}$$

to have components $s=\pm(1^4)$. The generalized GSO phases involving $\mathbf{S}$ then have to be chosen such that

$$C[{}^{\mathbf{S}}_{\mathbf{B}_a}] = C[{}^{\mathbf{S}}_{\mathbf{1}}] = C[{}^{\mathbf{S}}_{\mathbf{S}}] = -1 \ , \tag{3.27}$$

to preserve at least $\mathcal{N}=1$ supersymmetry. In particular, for basis vectors that do not overlap with $\mathbf{S}$ the opposite sign for GSO phases would kill all gravitino states. The second equality holds even when $\mathbf{1}$ is not part of the basis by (3.16b).

Chapter 4

Orbifold models

4.1 Geometrical lattices underlying symmetric orbifolds

One of the defining elements of any orbifold model is the underlying six-dimensional lattice that appears when we identify

$$X^i \sim X^i + 2\pi \varepsilon^i{}_{\underline{i}} n_{\underline{i}} \quad i = 1, \ldots, 6 \ , \tag{4.1}$$

where n is a vector of integers. The lattice,

$$\Lambda = \left\{ \varepsilon\, n = \varepsilon_{\underline{i}}\, n_{\underline{i}} \middle| n_{\underline{i}} \in \mathbb{Z} \right\} \ , \tag{4.2}$$

is spanned by a set of basis vectors $\varepsilon_{\underline{i}}$, $\underline{i} = 1, \ldots, 6$. The matrix ε, with these basis vectors as its columns, can be thought of as a vielbein associated with the metric,

$$G = \varepsilon^T \varepsilon \ , \tag{4.3}$$

on the six-torus. This metric carries all the information about the lengths and the angles of the lattice basis vectors. We refer to the vectors $\varepsilon_{\underline{i}}$ as the lattice basis. The lattice basis is in general not the standard orthogonal Euclidean basis; we reserve the notation e_i to denote the standard basis vectors of $\mathbb{R}^6$: $(e_i)_j = \delta_{ij}$ and write $e_{12} = e_1 + e_2$, etc. We offer further information on lattices in appendix A.

Sector	**Label**	**Description**
SUSY	X_L^i	Bosonic internal coordinates
(holomorphic)	ψ_L^μ, ψ_L^i	Real superpartners of the bosonic coordinates x^μ, X^i
Non-SUSY	X_R^i	Bosonic internal coordinates
(anti-holomorphic)	Y_R^I	Real bosons living on an internal torus T^{16} corresponding to the gauge degrees of freedom.

Table 4.1: This table gives the states that freely propagate on the string worldsheet: $\mu = 1, 2$, $i = 1, \ldots, 6$ and $I = 1, \ldots, 16$, are four-dimensional light-cone, six-dimensional internal and sixteen right-moving bosonic indices, respectively. The left-moving sector, labeled by L, is supersymmetric, while the right-moving sector, labeled by R, is not.

4.2 Orbifold actions

Let $\Gamma = \mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2} \times \ldots$ be a finite Abelian group, often referred to as the point group. The generators of this finite group on $\mathbb{R}^6$ are denoted $\theta_1, \theta_2, \ldots$, *i.e.* the action of a generic element of Γ can be written as $\theta^k := \theta_1^{k_1}\theta_2^{k_2}\ldots$ with $k_1 = 0, \ldots N_1 - 1$, etc. The action of the point group has to be compatible with the lattice Λ in the sense that

$$\theta^k\,\Lambda = \Lambda \;:\quad \theta^k\,\varepsilon = \varepsilon\,\rho^k\;,\quad \rho^k = \rho_1^{k_1}\rho_2^{k_2}\ldots\;,\quad \rho_s \in GL(n,\mathbb{Z})\;. \tag{4.4}$$

The order of ρ_s is at most N_s, but may be lower. The elements θ_s generate the point group Γ in the standard Euclidean basis. In the lattice basis, this group is generated by the matrices ρ_s. We normally first specify the point group in the Euclidean basis. If one also has a compatible lattice basis then one simply determines the point group generators in the lattice basis via $\rho_s = \varepsilon^{-1}\theta_s\varepsilon$. We denote the resulting symmetric orbifold with point group Γ as T^6/Γ.

The orbifold can be equivalently described as the quotient of $\mathbb{R}^6/S$ where S is the so-called space group. The space group S combines the elements of the lattice Λ and the point group Γ. It acts on the coordinates X of the covering space $\mathbb{R}^6$ as

$$h = (\theta^k, L_h) \in S\;:\quad X \mapsto h \circ X = \theta^k\,X + 2\pi\,L_h\;,\quad L_h = \ell\,k + \varepsilon\,n\;. \tag{4.5}$$

The vector $\ell = (\ell_s)$ that appears in the last equation encodes the information about the translation part of the space group element h. In particular, there is a vector $\ell_s \in \mathbb{R}^6$ associated with each generator θ_s of the point group, and the vector associated with a generic element θ^k will then be $\ell\,k = k_1\ell_1 + k_2\ell_2 + \cdots$. This realization induces the following group multiplication of space group elements:

$$h'\,h = (\theta^{k'}, \ell\,k' + \varepsilon\,n')\,(\theta^k, \ell\,k + \varepsilon\,n) = (\theta^{k'+k}, \theta^{k'}(\ell\,k + \varepsilon\,n) + \ell\,k' + \varepsilon\,n')\;. \tag{4.6}$$

To ensure that the orbifold elements have finite order, we need $N_s\,\ell_s \in \Lambda$. Depending on the choice of θ_s and ℓ_s for a given $\mathbb{Z}_{N_s}$ factor, we distinguish between pure twist, pure shift and roto-translational orbifold actions:

Orbifold action	**Characterization**
pure twist	$\theta_s \neq 1\;,\quad \ell_s = 0$
pure shift	$\theta_s = 1\;,\quad \ell_s \notin \Lambda$
roto-translation	$\theta_s \neq 1\;,\quad \ell_s \notin \Lambda$
true roto-translation	$\ell_s \notin \Lambda$ has components in directions in which $\theta_s \neq 1$ acts trivially.

In principle, for pure twist orbifolds we could allow for $\ell_s \in \Lambda$, but this can be absorbed by a redefinition of the vector $n \in \mathbb{Z}^6$. A pure shift orbifold can equivalently be thought of as a torus compactification with a new lattice in which some of the basis vectors e_i are replaced by the ℓ_s corresponding to the pure shift actions.

The distinction between a twist and a roto-translation is not always a coordinate independent statement: When the shift part of a roto-translation points only in directions where it also acts as a rotation, then one can change the origin and this action can look like a pure twist. On the other hand, when the shift of a roto-translation also has directions which are left inert by the twist part, the shift in these directions cannot be removed. We call this a true roto-translation. Note that even when a given roto-translation can be turned into a pure shift, it often happens that, at the same time, other pure twist actions become roto-translations. In such cases the effects of the roto-translations are also physical; they cannot be removed by a coordinate redefinition.

In the following we will also need the important concept of fixed points and fixed tori, because this is where additional so-called twisted matter typically arises. An orbifold fixed set arises as a solution to the fixed point equation $g \circ X = X$: Pure twist and roto-translations have fixed tori or points, depending on the twist action. A roto-translation, that has the same twist action as a pure twist, has its fixed points/tori simply shifted with respect to those of the pure twist. True roto-translations never leave any point inert, hence have an empty fixed set. Two space groups S_1 and S_2 belong to the same $\mathbb{Z}$-class if generators ρ_s and $\widetilde{\rho}_s$ of the corresponding point groups are related by

$$U^{-1}\rho_s U = \tilde{\rho}_s \ , \tag{4.7}$$

with $U \in \mathrm{GL}(6, \mathbb{Z})$. Two orbifolds with the same $\mathbb{Z}$-class means that they are defined on the same lattice. The structure of fixed points and/or tori is highly dependent both on the $\mathbb{Z}$-class of the lattice as well as on the orbifold action under consideration.

4.3 Conditions for supersymmetry

In this work we focus on six-dimensional orbifolds T^6/Γ which preserve (at least) $N = 1$ spacetime supersymmetry. Since the group Γ is Abelian, we can simultaneously diagonalize all elements of Γ using a complex basis, labeled by $\alpha = 1, 2, 3$, and write each element $\theta \in \Gamma$ in terms of the twist vector v as

$$\theta^k = e^{2\pi i\, v_h} \ , \quad v_h = k_s\, v_s \ , \quad v_s = \big(0, (v_s)_1, (v_s)_2, (v_s)_3\big) \ , \text{ etc. } , \tag{4.8}$$

(where the sum over s labels the different point group generators θ_s) for a space group element $h \in S$ with $N_1\, (v_1)_\alpha \, , \, N_2\, (v_2)_\alpha \, , \ldots = 0 \bmod 1$ to ensure that $\theta_s^{N_s} = 1$.

A positive chiral target space spinor in ten dimensions can be represented by vectors of the form $\frac{1}{2}(\underline{\pm 1^4})$ (*i.e.* all four entries can either be $+1/2$ or $-1/2$) with an even number of minus signs. The action of θ on a spinor state $|s_0, s_1, s_2, s_3\rangle$ reads

$$\theta^k \, |s_0, s_1, s_2, s_3\rangle = e^{2\pi i\, (v_h)_\alpha s_\alpha} \, |s_0, s_1, s_2, s_3\rangle \ , \tag{4.9}$$

($s_0, \ldots, s_3 = \pm 1/2$) where the sum is over the three complexified internal directions. Therefore, if we assume that the components of the surviving four-dimensional supersymmetry are represented by $\pm\frac{1}{2}(1^4)$, we have to require that

$$\sum_{\alpha} (v_s)_\alpha = 0 \bmod 2 \ . \tag{4.10}$$

In the heterotic orbifold literature, mostly twists that make the sum strictly zero are used in order to obtain a unique representation of the twist vectors.

The worldsheet supersymmetry generator is given by

$$T_F = \psi_\mu \, \partial x^\mu + \psi^i \, \partial X_L^i \tag{4.11}$$

in terms of the four-dimensional coordinate field x^μ and the fields given in table 4.1.

4.4 Shift embedding and discrete Wilson lines

In the bosonic orbifold description the gauge degrees of freedom are described by real right-moving coordinate fields Y_R that live on a sixteen dimensional torus $\mathbb{R}^{16}/2\pi\Lambda_{\text{gauge}}$ where the lattice Λ_{gauge} is either the root lattice $\Lambda_{8+8} = \Lambda_8 \oplus \Lambda_8$ of $\text{E}_8{\times}\text{E}_8$ or Λ_{16} of $\text{Spin}(32)/\mathbb{Z}_2$, where

$$\Lambda_{8\text{n}} = \bigoplus_{t=0,1} \left\{ u_{\text{sh}} = u + \tfrac{t}{2}\, \mathbf{1}_{8n} \;\middle|\; u \in \mathbb{Z}^{8n} \ , \ \mathbf{1}_{8n}^T u = 0 \bmod 2 \right\} \ , \tag{4.12}$$

with $\mathbf{1}_d = (1^d)$ (the vector with d entries equal to 1) for $n = 1, 2$. It consists of the direct sum of the root ($t = 0$) and spinorial ($t = 1$) lattices defined in appendix A. In particular, $\Lambda_{8\text{n}}$ is even and self-dual. We use α_I to denote the simple roots of these algebras. In the $\text{E}_8{\times}\text{E}_8$ case, we label the two spin-structures t_a for both Λ_8 lattices by $a = 1, 2$. In most orbifold models the action of the space group on these gauge degrees of freedom is assumed to be via the so-called shift embedding:

$$Y_R \mapsto h \circ Y_R = Y_R + 2\pi \, V_h \ , \quad V_h = k_s \, V_s + n_{\underline{i}} \, A_{\underline{i}} \ , \tag{4.13}$$

for any space group element h defined in (4.5). The vectors $A_{\underline{i}}$ are called discrete Wilson lines and compatibility with the group property (4.6) of the space group elements implies that

$$A \, \rho_s \cong A \ , \tag{4.14}$$

where $A \cong A'$ means that $A - A' \in \Lambda_{\text{gauge}}$. These conditions often relate various discrete Wilson lines to each other and strongly restrict the order $M_{\underline{i}}$ of the discrete Wilson lines $A_{\underline{i}}$:

$$N_s \, V_s \cong 0 \ , \qquad M_{\underline{i}} \, A_{\underline{i}} \cong 0 \ , \tag{4.15}$$

The gauge shift vectors V_s have the same order as the point group generators θ_s.

4.5 Narain moduli space

The starting point for orbifold models are torus compactifications which can conveniently be encoded in the Narain lattice description. This description starts from a Narain lattice [32,33] of dimensions $(6, 22)$ with Minkowskian signature defined by the metric

$$\eta = \begin{pmatrix} -\mathbb{1}_6 & 0 \\ 0 & \mathbb{1}_{22} \end{pmatrix} . \tag{4.16}$$

Points on the Narain lattice,

$$P = \begin{pmatrix} P_L \\ P_R \end{pmatrix} = E\, N\,, \qquad N \in \mathbb{Z}^{28}\,, \tag{4.17}$$

are the variables that appear in the untwisted sector partition function in the Hamilton representation

$$Z_{\text{Narain}}(\tau, \bar{\tau}) = \frac{1}{\eta^6 \bar{\eta}^{22}} \sum_P q^{\frac{1}{2} P_L^2}\, \bar{q}^{\frac{1}{2} P_R^2}\,, \tag{4.18}$$

where $q = e^{2\pi i \tau}$ and the Dedekind eta function $\eta = \eta(\tau) = q^{1/24} \prod_{n=1}^{\infty}(1 - q^n)$ are holomorphic functions of the Teichmüller parameter of the worldsheet torus τ and $\bar{q} = e^{-2\pi i \bar{\tau}}$ and $\bar{\eta} = \bar{\eta}(\bar{\tau})$ of its conjugate $\bar{\tau}$. This is the combined partition function of the six-torus and gauge lattice. A scaled version of this quantity will also appear in the untwisted sector of the orbifold. Schematically, this is because the untwisted sector with the orbifold projections inserted will be of the form

$$Z_{\text{untwisted}} = \frac{1 + g + \cdots + g^{N_g - 1}}{N_g} Z_{\text{Narain}} = \frac{1}{N_g} Z_{\text{Narain}} + \cdots \tag{4.19}$$

where N_g is the order of the orbifold. Identifying this term allows us to deduce the underlying Narain lattice from the partition function of a given orbifold. To that end, it is also useful to remember that a basis for the Narain lattice vectors is encoded in the columns of the so-called generalized vielbein

$$E = \frac{1}{\sqrt{2}} \begin{pmatrix} \varepsilon + \varepsilon^{-T} C^T & -\varepsilon^{-T} & \varepsilon^{-T} A^T \alpha \\ \varepsilon - \varepsilon^{-T} C^T & \varepsilon^{-T} & -\varepsilon^{-T} A^T \alpha \\ \sqrt{2}\, A & 0 & \sqrt{2}\, \alpha \end{pmatrix} . \tag{4.20}$$

The generalized vielbein contains the lattice vectors $\varepsilon_{\underline{i}}$ of the six-torus introduced in (4.2) and continuous Wilson lines $A_{\underline{i}}$, some of which will get frozen to discrete ones when the orbifold action is taken into account. Moreover, the anti-symmetric Kalb-Ramond tensor B is contained inside the matrix[1]: $C = B + \frac{1}{2} A^T A$. Finally, α

[1] In the literature there are various forms of (4.20) and the definition of C as they crucially depend on the string slope parameter α'; throughout this chapter we have set $\alpha' = 1$.

are the simple roots of a sixteen dimensional even, self-dual lattice and $g = \alpha^T\alpha$ the corresponding metric. For this, we can either choose the simple roots of $\mathrm{E}_8{\times}\mathrm{E}_8$ or $\mathrm{Spin}(32)/\mathbb{Z}_2$: The simple roots of $\mathrm{Spin}(32)/\mathbb{Z}_2$ and the corresponding Cartan matrix read

$$\alpha_{16} = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 & \frac{1}{2} \\ -1 & 1 & \cdots & 0 & 0 & \frac{1}{2} \\ 0 & -1 & \cdots & 0 & 0 & \frac{1}{2} \\ \vdots & \vdots & \ddots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 1 & \frac{1}{2} \\ 0 & 0 & \cdots & -1 & 1 & \frac{1}{2} \\ 0 & 0 & \cdots & 0 & 0 & \frac{1}{2} \end{pmatrix}_{16\times 16} , \qquad g_{16} = \alpha_{16}^T\alpha_{16} = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & & 2 & -1 & -1 & 0 \\ 0 & 0 & \cdots & & -1 & 2 & 0 & 0 \\ 0 & 0 & \cdots & & -1 & 0 & 2 & 1 \\ 0 & 0 & \cdots & & 0 & 0 & 1 & 4 \end{pmatrix}_{16\times 16} . \tag{4.21}$$

The simple roots of $\mathrm{E}_8{\times}\mathrm{E}_8$ and the corresponding Cartan matrix read

$$\alpha_{8\times 8} = \begin{pmatrix} \alpha_8 & 0 \\ 0 & \alpha_8 \end{pmatrix} , \qquad g_{8\times 8} = \begin{pmatrix} g_8 & 0 \\ 0 & g_8 \end{pmatrix} , \tag{4.22}$$

given here in terms of those of E_8:

$$\alpha_8 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & -\frac{1}{2} & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -\frac{1}{2} & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 \end{pmatrix}_{8\times 8} , \qquad g_8 = \alpha_8^T\alpha_8 = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 2 \end{pmatrix}_{8\times 8} . \tag{4.23}$$

It is possible to transform from the $\mathrm{E}_8{\times}\mathrm{E}_8$ to the $\mathrm{Spin}(32)/\mathbb{Z}_2$ description, see *e.g.* [34]; in this work we will indicate explicitly which description we are using.

The partition function (4.18) is modular invariant by virtue of the following constraint on the generalized vielbein

$$E^T\eta E = \hat{\eta} \ , \quad \text{where} \quad \hat{\eta} = \begin{pmatrix} 0 & \mathbb{1}_6 & 0 \\ \mathbb{1}_6 & 0 & 0 \\ 0 & 0 & g \end{pmatrix} : \tag{4.24}$$

In particular, under the modular transformation $\tau \to \tau + 1$ the partition function picks up a phase $\exp \pi i\,(P_L^2 - P_R^2)$ which is trivial by virtue of

$$-P_L^2 + P_R^2 = P^T\eta P = N^T\hat{\eta}N = 2\,m^T n + p^T g\, p \in 2\,\mathbb{Z} \ , \tag{4.25}$$

parameterizing $N^T = (m^T, n^T, p^T)$ where $m, n \in \mathbb{Z}^6$ and $p \in \mathbb{Z}^{16}$.

The associated Narain partition function (4.18) can be expressed in terms of the generalized vielbein,

$$Z_{\text{Narain}} = \frac{1}{\eta^6\bar{\eta}^{22}} \sum_{N\in\mathbb{Z}^{28}} q^{\frac{1}{4}\,N^T E^T\,(\mathbb{1}-\eta)\,EN}\; \bar{q}^{\frac{1}{4}\,N^T E^T\,(\mathbb{1}+\eta)\,EN} \ . \tag{4.26}$$

4.6 Orbifold partition functions

The general form of an orbifold one-loop partition function is given as a sum over commuting space group elements

$$Z(\tau,\overline{\tau}) = \sum_{[h,h']=0} c[{}^{h}_{h'}]\, Z[{}^{h}_{h'}](\tau,\overline{\tau}) \ , \tag{4.27}$$

where $c[{}^{h}_{h'}]$ are called generalized torsion phases and $Z[{}^{h}_{h'}]$ defines the partition function for a given sector, i.e. a set of boundary conditions, on the worldsheet torus, defined by the space group elements h and h'. The elements h are sometimes referred to as the constructing elements. They define the different sectors in the theory and affect the $q, \bar{q}$ expansions of the partition function. The elements h' are called projecting elements, as they only affect phases, *i.e.* the projection conditions in the partition function. We have restricted the sum to commuting constructing and projecting space group elements only; for non-commuting elements the corresponding partition function is simply zero.

The full one-loop partition function is required to be modular invariant: $Z(\tau+1) = Z(-1/\tau) = Z(\tau)$ (for brevity, we only indicate the τ dependence). The partition functions in the various sectors transform modular covariantly into each other, in the sense that

$$Z[{}^{h}_{h'}](-1/\tau) = Z[{}^{h'}_{h}](\tau) \ , \qquad Z[{}^{h}_{h'}](\tau+1) = Z[{}^{h}_{h'h}](\tau) \ , \tag{4.28}$$

without any additional phases (since we only sum over commuting elements the order of h' and h is irrelevant).

The partition function in a given sector, $(h;h')$, splits as a product of partition functions of the various worldsheet fields

$$Z[{}^{h}_{h'}](\tau,\overline{\tau}) = Z_x(\tau,\overline{\tau})\, Z_X[{}^{h}_{h'}](\tau,\overline{\tau})\, Z_\psi[{}^{h}_{h'}](\tau)\, Z_Y[{}^{h}_{h'}](\overline{\tau}) \ . \tag{4.29}$$

Let us briefly discuss the various factors in turn: The partition function $Z_x(\tau,\overline{\tau})$ is the partition function associated with the two non-compact coordinates x^μ in four dimensions in the light-cone gauge, already given in (3.14). The partition functions

$$Z_X[{}^{h}_{h'}](\tau,\overline{\tau}) = Z_\|[{}^{h}_{h'}](\tau,\overline{\tau})\, Z_\perp[{}^{h}_{h'}](\tau,\overline{\tau}) \tag{4.30}$$

correspond to the compactified internal directions parameterized by X^i: Here we need to distinguish between the directions in which the orbifold twist θ^k acts non-trivially and those which are left inert. To project on these subspaces we can define the projections

$$\mathcal{P}^k_\| = \frac{1}{N_k} \sum_{r=0}^{N_k-1} (\theta^k)^r \ , \qquad \mathcal{P}^k_\perp = \mathbb{1} - \mathcal{P}^k_\| \ , \tag{4.31}$$

where N_k is the order of θ^k (we will use similar notations to indicate other projected quantities). The dimensions of the corresponding subspaces are $D^k_\|$ and $D^k_\perp$, respectively, such that $D^k_\| + D^k_\perp = 6$. The orbifold action θ^k has fixed points in the subspace on which $\mathcal{P}_\perp^k$ projects, hence, in these directions, we only get contributions from the twisted excitations

$$Z_\perp[{}^{h}_{h'}](\tau,\overline{\tau}) = \left| \frac{\eta^{D_\perp^k/2}}{\vartheta_\perp^k\left[{}^{\mathbf{1}_4/2-v_h}_{\mathbf{1}_4/2-v_{h'}}\right]} \right|^2 . \tag{4.32}$$

Here the notation $\vartheta_\perp^k[{}^{v}_{v'}] = \prod \vartheta[{}^{v_\alpha}_{v'_\alpha}]$ signifies that we only take the product of the genus-one Jacobi theta function $\vartheta[{}^{a}_{a'}] = \vartheta[{}^{a}_{a'}](z=0;\tau)$, defined as

$$\vartheta[{}^{a}_{a'}](z;\tau) = \sum_{n\in\mathbb{Z}} q^{\frac{1}{2}(n-a)^2}\, e^{2\pi i\,(n-a)(z-a')} , \tag{4.33}$$

in the complexified directions where θ^k or $\theta^{k'}$ act non-trivially, *i.e. not* in the α directions where $(v_h)_\alpha = (v_{h'})_\alpha = 0$. In the directions where the twist acts as the identity, we have the usual lattice sums of the Narain partition function (4.18) restricted to the appropriate lower dimensional sublattice. For a symmetric orbifold, no further phases are needed to make these partition functions modular covariant.

The next partition function results from the superpartners $\psi = (\psi^\alpha)$ of the coordinate fields x^μ, X^i in a complex basis: $\alpha = 0$ corresponds to the four-dimensional light-cone coordinates x^μ and $\alpha = 1,2,3$ to the six internal directions. In a bosonized description it takes the form

$$Z_\psi[{}^{h}_{h'}](\tau) = \sum_{s,s'} e^{-2\pi i\,\frac{1}{2}{v_h}^T v_{h'}}\, \frac{1}{\eta^4}\,\frac{1}{2}\,(-)^{s's+s'+s} \sum_{p\in\mathbb{Z}^4} q^{\frac{1}{2}p_{\text{sh}}^2}\, e^{2\pi i\, s'\nu_L^T(p+s\,\nu_L)}\, e^{2\pi i\, v_{h'}^T p_{\text{sh}}} , \tag{4.34}$$

where the vector $p_{\text{sh}} = p + s\,\nu_L + v_h$ has four entries. The vector $\nu_L = \frac{1}{2}\,\mathbf{1}_4$ generates the left-moving spin structures labeled by $s,s' = 0,1$. The phase factor $(-)^{s's+s'+s}$ ensures that $p + s\,\nu_L$ lives on the direct sum lattice of the four-dimensional vectorial and spinorial lattices:

$$\Lambda_4 = \left\{ u \;\middle|\; u\in\mathbb{Z}^4 \;,\; \mathbf{1}_4^T u = 1 \text{ mod } 2 \right\} \oplus \left\{ u + \tfrac{1}{2}\,\mathbf{1}_4 \;\middle|\; u\in\mathbb{Z}^4 \;,\; \mathbf{1}_4^T u = 0 \text{ mod } 2 \right\} . \tag{4.35}$$

The next-to-last phase factor in (4.34) implements the appropriate projection on the so-called left-moving lattice momentum p. The phase factor in front, often referred to as the vacuum phase, ensures that these partition functions are modular covariant.

