Response determination of linear dynamical systems with singular matrices: A polynomial matrix theory approach

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Abstract

An approach is developed based on polynomial matrix theory for formulating the equations of motion and for determining the response of multi-degree-of-freedom (MDOF) linear dynamical systems with singular matrices and subject to linear constraints. This system modeling may appear for reasons such as utilizing redundant DOFs, and can be advantageous from a computational cost perspective, especially for complex (multi-body) systems. The herein developed approach can be construed as an alternative to the recently proposed methodology by Udwadia and coworkers, and has the significant advantage that it circumvents the use of pseudoinverses in determining the system response. In fact, based on the theoretical machinery of polynomial matrices, a closed form analytical solution is derived for the system response that involves non-singular matrices and relies on the use of a basis of the null space of the constraints matrix. Several structural/mechanical systems with singular matrices are included as examples for demonstrating the validity of the developed framework and for elucidating certain numerical aspects.

Keywords: Linear Constrained Structural/Mechanical Systems, Multibody Systems, Singular Matrix, Closed Form Solution, Polynomial Matrix Theory.

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1. Introduction

The problem of modeling and determining the response of mechanical and structural systems whose dynamics are subject to a number of constraints is, undoubtedly, a fundamental one in analytical dynamics [1]. Dating back to the 18th century, and based on pioneering work by Lagrange [17], Gauss [8], and others, the field of (constrained) analytical dynamics has been the subject of numerous studies ever since. The present paper focuses on a rather recent approach for the formulation and solution of the equations of motion of constrained systems with potentially singular matrices, proposed in a series of papers by Udwadia and coworkers [24, 25, 26, 27, 28, 29, 21].

Specifically, the technique developed by Udwadia and Phohomsiri [28] is adapted and reformulated herein for deriving and solving the equations of motion of a structural/mechanical system with singular matrices. In this regard, for a linear dynamical system subject to a number of constraints, the explicit determination of the corresponding constraint forces as well as the formulation of the respective governing equations can be a cumbersome task, especially for complex multi-body systems; see [21, 18, 7, 3]. To address the aforementioned challenge, an alternative approach [28] advocates modeling the system dynamics by utilizing, potentially, redundant (more than the minimum number) coordinates / degrees-of-freedom (DOFs), and neglecting, initially, the effect of the constraints. This can be advantageous particularly for complex (e.g. multi-body) systems where decomposing them into a number of independently modeled subsystems, makes the formulation of the equations of each of the constituent bodies an easier task. It is only at the second stage of the approach where the constraints are included in the augmented system of equations of motion. Note, however, that due to the utilization of redundant coordinates or when "half" degree of freedom is introduced, singular (mass, stiffness, damping, etc) matrices appear, and thus, a Moore-Penrose (pseudo) inverse matrix methodology needs to be applied for determining the response of such systems. Further, the approach provides with an explicit formula for the system response acceleration, without engaging any auxiliary variables such as Lagrange multipliers. It should be also noted that the method is applicable to systems subject to holonomic and non-holonomic constraints or their combination, as well as systems where the constraint forces may or may not be ideal.

In the present paper, an alternative formulation of the method introduced by Udwadia and Phohomsiri [28] is presented. Specifically, while retaining all the advantages of their method, our approach avoids the use of the Moore – Penrose (pseudo) inverse matrix for the derivation and solution of the constrained equations of motion, and accomplishes the same task by computing a basis for the null space of constraints related matrix. If the uniqueness condition, discussed in [28], is satisfied by the constraints, the herein developed approach has the significant advantage in comparison to [28] that it involves a square and non–singular mass matrix.

Another significant contribution of the paper is the analytical solution of the resulting second and higher order linear matrix differential equation, see

[15] and [19]. In this regard, we recall a number of facts and results from the theory of polynomial and rational matrices along with their use in the study of higher order linear multivariable systems, see for example [30] and [31]. Under this framework, a closed-form system response determination formula is derived, which notably is valid even in cases of systems with singular mass matrices.

The outline of the paper is as follows: In section 2 the proposed alternative approach for deriving/formulating the equations of motion is presented. In Section 3 the necessary mathematical background as well as several elements from polynomial matrix theory are delineated. In Section 4, a general closed form solution is provided for the response determination of MDOF structural/mechanical systems with singular matrices subject to constraints. Finally, in Section 5 concluding remarks and future research directions are provided.

In what follows $\mathbb{R}, \mathbb{C}$ denote the fields of real and complex numbers respectively, while the set of $m \times n$ matrices with elements from the aforementioned fields is denoted by $\mathbb{R}^{m\times n}$ and $\mathbb{C}^{m\times n}$.

2. Equations of Motion for Constrained Structural/Mechanical Systems with Singular Matrices

Following [28], in this section, a modified / alternative technique is presented to derive the equations of motion when singular matrices, such as the mass matrix, are present. The goal of the technique presented in [28] and its modified version proposed in the present section, is the formulation of the equations of motion for a discrete dynamical system, subject to holonomic or non – holonomic and ideal or non – ideal constraints, in terms of generalized coordinates that describe its configuration.

Let the equation of motion for the unconstrained system be

$$M(q,t)\ddot{q}(t) = Q(q,\dot{q},t), \tag{1}$$

with given initial conditions $q(0) = q_0$ and $\dot{q}(0) = \dot{q}_0$, where $q(t) \in \mathbb{R}^n$ is the vector of generalized coordinates / degrees-of-freedom (DOFs) and $Q(q,\dot{q},t) \in \mathbb{R}^n$ is a known $n-$vector depending on $q, \dot{q}$ and t, expressing the *external forces* acting on the system. The matrix $M(q,t) \in \mathbb{R}^{n\times n}$ is a symmetric positive semidefinite matrix, known in engineering dynamics as the *mass* or *inertia matrix*. In Eq. (1), the singularity of the mass matrix, may be, for instance, due to the use of redundant coordinates in the modeling process that are not in fact independent with each other, or when half degree of freedom is introduced. The former case has been addressed in [20], while the latter occurs when a massless mechanical element is used. Note, however, that if the minimum number of independent coordinates is utilized (i.e. generalized coordinates) the matrix $M(q,t)$ is symmetric positive definite.

As in [26, 27, 29] we assume that the system described by Eq. (1) is subject to $m = h + s$ constraints of the form

$$\varphi_i(q,t) = 0, \quad i = 1, 2, \ldots, h, \tag{2}$$

which are holonomic and

$$\psi_i(q, \dot{q}, t) = 0, \quad i = 1, 2, \ldots, s, \tag{3}$$

possibly non-holonomic. We assume further that the initial conditions $q(0) = q_0$ and $\dot{q}(0) = \dot{q}_0$ satisfy Eqs. (2) and (3) at $t = 0$. If $\varphi_i(q, t)$ and $\psi_i(q, \dot{q}, t)$ are smooth enough, differentiating twice both sides of Eqs. (2) and once those of Eqs. (3), with respect to time, yields a vector condition of the form

$$A(q, \dot{q}, t)\ddot{q}(t) = b(q, \dot{q}, t), \tag{4}$$

where $A(q, \dot{q}, t) \in \mathbb{R}^{m\times n}$ and $b(q, \dot{q}, t) \in \mathbb{R}^m$. When the constraints are imposed on the unconstrained system of Eq. (1), additional *constraint forces* $Q^c(q, \dot{q}, t)$ appear on the right hand side of Eq. (1) to ensure that the constraints are satisfied. Thus, the equation of motion becomes

$$M(q, t)\ddot{q}(t) = Q(q, \dot{q}, t) + Q^c(q, \dot{q}, t). \tag{5}$$

For our purposes we shall assume the constraint forces are not necessarily ideal, so they can be decomposed as follows

$$Q^c(q, \dot{q}, t) = Q^c_i(q, \dot{q}, t) + Q^c_{ni}(q, \dot{q}, t), \tag{6}$$

where Q^c_i is the ideal component of the constraint force producing zero virtual work, that is $w(t)^T Q^c_i(q, \dot{q}, t) = 0$ for any virtual displacement $w(t)$, while Q^c_{ni} is the non – ideal component assumed to produce non-zero virtual work. It was shown in [26] that the work produced by the constraint forces under virtual displacements satisfies

$$w(t)^T Q^c(q, \dot{q}, t) = w(t)^T Q^c_{ni}(q, \dot{q}, t) = w(t)^T C(q, \dot{q}, t), \tag{7}$$

where $C(q, \dot{q}, t)$ is an $n-$vector describing the nature of the non–ideal constraints which can be obtained from experimental results or simple observation. Note that in general $C(q, \dot{q}, t)$ does not have to coincide with the non-ideal constraint force $Q^c_{ni}(q, \dot{q}, t)$.

In most cases explicit knowledge of the constraint forces is not available. This fact dictates the need for an alternative to Eq. (5) formulation of the equations of motion, which avoids the explicit involvement of the term $Q^c(q, \dot{q}, t)$ and is the main goal of the present section.

Before we proceed with the presentation of the main result of this section, we prove the following Lemma, which will be instrumental in the sequel.

Lemma 1. *Let $M \in \mathbb{R}^{n\times n}$ be positive semi-definite, $A \in \mathbb{R}^{m\times n}$ and $V \in \mathbb{R}^{n\times k}$ be a matrix whose columns form a basis of* $\ker A$*. Then, $\bar{M} = \begin{bmatrix} V^T M \\ A \end{bmatrix}$ has full column rank if and only if $\begin{bmatrix} M \\ A \end{bmatrix}$ has full column rank.*

Proof. (a) Assume that $\bar{M} = \begin{bmatrix} V^T M \\ A \end{bmatrix}$ has full column rank and suppose there exists a vector x, such that $\begin{bmatrix} M \\ A \end{bmatrix} x = 0$ or equivalently that $Mx = 0$ and $Ax = 0$. Clearly, such vector would also satisfy $\bar{M}x = 0$, which in view of our initial assumption would imply $x = 0$. Hence, $\begin{bmatrix} M \\ A \end{bmatrix}$ has full column rank.

