

Asymptotic Results for a Markov-Modulated Risk Process with Stochastic Investment

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Abstract

In this paper we consider a Markov-modulated risk model, where the premium rates, claim frequency and the distribution of the claim sizes vary depending on the state of an external Markov chain. The free reserves of the insurer are invested in a risky asset whose prices are modelled by a geometric Brownian motion, with parameters that are also influenced according to the external Markov process. A system of integro-differential equations for the ruin probabilities and for the expected discounted penalty function is derived. Using Laplace transforms and regular variation theory, we investigate the asymptotic behaviour of both quantities for the case of light or heavy tailed claim size distributions. Specifically, within this set up (where we lose the strong Markov property of the risk process), we show that the ruin probabilities decrease asymptotically as a power function in the case of the light tailed claims, whilst for the heavy tails we show that the probabilities of ruin decay either like a power function, depending on the parameters of the investment, or behave asymptotically like the tails of the claim size distributions.

Keywords: Markov-modulated risk process, Investment, Integro-differential equation system, Ruin probabilities, Expected discounted penalty function, Regular variation, Frobenius method for systems.

1 Introduction

The investigation of insurance risk models with stochastic return on investments has attracted a lot of attention in recent years. Stimulated by the paper of Paulsen (1993) and

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25 Paulsen and Gjessing (1997), where continuous time risk processes in a stochastic eco-
26 nomic environment are introduced, many researchers have studied Poisson and renewal
27 risk models with risky investments. Lower and upper bounds, numerical solutions, asymp-
28 totics and analytic expressions for the probability of ruin (for some individual classes of
29 the aforementioned models), in the case where the wealth process of an insurance portfolio
30 is invested in a stock (whose prices follow a geometric Brownian motion or are Lévy pro-
31 cesses), have been derived by several authors. See for example, among others, Cai (2004),
32 Cai and Xu (2006), Paulsen (1998), Paulsen (2008), Tang and Tsitsiashvili (2003), Tang
33 and Tsitsiashvili (2004) and the references therein. More recently, another extension of the
34 aforementioned problem, where a general two sided jump-diffusion risk model that allows
35 correlation between the two Brownian motions driving the insurance risk and investment
36 return, has been investigated by Yin and Wen (2013) in the presence of a constant dividend
37 and a threshold barrier strategy.

38 With regards to the asymptotic results of risk models with investments, Paulsen (2002)
39 considers a Lévy risk process compounded by another independent Lévy process and shows
40 asymptotically that, as initial capital increases the ruin probability essentially behaves as a
41 power function of the initial capital. Moreover, Gaier and Grandits (2004) showed, within
42 the context of the classical risk model, that when the claim sizes are regularly varying,
43 then the probability of ruin is also regularly varying, whilst Wei (2009) extended these re-
44 sults into the context of the renewal risk model. More recently, Hult and Lindskog (2011)
45 studied the asymptotic decay of finite time ruin probabilities for an insurance portfolio in
46 the presence of heavy-tailed claims when the prices of the risky investments are given by
47 a quite general semimartingale. In this setting, the ruin problem corresponds to deter-
48 mining hitting probabilities for the solution to a randomly perturbed stochastic integral
49 equation. Additionally, Albrecher et al. (2012) considered a general class of renewal risk
50 models (where the inter-arrival claim times satisfy an ordinary differential equation with
51 constant coefficients) with geometrical Brownian motion investments and, using regular
52 variation theory, they derived a unified analytic method for the asymptotic behaviour of
53 the probability of ruin. For this general class of renewal risk models with investment,
54 explicit results for the asymptotic ruin probability are given in the case of both light and
55 heavy tailed claims.

56 The common idea that investing in an asset with stochastic returns proves too risky for
57 an insurance portfolio in the classical risk model, the renewal and the Lévy risk models,
58 can be justified mathematically by all the above papers. However, once we move to non-
59 renewal models (in the sense that the surplus process does not renew itself at the claim
60 time epochs), the strong Markov property is lost and the problem becomes cumbersome.
61 The Markov-modulated risk model was first introduced by Janssen (1980) and Reinhard
62 (1984) and has since received much attention in the risk theory literature, including ap-
63 plications in queueing theory, see among others Asmussen (1987), Asmussen et al. (1994)
64 and Asmussen and O’Cinneide (2002). The primary motive of these papers is to enhance
65 the flexibility of the models parameter setting. This is achieved by considering an exter-

66 nal Markovian environment process which influences both the claim frequencies and the
67 claim severities. The examples usually given are weather conditions, where the sojourns
68 of the external Markov process could be weather types, or in health insurance where the
69 sojourns of the environment process could be certain types of epidemics (see Asmussen
70 (1989)). Surprisingly, only a few authors have studied non-Poissonian risk models in the
71 presence of an investment strategy. Kötter and Bäuerle (2006) were the first to introduce
72 a Markov-modulated risk process where risk reserves, under a special investment strategy,
73 can be invested into a stock index following a geometric Brownian motion. Within this
74 set up, for a special class of investment policies, they derive results for the adjustment
75 coefficient. A second study within the Markov-modulated framework was made by Diko
76 and Usábel (2011), where they considered a risk model perturbed by diffusion in which
77 the reserves are invested into an asset whose return rate and volatility are time-dependent
78 Markov-modulated. For this model they used Chebyshev's polynomial approximation and
79 Laplace-Carson transforms to obtain a numerical solution for the integro-differential equa-
80 tion system for the risk quantity of interest.

81 In this paper, we consider a Markov-modulated risk model in which the reserves of the
82 insurance portfolio are continuously invested into an asset whose prices follow a geomet-
83 rical Brownian motion, which is also influenced by the external Markov chain. For the
84 aforementioned model the Markov property no longer holds and thus the ruin probability
85 is given in terms of an integro-differential equation system. Stimulated by Albrecher et al.
86 (2012), we extend their methodology (using Frobenius method for systems - see Barkatou
87 et al. (2010)) to obtain, using regular variation theory, an explicit asymptotic expression
88 of the ultimate ruin probability and the expected discounted penalty function. Within this
89 non-Poissonian model we are able to show that the ruin probability decreases asymptoti-
90 cally as a power function in the case of the light tailed claims, whilst for the heavy tails
91 we show that the probability of ruin decays either like a power function, depending on
92 the parameters of the investment, or behaves asymptotically like the tails of the claim size
93 distributions. The same kind of results hold for the Gerber-Shiu function. Note that the
94 above matrix based analysis holds for more general non-renewal risk models, such as the
95 Markov Arrival Process (MAP) risk models.

96 In more details the paper is organised as follows; in Section 2 we introduce a Markov-
97 modulated risk model where the reserves of the insurance portfolio are invested in a risky
98 asset whose price follows a geometrical Brownian motion, in which the drift and volatility
99 parameters are also influenced by the external Markov chain. In Section 3, using the
100 infinitesimal generator argument, we derive an integro-differential equation system for the
101 decompositions of the ruin probabilities. In Section 4, we use Laplace transforms to derive
102 an individual form for the system of ruin probabilities, that will allow an asymptotic
103 analysis in the later sections. In Section 5, we give the general solution for the Laplace
104 system and by using the Frobenius method for matrices, Tauberian theorems and Heaviside
105 principle, we derive explicit asymptotic expressions for the probabilities of ruin. Section 6
106 discusses an extension of the methodology used for the ruin probabilities to more general

107 ruin-related quantities, namely the Gerber-Shiu function.

108 **2 Markov-modulated risk process with stochastic investment**

109 In this section, we introduce the Markov-modulated Poisson risk model in the presence of
 110 risky asset investment, where the premium rate, the claim arrival rate, the distribution of
 111 the claim sizes and the parameters of the return on the surplus investment are influenced
 112 by an external Markov chain (see also Kötter and Bäuerle (2006) and Diko and Usábel
 113 (2011)).

Consider the external environment process $\{J(t)\}_{t \geq 0}$, which can be interpreted as the general economic conditions that govern the state of the economy. Suppose $\{J(t)\}_{t \geq 0}$ is a homogeneous, irreducible and recurrent continuous time Markov process, with finite state space $E = \{1, 2, \dots, m\}$. Let $\mathbf{Q} = (q_{ij})_{i,j=1}^m$, with $q_{ii} = -\sum_{j \neq i} q_{ij} = -q_i$, for $i \in E$, denote the intensity rate matrix of $\{J(t)\}_{t \geq 0}$, with a stationary distribution (which exists and is unique since $\{J(t)\}_{t \geq 0}$ is irreducible and has finite state space) given by

$$\pi = (\pi_1, \dots, \pi_m), \quad \pi_i \geq 0, \quad i \in E \quad \text{and} \quad \sum_{i \in E} \pi_i = 1.$$

114 Assume that when $J(t) = i \in E$, the number of claims, namely $N(t)$, occur according
 115 to a Poisson process with intensity rate $\lambda_i \in \mathbb{R}^+$. Further assume that the corresponding
 116 nonnegative claim amounts, $\{X_k\}_{k \geq 1}$, have common distribution function $F_i(x)$, with den-
 117 sity $f_i(x)$ and finite mean $\mu_i < \infty$. We will also assume that the premiums are received
 118 continuously at a rate $c_i > 0$ during the time when $\{J(t)\}_{t \geq 0}$ remains in the state $i \in E$.
 119 Under the above set up, the corresponding risk model is known as a Markov-modulated
 120 Poisson process.

Considering the above assumptions, the insurer's surplus process can be given by

$$U(t) = u + \int_0^t c_{J(s)} ds - \sum_{k=1}^{N(t)} X_k, \quad t \geq 0,$$

where $u \geq 0$ is the insurer's initial capital. Let us propose that the insurer invests its surplus into a risky asset, with returns process $\{R_i(t)\}_{t \geq 0}$, when $J(t) = i \in E$, which is also influenced by the external Markov process, $\{J(t)\}_{t \geq 0}$, and satisfies the stochastic differential equation

$$dR_{J(t)}(t) = a_{J(t)} dt + \sigma_{J(t)} dB(t),$$

121 where $\{a_{J(t)}\}_{t \geq 0}$ is the drift and $\{\sigma_{J(t)}\}_{t \geq 0}$ the volatility of the randomness produced by
 122 the standard Brownian motion $\{B(t)\}_{t \geq 0}$.