Finally, the partition function associated with the right-moving gauge lattice is given by

$$Z_Y[{}^{h}_{h'}](\overline{\tau}) = \sum_{t_u,t'_u} e^{2\pi i\,\frac{1}{2}{V_h}^T V_{h'}}\, \frac{1}{\overline{\eta}^{16}}\,\frac{1}{2} \sum_{P\in\mathbb{Z}^{16}} \overline{q}^{\frac{1}{2}P_{\text{sh}}^2}\, e^{-2\pi i\, t'_u \nu_{uR}^T(P+t_u\,\nu_{uR})}\, e^{-2\pi i\, V_{h'}^T P_{\text{sh}}} , \tag{4.36}$$

with

$$P_{\text{sh}} = P + t_u\,\nu_{uR} + V_h \tag{4.37}$$

where for the Spin(32)/$\mathbb{Z}_2$ theory the index u is obsolete and $\nu_R = \frac{1}{2}(1^{16})$; while the index $u = 1, 2$ is summed over and $\nu_{1R} = \frac{1}{2}(1^8, 0^8)$ and $\nu_{2R} = \frac{1}{2}(0^8, 1^8)$ for the $\mathrm{E}_8\times\mathrm{E}_8$ theory. The sums over the spin structures s', s in (4.34) and t'_u, t_u in (4.36) could be executed leading to slightly different expressions. If that was the case, we would write $p_{\text{sh}} = p + v_h$ and $P_{\text{sh}} = P + V_h$ with $p \in \Lambda_4$, $P \in \Lambda_{16}$ or $\Lambda_8 \oplus \Lambda_8$. To facilitate the comparison with the free fermionic formulation later, we choose to keep the sums over these spin structures explicit. The final phase factor in (4.36) implements the orbifold projection. Again, the vacuum phase factor ensures that these partition functions transform covariantly into each other. This lattice partition function can be obtained by assuming boundary conditions (4.13) for the right-moving coordinates Y_R in the sector h with spin structure(s) t_u.

Including the vacuum phases in front of the partition functions (4.34) and (4.36) makes them all modular covariant. However, it is not necessarily guaranteed that the full resulting partition function (4.27) has the proper orbifold and Wilson line projections built in, because of the factor of 1/2 in these phases. To ensure this, we need to require that:

$$\gcd(N_s, N_t)\, ({V_s}^T V_t - {v_s}^T v_t) \ , \ \gcd(N_s, N_{\underline{i}})\, {V_s}^T A_{\underline{i}} \ , \ \gcd(M_{\underline{i}}, M_{\underline{j}})\, {A_{\underline{i}}}^T A_{\underline{j}} = 0 \text{ mod } 2 \ , \tag{4.38}$$

(note there are no sums over repeated indices here). These conditions are commonly referred to as the modular invariance conditions.

4.7 Generalized discrete torsion phases

To ensure that the full partition function is modular invariant, the generalized torsion phases $c[{}^{h}_{h'}]$ satisfy the following conditions

$$c[{}^{h}_{h'}] = c[{}^{h'}_{h}] = c[{}^{h}_{h'h}] \ . \tag{4.39}$$

In particular, simply setting $c[{}^{h}_{h'}] = 1$ is an allowed solution, which is the typical choice for heterotic orbifolds unless otherwise stated. In general, we may parameterize these phases as

$$c[{}^{h}_{h'}] = c_{\text{anti}}[{}^{h}_{h'}]\, c_{\text{sym}}[{}^{h}_{h'}] \tag{4.40}$$

in terms of so-called generalized torsion phases. We distinguish between the symmetric and anti-symmetric phase factors: The anti-symmetric generalized torsion phases can be product expanded as

$$c_{\text{anti}}[{}^{h}_{h'}] = c_{st}[{}^{k_s}_{k'_t}]\, c_{\underline{i}\underline{j}}[{}^{n_{\underline{i}}}_{n'_{\underline{j}}}] c_{s\underline{i}}[{}^{k_s\, n_{\underline{i}}}_{k'_s\, n'_{\underline{i}}}] \ , \tag{4.41}$$

where appropriate products over different indices in the various factors are implied, *e.g.* over $t > s$. The factors, defined, for example, as

$$c_{st}[{}^{k_s}_{k'_t}] = e^{2\pi i\, c_{st}\, k_s k'_t} \ , \qquad c_{s\underline{i}}[{}^{k_s n_{\underline{i}}}_{k'_s n'_{\underline{i}}}] = e^{2\pi\, c_{s\underline{i}} (k_s n'_{\underline{i}} - k'_s n_{\underline{i}})} \ , \tag{4.42}$$

are characterized by the generalized torsion matrices c_{st}, $c_{s\underline{i}}$, etc.; their entries are anti-symmetric when they have two identical type indices, *e.g.* $c_{st} = -c_{ts}$. The generalized torsion matrices are subject to the quantization conditions to ensure that with these generalized torsion phases included one still has proper (orbifold) projections. They read, for instance, as

$$\gcd(N_s, N_t)\, c_{st}\ , \quad \gcd(N_s, M_{\underline{i}})\, c_{s\underline{i}}\ , \quad \gcd(M_{\underline{i}}, M_{\underline{j}})\, c_{\underline{i}\underline{j}} = 0 \bmod 1\ , \tag{4.43}$$

(no sums implied) and are characterized by the order of the respective elements to which the indices correspond. Here, and throughout this thesis, we will use the indices of the torsion matrices to indicate which torsion phases we are actually referring to: For example, c_{uv} refers to the possible torsion phase between the spin structure of the two E_8 factors; for the Spin(32)/$\mathbb{Z}_2$ theory, it is absent.

Furthemore, specifically for order-two elements we can admit additional symmetric phases:

$$c_{\text{sym}}\big[{}^{h}_{h'}\big] = c_s\big[{}^{k_s}_{k'_s}\big]\, c_{\underline{i}}\big[{}^{n_{\underline{i}}}_{n'_{\underline{i}}}\big]\ , \quad \text{where, for example:} \quad c_s\big[{}^{k_s}_{k'_s}\big] = (-)^{c_s(k_s+k'_s+k'_s k_s)}\ , \tag{4.44}$$

and the only allowed values are $c_s, c_{\underline{i}}, c_u = 0, 1$. These phases are symmetric under the interchange of primed and non-primed quantities. The phases c_s, c_u effectively select the spinorial lattice of the opposite chirality.

It should be emphasized that many of the generalized torsion phases introduced in (4.41) and (4.44) are normally not considered in the orbifold literature. The discrete torsion discussed by Vafa-Witten [35] only corresponds to the phase c_{st}. In [36] no symmetric torsion phases were introduced, only the anti-symmetric ones and in the current version of the `orbifolder` package [37] these symmetric torsion phases are not available. Moreover, one can introduce many additional symmetric and anti-symmetric generalized torsion phases that involve the spin structures ν_L and ν_{uR}:

$$c_{add} = c_L\big[{}^{s}_{s'}\big]\, c_u\big[{}^{t_u}_{t'_u}\big]\; c_{uv}\big[{}^{t_u}_{t'_v}\big]\, c_{Lu}\big[{}^{s\, t_u}_{s'\, t'_u}\big]\, c_{L\underline{i}}\big[{}^{s\, n_{\underline{i}}}_{s'\, n'_{\underline{i}}}\big]\, c_{su}\big[{}^{k_s\, t_u}_{k'_s\, t'_u}\big]\, c_{\underline{i}u}\big[{}^{n_{\underline{i}}\, t_u}_{n'_{\underline{i}}\, t'_u}\big]\ . \tag{4.45}$$

Brother models

Having fixed the orbifold geometry, the gauge shift and discrete Wilson lines, and the generalized torsion phases, one might hope that a heterotic orbifold model is uniquely specified. Unfortunately, this specification is somewhat redundant: Naively, one would think that by adding combinations of lattice vectors, $\Delta V_s, \Delta A_{\underline{i}} \in \Lambda_{\text{gauge}}$ to the defining gauge shifts V and discrete Wilson lines A:

$$\widetilde{V} = V + \Delta V\ , \quad \widetilde{A} = A + \Delta A\ , \tag{4.46}$$

would not change the model at all, as, for example, the resulting gauge group is typically unaffected by such changes. However, this is, in general, not true since adding such vectors leads to a whole family of so-called brother models [36]. Consequently, two heterotic orbifold brother models with gauge shift and Wilson lines satisfying (4.38) which are related via (4.46), can be viewed as two versions of the same orbifold model but with different generalized torsion phases [36]

$$\widetilde{c}[^{h}_{h'}] = e^{-2\pi i \, \frac{1}{2}\left(V_{h'}{}^T \Delta V_h - \Delta V_{h'}{}^T V_h + \Delta V_{h'}{}^T \Delta V_h \right)} \, c[^{h}_{h'}] \; . \tag{4.47}$$

The first two terms in the exponential are manifestly anti-symmetric, while the last term is not. To see that this term is in fact also anti-symmetric, one should realize that this term is always integral because ΔV_s and $\Delta A_{\underline{i}}$ are lattice vectors. In fact, for the diagonal part, *i.e.* $h' = h$, this term is even as Λ_{gauge} is even. For the off-diagonal parts, $h' \neq h$, we may flip the signs of the contributions because they are half-integral taking the factor of $1/2$ out front in the exponential into account. Finally, the conditions (4.38) ensure that the phase satisfies the quantization conditions of the generalized torsion (4.43).

4.8 Massless spectrum

Using the expressions for the partition functions for the various worldsheet fields, we can determine the complete spectrum of the orbifold theory. In many orbifold models (particularly those that make use of the `orbifolder` package [37]), one often restricts oneself to the massless spectrum only in a *generic* point of the moduli space. This means that one considers the compactification on orbifolds with arbitrary radii (as long as they are not set equal by the orbifold action). For such generic values of the orbifold radii, there is no "accidental" gauge symmetry enhancement, *i.e.* the lattice sum in (4.30) can be ignored as long as one is only interested in the massless spectrum.

The massless spectrum of an orbifold theory, in the sector $h \in S$ at a generic point of its moduli space, reads

$$M_L^2 = \tfrac{1}{2}\, p_{\text{sh}}^2 + \delta c - \tfrac{1}{2} \; , \qquad M_R^2 = \tfrac{1}{2}\, P_{\text{sh}}^2 + \delta c - 1 + N_R \; , \tag{4.48}$$

where N_R is the right-moving number operator and p_{sh} and P_{sh} the shifted momenta, defined below (4.34) and (4.36), respectively. The level matched massless states, of course, correspond to $M_L^2 = M_R^2 = 0$ (for supersymmetric orbifolds left-moving oscillator excitations will always lead to positive M_L^2, hence never constitute massless states). Here we have defined the shift δc in the zero point energy, given by

$$\delta c = \tfrac{1}{2}\, \omega^T (\mathbb{1}_4 - \omega) \; , \tag{4.49}$$

where the entries of $\omega_\alpha = (v_h)_\alpha \bmod 1$ are such that $0 \leq \omega_\alpha < 1$. The spectrum is subject to the orbifold projection condition

$$v_{h'}^T R - V_{h'}^T P_{\text{sh}} = \tfrac{1}{2} \left(v_{h'}^T v_h - V_{h'}^T V_h \right) \bmod 1 \tag{4.50}$$

for all projecting elements h' of the space group S that commute with the constructing elements h (only the standard generalized torsion phase $c[^{h}_{h'}] = 1$ is considered here for simplicity). Here we have defined

$$R^\alpha = p_{\text{sh}}^\alpha - N_R^\alpha + N_R^{\alpha *} \; , \tag{4.51}$$

which involves the shifted left-moving momentum and the number operators N_R^α and $N_R^{\alpha *}$ counting the bosonic oscillators, *e.g.* $\overline{\partial} X^\alpha$ and $\overline{\partial} X^{\alpha *}$. Note that the conditions (4.38) are essential for the projection conditions (4.50) to be well-defined.

4.9 Special features of $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifolds

So far our discussion has been for general orbifolds; in this section we make some statements that are specific to $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifolds which we will be using later.

Standard form of the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold twists

First of all, in this thesis we will use the following conventions to represent $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifolds. All $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifolds contain two twist elements combined with possible translations, *i.e.* roto-translations. The point group parts of the orbifolding elements are taken to be

$$\theta_1 = \begin{pmatrix} \mathbb{1}_2 & & \\ & -\mathbb{1}_2 & \\ & & -\mathbb{1}_2 \end{pmatrix}, \theta_2 = \begin{pmatrix} -\mathbb{1}_2 & & \\ & \mathbb{1}_2 & \\ & & -\mathbb{1}_2 \end{pmatrix}, \theta_3 = \theta_1\theta_2 = \begin{pmatrix} -\mathbb{1}_2 & & \\ & -\mathbb{1}_2 & \\ & & \mathbb{1}_2 \end{pmatrix} . \tag{4.52}$$

They define reflections in four of the six torus directions in the standard Euclidean basis, leaving the first, second and third two-torus inert, respectively. Their actions on the spinors (4.9) are defined by the vectors

$$v_1 = \big(0, 0, \tfrac{1}{2}, -\tfrac{1}{2}\big) , \quad v_2 = \big(0, -\tfrac{1}{2}, 0, \tfrac{1}{2}\big) . \tag{4.53}$$

Classification of $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifolds

The possible $\mathbb{Z}_2 \times \mathbb{Z}_2$ twist orbifolds were classified by Donagi and Faraggi in [40]. The classification was extended to include roto-translations by Donagi and Wendland in [39]. A full classification of all symmetric toroidal orbifolds that preserve at least $N = 1$ supersymmetry in four dimensions has been performed in [38]: This classification includes, but is not restricted to, $\mathbb{Z}_2 \times \mathbb{Z}_2$ or even Abelian orbifolds; most orbifolds turn out to possess non-Abelian point groups.

All these classifications are ultimately inspired by crystallography: The orbifold actions have to be compatible with a particular lattice; for given orbifold twists θ_s and lattice vectors ε_i, one needs to be able to fix the matrices $\rho_s \in \mathrm{GL}(6;\mathbb{Z})$ such that (4.4) is fulfilled. This, in turn, restricts the form of the metric G on the six-torus. Moreover, it determines the number and positions of two-tori and points that the various orbifold actions leave fixed. All $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifolds only possess fixed two-tori, which are either orbifolded by the second orbifold action or pairwise identified. All this information is encoded in the $\mathbb{Z}$-class (or arithmetic crystal class) of the six-dimensional lattice. The possible $\mathbb{Z}_2 \times \mathbb{Z}_2$ compatible lattices have been classified up to six dimensions [41].

FRTV label	DW label	twists / roto-translations	Hodge numbers
`CARAT` **$\mathbb{Z}$-class 1 :** $\{e_1, e_2, e_3, e_4, e_5, e_6\}$			
(1 - 1)	(0 - 1)	$(\theta_1, 0), (\theta_2, 0)$	$(51, 3)$
(1 - 2)	(0 - 2)	$(\theta_1, \frac{1}{2}e_2), (\theta_2, 0)$	$(19, 19)$
(1 - 3)	(0 - 3)	$(\theta_1, \frac{1}{2}e_{26}), (\theta_2, 0)$	$(11, 11)$
(1 - 4)	(0 - 4)	$(\theta_1, \frac{1}{2}e_{26}), (\theta_2, \frac{1}{2}e_4)$	$(3, 3)$
`CARAT` **$\mathbb{Z}$-class 2 :** $\{\frac{1}{2}e_{15}, e_2, e_3, e_4, e_5, e_6\}$			
(2 - 1)	(1 - 6)	$(\theta_1, 0), (\theta_2, 0)$	$(31, 7)$
(2 - 2)	(1 - 8)	$(\theta_1, \frac{1}{2}e_3), (\theta_2, 0)$	$(15, 15)$
(2 - 3)	(1 - 10)	$(\theta_1, \frac{1}{2}e_{36}), (\theta_2, 0)$	$(11, 11)$
(2 - 4)	(1 - 7)	$(\theta_1, 0), (\theta_2, \frac{1}{2}e_5)$	$(11, 11)$
(2 - 5)	(1 - 9)	$(\theta_1, \frac{1}{2}e_3), (\theta_2, \frac{1}{2}e_5)$	$(7, 7)$
(2 - 6)	(1 - 11)	$(\theta_1, \frac{1}{2}e_{36}), (\theta_2, \frac{1}{2}e_5)$	$(3, 3)$
`CARAT` **$\mathbb{Z}$-class 3 :** $\{\frac{1}{2}e_{15}, e_2, \frac{1}{2}e_{35}, e_4, e_5, e_6\}$			
(3 - 1)	(2 - 9)	$(\theta_1, 0), (\theta_2, 0)$	$(27, 3)$
(3 - 2)	(2 - 10)	$(\theta_1, \frac{1}{2}e_6), (\theta_2, 0)$	$(11, 11)$
(3 - 3)	(2 - 11)	$(\theta_1, \frac{1}{2}e_6), (\theta_2, \frac{1}{2}e_5)$	$(7, 7)$
(3 - 4)	(2 - 12)	$(\theta_1, \frac{1}{2}e_{46}), (\theta_2, \frac{1}{2}e_5)$	$(3, 3)$
`CARAT` **$\mathbb{Z}$-class 4 :** $\{\frac{1}{2}e_{15}, \frac{1}{2}e_{26}, e_3, e_4, e_5, e_6\}$			
(4 - 1)	(2 - 13)	$(\theta_1, 0), (\theta_2, 0)$	$(21, 9)$
(4 - 2)	(2 - 14)	$(\theta_1, 0), (\theta_2, \frac{1}{2}e_4)$	$(7, 7)$
`CARAT` **$\mathbb{Z}$-class 6 :** $\{\frac{1}{2}e_{15}, \frac{1}{2}e_{23}, e_3, e_4, e_5, e_6\}$			
(6 - 1)	(2 - 6)	$(\theta_1, 0), (\theta_2, 0)$	$(19, 7)$
(6 - 2)	(2 - 7)	$(\theta_1, 0), (\theta_2, \frac{1}{2}e_5)$	$(9, 9)$
(6 - 3)	(2 - 8)	$(\theta_1, \frac{1}{2}e_6), (\theta_2, \frac{1}{2}e_5)$	$(5, 5)$

FRTV label	DW label	twists / roto-translations	Hodge numbers
`CARAT` **$\mathbb{Z}$-class 5 :** $\{\frac{1}{2}e_{135}, e_2, e_3, e_4, e_5, e_6\}$			
(5 - 1)	(1 - 1)	$(\theta_1, 0), (\theta_2, 0)$	$(27, 3)$
(5 - 2)	(1 - 3)	$(\theta_1, \frac{1}{2}e_4), (\theta_2, 0)$	$(11, 11)$
(5 - 3)	(1 - 2)	$(\theta_1, \frac{1}{2}e_{23}), (\theta_2, 0)$	$(15, 15)$
(5 - 4)	(1 - 4)	$(\theta_1, \frac{1}{2}e_4), (\theta_2, \frac{1}{2}e_5)$	$(7, 7)$
(5 - 5)	(1 - 5)	$(\theta_1, \frac{1}{2}e_{46}), (\theta_2, \frac{1}{2}e_5)$	$(3, 3)$
`CARAT` **$\mathbb{Z}$-class 7 :** $\{\frac{1}{2}e_{15}, \frac{1}{2}e_{26}, \frac{1}{2}e_{36}, e_4, e_5, e_6\}$			
(7 - 1)	(3 - 3)	$(\theta_1, 0), (\theta_2, 0)$	$(17, 5)$
(7 - 2)	(3 - 4)	$(\theta_1, 0), (\theta_2, \frac{1}{2}e_6)$	$(7, 7)$
`CARAT` **$\mathbb{Z}$-class 8 :** $\{\frac{1}{2}e_{15}, \frac{1}{2}e_{26}, \frac{1}{2}e_{35}, \frac{1}{2}e_{46}, e_5, e_6\}$			
(8 - 1)	(4 - 1)	$(\theta_1, 0), (\theta_2, 0)$	$(15, 3)$
`CARAT` **$\mathbb{Z}$-class 9 :** $\{\frac{1}{2}e_{135}, \frac{1}{2}e_{26}, e_3, e_4, e_5, e_6\}$			
(9 - 1)	(2 - 3)	$(\theta_1, 0), (\theta_2, 0)$	$(17, 5)$
(9 - 2)	(2 - 5)	$(\theta_1, 0), (\theta_2, \frac{1}{2}e_6)$	$(7, 7)$
(9 - 3)	(2 - 4)	$(\theta_1, \frac{1}{2}e_{23}), (\theta_2, 0)$	$(11, 11)$
`CARAT` **$\mathbb{Z}$-class 10 :** $\{\frac{1}{2}e_{135}, \frac{1}{2}e_{26}, e_3, \frac{1}{2}e_{46}, e_5, e_6\}$			
(10 - 1)	(3 - 5)	$(\theta_1, 0), (\theta_2, 0)$	$(15, 3)$
(10 - 2)	(3 - 6)	$(\theta_1, \frac{1}{2}e_{12}), (\theta_2, 0)$	$(9, 9)$
`CARAT` **$\mathbb{Z}$-class 11 :** $\{\frac{1}{2}e_{14}, \frac{1}{2}e_{26}, \frac{1}{2}e_{35}, e_4, e_5, e_6\}$			
(11 - 1)	(3 - 1) ≡ (3 - 2)	$(\theta_1, 0), (\theta_2, 0)$	(12,6)
`CARAT` **$\mathbb{Z}$-class 12 :** $\{\frac{1}{2}e_{135}, \frac{1}{2}e_{246}, e_3, e_4, e_5, e_6\}$			
(12 - 1)	(2 - 1)	$(\theta_1, 0), (\theta_2, 0)$	$(15, 3)$
(12 - 2)	(2 - 2)	$(\theta_1, \frac{1}{2}e_{56}), (\theta_2, 0)$	$(9, 9)$

Table 4.2: Classification of all six-dimensional lattices that admit a $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold action according to [38] and [39] with the hodge numbers (h_{11}, h_{21}) indicated. Only models with $h_{11} \geq h_{21}$ appear on this list. Mirror partners with these two numbers interchanged can be obtained by switching on a discrete torsion phase between the two orbifold actions. We have grouped the geometries according to their `CARAT` $\mathbb{Z}$-classes and we give representative lattice choices for each of these $\mathbb{Z}$-classes. Here θ_1 and θ_2 denote the two $\mathbb{Z}_2$ reflections that leave the first and second two-torus fixed; e_i denotes the i-th standard Euclidean basis vector and $e_{ij} = e_i + e_j$, etc.

The required algorithms have been collected in the computer package `CARAT` [42]. This software provides a complete catalog of the $\mathbb{Z}$-classes.

The representations of both the lattice and the orbifold actions used in the classification are far from unique: For example, by scaling or permuting the torus directions and by shifting the origin on the six-torus, one obtains very different looking representations of the same orbifold. Moreover, the same lattice can be described in infinitely many bases.

We have given a compact representation of the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifolds in table 4.2. The data in this table are as follows: The first two columns give $\mathbb{Z}_2 \times \mathbb{Z}_2$ classifications following both Donagi,Wendland [39] and Fischer et al. [38]. The various `CARAT` $\mathbb{Z}$-classes following [42] are given with a representative lattice for each. The third column indicates a representation of the various orbifold actions on these lattices. The final column of this table displays the Hodge numbers of the various $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifolds. They can be determined as the number of generations and anti-generations when one uses the orbifold standard embedding, in which the orbifold shifts V_s are taken to be equal to v_s (completed with 13 zeros).

Chapter 5

The correspondence between orbifolds and free fermions

This chapter discusses the correspondence between orbifolds and free fermions from a model-builder's point of view and it is based on [3]. The idea that fermions and bosons in a 2-dimensional CFT are equivalent (fermionization/bosonization) dates back to the work of Sidney Coleman on the Sine-Gordon model in 1975 [43] and it starts being used in a string theory context in [44]. Aspects of the equivalence have often be used in a variety of settings [40, 45–47]. However, in many of these cases the exact steps of converting from one type of model to the other are often implicit and a concrete set of rules that would allow for a computational comparison appears to be missing from the literature. In this chapter, we explicitly present such a dictionary covering all aspects of the translation including the exact points in the moduli space in which various free fermionic models are found, free fermionic realizations of all possible 6-dimensional $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifolds and the conversion of the GSO phases. The discussion is geared at a level of detail that the output of the steps presented could be directly used as input for a computer program such as the the `orbifolder` [37].

5.1 Converting symmetric $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifolds to free fermionic models

In this section, we describe how one can associate a free fermionic model with the input data of a given symmetric orbifold model. This conversion takes an orbifold model, defined at a generic point, to a specific point in the geometrical moduli space; namely a point that actually admits a free fermionic description.

Heterotic symmetric orbifolds are defined as orbifolds of either the $\mathrm{E}_8\times\mathrm{E}_8$ or the $\mathrm{Spin}(32)/\mathbb{Z}_2$ string. A generic $\mathbb{Z}_2\times\mathbb{Z}_2$ symmetric orbifold model is defined by the two $\mathbb{Z}_2$ orbifold elements θ_s that can act as pure twists or as roto-translations on the geometry, accompanied by specific embeddings in the gauge degrees of freedom as encoded by the gauge shifts V_s. In addition, there are the Wilson lines $A_{\underline{i}}$, associated with the translations in the various lattice directions, $\varepsilon_{\underline{i}}$, that define the underlying torus or lattice. Finally, the model might possess some generalized torsion phases. This is the input data we need to translate into a collection of free fermionic basis vectors and generalized GSO phases.

