ON THE PRICE OF RISK UNDER A REGIME SWITCHING CGMY PROCESS

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ABSTRACT. In this paper, we study option pricing under a regime-switching exponential Lévy model. Assuming that the coefficients are time-dependent and modulated by a finite state Markov chain, we generalise the work in [20, 22], that is, we use a pricing method based on the Esscher transform conditional on the information available on the Markov chain. We also carry out numerical analysis, to show the impact of the risk induced by the underlying Markov chain on the price of the option.

1. INTRODUCTION

Empirical studies have suggested the need for modern financial modelling to move from the standard log-normal dynamics of the Black-Scholes model framework. This is primarily because in their works, the authors in [2, 17] assume that the price dynamic of the underlying risky asset are governed by geometric Brownian motion, an assumption which many researchers have challenged. There is evidence that the risky assets experience stochastic volatility overtime and therefore the assumption of constant volatility creates biases when an option is priced using the Black-Scholes model. Several models have been developed to provide more realistic ways to model empirical behaviour of option prices. Among them, we can list: the jump-diffusion models, the stochastic volatility and the regime switching models. In the latter case, economic cycles are described by a discrete, finite state Markov chain; See for example [12, 13] for more details. The states of the underlying Markov chain represent the different states of the economy and such model enable to incorporate the impact of changes in macro-economic conditions on the behaviour of the dynamics of the assets' prices.

The possibility of switching across induces an important source of risk that investors might want to hedge against. As pointed out in [8], in a regime switching Black-Scholes model, there exist at least two sources of risk that the investor needs to consider: the diffusion risk which can be considered as the market or financial risk and regime switching risk which can be thought as economic risk. In addition, when the underlying is driven by a Lévy process, one needs to consider the risk due to multiple jumps coming from Poisson random measures. There has been many works on option pricing under regime switching model, most of them assuming that the risk due to switching of regimes is zero. In [7, 20, 22], the importance of

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26 pricing the regime risk is shown, in the sense that, the authors show the impact of the change
27 in the regime on the option prices, hence addressing the problem of pricing the risk associated
28 to the regime. The work regime switching Black-Scholes model is discussed in [7, 22] whereas
29 [20] is an extension to the regime-switching Variance-Gamma model. See also the work [21]
30 where the author studies the price of the regime risk induced by the jumps in volatility.
31   One of the main characteristics of the regime-switching model is that they generate incom-
32 plete market and hence a family of Equivalent Martingales Measure (EMM). The first task is
33 to determine an equivalent martingale measure which will enable to price the different risks
34 efficiently. One may think of the martingale measure that minimises the "distance" between
35 the set of equivalent martingale measures and the real world probability measure. One of
36 such distances is given by the relative entropy and the associated minimiser is the *minimal*
37 *entropy martingale measure* (MEMM). In this work, we will use the regime switching Esscher
38 transform which was already used in [22] (see also [20]). The Esscher transform is taken
39 conditional on the information available on the Markov chain. The result by [19] can be used
40 to justify the choice of our pricing result by the minimal entropy martingale measure. It is
41 also worth mentioning that the work [11] introduces Esscher transform in actuarial science as
42 the pricing measure for option valuation and justify this choice by maximizing the expected
43 utility of power type of an investor. For other works on minimal entropy martingale measure,
44 the reader may consult [1, 9, 10, 18].
45   In this paper, we extend the works [20, 22], that is, we assume that the dynamic of the un-
46 derlying risky asset is governed by a regime switching Carr, Geman, Madan and Yor (CGMY)
47 process. We first study the option price under a general regime switching exponential Lévy
48 model. In this model, the parameters of the assets are assumed to be deterministic, time de-
49 pendent and are modulated by an observable continuous time, finite state Markov chain. For
50 example, one may interpret the time dependent interest rate as corresponding to the relative
51 frequent announcements or industry involving reasonably small shifts in the interest rates (see
52 for example [16]). One may also interpret the observable states of the chain as different stages
53 of the business cycle, for instance if the states of the Markov chain are two, they could be
54 interpreted as expansion and recession periods. As in [20, 22], we introduce a pricing model
55 to price the diffusion risk (for the time dependent regime switching Black-Scholes model), the
56 risk due to jumps and the regime-switching risk. To achieve this, we first adopt the regime
57 switching Esscher transform in order to determine a set of equivalent martingale measures
58 satisfying the martingale condition. The selection of the Esscher transform martingale mea-
59 sure is done by minimizing the maximum entropy between an equivalent martingale measure
60 and the real world probability measure over the different states of the economy (compare with
61 [20, 22]).
62   We conduct numerical experiments to show the impact of the risk induced by the underlying
63 Markov chain on the price of the option. This implies that in pricing options, a probable error
64 can be made when we chose to ignore the risk associated with the switching of regimes. Our
65 results extend those in [20, 22] to incorporate the time dependency of the parameters and to
66 the CGMY model. Another interesting observation in our model is the following: During the
67 lifetime of the option, its price is higher when the regime risk is priced than when it is not,
68 which is higher than the option price when there is no regime.
69   The remaining of the paper is organized as follows: In Section 2, we describe the model and
70 study the different pricing kernels and their associated martingale condition. These conditions
71 are explicitly given is the case of regime switching Black-Scholes model, Variance-Gamma and
72 CGMY model. Section 3 is devoted to numerical experiments to illustrate the effect of pricing

73 the regime switching risk are conducted and we find a significant difference between pricing
74 the risk and not.

2. The Model

76   In this section, we present a general regime switching exponential Lévy model. The model
77 is that of [20]. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space, where $\mathbb{P}$ is the reference measure.
78   The evolution of the states of the economy is modelled by an irreducible homoge-
79 neous continuous time Markov chain $X := \{X(t); t \in [0, T]\}$ with a finite state space
80 $\mathcal{X} = \{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_N\} \in \mathbb{R}^N$, where $N \in \mathbb{N}$, and the jth component of e_n is the Kronecker
81 delta δ_{nj} for each $n, j = 1, \ldots, D$. Denote by $\mathbf{A} := [a_{ij}]_{i,j=1,2,\ldots,N}$ the intensity matrix of
82 the Markov chain under $\mathbb{P}$. Then for each $i, j = 1, 2, \ldots, N$ with $i \neq j$, a_{ij} is the transition
83 intensity of the chain X jumping from state $\mathbf{e}_j$ to state $\mathbf{e}_i$ at time $t \in [0, T]$. Hence, for
84 $i \neq j$, $a_{ij} \geq 0$ and $\sum_{j=1}^{N} a_{ij} = 0$ i.e., $\lambda_{ii} \leq 0$. With the canonical representation of the state
85 space of the Markov chain, the following semimartingale decomposition for the Markov chain
86 X was given in [4]:

$$X(t) = X(0) + \int_0^t A(s)X(s)\,\mathrm{d}s + M(t), \qquad t \in [0, T]. \tag{2.1}$$

87 where $\{M(t); t \in [0, T]\}$ is an $\mathbb{R}^N$-valued martingale under the measure $\mathbb{P}$ with respect to the
88 filtration generated by X.

89   We consider a financial market with two primary securities, namely, a riskless asset B and
90 a risky stock S, which are traded continuously over the time horizon $[0, T]$. We model the
91 evolution of the instantaneous interest rate $r = \{r(t); t \in [0, T]\}$ of the money market account
92 B at time t as follows.

$$r(t) = r(t, X(t)) = \langle \mathbf{r}, X(t) \rangle = \sum_{i=1}^{N} r_i(t) \langle \mathbf{e}_i, X(t) \rangle, \tag{2.2}$$

93 where $\mathbf{r} := (r_1(t), r_2(t), \ldots, r_N(t))' \in \mathbb{R}^N$ for each $i = 1, 2, \ldots, N$ and $\langle \cdot, \cdot \rangle$ denotes the
94 inner product in $\mathbb{R}^N$. The i-th component $r_i(t)$ of the vector $\mathbf{r}$ is a deterministic function,
95 representing the value of the interest rate when the Markov chain is in state $\mathbf{e}_i$ that is when
96 $X(t) = \mathbf{e}_i$. The dynamics of $\{B(t); t \in [0, T]\}$ of the money market account B are given by

$$\mathrm{d}B(t) = r(t)B(t)\,\mathrm{d}t, \qquad B(0) = 1. \tag{2.3}$$

97 Denote by $\{\mu(t); t \in [0, T]\}$ and $\{\sigma(t); t \in [0, T]\}$ the appreciation rate and the volatility of
98 the stock S at the time t respectively. Using similar convention, we set

$$\mu(t) = \mu(t, X(t)) := \langle \boldsymbol{\mu}, X(t) \rangle = \sum_{i=1}^{N} \mu_i(t) \langle \mathbf{e}_i, X(t) \rangle, \tag{2.4}$$

99

$$\sigma(t) = \sigma(t, X(t)) := \langle \boldsymbol{\sigma}, X(t) \rangle = \sum_{i=1}^{N} \sigma_i(t) \langle \mathbf{e}_i, X(t) \rangle, \tag{2.5}$$

100   where $\boldsymbol{\mu} = (\mu_1(t), \mu_2(t), \ldots, \mu_N(t))' \in \mathbb{R}^N$ and $\boldsymbol{\sigma} = (\sigma_1(t), \sigma_2(t), \ldots, \sigma_N(t))' \in \mathbb{R}_+^N$.
101    $\mu_i(t)$ and $\sigma_i(t)$, $i = 1, 2 \ldots, N$ are deterministic functions representing respectively the
102 appreciation rate and volatility of S when the Markov chain is in state $\mathbf{e}_i$. The price dynamics

103 of the stock S is given by the following stochastic differential equation,

$$\mathrm{d}S(t) = S(t^-)\Big(\mu(t)\,\mathrm{d}t + \sigma(t)\,\mathrm{d}W(t) + \int_{\mathbb{R}_0} (e^z - 1)\tilde{N}^X(\,\mathrm{d}t;\,\mathrm{d}z)\Big),\ \ S(0) > 0, \tag{2.6}$$

104 where $\mathbb{R}_0 = \mathbb{R}\backslash\{0\}$, $W = \{W(t); t \in [0,T]\}$ is a Brownian motion and $\widetilde{N}^X(\mathrm{d}t, \mathrm{d}z) :=$
105 $N(\mathrm{d}z, \mathrm{d}t) - \rho^X(\mathrm{d}z)\,\mathrm{d}t$ is an independent compensated Markov regime-switching Poisson ran-
106 dom measure with $\rho^X(\mathrm{d}z)\,\mathrm{d}t$, the compensator (or dual predictable projection) of N, defined
107 by:

$$\rho^X(\mathrm{d}z)\mathrm{d}t := \sum_{j=1}^{D}\langle X(t-), e_j\rangle\rho_j(\mathrm{d}z)\mathrm{d}t. \tag{2.7}$$

108 For each $j \in \{1, 2, \ldots, D\}$, $\rho_j(\mathrm{d}z)$ is the conditional density of the jump size when the Markov
109 chain X is in state e_j and satisfies $\int_{\mathbb{R}_0} \min(1, z^2)\rho_j(\mathrm{d}z) < \infty$ and $\int_{|z|\geq 1}(e^z - 1)^2\rho_i(z)\,\mathrm{d}z < \infty$.