123 Within this framework, the surplus process under risky investment, is given by

$$U(t) = u + \int_0^t c_{J(s)} ds - \sum_{k=1}^{N(t)} X_k + \int_0^t U(s-) dR_{J(s)}(s), \quad t \geq 0. \quad (2.1)$$

124 This model extends the Markov-modulated risk process introduced by Reinhard (1984)
 125 and also the classical risk model, with investment, introduced by Paulsen (1993).

The first time the surplus process of the insurance portfolio falls below zero is referred to as the time of ruin and is denoted by

$$T = \inf\{t \geq 0 : U(t) < 0 | U(0) = u\}, \quad (\infty, \text{otherwise}).$$

The probability of ruin, given that the initial environment is in state $i \in E$, with initial capital $u \geq 0$, is described by

$$\psi_i(u) = \mathbb{P}\{T < \infty | U(0) = u, J(0) = i\}.$$

126 Then, the ultimate ruin probability, for the stationary case, is given by

$$\psi(u) = \sum_{k=1}^m \pi_k \psi_k(u), \quad u \geq 0. \quad (2.2)$$

127 3 An integro-differential equation system for the ruin 128 probabilities

The main aim of this section is to derive a system of integro-differential equations for the auxiliary function $\psi_i(u)$, $i \in E$. Before we proceed with the derivation, recall that if $\{X(t)\}_{t \geq 0}$ is an Itô diffusion, with $X(0) = x$, satisfying a stochastic differential equation of the form

$$dX(t) = \alpha(X(t)) dt + r(X(t)) dB(t),$$

129 then the infinitesimal generator of $X(t)$ is the operator \mathcal{A} , acting on suitable functions f ,
 130 given by

$$\mathcal{A}f(x) = \lim_{h \rightarrow 0} \frac{\mathbb{E}[f(X(h)) | X(0) = x] - f(x)}{h} = \alpha(x) \frac{\partial}{\partial x} f(x) + \frac{r^2(x)}{2} \frac{\partial^2}{\partial x^2} f(x). \quad (3.1)$$

131 Using an intuitive infinitesimal argument and methods similar to that in Cai and Xu (2006)
 132 and Lu and Tsai (2007), we get the following theorem.

133

Theorem 1. *For $u \geq 0$, the ruin probabilities, $\psi_i(u)$, $i \in E$, satisfy the following integro-differential equation system*

$$\begin{aligned} & \frac{1}{2} \sigma_i^2 u^2 \psi_i''(u) + (a_i u + c_i) \psi_i'(u) + \lambda_i \bar{F}_i(u) \\ & = (\lambda_i + q_i) \psi_i(u) - \lambda_i \int_0^u \psi_i(u-x) dF_i(x) - \sum_{j=1, j \neq i}^m q_{ij} \psi_j(u), \end{aligned} \quad (3.2)$$

134 with boundary conditions

$$\lim_{u \rightarrow \infty} \psi_i(u) = 0, \quad (3.3)$$

135 and

$$c_i \psi_i'(0) - (\lambda_i + q_i) \psi_i(0) + \sum_{j=1, j \neq i}^m q_{ij} \psi_j(0) + \lambda_i = 0, \quad (3.4)$$

136 where $\bar{F}_i(u) = 1 - F_i(u)$, $i \in E$.

137 *Proof.* Let

$$Y_i(t) = u + c_i t + \int_0^t Y_i(s-) dR_i(s), \quad i \in E, \quad (3.5)$$

138 be the income process under investment, given we start in state $i \in E$ and experience
 139 no claims up to time $t \geq 0$. In order to derive an integro-differential equation system
 140 for the ruin probabilities $\psi_i(u)$, $i \in E$, we consider the risk process $\{U(t)\}_{t \geq 0}$, defined by
 141 equation (2.1) in an infinitesimal time interval $(0, h]$. Moreover, given that $J(0) = i \in E$
 142 and $\{N(t)\}_{t \geq 0}$ is a Poisson process, there are four cases that could appear in $(0, h]$;

- 143 1. No claim and no change in state,
- 144 2. No change in state but a claim arrival,
- 145 3. No claim but a change in state of the external process,
- 146 4. Two or more events occur in the interval $(0, h]$.

Considering the possible events above and noticing, for the second case, it holds that $\psi_i(Y_i(h) - x) = 1$, for $x > Y_i(h)$, we have

$$\begin{aligned} \psi_i(u) = & (1 - \lambda_i h - q_i h) \mathbb{E}(\psi_i(Y_i(h))) \\ & + \lambda_i h \mathbb{E} \left[\int_0^{Y_i(h)} \psi_i(Y_i(h) - x) dF_i(x) + \bar{F}_i(Y_i(h)) \right] \\ & + h \mathbb{E} \left[\sum_{j=1, j \neq i}^m q_{ij} \psi_j(Y_i(h)) \right] + o(h), \end{aligned}$$

147 where $o(h)$ is such that, $o(h)/h \rightarrow 0$ as $h \rightarrow 0$.

Re-arranging the above equation, yields

$$\begin{aligned} (\lambda_i + q_i) \mathbb{E} [\psi_i(Y_i(h))] = & \frac{\mathbb{E} [\psi_i(Y_i(h))] - \psi_i(u)}{h} + \lambda_i \mathbb{E} \left[\int_0^{Y_i(h)} \psi_i(Y_i(h) - x) dF_i(x) + \bar{F}_i(Y_i(h)) \right] \\ & + \mathbb{E} \left[\sum_{j=1, j \neq i}^m q_{ij} \psi_j(Y_i(h)) \right] + \frac{o(h)}{h}. \end{aligned}$$

148 Now, letting $h \rightarrow 0$ in the equation above yields that

$$(\lambda_i + q_i)\psi_i(u) = \mathcal{A}\psi_i(u) + \lambda_i \left[\int_0^u \psi_i(u-x) dF_i(x) + \bar{F}_i(u) \right] + \sum_{j=1, j \neq i}^m q_{ij}\psi_j(u), \quad (3.6)$$

149 where \mathcal{A} is the infinitesimal generator, defined in equation (3.1), of the process $Y_i(t)$.

Rewriting equation (3.5) in the form of an Itô diffusion process, and using equation (3.1), we get that the generator of $Y_i(t)$ acting on $\psi_i(u)$ is given by

$$\mathcal{A}\psi_i(u) = (c_i + a_i u) \psi_i'(u) + \frac{1}{2} \sigma_i^2 u^2 \psi_i''(u).$$

150 After substituting this form of the generator into equation (3.6), we obtain the integro-
151 differential equation system (3.2). For the boundary condition (3.4), setting $u = 0$ in the
152 integro-differential equation system (3.2), the result follows immediately. □

153

Remark 1. For $m = 1$, we obtain the integro-differential equation for the classical risk model under risky investment

$$\frac{1}{2} \sigma^2 u^2 \psi''(u) + (au + c) \psi'(u) + \lambda \bar{F}(u) = \lambda \psi(u) - \lambda \int_0^u \psi(u-x) dF(x),$$

154 as it is given in Constantinescu and Thomann (2004).

155 4 Laplace transforms

156 The structure of the integro-differential equation system (3.2) suggests the use of Laplace
157 transforms for the asymptotic analysis of the probability of ruin. Thus, in this section,
158 we will derive a matrix closed form expression for the ruin probability, that will be vital
159 for our next section, where Karamata-Tauberian theorems will be applied to derive the
160 asymptotic ruin results.

161 Let $\widehat{\psi}_i(s)$, $\widehat{F}_i(s)$ and $\widehat{f}_i(s)$ be the Laplace transforms of $\psi_i(u)$, $\bar{F}_i(u)$ and $f_i(u)$, respec-
162 tively. Taking Laplace transforms on both sides of equation system (3.2), one can see that
163 $\widehat{\psi}_i(u)$ satisfies a second order non-homogeneous ordinary differential equation system, for
164 $i \in E$, given by

$$\begin{aligned} & \frac{s^2 \sigma_i^2}{2} \widehat{\psi}_i''(s) + [s(2\sigma_i^2 - a_i)] \widehat{\psi}_i'(s) \\ & + [\sigma_i^2 + c_i s - (a_i + q_i) - \lambda_i(1 - \widehat{f}_i(s))] \widehat{\psi}_i(s) + \sum_{j=1, j \neq i}^m q_{ij} \widehat{\psi}_j(s) \\ & = c_i \psi_i(0) - \lambda_i \widehat{F}_i(s), \end{aligned}$$

165 or in matrix form

$$s^2 \frac{d^2 \vec{\psi}(s)}{ds^2} + s \mathbf{A} \frac{d \vec{\psi}(s)}{ds} + \mathbf{B}(s) \vec{\psi}(s) = \mathbf{c} \vec{\psi}(0) - \mathbf{\Lambda} \vec{k}(s), \quad (4.1)$$

with

$$\mathbf{A} = \text{diag} \left(4 - \frac{2a_1}{\sigma_1^2}, \dots, 4 - \frac{2a_m}{\sigma_m^2} \right),$$

$$\vec{\psi}(s) = [\hat{\psi}_1(s), \dots, \hat{\psi}_m(s)]^T,$$

$$\vec{\psi}(0) = [\psi_1(0), \dots, \psi_m(0)]^T,$$

$$\mathbf{c} = \text{diag} \left(\frac{2c_1}{\sigma_1^2}, \dots, \frac{2c_m}{\sigma_m^2} \right),$$

$$\mathbf{\Lambda} = \text{diag} \left(\frac{2\lambda_1}{\sigma_1^2}, \dots, \frac{2\lambda_m}{\sigma_m^2} \right),$$

$$\vec{k}(s) = [\hat{F}_1(s), \dots, \hat{F}_m(s)]^T,$$

166 where the superscript, $(\cdot)^T$, denotes the transpose of a vector/matrix, and

$$\mathbf{B}(s) = \begin{pmatrix} \frac{2}{\sigma_1^2} Z_1(s) & \frac{2}{\sigma_1^2} q_{1,2} & \cdots & \frac{2}{\sigma_1^2} q_{1,m} \\ \frac{2}{\sigma_2^2} q_{2,1} & \ddots & & \vdots \\ \vdots & & \ddots & \frac{2}{\sigma_{m-1}^2} q_{m-1,m} \\ \frac{2}{\sigma_m^2} q_{m,1} & \cdots & \frac{2}{\sigma_m^2} q_{m,m-1} & \frac{2}{\sigma_m^2} Z_m(s) \end{pmatrix}, \quad (4.2)$$

167 with $Z_i(s) = \sigma_i^2 + c_i s - (a_i + q_i) - \lambda_i(1 - \hat{f}_i(s))$, $i \in E$.