To define such a set of basis vectors, we need to take into account both the Wilson lines as well as the free fermionic requirement that the vector $\mathbf{1}$ is in the additive set. To this end, we first observe that having an order $M_{\underline{i}}$ Wilson line, $A_{\underline{i}}$, associated with a certain translation $\varepsilon_{\underline{i}}$, can be thought of as a $\mathbb{Z}_{M_{\underline{i}}}$ pure shift orbifold. On a torus with a radius $M_{\underline{i}}$ times bigger than the original one, applying the Wilson line $M_{\underline{i}}$ times is like having standard periodicity. Since we are free to rescale the underlying torus, we can take this bigger six-torus as our starting point and assume that it is an orthonormal lattice with unit edges. Hence, we first define a standard set of basis vectors, B_0, that describes the $\mathrm{E}_8\times\mathrm{E}_8$ or the Spin(32)/$\mathbb{Z}_2$ theory on this orthonormal unit six-torus:

$$\mathsf{B}_0 = \big\{\mathbf{S}, \boldsymbol{\xi}_u, \mathbf{e}_1, \ldots, \mathbf{e}_6\big\} \ , \tag{5.1}$$

with $\boldsymbol{\xi}_u = \boldsymbol{\xi}$ (or $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2$) for the Spin(32)/$\mathbb{Z}_2$ (or $\mathrm{E}_8\times\mathrm{E}_8$) case, respectively.

Next we extend this set to include basis vectors $\widetilde{\mathbf{b}}_s$ and $\boldsymbol{\beta}_{\underline{i}}$ that correspond to the orbifold elements, θ_s, and the Wilson lines, $A_{\underline{i}}$, respectively. The resulting canonical basis set,

$$\mathsf{B} = \mathrm{B}_0 \cup \big\{\widetilde{\mathbf{b}}_1, \widetilde{\mathbf{b}}_2, \boldsymbol{\beta}_1, \ldots, \boldsymbol{\beta}_6\big\} = \big\{\mathbf{S}, \boldsymbol{\xi}_u, \mathbf{e}_1, \ldots, \mathbf{e}_6, \widetilde{\mathbf{b}}_1, \widetilde{\mathbf{b}}_2, \boldsymbol{\beta}_1, \ldots, \boldsymbol{\beta}_6\big\} \ , \tag{5.2}$$

contains up to 16 (or 17) elements for the Spin(32)/$\mathbb{Z}_2$ (or $\mathrm{E}_8\times\mathrm{E}_8$) case. Any element $\boldsymbol{\alpha}$ in the additive set $\boldsymbol{\Xi}$, associated with a given orbifold model, can therefore be expanded as

$$\boldsymbol{\alpha} = s\,\mathbf{S} + t_u\,\boldsymbol{\xi}_u + m_i\,\mathbf{e}_i + k_s\,\widetilde{\mathbf{b}}_s + n_{\underline{i}}\,\boldsymbol{\beta}_{\underline{i}} \ . \tag{5.3}$$

For the set of basis vectors in (5.2), we need a prescription for a choice of the generalized GSO phase matrix.

5.1.1 Defining the free fermionic basis vectors

Choice of ten-dimensional heterotic theory

Depending on whether the orbifolded string theory is the Spin(32)/$\mathbb{Z}_2$ or the $\mathrm{E}_8\times\mathrm{E}_8$ theory, the set of basis vectors B contains:

$$\text{Spin}(32)/\mathbb{Z}_2 : \quad \mathbf{S}, \boldsymbol{\xi} \in \mathsf{B} \ , \quad \text{or} \quad \mathrm{E}_8\times\mathrm{E}_8 : \quad \mathbf{S}, \boldsymbol{\xi}_1, \boldsymbol{\xi}_2 \in \mathsf{B} \ . \tag{5.4}$$

Encoding Wilson lines

Next, we turn to an order $M_{\underline{i}}$ Wilson line, $A_{\underline{i}}$ associated with a lattice translation $\varepsilon_{\underline{i}}$. Any of the lattice translations can be decomposed in the standard Euclidean basis e_i as: $\varepsilon_{\underline{i}} = (n_{\underline{i}})_i\, e_i/M_{\underline{i}}$, where we treat $n_{\underline{i}}$ as integral vectors. The associated fermionic basis vector, $\boldsymbol{\beta}_{\underline{i}}$, can then be taken to be:

$$\boldsymbol{\beta}_{\underline{i}} = \big\{0^8; \tfrac{n_{\underline{i}}}{M_{\underline{i}}}, \tfrac{n_{\underline{i}}}{M_{\underline{i}}} \,|\, \tfrac{n_{\underline{i}}}{M_{\underline{i}}}, \tfrac{n_{\underline{i}}}{M_{\underline{i}}}\big\}\big(2A_{\underline{i}}\big) \ . \tag{5.5}$$

The notation here means that no ψ^μ, χ^i fermions are involved and only the pairs of fermions y^i, w^i and $\bar{y}^i, \bar{w}^i$, in the Euclidean directions in which $\varepsilon_{\underline{i}}$ is pointing, appear. The latter part indicates that one completes the basis vector by two times the value of the discrete Wilson line in the orbifold formulation. As an illustrative example, the order-two Wilson lines, $A_{\underline{i}} = (0^7,1)(0^8)$ in the $\varepsilon_{\underline{i}} = \frac{1}{2}\, e_i$ direction in the $\mathrm{E}_8\times\mathrm{E}_8$ theory, become $\boldsymbol{\beta}_i = \{0^8; y^i, w^i \,|\, \bar{y}^i, \bar{w}^i\}(0^7,1,0^8)$. Also, the spin structure vector, say ν_L for the Spin(32)/$\mathbb{Z}_2$ theory defined under (4.37), which is a shift only in the gauge lattice, can be translated to a free fermionic basis vector using (5.5) to give $\boldsymbol{\xi}$ (similarly, ν_{1L} and ν_{2L} correspond to $\boldsymbol{\xi}_1$ and $\boldsymbol{\xi}_2$, respectively). Note that we did not include an extra factor of 2 in the y, w and $\bar{y}, \bar{w}$ parts of (5.5) since this element represents an order $M_{\underline{i}}$ vector w.r.t. the orthonormal lattice that was already generated by $\mathbf{e}_1, \ldots, \mathbf{e}_6$.

5.1.2 Orbifold elements in the free fermionic formulation

In the same way, we can associate the basis vectors $\mathbf{b}_1$ and $\mathbf{b}_2$ with the orbifold elements θ_1 and θ_2. Here the following complication arises: As discussed in section 4.2 there are different types of orbifold actions and their characterization is partially parameterization dependent. As can be inferred from the bosonization relation (see *e.g.* [6] or [8]):

$$-i\, y^i w^i \simeq i\, \partial X^i_L \ , \tag{5.6}$$

in order to represent twists or shifts, but not roto-translations, the fermionic basis vectors can be chosen as

$$\widetilde{\mathbf{b}}_1 = \big\{\chi^{34}, -\chi^{56}; z^{34}, z^{56} \,|\, \bar{z}^{34}, \bar{z}^{56}\big\}\big(2V_1\big) \ , \tag{5.7}$$

$$\widetilde{\mathbf{b}}_2 = \big\{-\chi^{12}, \chi^{56}; z^{12}, z^{56} \,|\, \bar{z}^{12}, \bar{z}^{56}\big\}\big(2V_2\big) \ , \tag{5.8}$$

where the signs in front of the complexified fermions, e.g. $\chi^{12} = \chi^1 + i\chi^2$, have been chosen such that they are compatible with the sign choices for the $\mathbb{Z}_2\times\mathbb{Z}_2$ actions on the spinor in the bosonic formulation in (4.53). (We use the same notation for the complexified z's as well.) The non-removable parts of the shifts in the true roto-translations can be taken into account by including the corresponding fermion pairs y^i, w^i and $\bar{y}^i, \bar{w}^i$ in their associated fermionic basis vectors in the same fashion as we did for the Wilson line elements, as in (5.5). Furthermore, each z^i, $i = 1, \ldots, 6$, equals either y^i or w^i and $\bar{z}^i$ either $\bar{y}^i$ or $\bar{w}^i$. Thus, a similar ambiguity is present in the fermionic description when defining the twist actions.

This seems to imply that there is also an ambiguity of how to associate definite fermionic basis vectors with their corresponding orbifold twist actions. To shed light on this issue, we compare the partition functions of the bosonic and fermionic descriptions of the orbifold twisted sectors. When doing so one notices some seemingly unrelated differences:

- In the bosonic description only commuting, constructing and projecting, elements give contributions to the partition function, while by definition all boundary conditions encoded in the additive set $\boldsymbol{\Xi}$ are allowed. Hence, the number of sectors on the worldsheet torus does not seem to be the same in both descriptions.

- Secondly, the bosonic twisted partition function, given in (4.29), involves ϑ-functions in the denominator as can be seen from (4.32). In contrast, the fermionic partition function (3.12) always has ϑ-functions in the numerator only. Moreover, for the geometrical part, the fermionic description involves twice as many ϑ-functions as the bosonic description, since each left-(right-)moving bosonic coordinate X_L^i corresponds to two fermions y^i, w^i.

But these issues are closely related and can, in fact, help us understand whether the twist-like $\mathbb{Z}_2$ elements are mutual twists or roto-translations:

Suppose the two twist-like elements $\widetilde{\mathbf{b}}_1$ and $\widetilde{\mathbf{b}}_2$ both contain a specified y^i or w^i. The part of the partition function in which one is the constructing and the other is the projecting element will vanish identically since this overlap leads to a (square root of) $\vartheta[^{1/2}_{1/2}] = 0$. This means that this sector does not give any contribution to the partition function; precisely as if we have two non-commuting space group elements. Hence, in the direction(s) where the overlapping y^i or w^i appear, one of the elements corresponds to a pure twist while the other acts as a roto-translation. Consequently, if the sector defined by one element is to have a proper projection from the other, then there should not be any overlap of any of the ys and ws. Note that this observation is merely used to make the connection between roto-translations and overlapping elements in the free fermionic basis vectors. We do not really advocate replacing $\vartheta[^{1/2}_{1/2}]$ with 0 whenever it appears in the partition function. Doing so would be unhelpful: at the end of the day the partition function iteself is zero for supersymmetric models. Furthermore, such a replacement would obfuscate the modular invariance nature of the partition function.

We can also see the same effect when we reverse the process: For commuting constructing and projecting space group elements, h and h', the geometrical twisted partition function is given in (4.32). Using the identity

$$\frac{\eta}{\vartheta\begin{bmatrix}\frac{1-a}{2}\\ \frac{1-a'}{2}\end{bmatrix}} = \frac{\vartheta\begin{bmatrix}\frac{a}{2}\\ 0\end{bmatrix}\vartheta\begin{bmatrix}0\\ \frac{a'}{2}\end{bmatrix}}{2\,\eta^2}\,, \tag{5.9}$$

for any $a, a' = 0, 1$, excluding $(a, a') = (0, 0)$, we can rewrite this partition function with twice the number of ϑ-functions in the numerator, just like one has in the fermionic formulation, for the ϑ-functions associated with the fermions y and w. Moreover, precisely as we noticed above, for elements that do not lead to a $\vartheta[^{1/2}_{1/2}]$ in the partition function, the characteristics in these ϑ-functions do not overlap.

Using these considerations it is always possible to find the appropriate choice of ys and ws (and their conjugates) in the two orbifold basis vectors $\widetilde{\mathbf{b}}_1, \widetilde{\mathbf{b}}_2$. In practice, figuring out the correct choices for given orbifold geometries can be quite tricky. Therefore, in table 5.2 in the example section, we provide specific choices of free fermionic basis vectors that can represent all 35 $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold geometries of table 4.2.

Some properties of the resulting set of basis vectors

If we translate orbifold twists, shifts and Wilson lines to basis vectors of the corresponding models, we will always obtain basis vectors which will satisfy the modular invariance conditions (3.15) in the free fermionic formulation, since the orbifold input satisfied (4.38). By adding appropriate multiples of 2 to some of the entries of these basis vectors, they can be brought to the specific range (3.4) as long as one remembers to modify the generalized GSO phases accordingly, once they have been determined.

It should be noted that the notion of order of the resulting basis vectors in the free fermionic model will be two times that of the orbifold theory for those orbifold shifts V_s or Wilson lines $A_{\underline{i}}$ that are built from spinorial roots. For example, $A_1 = (0^8)(\frac{1}{4}{}^8)$ has order two in the orbifold language since $2\,A_1 \in \Lambda_{8\times 8}$ while the corresponding $\boldsymbol{\beta}_1 = \{y^1, w^1|\bar{y}^1, \bar{w}^1; \frac{1}{2}\,\bar{\phi}^{1\dots 8}\}$ has order four. The reason for this difference is that in the free fermionic construction the order of the vectors is counted with respect to the orthogonal lattice while on the orbifold side it is counted with respect to the $\mathrm{E}_8\times\mathrm{E}_8$ or $\mathrm{Spin}(32)/\mathbb{Z}_2$ lattice.

We would also like to emphasize that when converting an orbifold to a free fermionic model we are forcing the theory to move to a very particular point in the moduli space, namely a free fermionic point. By the rules of the dictionary presented here this is automatically guaranteed. In particular, the vector $\mathbf{1}$ is always in the additive set. Moreover, we should mention that we can always find different lattice representations in the same $\mathbb{Z}$-class which are free fermionic points as well. Instead of starting from the basis vectors $\mathbf{e}_1, \dots, \mathbf{e}_6$ that define the standard Euclidean basis, we can also use more miminal (i.e. with fewer basis vectors) free fermionic realizations of the various orbifold geometries. Examples, for the different $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold geometries of table 4.2 are presented in table 5.2.

5.1.3 Determining the associated generalized GSO phases

The next step is to determine the generalized GSO phases from the partition function in the bosonic formulation. To do so, it is crucial to take into account all phases that appear in the partition functions on both the orbifold and the free fermionic sides. These phases in the orbifold description of chapter 4 get contributions from the bosonized superpartners of the coordinate fields (4.34), the gauge lattice (4.36), generalized torsion phases (4.41) and, finally, the additional symmetric phases (4.44). These phases should be compared with the generalized GSO phases in (3.11) taking into account the phases (3.13) included in the ϑ-functions, Θ. An important fact here is that the projection phase structure in both theories is not fully identical: In the free fermionic formulation, the projection phase, i.e. the final phases in (3.13), are fully factorized in the exponential. On the orbifold side, however, the phases in the exponential are not factorized: there are two projection phases in both (4.34) and (4.36): the last implement the orbifold and Wilson line projections while the next-to-last implement the various lattice constraints due to the spin structures.

Taking these observations into account, while comparing the various phases, we conclude that

$$(-)^{s's+s'+s}\, e^{-2\pi i\, \frac{1}{2}\left\{ v_h{}^T v_{h'} - V_h{}^T V_{h'}\right\}}\, c\big[{}^{h}_{h'}\big] = e^{-\pi i\, \frac{1}{2} \boldsymbol{\alpha}\cdot\boldsymbol{\alpha}'}\, e^{2\pi i\, (s'\nu_L^T v_h - t_u' \nu_{uR}^T V_h)}\, C\big[{}^{\boldsymbol{\alpha}}_{\boldsymbol{\alpha}'}\big]\ , \quad (5.10)$$

by simply setting the bosonic and fermionic phases equal, provided that we use the expansion in (5.3) for the vectors $\boldsymbol{\alpha}$ and $\boldsymbol{\alpha}'$. The second phase on the right-hand-side takes into account the fact that on the orbifold side the fully factorized exponentials are not present. Inserting the various definitions we find

$$C\big[{}^{\boldsymbol{\alpha}}_{\boldsymbol{\alpha}'}\big] = (-)^{s's+s'+s}\, e^{\pi i\, (v_h^T v_h' - V_h^T V_h')}\, e^{-2\pi i\, t_u\, \nu_{uL}^T V_h'}\, c\big[{}^{h}_{h'}\big]\ , \quad (5.11)$$

where we have used that $\nu_L^T v_s = 0$ strictly for all supersymmetric orbifolds.

If we make the identifications (5.3), we see that all the remaining phases also agree identically, hence, we can read off the generalized GSO phases of the free fermionic formulation from the orbifold input. For all phases involving $\mathbf{S}$ we find (3.27). For the remaining phases involving $\mathbf{e}_i$, we conclude that they are simply

$$C\big[{}^{\mathbf{e}_i}_{\mathbf{B}_a}\big] = 1\ , \quad (5.12a)$$

for all $\mathbf{B}_a \neq \mathbf{S}$. In addition, we find

$$\begin{aligned} C\begin{bmatrix}\tilde{\mathbf{b}}_1\\ \tilde{\mathbf{b}}_2\end{bmatrix} &= e^{\pi i\, (v_1^T v_2 - V_1^T V_2)}\, e^{2\pi i\, c_{st}}\ ,\\ C\begin{bmatrix}\boldsymbol{\beta}_{\underline{i}}\\ \boldsymbol{\beta}_{\underline{j}}\end{bmatrix} &= e^{-\pi i\, A_{\underline{i}}^T A_{\underline{j}}}\, e^{2\pi i\, c_{\underline{i}\underline{j}}}\ ,\\ C\begin{bmatrix}\tilde{\mathbf{b}}_s\\ \boldsymbol{\beta}_{\underline{i}}\end{bmatrix} &= e^{-\pi i\, V_s^T A_{\underline{i}}}\, e^{2\pi i\, c_{s\underline{i}}}\ . \end{aligned} \quad (5.12b)$$

As stressed in section 4.7, all other possible generalized discrete torsion phases are (mostly implicitly) taken to be trivial, i.e. $c = 0$, in the orbifold literature. Since any free fermionic construction is not complete without also specifying their values, we indicate the remaining phases here. We obtain

$$C\begin{bmatrix}\tilde{\mathbf{b}}_s\\ \tilde{\mathbf{b}}_s\end{bmatrix} = e^{\pi i\, (v_s^2 - V_s^2)}\, (-)^{c_s}\ , \qquad C\begin{bmatrix}\boldsymbol{\beta}_{\underline{i}}\\ \boldsymbol{\beta}_{\underline{i}}\end{bmatrix} = e^{-\pi i\, A_{\underline{i}}^2}\, (-)^{c_{\underline{i}}}\ , \qquad C\begin{bmatrix}\boldsymbol{\xi}_u\\ \boldsymbol{\xi}_u\end{bmatrix} = (-)^{c_u}\ , \quad (5.12c)$$

for the symmetric phases and

$$C\begin{bmatrix}\boldsymbol{\xi}_1\\ \boldsymbol{\xi}_2\end{bmatrix} = e^{2\pi i\, c_{uv}}\ , \qquad C\begin{bmatrix}\tilde{\mathbf{b}}_s\\ \boldsymbol{\xi}_u\end{bmatrix} = e^{2\pi i\, c_{su}}\ , \qquad C\begin{bmatrix}\boldsymbol{\beta}_{\underline{i}}\\ \boldsymbol{\xi}_u\end{bmatrix} = e^{2\pi i\, c_{\underline{i}u}}\ , \quad (5.12d)$$

$$C\begin{bmatrix}\boldsymbol{\xi}_u\\ \tilde{\mathbf{b}}_s\end{bmatrix} = e^{-2\pi i\, \nu_{uL}^T V_s}\, e^{-2\pi i\, c_{su}}\ , \qquad C\begin{bmatrix}\boldsymbol{\xi}_u\\ \boldsymbol{\beta}_{\underline{i}}\end{bmatrix} = e^{-2\pi i\, \nu_{uL}^T A_{\underline{i}}}\, e^{-2\pi i\, c_{\underline{i}u}}\ , \quad (5.12e)$$

for the anti-symmetric phases.

5.2 Converting free fermionic models to symmetric orbifolds

In this section we describe explicitly how to convert a free fermionic model to a symmetric orbifold model. In the proceeding subsection, the various steps are discussed in detail. In section 5.3 we then go through a number of examples to illustrate the general procedure.

Since the task of converting models is –in its fine print– rather involved, we first present a brief, non-technical outline of the steps involved. The general discussion that follows can be read in parallel with the examples in section 5.3, while the remaining parts of this section offer extensive explanations of the steps used.

1. Convert to a basis that admits an orbifold interpretation

As considered and described in chapter 3, a free fermionic model is defined by a set of basis vectors $\mathsf{B} = \{\mathbf{B}_a\}$, generating an additive set $\mathbf{\Xi}$, together with generalized GSO-phases that both satisfy a large set of consistency conditions.

The basis of a generic free fermionic model contains vectors whose role in the description of an orbifold geometry is rather obscure. For the subsequent identification of the properties of the orbifold model, it is necessary to go to a set of basis vectors that can be distinguished by the roles they play:

- supersymmetry vector $\mathbf{S}$,
- twist-like vectors $\widetilde{\mathbf{b}}_s$, $s = 1, 2$,
- Narain-like vectors $\boldsymbol{\beta}_x$,
- spin-structure vectors $\boldsymbol{\xi}_u$.

The twist-like generators, $\widetilde{\mathbf{b}}_1, \widetilde{\mathbf{b}}_2$, encode the two independent $\mathbb{Z}_2$ reflections, possibly combined with simultaneous shifts, *i.e.* the orbifold twists or roto-translations. The Narain-like basis vectors, $\boldsymbol{\beta}_x$, are characterized by the requirement that they do not act on the fermions $\{\psi^\mu, \chi^i\}$. Often one can identify one or two spin-structure basis vectors: either $\boldsymbol{\xi}$ or $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2$.

2a. Directly determine the orbifold twists, shifts and Wilson lines

If the spin-structure vectors, $\boldsymbol{\xi}$ or both $\boldsymbol{\xi}_1$ and $\boldsymbol{\xi}_2$, can be identified, then one can directly interpret the free fermionic model as an orbifold of the Spin(32)/$\mathbb{Z}_2$ or $\mathrm{E}_8 \times \mathrm{E}_8$ theories, respectively. If the set of remaining Narain-like vectors is not redundant, then one can directly read off the orbifold shifts and Wilson lines.

2b. Identify the geometrical Narain data

Unfortunately, often the spin-structure vectors are not present in the additive set $\mathbf{\Xi}$, or only one of the two $\boldsymbol{\xi}_u$'s is. In this case, we can only determine the orbifold data by comparison with the Narain description. This is possible because the Narain-like vectors, $\boldsymbol{\beta}_x$, define the untwisted sector of the orbifold. Their partition function can be represented as a lattice sum and from this we can, in principle, read off the geometrical parameters G, B, A that define a Narain torus compactification.

3. **Determine the generalized discrete torsion phases**

   We read off which generalized torsion phases are switched on for given generalized GSO phases. These relations are important since they affect the projection conditions on the spectra.

4. **Classify the orbifold geometry**

   Once the six-torus background is specified, we can identify the orbifold geometry which the free fermionic model corresponds to. To this end, we need to identify the space group associated with the two twist-like elements $\widetilde{\mathbf{b}}_s$ and the torus lattice identified above. The combination of these data fixes the $\mathbb{Z}$-class of the bosonic model. In particular, it determines whether $\widetilde{\mathbf{b}}_s$ should be thought of as $\mathbb{Z}_2$-twists and/or roto-translations. This will affect the number and type of fixed points of the orbifold and, consequently, the underlying geometry of the resolved manifold.

Before we go into the details, a couple of comments are in order:

When a complete set of spin-structure vector(s) can be identified, we suggest to use the direct route 2a to identify the Wilson lines. Of course, in that case, one can still follow the other route 2b: This gives more information as it does not only specify the topological data of the orbifold theory, but it also determines the value of all free moduli at the free fermionic point, where the free fermionic model is defined.

Especially via route 2b, one is confronted with the fact that the choice of twist-like vectors and Narain-like vectors out of the additive set is not unique. The representation of Wilson lines, or of the Narain lattice in general, is dependent on the choice of duality frame. In addition, one could keep some shift orbifold actions explicit in the description or absorb them, possibly including the associated generalized torsion phases, in a redefinition of the Narain lattice. To make the matching of free fermionic models with orbifold models as transparent as possible, it is often preferable to translate all generalized GSO phases of a free fermionic model to generalized torsion phases in the corresponding orbifold model. However, we will also encounter examples where this is simply not directly possible or where it would lead to other complications. Different choices could lead to seemingly different orbifold models that are associated with one and the same free fermionic model; consequently, these different orbifold models are equivalent descriptions of the same physics.

Whether a basis vector is of type $\mathbf{S}$, $\widetilde{\mathbf{b}}_s$ or $\boldsymbol{\beta}_x$ is determined by how it acts on the left-moving fermions only. Therefore, it is not automatically guaranteed that the twist-like elements $\widetilde{\mathbf{b}}_s$ have identical action on a certain set of right-moving fermions such that a symmetric orbifold interpretation is possible. Similarly, a Narain-like element might act as a twist on the right-moving coordinates, hence such Narain-like elements do not characterize the underlying Narain lattice of the construction. This is a subtle question because the pairing of the right-moving fermions with the left-moving y's and w's that correspond to the left-moving coordinates via (5.6) is, in fact, arbitrary; for different choices the interpretation of the model might be very different.

Similarly, Step 3 might also be a showstopper for the matching: In principle, the free fermionic description allows for more choice of generalized GSO phases than the orbifold description. As stressed in section 4.7, it is conventional in the orbifold literature to fix certain phases once and for all, even though not all these choices are strictly necessary. However, we have included additional generalized torsion phases that should correspond to the additional freedom of generalized GSO phases on the free fermionic side.