The dynamic of the stock S can also be written as

$$S(t) = S(0)e^{Y(t)},$$

110 where $Y(t)$ is given by:

$$Y(t) = Y(0) + \int_0^t \Big(\mu(s) - \frac{1}{2}\sigma^2(s) - \int_{\mathbb{R}\backslash\{0\}}(e^z - 1 - z)\rho^X(\,\mathrm{d}z)\Big)\,\mathrm{d}s$$
$$+ \int_0^t \sigma(s)\,\mathrm{d}W(s) + \int_0^t \int_{\mathbb{R}\backslash\{0\}} z\tilde{N}(\,\mathrm{d}s,\,\mathrm{d}z). \tag{2.8}$$

111   The model defined by (2.1)-(2.6) is referred to as a general regime switching exponential
112 Lévy model. Such model leads to incomplete markets i.e., there exists more than one equiv-
113 alent martingale measures (EMM) describing the risk-neutral price dynamic and compatible
114 with the no arbitrage requirement. In order to price contingent claim, we shall determine
115 EMM using regime switching Esscher transform introduced in [5, 22]. In fact, the classical
116 definition of Esscher transform based on the moment generating function of a random vari-
117 able is replaced by a conditional Esscher transform where the moment generating function
118 is conditional to a subset of information available on the Markov chain. This leads to two
119 different pricing kernels based on the conditional Esscher transform.

120 2.1. **Pricing Kernel I.** In this section, we construct a risk neutral measure assuming that
121 the whole path of the underlying Markov chain is known. This Esscher change of measure
122 produces a pricing kernel that does not take into account the risk associated with the Markov
123 chain.
124   We shall first specify the information structure of our model. Let $F^X := \{\mathcal{F}_t^X; t \in [0,T]\}$
125 and $F^S := \{\mathcal{F}_t^S; t \in [0,T]\}$ denote the $\mathbb{P}$-augmentation of natural filtrations generated by
126 $\{X(t); t \in [0,T]\}$ and $\{S(t); t \in [0,T]\}$ respectively. That is, for each $t \in [0,T]$, $\mathcal{F}_t^X$ and $\mathcal{F}_t^S$
127 are, respectively, the σ-fields generated by the histories of the chain X and the stock price S
128 up to and including time t. We define for $t \in [0,T]$, $\mathcal{G}_t$ to be the σ-algebra $\mathcal{F}_T^X \vee \mathcal{F}_t^S$. This
129 represents the information set generated by both histories of X and S up to and including

130   the time t. We write $\mathbf{G} := \{\mathcal{G}_t; t \in [0,T]\}$. We set

$$\Theta := \Big\{ \theta(t); t \in [0,T] | \theta(t) := \sum_{i=1}^{N} \theta_i(t)\langle X(t^-), \mathbf{e}_i\rangle, \text{ with } (\theta_1(t), \ldots, \theta_N(t)) \in \mathbb{R}^N,$$

such that $\theta_i, i = 1, \ldots, N$ are deterministic and $E^{\mathbb{P}}\Big[e^{-\int_0^t \theta(s)\,\mathrm{d}Y(s)}\Big|\mathcal{F}_T^X\Big] < \infty \Big\}.$ (2.9)

131   For $\theta := \{\theta(t); t \in [0,T]\} \in \Theta$, define the generalized Laplace transform of a $\mathbf{G}$-adapted
132   process Y by

$$M_Y(\theta) := E^{\mathbb{P}}\Big[e^{-\int_0^t \theta(s)\,\mathrm{d}Y(s)}\Big|\mathcal{F}_T^X\Big]. \tag{2.10}$$

133   We define the kernel of a generalized Esscher transform with respect to the parameter θ. Let
134    $\Lambda^\theta := \{\Lambda^\theta(t); t \in [0,T]\}$ denote a $\mathbf{G}$-adapted stochastic process defined as

$$\Lambda^\theta(t) = \frac{\exp\Big(-\int_0^t \theta(s)\,\mathrm{d}Y(s)\Big)}{M_Y(\theta)}, \qquad t \in [0,T], \quad \theta \in \Theta. \tag{2.11}$$

135   Then, the regime switching Esscher transform $\mathbb{Q} \sim \mathbb{P}$ on $\mathbf{G}$ with respect to a family of
136   parameters $\{\theta(s); s \in [0,t]\}$ is given by:

$$\Lambda^\theta(t) = \frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\bigg|_{\mathcal{G}_t} = \frac{\exp\Big(-\int_0^t \theta(s)\,\mathrm{d}Y(s)\Big)}{E_{\mathbb{P}}\Big[\exp\Big(-\int_0^t \theta(s)\,\mathrm{d}Y(s)\Big)\Big|\mathcal{F}_T^X\Big]}, \qquad t \in [0,T], \quad \theta \in \Theta. \tag{2.12}$$

137   Hence, as shown in [5], one has

$$\Lambda^\theta(t) = \exp\Big(-\int_0^t \theta(s)\sigma(s)\,\mathrm{d}W(s) - \frac{1}{2}\int_0^t (\theta(s))^2(\sigma(s))^2\,\mathrm{d}s$$
$$-\int_0^t \int_{\mathbb{R}_0} \theta(s^-)z\tilde{N}^X(\,\mathrm{d}s,\,\mathrm{d}z) - \int_0^t \int_{\mathbb{R}_0} (e^{-z\theta(s)} - 1 + \theta(s)z)\rho^X(\,\mathrm{d}z)\,\mathrm{d}s\Big). \tag{2.13}$$

138   For each $\theta \in \Theta$, Λ^θ is a density process (see [20, 22]), therefore a new equivalent probability
139   measure can be defined by setting

$$\frac{\mathrm{d}\mathbb{Q}^\theta}{\mathrm{d}\mathbb{P}}\bigg|_{\mathcal{G}_t} = \Lambda^\theta(t), \quad t \in [0,T]. \tag{2.14}$$

140   The pricing kernel associated to such measure shall then be defined by choosing θ adequately
141   (see Section 2.3.)

142   2.2. **Pricing Kernel II.** In this section, we construct a change of measure assuming that the
143   initial state of the underlying Markov chain is known. This assumption seems more realistic
144   since an investor can only observe the current and past information about the macro-economic
145   condition and then anticipate future evolution of the macro-economic conditions. The expec-
146   tation in the denominator of the regime switching Esscher transform is unconditional implying
147   that the risk due to the switching regimes is priced.

148   We introduce a new filtration, namely $\mathbb{G} := \{\overline{\mathcal{G}}_t = \mathcal{F}_t^X \vee \mathcal{F}_t^S; t \in [0, T]\}$ which denotes the
149 right continuous, $\mathbb{P}$-complete filtration generated by the bivariate process (X, S). Set

$$\Theta^* := \left\{\theta^*(t); t \in [0, T] | \theta^*(t) := \sum_{i=1}^{N} \theta_i^*(t) \langle X(t^-), \mathbf{e}_i \rangle, \text{ with} \right.$$

$$\left. (\theta_1^*(t), \ldots, \theta_N^*(t)) \in \mathbb{R}^N, \text{ such that } E^{\mathbb{P}}\left[e^{-\int_0^T \theta^*(s)\,\mathrm{d}Y(s)} \Big| X(0)\right] < \infty \right\} \qquad (2.15)$$

150 and define the generalized Laplace transform of a $\mathbb{G}$-adapted process Y as

$$M_Y(\theta^*) := E^{\mathbb{P}}\left[e^{-\int_0^T \theta(s)\,\mathrm{d}Y(s)} \Big| X(0)\right]. \qquad (2.16)$$

151 As in [20, 22], define the new kernel $\Lambda^{\theta^*} = \{\Lambda^{\theta^*}(t); t \in [0, T]\}$ as follows

$$\begin{cases} \Lambda^{\theta^*}(0) := 1 \\ \Lambda^{\theta^*}(t) := E[\Lambda^{\theta^*}(T) | \overline{\mathcal{G}}_t] = E^{\mathbb{P}}\left[\dfrac{e^{-\int_0^T \theta^*(s)\,\mathrm{d}Y(s)}}{E^{\mathbb{P}}[e^{-\int_0^T \theta^*(s)\,\mathrm{d}Y(s)}|X(0)]} \Big| \overline{\mathcal{G}}_t\right], \quad t \in (0, T]; \theta^* \in \Theta^*. \end{cases} \qquad (2.17)$$

Then $\{\Lambda^{\theta^*}(t); t \in [0, T]\}$ is a positive $(\mathbb{G}, \mathbb{P})$-martingale satisfying

$$E^{\mathbb{P}}[\Lambda^{\theta^*}] = 1, \ \ \forall t \in [0, T].$$

152 As for the first kernel, one can define a family of equivalent measures $\mathbb{Q}_{\theta^*}$ through

$$\frac{\mathrm{d}\mathbb{Q}_{\theta^*}}{\mathrm{d}\mathbb{P}} | \overline{\mathcal{G}}_t = \Lambda^{\theta^*}(t), \quad t \in [0, T]. \qquad (2.18)$$

153 and derive a pricing kernel by adequately choosing θ^* (see Section 2.3.)
154   The pricing kernel (2.14) and (2.18) and The knowledge of the whole path of the Markov
155 chain implies that there is no need for additional premium whereas the knowledge of only the
156 initial state of the Markov chain forces the need of additional premium that will take into
157 account the risk associated to the changes in the regime.