(b) Reversely, let $\begin{bmatrix} M \\ A \end{bmatrix}$ have full column rank and suppose there exists a vector $x \neq 0$, such that $\bar{M}x = 0$, which in turn implies $V^T Mx = 0$ and $Ax = 0$. The last equation states that $x \in \ker A$, thus there exists a vector $z \neq 0$, such that $x = Vz$ (note that $z \neq 0$, because V is assumed to have full column rank). Now, multiply $V^T Mx = 0$ on the left by z^T, to get $z^T V^T Mx = 0$ or $x^T Mx=0$. Since, M is positive semi-definite the vector x satisfies also $Mx = 0$. This would imply that the vector $x \neq 0$, satisfies $\begin{bmatrix} M \\ A \end{bmatrix} x = 0$ which contradicts our initial assumption. Hence, $\bar{M}$ has full column rank. □

We are now ready to state and prove the main result of the present section.

Theorem 1. *Consider the system described by Eq. (1) subject to $m \leq n$ independent constraints of the form Eq. (4) such that* $\text{rank}A(q, \dot{q}, t) = m$. *Then, the acceleration of the constrained system, is uniquely determined by*

$$\ddot{q}(t) = \bar{M}(q, \dot{q}, t)^{-1} \begin{bmatrix} V(q, \dot{q}, t)^T (Q(q, \dot{q}, t) + C(q, \dot{q}, t)) \\ b(q, \dot{q}, t) \end{bmatrix}, \tag{8}$$

if and only if the matrix $\begin{bmatrix} M(q, \dot{q}, t) \\ A(q, \dot{q}, t) \end{bmatrix}$ has full column rank, where $V(q, \dot{q}, t)$ is an $n \times (n - m)$ matrix whose columns form a basis of $\ker A(q, \dot{q}, t)$, $\bar{M}(q, \dot{q}, t) = \begin{bmatrix} V(q, \dot{q}, t)^T M(q, \dot{q}, t) \\ A(q, \dot{q}, t) \end{bmatrix}$ *and $C(q, \dot{q}, t)$ is an $n-$vector describing the virtual work done by the (possibly) non-ideal constraint force via Eq. (7).*

Proof. Since the system is subject to constraints we assume the presence of constraint forces $Q^c(q, \dot{q}, t)$ acting on the system and the equation of motion is (5). As shown in [25], virtual displacement vectors, $w(t)$, satisfy

$$A(q, \dot{q}, t)w(t) = 0. \tag{9}$$

Thus, since $V(q, \dot{q}, t)$ spans $\ker A(q, \dot{q}, t)$ there exists an $(n - m)$ – vector, $\gamma(t)$, such that

$$w(t) = V(q, \dot{q}, t)\gamma(t). \tag{10}$$

Substituting the above expression of $w(t)$ in Eq. (7) (omitting the middle part) gives

$$\gamma(t)^T V(q, \dot{q}, t)^T Q^c(q, \dot{q}, t) = \gamma(t)^T V(q, \dot{q}, t)^T C(q, \dot{q}, t). \tag{11}$$

Taking into account that Eq. (7) holds for every virtual displacement $w(t)$, and hence for arbitrary $\gamma(t)$, we drop the latter from Eq. (11) to get

$$V(q,\dot{q},t)^T Q^c(q,\dot{q},t) = V(q,\dot{q},t)^T C(q,\dot{q},t). \tag{12}$$

Next, premultiply Eq. (5) by $V(q,\dot{q},t)^T$, while making use of Eq. (12), and appending Eq. (4), the following equation of motion is obtained

$$\begin{bmatrix} V(q,\dot{q},t)^T M(q,\dot{q},t) \\ A(q,\dot{q},t) \end{bmatrix} \ddot{q}(t) = \begin{bmatrix} V(q,\dot{q},t)^T (Q(q,\dot{q},t) + C(q,\dot{q},t)) \\ b(q,\dot{q},t) \end{bmatrix}. \tag{13}$$

Obviously, the above equation describes uniquely the acceleration, $\ddot{q}(t)$, of the constrained system, if and only if the $n \times n$ matrix $\bar{M}(q,\dot{q},t)$ is invertible. Notably, according to Lemma 1, this is the case if and only if the matrix $\begin{bmatrix} M(q,\dot{q},t) \\ A(q,\dot{q},t) \end{bmatrix}$ has full column rank, as assumed in the statement of the theorem. Finally, the acceleration formula Eq. (8) is easily obtained by multiplying both sides of Eq. (13) on the left by $\bar{M}(q,\dot{q},t)^{-1}$. □

The above Theorem is directly comparable with the main result of [28]. However, in the present paper, the utilization of pseudo–inverses is avoided and the proposed methodology provides a more compact form for the equation of motion. If the matrix $V(q,\dot{q},t)$ is constant, the computation of the null space basis matrix $V(q,\dot{q},t)$ can be accomplished using well established numerical techniques, such as Gaussian elimination, QR decomposition or SVD (see [11]). Moreover, it is noted that in many cases, as in the linear one which is presented in the following section, the explicit inversion of $\bar{M}(q,\dot{q},t)^{-1}$ may not be even necessary to be calculated analytically.

A secondary goal which can be easily accomplished in view of the result of Theorem 1, is the recovery of the constraint forces. As mentioned earlier constraint forces are not explicitly known in most cases. An easy way to obtain a closed expression for $Q^c(q,\dot{q},t)$, if the assumptions of Theorem 1 hold, is to substitute $\ddot{q}(t)$ in Eq. (5) using the expression in Eq. (8) and in turn solve for Q^c, i.e.

$$Q^c = M\bar{M}^{-1} \begin{bmatrix} V^T(Q+C) \\ b \end{bmatrix} - Q. \tag{14}$$

Notably, similar formulae for the recovery of the constraint forces can be found in Udwadia & Kalaba approach (see [24, 26, 27]), but not in [28] where the results are directly comparable to ours.

It is also worth noting that the proposed approach is in a sense "parallel" to the one in [2] where an alternative proof of the method of Udwadia & Kalaba (see [24, 26, 27]) for the formulation of the equations of motion , is presented. In its original form the Udwadia & Kalaba formulation is based on Gauss's principle and makes use extensive use of Moore – Penrose generalized inverses. The approach taken in [2], avoids the use of pseudoinverses, by applying a decomposition of the constraint forces along the tangential and normal flat surface in the local coordinate space, providing this way more compact equations of motion and better insight into the geometry of the problem. On the other hand, as mentioned earlier, our method is essentially a modified version of the Udwadia & Phohomsiri [28], which is based on a different approach than the one [24, 26, 27], but still involves the computation of generalized inverses. The technique presented above bypasses the use of pseudoinverses, by computing a basis $V(q, \dot{q}, t)$ for the right null space of $A(q, \dot{q}, t)$, which at a fixed instant, is no other than the tangential flat surface in which virtual displacements lie. The equation of motion (13) is then composed by two subsets of equations, the first being equations corresponding to tangential component and the other (the acceleration constraints Eq. (4)), corresponding to the normal surface in the local coordinate space.

We demonstrate the proposed technique via the following example found in [23].

Example 1. [23] We consider a unit mass particle constrained to move along a vertical circular ring of radius R under the action of gravity. The Cartesian coordinate system used has its origin at the center of the ring. The equation of the unconstrained motion of the particle is

$$\begin{bmatrix} \ddot{x} \\ \ddot{y} \end{bmatrix} = \begin{bmatrix} 0 \\ -g \end{bmatrix}, \tag{15}$$

and the (ideal) constraint for the circular motion is described by

$$x^2 + y^2 = R^2. \tag{16}$$

To formulate the equation of the constrained motion we differentiate the above constraint twice, to obtain the equation of the form (4), which in our case reads

$$\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{y} \end{bmatrix} = -(\dot{x}^2 + \dot{y}^2). \tag{17}$$

With notation used in the present section $A(q, \dot{q}, t) = \begin{bmatrix} x & y \end{bmatrix}$ and $b(q, \dot{q}, t) = -(\dot{x}^2 + \dot{y}^2)$. The analysis used so far is identical to that in the illustrative example in [23].

From this point on instead of trying to obtain an explicit expression of the ideal constraint force $Q^c = Q^c_i$, as in [23], we shall focus on the direct formulation of the equation motion. The matrix

$$V(q, \dot{q}, t) = \begin{bmatrix} y \\ -x \end{bmatrix}, \tag{18}$$

clearly forms a basis of the right null space of $A(q, \dot{q}, t)$. Furthermore, $C(q, \dot{q}, t) = 0$ since the constraint is ideal and obviously

$$\begin{bmatrix} M(q, \dot{q}, t) \\ A(q, \dot{q}, t) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ x & y \end{bmatrix}, \tag{19}$$

has full column rank. Hence, the unique equation of motion is given by Eq. (13)

$$\begin{bmatrix} y & -x \\ x & y \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{y} \end{bmatrix} = \begin{bmatrix} gx \\ -(\dot{x}^2+\dot{y}^2) \end{bmatrix}, \tag{20}$$

or equivalently,

$$\begin{aligned} \begin{bmatrix} \ddot{x} \\ \ddot{y} \end{bmatrix} &= \frac{1}{x^2+y^2} \begin{bmatrix} y & x \\ -x & y \end{bmatrix} \begin{bmatrix} gx \\ -(\dot{x}^2+\dot{y}^2) \end{bmatrix} \\ &= \frac{1}{R^2} \begin{bmatrix} gxy - x(\dot{x}^2+\dot{y}^2) \\ -gx^2 - y(\dot{x}^2+\dot{y}^2) \end{bmatrix}. \end{aligned} \tag{21}$$

This last expression of the acceleration is equivalent to Eq. (34) in [23]. Notice that the constraint force Q_i^c can be easily recovered by substituting the acceleration given by Eq. (21) into Eq. (5) and solve for Q^c. In our case, this would give

$$Q_i^c = \frac{1}{R^2} \begin{bmatrix} gxy - x(\dot{x}^2+\dot{y}^2) \\ -gx^2 - y(\dot{x}^2+\dot{y}^2) \end{bmatrix} - \begin{bmatrix} 0 \\ -g \end{bmatrix} = \frac{gy - (\dot{x}^2+\dot{y}^2)}{R^2} \begin{bmatrix} x \\ y \end{bmatrix}, \tag{22}$$

which is identical to Eq. (40) in [23].