5.2.1 Convert to a basis that admits an orbifold interpretation

The first step in identifying an orbifold model that corresponds to a given free fermionic model is to bring the basis vectors into a form that makes interpreting them from the bosonic side easier.

Characterize different types of basis elements

As discussed in the previous section, any free fermionic model under consideration in this work possesses the supersymmetry vector $\mathbf{S}$ defined in table 3.2 as an element of the additive set $\mathbf{\Xi}$; conventionally, even as one of the basis vectors. For such models we can find two independent vectors $\widetilde{\mathbf{b}}_1$ and $\widetilde{\mathbf{b}}_2$ such that both of these vectors and their sum, $\widetilde{\mathbf{b}}_3 = \widetilde{\mathbf{b}}_1 + \widetilde{\mathbf{b}}_2$, all act on some of the χ^i but not on ψ^μ:

$$\mathbf{S} \cap \widetilde{\mathbf{b}}_s \neq \emptyset \ , \qquad \delta_{\widetilde{\mathbf{b}}_s} = 1 \ . \tag{5.13}$$

These basis vectors, $\widetilde{\mathbf{b}}_s$, are twist-like vectors since they act on the geometry at least as reflections and hence correspond to the orbifold elements as can be inferred from the bosonization relation (5.6). This can be obtained by comparing the supersymmetry currents in the fermionic and bosonic descriptions, given in (3.2) and (4.11), respectively, upon identifying the notation $\psi^i = \chi^i$.

For the remaining generators of the additive set, we construct linear combinations, $\boldsymbol{\beta}_x$, such that none of them acts on the fermions $\{\psi^\mu, \chi^i\}$, i.e.

$$\boldsymbol{\beta}_x \cap \mathbf{S} = \emptyset \ . \tag{5.14}$$

We refer to these vectors as Narain-like vectors. In this new basis,

$$\boldsymbol{\alpha} = s\,\mathbf{S} + \sum_{a \neq S} n_a\,\mathbf{B}_a = s\,\mathbf{S} + k_s\,\widetilde{\mathbf{b}}_s + n_x\,\boldsymbol{\beta}_x \ , \tag{5.15}$$

(with $s, k_s = 0, 1$ and n_a up to the order of the various elements $\mathbf{B}_a$) only the supersymmetry generator $\mathbf{S}$ has $\delta_{\mathbf{S}} = -1$. Notice that the two basis vectors $\widetilde{\mathbf{b}}_s$ are not uniquely defined because we can always combine them with arbitrary linear combinations of the basis vectors $\boldsymbol{\beta}_x$. A useful choice is to pick these linear combinations such that the overlap of the vectors $\widetilde{\mathbf{b}}_1$ and $\widetilde{\mathbf{b}}_2$ on the y's and w's is as small as possible.

Symmetric orbifold interpretation

Before we continue, we need to check that the fermionic model admits an interpretation as a symmetric orbifold at all: The free fermionic basis elements translated into the bosonic language should either act as a twist-like action or as a shift action on both left- and right-moving coordinates (but see section 5.3.6 for a brief discussion on asymmetric shifts). This is not guaranteed by the definitions of the twist-like and Narain-like basis vectors above as their characterizations involved their $\{\psi^\mu, \chi^i\}$-content only.

To understand the relation between fermionic and bosonic boundary conditions, it is helpful to make use of the bosonization relation (5.6). Since the supercurrent (3.2) has to be preserved by all basis elements of a free fermionic model, we infer that for any Narain-like element $\boldsymbol{\beta}_x$ the fermions y^i, w^i should always appear in pairs for any $i = 1, \ldots, 6$: Narain-like elements could act as translations on the coordinate fields but never as a twist, hence we see from (5.6) that precisely in these cases X_L does not change sign. For symmetric orbifolds, admissible Narain-like basis vectors should also contain $\bar{y}^j, \bar{w}^j$ pairwise.

Similarly, in any twist-like element, $\widetilde{\mathbf{b}}_s$, either y^i or w^i is present (but never both at the same time) whenever it contains χ^i; when it does not, the y^i, w^i's should appear pairwise. From (5.6) we see that, in this case, X_L at least changes sign, and so the interpretation of a twist-like element is justified. We demand that for a symmetric orbifold interpretation the same $\bar{y}^i$'s and $\bar{w}^i$'s should appear in the twist-like basis elements.

These criteria for having a symmetric orbifold interpretation are up to renaming of the right-moving real and complex fermions, since splitting in real $\bar{y}$ and $\bar{w}$ and complex $\bar{\lambda}$ fermions in table 3.1 is somewhat arbitrary. For a free fermionic model to admit a symmetric orbifold interpretation, there should be some choice for this such that these statements all hold.

By a reordering of the indices i we can ensure that we have chosen the twist-like elements such that

$$\widetilde{\mathbf{b}}_1 \supset \{\chi^{3,4}, \chi^{5,6}\} , \qquad \widetilde{\mathbf{b}}_2 \supset \{\chi^{1,2}, \chi^{5,6}\} . \tag{5.16}$$

Again, using the invariance of the supercurrent (3.2) this implies that $\widetilde{\mathbf{b}}_1$ and $\widetilde{\mathbf{b}}_2$ act as twist-like actions on the bosonic coordinates with point group actions given by (4.52). In the following, we are considering only free fermionic models that admit a symmetric orbifold interpretation and that the basis vectors $\mathbf{b}_s$ and $\boldsymbol{\beta}_x$ have been brought to the form defined here.

It is also possible to obtain some elements $\boldsymbol{\beta}_x$ that do not involve any y and w fermions; such elements may be associated with the gauge spin structures ν_{uR} in the bosonic language: If the model includes $\boldsymbol{\xi}_1$ and $\boldsymbol{\xi}_2$ then we can think of it as an orbifold of the ten dimensional heterotic $E_8 \times E_8$ theory, and when it only includes $\boldsymbol{\xi}$, of the Spin(32)/$\mathbb{Z}_2$ theory. It can also happen that there is no linear combination of the Narain-like basis vectors which equals $\boldsymbol{\xi}$; in particular it might be that only one of the two $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2$ is present. Given that the moduli space of Narain compactifications is connected, in such cases the free fermionic models correspond to orbifold theories at points in the moduli space other than the $\mathrm{E}_8\times\mathrm{E}_8$ or $\mathrm{Spin}(32)/\mathbb{Z}_2$ points. Some examples are given in table 5.1 in section 5.3.

If the additive set Ξ includes a set of spin-structure vectors, *i.e.* either $\boldsymbol{\xi}$ or both $\boldsymbol{\xi}_1$ and $\boldsymbol{\xi}_2$, and some further requirements are met, see below, we can continue either via route 2a or 2b. If this is not the case, only route 2b is available to us.

5.2.2a Directly determine the orbifold twists, shifts and Wilson lines

In this subsection we assume that we have a set of basis vectors

$$\mathsf{B} = \big\{\mathbf{B}_a\big\} = \big\{\mathbf{S}, \widetilde{\mathbf{b}}_s, \boldsymbol{\xi}_u, \boldsymbol{\beta}_x\big\} \ , \tag{5.17}$$

that admit a symmetric orbifold interpretation and has at most six remaining Narain-like basis vectors $\boldsymbol{\beta}_x$. In addition, we demand that they are strictly symmetric, *i.e.* each of them contains the same y^i, w^i as $\bar{y}^i, \bar{w}^i$-pairs. Finally, we require that they remain linearly independent when we restrict them to their geometrical action, characterized by the y, w-pairs only.

If these conditions are not satisfied by the basis vectors in question, then the methods described in this subsection cannot be applied. One could try to modify the input data of the free fermionic model, such that the new set of basis vectors do satisfy these conditions. Of course, alternatively, one can use the more general procedures of the next subsection corresponding to route 2b.

Free fermionic basis vectors and even lattice constraints

The defining data of an orbifold model, in particular the orbifold twists, shifts and Wilson lines, are assumed to satisfy some additional conventions: The gauge shifts and Wilson lines multiplied by their order should be lattice vectors in the appropriate gauge lattices. The orbifold twists were chosen to leave a standard choice for the four dimensional supersymmetry generators invariant. These conditions are technically enforced by requiring that the twists v_s satisfy (4.10) and the shifts V_s and the Wilson lines A_x multiplied by their orders are Λ_{gauge} lattice vectors (see the requirements (4.15)). In addition, the orbifold input data needs to satisfy the generalized modular invariant conditions (4.38). The conventions on the free fermionic basis vectors $\mathbf{B}_a$ are slightly different: their entries have to fulfill (3.8) and are conventionally chosen to lie in the range (3.4).

The additional specific lattice conditions on the orbifold input data translate in the free fermionic language as follows: The standard choice for supersymmetry under (4.10) requires that:

$$\mathbf{S} \cdot \widetilde{\mathbf{b}}_s = 0 \ , \tag{5.18}$$

(the conditions (4.15) are automatically fulfilled by (3.15)). If we have basis vectors that do not satisfy (5.18), then we can modify them as

$$\widetilde{\mathbf{b}}_s^{\text{orbi}} = \widetilde{\mathbf{b}}_s + \boldsymbol{\delta}_s \ , \tag{5.19}$$

where $\boldsymbol{\delta}_s$ are vectors with only even entries in the χ^i-directions, such that some signs in χ^i-entries of $\widetilde{\mathbf{b}}_s^{\text{orbi}}$ are flipped to satisfy (5.18): For example, we can take $\boldsymbol{\delta}_1 = \{-2\chi^{34}\}$ and $\boldsymbol{\delta}_2 = \{-2\chi^{12}\}$ so that $\widetilde{\mathbf{b}}_1^{\text{orbi}} \supset \{-\chi^{34}, \chi^{56}\}$ and $\widetilde{\mathbf{b}}_2^{\text{orbi}} \supset \{-\chi^{12}, \chi^{56}\}$. This does not modify the free fermionic model at all, provided that one modifies the generalized GSO phases accordingly using (3.18). In the orbifold language, this corresponds to the twists

$$v_1 = (0, 0, -\tfrac{1}{2}, \tfrac{1}{2}) \ , \qquad v_2 = (0, -\tfrac{1}{2}, 0, \tfrac{1}{2}) \ . \tag{5.20}$$

Up to possible brother phases (4.47) this corresponds to the most common choice (4.53) in the orbifold literature.

Characterizing the symmetric orbifold input data

We can now immediately read off the orbifold input: The orbifold twists and shifts are given by

$$v_s = \tfrac{1}{2}\, \widetilde{b}_s^{\text{orbi}}(\chi) \ , \qquad V_s = \tfrac{1}{2}\, \widetilde{b}_s(\bar{\lambda}) \ , \tag{5.21a}$$

taking care when going from a real to a complex basis for the fermions χ^i. Moreover, we can identify the Wilson lines

$$A_x = \tfrac{1}{2}\, \beta_x(\bar{\lambda}) \ , \tag{5.21b}$$

associated with translations in the directions $\varepsilon_x = \frac{1}{2}\, \beta_x(y) = \frac{1}{2}\, \beta_x(w)$.

5.2.2b Identify the geometrical Narain data

The Narain lattice corresponding to a free fermionic model can be determined in the following fashion. Not the whole fermionic partition function (3.12) admits a Narain lattice interpretation, therefore we only focus on the part of this partition function generated by the fermions $y^i, w^i, \bar{y}^i, \bar{w}^i, \bar{\lambda}^I$. Moreover, only the non-twist part of the fermionic partition function (3.12) should be considered, since the Narain description applies to torus compactifications. Hence, we further restrict to the basis vectors with $\boldsymbol{\beta} = n_x\, \boldsymbol{\beta}_x$ (*i.e.* setting $s = k_s = 0$):

$$Z_{\text{Narain}} = \frac{1}{N} \sum_{n,n'} \frac{\Theta\big[{\beta(y) \atop \beta'(y)}\big]}{\eta^6} \frac{\overline{\Theta}\big[{\beta(\overline{y}) \atop \beta'(\overline{y})}\big]\, \overline{\Theta}\big[{\beta(\bar{\lambda}) \atop \beta'(\bar{\lambda})}\big]}{\overline{\eta}^{22}} \ , \tag{5.22}$$

where N is the product of the orders of the elements $\boldsymbol{\beta}_x$. Here, we used that, for the non-twist elements, $\beta(w) = \beta(y)$ and similarly for their conjugates. Using the sum representation (3.13), this is immediately written in the form of a Narain lattice sum (4.26) and hence one can read off a basis for the Narain lattice. An example illustrating this procedure in detail is given in section 5.3.1.

Narain standard form

With either of the above methods, one obtains a basis for the Narain lattice. The collection of basis vectors may be interpreted as the generalized vielbein E'. However, when we compute

$$E'^T \eta E' = \hat{\eta}' \ , \tag{5.23}$$

we generically do not find the metric $\hat{\eta}$ generated in (4.24), but a matrix $\hat{\eta}'$ that is related to this via a transformation $M \in \mathrm{GL}(28;\mathbb{Z})$:

$$\hat{\eta} = M^T \hat{\eta}' M \ . \tag{5.24}$$

It is important to realize that the determination of the Narain moduli strongly depends on the form of $\hat{\eta}'$. Hence, it is not sufficient to know the generalized vielbein E' in some arbitrary basis, but it is crucial to find a matrix M that brings it to a standard form. Unfortunately, as far as we are aware, no generic algorithm is known about how to determine such a transformation. However, this is not a problem of encoding a free fermionic model in the orbifold description, but rather an issue of how to practically work with Narain moduli spaces.

5.2.3 Determine the generalized torsion phases

We have seen in the previous subsections that we can distinguish two types of free fermionic constructions: those that can be thought of as orbifolds of the $\mathrm{Spin}(32)/\mathbb{Z}_2$ or $\mathrm{E}_8\times\mathrm{E}_8$ theories and the others. This distinction is also important for how concretely one can describe the translation of the generalized GSO phases to the generalized torsion phases on the bosonic side.

Orbifolds of the Spin(32)/$\mathbb{Z}_2$ or $\mathbf{E_8}\times\mathbf{E_8}$ theories

Modulo the fact that one, in general, needs to add even entries to some of the basis vectors, *i.e.* (5.19), we see that the translation of the free fermionic to the orbifold data in (5.21) is essentially identical to that in the opposite direction, see (5.5) and (5.7) (up to a factor of 1/2 in (5.5), which we included since all vectors $\mathbf{e}_i$ were taken to be in the basis vector set. Substituting the translations into each other, one gets the original input data back). Hence, to determine translation of the phases, we can also simply invert the phase relations (5.12).

Since free fermionic data do not necessarily satisfy (5.18), we may need some sign flips in $\widetilde{\mathbf{b}}_s$. Via (3.18), we have

$$e^{2\pi i\, c_{st}} = e^{-\frac{1}{4}\pi i\, (\widetilde{\mathbf{b}}_1 - \boldsymbol{\delta}_1)\cdot(\widetilde{\mathbf{b}}_2 + \boldsymbol{\delta}_2)}\, C \begin{bmatrix} \widetilde{\mathbf{b}}_1 \\ \widetilde{\mathbf{b}}_2 \end{bmatrix} \ . \tag{5.25a}$$

In addition, we obtain:

$$e^{2\pi i\, c_{\underline{i}\underline{j}}} = e^{-\frac{1}{4}\pi i\, \boldsymbol{\beta}_{\underline{i}}\cdot\boldsymbol{\beta}_{\underline{j}}}\, C \begin{bmatrix} \boldsymbol{\beta}_{\underline{i}} \\ \boldsymbol{\beta}_{\underline{j}} \end{bmatrix} \ , \qquad e^{2\pi i\, c_{s\underline{i}}} = e^{-\frac{1}{4}\pi i\, \widetilde{\mathbf{b}}_s\cdot\boldsymbol{\beta}_{\underline{i}}}\, C \begin{bmatrix} \widetilde{\mathbf{b}}_s \\ \boldsymbol{\beta}_{\underline{i}} \end{bmatrix} \ , \tag{5.25b}$$

$$(-)^{c_s} = e^{-\frac{1}{4}\pi i \,\widetilde{\mathbf{b}}_s^2} \, C \begin{bmatrix} \widetilde{\mathbf{b}}_s \\ \widetilde{\mathbf{b}}_s \end{bmatrix} , \qquad (-)^{c_{\underline{i}}} = e^{-\frac{1}{4}\pi i \,\boldsymbol{\beta}_{\underline{i}}^2} \, C \begin{bmatrix} \boldsymbol{\beta}_{\underline{i}} \\ \boldsymbol{\beta}_{\underline{i}} \end{bmatrix} , \qquad (-)^{c_u} = C \begin{bmatrix} \boldsymbol{\xi}_u \\ \boldsymbol{\xi}_u \end{bmatrix} , \quad (5.25c)$$

$$e^{2\pi i \, c_{uv}} = C \begin{bmatrix} \boldsymbol{\xi}_1 \\ \boldsymbol{\xi}_2 \end{bmatrix} , \qquad e^{2\pi i \, c_{su}} = C \begin{bmatrix} \widetilde{\mathbf{b}}_s \\ \boldsymbol{\xi}_u \end{bmatrix} , \qquad e^{2\pi i \, c_{\underline{i}u}} = C \begin{bmatrix} \boldsymbol{\beta}_{\underline{i}} \\ \boldsymbol{\xi}_u \end{bmatrix} . \qquad (5.25d)$$

General Narain orbifolds

If one has determined the Narain lattice associated with the Narain-like elements following route 2b, then one has absorbed some of the original generalized GSO phases into the Narain lattice. This will typically mean that the geometrical part of the lattice has changed, *i.e.* the ε in the generalized vielbein (4.20) is not the same as the one we started with. Therefore, the Wilson lines that are read off from it, are related, in a complicated way, to the original ones, hence unfortunately, it is very difficult to describe the relation between the original phases of the free fermionic model and the remaining ones after rewriting the underlying torus compactification in the Narain form. In light of this, the most systematic approach seems to be to simply scan a variety of generalized torsions for the translated orbifold model.

5.2.4 Identifying the orbifold geometry

Above, we obtained a basis of generators of the additive set which are divided into Narain-like and twist-like elements. The twist-like elements, $\widetilde{\mathbf{b}}_1$ and $\widetilde{\mathbf{b}}_2$, can either be interpreted as pure twists or roto-translations. However, reversing the logic presented in section 5.1.2, we are able to determine how to interpret their actions geometrically.

Consequently, any free fermionic model that admits an interpretation as a symmetric $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold model should correspond to one of the geometries given in table 4.2. When the orbifold actions and the six-torus lattice ε have been identified, the corresponding $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold can be determined by referring to the program `CARAT`. In particular, using this code, one determines the $\mathbb{Z}$-class of the lattice, simply by calculating the matrices $\varepsilon^{-1}\theta_1\varepsilon$ and $\varepsilon^{-1}\theta_2\varepsilon$ and using the `CARAT` command: `Name`.

5.3 Examples

5.3.1 Narain torus compactification models

The SO(12)×SO(32) model

Our review of free fermionic models in chapter 3 indicated that all free fermionic models contain at least the vectors: $\{\mathbf{1}, \mathbf{S}\}$. For simplicity, the first example we consider here is the free fermionic model obtained from this set augmented with the vector $\boldsymbol{\xi}$ given in table 3.2, *i.e.* defined by the set of basis vectors $\{\mathbf{1}, \mathbf{S}, \boldsymbol{\xi}\}$. The resulting model possesses $N = 4$ supersymmetry in four dimensions and has an SO(12)×SO(32) gauge group.

To translate this free fermionic model to the bosonic description, the first step is to define the orbifold interpretable basis. To this end, we make a change of basis such that the new basis vectors do not have any overlap: $\{\mathbf{S}, \mathbf{e}_{123456}, \boldsymbol{\xi}\}$: $\boldsymbol{\xi}$ is already a Narain-like basis vector. Since we have the basis vector $\mathbf{S}$ explicitly, the other element which does not contain ψ^μ and has no overlap with $\boldsymbol{\xi}$ is

$$\mathbf{e}_{123456} = \mathbf{1} - \mathbf{S} - \boldsymbol{\xi} = \left(0^8, 1^{12} \,|\, 1^{12}; 0^{16}\right) . \tag{5.26}$$

As there is no overlap with $\mathbf{S}$, this is also a Narain-like basis vector. In addition, due to there being no overlap between the basis vectors $\mathbf{e}_{123456}$ and $\boldsymbol{\xi}$, the resulting Narain part of the partition function (5.22) factorizes as

$$Z_{\text{Narain}} = \frac{1}{4\,\eta^6 \bar{\eta}^{22}} \sum_{s',s=0,1} \Theta[{}^{s}_{s'}]^6 \overline{\Theta}[{}^{s}_{s'}]^6 \sum_{t',t=0,1} \overline{\Theta}[{}^{t}_{t'}]^{16} . \tag{5.27}$$

Using the sum representation of the Θ function (3.13), we can read off the projection conditions on the summation variables, $m'', n'' \in \mathbb{Z}^6$ and $p'' \in \mathbb{Z}^{16}$, to obtain

$$Z_{\text{Narain}} = \frac{1}{4\,\eta^6 \bar{\eta}^{22}} \sum_{\substack{s=0,1,\ m'',n''\in\mathbb{Z}^6,\\ \sum(m''_i+n''_i)=0 \text{ mod } 2}} \bar{q}^{\frac{1}{2}\sum_i (m''_i+\frac{s}{2})^2} q^{\frac{1}{2}\sum_j (n''_j+\frac{s}{2})^2} \sum_{\substack{t=0,1,\ p''\in\mathbb{Z}^{16}\\ \sum p''_k=0 \text{ mod } 2}} q^{\frac{1}{2}\sum_k (p''_k+\frac{t}{2})^2}. \tag{5.28}$$

We define new variables m', n' and p' as

$$m'_i = m''_i + \tfrac{s}{2} \ , \quad n'_i = n''_i + \tfrac{s}{2} \ , \qquad p'_k = p''_k + \tfrac{t}{2} \ . \tag{5.29}$$

Note that for $s = 0$ or 1 the new variables m'_i's and n'_i's are all integral or all half-integral. The same holds for the new variables p'_k's. Furthermore, the fact that

$$\sum (m''_i + n''_i) = \text{even} \ , \qquad \sum p''_k = \text{even} \ , \tag{5.30}$$

implies that

$$\sum (m'_i + n'_i) = \text{even} \ , \qquad \sum p'_k = \text{even} \ . \tag{5.31}$$

Knowing that p'_k's are all integral or all half-integral and that their sum is even implies that $p \in \mathcal{D}_{16}$. Likewise, knowing that the m'_i's and n'_i's are all integral or all half-integral simultaneously and that the sum (involving all of them) is even, implies that the combined vector $(m', n') \in \mathcal{D}_{12}$. Here the lattice $\mathcal{D}_D$ in D dimensions is defined as

$$\mathcal{D}_D = \mathcal{R}_D + \mathcal{S}_D \ , \tag{5.32}$$

where $\mathcal{R}_D$ and $\mathcal{S}_D$ are the root and spinor lattices of $SO(2D)$ described in appendix A.2.1. In particular, $\mathcal{D}_8$ is the E_8 root lattice. Hence, we can write the lattice sum as

$$Z_{\text{Narain}} = \frac{1}{4\,\eta^6 \bar{\eta}^{22}} \sum_{(m',n')\in\mathcal{D}_{12}} \bar{q}^{\frac{1}{2}m'^2} q^{\frac{1}{2}n'^2} \sum_{p'\in\mathcal{D}_{16}} q^{\frac{1}{2}p'^2} . \tag{5.33}$$

Note that the above expression is not quite in the standard form yet. In particular, the appearance of the $\mathcal{D}_{12}$ lattice despite resulting from a useful mathematical trick does not imply that the gauge group includes an $SO(24)$ factor. To identify this partition

Basis vectors	Gauge group	Six-torus lattice	Narain moduli
$\{\mathbf{S}, \mathbf{e}_{1\ldots6}+\boldsymbol{\xi}\}$	SO(44)	$\{\frac{1}{2}e_{1\ldots6}, e_2, e_3, \ldots, e_6\}$	$\varepsilon_{\mathbb{1}},\ B_{\mathbb{1}},\ A_{16},\ \alpha_{16}$
$\{\mathbf{S}, \mathbf{e}_{1\ldots6}, \boldsymbol{\xi}\}$	SO(12) × SO(32)		$\varepsilon_{so},\ B_G,\ A=0,\ \alpha_{16}$
$\{\mathbf{S}, \mathbf{e}_{1\ldots6}+\boldsymbol{\xi}_1, \boldsymbol{\xi}_2\}$	SO(24) × E_8		$\varepsilon_{\mathbb{1}},\ B_{\mathbb{1}},\ A_8,\ \alpha_{8\times8}$
$\{\mathbf{S}, \mathbf{e}_{1\ldots6}, \boldsymbol{\xi}_1, \boldsymbol{\xi}_2\}$	SO(12) × E_8 × E_8		$\varepsilon_{so},\ B_G,\ A=0,\ \alpha_{8\times8}$
$\{\mathbf{S}, \mathbf{e}_1, \ldots, \mathbf{e}_6, \boldsymbol{\xi}\}$	$\mathrm{U}(1)^6$ × SO(32)	$\{\frac{1}{2}e_1, \ldots, \frac{1}{2}e_6\}$	$\varepsilon_{\mathbb{1}},\ B=0,\ A=0,\ \alpha_{16}$
$\{\mathbf{S}, \mathbf{e}_1, \ldots, \mathbf{e}_6, \boldsymbol{\xi}_1, \boldsymbol{\xi}_2\}$	$\mathrm{U}(1)^6$ × E_8 × E_8		$\varepsilon_{\mathbb{1}},\ B=0,\ A=0,\ \alpha_{8\times8}$

Table 5.1: This table summarizes the most prominent free fermionic models that can be interpreted as Narain compactifications. The explicit moduli were derived for the standard choice of the GSO phases (5.39). The notation for the Narain moduli fields is defined in section 5.3.1.

function (5.33) with the Narain partition function given in (4.18), one needs to find a change of variables, $N' = (m', n', p') = E\,N$, that solves the constraints and allows us to write the sum over all of $\mathbb{Z}^{28}$ instead of the restricted set $\mathcal{D}_{12} \oplus \mathcal{D}_{16}$. This change of variables is precisely of the form of the Narain momentum vector (4.17), hence the matrix E can be taken in the form of the generalized vielbein (4.20). For the case at hand, a possible choice for this is given by

$$\varepsilon = \varepsilon_{so}\ , \qquad G = \varepsilon^T \varepsilon\ , \qquad B = B_G\ , \qquad A = 0_{16\times6}\ , \qquad \alpha = \alpha_{16} \tag{5.34}$$

using the notation defined in (5.36) and (5.37). Performing this change of variables the partition function becomes

$$Z_{\text{Narain}} = \frac{1}{\eta^6 \bar{\eta}^{22}} \sum_{N \in \mathbb{Z}^{28}} q^{\frac{1}{4} N^T E^T (\mathbb{1}-\eta)\, EN}\, \bar{q}^{\frac{1}{4} N^T E^T (\mathbb{1}+\eta)\, EN}\ , \tag{5.35}$$

with E related to the quantities appearing in (5.34) via (4.20). This is our final result in standard form. In particular, the appearance of the ε_{so} and α_{16} matrices in (5.34) proves that the gauge group of this model is $SO(12) \times SO(32)$.