168 The form of the non-homogeneous matrix equation (4.1) will be used in the sequel
 169 so as to derive asymptotic expressions for $\vec{\psi}(s)$ and consequently for the ultimate ruin
 170 probability, namely $\psi(u)$.

171 5 Asymptotic results for arbitrary claim size distributions

172 In this section we analyse the asymptotic behaviour of the Laplace transform vectors,
 173 for the ruin probabilities, and derive asymptotic expressions using Karamata-Tauberian
 174 theorems and Heaviside Principle. In order to achieve this, we first need to draw the
 175 solution of the Laplace transform vector, satisfying equation (4.1), in the neighbourhood
 176 of their singularities.

177 Let us define the following m -dimensional vector $\vec{\psi}(s) = \vec{y}(s) = (y_1(s), \dots, y_m(s))^T$,
 178 with $\frac{d}{ds}\vec{\psi}(s) = \vec{y}'(s)$ and $\frac{d^2}{ds^2}\vec{\psi}(s) = \vec{y}''(s)$ denoting the first and second derivatives re-
 179 spectively, of every element of the vector $\vec{y}(s)$. Then, we can rewrite equation (4.1), as
 180 follows

$$s^2\vec{y}''(s) + s\mathbf{A}\vec{y}'(s) + \mathbf{B}(s)\vec{y}(s) = \vec{h}(s), \quad (5.1)$$

181 where $\vec{h}(s)$ is the m -dimensional vector, given by

$$\vec{h}(s) = \mathbf{c}\vec{\Psi}(0) - \mathbf{\Lambda}\vec{k}(s). \quad (5.2)$$

182 By the general methodology of differential equations, equation (5.1) has a general solution
 183 of the form

$$\vec{y}(s) = \vec{y}_h(s) + \vec{y}_p(s), \quad (5.3)$$

184 where $\vec{y}_p(s)$ is a particular solution vector and $\vec{y}_h(s)$ is the associated homogeneous solution
 185 vector to the corresponding homogeneous matrix equation of (5.1).

186 **Remark 2.** *The corresponding homogeneous equation system of (4.1) has a regular sin-*
 187 *gular point at zero and will play a vital role in the formulation of its solution, while the*
 188 *extra term in the non-homogeneous system depends on the Laplace transform of the tail of*
 189 *the claim size distribution.*

190 For the rest of this section let us consider the analysis of the associated homogeneous
 191 equation system and the analysis of the particular solution to the matrix equation (5.1)
 192 separately. First, let us consider the associated homogeneous equation of (5.1), which has
 193 the form

$$s^2\vec{y}''(s) + s\mathbf{A}\vec{y}'(s) + \mathbf{B}(s)\vec{y}(s) = \vec{0}, \quad (5.4)$$

where $\vec{0}$ is an m -dimensional vector of zero elements. The form of the second order linear
 homogeneous differential matrix equation (5.4) and the presence of the regular singular
 point at $s = 0$, requires that the Frobenius method should be employed to determine the
 solution. Thus, using similar arguments to Barkatou et al. (2010) we consider that the
 vector solution to (5.4) is in a Frobenius form for systems, i.e.

$$\vec{y}(s, r) = \sum_{k=0}^{\infty} \vec{g}_k(r) s^{r+k},$$

194 where $\vec{y}(s, r) = (y_1(s, r), \dots, y_m(s, r))^T$ and $\vec{g}_k(r) = (g_{1,k}(r), \dots, g_{m,k}(r))^T$, $k \geq 0$, are m -
 195 dimensional vectors, where \vec{g}_0 is non-zero and the exponent r may be real or complex.
 196 Differentiating the above form of the solution vector twice, with respect to s , gives

$$\begin{aligned}\vec{y}'(s, r) &= \sum_{k=0}^{\infty} (r+k)\vec{g}_k(r)s^{r+k-1}, \\ \vec{y}''(s, r) &= \sum_{k=0}^{\infty} (r+k)(r+k-1)\vec{g}_k(r)s^{r+k-2}.\end{aligned}$$

Substituting the above forms of the vectors for $\vec{y}(s, r)$, $\vec{y}'(s, r)$ and $\vec{y}''(s, r)$ into the homogeneous second order matrix differential equation (5.4), yields

$$\begin{aligned}\sum_{k=0}^{\infty} (r+k)(r+k-1)\vec{g}_k(r)s^{r+k} \\ + \mathbf{A} \sum_{k=0}^{\infty} (r+k)\vec{g}_k(r)s^{r+k} + \mathbf{B}(s) \sum_{k=0}^{\infty} \vec{g}_k(r)s^{r+k} = \vec{0}.\end{aligned}$$

197 Analysing the above equation, we can see that by dividing through by the common term
198 s^r and then setting $s = 0$, all terms with $k > 0$ vanish. Thus, we can deduce that r is the
199 solution of the indicial matrix equation

$$r(r-1)\vec{g}_0(r) + \mathbf{A}r\vec{g}_0(r) + \mathbf{B}(0)\vec{g}_0(r) = \vec{0}. \quad (5.5)$$

Recalling the definition of $\mathbf{B}(s)$ and noting that $\hat{f}_i(0) = 1$ for all $i \in E$, we can easily see that $\mathbf{B}(0)$ is an $m \times m$ matrix with constant elements, given by

$$\mathbf{B}(0) = \begin{pmatrix} \frac{2}{\sigma_1^2}(\sigma_1^2 - (a_1 + q_1)) & \frac{2}{\sigma_1^2}q_{1,2} & \cdots & \cdots & \frac{2}{\sigma_1^2}q_{1,m} \\ \frac{2}{\sigma_2^2}q_{2,1} & & \ddots & & \vdots \\ \vdots & & & & \vdots \\ \vdots & & & & \frac{2}{\sigma_{m-1}^2}q_{m-1,m} \\ \frac{2}{\sigma_m^2}q_{m,1} & \cdots & \cdots & \frac{2}{\sigma_m^2}q_{m,m-1} & \frac{2}{\sigma_m^2}(\sigma_m^2 - (a_m + q_m)) \end{pmatrix}.$$

200 Alternatively, the indicial matrix equation (5.5), may be written as

$$\mathbf{L}(r)\vec{g}_0(r) = \vec{0}, \quad (5.6)$$

201 where

$$\mathbf{L}(r) = r^2\mathbf{I} + (\mathbf{A} - \mathbf{I})r + \mathbf{B}(0), \quad (5.7)$$

202 is an $m \times m$ matrix and \mathbf{I} is the m -dimensional identity matrix.

203 An equation of this form has non-trivial solutions, $\vec{g}_0(r)$, only for $\det(\mathbf{L}(r)) = 0$, known
204 as the characteristic equation. Since the determinant of $\mathbf{L}(r)$ gives a polynomial of degree
205 $2m$, with leading coefficient 1, we have the following Lemma.

206

207 **Lemma 1.** For $r \in \mathbb{C}$, the characteristic equation, $\det(\mathbf{L}(r)) = 0$, has exactly $2m$ solutions
 208 r_1, r_2, \dots, r_{2m} .

209 Now, referring back to Frobenius' method, any set of fundamental solutions to the homo-
 210 geneous matrix equation (5.4), may be written

$$\vec{y}_i(s) = s^{r_i} \sum_{k=0}^{\infty} \vec{g}_k(r_i) s^k = s^{r_i} \vec{\gamma}_i(s), \quad i = 1, \dots, 2m, \quad (5.8)$$

211 where $\vec{y}_i(s) = (y_{i,1}(s), \dots, y_{i,m}(s))^T = \vec{y}(s, r_i)$ and $\vec{\gamma}_i(s) = (\gamma_{i,1}(s), \dots, \gamma_{i,m}(s))^T$ are vectors
 212 of holomorphic functions with $\vec{\gamma}_i(0) = \vec{g}_0(r_i) \neq 0$. Then, as it will be shown later, since
 213 the vector solutions $\vec{y}_i(s)$ are linearly independent, the general solution to equation (5.4)
 214 is given by

$$\vec{y}_h(s) = \sum_{i=1}^{2m} \eta_i \vec{y}_i(s) = \eta_1 s^{r_1} \vec{\gamma}_1(s) + \dots + \eta_{2m} s^{r_{2m}} \vec{\gamma}_{2m}(s), \quad (5.9)$$

215 where η_i are constant coefficients and r_i , $i = 1, \dots, 2m$ are the solutions to the characteristic
 216 equation $\det(\mathbf{L}(r)) = 0$. The linear independence of the solution vectors will be made more
 217 apparent in a later section.

218 In particular, the j -th element of the solution vector, $\vec{y}_h(s)$, is given by

$$y_{h,j}(s) = \sum_{i=1}^{2m} \eta_i y_{i,j}(s) = \eta_1 s^{r_1} \gamma_{1,j}(s) + \dots + \eta_{2m} s^{r_{2m}} \gamma_{2m,j}(s). \quad (5.10)$$

219 Having obtained a general solution for the homogeneous solution, it remains to determine
 220 the contribution of the particular solution $\vec{y}_p(s)$ of equation (5.3).