Following [23] we assume now that the constraint is non-ideal, due to the presence of sliding friction between the ring and particle. Let the nature of the non-ideal constraint generated by sliding friction between the ring and the mass be described by

$$w^T Q^c = w^T C = -w^T \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} \frac{\mu\,|Q_i^c|}{\sqrt{\dot{x}^2+\dot{y}^2}}, \tag{23}$$

where w is the virtual displacement vector and μ is the coefficient of friction and $|Q_i^c|$ is the magnitude of the ideal component of the constraint force given by Eq. (22). The above relation states that frictional force acts on the opposite direction of the motion and its magnitude is $\mu\,|Q_i^c|$. As in [23], taking into account that due to the constraint, $x\dot{x} = -y\dot{y}$, we get

$$C = \frac{\mu\,|Q_i^c|}{\sqrt{\dot{x}^2+\dot{y}^2}} \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = -\frac{\mu\,|Q_i^c|}{R} \begin{bmatrix} -y\,\mathrm{sgn}(x) \\ |x| \end{bmatrix} \mathrm{sgn}(\dot{y}). \tag{24}$$

Using the proposed method the unique equation of motion is now given by Eq. (13)

$$\begin{bmatrix} y & -x \\ x & y \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{y} \end{bmatrix} = \begin{bmatrix} gx + \frac{\mu|Q_i^c|}{R}(y^2\mathrm{sgn}(x) + x\,|x|)\mathrm{sgn}(\dot{y}) \\ -(\dot{x}^2+\dot{y}^2) \end{bmatrix} \tag{25}$$

$$= \begin{bmatrix} gx + \mu\,|Q_i^c|\,R\,\mathrm{sgn}(x)\,\mathrm{sgn}(\dot{y}) \\ -(\dot{x}^2+\dot{y}^2) \end{bmatrix}, \tag{26}$$

hence,

$$\begin{aligned} \begin{bmatrix} \ddot{x} \\ \ddot{y} \end{bmatrix} &= \frac{1}{x^2+y^2} \begin{bmatrix} y & x \\ -x & y \end{bmatrix} \begin{bmatrix} gx + \mu\,|Q_i^c|\,R\,\mathrm{sgn}(x)\,\mathrm{sgn}(\dot{y}) \\ -(\dot{x}^2+\dot{y}^2) \end{bmatrix} \\ &= \frac{1}{R^2} \begin{bmatrix} gxy - x(\dot{x}^2+\dot{y}^2) + \mu y\,|Q_i^c|\,R\,\mathrm{sgn}(x)\,\mathrm{sgn}(\dot{y}) \\ -gx^2 - y(\dot{x}^2+\dot{y}^2) - \mu x\,|Q_i^c|\,R\,\mathrm{sgn}(x)\,\mathrm{sgn}(\dot{y}) \end{bmatrix}, \end{aligned} \tag{27}$$

which can be easily verified to be equivalent to Eq. (39) in [23].

3. Dynamics of Second and Higher Order LTI systems

In this section the tools required to obtain a closed form solution for linear MDOF systems are developed. Before proceeding with the presentation of the results regarding the dynamics of second and higher order linear time-invariant (LTI) systems, we review some terminology from the theory of polynomial matrices which play an instrumental role in the ensuing analysis. For more details on the subject we refer the reader to [30], [10], [14].

Let $\mathbb{R}[s]$ denote the ring of polynomials in the indeterminate s with real coefficients. The quotient field of the ring of real polynomials, i.e., the field of real rational functions, will be denoted by $\mathbb{R}(s)$. A real *polynomial matrix*, is a matrix whose elements are polynomials in some indeterminate s, i.e., a matrix of the form $P(s) = [p_{ij}(s)]_{m\times n}$, where $p_{ij}(s) \in \mathbb{R}[s]$. Notably, a polynomial matrix can be alternatively written in the form of a polynomial in the indeterminate s having as coefficients real constant matrices, i.e.

$$P(s) = \sum_{i=0}^{r} P_i s^i, \tag{28}$$

where $P_i \in \mathbb{R}^{m\times n}$ and $P_r \neq 0$. The degree of $P(s)$ is defined as the greatest amongst the degrees of s in the entries of the matrix, hence considering the above notation yields $\deg P(s) = r$. The set of $n \times n$ real polynomial matrices endowed with the usual matrix operations is a ring denoted by $\mathbb{R}[s]^{n\times n}$. The *normal rank* of a polynomial matrix $P(s) \in \mathbb{R}[s]^{m\times n}$, denoted by $\text{rank}_{\mathbb{R}(s)}P(s)$, is the maximum rank of $P(s)$ over all $s \in \mathbb{C}$. A polynomial matrix $P(s)$ is termed *regular*, if it is square and $\det P(s) \neq 0$ for some $s \in \mathbb{C}$, or equivalently if it is square and of full normal rank. A regular polynomial matrix for which $\det P(s) \neq 0$ for every $s \in \mathbb{C}$ is called *unimodular*, or equivalently if and only if $\det P(s) = c \neq 0$. Unimodular matrices are units of the ring of square polynomial matrices, in the sense that their inverses are polynomial matrices themselves.

A rational function with a numerator degree less than or equal (less than) to the denominator degree is called *proper* (*strictly proper*). The sets of proper and strictly proper rational functions are rings denoted by $\mathbb{R}_{pr}(s)$, $\mathbb{R}_{sp}(s)$, respectively. Clearly, $r(s) \in \mathbb{R}_{pr}(s)$ if and only if $\lim_{s\to\infty} r(s) = c \in \mathbb{R}$. Moreover, if $c = 0$, $r(s)$ is strictly proper, while in case $c \neq 0$ the function $r(s)$ is called *biproper*. The set of $m \times n$ matrices with elements in the sets $\mathbb{R}(s)$, $\mathbb{R}_{pr}(s)$ and $\mathbb{R}_{sp}(s)$ will be denoted by $\mathbb{R}(s)^{m\times n}$, $\mathbb{R}_{sp}(s)^{m\times n}$ and $\mathbb{R}_{sp}(s)^{m\times n}$. A square proper rational matrix $R(s)$ is called *biproper* if $\lim_{s\to\infty} |R(s)| = c \neq 0$. Biproper matrices serve as units on the ring of square proper rational matrices, in the sense that their inverses are proper rational matrices as well.

We are now in position to present a brief review of a series of results regarding the dynamical interpretation of the structural invariants of polynomial matrices associated to LTI second and higher order systems. The majority of the results presented below can be found in [30]. Next, consider systems of the form

$$P(\rho)\xi(t) = f(t), \tag{29}$$

for $t \geq 0$, where in general $P(s) = \sum_{i=0}^{r} P_i s^i \in \mathbb{R}[s]^{n\times n}$ and $\text{rank}_{\mathbb{R}(s)} P(s) = n$, $\xi(t)$ is the *pseudo-state* vector, $f(t)$ is a *exogenous input* or *excitation* and $\rho = \frac{d}{dt}$ is the differential operator (using right hand side differentiation at the origin). It is also assumed that both ξ, f are smooth functions on $\mathbb{R}^+$, that is they are arbitrarily often differentiable on $\mathbb{R}^+ = [0, +\infty)$. In the special case where $f(t) = 0$, we shall refer to Eq. (29) as *homogeneous*.

A distinctive feature of the response of polynomial systems is without doubt the possibility of impulsive behavior. The presence of non-trivial zero structure at $s = \infty$ in a polynomial matrix $P(s)$, associated with a LTI system, gives rise to *impulsive behavior*. Impulsive behavior refers to solutions which are not functional, but rather distributional and, in particular, linear combinations of the Dirac δ generalized function and its distributional derivatives. The detailed presentation of the distributional framework is beyond the scope of the present paper. For more details on the subject we refer the reader to [9], [12], [13] and references therein. Further, the purpose of the present paper is to exploit and apply polynomial matrix theory concepts and tools to the case of structural dynamical systems. In this regard, necessary and sufficient conditions for the avoidance of the, generally undesirable, discontinuous or even impulsive behavior are provided as well.