Other toroidal Narain models

To describe the previous and some other free fermionic models which correspond to purely Narain compactifications, we define: the six-dimensional vielbeins,

$$\varepsilon_{\mathbb{1}} = \frac{1}{\sqrt{2}} \mathbb{1}_6, \qquad \varepsilon_{so} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{pmatrix}_{6\times6} \ ; \tag{5.36}$$

Kalb-Ramond B-fields,

$$B_{\mathbb{1}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -1 & \cdots & -1 \\ 1 & \ddots & \ddots & \vdots \\ \vdots & & \ddots & -1 \\ 1 & \cdots & 1 & 0 \end{pmatrix}_{6\times 6} \quad , \quad B_G = \begin{cases} G_{ij} & \text{if } i < j \\ 0 & \text{if } i = j \\ -G_{ij} & \text{if } i > j \end{cases} ; \tag{5.37}$$

and Wilson lines,

$$A_i = \begin{pmatrix} 0 \\ 1\,1\,1\,1\,1\,1 \\ 0 \end{pmatrix}_{16\times 6} \quad \leftarrow i^{\text{th}} \text{ row} \ . \tag{5.38}$$

Using these definitions, we can express the moduli of a number of pure Narain free fermionic models given in table 5.1. They have been derived following the procedure in the previous subsection. For all of them we have made the standard choice of GSO phases, given by

$$C[^{\mathbf{S}}_{\mathbf{S}}] = C[^{\mathbf{S}}_{\mathbf{B}_a}] = -1 \ , \qquad C[^{\mathbf{B}_a}_{\mathbf{B}_b}] = 1 \ , \tag{5.39}$$

for all basis vectors $\mathbf{B}_a, \mathbf{B}_b \neq \mathbf{S}$. Certain phases do not change the gauge group, but only the lattices. A simple example of this effect is to set $C[^{\boldsymbol{\xi}_2}_{\boldsymbol{\xi}_2}] = -1$ leading to a change of the spinor lattice to the co-spinor lattice $\mathcal{D}_8$ in (5.32) for the second E_8 factor.

5.3.2 A simple free fermionic $\mathbb{Z}_2 \times \mathbb{Z}_2$ model

We will start our analysis of free fermionic models that include orbifold twists by considering the free fermionic model with basis vectors

$$\Big\{ \mathbf{S}, \mathbf{b}_1, \mathbf{b}_2, \mathbf{e}_{1\ldots6}, \boldsymbol{\xi}_1, \boldsymbol{\xi}_2 \Big\} \ , \tag{5.40}$$

introduced in table 3.2. The upper triangular part of the generalized GSO phase matrix, including the diagonal is taken to be:

$$C[^{\mathbf{B}_a}_{\mathbf{B}_b}] = \begin{array}{c|cccccc} \mathbf{B}_a\backslash\mathbf{B}_b & \mathbf{S} & \mathbf{b}_1 & \mathbf{b}_2 & \boldsymbol{\xi}_1 & \boldsymbol{\xi}_2 & \mathbf{e}_{1\ldots6} \\ \hline \mathbf{S} & -1 & -1 & -1 & -1 & -1 & -1 \\ \mathbf{b}_1 & 1 & -1 & 1 & 1 & 1 & 1 \\ \mathbf{b}_2 & 1 & 1 & -1 & 1 & 1 & 1 \\ \boldsymbol{\xi}_1 & 1 & -1 & -1 & 1 & 1 & 1 \\ \boldsymbol{\xi}_2 & 1 & 1 & 1 & 1 & 1 & 1 \\ \mathbf{e}_{1\ldots6} & -1 & 1 & 1 & 1 & 1 & 1 \end{array} \ . \tag{5.41}$$

To emphasize that the entries in the lower triangular part cannot be chosen arbitrarily, but are fixed via (3.16c), we have drawn these entries in a lighter grey colour.

In this model, the interpretation of the basis vector elements is immediate: $\mathbf{S}$ is the target space supersymmetry element; $\mathbf{b}_1, \mathbf{b}_2$, the twist-like elements; and $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \mathbf{e}_{1\ldots6}$, Narain-like elements. Since the twist-like elements involve the fermions χ^i as dictated in (5.16), we can associate $\mathbf{b}_s$ with the orbifold twists θ_s defined in (4.52). Moreover, since these twists do not have any y or w overlap, we know we can interpret them both as generating pure twists, as discussed in section 5.2.4.

In more detail, by the multiplication in table 3.2 we notice that the inner products

$$\mathbf{b}_s \cdot \mathbf{S} = 2 \bmod 4 \ . \tag{5.42}$$

Hence, the twist-like elements do not satisfy (5.18). Therefore, when we want to read off the associated orbifold twists and gauge shifts according to (5.21a), we need to flip some signs (see (5.19)):

$$\mathbf{b}_1 \ : \quad v_1 = \tfrac{1}{2}\left(\ 0, -1,\ \ 1\right)\ , \quad V_1^{\mathrm{SE}} = \tfrac{1}{2}\left(0^5,\ \ 0,\ \ 1,\ \ 1\right)\left(0^8\right)\ , \tag{5.43a}$$

$$\mathbf{b}_2 \ : \quad v_2 = \tfrac{1}{2}\left(-1,\ \ 0,\ \ 1\right)\ , \quad V_2^{\mathrm{SE}} = \tfrac{1}{2}\left(0^5,\ \ 1,\ \ 0,\ \ 1\right)\left(0^8\right)\ , \tag{5.43b}$$

which we can see with the help of (3.18), do not modify the phases. Hence, we can keep using (5.41) in its current form. Since the model includes the basis vectors $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2$, we can interpret it as an orbifold of the $\mathrm{E}_8\times\mathrm{E}_8$ theory. Moreover, since V_s^{SE} contains v_s, this model corresponds to the standard embedding. Consequently, we can use the number of **16**-plet generations and anti-generations to determine the Hodge numbers of the orbifold geometry.

The orbifold phases can be read from the matrix in (5.41) using (5.25). We find that all the orbifold torsion phases are trivial, i.e.

$$c_s = c_{\underline{i}} = c_u = 0\ , \qquad c_{st} = c_{\underline{i}\underline{j}} = c_{s\underline{i}} = c_{uv} = c_{su} = c_{\underline{i}u} = 0\ . \tag{5.44}$$

In particular, the spin-structure projections are the standard ones used in the orbifold literature. Since, all the other possible generalized torsion phases (4.40) are also zero, this model can be directly understood as a standard orbifold model. Furthermore, the non-twist-like basis vectors, $\left\{\mathbf{S}, \mathbf{e}_{1\ldots 6}, \boldsymbol{\xi}_1, \boldsymbol{\xi}_2\right\}$, are the same as the set of basis vectors on the fourth row of table 5.1. Hence, given that the relevant phases are also chosen identically, we can immediately read off the moduli from that row of the table.

To summarize, we have found that this simple free fermionic model corresponds to the standard $\mathbb{Z}_2 \times \mathbb{Z}_2$ pure twist orbifold on the $SO(12)$ lattice with the standard embedding. This corresponds to the DW(1 - 1) geometry.

5.3.3 Free fermionic realizations of the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold geometries

In this subsection, we would like to give explicit examples of free fermionic models corresponding to each of the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold geometries. The results of this analysis have been collected in table 5.2. (They are independent of the gauge structure and therefore apply to both the $\mathrm{E}_8\times\mathrm{E}_8$ and the $\mathrm{Spin}(32)/\mathbb{Z}_2$ cases.) In principle, we can directly use the results of section 5.1 to translate each of these geometries in the free fermionic language. This way one obtains a large set of basis vectors which can be computationally inconvenient. In table 5.2 we give free fermionic realizations of each of the $\mathbb{Z}_2 \times \mathbb{Z}_2$ geometries that are minimal in their number of basis vectors.

To determine these results we started from the explicit parameterization of the orbifold geometries given in [39]: In particular, the periodicity of the target space two-tori in terms of a modular parameter is taken to be 2τ (not to be confused with the Teichmüller parameter of the worldsheet torus defined under (4.18)). Whenever possible, we modified the shift elements indicated there such that they can be represented by free fermionic translational elements $\mathbf{e}_i$, $\mathbf{e}_{ij}$, etc., so that the sum of all these elements is identical to $\mathbf{e}_{123456}$ (combined with $\mathbf{S}$, $\boldsymbol{\xi}_1$ and $\boldsymbol{\xi}_2$, this ensures that $\mathbf{1}$ is part of the additive set). To that effect, we sometimes change 1 or τ to $1+\tau$ throughout an orbifold geometry: *i.e.* both in the shift elements as well as in the twists/roto-translations. For all geometries, we extend the resulting elements such that we get a set of shift elements that sum to $\mathbf{e}_{123456}$.

We took the standard $\mathbb{Z}_2 \times \mathbb{Z}_2$ action to be the one that leads to chirality in the standard embedding in the first E_8 of the $E_8 \times E_8$ theory. This means that the twist-like elements in this case are simply $\mathbf{b}_1$ and $\mathbf{b}_2$, given in table 3.2. The related non-chiral geometries in the same class have one or both twist elements replaced by roto-translations. These roto-translations can be represented in the fermionic language by combining the twist elements with the appropriate translational basis vectors $\mathbf{e}_i$. We have tried to choose the free fermionic representations of the lattice and the twists/roto-translations such that they are all manifestly order two free fermionic elements. It was only for the DW geometry (2 - 12) that we were unable to find such a representation and resorted to a seemingly order four twist $\mathbf{b}_1 + \frac{1}{2}\,\mathbf{e}_2$.

The standard choice of generalized GSO phases we use in table 5.2 is given by:

$$C\begin{bmatrix}\mathbf{B}_a\\ \mathbf{B}_b\end{bmatrix} = \begin{array}{c} \mathbf{B}_a\backslash\mathbf{B}_b \\ \mathbf{S} \\ \mathbf{B}_1 \\ \mathbf{B}_2 \\ \boldsymbol{\beta}_y \end{array} \begin{array}{c} \begin{array}{cccc} \mathbf{S} & \mathbf{B}_1 & \mathbf{B}_2 & \boldsymbol{\beta}_x \end{array} \\ \begin{pmatrix} -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ -1 & \delta_y & \delta_y & 1 \end{pmatrix} \end{array} . \tag{5.45}$$

Here we define

$$\delta_y = \begin{cases} -1 & \boldsymbol{\beta}_y = \boldsymbol{\xi}_1 \ , \\ +1 & \text{otherwise} \ . \end{cases} \tag{5.46}$$

Moreover, $\mathbf{B}_s$, $s = 1, 2$, stands for the twist elements given in the next-to-last column of table 5.2, $\boldsymbol{\beta}_x, \boldsymbol{\beta}_y$ for $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2$ and the shift elements given in the last column of that table.

DW Label	Hodge #	Twists / roto-translations	Shifts elements	Free fermionic realization in the standard embedding: $\mathbf{S}, \boldsymbol{\xi}_1, \boldsymbol{\xi}_2$ and	
(0 - 1)	$(51,3)$	$(0+,0-,0-),(0-,0+,0-)$	none	$\mathbf{b}_1, \mathbf{b}_2,$	$\mathbf{e}_{12}, \mathbf{e}_{34}, \mathbf{e}_{56}$
(0 - 2)	$(19,19)$	$(0+,0-,0-),(0-,0+,1-)$	none	$\mathbf{b}_1, \mathbf{b}_2+\mathbf{e}_5,$	$\mathbf{e}_{12}, \mathbf{e}_{34}, \mathbf{e}_{56}$
(0 - 3)	$(11,11)$	$(0+,0-,0-),(0-,1+,1-)$	none	$\mathbf{b}_1, \mathbf{b}_2+\mathbf{e}_{35},$	$\mathbf{e}_{12}, \mathbf{e}_{34}, \mathbf{e}_{56}$
(0 - 4)	$(3,3)$	$(1+,0-,0-),(0-,1+,1-)$	none	$\mathbf{b}_1+\mathbf{e}_1, \mathbf{b}_2+\mathbf{e}_{35},$	$\mathbf{e}_{12}, \mathbf{e}_{34}, \mathbf{e}_{56}$
(1 - 1)	$(27,3)$	$(0+,0-,0-),(0-,0+,0-)$	(τ,τ,τ)	$\mathbf{b}_1, \mathbf{b}_2,$	$\mathbf{e}_{123456}$
(1 - 2)	$(15,15)$	$(0+,0-,0-),(0-,0+,\tau-)$	(τ,τ,τ)	$\mathbf{b}_1, \mathbf{b}_2+\mathbf{e}_{56},$	$\mathbf{e}_{123456}$
(1 - 3)	$(11,11)$	$(0+,0-,0-),(0-,0+,1-)$	(τ,τ,τ)	$\mathbf{b}_1, \mathbf{b}_2+\mathbf{e}_5,$	$\mathbf{e}_{123456}$
(1 - 4)	$(7,7)$	$(0+,0-,0-),(0-,1+,1-)$	(τ,τ,τ)	$\mathbf{b}_1, \mathbf{b}_2+\mathbf{e}_{35},$	$\mathbf{e}_{123456}$
(1 - 5)	$(3,3)$	$(1+,0-,0-),(0-,1+,1-)$	(τ,τ,τ)	$\mathbf{b}_1+\mathbf{e}_1, \mathbf{b}_2+\mathbf{e}_{35},$	$\mathbf{e}_{123456}$
(1 - 6)	$(31,7)$	$(0+,0-,0-),(0-,0+,0-)$	$(\tau,\tau,0)$	$\mathbf{b}_1, \mathbf{b}_2,$	$\mathbf{e}_{1234}, \mathbf{e}_{56}$
(1 - 7)	$(11,11)$	$(0+,0-,0-),(0-,0+,1-)$	$(\tau,\tau,0)$	$\mathbf{b}_1, \mathbf{b}_2+\mathbf{e}_5,$	$\mathbf{e}_{1234}, \mathbf{e}_{56}$
(1 - 8)	$(15,15)$	$(0+,0-,0-),(0-,1+,0-)$	$(\tau,\tau,0)$	$\mathbf{b}_1, \mathbf{b}_2+\mathbf{e}_3,$	$\mathbf{e}_{1234}, \mathbf{e}_{56}$
(1 - 9)	$(7,7)$	$(0+,0-,0-),(0-,1+,1-)$	$(\tau,\tau,0)$	$\mathbf{b}_1, \mathbf{b}_2+\mathbf{e}_{35},$	$\mathbf{e}_{1234}, \mathbf{e}_{56}$
(1 - 10)	$(11,11)$	$(1+,0-,0-),(0-,1+,0-)$	$(\tau,\tau,0)$	$\mathbf{b}_1+\mathbf{e}_1, \mathbf{b}_2+\mathbf{e}_3,$	$\mathbf{e}_{1234}, \mathbf{e}_{56}$
(1 - 11)	$(3,3)$	$(1+,0-,0-),(0-,1+,1-)$	$(\tau,\tau,0)$	$\mathbf{b}_1+\mathbf{e}_1, \mathbf{b}_2+\mathbf{e}_{35},$	$\mathbf{e}_{1234}, \mathbf{e}_{56}$
(2 - 1)	$(15,3)$	$(0+,0-,0-),(0-,0+,0-)$	$(1,1,1),(\tau,\tau,\tau)$	$\mathbf{b}_1, \mathbf{b}_2,$	$\mathbf{e}_{135}, \mathbf{e}_{246}$
(2 - 2)	$(9,9)$	$(0+,0-,0-),(0-,0+,1-)$	$(1,1,1),(\tau,\tau,\tau)$	$\mathbf{b}_1, \mathbf{b}_2+\mathbf{e}_5,$	$\mathbf{e}_{135}, \mathbf{e}_{246}$
(2 - 3)	$(17,5)$	$(0+,0-,0-),(0-,0+,0-)$	$(1,1,1),(\tau,\tau,0)$	$\mathbf{b}_1, \mathbf{b}_2,$	$\mathbf{e}_{1356}, \mathbf{e}_{24}$
(2 - 4)	$(11,11)$	$(0+,0-,0-),(0-,0+,1-)$	$(1,1,1),(\tau,\tau,0)$	$\mathbf{b}_1, \mathbf{b}_2+\mathbf{e}_{56},$	$\mathbf{e}_{1356}, \mathbf{e}_{24}$
(2 - 5)	$(7,7)$	$(0+,0-,0-),(0-,0+,\tau-)$	$(1,1,1),(\tau,\tau,0)$	$\mathbf{b}_1, \mathbf{b}_2+\mathbf{e}_6,$	$\mathbf{e}_{1356}, \mathbf{e}_{24}$
(2 - 6)	$(19,7)$	$(0+,0-,0-),(0-,0+,0-)$	$(1,1,1),(\tau,1,0)$	$\mathbf{b}_1, \mathbf{b}_2,$	$\mathbf{e}_{156}, \mathbf{e}_{234}$
(2 - 7)	$(9,9)$	$(0+,0-,0-),(0-,0+,\tau-)$	$(1,1,1),(\tau,1,0)$	$\mathbf{b}_1, \mathbf{b}_2+\mathbf{e}_6,$	$\mathbf{e}_{156}, \mathbf{e}_{234}$
(2 - 8)	$(5,5)$	$(0+,0-,0-),(0-,\tau+,\tau-)$	$(1,1,1),(\tau,1,0)$	$\mathbf{b}_1, \mathbf{b}_2+\mathbf{e}_{46},$	$\mathbf{e}_{156}, \mathbf{e}_{234}$
(2 - 9)	$(27,3)$	$(0+,0-,0-),(0-,0+,0-)$	$(0,1,1),(1,0,1)$	$\mathbf{b}_1, \mathbf{b}_2,$	$\mathbf{e}_{12}, \mathbf{e}_{134}, \mathbf{e}_{156}$
(2 - 10)	$(11,11)$	$(0+,0-,0-),(0-,0+,\tau-)$	$(0,1,1),(1,0,1)$	$\mathbf{b}_1, \mathbf{b}_2+\mathbf{e}_6,$	$\mathbf{e}_{12}, \mathbf{e}_{134}, \mathbf{e}_{156}$
(2 - 11)	$(7,7)$	$(0+,0-,0-),(0-,\tau+,\tau-)$	$(0,1,1),(1,0,1)$	$\mathbf{b}_1, \mathbf{b}_2+\mathbf{e}_{46},$	$\mathbf{e}_{12}, \mathbf{e}_{134}, \mathbf{e}_{156}$
(2 - 12)	$(3,3)$	$(\tau+,0-,0-),(0-,\tau+,\tau-)$	$(0,1,1),(1,0,1)$	$\mathbf{b}_1+\frac{1}{2}\mathbf{e}_2, \mathbf{b}_2+\mathbf{e}_{46},$	$\mathbf{e}_{12}, \mathbf{e}_{134}, \mathbf{e}_{156}$
(2 - 13)	$(21,9)$	$(0+,0-,0-),(0-,0+,0-)$	$(1,1,0),(\tau,\tau,0)$	$\mathbf{b}_1, \mathbf{b}_2,$	$\mathbf{e}_{13}, \mathbf{e}_{24}, \mathbf{e}_{56}$
(2 - 14)	$(7,7)$	$(0+,0-,0-),(0-,0+,1-)$	$(1,1,0),(\tau,\tau,0)$	$\mathbf{b}_1, \mathbf{b}_2+\mathbf{e}_5,$	$\mathbf{e}_{13}, \mathbf{e}_{24}, \mathbf{e}_{56}$
(3 - 1)	$(12,6)$	$(0+,0-,0-),(0-,0+,0-)$	$(0,\tau,1),(\tau,1,0),(1,0,\tau)$	$\mathbf{b}_1, \mathbf{b}_2,$	$\mathbf{e}_{45}, \mathbf{e}_{23}, \mathbf{e}_{16}$
(3 - 3)	$(17,5)$	$(0+,0-,0-),(0-,0+,0-)$	$(1,1,0),(\tau,\tau,0),(1,\tau,1)$	$\mathbf{b}_1, \mathbf{b}_2,$	$\mathbf{e}_{134}, \mathbf{e}_{124}, \mathbf{e}_{1456}$
(3 - 4)	$(7,7)$	$(0+,0-,0-),(0-,0+,\tau-)$	$(1,1,0),(\tau,\tau,0),(1,\tau,1)$	$\mathbf{b}_1, \mathbf{b}_2+\mathbf{e}_6,$	$\mathbf{e}_{134}, \mathbf{e}_{124}, \mathbf{e}_{1456}$
(3 - 5)	$(15,3)$	$(0+,0-,0-),(0-,0+,0-)$	$(0,1,1),(1,0,1),(\tau,\tau,\tau)$	$\mathbf{b}_1, \mathbf{b}_2,$	$\mathbf{e}_{35}, \mathbf{e}_{15}, \mathbf{e}_{246}$
(3 - 6)	$(9,9)$	$(0+,0-,0-),(0-,0+,\tau-)$	$(0,1,1),(1,0,1),(\tau,\tau,\tau)$	$\mathbf{b}_1, \mathbf{b}_2+\mathbf{e}_{56},$	$\mathbf{e}_{35}, \mathbf{e}_{15}, \mathbf{e}_{246}$
(4 - 1)	$(15,3)$	$(0+,0-,0-),(0-,0+,0-)$	$(0,\tau,1),(\tau,1,0)$, $(1,0,\tau),(1,1,1)$	$\mathbf{b}_1, \mathbf{b}_2,$	$\mathbf{e}_{45}, \mathbf{e}_{23}, \mathbf{e}_{16}, \mathbf{e}_{135}$

Table 5.2: Free fermionic realizations of all inequivalent $\mathbb{Z}_2\times\mathbb{Z}_2$ orbifold geometries [39] are suggested. In this notation $0, 1, \tau$ signify no shift, shift along the 1 direction of the torus and shift along the τ direction of the torus correspondingly, while $+/-$ signifies no twist/twist.

5.3.4 The NAHE set

Maybe the most famous free fermionic construction is the so-called NAHE set, which was first introduced in [48–51]. This set has been the basis for many phenomenological explorations of free fermionic models. It reads:

$$\left\{\mathbf{1}, \mathbf{S}, \mathbf{b}'_1, \mathbf{b}'_2, \mathbf{b}'_3\right\} , \tag{5.47}$$

The vectors $\mathbf{1}$ and $\mathbf{S}$ were defined in table 3.2; the vectors $\mathbf{b}'_s$ are given by:

$$\mathbf{b}'_1 = \left\{\psi^\mu, \chi^{1,2}, y^{3,\dots,6} \,|\, \overline{y}^{3,\dots,6}, \overline{\psi}^{1,\dots,5}, \overline{\eta}^1\right\} , \tag{5.48a}$$

$$\mathbf{b}'_2 = \left\{\psi^\mu, \chi^{3,4}, y^{1,2}, w^{5,6} \,|\, \overline{y}^{1,2}, \overline{w}^{5,6}, \overline{\psi}^{1,\dots,5}, \overline{\eta}^2\right\} , \tag{5.48b}$$

$$\mathbf{b}'_3 = \left\{\psi^\mu, \chi^{5,6}, w^{1,\dots,4} \,|\, \overline{w}^{1,\dots,4}, \overline{\psi}^{1,\dots,5}, \overline{\eta}^3\right\} . \tag{5.48c}$$

These can be expanded as

$$\mathbf{b}'_1 = \mathbf{b}_1 + \mathbf{S} + \boldsymbol{\xi}_1 , \quad \mathbf{b}'_2 = \mathbf{b}_2 + \mathbf{S} + \boldsymbol{\xi}_1 , \quad \mathbf{b}'_3 = \mathbf{b}_1 + \mathbf{b}_2 + \mathbf{e}_{1\dots6} + \mathbf{S} + \boldsymbol{\xi}_1 , \tag{5.49}$$

in terms of the basis vectors given in table 3.2. In accordance with (3.27) the generalized GSO projection phases are chosen such that

$$C\big[{}^{\mathbf{B}_a}_{\mathbf{B}_b}\big] = \begin{array}{c|ccccc} \mathbf{B}_a\backslash\mathbf{B}_b & \mathbf{1} & \mathbf{S} & \mathbf{b}'_1 & \mathbf{b}'_2 & \mathbf{b}'_3 \\ \mathbf{1} & 1 & -1 & -1 & -1 & -1 \\ \mathbf{S} & -1 & -1 & -1 & -1 & -1 \\ \mathbf{b}'_1 & -1 & -1 & -1 & -1 & -1 \\ \mathbf{b}'_2 & -1 & -1 & -1 & -1 & -1 \\ \mathbf{b}'_3 & -1 & -1 & -1 & -1 & -1 \end{array} . \tag{5.50}$$

With these input parameters, the gauge group is SO(10)×SO(6)^3×E$_8$: In particular, the SO(16) gauge fields correspond to the states $\psi^\mu\,|0\rangle_L^{\rm NS} \otimes \overline{\phi}^A\overline{\phi}^B\,|0\rangle_R^{\rm NS}$. Additional gauge bosons arise in the $\boldsymbol{\xi}_2 = \mathbf{1} + \mathbf{b}'_1 + \mathbf{b}'_2 + \mathbf{b}'_3$ sector; transforming in the **128** representation of SO(16). This enhances the gauge group to E$_8$. The charged matter consists of 48 generations of **16**-plets of $SO(10)$; 16 originating in each of the $\mathbf{b}'_i$.