To find the particular solution of the differential equation system (5.1), we use the
 method of variation of parameters, similar to Albrecher et al. (2012). Hence, the particular
 solution has the following form

$$\vec{y}_p(s) = \sum_{i=1}^{2m} v_i(s) \vec{y}_i(s),$$

221 where $\vec{y}_i(s)$, $i = 1, \dots, 2m$ are the solution vectors to the homogeneous equation (5.4), given
 222 by equation (5.8), and $v_i(s)$ are scalar coefficients that need to be determined.

223 By the method of variation of parameters and the use of Cramer's rule, the variables
 224 $v_i(s)$, $i = 1 \dots, 2m$, have the following form

$$v_i(s) = \int_{s_0}^s \frac{W_i(t)}{t^2 W(t)} dt, \quad (5.11)$$

where s_0 is a small positive constant, $W(s) (\neq 0)$ is the Wronskian (block) determinant given by

$$W(s) = \begin{vmatrix} \vec{y}_1(s) & \vec{y}_2(s) & \dots & \vec{y}_{2m}(s) \\ \vec{y}'_1(s) & \vec{y}'_2(s) & \dots & \vec{y}'_{2m}(s) \end{vmatrix},$$

and $W_i(s)$ is a consequence of $W(s)$, with the i -th column replaced with $(\vec{0}, \vec{h}(s))^T$. For example, for $i = 1$, $W_1(s)$ is given by

$$W_1(s) = \begin{vmatrix} \vec{0} & \vec{y}_2(s) & \dots & \vec{y}_{2m}(s) \\ \vec{h}(s) & \vec{y}'_2(s) & \dots & \vec{y}'_{2m}(s) \end{vmatrix} = \begin{vmatrix} 0 & y_{2,1} & \dots & y_{2m,1} \\ \vdots & \vdots & \dots & \vdots \\ 0 & y_{2,m} & \dots & y_{2m,m} \\ h_1(s) & y'_{2,1} & \dots & y'_{2m,1} \\ \vdots & \vdots & \dots & \vdots \\ h_m(s) & y'_{2,m} & \dots & y'_{2m,m} \end{vmatrix}.$$

225 **Remark 3.** $W(s) \neq 0$ implies the linear independence of the solutions $\vec{y}_i(s)$.

226 Equation (5.11) can be re-written as

$$v_i(s) = \int_{s_0}^s \frac{\vec{h}(t)^T \vec{W}_i(t)}{t^2 W(t)} dt, \quad (5.12)$$

227 where $\vec{h}(s)^T = (h_1(s), \dots, h_m(s))$ is the transpose vector of that given in (5.2) and $\vec{W}_i(s) =$
 228 $(W_{i,1}(s), \dots, W_{i,m}(s))^T$ is a vector of corresponding Wronskian determinants, namely $W_{i,j}(s)$,
 229 which are a consequence of $W(s)$, with the i -th column replaced by $(0, \dots, 0, 1, 0, \dots, 0)^T$,
 230 where the unit is in the $(m + j)$ -th row.

231 After algebraic manipulations, the above equation can be written as

$$v_i(s) = \sum_{k=1}^m \int_{s_0}^s t^{-r_i-1} h_k(t) \frac{\xi_{i,k}(t)}{\xi(t)} dt, \quad (5.13)$$

232 where $\xi(t)$ and $\xi_{i,k}(t)$, $i = 1, \dots, 2m$, are holomorphic functions, with $\xi(0) \neq 0 \neq \xi_{i,k}(0)$
 233 (as they are linear combinations of $\gamma_{i,j}(s)$, $i = 1, \dots, 2m$, $j \in E$ and their derivatives, for
 234 which $\vec{\gamma}_i(0) = \vec{g}_0(r_i) \neq 0$ holds).

Recalling the definition of $\vec{h}(s)$ from equation (5.2), we see that $h_k(s)$ has the form

$$h_k(s) = \frac{2}{\sigma_k} \left(c_k \psi_k(0) - \lambda_k \widehat{F}_k(s) \right), \quad k \in E,$$

and thus we can write the particular solution to the non-homogeneous second order differ-

ential equation system (5.1) as

$$\begin{aligned}
\vec{y}_p(s) &= \sum_{i=1}^{2m} s^{r_i} \vec{\gamma}_i(s) v_i(s) \\
&= \sum_{i=1}^{2m} s^{r_i} \vec{\gamma}_i(s) \sum_{k=1}^m \int_{s_0}^s t^{-r_i-1} h_k(t) \frac{\xi_{i,k}(t)}{\xi(t)} dt \\
&= \sum_{i=1}^{2m} s^{r_i} \vec{\gamma}_i(s) \sum_{k=1}^m \frac{2c_k \psi_k(0)}{\sigma_k^2} \int_{s_0}^s t^{-r_i-1} \frac{\xi_{i,k}(t)}{\xi(t)} dt \\
&\quad - \sum_{i=1}^{2m} s^{r_i} \vec{\gamma}_i(s) \sum_{k=1}^m \frac{2\lambda_k}{\sigma_k^2} \int_{s_0}^s t^{-r_i-1} \widehat{F}_k(t) \frac{\xi_{i,k}(t)}{\xi(t)} dt. \tag{5.14}
\end{aligned}$$

From this equation, we can see that the particular solution, for each element $y_{p,j}(s)$, $j \in E$ of $\vec{y}_p(s)$, has the form

$$\begin{aligned}
y_{p,j}(s) &= \sum_{i=1}^{2m} s^{r_i} \gamma_{i,j}(s) \sum_{k=1}^m \frac{2c_k \psi_k(0)}{\sigma_k^2} \int_{s_0}^s t^{-r_i-1} \frac{\xi_{i,k}(t)}{\xi(t)} dt \\
&\quad - \sum_{i=1}^{2m} s^{r_i} \gamma_{i,j}(s) \sum_{k=1}^m \frac{2\lambda_k}{\sigma_k^2} \int_{s_0}^s t^{-r_i-1} \widehat{F}_k(t) \frac{\xi_{i,k}(t)}{\xi(t)} dt. \tag{5.15}
\end{aligned}$$

235 From the form of the above equations and using equations (5.3) and (5.10), it is clear
236 that the asymptotic behaviour of $\widehat{\psi}_j(s)$, and thus of $\psi_j(s)$, heavily depends on the roots
237 r_i , $i = 1, \dots, 2m$, of the characteristic equation $\det(\mathbf{L}(r)) = 0$, and the behaviour of
238 $\widehat{F}_k(s)$, $k \in E$.

239 Having determined the general solution of the matrix equation (5.1) (given by equa-
240 tions (5.3), (5.9) and (5.14)), in the subsequent work we will perform an asymptotic analysis
241 using Karamata-Tauberian theorems and the Heaviside Operational principle, for the ho-
242 mogeneous solution and particular solutions respectively. We separate the cases for $y_{h,j}(s)$
243 and $y_{p,j}(s)$ and consequently $\widehat{\psi}_{h,j}(s)$ and $\widehat{\psi}_{p,j}(s)$ respectively, as follows.

Now, since the Karamata-Tauberian theorems correspond to the asymptotic behaviour of the Laplace-Stieltjes transform of a function, then, for the analysis of $y_{h,j}(s)$ similarly to Albrecher et al. (2012), we introduce the auxiliary functions

$$U_j(u) = \begin{cases} 0 & \text{if } u < 0 \\ \int_0^u \psi_{h,j}(x) dx & \text{if } u \geq 0. \end{cases}$$

Let $\widetilde{U}_j(s)$ be the Laplace-Stieltjes transform of $U_j(u)$. Note that the Laplace transform of the ruin probabilities $\psi_{h,j}(u)$, defined as $\widehat{\psi}_{h,j}(s)$, is equivalent to the Laplace-Stieltjes

transform of the function $U_j(u)$, i.e.

$$\widehat{\psi}_{h,j}(s) = \mathcal{L}(\psi_{h,j}(u))(s) = \int_0^\infty e^{-su} \psi_{h,j}(u) du = \int_0^\infty e^{-su} dU_j(u) = \widetilde{U}_j(s).$$

244 The asymptotic behaviour at zero of the homogeneous solutions, given by equation (5.10),
 245 describes the asymptotic behaviour at zero of $\widehat{\psi}_{h,j}(s)$, consequently of $\widetilde{U}_j(s)$. The slowest
 246 decaying power of this linear combination dictates the asymptotic behaviour of the solution
 247 as $s \rightarrow 0$. In general, this power can be found numerically by evaluating all roots r_i , $i \in E$,
 248 to the characteristic equation $\det(\mathbf{L}(r)) = 0$, however, in order to explicitly determine the
 249 leading power of this equation we must restrict ourselves to the case where the drift and
 250 volatility parameters of the investment process are all equal, i.e. $a_i = a, \sigma_i = \sigma$ for all
 251 $i \in E$. Note that this restriction does not affect the Markov-modulated environment of
 252 the arrival process which is still influenced by the external environment process. Adopting
 253 this modification and using the following two Lemmas, we are able to show that the rate of
 254 decay, of the homogeneous solution, is driven by the slowest decaying power, corresponding
 255 to the leading power of equation (5.10), which will be determined.

256
 257 **Lemma 2.** *The transition rate matrix \mathbf{Q} has 0 as an eigenvalue and the remaining eigen-*
 258 *values have negative real parts.*

Proof. Let η be a real positive number greater than the absolute value of all entries of \mathbf{Q} ,
 i.e. $\eta > |q_{ij}|$, $\forall i, j \in E$. Now, define the matrix

$$\mathbf{P} = \frac{1}{\eta} \mathbf{Q} + \mathbf{I},$$

with elements

$$p_{ij} = \frac{1}{\eta} q_{ij} + \mathbb{I}_{(i=j)},$$

where $\mathbb{I}_{(\cdot)}$ is an indicator function. Now, since

$$\sum_{j \in E} p_{ij} = \sum_{j \in E} \left(\frac{1}{\eta} q_{ij} + \mathbb{I}_{(i=j)} \right) = \frac{1}{\eta} \sum_{j \in E} q_{ij} + \sum_{j \in E} \mathbb{I}_{(i=j)} = 1, \quad i \in E,$$

and

$$p_{ij} = \frac{1}{\eta} q_{ij} \geq 0, \quad i \neq j \in E,$$

$$p_{ii} = \frac{1}{\eta} q_{ii} + 1 \geq 1 - \frac{1}{\eta} |q_{ii}| > 1 - 1 = 0, \quad \text{since } \eta > |q_{ii}|,$$

259 the matrix \mathbf{P} is a stochastic matrix.