158 **2.3. Martingale condition.** Denote by $\{S^*(t) := \frac{S(t)}{B(t)}; t \in [0, T]\}$ the discounted price
159 process. Therefore, by the fundamental theorem of asset pricing (see [14, 15]), the no-arbitrage
160 price of any contingent claim written on S in this market is given by

$$E^{\mathbb{Q}}\left[S^*(t) \Big| \mathcal{G}_0\right] = S^*(0), \qquad (2.19)$$

161 with $\mathbb{Q} \in \{\mathbb{Q}^\theta, \mathbb{Q}_{\theta^*}\}$. Eq. (2.19) implicitly gives the condition on the process θ and θ^* that
162 determine an EMM within the families $\{\mathbb{Q}^\theta : \theta \in \Theta\}$ and $\{\mathbb{Q}_{\theta^*} : \theta^* \in \Theta^*\}$.
163   The following theorem gives necessary and sufficient conditions for $\mathbb{Q}^\theta$ to be an EMM.

164 **Theorem 2.1.** *Consider the Lévy regime-switching market defined in (2.3) and (2.6). An*
165 *equivalent probability measure $\mathbb{Q}^\theta$ defined through (2.14) is an equivalent martingale measure*
166 *on $(\Omega, \mathcal{G}_T)$, i.e., it satisfies the condition (2.19), if and only if θ satisfies the following equation*
167

$$\mu_i(t) - r_i(t) - \theta_i(t)\sigma_i^2(t) + \int_{\mathbb{R}} (e^z - 1)(e^{-z\theta_i(t)} - 1)\rho_i(z)\,\mathrm{d}z = 0, \ t\text{-a.e.}, \ \forall \ t \in [0, T] \qquad (2.20)$$

168 *for $i = 1, \ldots, N$.*

169 *Proof.* It easily follows using the martingale condition under the enlarged filtration
170 $\mathbf{G} = \{\mathcal{G}_t; 0 \le t \le T\}$ and Bayes rules. $\square$

171   Next, we shall discuss the necessary and sufficient condition for $\mathbb{Q}_{\theta^*}$ to be an equivalent
172 martingale on $(\Omega, \mathcal{G}_T)$. We begin by presenting, without proof, a lemma which gives an explicit
173 form of the moment generating function of the Markov chain in terms of the occupation times.

174 **Lemma 2.2.** *Consider an irreducible homogeneous continuous-time Markov chain*
175 $X := \{X(t); t \in [0, T]\}$ *on* $(\Omega, \mathcal{G}_T, \mathbb{G}, \mathbb{P})$ *with a finite state space* $\mathcal{X}$ *of size* $N \in \mathbb{N}$ *and with*
176 *an intensity matrix* $\boldsymbol{A} := \{a_{ij} : 1 \le i, j \le N\}$. *Let*

$$\underline{\mathbf{J}}(u, v) := (J_1(u, v), J_2(u, v), \ldots, J_N(u, v)) \tag{2.21}$$

177 *denote the vector of the occupation times of* X *during a period of time* $[u, v] \subset [0, T]$. *We*
178 *have*

$$J_k(u, v) = \int_u^v \langle X(s), \mathbf{e}_k \rangle \, \mathrm{d}s.$$

179 *The conditional moment generating function of* $\underline{\mathbf{J}}(u, v)$ *is given by*

$$E^{\mathbb{P}}\big[e^{\sum_{k=1}^N \int_u^t \zeta_k(v) \, \mathrm{d}J_k(u,v)}\big|\mathcal{G}_u\big] = \big\langle e^{\int_u^t (\boldsymbol{A} + Diag(\underline{\zeta}_k(r))) \, \mathrm{d}r} X(u), \underline{\mathbf{1}} \big\rangle, \quad \underline{\zeta} \in \mathbb{R}^N, \tag{2.22}$$

180 *where* $\underline{\mathbf{1}} = (1, 1, \ldots, 1)' \in \mathbb{R}^N$, $\langle \cdot, \cdot \rangle$ *is the scalar product in* $\mathbb{R}^N$ *and* $Diag(\underline{\zeta})$ *is an* $N \times N$
181 *diagonal matrix of the form*

$$Diag(\underline{\zeta}) = \begin{pmatrix} \zeta_1 & 0 & \ldots & 0 & 0 \\ 0 & \zeta_2 & 0 & \ldots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \zeta_{N-1} & 0 \\ 0 & \ldots & 0 & 0 & \zeta_N \end{pmatrix}.$$

182 *Proof.* Follows in the same way as in the proof of [6, Proposition 2] $\qquad\qquad\square$

183   We can now state the necessary and sufficient condition for $\mathbb{Q}_{\theta^*}$ to be an equivalent mar-
184 tingale measure on $(\Omega, \mathcal{G}_T)$. This result is adapted from Siu and Yang [22].

185 **Theorem 2.3.** *Consider the Lévy regime-switching market defined in* (2.3) *and* (2.6). *An*
186 *equivalent measure* $\mathbb{Q}_{\theta^*}$ *defined through* (2.18) *is an equivalent martingale measure on* $(\Omega, \mathcal{G}_T)$,
187 *i.e., condition* (2.19) *holds if and only if* θ^* *satisfies the following equation*

$$\Big\langle e^{\int_0^t (\boldsymbol{A} + Diag(\tilde{\underline{\xi}}(\theta^*(r)))) \, \mathrm{d}r} X(0), \underline{\mathbf{1}} \Big\rangle - \Big\langle e^{\int_0^t (\boldsymbol{A} + Diag(\underline{\xi}(\theta^*(r)))) \, \mathrm{d}r} X(0), \underline{\mathbf{1}} \Big\rangle = 0, \tag{2.23}$$

188 *where*

$$\underline{\xi}(\theta^*) = (\xi_1(\theta_1^*(t)), \xi_2(\theta_2^*(t)), \ldots, \xi_N(\theta_N^*(t))),$$
$$\tilde{\underline{\xi}}(\theta^*) = (\tilde{\xi}_1(\theta_1^*(t)), \tilde{\xi}_2(\theta_2^*(t)), \ldots, \tilde{\xi}_N(\theta_N^*t)),$$

189 *with*

$$\xi_i(\theta_i^*(t)) = -\theta_i^*(t)\big(\mu_i(t) - \frac{1}{2}\sigma_i^2(t)\big) + \frac{1}{2}(\theta_i^*(t))^2 \sigma_i^2(t)$$
$$+ \int_{\mathbb{R}} (e^{-z\theta_i^*(t)} - 1 + \theta_i^*(t)(e^z - 1))\rho_i(z) \, \mathrm{d}z, \ t\text{-}a.e., \tag{2.24}$$

$$\tilde{\xi}_i(\theta_i^*(t)) = -r_i(t) - (\theta_i^*(t) - 1)(\mu_i(t) - \frac{1}{2}\sigma_i^2(t)) + \frac{1}{2}(\theta_i^*(t) - 1)^2 \sigma_i^2(t)$$
$$+ \int_{\mathbb{R}} (e^{-z(\theta_i^*(t)-1)} - 1 + (\theta_i^*(t) - 1)(e^z - 1))\rho_i(z) \, \mathrm{d}z, \ t\text{-}a.e. \tag{2.25}$$

190 *for $i = 1, 2, \ldots, N$.*

191   In order to prove this theorem, we will need the following lemma, which is a extension of
192 As in [20, Lemma4.2] and [22, Lemma 3.1].

193 **Lemma 2.4.** *Under Assumptions of Theorem 2.3, for all $u, v \in [0, T]$ such that $u \leq v$, we*
194 *have that*

$$E^{\mathbb{Q}_{\theta^*}}\left[S^*(v)\Big|\mathcal{G}_u\right] = \frac{\left\langle e^{\int_u^v (\boldsymbol{A} + Diag(\tilde{\underline{\xi}}(\theta^*(r)))) \, \mathrm{d}r} X(u), \mathbf{1}\right\rangle}{\left\langle e^{\int_u^v (\boldsymbol{A} + Diag(\underline{\xi}(\theta^*(r)))) \, \mathrm{d}r} X(u), \mathbf{1}\right\rangle} S^*(u), \qquad (2.26)$$

195 *where $\tilde{\underline{\xi}}(\theta^*(r))$ and $\underline{\xi}(\theta^*(r))$ are given in Theorem 2.3.*

196 *Proof.* Choose $u, v \in [0, T]$ such that $v \geq u$. Then the discounted stock price is given by
197 $S^*(v) := S(v) e^{-\int_u^v r(s) \, \mathrm{d}s}$. using this and a version of the Bayes's rule, we get

$$
\begin{aligned}
E^{\mathbb{Q}_{\theta^*}}\left[S^*(v)\Big|\mathcal{G}_u\right] &= S^*(u) E^{\mathbb{Q}_{\theta^*}}\left[e^{-\int_u^v r(s) \, \mathrm{d}s} e^{\int_u^v \, \mathrm{d}Y(s)}\Big|\mathcal{G}_u\right] \\
&= S^*(u) \frac{E^{\mathbb{P}}\left[e^{-\int_u^v r(s) \, \mathrm{d}s} e^{\int_u^v \, \mathrm{d}Y(s)} \Lambda^{\theta^*}(v)\Big|\mathcal{G}_u\right]}{E^{\mathbb{P}}\left[\Lambda^{\theta^*}(v)\Big|\mathcal{G}_u\right]} \\
&= S^*(u) \frac{E^{\mathbb{P}}[e^{-\int_u^v r(s) \, \mathrm{d}s} e^{\int_u^v \, \mathrm{d}Y(s)} \Lambda^{\theta^*}(v)|\mathcal{G}_u]}{E^{\mathbb{P}}[\Lambda^{\theta^*}(v)|\mathcal{G}_u]} \\
&= S^*(u) \frac{E^{\mathbb{P}}\left[e^{-\int_u^v r(s) \, \mathrm{d}s} e^{-\int_u^v (\theta^*(s)-1) \, \mathrm{d}Y(s)} E^{\mathbb{P}}\left[e^{-\int_v^T \theta^*(s) \, \mathrm{d}Y(s)}\Big|\mathcal{G}_v\right]\Big|\mathcal{G}_u\right]}{E^{\mathbb{P}}\left[e^{-\int_u^T \theta^*(s) \, \mathrm{d}Y(s)}\Big|\mathcal{G}_u\right]}
\end{aligned} \qquad (2.27)
$$