Since

$$\mathbf{b}'_1 \cap \mathbf{S} = \left\{\psi^\mu, \chi^{1,2}\right\} , \qquad \mathbf{b}'_2 \cap \mathbf{S} = \left\{\psi^\mu, \chi^{3,4}\right\} , \tag{5.51}$$

the $\mathcal{N} = 4$ spacetime SUSY generated by $\mathbf{S}$ is indeed reduced to $\mathcal{N} = 1$. The phases $C\big[{}^{\mathbf{S}}_{\mathbf{b}'_s}\big] = -1$ are chosen such that the remaining gravitino is not projected out.

We begin the translation of this NAHE model to the orbifold language by taking linear combinations of the basis vectors, so that it is clear which basis vectors are Narain-like and which impose the $\mathbb{Z}_2$ orbifold actions. We can identify two Narain-like vectors via

$$\boldsymbol{\beta} = \mathbf{b}'_1 + \mathbf{b}'_2 + \mathbf{b}'_3 - \mathbf{S} = \mathbf{e}_{1\dots6} + \boldsymbol{\xi}_1 , \qquad \boldsymbol{\xi}_2 = \mathbf{1} - \mathbf{b}'_1 - \mathbf{b}'_2 - \mathbf{b}'_3 . \tag{5.52}$$

In addition, we define the twist-like elements

$$\mathbf{B}_1 = \mathbf{S} + \mathbf{b}_1' = \left\{\chi^{3,4,5,6}, y^{3,\dots,6} \,|\, \overline{y}^{3,\dots,6}, \overline{\psi}^{1,\dots,5}, \overline{\eta}^1\right\} , \tag{5.53a}$$

$$\mathbf{B}_2 = \mathbf{S} + \mathbf{b}_2' = \left\{\chi^{1,2,5,6}, y^{1,2}, w^{5,6} \,|\, \overline{y}^{1,2}, \overline{w}^{5,6}, \overline{\psi}^{1,\dots,5}, \overline{\eta}^2\right\} , \tag{5.53b}$$

which are associated with the twists θ_1 and θ_2, respectively. Since they do not involve pairs of y's and w's and they do not overlap, they can be thought of as pure twist elements with the shift gauge embeddings:

$$V_1 = \tfrac{1}{2}\,(1^5, 1, 2, 0)(0^8) \ , \qquad V_2 = \tfrac{1}{2}\,(1^5, 0, 1, 2)(0^8) \ , \tag{5.54}$$

where we have taken into account that $\mathbf{B}_s$ do not fulfill (5.18). We arrive at this form by flipping signs and adding lattice vectors. Notice that these elements are related to the standard embedding choices (5.43) as $V_s = V_s^{\mathrm{SE}} + \frac{1}{2}\,(1^8)(0^8)$.

The separation of the twists in two bunches of eight entries is possible because we have the element $\boldsymbol{\xi}_2$ which distinguishes the second eight entries from the first eight. Notice that in this case, the gauge shifts are not in the standard embedding, hence, the number of SO(10) generations does not necessarily correspond to the Hodge numbers.

In the new basis, $\left\{\mathbf{S}, \mathbf{B}_1, \mathbf{B}_2, \boldsymbol{\beta}, \boldsymbol{\xi}_2\right\}$, the generalized GSO matrix (5.50) takes the form

$$C\big[\begin{smallmatrix}\mathbf{B}_a\\ \mathbf{B}_b\end{smallmatrix}\big] = \begin{array}{c|ccccc} \mathbf{B}_a\backslash\mathbf{B}_b & \mathbf{S} & \mathbf{B}_1 & \mathbf{B}_2 & \boldsymbol{\beta} & \boldsymbol{\xi}_2 \\ \hline \mathbf{S} & -1 & -1 & -1 & -1 & -1 \\ \mathbf{B}_1 & 1 & -1 & -1 & -1 & 1 \\ \mathbf{B}_2 & 1 & -1 & -1 & -1 & 1 \\ \boldsymbol{\beta} & 1 & -1 & -1 & -1 & 1 \\ \boldsymbol{\xi}_2 & 1 & -1 & -1 & -1 & 1 \end{array} \tag{5.55}$$

from which all the orbifold phases can be read using (5.12).

The final step is to identify the Narain moduli, which are given in the third row of table 5.1. Note that even though the vectors $\boldsymbol{\xi}_1$ and $\boldsymbol{\xi}_2$ do not both appear, we can still consider the $\mathrm{E}_8\times\mathrm{E}_8$ model as the starting point of the construction because of the appearance of $\alpha_{8\times 8}$. The particular values of the rest of the moduli then place this model at a point of enhanced symmetry in the moduli space, where the lattice between the 6d and the gauge degrees of freedom is not fully factorized anymore.

5.3.5 Semi-realistic free fermionic $\mathbb{Z}_2 \times \mathbb{Z}_2$ models

In [52] a class of free fermionic models is considered. The twelve defining basis vectors are

$$\left\{\mathbf{1}, \mathbf{S}, \mathbf{e}_1, \dots, \mathbf{e}_6, \mathbf{B}_1, \mathbf{B}_2, \mathbf{z}_1, \mathbf{z}_2\right\} \tag{5.56}$$

where the first eight were defined in table 3.2; the remaining read

$$\mathbf{B}_1 = \mathbf{b}_1 + \boldsymbol{\xi}_1 \ , \quad \mathbf{B}_2 = \mathbf{b}_2 + \mathbf{e}_{56} + \boldsymbol{\xi}_1 \ , \qquad \mathbf{z}_2 = \boldsymbol{\xi}_2 - \mathbf{z} \ , \quad \mathbf{z}_1 = \mathbf{z} = \left\{\overline{\phi}^{1\dots4}\right\} \ . \tag{5.57}$$

This set spans the same additive set as our standard choice

$$\Big\{ \mathbf{S}, \mathbf{b}_1, \mathbf{b}_2, \mathbf{e}_1, \ldots, \mathbf{e}_6, \boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \mathbf{z} \Big\} \; . \tag{5.58}$$

Since we have all the elements $\mathbf{e}_i$ separately, we know that we have moved away from the special free fermionic point with enhanced gauge symmetry. Since we have the basis vectors $\mathbf{b}_s$, these models can be interpreted as the (0-1) orbifold for the standard choice of phases, like in (5.45).

The new ingredient in this model is the basis vector $\mathbf{z}$. Note that it can be combined with any of the $\mathbf{e}_i$'s of the model to be interpreted as a Wilson line. Its effect does indeed reduce the gauge symmetry of the model.

5.3.6 Free fermonic MSSM-like constructions

In this subsection, we consider some more complicated free fermionic models that were constructed in the past, and have a rich phenomenology.

An MSSM model with a symmetric orbifold interpretation

One of the earliest MSSM-like constructions in string theory was the model constructed in [53] (closely related MSSM-like models were constructed in [54]). This free fermionic model is an extension of the NAHE model discussed in section 5.3.4 with three additional basis elements:

$$\mathbf{b}_4' = \left\{ \psi^\mu, \chi^{12}, y^{36}, w^{45} \,|\, \bar{y}^{36}, \bar{w}^{45}; \bar{\psi}^{1\ldots5}, \bar{\eta}^1 \right\} \; , \tag{5.59a}$$

$$\boldsymbol{\alpha} = \left\{ \psi^\mu, \chi^{56}, y^2, w^{134} \,|\, \bar{y}^{1236}, \bar{w}^{46}; \bar{\psi}^{123}, \bar{\eta}^{12}, \bar{\phi}^{1\ldots4} \right\} \; , \tag{5.59b}$$

$$\boldsymbol{\beta} = \left\{ \psi^\mu, \chi^{34}, y^{15}, w^{26} \,|\, \bar{y}^{15}, \bar{w}^{26}; \tfrac{1}{2}\bar{\psi}^{1\ldots5}, \tfrac{1}{2}\bar{\eta}^{123}, \bar{\phi}^{34}, \tfrac{1}{2}\bar{\phi}^{1567} \right\} \; . \tag{5.59c}$$

We notice that these three elements can be modified to $\mathbf{e}_{45} = \mathbf{b}_4' - \mathbf{b}_1'$ and

$$\boldsymbol{\alpha}' = \boldsymbol{\alpha} + \mathbf{b}_3' = \left\{ y^2 w^2 \,|\, \bar{y}^{1236}, \bar{w}^{1236}; \bar{\psi}^{45}, \bar{\eta}^{123}, \bar{\phi}^{1\ldots4} \right\} \; , \tag{5.60a}$$

$$\boldsymbol{\beta}' = \boldsymbol{\beta} + \mathbf{b}_2' = \left\{ y^{25}, w^{25} \,|\, \bar{y}^{25}, \bar{w}^{25}; \tfrac{3}{2}\bar{\psi}^{1\ldots5}, \tfrac{1}{2}\bar{\eta}^{13}\, \tfrac{3}{2}\bar{\eta}^{2}, \bar{\phi}^{34}, \tfrac{1}{2}\bar{\phi}^{1567} \right\} \; , \tag{5.60b}$$

which all are Narain-like elements. Hence we see that this model admits a symmetric orbifold interpretation, in the sense that the orbifold actions act symmetrically. On the other hand, we see that the basis vectors $\boldsymbol{\alpha}'$ and $\boldsymbol{\beta}'$ are asymmetric shifts, accompanied by Wilson lines. The machinery we have developed should also apply to such models. Nevertheless, even though we can use the basis vectors above to read off the generalized vielbein E, this is one of the cases discussed in Section 5.2.2b for which it is not straightforward to bring it to a basis in which it will have the form (4.20).

A non-geometric MSSM model

Another free fermionic MSSM-like realization was constructed in [55]. This model also starts from the NAHE set and adds

$$\boldsymbol{\alpha} = \left\{y^{36}, w^{36} \,|\, \bar{y}^{1}, \bar{w}^{23456}; \bar{\psi}^{123}, \bar{\phi}^{1\ldots4}\right\} , \tag{5.61a}$$

$$\boldsymbol{\beta} = \left\{y^{15}, w^{15} \,|\, \bar{y}^{356}, \bar{w}^{124}; \bar{\psi}^{123}, \tfrac{1}{2}\bar{\eta}^{123}, \bar{\phi}^{1\ldots4}\right\} , \tag{5.61b}$$

$$\boldsymbol{\gamma} = \left\{y^{24}, w^{24} \,|\, \bar{y}^{12346}, \bar{w}^{4}; \tfrac{1}{2}\bar{\psi}^{1\ldots5}, \tfrac{1}{2}\bar{\eta}^{123}, \tfrac{1}{2}\bar{\phi}^{1567}, \bar{\phi}^{34}\right\} . \tag{5.61c}$$

All three elements are shift elements on the left-moving side: the fermions y^i and w^i appear in pairs. From the right-moving side these elements act as twists and roto-translations with twist parts that act in all six torus directions: All three elements either have only $\bar{y}^i$ or only $\bar{w}^i$ for each of the six directions. In fact, the differences $\boldsymbol{\beta} - \boldsymbol{\alpha}$ and $\boldsymbol{\gamma} - \boldsymbol{\alpha}$ are ordinary Narain-like elements. They can be understood as modifying the Narain moduli of the underlying torus compactification. Hence, there is really only one element, say $\boldsymbol{\alpha}$, that does not admit a symmetric orbifold interpretation; this model corresponds to an asymmetric orbifold and is therefore beyond the scope of this work.

5.3.7 The Blaszczyk model at the free fermionic point

Our final example considers an interesting MSSM-like model construction on a $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold of the $\mathrm{E}_8 \times \mathrm{E}_8$ string, the so-called Blaszczyk model [56]. This model was defined in two steps:

1. A six generation GUT model was constructed on the standard $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold with a specific choice of gauge shifts V_s and discrete Wilson lines A_i in the six torus directions.

2. By a freely acting $\mathbb{Z}_2$ shift, in all three two-tori simultaneously, with an accompanying Wilson line A, the GUT group was broken to the SM group and the number of generations halved.

Upstairs model matching

In detail, the upstairs model was defined by the gauge shifts

$$V_1 = \left(\tfrac{5}{4}, -\tfrac{3}{4}, -\tfrac{7}{4}, \tfrac{1}{4}, \tfrac{1}{4}, -\tfrac{3}{4}, -\tfrac{3}{4}, \tfrac{1}{4}\right)\left(0, 1, 1, 0, 1, 0, 0, -1\right) , \tag{5.62a}$$

$$V_2 = \left(-\tfrac{1}{2}, -\tfrac{1}{2}, -\tfrac{1}{2}, \tfrac{1}{2}, -\tfrac{1}{2}, -\tfrac{1}{2}, -\tfrac{1}{2}, -\tfrac{1}{2}\right)\left(\tfrac{1}{2}, \tfrac{1}{2}, 0, 0, 0, 0, 0, 4\right) , \tag{5.62b}$$

and the discrete Wilson lines

$$A_1 = \left(0^8\right)\left(0^8\right) , \tag{5.63a}$$

$$A_{2k} = \left(\tfrac{5}{4}, \tfrac{1}{4}, \tfrac{3}{4}, -\tfrac{1}{4}, -\tfrac{1}{4}, \tfrac{3}{4}, \tfrac{3}{4}, \tfrac{3}{4}\right)\left(-\tfrac{1}{4}, \tfrac{3}{4}, \tfrac{5}{4}, \tfrac{5}{4}, \tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{4}\right) , \tag{5.63b}$$

$$A_3 = \left(-\tfrac{3}{4}, -\tfrac{1}{4}, \tfrac{1}{4}, \tfrac{7}{4}, -\tfrac{1}{4}, -\tfrac{1}{4}, -\tfrac{1}{4}, -\tfrac{1}{4}\right)\left(\tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{4}, \tfrac{5}{4}, -\tfrac{3}{4}, \tfrac{1}{4}, -\tfrac{3}{4}, \tfrac{1}{4}\right) , \tag{5.63c}$$

$$A_5 = \left(-\tfrac{1}{2}, -\tfrac{1}{2}, \tfrac{1}{2}, -\tfrac{1}{2}, \tfrac{1}{2}, -\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}\right)\left(\tfrac{1}{2}, \tfrac{1}{2}, 0, 0, 0, 0, -\tfrac{1}{2}, -\tfrac{1}{2}\right) , \tag{5.63d}$$

with $k = 1, 2, 3$.

To translate this model into the free fermionic language, we begin by observing that it is an orbifold of the $E_8 \times E_8$ theory on the standard orthogonal lattice, hence the free fermionic analogue has to have the basis vectors: $\{\mathbf{S}, \mathbf{e}_1, \ldots, \mathbf{e}_6, \boldsymbol{\xi}_1, \boldsymbol{\xi}_2\}$. Since the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold actions do not involve any roto-translations, we augment the standard pure twist basis vectors $\mathbf{b}_1$ and $\mathbf{b}_2$ of table 3.2 with $2V_1$ and $2V_2$:

$$\widetilde{\mathbf{b}}_1 = \{\chi^{34}, -\chi^{56}; y^{34}, y^{56} \,|\, \bar{y}^{34}, \bar{y}^{56}\}(2V_1) \ , \tag{5.64a}$$

$$\widetilde{\mathbf{b}}_2 = \{-\chi^{12}, \chi^{56}; y^{12}, w^{56} \,|\, \bar{y}^{12}, \bar{w}^{56}\}(2V_2) \ , \tag{5.64b}$$

$$\boldsymbol{\beta}_i = \{\tfrac{1}{2}y^i, \tfrac{1}{2}w^i \,|\, \tfrac{1}{2}\bar{y}^i, \tfrac{1}{2}\bar{w}^i\}(2A_i) \ , \quad i = 1, \ldots, 6 \ . \tag{5.64c}$$

Note that we have included some minus signs in front of some of the χ^i to ensure that we satisfy the conditions (3.15), as they then precisely correspond to the orbifold consistency conditions (4.38).

There are no discrete torsion phases turned on in the orbifold description of this model, so we can make the standard choice (5.45) for the resulting free fermionic model. The only subtlety here is that in the free fermionic language not all of the above basis vectors are independent (mod 2), because $2\widetilde{\mathbf{b}}_1 = \boldsymbol{\xi}_1$. This is easily rectified by removing $\boldsymbol{\xi}_1$ from the set of basis vectors, to get a minimal set.

Downstairs model matching

The downstairs model is obtained by modding out a freely acting Wilson line, which acts in the $\frac{1}{2}(e_2 + e_4 + e_6)$ direction with

$$A = \tfrac{1}{2}\left(A_2 + A_4 + A_6\right) \ . \tag{5.65}$$

Before the freely acting shift, the model lives on the DW(0 - 1) geometry; after the freely acting element is applied, the underlying geometry is DW(1 - 1). Similarly, in the free fermionic language we have to include the element

$$\boldsymbol{\beta} = \tfrac{1}{2}\left(\boldsymbol{\beta}_2 + \boldsymbol{\beta}_4 + \boldsymbol{\beta}_6\right) \ , \tag{5.66}$$

and then select an appropriate minimal set of independent vectors.

Chapter 6

Spectral flow as a map between models

6.1 Introduction

Having explored the equivalence between free fermionic models and orbifolds, we now turn our attention to investigating more abstract constructions based on the properties of their internal CFTs. This chapter is based on [1, 2]. It discusses how a new kind of duality that was observed in $\mathbb{Z}_2 \times \mathbb{Z}_2'$ orbifold models [57–62] (spinor-vector duality) generalizes to models with an arbitrary internal CFT.

The spinor-vector duality is a duality of the massless spectra of two such orbifold models under the exchange of vectorial and spinorial representations of the $SO(10)$ GUT gauge group. This means that for a model with N_1 massless states in the **10** (vectorial) representation of $SO(10)$ and N_2 massless states in the **16** and $\overline{\mathbf{16}}$ (spinorial and anti-spinorial) representations, there is another model that has N_2 states in the **10** and N_1 in the **16** and $\overline{\mathbf{16}}$. The origin of the duality is apparent when we consider models with E_6 symmetry. The representations with respect to the $SO(10)$ subgroup are

$$\begin{aligned} \mathbf{27} &= \mathbf{16} + \mathbf{10} + \mathbf{1} \\ \text{and} \quad \overline{\mathbf{27}} &= \overline{\mathbf{16}} + \mathbf{10} + \mathbf{1} \ . \end{aligned}$$

In this case the number of **16** plus $\overline{\mathbf{16}}$ is equal to the number of **10**s so the model is self-dual under the exchange. The spinor-vector duality arises from the breaking of E_6. It is a discrete remnant of the breaking of the enhanced symmetry at the self-dual point, a feature also familiar from T-duality.

This idea appears to have some deep consequences that merit further investigation and one of the first applications was to use it to find extra Z' models where E_6 is broken without spoiling the low scale gauge coupling data [63]. The difficulty in constructing extra $U(1)$ models from string theory arises because the symmetry breaking pattern $E_6 \to SO(10) \times U(1)$ forces the $U(1)$ to be anomalous. The primary reason is that the charge assignment of the Standard Model states under anomaly free family universal extra $U(1)$ symmetries does not admit an E_6 embedding. On the other hand, this is a necessary ingredient to accommodate the gauge coupling unification data [63]. The way around the problem is to maintain in the spectrum massless **10**s and **16**s which are however located at different fixed points. These models are self-dual under the spinor-vector duality. One can then also study what happens when considering models that are not self-dual.

The connection between the spinor-vector duality and the underlying spectral flow operator was explicitly made in [62] (but the idea seems to be implicit in [59] as well). It was known from the work in [57–61] that a pair of dual models (call them $\mathcal{M}_0$ and $\mathcal{M}_1$) can be thought of as orbifolds of a single parent model $\mathcal{M}_P$ in which E_6 is not broken. The value of the discrete torsion between the $\mathbb{Z}_2$ orbifold actions effectively controls if the resulting model will be $\mathcal{M}_0$ or $\mathcal{M}_1$. The reason behind this is that one particular choice of discrete torsion maintains the massless vectorials in the spectrum while projecting out the massless spinorials, whereas the situation is reversed for a different values of the discrete torsion. In [62] it was emphasized that in the parent model, *i.e.* before any orbifold projections, all these states belong to the **27** of E_6 and therefore there is a spectral flow operator that transforms one to the other. This operator acts and relates states between a given model, in a way very similar to how the supercharge operator acts and relates states within a single model. However, no suggestion was made as to how the spectral flow could be used for the actual construction of the descendant models $\mathcal{M}_0$ and $\mathcal{M}_1$.

This was the motivation behind the investigation in [1] in which we found a concrete function (or map) f that allows the direct construction of model $\mathcal{M}_0$ and $\mathcal{M}_1$ from $\mathcal{M}_P$. Schematically:

$$\mathcal{M}_P \xrightarrow{f_0} \mathcal{M}_0, \qquad \mathcal{M}_P \xrightarrow{f_1} \mathcal{M}_1$$

As we have already discussed, one of the ways to achieve this effect is to say that f_0 is associated with one value of discrete torsion while f_1 is associated with another. However, knowing that a spectral flow operator exists within the parent model, it would be interesting to find a new way of thinking about f_0 and f_1 in a way that uses the spectral flow. The toolkit that allowed us to formalize this kind of thinking was developed by Schellekens and Yankielowicz [64] and comes under the name *simple current formalism*. We will describe it in section 6.3. In that language, and using the notation of section 6.5, the answer is $f_0 = \beta$ and $f_1 = \beta + \beta_0$, where β_0 is the (simple current associated with) the spectral flow operator. The innovative part of our work was to combine the ideas in [62] and [64] to present a method that uses the spectral flow to construct models, not just relate states within a model.

This conceptual change also clarified the way the spinor-vector duality operates and allowed us to describe how the same idea works in more general cases, such as when the models have an arbitrary rational internal CFT. The usefulness of our results is that it opens the road to investigations of $(2,0)$ models on generic Calabi-Yau compactifications and not just toroidal compactifications. It can potentially serve as a tool to make statements about the vector bundles on these manifolds.

$(2,0)$ models are of particular interest because it is known that $N=1$ spacetime supersymmetry requires (at least) $(2,0)$ world-sheet supersymmetry and because they can accommodate $SO(10)$ unification. The problem is that the space of these models is huge. For example, even though the number of $(2,2)$ Gepner models is quite tractable and they have been studied in detail [18, 64], the number of $(2,0)$ models that arise is much greater [64]. This re-enforces the idea that it would be very useful to discover relations in the space of such models.

6.2 The spectral flow

Our starting point here is generic (2,2) heterotic models with an internal CFT with c=9. The standard examples of interacting constructions are the Gepner models introduced in section 2.6.3, in which the internal CFT is a product of minimal models, but all our arguments are completely general. A general state in such a model is of the form:

$$\Phi_L \otimes \Phi_R \tag{6.1}$$

and the right-moving part which we wish to focus on is of the form

$$\Phi_R = (w)(h, Q)(p), \tag{6.2}$$

where w is an $SO(10)$ weight (o, v, s, c) and p an E_8 weight. The appearance of the $SO(10)$ and E_8 weights is because of the bosonic string map which is used to construct a modular invariant heterotic-string theory from a type II theory. It replaces the $\widehat{so}(2)_1$ Kac-Moody algebra with an $\widehat{so}(10)_1 \times (\widehat{e}_8)_1$ one [6].

The mass formula is

$$\begin{aligned} \frac{\alpha' M_R^2}{2} &= h_{\text{TOT}} - \frac{c}{24} \\ &= \frac{w^2}{2} + h + \frac{p^2}{2} + N_R - 1 \ , \end{aligned} \tag{6.3}$$

where we have used the fact that $c = 24$ for the bosonic string and we have also included the contribution N_R from the oscillators corresponding to the spacetime bosons.