In order to obtain the response formula of Eq. (29), it is instrumental to introduce some algebraic tools which will be useful in the ensuing analysis. Since $P(s)$ is assumed to be regular, its determinant which is a non-zero finite degree polynomial vanishes only finite on a finite set of points in $\mathbb{C}$. Thus, its inverse exists for almost every $s \in \mathbb{C}$, that is for all but a finite number of points in $\mathbb{C}$, and it is in general a rational matrix. In case $P(s)$ is non-regular, existence and uniqueness of the solution of Eq. (29) for given initial conditions and excitation is not guaranteed (see [22] for a recent survey on the subject), indicating thus insufficient or poor modeling of the underlying system. Applying polynomial divisions between the numerators, and the denominators of the elements of $P(s)^{-1}$, we can obtain the following decomposition

$$P(s)^{-1} = H_{pol}(s) + H_{sp}(s), \tag{30}$$

where $H_{pol}(s) \in \mathbb{R}[s]^{n\times n}$ is a polynomial matrix and $H_{sp}(s) \in \mathbb{R}_{sp}(s)^{n\times n}$ is a strictly proper rational matrix. It is worth noticing that the presence of the non–zero polynomial part $H_{pol}(s)$ of $P(s)^{-1}$, is strongly related to the presence of zeros at $s = \infty$ in $P(s)$ (see [30]). Considering the above decomposition, the Laurent expansion of $P(s)^{-1}$ about $s = \infty$ will be as follows

$$P(s)^{-1} = \sum_{i=-\infty}^{\nu} H_i s^i. \tag{31}$$

The terms H_i in the above expansion are known in the literature as the *Markov parameters* of the system (see for instance [14]) defined in Eq. (29), and they can be effectively computed using the technique shown in [6].

The next important tool for deriving the response formula of Eq. (29) is the (generalized) state space realization of $H_{sp}(s)$ and $H_{pol}(s)$. Starting with $H_{sp}(s)$, there are many techniques available in the literature (see for instance [14] or [30]) for constructing a *minimal state space realization* of a strictly proper transfer function, defined as follows

Definition 1. [14, Sec.6.2.2] Let $G(s) \in \mathbb{R}_{sp}(s)^{m\times n}$ with $\mathrm{rank}_{\mathbb{R}(s)} = \min\{m, n\}$ be a strictly proper matrix. A of $G(s) \in \mathbb{R}_{sp}(s)^{m\times n}$ is a triple of matrices $(A_F, B_F, C_F) \in \mathbb{R}^{\sigma\times\sigma} \times \mathbb{R}^{\sigma\times n} \times \mathbb{R}^{m\times\sigma}$ such that

$$G(s) = C_F(sI - A_F)^{-1}B_F. \tag{32}$$

Such a state space realization is called if A_F has the smallest possible dimensions amongst all realizations of $G(s)$.

Minimal state space realizations of a given strictly proper rational matrix are characterized by the following result.

Proposition 1. *[30, Prop. 1.88] A state space realization $(A_F, B_F, C_F) \in \mathbb{R}^{\sigma\times\sigma} \times \mathbb{R}^{\sigma\times n} \times \mathbb{R}^{m\times\sigma}$ of $G(s) \in \mathbb{R}_{sp}(s)^{m\times n}$ is minimal if and only if*

$$rank \begin{bmatrix} B_F & A_F B_F & \cdots & A_F^{\sigma-1}B_F \end{bmatrix} = rank \begin{bmatrix} C_F \\ C_F A_F \\ \vdots \\ C_F A_F^{\sigma-1} \end{bmatrix} = \sigma. \tag{33}$$

It is worth noticing that the dimension of A_F in a minimal state space realization of $H_{sp}(s)$ is equal to $\sigma = \deg |P(s)|$. It can be shown [14, Sec. 6.3.3], that the spectrum of the matrix A_F incorporates the finite zero structure of $P(s)$. Moreover, it can be easily verified (see [30, Sec. 4.2.3]) that

$$H_{-i} = C_F A_F^{i-1} B_F, \quad i = 1, 2, 3, \ldots \tag{34}$$

The polynomial part $H_{pol}(s)$ can be "realized" following a similar procedure, by applying a cumbersome manipulation (see [30, Sec. 4.2.3], [32]). As above, consider a minimal state space realization $(A_\infty, B_\infty, C_\infty) \in \mathbb{R}^{\mu\times\mu} \times \mathbb{R}^{\mu\times n} \times \mathbb{R}^{m\times\mu}$ of the strictly proper matrix $s^{-1}H_{pol}(s^{-1})$. Thus

$$s^{-1}H_{pol}(s^{-1}) = C_\infty(sI - A_\infty)^{-1}B_\infty,$$

or equivalently,

$$H_{pol}(s) = C_\infty(I - sA_\infty)^{-1}B_\infty, \tag{35}$$

with the minimality conditions

$$rank \begin{bmatrix} B_\infty & A_\infty B_\infty & \cdots & A_\infty^{\mu-1}B_\infty \end{bmatrix} = rank \begin{bmatrix} C_\infty \\ C_\infty A_\infty \\ \vdots \\ C_\infty A_\infty^{\mu-1} \end{bmatrix} = \mu \tag{36}$$

satisfied. Note that the matrix A_∞ is nilpotent by construction. A triple $(A_\infty, B_\infty, C_\infty)$ satisfying Eqs. (35) and (36) is termed *irreducible at* $s = \infty$ *generalized state space realization* of $H_{pol}(s)$. An irreducible generalized state space realization of $H_{pol}(s)$ captures the zero structure at infinity of the polynomial matrix $P(s)$ (see [30, Sec. 4.2], [32]) and considering Eqs. (35) and (31) it is easy to verify that

$$H_i = C_\infty A_\infty^i B_\infty, \quad i = 0, 1, \ldots, \nu. \tag{37}$$

In the light of the above results, the smooth response of Eq. (29) is described by the following.

Theorem 2. *Consider the system described by Eq. (29), where* $P(s) \in \mathbb{R}[s]^{n\times n}$ *and* $\det P(s) \neq 0$ *for some* $s \in \mathbb{C}$*. Then, for initial conditions*

$$x_0 = \begin{bmatrix} \xi(0^-) \\ \xi^{(1)}(0^-) \\ \vdots \\ \xi^{(r-1)}(0^-) \end{bmatrix}, \qquad f_0 = \begin{bmatrix} f(0^-) \\ f^{(1)}(0^-) \\ \vdots \\ f^{(\nu-1)}(0^-) \end{bmatrix}, \tag{38}$$

satisfying

$$\bar{H}\mathcal{Y}x_0 = \hat{H}f_0, \tag{39}$$

and smooth on $\mathbb{R}^+$ *excitation* $f(t)$*, the response of Eq. (29) is given by*

$$\xi(t) = C_F e^{A_F t} Q_F \mathcal{X} x_0 + \int_0^t C_F e^{A_F \tau} B_F f(t-\tau) d\tau + \sum_{i=0}^{\nu} C_\infty A_\infty^i B_\infty f^{(i)}(t), \tag{40}$$

where the triples (A_F, B_F, C_F) *and* $(A_\infty, B_\infty, C_\infty)$ *are respectively, a minimal state space realization of* $H_{sp}(s)$*, and an irreducible at* $s = \infty$ *generalized state space realization of the* $H_{pol}(s)$*, and*

$$\mathcal{X} = \begin{bmatrix} P_r & 0 & \ldots & 0 \\ P_{r-1} & P_r & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ P_1 & P_2 & \ldots & P_r \end{bmatrix}, \quad \mathcal{Y} = \begin{bmatrix} P_0 & P_1 & \cdots & P_{r-1} \\ 0 & P_0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & P_1 \\ 0 & \cdots & 0 & P_0 \end{bmatrix}, \tag{41}$$

$$\bar{H} = \begin{bmatrix} H_v & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & H_v \\ H_2 & \ddots & \ddots & \vdots \\ H_1 & H_2 & \cdots & H_r \end{bmatrix}, \quad \hat{H} = \begin{bmatrix} H_\nu & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ H_2 & \ddots & \ddots & 0 \\ H_1 & H_2 & \cdots & H_\nu \end{bmatrix}, \tag{42}$$

$$Q_F = \left[\begin{array}{cccc} A_F^{r-1} B_F, & A_F^{r-2} B_F, & \ldots, & B_F \end{array} \right]. \tag{43}$$

Proof. Following similar lines with [30, Sec. 4.2, 4.3] to derive the response formula Eq. (39), we apply Laplace transform on both sides of Eq. (29), to take

$$P(s)\hat{\xi}(s) - \left[s^{r-1}I, \ldots, sI, I\right] \mathcal{X} x_0 = \hat{f(s)}, \tag{44}$$

where $\hat{\xi}(s)$, $\hat{f}(s)$ are Laplace transforms of $\xi(t)$ and $f(t)$ respectively. Next, Eq. (44) is solved with respect to $\hat{\xi}(s)$ by premultiplying both sides by $P(s)^{-1}$. This results in

$$\hat{\xi}(s) = \underbrace{P(s)^{-1} \left[s^{r-1}I, \ldots, sI, I\right] \mathcal{X} x_0}_{\hat{\xi}_{free}(s)} + \underbrace{P(s)^{-1} \hat{f(s)}}_{\hat{\xi}_{dyn}(s)}, \tag{45}$$

from which it is clear that the Laplace transformed response $\hat{\xi}(s)$ consists of two parts, the free response $\hat{\xi}_{free}(s)$ which is due to the presence of non-zero initial conditions, and the dynamic response $\hat{\xi}_{dyn}(s)$ due to presence of the excitation $f(t)$.