198   Using the occupation times as in Lemma 2.2.

$$
\begin{aligned}
&E^{\mathbb{Q}_{\theta^*}}\left[S^*(v)\Big|\mathcal{G}_u\right] \\
&= S^*(u) \frac{E^{\mathbb{P}}\left[\exp\left(\sum_{i=1}^N \int_u^v \tilde{\xi}_i(\theta_i^*(t)) \, \mathrm{d}J_i(u,t)\right) E^{\mathbb{P}}\left[\exp\left(\sum_{i=1}^N \int_v^T \xi_i(\theta_i^*(t)) \, \mathrm{d}J_i(v,t)\right)\Big|\mathcal{G}_v\right]\Big|\mathcal{G}_u\right]}{E^{\mathbb{P}}\left[\exp\left(\sum_{i=1}^N \int_u^v \xi_i(\theta_i^*(t)) \, \mathrm{d}J_i(u,t)\right) E^{\mathbb{P}}\left[\exp\left(\sum_{i=1}^N \int_v^T \xi_i(\theta_i^*(t)) \, \mathrm{d}J_i(v,t)\right)\Big|\mathcal{G}_v\right]\Big|\mathcal{G}_u\right]}.
\end{aligned} \qquad (2.28)
$$

199 Using the following property of homogeneous Markov chains

$$
\begin{aligned}
Law(J_1(v,T), \ldots, J_N(v,T)|\mathcal{G}(v)) &= Law(J_1(v,T), \ldots, J_N(v,T)|X(v)) \\
&= Law(J_1(0, T-v), \ldots, J_N(0, T-v)|X(0)),
\end{aligned}
$$

200 (2.28) becomes

$$
\begin{aligned}
&E^{\mathbb{Q}_{\theta^*}}\left[S^*(v)\Big|\mathcal{G}_u\right] \\
&= S^*(u) \frac{E^{\mathbb{P}}\left[\exp\left(\sum_{i=1}^N \int_0^{T-v} \xi_i(\theta_i^*(t)) \, \mathrm{d}J_i(0,t)\right)\Big|X(0)\right] E^{\mathbb{P}}\left[\exp\left(\sum_{i=1}^N \int_u^v \tilde{\xi}_i(\theta_i^*(t)) \, \mathrm{d}J_i(u,t)\right)\Big|\mathcal{G}_u\right]}{E^{\mathbb{P}}\left[\exp\left(\sum_{i=1}^N \int_0^{T-v} \xi_i(\theta_i^*(t)) \, \mathrm{d}J_i(0,t)\rangle\right)\Big|X(0)\right] E^{\mathbb{P}}\left[\exp\left(\sum_{i=1}^N \int_u^v \xi_i(\theta_i^*(t)) \, \mathrm{d}J_i(u,t)\right)\Big|\mathcal{G}_u\right]}.
\end{aligned}
$$

201 This implies

$$E^{\mathbb{Q}_{\theta^*}}\Big[S^*(v)\Big|\mathcal{G}_u\Big] = S^*(u)\frac{E^{\mathbb{P}}\Big[\exp\Big(\sum_{i=1}^{N}\int_u^v \tilde{\xi}_i(\theta_i^*(t))\,\mathrm{d}J_i(u,t)\Big)\Big|\mathcal{G}_u\Big]}{E^{\mathbb{P}}\Big[\exp\Big(\sum_{i=1}^{N}\int_u^v \xi_i(\theta_i^*(t))\,\mathrm{d}J_i(u,t)\Big)\Big|\mathcal{G}_u\Big]}.$$

202 Hence, using Lemma 2.2, we get

$$E^{\mathbb{Q}_{\theta^*}}\Big[S^*(v)\Big|\mathcal{G}_u\Big] = S^*(u)\frac{\Big\langle e^{\int_u^v (\mathbf{A}+\mathrm{Diag}(\underline{\tilde{\xi}}(\theta^*(r))))\,\mathrm{d}r}X(u),\underline{\mathbf{1}}\Big\rangle}{\Big\langle e^{\int_u^v (\mathbf{A}+\mathrm{Diag}(\underline{\xi}(\theta^*(r))))\,\mathrm{d}r}X(u),\underline{\mathbf{1}}\Big\rangle}. \tag{2.29}$$

203 $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\Box$

204 *Proof of Theorem 2.3.* This follows directly from the previous lemma by setting $v = t$ and
205 $u = 0$ in (2.26). In fact, we have that the martingale condition (2.19) is equivalent to (2.23).
206 $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\Box$

207   We turn our main focus on the condition for the family $\{\mathbb{Q}_{\theta^*} : \theta^* \in \Theta^*\}$ because through a
208 standard approximation for the matrix exponential in (2.23), we shall deduce the martingale
209 condition for the family $\{\mathbb{Q}^{\theta} : \theta \in \Theta\}$; See [20, 22].

210 2.4. **Approximations.** Here, we analyse the two families of equivalent martingale measures
211 $\mathbb{Q}^{\theta}$ and $\mathbb{Q}_{\theta^*}$ via certain types of approximations for the martingale condition (2.23). The
212 exponential of a $N \times N$ matrix $\mathbf{E}$ is defined as

$$\exp(\mathbf{E}) := \sum_{n=0}^{\infty}\frac{E^n}{n!}, \tag{2.30}$$

213 where $\mathbf{E}^0 = \mathbf{I}$ is the identity matrix and by convention $0! = 1$. Replacing $X(0)$ by $\mathbf{e}_i$ for
214 $i = 1,\ldots,N$ in (2.23) yields,

$$\Big\langle e^{\int_0^t (\mathbf{A}+\mathrm{Diag}(\underline{\tilde{\xi}}(\theta^*(r))))\,\mathrm{d}r}\mathbf{e}_i,\underline{\mathbf{1}}\Big\rangle - \Big\langle e^{\int_0^t (\mathbf{A}+\mathrm{Diag}(\underline{\xi}(\theta^*(r))))\,\mathrm{d}r}\mathbf{e}_i,\underline{\mathbf{1}}\Big\rangle = 0. \tag{2.31}$$

215 This is a system of N equations and in practice to solve it, one needs to adopt a finite number
216 of terms in the series' expansion of $\exp(\mathbf{E})$. Using the first-order approximation of $\exp(\mathbf{E})$
217 (i.e., $\exp(\mathbf{E}) \approx \mathbf{I} + \mathbf{E}$) in (2.31), we have

$$\Big\langle \Big(\mathbf{I} + \int_0^t (\mathbf{A} + \mathrm{Diag}(\underline{\tilde{\xi}}(\theta^*(r))))\,\mathrm{d}r\Big)\mathbf{e}_i,\underline{\mathbf{1}}\Big\rangle - \Big\langle \Big(\mathbf{I} + \int_0^t (\mathbf{A} + \mathrm{Diag}(\underline{\xi}(\theta^*(r))))\,\mathrm{d}r\Big)\mathbf{e}_i,\underline{\mathbf{1}}\Big\rangle = 0.$$

218 This yields

$$\Big(\sum_{k=1,k\neq i}^{N} ta_{ki} + 1 + a_{ii}t + \int_0^t \tilde{\xi}_i(\theta_i^*(r))\,\mathrm{d}r\Big) - \Big(\sum_{k=1,k\neq i}^{N} ta_{ki} + 1 + a_{ii}t + \int_0^t \xi_i(\theta_i^*(r))\,\mathrm{d}r\Big) = 0,$$

219 i.e.,

$$\int_0^t \tilde{\xi}_i(\theta_i^*(r))\,\mathrm{d}r - \int_0^t \xi_i(\theta_i^*(r))\,\mathrm{d}r = 0, \quad \text{for } i = 1,2,\ldots,N,$$

220 which simplifies to

$$\mu_i(t) - r_i(t) - \theta_i(t)\sigma_i^2(t) + \int_{\mathbb{R}}(e^z - 1)(e^{-z\theta_i(t)} - 1)\rho_i(z)\,\mathrm{d}z = 0, \quad t\text{-a.e.}, \ \forall\, t \in [0,T]. \tag{2.32}$$

221 Eq. (2.32) coincides with the martingale condition for the family $\{\mathbb{Q}^\theta : \theta \in \Theta\}$ as given
222 in (2.20). Hence, the martingale condition for the family $\{\mathbb{Q}^\theta : \theta \in \Theta\}$ is a first order
223 approximation of the martingale condition for $\{\mathbb{Q}_{\theta^*} : \theta^* \in \Theta^*\}$. We can think of the pricing
224 kernel Λ^{θ^*} as having more information than the kernel Λ^θ with θ^* been more realistic.
225   We will now as in [20, 22] derive the martingale condition for $\mathbb{Q}_{\theta^*}$ by taking a two-order
226 approximation for the matrix exponential in (2.30). This will enable to move from the less
227 realistic assumption where the whole path of the Markov chain is known to the more realistic
228 one where only the initial state in know. The approximation is given by

$$\exp(\mathbf{E}) \approx \mathbf{I} + \mathbf{E} + \frac{1}{2}\mathbf{E}^2. \tag{2.33}$$

229 For simplicity, we consider two regimes i.e, $N = 2$ and we set $a_{11} = -a_{12} = -a$ and
230 $a_{21} = -a_{22} = a$; $a \geq 0$ and $t > 0$. In this case, we need to solve the following pair of equations:

$$\left\langle e^{\int_0^t (\mathbf{A} + \mathrm{Diag}(\underline{\tilde{\xi}}(\theta^*(r)))) \, \mathrm{d}r} \mathbf{e}_1, \underline{\mathbf{1}} \right\rangle - \left\langle e^{\int_0^t (\mathbf{A} + \mathrm{Diag}(\underline{\xi}(\theta^*(r)))) \, \mathrm{d}r} \mathbf{e}_1, \underline{\mathbf{1}} \right\rangle = 0, \tag{2.34}$$

$$\left\langle e^{\int_0^t (\mathbf{A} + \mathrm{Diag}(\underline{\tilde{\xi}}(\theta^*(r)))) \, \mathrm{d}r} \mathbf{e}_2, \underline{\mathbf{1}} \right\rangle - \left\langle e^{\int_0^t (\mathbf{A} + \mathrm{Diag}(\underline{\xi}(\theta^*(r)))) \, \mathrm{d}r} \mathbf{e}_2, \underline{\mathbf{1}} \right\rangle = 0 \tag{2.35}$$