Now, note that the eigenvalues of \mathbf{P} and \mathbf{Q} are related as follows. If λ is an eigenvalue of \mathbf{P} , with right eigenvector \vec{y} , then

$$\mathbf{P}\vec{y} = \lambda\vec{y},$$

giving that

$$\mathbf{Q}\vec{y} = (\eta\mathbf{P} - \eta\mathbf{I})\vec{y} = \eta\mathbf{P}\vec{y} - \eta\vec{y} = (\eta\lambda - \eta)\vec{y} = \eta(\lambda - 1)\vec{y},$$

which implies that $\eta(\lambda - 1)$ is an eigenvalue of \mathbf{Q} . Now, by the Perron-Frobenius theorem, we have that $\lambda_{max} = 1$ is the maximum eigenvalue of \mathbf{P} and the remaining eigenvalues λ are such that $|\lambda| < 1$. Based on the connection between the eigenvalues of \mathbf{P} and \mathbf{Q} , for the maximum eigenvalue, namely $\lambda_{max} = 1$, the corresponding eigenvalue of \mathbf{Q} is equal to 0. Thus, in order to complete the Lemma, it remains to prove that the remaining eigenvalues of \mathbf{Q} have negative real parts. The remaining eigenvalues of \mathbf{P} are λ such that $|\lambda| < 1$, which for complex λ implies its real part has absolute value less than 1. Thus, since the eigenvalues of \mathbf{Q} are of the form $\eta(\lambda - 1)$, we have

$$\Re(\eta(\lambda - 1)) = \eta(\Re(\lambda) - 1) < 0,$$

260 since η is real and positive. □

261 **Lemma 3.** For $a_i = a$ and $\sigma_i = \sigma$, for all $i \in E$, the characteristic equation $\det(\mathbf{L}(r)) = 0$
 262 has two roots, $r_1 = -1$ and $r_2 = \frac{2a}{\sigma^2} - 2 = \rho - 1$. The remaining roots all have real parts
 263 that lie outside the interval determined by r_1 and r_2 .

Proof. In order to find the roots of the characteristic equation, $\det(\mathbf{L}(r)) = 0$, where $\mathbf{L}(r) = r^2\mathbf{I} + (\mathbf{A} - \mathbf{I})r + \mathbf{B}(0)$, we need to rewrite $\mathbf{L}(r)$ in a slightly different form. Recalling the forms of the matrices \mathbf{A} and $\mathbf{B}(0)$, and after some algebraic manipulations we have that

$$\mathbf{L}(r) = \alpha(r)\mathbf{I} + \frac{2}{\sigma^2}\mathbf{Q},$$

264 where $\alpha(r) = r^2 + (3 - \frac{2a}{\sigma^2})r + 2 - \frac{2a}{\sigma^2} = (r + 1)(r + 2 - \frac{2a}{\sigma^2})$.

Recalling the indicial matrix equation (5.6), and using the above expression of $\mathbf{L}(r)$, equation (5.6) may be written

$$\left(\alpha(r)\mathbf{I} + \frac{2}{\sigma^2}\mathbf{Q}\right)\vec{g}_0(r) = \vec{0},$$

265 or equivalently

$$\frac{2}{\sigma^2}\mathbf{Q}\vec{g}_0(r) = -\alpha(r)\vec{g}_0(r). \tag{5.16}$$

From the above equation we see that $-\alpha(r)$ forms an eigenvalue with respect to the matrix $\frac{2}{\sigma^2}\mathbf{Q}$. Thus, solving

$$\det(\mathbf{L}(r)) = \det\left(\alpha(r)\mathbf{I} + \frac{2}{\sigma^2}\mathbf{Q}\right) = 0,$$

266 is equivalent to finding the eigenvalues of the matrix $\frac{2}{\sigma^2}\mathbf{Q}$, which are of the form $-\alpha(r)$.

Using Lemma 2, and since $2/\sigma^2$ is real and positive, it follows that the matrix $\frac{2}{\sigma^2}\mathbf{Q}$ also has 0 as an eigenvalue with remaining eigenvalues having negative real parts. Now, given that $-\alpha(r)$ forms an eigenvalue of $\frac{2}{\sigma^2}\mathbf{Q}$, we have for the 0 eigenvalue that

$$-\alpha(r) = 0,$$

267 which implies

$$\alpha(r) = (r + 1) \left(r + 2 - \frac{2a}{\sigma^2} \right) = 0, \quad (5.17)$$

268 giving the two roots $r_1 = -1$ and $r_2 = 2a/\sigma^2 - 2 = \rho - 1$. Consequently, $r = r_1$ and $r = r_2$
269 are two roots of the characteristic equation $\det(\mathbf{L}(r)) = 0$.

270 To complete our Lemma it remains to prove that the real parts of the remaining roots
271 lie outside the interval determined by r_1 and r_2 .

Consider that the eigenvalues of the matrix $\frac{2}{\sigma^2}\mathbf{Q}$ have complex form i.e. they are given by

$$-\alpha_k(r) = u_k + iv_k, \quad k = 1, 2, \dots, m, \quad k \neq j,$$

272 where u_k and v_k are real numbers and $-\alpha_j(r)$ is an individual eigenvalue corresponding to
273 the 0 eigenvalue (without the loss of generality).

Using the form of $\alpha(r)$ given in equation (5.17), and the fact that $r_1 = -1$ and $r_2 = \rho - 1$ satisfy equation (5.17), then $\alpha_k(r)$ could be written

$$\alpha_k(r) = (r - r_1)(r - r_2) = -(u_k + iv_k).$$

Since r can also be complex, i.e. $r = x + iy$, the above equation becomes

$$(x - r_1 + iy)(x - r_2 + iy) = -(u_k + iv_k).$$

Equating the real terms gives

$$(x - r_1)(x - r_2) - y^2 = -u_k,$$

or alternatively

$$(x - r_1)(x - r_2) = y^2 - u_k.$$

274 Now, from Lemma 2, we have that the non-zero eigenvalues have negative real parts im-
275 plying that $u_k < 0$, for $k \neq j$. Therefore $(x - r_1)(x - r_2) > 0$, from which it follows that
276 $(x - r_1)$ and $(x - r_2)$ have the same sign. That is, x is either larger or smaller than both
277 r_1 and r_2 .

278 Note that in the case that r has no imaginary part, i.e. $y = 0$, the same argument
279 holds, meaning that the other real solutions also lie outside the interval determined by r_1
280 and r_2 . This completes our proof. \square

Remark 4. Since r_1 and r_2 correspond to the 0 eigenvalue of $\frac{2}{\sigma^2}\mathbf{Q}$ and hence \mathbf{Q} , it follows from equation (5.16) that

$$\mathbf{Q}\vec{g}_0(r_k) = \vec{0}, \quad k = 1, 2.$$

281 Using the fact that the elements in each row of \mathbf{Q} sum to 0, it is not difficult to see that
 282 $\vec{g}_0(r_k) = \beta\vec{e}$, $k = 1, 2$, where \vec{e} is an m -dimensional vector of units and β is arbitrary, let's
 283 say $\beta = 1$.

Proposition 1. Consider the model given by (2.1) and assume that $\sigma > 0$. Then, if the ruin probability $\psi(u)$ decays at infinity, we have

$$\rho = \frac{2a}{\sigma^2} - 1 > 0.$$

284 *Proof.* The proof of this proposition will become apparent towards the end of this section.
 285 □

286 Using the two Lemmas above, we can determine the slowest decaying power of the ho-
 287 mogeneous solution to the vector equation (5.3), given by equation (5.9). Note that, by
 288 Proposition 1 we have $r_1 < r_2$. Now, the boundary condition $\lim_{u \rightarrow \infty} \psi_i(u) = 0$, and the
 289 use of final value theorem, implies that the coefficients of terms with powers that have real
 290 part less than r_1 in equation (5.10), must be zero. Consequently this makes r_1 the slowest
 291 decaying power.

292 Next, we will apply Karamata-Tauberian theorems to find the asymptotic behaviour
 293 of the homogeneous solution. It is crucial to observe that by applying the Karamata-
 294 Tauberian theorem, in the case that the slowest decaying power of equation (5.9) is r_1 ,
 295 results in the fact that the ruin probabilities converge to a constant, which is in contradic-
 296 tion with the boundary condition (3.3). Hence, it should be clear that the coefficient of
 297 s^{r_1} , namely η_1 , vanishes.

Based on the above observation, we conclude that eventually the slowest decaying power is r_2 . Thus, we are ready now to apply Karamata-Tauberian theorem and the Monotone Density theorem to find the asymptotic behaviour of the homogeneous solution, given by equation (5.9). Since we have concluded the root r_2 represents the slowest decaying power, we have that the individual elements of the homogeneous solution vector, $\vec{y}_h(s)$, behave like

$$\tilde{U}_j(s) \sim \eta_2 s^{\rho-1} \gamma_{2,j}(s), \quad s \rightarrow 0,$$

which is equivalent to

$$U_j(u) \sim \frac{\eta_2 u^{1-\rho} \gamma_{2,j}(1/u)}{\Gamma(2-\rho)}, \quad u \rightarrow \infty,$$

by the application of Karamata-Tauberian theorem. Finally, applying the Monotone Density theorem gives

$$\psi_{h,j}(u) \sim \frac{\eta_2(1-\rho)u^{-\rho} \gamma_{2,j}(1/u)}{\Gamma(2-\rho)}, \quad u \rightarrow \infty.$$

298 Note that, since $\rho > 0$ by Proposition 1, $\psi_{h,j}(u)$ decays to zero, as required, and the
 299 conclusion is that

$$\psi_{h,j}(u) \sim C u^{-\rho} \gamma_{2,j}(1/u), \quad u \rightarrow \infty \quad (5.18)$$

where $C = \frac{\eta_2(1-\rho)}{\Gamma(2-\rho)}$. Alternatively, we have

$$\lim_{u \rightarrow \infty} \psi_{h,j}(u) u^\rho = C$$

300 since $\gamma_{2,j}(0) = g_{j,0}(r_2) = 1$ (see Remark 4).