We deal first with the dynamic response $\hat{\xi}_{dyn}(s)$. In view of Eq. (30) we may write

$$\hat{\xi}_{dyn}(s) = H_{sp}(s)\hat{f}(s) + H_{pol}(s)\hat{f}(s). \tag{46}$$

Now, substitute in the above expression $H_{sp}(s)$ using its minimal realization (A_F, B_F, C_F) as in Eq. (32) and $H_{pol}(s)$ using its irreducible at $s = \infty$ realization $(A_\infty, B_\infty, C_\infty)$ as in Eq. (37), to obtain

$$\hat{\xi}_{dyn}(s) = C_F(sI - A_F)^{-1} B_F \hat{f}(s) + \sum_{i=0}^{v} s^i C_\infty A_\infty^i B_\infty \hat{f}(s). \tag{47}$$

Finally, apply inverse Laplace transform on both sides of Eq. (47), yielding

$$\xi_{dyn}(t) = \int_0^t C_F e^{A_F \tau} B_F f(t-\tau) d\tau + \sum_{i=0}^{\nu} C_\infty A_\infty^i B_\infty f^{(i)}(t) + \left[\delta^{(v-1)}(t)I, \ldots, \delta^{(1)}(t)I, \delta(t)I\right] \hat{H} f_0, \tag{48}$$

where $\delta^{(i)}(t)$ denotes the $i-$th (distributional) derivative of Dirac's δ distribution.

The free response $\hat{\xi}_{free}(s)$ is in general a rational vector, thus it can be decomposed into the sum of a polynomial and a strictly proper part. This decomposition is crucial for the computation of the free response in the time domain, since polynomial functions in the frequency domain give rise to impulses, i.e. Dirac δ distributions and its derivatives in the time domain.

As a first step we substitute the Laurent expansion of $P(s)^{-1}$ at $s = \infty$, given in Eq. (31) in

$$\hat{\xi}_{free}(s) = \left(\sum_{i=-\infty}^{\nu} H_i s^i\right)\left[s^{r-1}I, \ldots, sI, I\right]\mathcal{X}x_0 \tag{49}$$

$$= \left[\sum_{i=-\infty}^{\nu} H_i s^{i+r-1}, \ldots, \sum_{i=-\infty}^{\nu} H_i s^{i+1}, \sum_{i=-\infty}^{\nu} H_i s^i\right]\mathcal{X}x_0 \tag{50}$$

$$= \sum_{i=-\infty}^{\nu+r-1} s^i \left[H_{i-r+1}, \ldots, H_{i-1}, H_i\right]\mathcal{X}x_0 \tag{51}$$

$$= \underbrace{\sum_{i=0}^{\nu+r-1} s^i \left[H_{i-r+1}, \ldots, H_i\right]\mathcal{X}x_0}_{\hat{\xi}^{pol}_{free}(s)} + \underbrace{\sum_{i=1}^{\infty} s^{-i} \left[H_{-i-r+1}, \ldots, H_{-i}\right]\mathcal{X}x_0}_{\hat{\xi}^{sp}_{free}(s)}. \tag{52}$$

In view of Eq. (34) the strictly proper part takes the form

$$\hat{\xi}^{sp}_{free}(s) = \sum_{i=1}^{\infty} s^{-i}\left[C_F A_F^{i+r-2} B_F, \ldots, C_F A_F^{i-1} B_F\right]\mathcal{X}x_0 \tag{53}$$

$$= \sum_{i=1}^{\infty} s^{-i} C_F A_F^{i-1}\left[A_F^{r-1} B_F, \ldots, B_F\right]\mathcal{X}x_0 \tag{54}$$

$$= \sum_{i=1}^{\infty} s^{-i} C_F A_F^{i-1} Q_F \mathcal{X}x_0 \tag{55}$$

$$= C_F(sI - A_F)^{-1} Q_F \mathcal{X}x_0. \tag{56}$$

Applying inverse Laplace transform on the above expression we obtain

$$\xi^{sp}_{free}(t) = C_F e^{A_F t} Q_F \mathcal{X}x_0. \tag{57}$$

The polynomial part $\hat{\xi}^{pol}_{free}(s)$ can be simplified by taking into account (for more details see [30, Sec. 4.2] or [31]) that

$$\left[H_{i-r+1}, \ldots, H_i\right]\mathcal{X} + \left[H_{i+1}, \ldots, H_{i+r}\right]\mathcal{Y} = 0, \quad \text{for } i = 0, 1, \ldots, v + r - 1, \tag{58}$$

and particularly that

$$\left[H_{i-r+1}, \ldots, H_i\right]\mathcal{X} = 0, \text{ for } i = v, v+1, \ldots, v + r - 1, \tag{59}$$

because $H_k = 0$, for $k > v$. Thus, we can write

$$\hat{\xi}^{pol}_{free}(s) = -\sum_{i=0}^{\nu-1} s^i \left[H_{i+1}, \ldots, H_{i+r}\right]\mathcal{Y}x_0, \tag{60}$$

while the corresponding time domain response is

$$\xi^{pol}_{free}(t) = -\sum_{i=0}^{\nu-1} \delta^{(i)}(t) \left[H_{i+1}, \ldots, H_{i+r} \right] \mathcal{Y} x_0$$
$$= -\left[\delta^{(v-1)}(t)I, \ldots, \delta^{(1)}(t)I, \delta(t)I \right] \bar{H} \mathcal{Y} x_0. \tag{61}$$

The free response formula can be found in [30, Sec. 4.2] or [31], using slightly different notation.

With the above setup, the overall response of Eq. (29) is

$$\xi(t) = C_F e^{A_F t} Q_F \mathcal{X} x_0 - \left[\delta^{(v-1)}(t)I, \ldots, \delta^{(1)}(t)I, \delta(t)I \right] \bar{H} \mathcal{Y} x_0 +$$
$$+ \int_0^t C_F e^{A_F \tau} B_F f(t-\tau) d\tau + \sum_{i=0}^{\nu} C_\infty A_\infty^i B_\infty f^{(i)}(t) +$$
$$+ \left[\delta^{(v-1)}(t)I, \ldots, \delta^{(1)}(t)I, \delta(t)I \right] \hat{H} f_0. \tag{62}$$

Clearly the response given by Eq. (62) may involve impulses. Since we are interested only in functional solutions of Eq. (29), the initial condition vectors x_0, f_0 have to be appropriately chosen, so that the terms involving Dirac delta's in Eq. (62) are eliminated, that is, if and only if

$$- \left[\delta^{(v-1)}(t)I, \ldots, \delta^{(1)}(t)I, \delta(t)I \right] \bar{H} \mathcal{Y} x_0 +$$
$$+ \left[\delta^{(v-1)}(t)I, \ldots, \delta^{(1)}(t)I, \delta(t)I \right] \hat{H} f_0 = 0, \tag{63}$$

which is satisfied if and only if Eq. (39) holds. Under this condition, the response formula Eq. (62) simplifies to Eq. (40). We should note that both the response formula Eq. (62) and the compatibility condition (40) can be found under slightly different formulation in [16]. □

Another point worth noticing, which plays a key role in the study of the response of constrained linear mechanical systems, is that when the leading coefficient matrix of $P(s)$ is invertible, then $H_i = 0$ for all $i = -r+1, -r+2, \ldots, \nu$ (see [31], Remark 3). If this is the case, both matrices $\bar{H}$, $\hat{H}$ in Eq. (39) vanish. As a result, the response of Eq. (29) is smooth for any choice of the initial conditions vectors x_0 and f_0.

4. Application to Constrained LTI Structural Systems with Singular Matrices

In this subsection, we focus on LTI structural systems subject to constraints of a particular linear form. Our aim is to provide an explicit formulation of the constrained equations of motion presented in Section 2, for this case. In this regard, consider an $n-$DOF structural system, which is described by the second – order linear differential equations

$$M\ddot{q}(t) + C\dot{q}(t) + Kq(t) = f(t), \tag{64}$$

with given initial conditions $q(0) = q_0$ and $\dot{q}(0) = \dot{q}_0$, where M, C and K are $n \times n$ positive semi-definite matrices representing the mass, (viscous) damping, and stiffness coefficients of the system, respectively. The $n-$vector $q(t)$ represents the coordinates of the system, and $f(t)$ is the externally applied force vector. As discussed previously singular M, C, and K matrices may appear for various reasons such as the use of redundant coordinates/DOFs; see also [3, 28] for a related discussion. It is further assumed that the system is subject to ideal constraints which after appropriate differentiation with respect to t (see discussion in Section 2) take the form

$$A_2\ddot{q}(t) + A_1\dot{q}(t) + A_0q(t) = g(t), \tag{65}$$

where $A_i \in \mathbb{R}^{m\times n}, i = 0, 1, 2$, with $\text{rank}A_2 = m$ and $g(t) \in \mathbb{R}^m$. Obviously, Eqs. (64) and (65) are special cases of Eqs. (1) and (4), respectively. Let V be a $n \times (n - m)$ matrix whose columns form a basis of $\ker(A_2)$. Then, according to the discussion in the previous section, we can pre-multiply Eq. (64) by V^T and append Eq. (65) to obtain the constrained equations of motion

$$\bar{M}\ddot{q}(t) + \bar{C}\dot{q}(t) + \bar{K}q(t) = \bar{f}(t), \tag{66}$$

where

$$\bar{M} = \begin{bmatrix} V^TM \\ A_2 \end{bmatrix}, \qquad \bar{C} = \begin{bmatrix} V^TC \\ A_1 \end{bmatrix}, \qquad \bar{K} = \begin{bmatrix} V^TK \\ A_0 \end{bmatrix}, \tag{67}$$

are square $n \times n$, real matrices and

$$\bar{f}(t) = \begin{bmatrix} V^Tf(t) \\ g(t) \end{bmatrix}. \tag{68}$$

Notice that in this particular case, we have assumed that the constraints are ideal, so the term $C(q, \dot{q}, t)$ in Eq. (13) vanishes. For ease of notation we may set $P(s) = s^2\bar{M} + s\bar{C} + \bar{K}$, and rewrite Eq. (66) in a more compact form as

$$P(\rho)q(t) = \bar{f}(t). \tag{69}$$

Now Theorem 2 can be applied to obtain smoothness conditions for the response and determine it analytically. In this regard, to have a smooth response, the initial conditions must satisfy Eq. (39), which in our case takes the form

$$\begin{bmatrix} H_v & 0 \\ \ddots & H_v \\ H_2 & \ddots \\ H_1 & H_2 \end{bmatrix} \begin{bmatrix} \bar{K} & \bar{C} \\ 0 & \bar{K} \end{bmatrix} \begin{bmatrix} q(0^-) \\ \dot{q}(0^-) \end{bmatrix} =$$

$$= \begin{bmatrix} H_\nu & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ H_2 & \ddots & \ddots & 0 \\ H_1 & H_2 & \cdots & H_\nu \end{bmatrix} \begin{bmatrix} f(0^-) \\ f^{(1)}(0^-) \\ \vdots \\ f^{(\nu-1)}(0^-) \end{bmatrix}. \quad (70)$$