231 for $\underline{\mathbf{1}} = (1, 1)' \in \mathbb{R}^2$. But

$$E = \int_0^t \left(\mathbf{A} + \mathrm{Diag}(\underline{\tilde{\xi}}(\theta^*(r)))\right) \mathrm{d}r, \tag{2.36}$$

232 or

$$E = \begin{pmatrix} \int_0^t (-a + \tilde{\xi}_1(\theta_1^*(r))) \, \mathrm{d}r & at \\ at & \int_0^t (-a + \tilde{\xi}_2(\theta_2^*(r))) \, \mathrm{d}r \end{pmatrix}. \tag{2.37}$$

233 Substituting (2.33) in (2.34), the martingale condition (2.23), for $X(0) = \mathbf{e}_1 = (1, 0)'$ becomes

$$\int_0^t (\tilde{\xi}_1(\theta_1^*(r)) - \xi_1(\theta_1^*(r))) \, \mathrm{d}r - at \int_0^t (\tilde{\xi}_1(\theta_1^*(r)) - \xi_1(\theta_1^*(r))) \, \mathrm{d}r$$

$$+ \frac{1}{2}\left\{ \left[\int_0^t (\tilde{\xi}_1(\theta_1^*(r)) - \xi_1(\theta_1^*(r))) \, \mathrm{d}r \right] \left[\int_0^t (\tilde{\xi}_1(\theta_1^*(r)) + \xi_1(\theta_1^*(r))) \, \mathrm{d}r \right]\right.$$

$$\left. + at \int_0^t (\tilde{\xi}_2(\theta_2^*(r)) - \xi_2(\theta_2^*(r))) \, \mathrm{d}r \right\} = 0. \tag{2.38}$$

234 Similarly, for $X(0) = \mathbf{e}_2 = (0, 1)$, substituting (2.33) in (2.35), we get

$$\int_0^t (\tilde{\xi}_2(\theta_2^*(r)) - \xi_2(\theta_2^*(r))) \, \mathrm{d}r - at \int_0^t (\tilde{\xi}_2(\theta_2^*(r)) - \xi_2(\theta_2^*(r))) \, \mathrm{d}r$$

$$+ \frac{1}{2}\left\{ \left[\int_0^t (\tilde{\xi}_2(\theta_2^*(r)) - \xi_2(\theta_2^*(r))) \, \mathrm{d}r \right] \left[\int_0^t (\tilde{\xi}_2(\theta_2^*(r)) + \xi_2(\theta_2^*(r))) \, \mathrm{d}r \right]\right.$$

$$\left. + at \int_0^t (\tilde{\xi}_1(\theta_1^*(r)) - \xi_1(\theta_1^*(r))) \, \mathrm{d}r \right\} = 0. \tag{2.39}$$

235 Here

$$\tilde{\xi}_i(\theta_i^*(t)) - \xi_i(\theta_i^*(t)) = \mu_i(t) - r_i(t) - \theta_i^*(t)\sigma_i^2(t) + \int_{\mathbb{R}} (e^z - 1)(e^{-z\theta_i^*(t)} - 1)\rho_i(z) \, \mathrm{d}z, \ t\text{-a.e} \tag{2.40}$$

236   and

$$\tilde{\xi}_i(\theta_i^*(t)) + \xi_i(\theta_i^*(t)) = \mu_i(t) - r_i(t) - 2\theta_i^*(t)\mu_i(t) + (\theta_i^*(t))^2\sigma_i^2(t)$$
$$+ \int_{\mathbb{R}} (e^{-z(\theta_i^*(t)-1)} + e^{-z\theta_i^*(t)} - 2) + (2\theta_i^*(t) - 1)(e^{-z} - 1)\rho_i(z)\,\mathrm{d}z,\ t\text{-a.e.}$$

(2.41)

237   for $i = 1, 2$. (2.38) and (2.39) are more tractable than (2.23) and we shall use them to
238   determine the EMM parameters $(\theta_1^*(t), \theta_2^*(t))$ for the numerical illustrations.

239   2.5. **Particular cases.** In this section, we present the developments made in the previous
240   section for particular models. In the sequel, we take $N = 2$, i.e., the Markov chain X
241   moves only between the two states $\mathbf{e}_1 = (1,0)^T$ and $\mathbf{e}_2 = (0,1)^T$. We shall give explicit
242   martingale conditions for regime-switching Black-Scholes, Variance Gamma (VG) and Carr
243   Geman Madan and Yor (CGMY) models when the coefficient are constants. Note that the
244   former cases of regime-switching Black-Scholes and Variance Gamma models were already
245   derived in [22] and [20].

246   2.5.1. *The regime-switching Black-Scholes model.* In this section, we present the regime
247   switching Black-Scholes model. The dynamic of price of the risky asset in this case is given
248   by

$$S(t) = S(0)\exp\left\{ \int_0^t \left(\mu(s) - \frac{1}{2}\sigma^2(s)\right)\mathrm{d}s + \int_0^t \sigma(s)\,\mathrm{d}W(s)\right\}.$$

(2.42)

249   In the following theorem, we give (without proof) the equation satisfied by the state price
250   density θ_i and θ_i^*.

251   **Theorem 2.5.**
252   *Assume that the dynamic of the stock price is given by (2.42). Then the values of θ_i satisfying*
253   *the martingale condition (2.20) are reduced to*

$$\theta_i = \frac{\mu_i - r_i}{\sigma_i^2}\ \text{for } i = 1, 2.$$

(2.43)

254   *Moreover, θ_i^* in (2.23) satisfy the following system of nonlinear equations in (θ_1^*, θ_2^*),*

$$\frac{\sigma_1^4 t^2}{2}(\theta_1^*)^3 - \frac{(3\mu_1 - r_1)\sigma_1^2 t^2}{2}(\theta_1^*)^2 + \left(\sigma_1^2 t + \frac{(\mu_1 - r_1)(\sigma_1^2(t) + 2\mu_1)t^2 - a\sigma_1^2 t^2}{2}\right)\theta_1^*(t)$$
$$+ \frac{a\sigma_2^2 t^2}{2}\theta_2^* - \left(\frac{(\mu_1 - r_1)^2 - a(\mu_1 - r_1) + a(\mu_2 - r_2)}{2}\right)t^2 - (\mu_1 - r_1)t = 0,\ t \in [0, T] \quad (2.44)$$

255   *and*

$$\frac{\sigma_2^4 t^2}{2}(\theta_2^*)^3 - \frac{(3\mu_2 - r_2)\sigma_2^2 t^2}{2}(\theta_2^*)^2 + \left(\sigma_2^2 t + \frac{(\mu_2 - r_2)(\sigma_2^2 + 2\mu_2)t^2 - a\sigma_2^2 t^2}{2}\right)\theta_2^*$$
$$+ \frac{a\sigma_1^2 t^2}{2}\theta_1^* - \left(\frac{(\mu_2 - r_2)^2 - a(\mu_2 - r_2) + a(\mu_1 - r_1)}{2}\right)t^2 - (\mu_2 - r_2)t = 0,\ t \in [0, T]. \quad (2.45)$$

256   *Proof.* See [22]. □

257 2.5.2. *The regime-switching Variance-Gamma model.* In this section we present the regime
258 switching variance-gamma model. We obtain this model from the general model for the risky
259 asset described in equation (2.6) by setting the dynamics of the process as

$$
S(t) = S(0) \exp \Big[\int_0^t \mu(s)\, \mathrm{d}s + \int_0^t \int_{\mathbb{R}\setminus\{0\}} z \tilde{N}_{VG}^X(\, \mathrm{d}s,\, \mathrm{d}z)
$$
$$
- \int_0^t \int_{\mathbb{R}\setminus\{0\}} (e^z - 1 - z)\rho_{VG}^X(\, \mathrm{d}z)\, \mathrm{d}s \Big], \tag{2.46}
$$

260 where the jump process $N_{VG}(\cdot,\cdot)$ has the predictable compensator

$$
\rho_{VG}^X(\, \mathrm{d}z)\, \mathrm{d}t = \sum_{i=1}^{2} \langle \mathbf{e}_i, X(t^-) \rangle \rho_i^{VG}(z)\, \mathrm{d}t, \tag{2.47}
$$

261 with the Lévy measure associated to the variance gamma process as

$$
\rho_i^{VG}(z) = C_i \frac{e^{-G_i|z|}}{|z|} 1_{z<0} + C_i \frac{e^{-M_i|z|}}{|z|} 1_{z>0}. \tag{2.48}
$$

262 We then have the following martingale conditions theorem

263 **Theorem 2.6.** *Assume that the dynamic of the stock price is given by* (2.46). *Then the*
264 *values of θ_i satisfying the martingale condition* (2.20) *are reduced to*

$$
\mu_i - r_i - C_i \log \Big(\frac{G_i M_i}{(G_i+1)(M_i-1)} \Big) + C_i \log \Big(\frac{(G_i - \theta_i)(M_i + \theta_i)}{(G_i - \theta_i + 1)(M_i + \theta_i - 1)} \Big) = 0 \tag{2.49}
$$

265 *for $i = 1, 2$. Moreover, θ_i^* in* (2.23) *satisfy the following system of nonlinear equations in*
266 (θ_1^*, θ_2^*)

$$
\Big\{ \mu_1 - r_1 - C_1 \log \Big(\frac{G_1 M_1}{(G_1+1)(M_1-1)} \Big) + C_1 \log \Big(\frac{(G_1 - \theta_1^*)(M_1 + \theta_1^*)}{(G_1 - \theta_1^* + 1)(M_1 + \theta_1^* - 1)} \Big) \Big\}
$$
$$
\times \Big\{ t + \frac{1}{2} t^2 \Big[\mu_1 - r_1 - 2\theta_1^* \mu_1 + C_1 \log \Big(\frac{G_1 M_1}{(G_1 - \theta_1^*)(M_1 + \theta_1^*)} \Big) \tag{2.50}
$$
$$
+ C_1 \log \Big(\frac{G_1 M_1}{(G_1 - \theta_1^* + 1)(M_1 + \theta_1^* - 1)} \Big) + (2\theta_1^* - 1) C_1 \log \Big(\frac{G_1 M_1}{(G_1+1)(M_1-1)} \Big) \Big] - a \Big\}
$$
$$
+ \frac{1}{2} a t^2 \Big\{ \mu_2 - r_2 - C_2 \log \Big(\frac{G_2 M_2}{(G_2+1)(M_2-1)} \Big) + C_2 \log \Big(\frac{(G_2 - \theta_2^*)(M_2 + \theta_2^*)}{(G_2 - \theta_2^* + 1)(M_2 + \theta_2^* - 1)} \Big) \Big\} = 0
$$