301

302 Having completed the asymptotic analysis of the homogeneous part of equation (5.3), it re-
 303 mains to analyse the asymptotic behaviour of the particular solution of the aforementioned
 304 equation, namely $\vec{y}_p(s)$. Noticing that the elements of the vector $\vec{y}_p(s)$, given by equation
 305 (5.15), strongly depend on the tail of the claim size distribution, below we consider two
 306 separate cases.

307 Depending on the distribution of $\widehat{F}_k(s)$ we can identify two cases, similarly to Albrecher
 308 et al. (2012):

309 **A.** Light tailed claims with exponentially bounded tails. Assume $\widehat{F}_k(s)$ has a rightmost
 310 singularity at $-\mu_k < 0$, $k \in E$, and $\widehat{F}_k(-\mu_k) = \infty$ for each $k \in E$.

311 **B.** Heavy tailed claims $\widehat{F}_k(-\epsilon) = \infty$, for $\epsilon > 0$, $k \in E$.

312 **Light Tailed claims.** Let us first note that if $-\mu_k$ is the rightmost singularity of each
 313 $\widehat{F}_k(s)$, $k \in E$, then $-\delta$, where $\delta = \min_{k \in E}(\mu_k)$, is the rightmost singularity of the summa-
 314 tion of $\widehat{F}_k(s)$, $k \in E$. Now, using L'Hopital's rule, we have

$$\begin{aligned} \lim_{s \rightarrow -\delta} \frac{\sum_{k=1}^m \lambda_k \int_{s_0}^s t^{-r_i-1} \widehat{F}_k(t) dt}{s^{-r_i} \sum_{k=1}^m \lambda_k \widehat{F}_k(s)} &= \lim_{s \rightarrow -\delta} \frac{\sum_{k=1}^m \lambda_k s^{-r_i-1} \widehat{F}_k(s)}{-r_i s^{-r_i-1} \sum_{k=1}^m \lambda_k \widehat{F}_k(s) + s^{-r_i} \sum_{k=1}^m \lambda_k \frac{d}{ds} \widehat{F}_k(s)} \\ &= \lim_{s \rightarrow -\delta} \frac{1}{-r_i + s \frac{\sum_{k=1}^m \lambda_k \frac{d}{ds} \widehat{F}_k(s)}{\sum_{k=1}^m \lambda_k \widehat{F}_k(s)}} = \frac{1}{-r_i}. \end{aligned}$$

Thus,

$$\sum_{k=1}^m \lambda_k \int_{s_0}^s t^{-r_i-1} \widehat{F}_k(t) dt \sim \frac{1}{-r_i} s^{-r_i} \sum_{k=1}^m \lambda_k \widehat{F}_k(s), \quad \text{as } s \rightarrow -\delta.$$

Then, from equation (5.15), we have

$$\begin{aligned} \widehat{\psi}_{p,j}(s) &\sim \sum_{i=1}^{2m} \left(\frac{1}{-r_i} \right) \gamma_{i,j}(-\delta) \sum_{k=1}^m \frac{2c_k \psi_k(0)}{\sigma^2} \frac{\xi_{i,k}(-\delta)}{\xi(-\delta)} \\ &\quad - \sum_{i=1}^{2m} \left(\frac{1}{-r_i} \right) \gamma_{i,j}(-\delta) \sum_{k=1}^m \frac{2\lambda_k}{\sigma^2} \frac{\xi_{i,k}(-\delta)}{\xi(-\delta)} \widehat{F}_k(s), \quad s \rightarrow -\delta. \end{aligned}$$

315 Normalise $\xi_{i,k}(-\delta)$ such that $\gamma_{i,j}(-\delta) \frac{\xi_{i,k}(-\delta)}{\xi(-\delta)} = 1$, for all $i = 1, \dots, 2m$, $k = 1, \dots, m$. Since
 316 $-\delta$ is the rightmost singularity of $\widehat{\psi}_{p,j}(s)$ and the first term of the above equation is analytic
 317 in $-\delta$, one can apply the Heaviside Operational Principle (see Abate and Whitt (1997)) to
 318 deduce

$$\psi_{p,j}(u) \sim \frac{2}{\sigma^2} \sum_{i=1}^{2m} \frac{1}{r_i} \sum_{k=1}^m \lambda_k \overline{F}_k(u), \quad u \rightarrow \infty. \quad (5.19)$$

319 **Heavy Tailed claims.** Using L'Hopital's rule and other limit properties, we have, for
 320 each $k \in E$

$$\begin{aligned} \lim_{s \rightarrow 0} \frac{\sum_{k=1}^m \lambda_k \int_{s_0}^s t^{-r_i-1} \widehat{F}_k(t) dt}{s^{-r_i} \sum_{k=1}^m \lambda_k \widehat{F}_k(s)} &= \lim_{s \rightarrow 0} \frac{\sum_{k=1}^m \lambda_k s^{-r_i-1} \widehat{F}_k(s)}{-r_i s^{-r_i-1} \sum_{k=1}^m \lambda_k \widehat{F}_k(s) + s^{-r_i} \sum_{k=1}^m \lambda_k \frac{d}{ds} \widehat{F}_k(s)} \\ &= \lim_{s \rightarrow 0} \frac{1}{-r_i + s \frac{\sum_{k=1}^m \lambda_k \frac{d}{ds} \widehat{F}_k(s)}{\sum_{k=1}^m \lambda_k \widehat{F}_k(s)}} = \frac{1}{-r_i}. \end{aligned}$$

Thus,

$$\sum_{k=1}^m \lambda_k \int_{s_0}^s t^{-r_i-1} \widehat{F}_k(t) dt \sim \frac{1}{-r_i} s^{-r_i} \sum_{k=1}^m \lambda_k \widehat{F}_k(s), \quad \text{as } s \rightarrow 0.$$

Then,

$$\begin{aligned} \widehat{\psi}_{p,j}(s) &\sim \sum_{i=1}^{2m} \left(\frac{1}{-r_i} \right) \gamma_{i,j}(0) \sum_{k=1}^m \frac{2c_k \psi_k(0)}{\sigma^2} \frac{\xi_{i,k}(0)}{\xi(0)} \\ &\quad - \sum_{i=1}^{2m} \left(\frac{1}{-r_i} \right) \gamma_{i,j}(0) \sum_{k=1}^m \frac{2\lambda_k}{\sigma^2} \frac{\xi_{i,k}(0)}{\xi(0)} \widehat{F}_k(s), \quad s \rightarrow 0. \end{aligned}$$

321 Normalise $\xi_{i,k}(0)$ such that $\gamma_{i,j}(0) \frac{\xi_{i,k}(0)}{\xi(0)} = 1$, for all $i = 1, \dots, 2m$, $k = 1, \dots, m$. Similarly
 322 to previous, the first term is analytic in zero, thus one can apply the Heaviside Operational
 323 Principle to deduce

$$\psi_{p,j}(u) \sim \frac{2}{\sigma^2} \sum_{i=1}^{2m} \frac{1}{r_i} \sum_{k=1}^m \lambda_k \overline{F}_k(u), \quad u \rightarrow \infty. \quad (5.20)$$

324 Now that we have completed the analysis of both the homogeneous and non-homogeneous
 325 parts of equation (5.3) we can present the asymptotic behaviour of the general solution,
 326 for each $j \in E$, namely $\psi_j(u)$. By combining equations (5.18), (5.19) and (5.20), we have

$$\psi_j(u) \sim Cu^{-\rho} \gamma_{2,j}(1/u) + \frac{2}{\sigma^2} \sum_{i=1}^{2m} \frac{1}{r_i} \sum_{k=1}^m \lambda_k \bar{F}_k(u), \quad u \rightarrow \infty. \quad (5.21)$$

327 Consequently, by equation (2.2), we can derive the asymptotic behaviour for the ultimate
 328 ruin probability, $\psi(u)$, given by

$$\psi(u) \sim Cu^{-\rho} \sum_{j=1}^m \pi_j \gamma_{2,j}(1/u) + \frac{2}{\sigma^2} \sum_{i=1}^{2m} \frac{1}{r_i} \sum_{k=1}^m \lambda_k \bar{F}_k(u), \quad u \rightarrow \infty. \quad (5.22)$$

Remark 5. *On the right hand side of equation (5.22) we have a summation of light and/or heavy tailed distributions. Now, since for some positive constants r, n, α_k and c_k ($k = 1, \dots, n$)*

$$\lim_{u \rightarrow \infty} \frac{\sum_{k=1}^n c_k e^{-\alpha_k u}}{u^{-r}} = \sum_{k=1}^n \lim_{u \rightarrow \infty} \frac{c_k e^{-\alpha_k u}}{u^{-r}} = 0,$$

329 *we have that the particular solution does not represent a significant asymptotic decay in*
 330 *the case of light tails. However, in the case of heavy tails we have to compare the decay of*
 331 *the power function and the tail of the claim size distributions to determine which one is*
 332 *slower.*

333 Considering all of the above, we obtain the following theorem.

334

Theorem 2. *Let $a_i = a$ and $\sigma_i = \sigma$, for all $i \in E$. Then, if $\rho = \frac{2a}{\sigma^2} - 1 > 0$, the ultimate ruin probability behaves asymptotically as*

$$\psi(u) \sim Cu^{-\rho} \sum_{j=1}^m \pi_j \gamma_{2,j}(1/u) + \frac{2}{\sigma^2} \sum_{i=1}^{2m} \frac{1}{r_i} \sum_{k=1}^m \lambda_k \bar{F}_k(u), \quad u \rightarrow \infty,$$

335 *where $C = \frac{\eta_2(1-\rho)}{\Gamma(2-\rho)}$.*

336 6 Asymptotic results for the Gerber-Shiu function

337 In this section our aim is to derive asymptotic results with respect to the expected dis-
 338 counted penalty function, introduced first by Gerber and Shiu (1998). The expected dis-
 339 counted penalty function, also called the Gerber-Shiu function, has been extensively studied
 340 in ruin theory since it unifies many risk-related quantities into a single function. In more

341 details, quantities such as the time of ruin T , the deficit at ruin $|U(T)|$, the surplus immedi-
 342 ately prior to ruin $U(T-)$ and many others can be explicitly derived from the Gerber-Shiu
 343 function [see among others Albrecher et al. (2012), Cai (2004) and Lu and Tsai (2007)].