With the above setup, we may determine the response using Eq. (40), which takes the form

$$q(t) = C_F e^{A_F t} \begin{bmatrix} A_F B_F & B_F \end{bmatrix} \begin{bmatrix} \bar{M} & 0 \\ \bar{C} & \bar{M} \end{bmatrix} \begin{bmatrix} q_0 \\ \dot{q}_0 \end{bmatrix} +$$

$$+ \int_0^t C_F e^{A_F \tau} B_F \bar{f}(t-\tau) d\tau + \sum_{i=0}^{\nu} C_\infty A_\infty^i B_\infty \bar{f}^{(i)}(t). \quad (71)$$

Further, as already noted in section 2, the matrix $\bar{M}$ is non–singular if and only if $\begin{bmatrix} M \\ A_2 \end{bmatrix}$ has full column rank. The above developed response determination technique for systems with singular matrices is demonstrated in the following via numerical examples pertaining to structural/mechanical systems. In both examples the systems presented are with singular mass matrix, since the first example is modeled with using redundant coordinates, and the second because of the presence of a "half" oscillator.

Example 2. Consider the simplified model of a quarter car suspension system shown in Figure 1.(a), where m_2 is 1/4 of the car mass, m_1 is the wheel mass, k_1, k_2 are the stiffness coefficients of the tire and the suspension respectively and c is the damping coefficient of the suspension. It is assumed that the displacements $x(t), y(t)$ are zero when the system is in static equilibrium, whereas $u(t)$ represents the road profile.

The following parameters values are considered, i.e.

$$m_1 = 30\,\text{Kg}, \qquad m_2 = 250\,\text{Kg}, \qquad c = 100\,\text{Ns/m}, \quad (72)$$

$$k_1 = 2000\,\text{N/m}, \qquad k_2 = 1000\,\text{N/m}. \quad (73)$$

Typically, the equations of motion for the structural/mechanical system are formulated based on a Newtonian or a Lagrangian formulation. For the above system this would lead to the equations

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \ddot{\bar{q}} + \begin{bmatrix} c & -c \\ -c & c \end{bmatrix} \dot{\bar{q}} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \bar{q} = \begin{bmatrix} k_1 u(t) \\ 0 \end{bmatrix}, \quad (74)$$

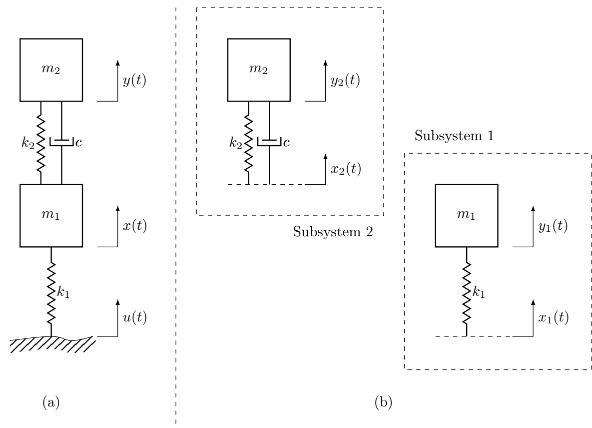

Figure 1: Quarter car suspension (a) modeled as one system (b) separated into two subsystems.

where $\bar{q}(t)^T = \begin{bmatrix} x(t) & y(t) \end{bmatrix}$. Taking into account that the "constraints" in this case are already incorporated in the equations of motion, the response can be computed using the closed form solution of Eq. (71). Note that since the mass matrix is invertible, the inverse of the matrix $P(s) = s^2M + sC + K$ will be strictly proper, thus

$$(A_\infty, B_\infty, C_\infty) = (0, 0, 0). \tag{75}$$

Furthermore, since $P(s)^{-1}$ is strictly proper, a minimal state space realization (A_F, B_F, C_F), which satisfies

$$P(s)^{-1} = C_F(sI_n - A_F)^{-1}B_F, \tag{76}$$

can be computed using any of the widely used techniques and algorithms available for this purpose (see for instance [14, Ch.8] or [30, Sec.1.11]). In the present example Wolfram's Mathematica functions StateSpaceModel[] and MinimalStateSpaceModel[] have been employed to obtain the following minimal realization

$$A_F = \begin{bmatrix} 0.182 & 1.04 & 0 & 0 \\ -2.38 & -0.392 & 0.852 & 0 \\ 0.103 & 0.0115 & 0.0514 & 0.997 \\ -27.2 & 3.35 & -101. & -3.57 \end{bmatrix},$$

$$B_F = \begin{bmatrix} 0 & -0.00097 \\ 0 & 0.0095 \\ 0 & -0.000335 \\ 0.366 & -0.0175 \end{bmatrix},$$

$$C_F = \begin{bmatrix} 0.168 & 0.0204 & 0.0913 & 0 \\ 0.42 & 0.0429 & 0 & 0 \end{bmatrix}.$$

Equivalent results can be obtained using Matlab's functions and minreal(). Next, the response given by Eq. (71), for initial conditions

$$x(0^-) = y(0^-) = \dot{x}(0^-) = \dot{y}(0^-) = 0, \tag{77}$$

and ground profile $u(t) = \sin(5t)$ is depicted in Figure 2. In the same figure the numerical solution of Eq. (74) has been computed using the standard Runge - Kutta method (dashed lines). Clearly, the results of the two approaches coincide.

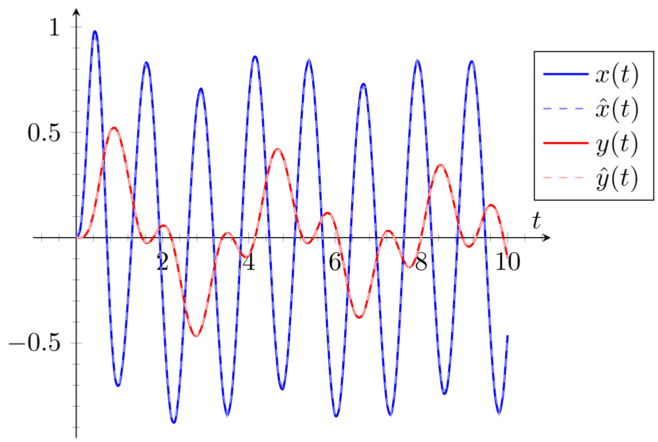

Figure 2: The response of the constrained system using the classical approach- Case (a). Responses $x(t), y(t)$ obtained from Eq. (71) and $\hat{x}(t), \hat{y}(t)$ using the standard Runge - Kutta numerical scheme.

Alternatively, adopting the approach proposed in [28] each of the constituent subsystems is considered separately as a 2-DOF system as depicted in Figure 1.(b) for deriving the equations of motion. Then, appropriate constraints are imposed. In particular, the equations of motion for both subsystems have the form

$$M_i \ddot{q}_i + C_i \dot{q}_i + K_i q_i = 0, \quad i = 1, 2, \tag{78}$$

where $q_i(t)^T = \begin{bmatrix} x_i(t) & y_i(t) \end{bmatrix}$,

$$M_i = \begin{bmatrix} 0 & 0 \\ 0 & m_i \end{bmatrix}, \qquad K_i = \begin{bmatrix} k_i & -k_i \\ -k_i & k_i \end{bmatrix}, \tag{79}$$

for $i = 1, 2$ and

$$C_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \qquad C_2 = \begin{bmatrix} c & -c \\ -c & c \end{bmatrix}. \tag{80}$$

Thus, the overall equation of the two, still unconnected, subsystems will have the form

$$M\ddot{q} + C\dot{q} + Kq = 0, \tag{81}$$

where $q(t)^T = \begin{bmatrix} q_1(t) & q_2(t) \end{bmatrix}$ and

$$M = \begin{bmatrix} M_1 & 0 \\ 0 & M_2 \end{bmatrix}, \quad C = \begin{bmatrix} C_1 & 0 \\ 0 & C_2 \end{bmatrix}, \quad K = \begin{bmatrix} K_1 & 0 \\ 0 & K_2 \end{bmatrix}. \tag{82}$$

In order to model the original system we need to take into account the (ideal) constraints

$$y_1(t) = x_2(t), \tag{83}$$
$$x_1(t) = u(t), \tag{84}$$

where $u(t)$ is the ground profile. Differentiating twice with respect to time the constraints Eqs. (83) and (84), yields a matrix equation of the form of Eq. (65), i.e.

$$\begin{bmatrix} 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \ddot{q}(t) = \begin{bmatrix} 0 \\ \ddot{u}(t) \end{bmatrix}. \tag{85}$$

Considering Eq. (65), $A_2 = \begin{bmatrix} 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$, $A_0 = A_1 = 0$ and $g(t)^T = \begin{bmatrix} 0 & \ddot{u}(t) \end{bmatrix}$. Notice that the compound matrix $\begin{bmatrix} M^T & A_2^T \end{bmatrix}^T$ has full column rank, hence the two constraints are sufficient to uniquely determine the equations of motion. The next step for the formation of the equations of the constrained motion is to compute a basis of the null space of A_2. Such a basis is given by the columns of

$$V = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}. \tag{86}$$

The matrix V has been computed here using Gaussian elimination on the columns of A_2. In more complex systems, where the matrix A_2 is larger in dimensions, the columns of V can be recovered using more numerically efficient techniques, such as the singular value decomposition (SVD).