The definition of an $N = 2$ SCFT is given in section 2.4 but we repeat it here for convenience. A CFT is said to have $N = 2$ world-sheet supersymmetry if it includes four fields:

$$T(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2} \ , \tag{6.4}$$

$$G^{\pm}(z) = \sum_{n\in\mathbb{Z}} G^{\pm}_{n\pm a} z^{-n-\frac{3}{2}\mp a} \ , \tag{6.5}$$

$$J(z) = \sum_{n\in\mathbb{Z}} J_n z^{-n-1} \ , \tag{6.6}$$

that satisfy the algebra:

$$\begin{aligned}
[L_m, L_n] &= (m-n)L_{m+n} + \frac{c}{12}(m^3-m)\delta_{m+n,0} \ , \\
[L_m, G^{\pm}_{n\pm a}] &= (\frac{m}{2} - n \mp a)G^{\pm}_{m+n\pm a} \ , \\
[L_m, J_n] &= -nJ_{m+n} \ , \\
[J_m, J_n] &= \frac{c}{3}m\delta_{m+n,0} \ , \\
[J_m, G^{\pm}_{n\pm a}] &= \pm G^{\pm}_{m+n\pm a} \ , \\
\{G^{+}_{m+a}, G^{-}_{n-a}\} &= 2L_{m+n} + (m-n+2a)J_{m+n} + \frac{c}{3}\Big((m+a)^2 - \frac{1}{4}\Big)\delta_{m+n,0} \ , \\
\{G^{+}_{m+a}, G^{+}_{n+a}\} &= \{G^{-}_{m-a}, G^{-}_{n-a}\} = 0 \ ,
\end{aligned} \tag{6.7}$$

where a is a real parameter that describes how the fermionic superpartners $G^{\pm}$ of T transform:

$$G^{\pm}(e^{2\pi i}z) = -e^{\mp 2\pi i a}G^{\pm}(z). \tag{6.8}$$

The algebras for a and $a+1$ are isomorphic. $a \in \mathbb{Z}$ corresponds to the R sector and $a \in \mathbb{Z} + \frac{1}{2}$ corresponds to the NS sector. A state is completely described by the eigenvalues h (called the conformal dimension) and Q (called the $U(1)$ charge) of the operators L_0 and J_0 that form the Cartan subalgebra:

$$|\phi\rangle = |h, Q\rangle. \tag{6.9}$$

We also note that the algebra is invariant under the following transformation which is known as the *spectral flow*:

$$\begin{aligned}
L^{\eta}_n &= L_n + \eta J_n + \frac{c}{6}\eta^2\delta_{n,0} \ , \\
G^{\eta\pm}_{n\pm a} &= G^{\eta\pm}_{n\pm(a+\eta)} \ , \\
J^{\eta}_n &= J_n + \frac{c}{3}\eta\delta_{n,0}.
\end{aligned} \tag{6.10}$$

This also implies the existence of a *spectral flow operator* U_η that acts on states in the following way:

$$U_\eta|h, Q\rangle = |h_\eta, Q_\eta\rangle = \big|h - \eta Q + \frac{\eta^2 c}{6}, Q - \frac{c\eta}{3}\rangle. \tag{6.11}$$

Of particular interest are the states

$$\big|\frac{3}{8}, \pm\frac{3}{2}\rangle_{\rm R} = U_{\mp\frac{1}{2}}|0,0\rangle_{\rm NS} \ , \tag{6.12}$$

because they can be combined with the s and c weight vectors of $SO(10)$ with the smallest possible length to give massless states. Indeed, such vectors are of the form

$$w = (\pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}) \tag{6.13}$$

and have $w^2 = \frac{5}{4}$. An even number of minus signs corresponds to s and an odd number of minus signs to c, as defined in appendix A.2.1. We then note from (6.3) that whenever the internal CFT has $N=2$ world-sheet supersymmetry the states

$$\pm\beta_0 = (\pm c)(\frac{3}{8}, \pm\frac{3}{2})(0). \tag{6.14}$$

will be part of the massless spectrum. These states describe gauge bosons in the **16** and $\overline{\mathbf{16}}$ of $SO(10)$ and, in conjunction with the $U(1)$ symmetry of the $N=2$ algebra, they extend $SO(10)$ to E_6. This shows that the $N=2$ superconformal algebra on the bosonic sector is associated with E_6 gauge symmetry. The states in (6.14) are an extension of the spectral flow operator of the internal CFT. We call these states the spectral flow operator as well.

6.3 The simple current formalism

In this section we revise a formalism introduced by Schellekens and Yankielowicz in [64] (see also [65]) which allows new models to be built from a given model using simple currents.

Since we already started from a (2,2) model, there will be a modular invariant partition function (MIPF) describing it. It will be of the form

$$Z[\tau,\bar{\tau}] = \sum_{i,j} \chi_i(\tau) M_{ij} \chi_j(\bar{\tau}), \tag{6.15}$$

where χ_i are the characters of the chiral algebra and M_{ij} a modular invariant. For our examples, we take this to be the partition function of the usual Gepner models, *i.e.* after the projections of the universal simple currents β_0 and β_i have been applied to ensure spacetime supersymmetry [16, 17, 21]. Nevertheless, the approach is very general and valid whenever the simple current method can be used to construct modular invariants. This includes any rational conformal field theory (RCFT)[1] and potentially some non-rational CFTs in which the chosen simple current defines a finite orbit as well. To avoid this complication we restrict ourselves to RCFTs throughout this thesis.

As explained in the introduction we are not interested in the $(2,2)$ models *per se* but rather in the $(2,0)$ that we get after breaking the E_6 symmetry on the right. A consistent and modular invariant $(2,0)$ model can be derived from a $(2,2)$ model through the simple current construction [24,25]. This is the same as the beta method for Gepner models and it practically amounts to orbifolding the original $(2,2)$ model. The result is that states not invariant under the action of the simple current are projected out and new states appear in twisted sectors. We will use both notations J and β for a simple current[2] and we will focus on simple currents that break E_6 on the right to $SO(10)$. The MIPF for the resulting model is then

$$Z[\tau,\bar{\tau}] = \sum \chi_i(\tau) M_{ik} M_{kj}(J) \chi_j(\bar{\tau}), \tag{6.16}$$

where

$$M_{kj}(J) = \frac{1}{N} \sum_{n=1}^{N_J} \delta(\Phi_k, J^n \Phi_j) \delta_{\mathbb{Z}}(Q_J(\Phi_k) + \frac{n}{2} Q_J(J)) \tag{6.17}$$

[1] A CFT is rational if it has a finite number of primary fields. In these theories, all dimensions (and the central charge) are rational numbers.

[2] Using multiplicative notation for the action of J and additive notation for the action of β.

is called a simple current modular invariant (SCMI) and N is a normalization constant ensuring that the vacuum only appears once. In practical terms, the above formula means that:

i) Only states whose left part is connected to the right through J will appear in the partition function, *i.e.* states with $\Phi_L = J^n \Phi_R = \Phi_R + n\beta$. This defines the n^{th} J-twisted sector.

ii) Only states invariant under the projection will appear in the partition function. This is expressed in the constraint $Q_J(\Phi) + \frac{n}{2} Q_J(J) \in \mathbb{Z}$. Q_J is called the monodromy charge and is defined as

$$Q_J(\Phi) = h(\Phi) + h(J) - h(J\Phi) \qquad \text{mod } 1. \tag{6.18}$$

The easiest way to see that this is the appropriate condition for invariance under the J projection is to note that the monodromy charge is conserved modulo 1 in operator products and thus implies the existence of a phase symmetry $\Phi \to e^{-2\pi i Q_J(\Phi)}\Phi$. This induces a cyclic group of order N_J. N_J is called the order of J and it can also be proven that $Q_J(\Phi)$ is quantized in units of $1/N_J$ [25].

The definition (6.18) is for any general RCFT. For Gepner models, where $\Phi = (w_\Phi)(\vec{l}_\Phi, \vec{q}_\Phi, \vec{s}_\Phi)(p_\Phi)$ and $J = (w_J)(\vec{l}_J, \vec{q}_J, \vec{s}_J)(p_J)$, it takes the explicit form:

$$Q_J(\Phi) = -w_J \cdot w_\Phi - p_J \cdot p_\Phi + \sum_{i=1}^{r} \left(\frac{-l^i_\Phi l^i_J + q^i_\Phi q^i_J}{2(k_i+2)} - \frac{s^i_\Phi s^i_J}{4} \right). \tag{6.19}$$

In this form it is easy to see that

$$Q_\beta(\Phi) = Q_\Phi(\beta) \quad \text{and} \quad Q_{\beta_1+\beta_2}(\Phi) = Q_{\beta_1}(\Phi) + Q_{\beta_2}(\Phi), \tag{6.20}$$

i.e. the monodromy charge is symmetric and linear with respect to its arguments. These properties are true in general [25].

Another thing to note is that if J and J' are simple currents then JJ' is a simple current as well. In fact, we can generalize (6.17) to the case where we orbifold by $J_1, \cdots, J_i, \cdots$ simultaneously. To simplify the notation let $\vec{n}$ label the twisted sectors and define

$$[\vec{n}]k \equiv J_1^{n_1} \cdots J_i^{n_i} \cdots \Phi_k \equiv \Phi_k + \sum_i n_i \beta_i.$$

Then the most general SCMI is [65]:

$$M_{k,[\vec{n}]k} = \frac{1}{N} \prod_i \delta_{\mathbb{Z}}(Q_{J_i}(\Phi_k) + X_{ij} n^j). \tag{6.21}$$

The matrix X is defined modulo 1 and its elements are quantized as $X_{ij} = \frac{n_{ij} \in \mathbb{Z}}{\gcd(N_i, N_j)}$. It also satisfies $X_{ij} + X_{ji} = Q_{J_i}(J_j)$. This fixes its symmetric part completely. The remaining freedom in choosing the antisymmetric part corresponds to discrete torsion [65].

6.4 Outline of the idea

We start with a particular simple current J. Any J would do, but for the reasons explained in the introduction the simple currents that we have in mind will break E_6, thus giving a $(2,0)$ model. We call the $(2,0)$ model that is derived this way $\mathcal{M}_0$. We also know that J_0 (β_0) is generically a simple current of every $(2,2)$ model since it is the spectral flow operator that enhances the symmetry to E_6 on the right. This naturally defines a whole family of models $\{\mathcal{M}\}_\alpha$ that are derived through the simple currents J, J_0 and linear combinations of them with and without discrete torsion.

The task of examining how the spectra of these models are related to each other is very fascinating and daunting at the same time. We will not attempt to carry out the analysis in its full generality here. Instead, we will restrict ourselves to the more modest goal of explaining how the mapping induced by the spectral flow J_0 (β_0) works.

6.5 Mapping induced by the spectral flow

Here we focus on the family of models $\mathcal{M}_0, \cdots, \mathcal{M}_m$ that are derived through the simple currents $J, JJ_0, \cdots, JJ_0^m$ or equivalently $\beta, \beta+\beta_0, \cdots, \beta+m\beta_0$. This family will have N_{β_0} members where N_{β_0} is the order of β_0. Our goal is to study how the massless spectra in these models are related. To that end, we take a closer look at the model $\mathcal{M}_m$.

We start by examining the untwisted sector[3]. Massless states in the original $(2,2)$ model will also belong to the $\mathcal{M}_m$ model if they survive the invariance projections. Note that

$$Q_{\beta+m\beta_0}(\Phi) = Q_\beta(\Phi) + mQ_{\beta_0}(\Phi) = Q_\beta(\Phi) \mod 1, \tag{6.22}$$

where in the last step we used the fact that $Q_{\beta_0}(\Phi) \in \mathbb{Z}$ because Φ belongs to the original $(2,2)$ model. This proves that $Q_{\beta+m\beta_0}(\Phi) \in \mathbb{Z} \Leftrightarrow Q_\beta(\Phi) \in \mathbb{Z}$ and therefore the untwisted sectors of all models in the $\mathcal{M}$ family are identical.

Let us now consider the twisted sectors. Note that models $\mathcal{M}_{m_1}$ and $\mathcal{M}_{m_2}$ will in general have a different number of twisted sectors since $\beta+m_1\beta_0$ and $\beta+m_2\beta_0$ will be of different order. Let us analyze the n^{th}-twisted sector of the $\mathcal{M}_m$ model. A very useful formula can be found by rearranging (6.18) as

$$h(\Phi+\beta) = h(\Phi) + h(\beta) - Q_\beta(\Phi) \tag{6.23}$$

and by induction:

$$h(\Phi+m\beta) = h(\Phi) + mh(\beta) - mQ_\beta(\Phi) - \frac{m(m-1)}{2}Q_\beta(\beta)\ , \tag{6.24}$$

where the equations are understood mod 1. Massless states in the n^{th}-twisted sector of $\mathcal{M}_m$ are of the form

$$\Phi_L \otimes (\tilde{\Phi}_L + n(\beta+m\beta_0))\ , \tag{6.25}$$

[3]Here and in what follows untwisted sector means untwisted with respect to the simple current that defines the model, *i.e.* states with $n=0$ in (6.25). The states might be twisted with respect to other simple currents that were present in the original (2,2) model but this does not affect our argument.

where this time we have written the tilde explicitly to remind us that we have applied the bosonic string map. In the notation of equation (6.2) this is simply [16, 17, 21]:

$$\tilde{\Phi}_L = \Phi_L + (v)(0,0)(0). \tag{6.26}$$

The massless condition gives

$$h(\Phi_L) = \frac{1}{2}, \quad h(\tilde{\Phi}_L) = 1 \quad \text{and} \quad h(\tilde{\Phi}_L + n\beta + nm\beta_0) = 1. \tag{6.27}$$

Furthermore, as explained before and as can be seen from (6.17), the states must also satisfy the invariance condition

$$Q_{\beta+m\beta_0}(\tilde{\Phi}_L) + \frac{n}{2}Q_{\beta+m\beta_0}(\beta + m\beta_0) \in \mathbb{Z}. \tag{6.28}$$

Using linearity of the monodromy charge and the fact that $Q_{\beta_0}(\tilde{\Phi}_L) \in \mathbb{Z}$ and $Q_{\beta_0}(\beta_0) \in 2\mathbb{Z}$ because $\tilde{\Phi}_L$ and β_0 belonged to the massless spectrum of the original $(2,2)$ model, the invariance condition becomes

$$Q_\beta(\tilde{\Phi}_L) + \frac{n}{2}Q_\beta(\beta) + mnQ_{\beta_0}(\beta) \in \mathbb{Z}. \tag{6.29}$$

We can also further manipulate (6.27) to derive another condition. Bearing in mind that in what follows all the calculations are mod 1, we get:

$$\begin{aligned}
0 = 1 \quad &= \quad h(\tilde{\Phi}_L + n\beta + nm\beta_0) \\
&\overset{(6.24)}{=} \quad h(\tilde{\Phi}_L + n\beta) + \underbrace{nmh(\beta_0)}_{\in\mathbb{Z}} - \underbrace{nmQ_{\beta_0}(\tilde{\Phi}_L)}_{\in\mathbb{Z}} \\
&\qquad -n^2mQ_{\beta_0}(\beta) - \underbrace{\frac{nm(nm-1)}{2}}_{\in\mathbb{Z}}\underbrace{Q_{\beta_0}(\beta_0)}_{\in\mathbb{Z}} \\
&= \quad h(\tilde{\Phi}_L + n\beta) - n^2mQ_{\beta_0}(\beta) \\
&\overset{(6.24)}{=} \quad \underbrace{h(\tilde{\Phi}_L)}_{=1=0} + nh(\beta) - nQ_\beta(\tilde{\Phi}_L) - \frac{n(n-1)}{2}Q_\beta(\beta) - n^2mQ_{\beta_0}(\beta) \\
&\overset{(6.29)}{=} \quad nh(\beta) + \frac{n}{2}Q_\beta(\beta)
\end{aligned} \tag{6.30}$$

Or in other words,

$$n\Big(h(\beta) + \frac{1}{2}Q_\beta(\beta)\Big) \in \mathbb{Z}. \tag{6.31}$$

(6.29) and (6.31) are the main results of this section. In general, these conditions are necessary but not sufficient because of the inherent uncertainty in the definition of the monodromy charge which is given mod 1. Nevertheless, the beauty of this general argument is that starting from an arbitrary $(2,0)$ model we get a handle on the massless spectrum in any twisted sector of any model in the family.

6.6 An example

The fact that these conditions are necessary provides a prime test for where *not* to look for massless states in a particular model. This can be of great importance when performing a computer scan in the space of models, so we give an example below.

Our starting point is the Gepner model $k^r = 2^6$, which is a (2,2) model. In this model the internal CFT is a product of 6 minimal models each of which has central charge $c = \frac{3k}{k+2} = \frac{3}{2}$. All states will be of the form (6.1) but this time the internal CFT state is completely described by three vectors $\vec{l}, \vec{q}$ and $\vec{s}$ so we will be using the notation $\Phi_R = (w)(\vec{l}, \vec{q}, \vec{s})(p = 0)$ instead. For the sake of the argument let us focus our attention on the massless charged spectrum in this model, which of course will fall into the fundamental (**27**) or anti-fundamental ($\overline{\mathbf{27}}$) representation of E_6. Without loss of generality, we will study states in the **27**, which under the $SO(10)$ group decomposes into $\mathbf{10} + \mathbf{16} + \mathbf{1}$. Let us briefly remind the reader that the right-moving part of such massless states will then be of the form:

- **10**s: $\Phi_R = (v)(\Phi^{\mathrm{I}})(p = 0)$ with

$$\Phi^{\mathrm{I}} \in \left\{ \underline{(0,0,0)^4(0,2,2)^2},\ \underline{(0,0,0)^2(1,-1,0)^4},\ \underline{(0,0,0)^3(0,2,2)(1,-1,0)^2} \right\} ,$$

- **16**s: $\Phi_R = (c)(\Phi^{\mathrm{II}})(p = 0)$ with

$$\Phi^{\mathrm{II}} \in \left\{ \begin{array}{c} \underline{(0,-1,-1)^4(0,1,1)^2},\ \underline{(0,-1,-1)^2(1,-2,-1)^4}, \\ \underline{(0,-1,-1)^3(0,1,1)(1,-2,-1)^2} \end{array} \right\} ,$$

- **1**s: $\Phi_R = (w = 0)(\Phi^{\mathrm{III}})(p = 0)$ with

$$\Phi^{\mathrm{III}} \in \left\{ \begin{array}{c} \underline{(0,-2,-2)^4(0,0,0)^2},\ \underline{(0,-2,-2)^2(1,-3,-2)^4}, \\ \underline{(0,-2,-2)^3(0,0,0)(1,-3,-2)^2} \end{array} \right\} ,$$

where underlining means permutations. This is found using the known h and Q values for such states from section 2.6.5 and then solving (2.77) and (2.78) for $(\vec{l}, \vec{q}, \vec{s})$.

In this model β_0 has the usual form

$$\beta_0 = (c)(0,1,1)^6(p = 0) \tag{6.32}$$

and is of order $N_{\beta_0} = 8$. We choose the simple current with which we will orbifold our theory to be

$$\beta = \ (w = 0)(2,1,-1)(0,0,0)^5(p = 0) \ , \tag{6.33}$$

which is also of order $N_\beta = 8$ and we note that $Q_\beta(\beta_0) = \frac{3}{8} \notin \mathbb{Z}$. Therefore the gauge bosons extending $SO(10)$ to E_6 are indeed projected out and we end up with a $(2,0)$ model. As explained in the previous section, this process naturally induces a whole family of models $\mathcal{M}_0, \cdots, \mathcal{M}_7$ that arise if we orbifold by $\beta, \cdots, \beta + 7\beta_0$ respectively.

The untwisted sector in all of these models will be the same and it will consist of all the states mentioned above that satisfy the invariance condition

$$Q_\beta(\Phi_R) \in \mathbb{Z} \quad \Leftrightarrow \quad \frac{-2l_1 + q_1 + 2s_1}{8} \in \mathbb{Z}. \tag{6.34}$$

For the n^{th}-twisted sector we will use equation (6.31). $h(\beta)$ can be readily calculated from the known formula for Gepner models [16, 17, 21]:

$$h = \sum_{i=1}^{r} \left(\frac{l_i(l_i + 2) - q_i^2}{4(k_i + 2)} + \frac{s_i^2}{8} \right) \tag{6.35}$$

and we find that

$$n\Big(h(\beta) + \frac{1}{2}Q_\beta(\beta)\Big) = n\Big(\frac{9}{16} + \frac{1}{2}(-\frac{5}{8})\Big) = \frac{n}{4} \in \mathbb{Z}. \tag{6.36}$$

This means that massless states can only arise in the untwisted $n = 0$ sector, which we have already studied, or in the $n = 4$ twisted sector. In the latter sector the right-moving part of the states will be of the form

$$\begin{aligned}
\Phi_R &= \tilde{\Phi}_L + 4(\beta + m\beta_0) \\
&= \tilde{\Phi}_L + 4\beta + 4m\beta_0 \\
&= \tilde{\Phi}_L + (w = 0)(0, 4, 0)(0, 0, 0)^5(p = 0) + m(w = 0)(0, 4, 0)^6(p = 0) \\
&= \begin{cases} \Phi_L + (w = 0)(0, 4, 0)(0, 0, 0)^5(p = 0) & \text{if m even} \\ \Phi_L + (w = 0)(0, 0, 0)(0, 4, 0)^5(p = 0) & \text{if m odd} \end{cases} ,
\end{aligned} \tag{6.37}$$

where we have used the properties/identifications (2.76) multiple times. A quick comparison with Φ^{I}, Φ^{II} and Φ^{III} given above shows that states of the form (6.37) cannot be massless charged states, so the spectrum consists of the states in the untwisted sector only.

Once more, the power of this method is that it allowed us to check only one twisted sector ($n = 4$) for massless states, as opposed to checking as many as seven of them for each model that we would *a priori* expect in this example.

6.7 Some further generalizations

There are many ways to generalize the above ideas to generate even more relationships in the space of $(2, 0)$ models. For example, we are not restricted to using only β_0 but the natural splitting of the states into an $SO(10)$ part, an internal $N = 2$ CFT and an E_8 part suggests that any

$$\beta_{0'} = (w)(\beta_0^{\text{CFT}})(p)$$

would generate its own orbit of $(2, 0)$ models. Furthermore, when the internal CFT can be written as a tensor product of $N = 2$ superconformal theories each term comes with a spectral flow operator β_0^i. We can then go one step further and use only some of the β_0^i's instead of the entire β_0^{CFT}.

Finally, as explained earlier, the presence of a simple current J that breaks (2,2) to $(2, 0)$ increases the possibilities even further. We can now have any linear combination of J, with any of the β's mentioned above, with or without discrete torsion, and any such simple current will create its own orbit in the space of $(2, 0)$ models.

In this chapter, we have shown explicitly how to use one of these mappings, the spectral flow β_0, to generate an entire family of models and we have derived useful expressions for the analysis of the spectra of these models.

Chapter 7

Conclusions

Understanding better the landscape of string models and eventually finding, if possible, a dynamical way to select among them is one of the most interesting, open problems in string theory. In this thesis, we have investigated aspects of the heterotic landscape and discussed relations among large classes of vacua.

A large part of the thesis was devoted to the equivalence between free fermionic models and orbifolds. Having different formulations that overlap, covering the same part of the landscape can be useful in a variety of ways. Firstly, because different groups have been performing independent computer scans of these models, it allows for a cross check between their results. Secondly, it allows us to solve certain problems in one formalism using tools from the other. Being versatile in the interplay between the two languages combines the best of the two worlds. For example, one might typically start from a free fermionic model in which it is easier to see the gauge group and then (convert to the bosonic language and) deform away from the special point to make more general statements.

In the second half of the thesis, we turned our attention to a novel idea called spinor-vector duality. In its original form, spinor-vector duality was limited to $\mathbb{Z}_2$ structures. Here, we used the language of simple currents to generalize this idea to theories with arbitrary internal RCFTs. We also elucidated the underlying spectral flow structure. Even though the spectral flow has been traditionally used to relate states within a single model, we offered a new way to look at it, allowing relations between different models. Contrary to the equivalence between free fermionic models and orbifolds discussed already, many of the models related by the spectral flow are not physically equivalent. Nevertheless, the idea of grouping together models into families according to the spectral flow orbit is quite important: the spectra of the models, though not identical, are related and we can make statements about models in the entire family by examining one representative. The grouping also offers a conceptual handle, acting as an organization principle in a vast landscape of models. We believe that having not just one, but a big selection of such mappings, as explained in the previous chapter, will prove to be an important tool in the classification of $(2,0)$ models.

Appendix A

Lattices, theta functions and other mathematical tools

A.1 Lattices

Definition. A *lattice* Λ in d dimensions is the $\mathbb{Z}$-span of d linearly independent vectors ε_i, $i = 1, \ldots, d$:

$$\Lambda = \langle \varepsilon \rangle_{\mathbb{Z}} = \left\{ \varepsilon_i \, n_i \middle| n_i \in \mathbb{Z} \right\} , \tag{A.1}$$

where in the above notation the matrix ε has the basis vectors ε_i as its columns.

Proposition. *From the above definition it is easy to see that a lattice is a subgroup of* $\mathbb{R}^d$ *which is isomorphic to* $\mathbb{Z}^d$.[1]

Definition. The *dual lattice* Λ^* of a lattice Λ is the $\mathbb{Z}$-span of the dual basis ε_i^*, $i = 1, \ldots, d$:

$$\Lambda^* = \langle \varepsilon^* \rangle_{\mathbb{Z}} = \left\{ \varepsilon_i^* \, n_i \middle| n_i \in \mathbb{Z} \right\} = \left\{ x \in \mathbb{R}^d \;\middle|\; \langle x | l \rangle \in \mathbb{Z} \; \forall l \in \Lambda \right\} . \tag{A.2}$$

The dual basis matrix ε^* satisfies $\varepsilon^* \varepsilon = \mathbb{1}_d$ and, as usual for dual descriptions, the basis vectors for the dual lattice are now the rows of ε^*. Note that for the usual Euclidean inner product, the dual matrix is simply the inverse matrix.