267 *and*

$$
\Big\{ \mu_2 - r_2 - C_2 \log \Big(\frac{G_2 M_2}{(G_2+1)(M_2-1)} \Big) + C_2 \log \Big(\frac{(G_2 - \theta_2^*)(M_2 + \theta_2^*)}{(G_2 - \theta_2^* + 1)(M_2 + \theta_2^* - 1)} \Big) \Big\}
$$
$$
\times \Big\{ t + \frac{1}{2} t^2 \Big[\mu_2 - r_2 - 2\theta_2^* \mu_2 + C_2 \log \Big(\frac{G_2 M_2}{(G_2 - \theta_2^*)(M_2 + \theta_2^*)} \Big) \tag{2.51}
$$
$$
+ C_2 \log \Big(\frac{G_2 M_2}{(G_2 - \theta_2^* + 1)(M_2 + \theta_2^* - 1)} \Big) + (2\theta_2^* - 1) C_2 \log \Big(\frac{G_2 M_2}{(G_2+1)(M_2-1)} \Big) \Big] - a \Big\}
$$
$$
+ \frac{1}{2} a t^2 \Big\{ \mu_1 - r_1 - C_1 \log \Big(\frac{G_1 M_1}{(G_1+1)(M_1-1)} \Big) + C_1 \log \Big(\frac{(G_1 - \theta_1^*)(M_1 + \theta_1^*)}{(G_1 - \theta_1^* + 1)(M_1 + \theta_1^* - 1)} \Big) \Big\} = 0.
$$

268 *Proof.* See [20]. □

269 **2.5.3. The regime-Switching CGMY Model.** In this section we present the regime switching
270 CGMY. This model is obtained from the general case by setting the dynamics of the risky
271 process S as

$$S(t) = S(0) \exp\left[\int_0^t \mu(s)\,\mathrm{d}s + \int_0^t \int_{\mathbb{R}\setminus\{0\}} z\tilde{N}_{CGMY}^X(\,\mathrm{d}s,\,\mathrm{d}z)\right.$$
$$\left. - \int_0^t \int_{\mathbb{R}\setminus\{0\}} (e^z - 1 - z)\rho_{CGMY}^X(\,\mathrm{d}z)\,\mathrm{d}s\right], \tag{2.52}$$

272 where the jump process $N_{CGMY}^X(t;\cdot)$ has the predictable compensator

$$\rho_{CGMY}^X(\,\mathrm{d}z)\,\mathrm{d}t = \sum_{i=1}^2 \langle \mathbf{e}_i, X(t^-)\rangle \rho_i^{CGMY}(z)\,\mathrm{d}t, \tag{2.53}$$

273 with the Lévy measure associated to the CGMY process as

$$\rho_i^{CGMY}(z) = C_i \frac{e^{-G_i|z|}}{|z|^{1+Y}}\mathbf{1}_{z<0} + C_i \frac{e^{-M_i|z|}}{|z|^{1+Y}}\mathbf{1}_{z>0}. \tag{2.54}$$

274 In the following theorem, we derive the equation satisfied by the state price density θ_i of
275 the equivalent martingale measure $\mathbb{Q}^{\theta_i}$ when the price of risk in the regime switching model
276 is not taken into account.

277 **Theorem 2.7** (Martingale condition without price of risk)**.** *Assume that the dynamic of the*
278 *stock price is given by* (2.52)*. Moreover assume that the state price density θ_i is such that*
279 $0 < \theta_i < G_i$ *and* $M_i > 1$*. Then $\theta_i(t)$ satisfies the following system of equations*

$$\mu_i - r_i + C_i\Gamma(-Y_i)\Big[(G_i - (\theta_i - 1))^{Y_i} - (G_i + 1)^{Y_i} - (G_i - \theta_i)^{Y_i}$$
$$+ G_i^{Y_i} + M_i^{Y_i} + (M_i + \theta_i - 1)^{Y_i} - (M_i - 1)^{Y_i} - (M_i + \theta_i)^{Y_i}\Big] = 0$$
$$\textit{for } i = 1, 2. \tag{2.55}$$

280 *Proof.* Assume that S satisfies (2.52), then (2.20) is reduced to

$$\mu_i - r_i + \int_{\mathbb{R}} (e^z - 1)(e^{-z\theta_i} - 1)\nu_i(z)\,\mathrm{d}z = 0, \ \ i = 1, 2. \tag{2.56}$$

281 The integral term involved in equation (2.56) is computed as follows

$$\int_{\mathbb{R}} (e^z - 1)(e^{-z\theta_i} - 1)\nu_i(z)\,\mathrm{d}z = \int_{\mathbb{R}} (e^z - 1)(e^{-z\theta_i} - 1)\Big(C_i\frac{e^{-G_i|z|}}{|z|^{1+Y}}\mathbf{1}_{z<0} + C_i\frac{e^{-M_i|z|}}{|z|^{1+Y}}\mathbf{1}_{z>0}\Big)\,\mathrm{d}z$$
$$= \int_{-\infty}^0 (e^z - 1)(e^{-z\theta_i} - 1)\frac{C_i\exp(G_iz)}{(-z)^{Y_i+1}}\,\mathrm{d}z$$
$$+ \int_0^\infty (e^z - 1)(e^{-z\theta_i} - 1)\frac{C_i\exp(-M_iz)}{(z)^{Y_i+1}}\,\mathrm{d}z$$
$$= I_1 + I_2. \tag{2.57}$$

282 We shall now consider different cases
283 **Case 1;** $\quad Y = 0$

284 This now becomes the variance gamma case. This case was discussed in the previous section.

285 **Case 2;** $Y \neq 0$ We have that

$$
\begin{aligned}
I_1 &= \int_{-\infty}^{0} \left(e^{-z(\theta_i - 1)} - e^z - e^{-z\theta_i} + 1 \right) \frac{C_i e^{G_i z}}{(-z)^{Y_i + 1}} \, dz \\
&= C_i \Big[\int_{-\infty}^{0} \left(e^{G_i - z(\theta_i - 1)} (-z)^{-1-Y_i} \right) dz - \int_{-\infty}^{0} \left(e^{(G_i + 1)z} (-z)^{-1-Y_i} \right) dz \\
&\quad - \int_{-\infty}^{0} \left(e^{(G_i - \theta_i)z} (-z)^{-1-Y_i} \right) dz - \int_{-\infty}^{0} \left(e^{G_i z} (-z)^{-1-Y_i} \right) dz \Big].
\end{aligned}
\tag{2.58}
$$

286 Put $w = -(G_i - (\theta_i - 1))z$, $w = -(G_i + 1)z$, $w = -(G_i - \theta_i)z$, $w = -G_i z$ in the first, second,
287 third and fourth integral respectively, then using the definition of the gamma function, we
288 get

$$
I_1 = C_i \Gamma(-Y_i) \Big[(G_i - (\theta_i - 1))^{Y_i} - (G_i + 1)^{Y_i} - (G_i - \theta_i)^{Y_i} + G^{Y_i} \Big].
\tag{2.59}
$$

289 In the same way, I_2 is solved explicitly using change of variable and the definition of the
290 gamma function to get

$$
I_2 = C_i \Gamma(-Y_i) \Big[(M_i + \theta_i - 1)^{Y_i} - (M_i - 1)^{Y_i} - (M_i + \theta_i)^{Y_i} + M_i^{Y_i} \Big].
\tag{2.60}
$$

291 Combining (2.59) and (2.60), we get

$$
\begin{aligned}
\int_{\mathbb{R}} (e^z - 1)(e^{-z\theta_i} - 1)\rho_i(z) \, dz = C_i \Gamma(-Y_i) \Big[&(G_i - (\theta_i - 1))^{Y_i} - (G_i + 1)^{Y_i} \\
&+ G_i^{Y_i} - (G_i - \theta_i)^{Y_i} + (M_i + \theta_i - 1)^{Y_i} \\
&- (M_i - 1)^{Y_i} - (M_i + \theta_i)^{Y_i} + M_i^{Y_i} \Big].
\end{aligned}
\tag{2.61}
$$

292 Substituting this into equation (2.56) gives us the desired result.   □

293   In the following theorem, we derive the equation satisfied by the state price density θ_i^* of
294 the equivalent martingale measure $\mathbb{Q}_{\theta_i^*}$ when the price of risk in the regime switching model
295 is taken into account.