The equations of the constrained motion resulting from the application of Eq. (66) have the form

$$\bar{M}\ddot{q}(t) + \bar{C}\dot{q}(t) + \bar{K}q(t) = \bar{f}(t), \tag{87}$$

where

$$\bar{M} = \begin{bmatrix} 0 & 0 & 0 & m_2 \\ 0 & m_1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \qquad \bar{C} = \begin{bmatrix} 0 & 0 & -c & c \\ 0 & 0 & c & -c \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{88}$$

$$\bar{K} = \begin{bmatrix} 0 & 0 & -k_2 & k_2 \\ -k_1 & k_1 & k_2 & -k_2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad \bar{f}(t) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \ddot{u}(t) \end{bmatrix}. \tag{89}$$

It can be easily verified that the mass matrix $\bar{M}$ is non–singular. Consequently, the inverse of the matrix $\bar{P}(s) = s^2\bar{M} + s\bar{C} + \bar{K}$ will be strictly proper, hence as above the triple $(A_\infty, B_\infty, C_\infty)$ vanishes and a minimal state space realization of $\bar{P}(s)^{-1}$, computed using Mathematica's above mentioned functions, is given by

$$A_F = \begin{bmatrix} 0.24 & 1.06 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2.33 & -0.744 & 2.21 & 0 & 0 & 0 & 0 & 0 \\ -7.16 & -2.56 & 6.6 & 3.21 & 0 & 0 & 0 & 0 \\ 16.4 & 14. & -49.6 & -9.83 & 0 & 0 & 0 & 0 \\ 1.64 & 0.564 & -1.41 & -0.654 & -0.00346 & 1.02 & 0 & 0 \\ -1.13 & -1.74 & 6.35 & 1.12 & -0.0000117 & 0.00346 & 0 & 0 \\ 1.56 & 0.537 & -1.34 & -0.623 & -0.00324 & 0.0464 & -0.00306 & 1.02 \\ -2.38 & -1.57 & 5.7 & 1.01 & -0.0000202 & 0.00323 & 0 & 0.00306 \end{bmatrix},$$

$$B_F = \begin{bmatrix} 0 & -0.0000409 & 0 & 0.00123 \\ 0 & 0.000397 & 0 & -0.0119 \\ 0 & 0.00122 & 0 & -0.0366 \\ 0.000364 & -0.00865 & 0.0911 & 0.168 \\ 0 & -0.000275 & -0.000254 & 0.00812 \\ -0.0000564 & 0.0011 & -0.0105 & -0.0188 \\ 0 & -0.000261 & -0.000249 & 0.00773 \\ -0.0000367 & 0.000995 & -0.00937 & -0.017 \end{bmatrix},$$

$$C_F = \begin{bmatrix} -76.1 & -17.8 & 64. & 3.62 & 2.36 & 0.595 & 268. & 27.4 \\ -26. & -12.6 & 64. & 3.62 & 2.36 & 0.595 & 268. & 27.4 \\ -96.9 & 1.14 & -6.15 & -0.28 & -265. & -26.9 & 268. & 27.4 \\ -2.73 & 0.364 & -2.73 & -0.28 & -265. & -26.9 & 268. & 27.4 \end{bmatrix}.$$

To obtain a response comparable to the one obtained in case (a), where the system was modeled using classical techniques, we need to take into account the relations $x_1(t) = u(t)$, $y_1(t) = x_2(t) = x(t)$ and $y_2(t) = y(t)$ and use the same ground profile, $u(t) = \sin(5t)$, as above. This dictates the choice of initial conditions for the new coordinates and their derivatives, which in view of Eq. (77) and the particular choice of $u(t)$, should be

$$\begin{aligned} &x_1(0^-) = 0, && y_1(0^-) = 0, && x_2(0^-) = 0, && y_2(0^-) = 0, \\ &\dot{x}_1(0^-) = 5, && \dot{y}_1(0^-) = 0, && \dot{x}_2(0^-) = 0, && \dot{y}_2(0^-) = 0 \end{aligned}$$

With this setup, $q(t)$ can be computed by Eq. (71) and the resulting response is shown in Figure 3.

It can be readily seen that the computed $x_1(t)$ coincides with the ground profile $u(t) = \sin(5t)$, while the coordinates $y_1(t), x_2(t)$ coincide as required by the constraint Eqs. (84) and (83). Moreover, from the comparison of the plots in Figures (2) and (3), it is clear that $x(t) = y_1(t)$ and $y(t) = y_2(t)$ as expected. □

Example 3. In this example the system shown in Figure 4.(a) is considered with the parameters values.

$$m = 10\,\mathrm{Kg}, \qquad c_1 = 1\,\mathrm{Ns/m}, \qquad k_1 = 2\,\mathrm{N/m}, \qquad k_2 = 1\,\mathrm{N/m}. \qquad (90)$$

Using either Newton's or Lagrange's formulation the equation of motion for the system in Figure 4.(a) becomes

$$\begin{bmatrix} 0 & 0 \\ 0 & m \end{bmatrix} \ddot{\bar{q}} + \begin{bmatrix} c_1 & 0 \\ 0 & 0 \end{bmatrix} \dot{\bar{q}} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \bar{q} = \begin{bmatrix} 0 \\ F(t) \end{bmatrix}, \qquad (91)$$

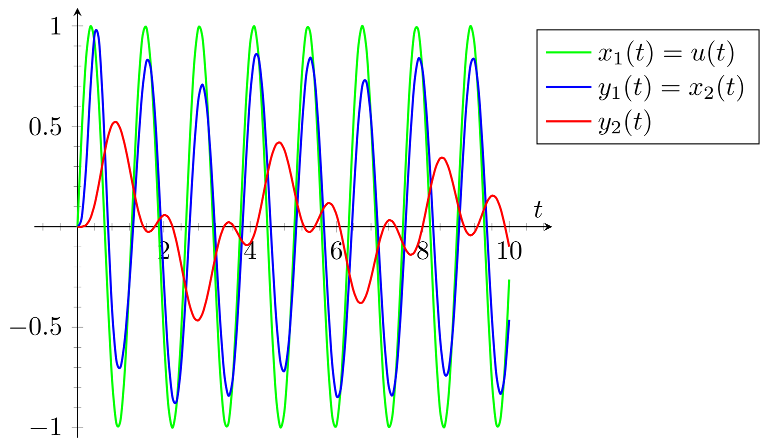

Figure 3: The response of the constrained system using the proposed approach - Case (b).

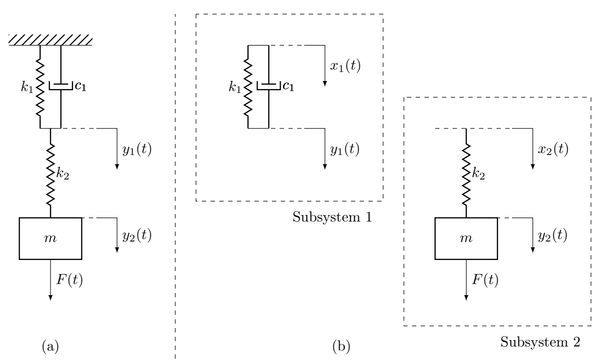

Figure 4: Singular mass matrix system (a) modeled as one system (b) separated into two subsystems.

where $\bar{q}(t)^T = \begin{bmatrix} y_1(t) & y_2(t) \end{bmatrix}$ and $F(t) = \cos t$ is the external force applied to the mass m. In a similar manner as in the previous example, the response can be computed using the closed form solution given by Eq. (71).

Although the mass matrix is singular, the inverse of the matrix $P(s) = s^2M + sC + K$ is strictly proper, thus

$$(A_\infty, B_\infty, C_\infty) = (0, 0, 0). \tag{92}$$

Furthermore, since $P(s)^{-1}$ is strictly proper we can compute a minimal state realization (using Mathematica), which satisfies

$$P(s)^{-1} = C_F(sI_n - A_F)^{-1}B_F, \tag{93}$$

where

$$A_F = \begin{bmatrix} 0.214 & 0.977 & 0 \\ -0.106 & -0.246 & 0.549 \\ -0.117 & -0.0993 & -2.97 \end{bmatrix},$$

$$B_F = \begin{bmatrix} 0 & 0 \\ 0 & 0.274 \\ 0.997 & 0.016 \end{bmatrix},$$

$$C_F = \begin{bmatrix} 0.267 & -0.0586 & 1 \\ 0.374 & 0 & 0 \end{bmatrix},$$

Notice that the dimension of the matrix A_F, and hence the state of the system is now three. Further, in this case the initial conditions vectors can be arbitrarily chosen, since Eq. (70) vanishes. Assuming that the system is at rest at $t = 0^-$, yields

$$y_1(0^-) = y_2(0^-) = \dot{y_1}(0^-) = \dot{y_2}(0^-) = 0.$$

With this setup the response given by Eq. (71) is depicted in Figure 5. In order to validate the response obtained by Eq. (71), the equations of motion (91) have been also solved numerically using the standard Runge - Kutta method (dashed lines).