344 Let $\mathbb{P}_i(\cdot) = \mathbb{P}(\cdot | J(0) = i)$ and $\mathbb{E}_i(\cdot)$ be the expectation with respect to \mathbb{P}_i , $i \in E$. Also,
 345 let $w(x, y)$, for $x, y \geq 0$, be an arbitrary non-negative function representing the penalty at
 346 ruin. Then, the Gerber-Shiu function, for $\delta, u \geq 0$, is given by

$$\phi_i(u) = \mathbb{E}_i \left[e^{-\delta T} w(U(T-), |U(T)|) \mathbb{I}_{(T < \infty)} | U(0) = u \right], \quad i \in E, \quad (6.1)$$

where δ can be considered as a constant force of interest. In particular, when $\delta = 0$ and
 $w(x, y) = 1$, we have

$$\phi_i(u) = \mathbb{E}_i \left[\mathbb{I}_{(T < \infty)} | U(0) = u \right] = \psi_i(u).$$

347 In a similar way to the ruin probability we can define the ultimate discounted penalty at
 348 ruin, in the stationary case, by

$$\phi(u) = \sum_{j=1}^m \pi_j \phi_j(u), \quad j \in E. \quad (6.2)$$

349 Using similar arguments as in Theorem 2, we have the following theorem.

350

Theorem 3. *The system of Gerber-Shiu functions, $\phi_i(u)$, satisfy the following integro-differential equation system*

$$\begin{aligned} & \frac{1}{2} \sigma_i^2 u^2 \phi_i''(u) + (a_i u + c_i) \phi_i'(u) \\ & = (\lambda_i + q_i + \delta) \phi_i(u) - \lambda_i \left[\int_0^u \phi_i(u-x) dF_i(x) + w_i(u) \right] - \sum_{j=1, j \neq i}^m q_{ij} \phi_j(u), \end{aligned} \quad (6.3)$$

351 where $w_i(u) = \int_u^\infty w(u, x-u) dF_i(x)$, with boundary conditions

$$\lim_{u \rightarrow \infty} \phi_i(u) = 0, \quad (6.4)$$

352 and

$$c_i \phi_i'(0) - (\lambda_i + q_i + \delta) \phi_i(0) + \lambda_i \int_0^\infty w(0, x) dF_i(x) + \sum_{j=1, j \neq i}^m q_{ij} \phi_j(0) = 0. \quad (6.5)$$

Next, we investigate the asymptotic behaviour of the Gerber-Shiu function using a similar methodology as the one used for the analysis of the ruin probabilities. Thus, letting $\hat{\phi}_i(s)$

and $\widehat{w}_i(s)$ be the Laplace transforms of $\phi_i(u)$ and $w_i(u)$ respectively, taking the Laplace transforms on both sides of equation (6.3), yields

$$\begin{aligned} & \frac{s^2\sigma_i^2}{2}\widehat{\phi}_i''(s) + [s(2\sigma_i^2 - a_i)]\widehat{\phi}_i'(s) \\ & + \left[\sigma_i^2 + c_i s - (a_i + q_i + \delta) - \lambda_i(1 - \widehat{f}_i(s)) \right] \widehat{\phi}_i(s) + \sum_{j=1, j \neq i}^m q_{ij}\widehat{\phi}_j(s) \\ & = c_i\phi_i(0) - \lambda_i\widehat{w}_i(s), \quad i \in E. \end{aligned} \tag{6.6}$$

353 In matrix form, the above equation can be written as

$$s^2 \frac{d^2 \vec{\phi}(s)}{ds^2} + s\mathbf{A} \frac{d\vec{\phi}(s)}{ds} + \mathbf{V}(s)\vec{\phi}(s) = \mathbf{c}\vec{\phi}(0) - \mathbf{\Lambda}\vec{w}(s), \tag{6.7}$$

where

$$\begin{aligned} \vec{\phi}(s) &= [\widehat{\phi}_1(s), \dots, \widehat{\phi}_m(s)]^T \\ \vec{\phi}(0) &= [\phi_1(0), \dots, \phi_m(0)]^T \\ \vec{w}(s) &= [\widehat{w}_1(s), \dots, \widehat{w}_m(s)]^T, \end{aligned}$$

354 $\mathbf{V}(s) = \mathbf{B}(s) - \text{diag}(\frac{2\delta}{\sigma_1^2}, \dots, \frac{2\delta}{\sigma_m^2})$, with $\mathbf{B}(s)$, \mathbf{A} , \mathbf{c} , $\mathbf{\Lambda}$ all defined as in Section 4.

355 Note that, the matrix equation (6.7) is of a similar form as the matrix equation (4.1).
 356 Therefore, this equation can be solved using similar arguments as the ones used for the
 357 analysis of the ruin probabilities, i.e. using the Frobenius method for systems [similar to
 358 Barkatou et al. (2010)].

359 Letting $\vec{\phi}(s) = \vec{x}(s) = (x_1(s), \dots, x_m(s))^T$ (with corresponding first and second deriva-
 360 tive as in the previous section), then equation (6.7) has the form

$$s^2 \vec{x}''(s) + s\mathbf{A}\vec{x}'(s) + \mathbf{V}(s)\vec{x}(s) = \vec{g}(s), \tag{6.8}$$

where $\vec{g}(s) = (g_1(s), \dots, g_m(s))^T$ is the m -dimensional vector, given by

$$\vec{g}(s) = \mathbf{c}\vec{\phi}(0) - \mathbf{\Lambda}\vec{w}(s).$$

By the general theory of ordinary differential equations, equation (6.8) has a general solution of the following form

$$\vec{x}(s) = \vec{x}_h(s) + \vec{x}_p(s),$$

361 where $\vec{x}_h(s)$ is the solution to the corresponding homogeneous matrix equation and $\vec{x}_p(s)$
 362 is the associated particular solution.

363

364 In the following, the particular vector solution $\vec{x}_p(s)$ and the vector solution $\vec{x}_h(s)$ of
 365 the corresponding homogeneous matrix equation of (6.8) will be analysed separately. The
 366 associated homogeneous equation system is given by

$$s^2 \vec{x}''(s) + s \mathbf{A} \vec{x}'(s) + \mathbf{V}(s) \vec{x}(s) = \vec{0}, \quad (6.9)$$

367 and hence, by the Frobenius method, we adopt a solution of the form

$$\vec{x}(s, r_\delta) = \sum_{k=0}^{\infty} \vec{b}_k(r_\delta) s^{r_\delta+k}, \quad (6.10)$$

where $\vec{b}_k(r_\delta) = (b_{k,1}(r_\delta), \dots, b_{k,m}(r_\delta))^T$ is an m -dimensional vector of constants with
 $\vec{b}_0(r_\delta) \neq \vec{0}$, and r_δ is a solution to the characteristic equation

$$\det \left(\mathbf{L}(s) - \text{diag} \left(\frac{2\delta}{\sigma_1^2}, \dots, \frac{2\delta}{\sigma_m^2} \right) \right) = 0,$$

368 where $\mathbf{L}(s)$ is defined in equation (5.7).

369 Following the same arguments as in Lemma 1, one can see that the characteristic
 370 equation has $2m$ roots, namely $r_{\delta,1}, \dots, r_{\delta,2m}$, therefore the solution to the homogeneous
 371 equation system (6.9), by the linear independence of solution vectors, is

$$\vec{x}_h(s) = \sum_{i=1}^{2m} p_i s^{r_{\delta,i}} \vec{\beta}_i(s), \quad (6.11)$$

372 where p_i 's are constant coefficients and $\vec{\beta}_i(s)$ are vectors of holomorphic functions with
 373 $\vec{\beta}_i(0) = \vec{b}_0(r_{\delta,i}) \neq \vec{0}$.

To complete the solution of the second order differential equation system (6.8), it re-
 mains to find the contribution of the particular solution $\vec{x}_p(s)$. For the particular solution,
 we again use variation of parameters to obtain

$$\begin{aligned} \vec{x}_p(s) &= \sum_{i=1}^{2m} s^{r_{\delta,i}} \vec{\beta}_i(s) v_i(s) \\ &= \sum_{i=1}^{2m} s^{r_{\delta,i}} \vec{\beta}_i(s) \sum_{k=1}^m \int_{s_0}^s t^{-r_{\delta,i}-1} g_k(t) \frac{\theta_{i,k}(t)}{\theta(t)} dt \\ &= \sum_{i=1}^{2m} s^{r_{\delta,i}} \vec{\beta}_i(s) \sum_{k=1}^m \frac{2c_k \phi_k(0)}{\sigma_k^2} \int_{s_0}^s t^{-r_{\delta,i}-1} \frac{\theta_{i,k}(t)}{\theta(t)} dt \\ &\quad - \sum_{i=1}^{2m} s^{r_{\delta,i}} \vec{\beta}_i(s) \sum_{k=1}^m \frac{2\lambda_k}{\sigma_k^2} \int_{s_0}^s t^{-r_{\delta,i}-1} \widehat{w}_k(t) \frac{\theta_{i,k}(t)}{\theta(t)} dt, \end{aligned} \quad (6.12)$$

from which we get the following form for the j -th element of $\vec{x}_p(s)$

$$x_{p,j}(s) = \sum_{i=1}^{2m} s^{r_{\delta,i}} \beta_{i,j}(s) \sum_{k=1}^m \frac{2c_k \phi_k(0)}{\sigma_k^2} \int_{s_0}^s t^{-r_{\delta,i}-1} \frac{\theta_{i,k}(t)}{\theta(t)} dt - \sum_{i=1}^{2m} s^{r_{\delta,i}} \beta_{i,j}(s) \sum_{k=1}^m \frac{2\lambda_k}{\sigma_k^2} \int_{s_0}^s t^{-r_{\delta,i}-1} \widehat{w}_k(t) \frac{\theta_{i,k}(t)}{\theta(t)} dt, \quad (6.13)$$

374 for each $j \in E$, where $\theta(t)$ and $\theta_{i,k}(t)$, $i = 1, \dots, 2m$ are holomorphic functions, with
 375 $\theta(0) \neq 0 \neq \theta_{i,k}(0)$ (as they are linear combinations of $\beta_{i,j}(s)$, $i = 1, \dots, 2m$, $j \in E$ and their
 376 derivative, for which $\vec{\beta}_i(0) = \vec{b}_0(r_{\delta,i}) \neq \vec{0}$ holds).