296 **Theorem 2.8** (Martingale condition with price of risk). *Assuming that conditions of theorem*
297 *(2.7) are satisfied. Then the state price densities $\theta_i^*(t)$ in (2.23) satisfy the following system*

298 *of non linear equations in (θ_1^*, θ_2^*)),*

$$\left\{\mu_1 - r_1 + C_1\Gamma(-Y_1)\Big[(G_1 - (\theta_1 - 1))^{Y_1} - (G_1 + 1)^{Y_1} - (G_1 - \theta_1)^{Y_1}\right.$$
$$\left.+ G_1^{Y_1} + M_1^{Y_1} + (M_1 + \theta_1 - 1)^{Y_1} - (M_1 - 1)^{Y_1} - (M_1 + \theta_1)^{Y_1}\Big]\right\}$$
$$\times \left\{t + \frac{1}{2}t^2\Big(\mu_1 - r_1 - 2\theta_1^*\mu_1 + C_1\Gamma(-Y_1)\Big[(G_1 - (\theta_1^* - 1))^{Y_1}\right.$$
$$+ (G_1 - \theta_1^*)^{Y_1} + (2\theta_1^* - 1)(G_1 + 1)^{Y_1} - (2\theta_1^* + 1)G_1{}^{Y_1} + (M_1 + \theta_1^* - 1)^{Y_1}$$
$$\left.+ (M_1 + \theta_1)^{Y_1} + (2\theta_1^* - 1)(M_1 - 1)^{Y_1} - (2\theta_1^* + 1)M_1{}^{Y_1}\Big]\Big) - a\right\}$$
$$+ \frac{1}{2}t^2 a\Big\{\mu_2 - r_2 + C_2\Gamma(-Y_2)\Big[(G_2 - (\theta_2 - 1))^{Y_2} - (G_2 + 1)^{Y_2} + G_2^{Y_2} + M_2^{Y_2}$$
$$- (G_2 - \theta_2)^{Y_2} + (M_2 + \theta_2 - 1)^{Y_2} - (M_2 - 1)^{Y_2} - (M_2 + \theta_2)^{Y_2}\Big]\Big\} = 0 \qquad (2.62)$$

299 *and*

$$\left\{\mu_2 - r_2 + C_2\Gamma(-Y_2)\Big[(G_2 - (\theta_2 - 1))^{Y_2} - (G_2 + 1)^{Y_2} - (G_2 - \theta_2)^{Y_2}\right.$$
$$\left.+ G_2^{Y_2} + M_2^{Y_2} + (M_2 + \theta_2 - 1)^{Y_2} - (M_2 - 1)^{Y_2} - (M_2 + \theta_2)^{Y_2}\Big]\right\}$$
$$\times \left\{t + \frac{1}{2}t^2\Big(\mu_2 - r_2 - 2\theta_2^*\mu_2 + C_2\Gamma(-Y_2)\Big[(G_2 - (\theta_2^* - 1))^{Y_2}\right.$$
$$+ (G_2 - \theta_2^*)^{Y_2} + (2\theta_2^* - 1)(G_2 + 1)^{Y_2} - (2\theta_2^* + 1)G_2{}^{Y_2} + (M_2 + \theta_2^* - 1)^{Y_2}$$
$$\left.+ (M_2 + \theta_2)^{Y_2} + (2\theta_2^* - 1)(M_2 - 1)^{Y_2} - (2\theta_2^* + 1)M_2{}^{Y_2}\Big]\Big) - a\right\}$$
$$+ \frac{1}{2}t^2 a\Big\{\mu_1 - r_1 + C_1\Gamma(-Y_1)\Big[(G_1 - (\theta_1 - 1))^{Y_1} - (G_1 + 1)^{Y_1} + G_1^{Y_1} + M_1^{Y_1}$$
$$- (G_1 - \theta_1)^{Y_1} + (M_1 + \theta_1 - 1)^{Y_1} - (M_1 - 1)^{Y_1} - (M_1 + \theta_1)^{Y_1}\Big]\Big\} = 0. \qquad (2.63)$$

300 *Proof.* In this case, (2.40) and (2.41) are reduced to

$$\tilde{\xi}_i(\theta_i^*(t)) - \xi_i(\theta_i^*(t)) = \mu_i(t) - r_i(t) + C_i\Gamma(-Y_i)\Big[(G_i - (\theta_i^*(t) - 1))^{Y_i} + G^{Y_i}$$
$$- (G_i + 1)^{Y_i} + M^{Y_i} - (G_i - \theta_i^*(t))^{Y_i} + (M_i + \theta_i^*(t) - 1)^{Y_i}$$
$$- (M_i - 1)^{Y_i} - (M_i + \theta_i^*(t))^{Y_i}\Big], \qquad (2.64)$$

301 and

$$\tilde{\xi}_i(\theta_i^*(t)) + \xi_i(\theta_i^*(t)) = \mu_i(t) - r_i(t) - 2\theta_i^*(t)\mu_i(t) + C_i\Gamma(-Y_i)\Big[(G_i - \theta_i^*(t))^{Y_i}$$
$$+ (G_i - (\theta_i^*(t) - 1))^{Y_i} + (2\theta_i^*(t) - 1)(G_i + 1)^{Y_i}$$
$$+ (M_i + \theta_i^*(t) - 1)^{Y_i} + (M_i + \theta_i(t))^{Y_i} - (2\theta_i^*(t) + 1)G_i{}^{Y_i}$$
$$+ (2\theta_i^*(t) - 1)(M_i - 1)^{Y_i} - (2\theta_i^*(t) + 1)M_i{}^{Y_i}\Big], \qquad (2.65)$$

302 respectively and the result follows. $\qquad \square$

303   The solutions to the martingale condition for $\mathbb{Q}_{\theta^*}$ are generally not unique and therefore
304 we need to use some criteria to select the final Esscher parameters. These criteria a discussed
305 in the Appendix.

3. Numerical results and discussions

In this section, we conduct numerical experiments for the models discussed in the previous
sections; the regime switching Black-Scholes (Model I) and CGMY (Model II). We shall
assume that there are two states of the economy i.e., $N = 2$. State 1 represents an expansion
period while state 2 represents a recession period. We assume that the transition probability
matrix is

$$\mathbf{A} = \begin{pmatrix} -a_1 & a_1 \\ a_2 & -a_2 \end{pmatrix}, \text{ with } a_1 = a_2 = 0.5$$

307 .

308 **3.1. Model I.** We assume that the stock price is driven by a regime switching geometric
309 Brownian motion.

310 *Specific forms of time dependent interest rate and volatility.* Here, we will extend the results
311 and analysis in [22] to the time dependent interest rate and volatility that is, there are both
312 functions of time. We refer the reader to [22] (see also [20]) in the case of constant parameters
313   In the following graphs, it is assumed that the exercise price is 100, the value of the asset
314 is 120, and the expiry date is one year in the future. $t = T$ is known as the remaining life of
315 an option. It is also assumed that there is a gradual trend for the parameter to move in a
316 decreasing or increasing manner which might conveniently be regarded as continuous.
317   We write the two forms as,

(a) Constant model. The interest rates in the two regimes are given by

$$r_1(t) = a_1 \text{ and } r_2(t) = a_2.$$

(b) Linear model. The interest rates are given by

$$r_1(t) = a_1 + b_1 t \text{ and } r_2(t) = a_2 - b_2 t,$$

318   where a_1, a_2, b_1, b_2 are constants with $a_1 = a_2 = b_1 = b_2 = 0.05$.

319   We define the forms of volatility as;

(a) Constant model. Volatility in the two regimes are given by

$$\sigma_1(t) = b_1 \text{ and } \sigma_2(t) = b_4.$$

(b) Decaying model. The volatility is given by

$$\sigma_1(t) = b_1 + b_2 e^{-b_3 t} \text{ and } \sigma_2(t) = b_4 + b_5 e^{-b_6 t},$$

320   where b_1, b_2, b_3, b_4, b_5, b_6 are constants with $b_1 = 0.15$, $b_2 = b_5 = b_4 = 0.25$ and
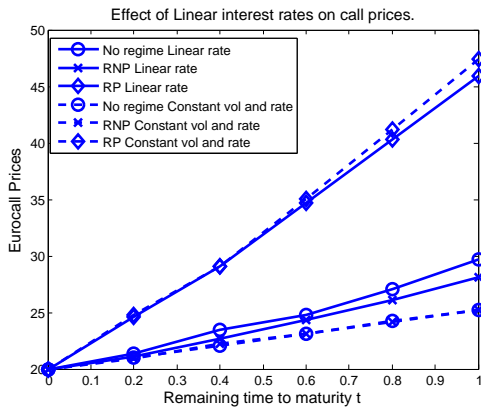321 $b_3 = b_6 = 4$.

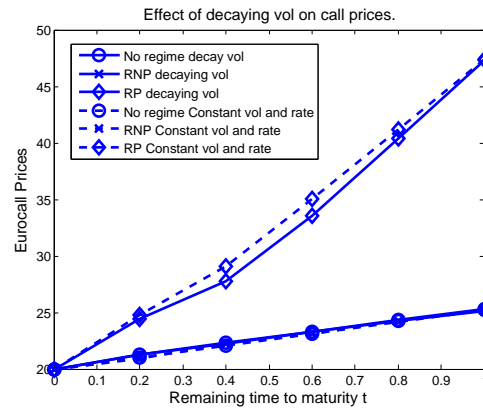FIGURE 1. Effect of linear interest rates

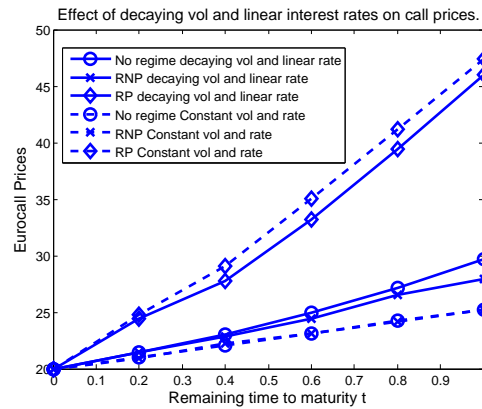FIGURE 2. Effect of decaying volatility

FIGURE 3. Effect of linear interest and decaying volatility

322   In Figure 1, while keeping the volatility constant, we investigate the impact on the option
323 price of a variation in the form of interest rate when there is no regime (NR), the regime
324 risk not priced (RNP) and the regime risk priced (RP), respectively. In Figure 2, the same
325 study is made assuming that the interest rate is constant and the form of the volatility can
326 change. Finally, in Figure 3, we looked at the impact of both linear interest rate and decaying
327 volatility on the option prices in the case of NR, RNP and RP.
328   As shown in the graphs, the same qualitative results are observed over the lifetime of the
329 option. The initial price of the option is affected in all the situations (NR, RNP and RP)
330 by the change in the form of interest rate and volatility. When the interest is constant, the
331 option price values are very closed during the option's lifetime irrespective of the form of
332 volatility. Note also that, when the regime risk is priced, the option prices are lower when

333 the parameters are time dependent than those with constant parameters. The graphs also
334 show that taking only into account the impact of the regime on the option prices leads to
335 a completely different overall result. For example, the initial value of the option prices are
336 increased substantially when the regime risk is priced. Moreover, during the lifetime of the
337 option, the option prices with the regime risk priced are higher than those with regime risk
338 not priced which are higher than those without regime risk considered.

339 3.2. **Model II.** In this section, we discuss the regime switching CGMY model. We cover in
340 particular two cases: $Y = 0$ (known as the variance gamma (VG) model) and $Y \neq 0$. We
341 refer the reader to [20] for the case $Y = 0$ with constants coefficients.