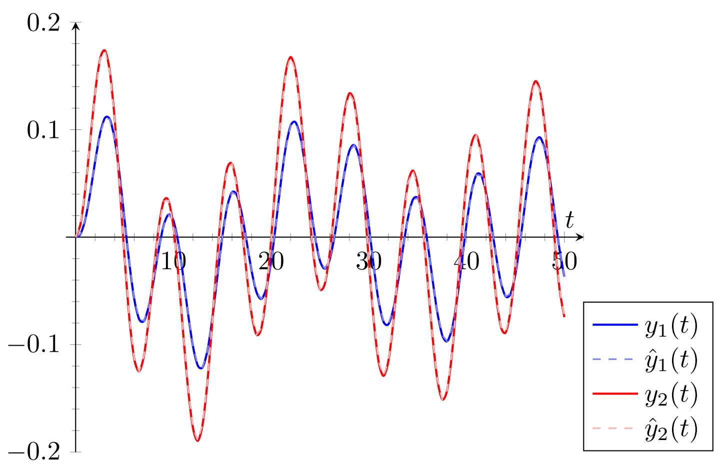

Figure 5: The response of the constrained system using the classical approach - Case (a). Responses $y_1(t), y_2(t)$ obtained through Eq. (71) and $\hat{y}_1(t), \hat{y}_2(t)$ using the Runge - Kutta method.

Next, an alternative formulation [28] that considers the constituent subsystems separately as 2-DOF systems (see Figure 4.(b)) is utilized, where their interaction is realized by imposing appropriate constraints. Specifically, the equations of motion for both subsystems have the form

$$M_i\ddot{q}_i + C_i\dot{q}_i + K_i q_i = f(t), \quad i = 1, 2, \tag{94}$$

where $q_i(t)^T = \begin{bmatrix} x_i(t) & y_i(t) \end{bmatrix}$,

$$M_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \qquad M_2 = \begin{bmatrix} 0 & 0 \\ 0 & m \end{bmatrix}, \tag{95}$$

$$K_i = \begin{bmatrix} k_i & -k_i \\ -k_i & k_i \end{bmatrix},$$

for $i = 1, 2$ and

$$C_1 = \begin{bmatrix} c_1 & -c_1 \\ -c_1 & c_1 \end{bmatrix}, \qquad C_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \tag{96}$$

In view of the above, the overall equation of the two, still unconnected, subsystems takes the form

$$M\ddot{q} + C\dot{q} + Kq = f(t), \tag{97}$$

where $q(t)^T = \begin{bmatrix} q_1(t) & q_2(t) \end{bmatrix}$ and

$$M = \begin{bmatrix} M_1 & 0 \\ 0 & M_2 \end{bmatrix}, \qquad D = \begin{bmatrix} C_1 & 0 \\ 0 & C_2 \end{bmatrix}, \qquad K = \begin{bmatrix} K_1 & 0 \\ 0 & K_2 \end{bmatrix}, \tag{98}$$

and $f(t) = [0, 0, 0, F(t)]$. Similarly as in Example 2, to model the composite system we take into account the (ideal) constraints

$$y_1(t) = x_2(t), \tag{99}$$
$$x_1(t) = 0, \tag{100}$$

Differentiating twice with respect to time the constraints Eqs. (99) and (100), yields a matrix equation of the form of Eq. (65), that is

$$\begin{bmatrix} 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \ddot{q}(t) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{101}$$

Thus, with the notation of Eq. (65), $A_2 = \begin{bmatrix} 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$, $A_0 = A_1 = 0$ and $g(t)^T = \begin{bmatrix} 0 & \ddot{u}(t) \end{bmatrix}$. Next, to formulate the equations of the constrained motion we compute a basis of the null space of A_2. Such a basis can be easily computed using Gaussian elimination on the columns of A_2, which in our case gives

$$V = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}. \tag{102}$$

The equations of the constrained motion given by Eq. (66), take the form

$$\bar{M}\ddot{q}(t) + \bar{C}\dot{q}(t) + \bar{K}q(t) = \bar{f}(t), \tag{103}$$

where

$$\bar{M} = \begin{bmatrix} 0 & 0 & 0 & m \\ 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \qquad \bar{C} = \begin{bmatrix} 0 & 0 & -c & c \\ 0 & 0 & c & -c \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{104}$$

$$\bar{K} = \begin{bmatrix} 0 & 0 & -k_2 & k_2 \\ -k_1 & k_1 & k_2 & -k_2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad \bar{f}(t) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ F(t) \end{bmatrix}. \tag{105}$$

Despite the fact that the mass matrix $\bar{M}$ is singular, the inverse of the matrix $\bar{P}(s) = s^2\bar{M}+s\bar{C}+\bar{K}$ is strictly proper, hence as above the triple $(A_\infty, B_\infty, C_\infty)$ vanishes, and a minimal state space realization of $\bar{P}(s)^{-1}$ (using Mathematica) is given by

$$A_F = \begin{bmatrix} -0.335 & 1.33 & 0 & 0 & 0 & 0 & 0 \\ 0.671 & -1.16 & 1.66 & 0 & 0 & 0 & 0 \\ -0.667 & 0.931 & -1.5 & 0 & 0 & 0 & 0 \\ 0.0112 & -0.0183 & 0.025 & -0.498 & 0.731 & 0 & 0 \\ 0.06 & 0.0387 & -0.0419 & -0.34 & 0.498 & 0 & 0 \\ -0.107 & 0.176 & -0.24 & -0.0215 & 0.000446 & -0.36 & 0.806 \\ -0.501 & -0.366 & 0.404 & -0.0227 & 0.0194 & -0.16 & 0.36 \end{bmatrix},$$

$$B_F = \begin{bmatrix} 0 & 0.332 & 0 & 0 \\ 0 & -0.665 & 0 & 0 \\ 0.03 & 0.601 & 0.3 & -0.3 \\ 0.0191 & 0.00246 & -0.201 & -0.191 \\ -0.0131 & 0.00163 & 0.138 & 0.131 \\ -0.183 & -0.0235 & 0.0291 & -0.0646 \\ 0.145 & -0.01 & -0.0258 & 0.0568 \end{bmatrix},$$

$$C_F = \begin{bmatrix} -1.99 & -1. & 0.00299 & 1.24 & 1.81 & 0.337 & 0.425 \\ 1.02 & -1. & 0.00299 & 1.24 & 1.81 & 0.337 & 0.425 \\ -0.969 & -2. & 0.00597 & -0.0354 & -0.0559 & 0.337 & 0.425 \\ 0.0332 & 0.00336 & 0.00597 & -0.0354 & -0.0559 & 0.337 & 0.425 \end{bmatrix}.$$

Note that the compound matrix $\begin{bmatrix} M^T & A_2^T \end{bmatrix}^T$ is rank deficient. Since $P(s)^{-1}$ is strictly proper, Eq. (71) vanish and the initial conditions vector can be chosen arbitrary to be

$$x_1(0) = 0, \qquad y_1(0) = 0, \qquad x_2(0) = 0, \qquad y_2(0) = 0,$$
$$\dot{x}_1(0) = 0, \qquad \dot{y}_1(0) = 0, \qquad \dot{x}_2(0) = 0, \qquad \dot{y}_2(0) = 0,$$

then the motion of the system can be uniquely determined using the proposed approach. Note also that the above choice of initial conditions satisfies also the constraints imposed by Eqs. (99) and (100) as well. With this setup $q(t)$ can be computed using Eq. (66) and the resulting response is shown in Figure 6. Comparing the plots in Figures 5 and 6, it is easy to see that the components $y_1(t), y_2(t)$ of the system described by Eq. (103), coincide with their counterparts in Eq. (91) □

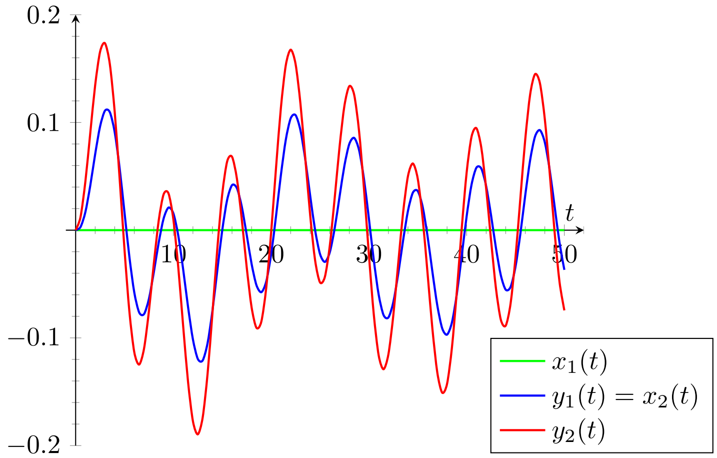

Figure 6: The response of the constrained system using the proposed approach - Case (b).

5. Conclusions

An approach has been developed based on polynomial matrix theory for formulating the equations of motion and for determining the response of multi-degree-of-freedom (MDOF) linear dynamical systems with singular matrices and subject to linear constraints. The herein developed approach can be construed as an alternative to the methodology proposed by Udwadia and coworkers [28], and has the significant advantage that, under the same uniqueness conditions as in [28], it circumvents the use of pseudoinverses in determining the system response. In fact, based on the theoretical machinery of polynomial matrices, a closed form analytical solution has been derived for the system response that involves square and non-singular matrices, and relies on the use of a basis of the null space of the constraints matrix. Several structural/mechanical systems with singular matrices have been included as examples for demonstrating the validity of the developed framework and for elucidating certain numerical aspects. Regarding potential future work, the approach can be extended to account for stochastic excitations as well based on recent work by Fragkoulis et al. [5, 3, 4].

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