Definition.

- The *lattice metric*, also known as the *Gram matrix*, is

$$g = \varepsilon^T \varepsilon . \tag{A.3}$$

- Let $\mathrm{Vol}(\Lambda)$ stand for the volume of a unit cell of Λ. This is basis independent and given by

$$\mathrm{Vol}(\Lambda) = \det(\varepsilon) = \sqrt{\det(g)} . \tag{A.4}$$

- A lattice Λ is called
  - *unimodular*, if $\det(\varepsilon) = 1$,

[1] Note that some authors allow lattices to be less than full rank, *i.e.* isomorphic to $\mathbb{Z}^n$ for some $n \leq d$, but we will not follow this convention here.

– *integral*, if $\langle u|v\rangle \in \mathbb{Z}\ \forall u, v \in \Lambda$, (note that this does not imply that all entries are integers),

– *even*, if it is integral and $\langle u|u\rangle$ is even for all $u \in \Lambda$,

– *self-dual*, if $\Lambda^* = \Lambda$.

For any lattice, it is often useful to keep track of how many lattice points lie at a given distance from the origin. The generating function providing this information is the lattice theta function.

Definition. The *lattice theta function* Θ_Λ is defined as:

$$\Theta_\Lambda = \sum_{u\in\Lambda} q^{\frac{1}{2}\langle u|u\rangle} = \sum_{k\geq 0} d_k q^k \tag{A.5}$$

and the coefficients d_k give the number of points of the lattice at distance k from the origin.

A.2 Lattices and Lie algebras

Lattices have applications in many different fields but one of them that particularly merits mentioning is the connection with Lie algebras. Following Dynkin's ideas there is a one-to-one correspondence between an algebra and its root vectors (which form a lattice) and also between the representation of an algebra and its weight vectors (which also form a lattice).

We will not review this construction here, redirecting the interested reader to [66], but we will discuss the lattices derived from $SO(2n)$ and E_8 because of their particular importance for the heterotic string. To that end, we state without proof the following:

- The weights characterize a representation of a Lie algebra. The weights in a given representation differ by vectors in the root lattice (Λ_R).
- Irreducible representations (irreps) fall into **conjugacy classes**. Two different irreps are in the same conjugacy class iff

$$w_1 - w_2 \in \Lambda_R,$$

where $w_1 \in \mathcal{R}_1$, $w_2 \in \mathcal{R}_2$.

A.2.1 Conjugacy classes of $SO(2n)$

$SO(2n)$ weights belong to any of the following conjugacy classes: scalar (o), vector (v), spinor (s) and conjugate spinor (c). In what follows we describe these classes and give alternative notations that might be used for them:

- The scalar conjugacy class (root lattice):

$$(o) = \mathcal{R} = \Lambda_R = \{(k_1, \dots, k_n) : k_i \in Z \text{ and } \sum_{i=1}^{n} k_i = 0 \mod 2\}$$

Note that

$$\vec{o}^2 = 0 \mod 2 \quad \forall \vec{o} \in (o).$$

When searching for massless states we will be particularly interested in the elements with the smallest length. In this case, the smallest one is obviously $0 = (0, \ldots, 0)$ with $0^2 = 0$ and then follow the $2n(n-1)$ roots of $SO(2n)$ $w = \underline{(\pm 1, \pm 1, 0 \ldots, 0)}$ with $w^2 = 2$, where underlying means permutations.

- The vector conjugacy class:

$$(v) = \mathcal{V} = (1, 0, \ldots, 0) + \Lambda_R = \{(k_1, \ldots, k_n) : k_i \in Z \text{ and } \sum_{i=1}^{n} k_i = 1 \mod 2\}$$

Note that

$$\vec{v}^2 = 1 \mod 2 \quad \forall \vec{v} \in (v).$$

There are $2n$ elements with the smallest length: $v_0 = \underline{(\pm 1, 0, \ldots, 0)}$. They have $v_0^2 = 1$.

- The spinor conjugacy class:

$$\begin{aligned}(s) = \mathcal{S} &= (\frac{1}{2}, \ldots, \frac{1}{2}) + \Lambda_R \\ &= \{(k_1 + \frac{1}{2}, \ldots, k_n + \frac{1}{2}) : k_i \in Z \text{ and } \sum_{i=1}^{n} k_i = 0 \mod 2\}\end{aligned}$$

Note that

$$\vec{s}^2 = \frac{n}{4} \mod 2 \quad \forall \vec{s} \in (s).$$

There are 2^{n-1} elements with the smallest length: $s_0 = (\pm\frac{1}{2}, \ldots, \pm\frac{1}{2})$, with an even number of (-) signs. Their length is $s_0^2 = \frac{n}{4}$.

- The conjugate-spinor conjugacy class:

$$\begin{aligned}(c) = \mathcal{C} &= (-\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}) + \Lambda_R \\ &= \{(k_1 + \frac{1}{2}, \ldots, k_n + \frac{1}{2}) : k_i \in Z \text{ and } \sum_{i=1}^{n} k_i = 1 \mod 2\}\end{aligned}$$

Note that

$$\vec{c}^2 = \frac{n}{4} \mod 2 \quad \forall \vec{c} \in (c).$$

There are 2^{n-1} elements with the smallest length: $c_0 = (\pm\frac{1}{2}, \ldots, \pm\frac{1}{2})$, with an odd number of (-) signs. Their length is $c_0^2 = \frac{n}{4}$.

It is easy to see, using particular representatives from each class, how the conjugacy classes behave under addition. For example, from the definitions above we see that $(v) + (s) = (c)$. In general, the conjugacy classes form a group under addition. This group is isomorphic to the center of the Lie algebra. For $SO(2n)$ this is $\mathbb{Z}_4$ if n odd and $\mathbb{Z}_2 \times \mathbb{Z}_2$ if n even.

A.2.2 Conjugacy classes of E_8

E_8 only has one conjugacy class given by

$$\begin{aligned}(o)_{E_8} &= \Lambda_R = (o)_{SO(16)} \cup (s)_{SO(16)} \\ &= \{(k_1, \ldots, k_8) : k_i \in Z \text{ and } \sum_{i=1}^{8} k_i = 0 \mod 2\} \bigcup \\ &\quad \{(k_1 + \frac{1}{2}, \ldots, k_8 + \frac{1}{2}) : k_i \in Z \text{ and } \sum_{i=1}^{8} k_i = 0 \mod 2\} \end{aligned} \tag{A.6}$$

Note that

$$p^2 = 0 \mod 2 \quad \forall p \in (o)_{E_8}.$$

The element with the smallest length is obviously $p = (0, \ldots, 0)$ with $p^2 = 0$ and then follow the 248 roots of E_8 with $p^2 = 2$.

A.3 Hermite Normal Form

In general it is not immediately obvious if two different matrices ε and ε' generate the same lattice or not. ε and ε' are two different basis of the same lattice if and only if

$$\varepsilon = \varepsilon' U \ , \tag{A.7}$$

where U is an integral unimodular transformation. This can be easily seen because we expect the following operations, generating the unimodular transformations, to not change a lattice:

1. swap two columns of ε
2. multiply a column by -1
3. add an integer multiple of a column to another

For matrices with integer entries (and also trivially generalized for rational entries), it is easy to avoid any uncertainty by using the above operations to bring the matrix in Hermite Normal Form. This form is unique and it is defined by the following requirements:

Definition. A lattice with integer entries is in (column) *Hermite Normal Form (HNF)* if and only if:

- it is lower triangular,
- its diagonal entries are positive,
- in every row, the entries on the left of the diagonal are non-negative and smaller than the entry on the diagonal.

Every lattice with integer entries can be brought to HNF by multiplication from the right by a unimodular matrix. For example

$$\varepsilon = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 1 \\ 0 & -1 & 1 \end{pmatrix} \quad \sim \quad \varepsilon_{\rm HNF} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{pmatrix} \tag{A.8}$$

and

$$\varepsilon_{\rm HNF} = \varepsilon U \quad \text{where} \quad U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 1 & 1 & 1 \end{pmatrix} . \tag{A.9}$$

A tangential, yet interesting, application of the HNF is that it provides an algorithmic way to write down all sublattices $\Lambda \subset \mathbb{Z}^n$ that have a given volume $\mathrm{Vol}(\Lambda) = N$. The algorithm is the following:

1. Split N as a product of exactly n integers in all possible ways without ignoring the order (*i.e.* $1 * 2$ and $2 * 1$ are different).

2. Write the above as the diagonal elements of a lower triangular matrix, with the lower elements to be determined.

3. Fill in the lower elements in all possible ways consistent with the HNF.

Note that by examining all lattices with Vol $= 1, 2, 3, \ldots$ the above algorithm also allows us to enumerate in a consistent way, and write a basis for, all the countably infinitely many sublattices of $\mathbb{Z}^n$.

Hermite Decomposition (the procedure to bring a matrix to HNF) works even for matrices with more columns than their rank. In this case the result is simply of the form

$$\begin{pmatrix} H | 0 \end{pmatrix} , \tag{A.10}$$

with H a square matrix in HNF.

The following algorithm gives a basis for the lattice $\Lambda_1 \cap \Lambda_2$ if we have a basis B_1 of Λ_1 and B_2 of Λ_2. It is based on the fact that if $\Lambda = \Lambda_1 \cap \Lambda_2$ then $\Lambda^* = \Lambda_1^* \cup \Lambda_2^*$.

1. Find a basis B_d for Λ^* by performing a Hermite Decomposition to the extended matrix
$$\begin{pmatrix} B_1^{-1} | B_2^{-1} \end{pmatrix} , \tag{A.11}$$

2. Dualize this to obtain a basis for Λ, *i.e.* $B = B_d^{-1}$.

A.4 The Poisson resummation formula

Another aspect of lattices that is worth discussing in some detail is the interplay between Λ and Λ^*. The first simple observation is that since

$$\mathrm{Vol}(\Lambda)\mathrm{Vol}(\Lambda^*) = \det(\varepsilon)\det(\varepsilon^*) = \det(\mathbb{1}) = 1 , \tag{A.12}$$

whenever the volume of one lattice grows, the volume of the other shrinks. It is also easy to see from the definitions that

$$\Lambda_1 \subset \Lambda_2 \quad \Rightarrow \quad \Lambda_2^* \subset \Lambda_1^* \ . \tag{A.13}$$

The second observation is a much deeper formula, relating functions on Λ and Λ^*. Let

$$I(x) = \sum_{p\in\Lambda} f(p+x) = \sum_{p\in\Lambda} f(p) = I(0) \ , \quad \text{for any} \quad x \in \Lambda \ . \tag{A.14}$$

The above periodicity implies we can expand $I(x)$ in a Fourier series as

$$I(x) = \sum_{q\in\Lambda^*} e^{2\pi i x\cdot q} I^*(q) \ , \tag{A.15}$$

where

$$I^*(q) = \frac{1}{\mathrm{Vol}(\Lambda)} \int\limits_{\text{unit cell}} d^n x \ e^{-2\pi i q\cdot x} I(x) \ . \tag{A.16}$$

Combining the above we obtain

$$\sum_{p\in\Lambda} f(p) = \sum_{q\in\Lambda^*} I^*(q) \ , \tag{A.17}$$

which is known as the *Poisson resummation formula*. It allows the calculation of a sum over all lattice points using the dual lattice instead.

A version of this formula, that will be particularly useful in the following, can be derived by considering the $\mathbb{Z}^n$ lattice and a Gaussian function. It reads:

$$\sum_{p\in\mathbb{Z}^n} e^{-\pi p^T A p} = \frac{1}{\sqrt{\det(A)}} \sum_{p\in\mathbb{Z}^n} e^{-\pi p^T A^{-1} p} \tag{A.18}$$

Proof. For the $\mathbb{Z}^n$ lattice $\mathrm{Vol}(\Lambda) = 1$ and:

$$\begin{aligned} I^*(q) &= \int\limits_{\text{unit cell}} d^n x \ e^{-2\pi i q\cdot x} I(x) \\ &= \int\limits_{\text{unit cell}} d^n x \ e^{-2\pi i q\cdot x} \sum_{p\in\mathbb{Z}^n} e^{-\pi (p+x)^T A (p+x)} \\ &= \int d^n x \ e^{-2\pi i q\cdot x} e^{-\pi x^T A x} \\ &= \frac{1}{\det(A)} e^{-\pi q^T A^{-1} q} \ , \end{aligned}$$

where in the third line we combined the integration over the unit cell with the summation over $\mathbb{Z}^n$ to obtain an integral over the entire space. □

A.5 Lattices and modular invariance

As discussed in section 2.7, all physical quantities in string theory must be modular invariant. When the string construction is a lattice compactification, this is achieved by demanding that the lattice is even and self-dual. Indeed, let Λ be such a lattice and let Θ_Λ be its theta function as defined in (A.5) with $q = e^{2\pi i\tau}$. Then

$$\Theta_\Lambda(\tau+1) = \sum_{u\in\Lambda} q^{\frac{1}{2}\langle u|u\rangle} e^{\pi i\langle u|u\rangle} = \Theta_\Lambda(\tau) \ , \tag{A.19}$$

because $\langle u|u\rangle$ is an even integer. Now let B be a basis for the lattice such that $u = Bm$ with $u \in \Lambda$ and $m \in \mathbb{Z}^n$. Then

$$\begin{aligned}
\Theta_\Lambda(\tau) \quad &= \quad \sum_{u\in\Lambda} e^{\pi i\tau\langle u|u\rangle} = \sum_{m\in\mathbb{Z}^n} e^{\pi i\tau m^T B^T Bm} && \text{(A.20)}\\
&\overset{\text{(A.18)}}{=} \quad \frac{1}{\sqrt{(-i\tau)^n}\sqrt{B^T B}} \sum_{m\in\mathbb{Z}^n} e^{\frac{\pi}{i\tau} m^T B^{-1} B^{-T} m} && \text{(A.21)}\\
&\overset{\text{(A.4)}}{=} \quad \frac{1}{\sqrt{(-i\tau)^n}} \frac{1}{\mathrm{Vol}(\Lambda)} \Theta_{\Lambda^*}(-\frac{1}{\tau}) && \text{(A.22)}\\
&= \quad \frac{1}{\sqrt{(-i\tau)^n}} \Theta_{\Lambda^*}(-\frac{1}{\tau}) \ , && \text{(A.23)}
\end{aligned}$$

where in the last step we used that for self-dual lattices

$$\mathrm{Vol}(\Lambda)\mathrm{Vol}(\Lambda^*) = 1 \Rightarrow \mathrm{Vol}(\Lambda) = 1 \ . \tag{A.24}$$

The prefactor of the theta function will be canceled by the contributions of the Jacobi eta function yielding a truly modular invariant string partition function.

Once a lattice is defined, new lattices can be obtained from it in a variety of ways. One method of particular interest is to include an extra *shift*. For example, starting from a lattice with basis

$$\varepsilon = [\varepsilon_1 \ \varepsilon_2 \ \cdots \ \varepsilon_n] \ , \tag{A.25}$$

we can perform *e.g.* a shift by 1/2 in the i direction and obtain the lattice with basis

$$\varepsilon = [\varepsilon_1 \ \cdots \ \frac{1}{2}\varepsilon_i \ \cdots \ \varepsilon_n] \ . \tag{A.26}$$

The spacing of lattice points in the i direction of the new lattice is half of the original one. From our discussion in the previous section, we then expect that the spacing in the dual lattice in certain directions will have increased.

In the previous example, it was trivial to see what the new basis will be after a shift was performed. For more complicated cases, such as a shift by $\frac{1}{2}e_1 + e_2 + \frac{1}{3}e_3$, this will not be the case. Nevertheless, the following simple algorithm, valid for an arbitrary number of shifts $\{s_i\}$, gives the answer:

1. Append the shifts as columns to the original matrix ε to obtain an extended matrix

$$\tilde{\varepsilon} = [\varepsilon_1 \ \varepsilon_2 \ \cdots \ \varepsilon_n \ s_1 \ \cdots \ s_m] \ , \tag{A.27}$$

2. Bring $\tilde{\varepsilon}$ to Hermite Normal Form. It will be of the form

$$\tilde{\varepsilon}_{\text{HNF}} = [\tilde{\varepsilon}_1 \ \tilde{\varepsilon}_2 \ \cdots \ \tilde{\varepsilon}_n \ 0 \ \cdots \ 0] \ , \tag{A.28}$$

with the first n non-zero columns of the matrix providing the new basis.

Note that if we start from an even self-dual lattice and add a shift by $\frac{1}{2}\varepsilon_i$ like in the example above, then the resulting matrix will not be even and self-dual anymore (easy to see using (A.24)). The question then is: How can we use shifts to create new lattices that are still even and self-dual?

Since a shift can be thought of as a specific orbifold action with no fixed points, we know from the orbifold construction that introducing a shift will project out some states and will also create new sectors. The interesting fact is that, for such an action, these two effects combine in a way that the resulting object is still a lattice. In the following, we will study these ideas using the lattice language exclusively and see how they come about, but first we need to introduce a few more concepts.

A.6 Q-arry lattices

In this section we will focus for convenience on the archetypical lattice $\mathbb{Z}^m$. Since any lattice Λ with a basis B satisfies $\Lambda = B\mathbb{Z}^m$, *i.e.*

$$\vec{\lambda} \in \Lambda \quad \Leftrightarrow \quad \vec{\lambda} = B\vec{n} \quad \text{with} \quad \vec{n} \in \mathbb{Z}^m \ , \tag{A.29}$$

our results from $\mathbb{Z}^m$ will carry straightforwardly to any other lattice.

Our task here is to understand how to find a basis for lattices that are defined implicitly through some restrictions, *i.e.* for lattices defined as

$$\Lambda = \left\{ p \in \mathbb{Z}^m | \quad f(p) = 0 \right\} \subset \mathbb{Z}^m \ , \tag{A.30}$$

for some function f that prescribes which lattice points to keep and which not. Since

$$0 \in \Lambda \quad \text{and} \quad p_1, p_2 \in \Lambda \Rightarrow p_1 + p_2 \in \Lambda \ , \tag{A.31}$$

f has to be linear (over some appropriate field), so the most general option is

$$f(p) = Ap \bmod q \tag{A.32}$$

for some $n \times m$ matrix A with integer entries and a prime number q.

Definition. Let $A \in M(\mathbb{Z})^{n\times m}$ and q be a prime number. The lattices

$$\Lambda_q(A) = \left\{ p \in \mathbb{Z}^m | \quad A^T s = p \bmod q \quad \text{for some } s \in \mathbb{Z}^n \right\} \subset \mathbb{Z}^m \tag{A.33}$$

and

$$\Lambda_q^{\perp}(A) = \left\{ p \in \mathbb{Z}^m | \quad Ap = 0 \bmod q \right\} \subset \mathbb{Z}^m \tag{A.34}$$

are called *q-arry lattices*.

Note that

$$\Lambda_q^{\perp}(A) = q \cdot (\Lambda_q(A))^* \ . \tag{A.35}$$

The problem of finding a basis for the lattice in (A.30) is equivalent to finding a basis for the lattice $\Lambda_q^{\perp}(A)$ in (A.34). The problem is then further reduced to finding a basis B for $\Lambda_q(A)$; because of (A.35) a basis for $\Lambda_q^{\perp}(A)$ will be given as qB^{-1}.

It is easy to see by inspection (setting $s = \vec{0}$ and $s = \vec{e}_i$) that the lattices of (A.33) are generated by the columns of the matrix

$$\left(q\mathbb{1}|A^T\right) . \tag{A.36}$$

Performing a Hermite Decomposition for this matrix results to the basis B of $\Lambda_q(A)$, which is what we were after.

As a direct application of the above, we present here a concrete basis for the lattices generated by the different conjugacy classes of $SO(2n)$ as described in section A.2.1. We provide both the form used most commonly in the literature, as well as the HNF form that comes out of our algorithm:

$$B_o = \begin{pmatrix} 1 & 0 & \cdots & & 0 \\ -1 & 1 & \cdots & & 0 \\ 0 & -1 & \cdots & & 0 \\ \vdots & \vdots & \ddots & & \vdots \\ 0 & 0 & \cdots & 1 & 1 \\ 0 & 0 & \cdots & -1 & 1 \end{pmatrix}_{n\times n} \overset{\text{HNF}}{\sim} \begin{pmatrix} 1 & 0 & 0 & \cdots & & 0 \\ 0 & 1 & 0 & \cdots & & 0 \\ 0 & 0 & 1 & \cdots & & 0 \\ \vdots & \vdots & & \ddots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 1 & 1 & 1 & \cdots & 1 & 2 \end{pmatrix}_{n\times n} \tag{A.37}$$

$$B_s = \begin{pmatrix} 1 & 0 & \cdots & & 0 & 1/2 \\ -1 & 1 & \cdots & & 0 & 1/2 \\ 0 & -1 & \cdots & & 0 & 1/2 \\ \vdots & \vdots & \ddots & & & \vdots \\ 0 & 0 & \cdots & 1 & 1 & 1/2 \\ 0 & 0 & \cdots & -1 & 1 & 1/2 \\ 0 & 0 & \cdots & 0 & 0 & 1/2 \end{pmatrix}_{n\times n} \overset{\text{HNF}}{\sim} \begin{pmatrix} 1/2 & 0 & 0 & \cdots & & 0 \\ 1/2 & 1 & 0 & \cdots & & 0 \\ 1/2 & 0 & 1 & \cdots & & 0 \\ \vdots & \vdots & & \ddots & & \vdots \\ 1/2 & 0 & 0 & \cdots & 1 & 0 \\ 1/2 & 1 & 1 & \cdots & 1 & 2 \end{pmatrix}_{n\times n} \tag{A.38}$$

$$B_c = \begin{pmatrix} 1 & 0 & \cdots & & 0 & 1/2 \\ -1 & 1 & \cdots & & 0 & 1/2 \\ 0 & -1 & \cdots & & 0 & 1/2 \\ \vdots & \vdots & \ddots & & & \vdots \\ 0 & 0 & \cdots & 1 & 1 & 1/2 \\ 0 & 0 & \cdots & -1 & 1 & 1/2 \\ 0 & 0 & \cdots & 0 & 0 & -1/2 \end{pmatrix}_{n\times n} \overset{\text{HNF}}{\sim} \begin{pmatrix} 1/2 & 0 & 0 & \cdots & & 0 \\ 1/2 & 1 & 0 & \cdots & & 0 \\ 1/2 & 0 & 1 & \cdots & & 0 \\ \vdots & \vdots & & \ddots & & \vdots \\ 1/2 & 0 & 0 & \cdots & 1 & 0 \\ 3/2 & 1 & 1 & \cdots & 1 & 2 \end{pmatrix}_{n\times n} \tag{A.39}$$

A.7 Lattices in string theory

We are now ready to use the machinery developed in the previous sections to offer some insights about lattices in string model building in general and the heterotic free fermionic construction in particular.

We begin by focusing on the lattice of the gauge degrees of freedom which is 16-dimensional, so we would like to find even and self-dual 16-dimensional lattices. The idea is the following: We start with any lattice Λ_0, we find an even sublattice $\Lambda_E \subset \Lambda_0$ and finally we find a way to make a self-dual lattice while maintaining the property of being even.

As an example, let us simply start with $\Lambda_0 = \mathbb{1}_{16}$. Even sublattices of $\mathbb{1}_{16}$ will have $\text{Vol} = 0 \text{ mod} 2$. Using the algorithm after (A.9) we can write down all sublattices with a given volume and check afterwards which are even. There is only one even sublattice with Vol $= 2$ generated by:

$$b_{16} = \begin{pmatrix} 1 & 0 & \cdots & & 0 \\ 0 & 1 & \cdots & & 0 \\ \vdots & \vdots & \ddots & & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 1 & 1 & \cdots & 1 & 2 \end{pmatrix}_{16\times 16} , \tag{A.40}$$

which we recognize as the HNF of the $SO(32)$ simple roots. For reference, the HNF of the dual lattice (with the basis read as the rows) is:

$$b_{16}^{-1} = \begin{pmatrix} 1/2 & 1/2 & \cdots & & 1/2 \\ 0 & 1 & \cdots & & 0 \\ \vdots & \vdots & \ddots & & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}_{16\times 16} . \tag{A.41}$$

The lattice generated by b_{16} is not self-dual because $\det(b) = 2 \neq 1$. This means that to achieve self-duality we will have to include some shift vectors that will reduce the lattice spacing in some directions, hence reducing the volume. The simplest option is to include only one shift vector $s = \{1/2, \cdots, 1/2\}$, leading to

$$\tilde{b}_{16} = \begin{pmatrix} 1/2 & 0 & \cdots & & 0 \\ 1/2 & 1 & \cdots & & 0 \\ \vdots & \vdots & \ddots & & \vdots \\ 1/2 & 0 & \cdots & 1 & 0 \\ 1/2 & 1 & \cdots & 1 & 2 \end{pmatrix}_{16\times 16} , \tag{A.42}$$

which we recognize as the HNF of the $SO(32)$ spinor lattice, and this lattice is indeed even and self-dual.

This is the process that is inherently built-in in the AB rules of the free fermionic construction [27], even though in that case the relevant lattices are 28-dimensional. Including the vector $\mathbb{1}$ as dictated by the rules is required by modular invariance because, as we just discussed, it is equivalent to starting from an even self-dual lattice.

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