377 Following a similar line of logic as in Section 5, we will use Karamata-Tauberian theorem
 378 to get an asymptotic expression for $\vec{x}_h(s)$ and Heaviside Principle for $\vec{x}_p(s)$, respectively.
 379 For the application of the Karamata-Tauberian theorem, we have to identify the slowest
 380 decaying power in equation (6.11). To do this explicitly we will have to adopt the same
 381 idea as Section 5. Let $a_i = a$ and $\sigma_i = \sigma$, for all $i \in E$, with no change in the Markovian
 382 environment of the claim arrival process.

Note that we now have

$$\mathbf{L}(s) - \text{diag} \left(\frac{2\delta}{\sigma^2}, \dots, \frac{2\delta}{\sigma^2} \right) = \mathbf{L}(s) - \frac{2\delta}{\sigma^2} \mathbf{I} = \alpha_{\delta}(s) \mathbf{I} + \frac{2}{\sigma^2} \mathbf{Q},$$

383 with $\alpha_{\delta}(s) = \alpha(s) - \frac{2\delta}{\sigma^2} = s^2 + (3 - \frac{2a}{\sigma^2})s + 2 - \frac{2(a+\delta)}{\sigma^2}$.

384

Following the same arguments as in Lemmas 2 and 3 of Section 5 and noticing that
 $\alpha_{\delta}(s) = 0$ has two roots, namely $r_{\delta,i}$, given by

$$r_{\delta,i} = -\frac{2-\rho}{2} \pm \sqrt{\left(\frac{-\rho}{2}\right)^2 + \frac{2\delta}{\sigma^2}}, \quad i = 1, 2,$$

385 where $\rho = \frac{2a}{\sigma^2} - 1$, we have the following Lemma.

386

387 **Lemma 4.** For $a_i = a$, $\sigma_i = \sigma$, for all $i \in E$, the characteristic equation $\det(\mathbf{L}(s) - \frac{2}{\sigma^2} \mathbf{I}) =$
 388 0 has two roots, $r_{\delta,i}$, $i = 1, 2$ and all remaining roots have real parts that lie outside the
 389 interval determined by $r_{\delta,1}$ and $r_{\delta,2}$.

390

391 **Remark 6.** It should be clear that for $\delta = 0$, $r_{\delta,1}$ and $r_{\delta,2}$ reduce to r_1 and r_2 , respectively
 392 of Lemma 2.

393

The power $r_{\delta,1} = -\frac{2-\rho}{2} + \sqrt{\left(\frac{-\rho}{2}\right)^2 + \frac{2\delta}{\sigma^2}}$ would not produce a decay to zero at infinity resulting in the corresponding coefficient in equation (6.11), namely p_1 , vanishing. Thus, the slowest decay power is given by

$$r_{\delta,2} = -\frac{2-\rho}{2} - \sqrt{\left(\frac{-\rho}{2}\right)^2 + \frac{2\delta}{\sigma^2}}.$$

The slowest asymptotic behaviour of the solutions to the homogeneous part, given in equation (6.11), for some $j \in E$, is then given by

$$\widehat{\phi}_{h,j}(s) \sim p_2 s^{-\frac{2-\rho}{2} - \sqrt{\left(\frac{-\rho}{2}\right)^2 + \frac{2\delta}{\sigma^2}}} \beta_{2,j}(s), \quad s \rightarrow 0,$$

which, by Karamata-Tauberian theorem is equivalent to

$$\phi_{h,j}(u) \sim K u^{-\frac{\rho}{2} + \sqrt{\left(\frac{-\rho}{2}\right)^2 + \frac{2\delta}{\sigma^2}}} \beta_{2,j}(1/u), \quad u \rightarrow \infty,$$

394 where K is a constant.

395 It remains to analyse the asymptotic behaviour of the particular solution, given by
 396 equation (6.13). As before we have to deal with the two cases of light and heavy tailed
 397 distributions.

398

Light tailed claims. We again consider the right most singularity of $\widehat{w}_k(s)$, namely $-\mu_k$, then, in a similar way to the previous section, we can define the rightmost singularity $-\Delta$, where $\Delta = \min_{k \in E}(\mu_k)$, of the summation of $\widehat{w}_k(s)$, $k \in E$. Applying L'Hopital's rule we have the following:

$$\begin{aligned} \lim_{s \rightarrow -\Delta} \frac{\sum_{k=1}^m \lambda_k \int_{s_0}^s t^{-r_{\delta,i}-1} \widehat{w}_k(t) dt}{s^{-r_{\delta,i}} \sum_{k=1}^m \lambda_k \widehat{w}_k(s)} \\ &= \lim_{s \rightarrow -\Delta} \frac{s^{-r_{\delta,i}-1} \sum_{k=1}^m \lambda_k \widehat{w}_k(s)}{-r_{\delta,i} s^{-r_{\delta,i}-1} \sum_{k=1}^m \lambda_k \widehat{w}_k(s) + s^{-r_{\delta,i}} \sum_{k=1}^m \lambda_k \frac{d}{ds} \widehat{w}_k(s)} \\ &= \lim_{s \rightarrow -\Delta} \frac{1}{-r_{\delta,i} + s \frac{\sum_{k=1}^m \lambda_k \frac{d}{ds} \widehat{w}_k(s)}{\sum_{k=1}^m \lambda_k \widehat{w}_k(s)}} = \frac{1}{-r_{\delta,i}}, \end{aligned}$$

Then, we have

$$\begin{aligned} \widehat{\phi}_{p,j}(s) \sim \frac{2}{\sigma^2} \sum_{i=1}^{2m} \left(\frac{1}{-r_{\delta,i}} \right) \sum_{k=1}^m c_k \phi_k(0) \\ + \frac{2}{\sigma^2} \sum_{i=1}^{2m} \frac{1}{r_{\delta,i}} \sum_{k=1}^m \lambda_k \widehat{w}_k(s), \quad s \rightarrow -\Delta, \end{aligned} \quad (6.14)$$

399 after normalising the value of $\theta_{i,k}$ such that $\beta_{i,j}(-\Delta) \frac{\theta_{i,k}(-\Delta)}{\theta(-\Delta)} = 1, i = 1, \dots, 2m, k =$
 400 $1 \dots, m$. Applying the Heaviside Operational Principle, we have

$$\phi_{p,j}(u) \sim \frac{2}{\sigma^2} \sum_{i=1}^{2m} \frac{1}{r_{\delta,i}} \sum_{k=1}^m \lambda_k w_k(u), \quad u \rightarrow \infty. \quad (6.15)$$

401 **Heavy Tailed Claims.** A similar argument can be given for the case of heavy tailed
 402 claim size distributions to obtain

$$\phi_{p,j}(u) \sim \frac{2}{\sigma^2} \sum_{i=1}^{2m} \frac{1}{r_{\delta,i}} \sum_{k=1}^m \lambda_k w_k(u), \quad u \rightarrow \infty, \quad (6.16)$$

403 as long as $-\infty < \frac{d}{ds} \ln (\sum_{k=1}^m \lambda_k \hat{w}_k(s)) |_{s=0} < \infty, k \in E$.

404 Finally, by the same method as in Section 5, we can combine the homogeneous and
 405 corresponding particular solutions of both light and heavy tailed distributions to obtain the
 406 asymptotic behaviour of the ultimate Gerber-Shiu function, $\phi(u)$, given in the following
 407 theorem.

408

Theorem 4. *Let $a_i = a$ and $\sigma_i = \sigma$, for all $i \in E$. Consider that $\rho = \frac{2a}{\sigma^2} - 1 > 0$, assume that $\hat{w}_i(s)$ exists and that $|\frac{d}{ds} \ln (\sum_{k=1}^m \lambda_k \hat{w}_i(s)) |_{s=0} < \infty, i \in E$. Then, the ultimate Gerber-Shiu function, $\phi(u)$, behaves asymptotically as*

$$\phi(u) \sim K u^{-\frac{\rho}{2} + \sqrt{(\frac{-\rho}{2})^2 + \frac{2\delta}{\sigma^2}}} + \frac{2}{\sigma^2} \sum_{i=1}^{2m} \frac{1}{r_{\delta,i}} \sum_{k=1}^m \lambda_k w_k(u), \quad u \rightarrow \infty, \quad (6.17)$$

409 for some strictly positive constant K .

410

411 **Remark 7.** *From the above theorem we deduce that the asymptotic decay will be given by*
 412 *the slower of the power function or the sum of functions $w_i(u)$, $i \in E$. By the definition*
 413 *of the these functions $w_i(u)$, it is clear that the asymptotic behaviour is dependent on*
 414 *the combination of the penalty function and the claim size distributions, which has been*
 415 *discussed in the previous literature.*

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