342 3.2.1. *VG Case.* We consider linear interest rates and analyse their effects on the call prices.
343 We set

$$r_1 = 0.05 + 0.05t \text{ and } r_2 = 0.01 - 0.005t,$$
$$C = [3, 4], \ G = [5, 6], \ M = [10, 8],$$
$$S(0) = 100, \ X(0) = \mathbf{e}_1, \ \mu = [0.35, 0.05].$$

344 We use the constant parameter case i.e., constant interest rates, as a marker. We define $t = T$
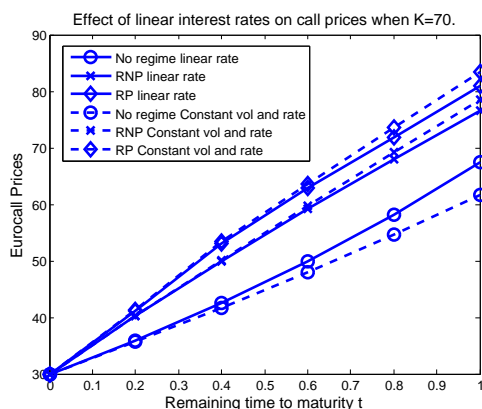345 as the remaining time to maturity. We present the results of our simulation below.

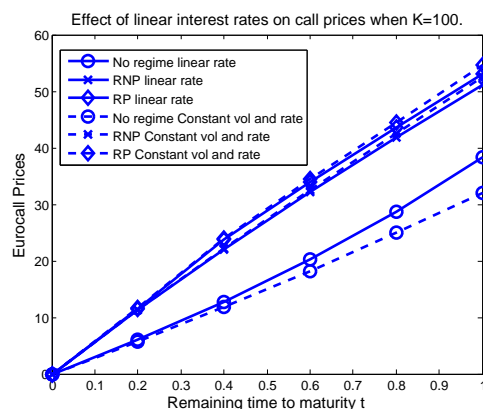FIGURE 4. Effect of Linear rates on call prices when K=70

FIGURE 5. Effect of Linear rates on call prices when K=100

346 In Figure 4 and 5, we investigate the impact of a variation in the form of interest rate on
347 the option price in three cases: no regime (NR), regime risk not priced (RNP) and regime
348 risk priced (RP). The same conclusions as in the Black-Scholes regime switching model hold
349 concerning the impact of the regime risk on the option prices. Note however that during the
350 life time of the option, the difference in option prices when the regime is priced and when it
351 is not are not significant.

352 3.2.2. *CGMY Case.* The simulation of this case proved to be more difficult than the former
353 case. We have simulated the CGMY's with $Y > 0$. We shall give the results of our simulation
354 in two examples. An algorithm for the simulation of the CGMY process can be found in [3].

355   **(1):** We assume that $Y \in (0,1)$ and set the parameters to be

$$r = [0.05, 0.01], \ \mu = [0.35, 0.05],$$
$$C = [3, 4], \ G = [5, 6], \ M = [10, 8], \ Y = [0.5, 0.5]$$
$$S(0) = 100, \ X(0) = \mathbf{e}_1, \ K = \{70, 80, 90, 100, 110, 120, 130, 140, 150\}.$$

356   We plot graphs of Call prices across different strikes.
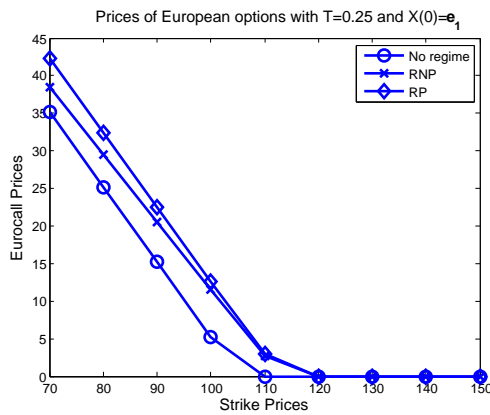
FIGURE 6. Call prices across strikes when $T = 0.25$
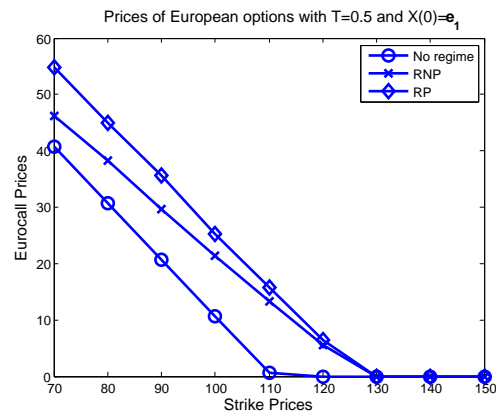
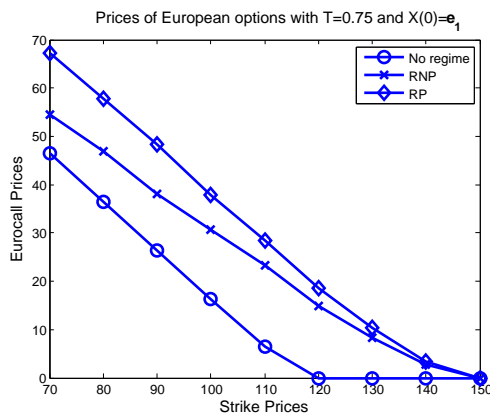FIGURE 7. Call prices across strikes when $T = 0.5$
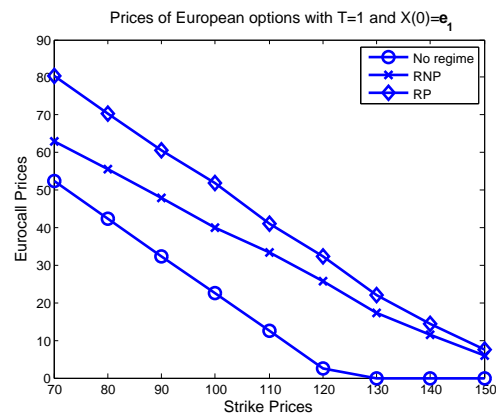

FIGURE 8. Call prices across strikes when T=0.75

FIGURE 9. Call prices across strikes when T=1

357   Figures 6- 9, depict the impact of a change in the regime on the option prices when the
358 strike price changes and the interest rate is constant in three situations: no regime (NR),
359 regime risk not priced (RNP) and regime risk priced (RP). The effect of the parameter Y is
360 seen in this case. As shown in the graph, when the exercised time increases, the initial price
361 of the option is substantially affected. For each time to maturity, as the strike price increases,
362 the value of the option decreases. Contrarily to the Black-Scholes regime switching model
363 (see [22]), the option prices with regime risk priced are higher than those with regime risk
364 not priced regardless of the option maturity. .
365   Assume now that the interest rates are linear and set

$$r_1 = 0.05 + 0.05t \text{ and } r_2 = 0.01 - 0.005t.$$

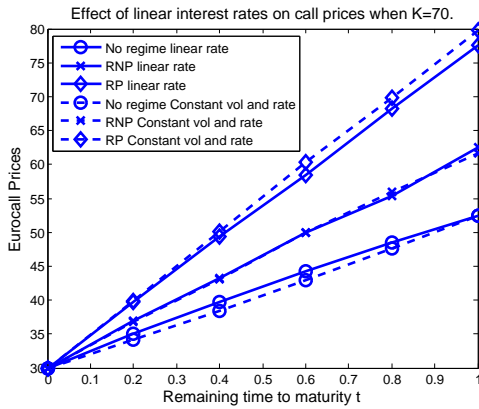366 We use the constant parameter case., i.e constant interest rates, as a marker.

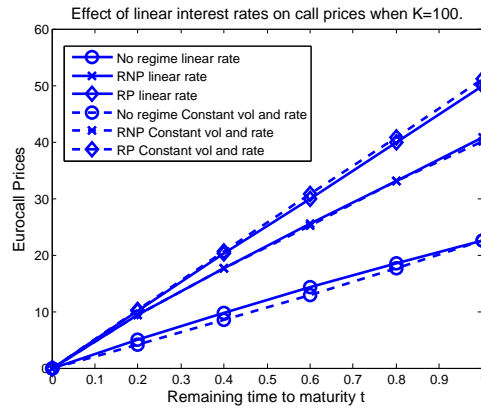FIGURE 10. Effect of
Linear rates on call
prices when K=70

FIGURE 11. Effect of
Linear rates on call
prices when K=100

367   In Figures 10 and 11, we examine the impact that a change in the form of interest rate has
368 on the option price. It can be seen that, there is no substantial impact of the form of interest
369 rate in the three cases. However, there is a significant difference in the option prices when
370 considering the impact of the regime risk. Once again, the initial value of the option price is
371 considerably increased when the regime risk is priced, and during the lifetime of the option,
372 its price when the regime risk is priced is higher than that when when the regime risk is not
373 priced which is higher than that when there is no regime.

374 **Remark 3.1.** *When $Y \in (0,1)$, the CGMY process is an infinite activity and finite variation*
375 *process. This means that the path of the process has a similar behaviour to the path of the*
376 *VG process.*

377   4. CONCLUSION

378   In this paper, we use the pricing method developed in [22] to price options when the
379 underlying assets are driven by a regime switching CGMY process with time dependent
380 parameters. The theoretical results are given for general regime switching exponential Lévy
381 model with time dependent parameters. The choice of the martingale pricing measure is

382 justified by the minimization of the maximum entropy. We conduct numerical experiments
383 to investigate the effect of pricing regime-switching risk and the analysis shows a significant
384 difference of values between prices of an European call when the regime-risk priced and when
385 the regime risk not priced. We also observe that the regime risk is sensitive to market
386 parameters like time dependent interest rates and volatilities with the sensitivity higher in
387 the case of the Black-Scholes than in the Variance Gamma or CGMY cases.

388 We may explore the applications of our models to other types of options such as American
389 options, barrier options, look back options, Asian options, Exotic options, option-embedded
390 insurance products, etc. We may also extend our framework to include stochastic interest
391 rates and volatility which would probably give higher values of the option prices.

Appendix A. Criteria for selecting Esscher parameters

393 As already mentioned systems of equations characterizing martingale condition for $\mathbb{Q}_{\theta^*}$
394 have in general more than one solution in $(\theta_1^*(t), \theta_2^*(t))$. Here, we present the selection criteria
395 of the set of neutral Esscher parameters $(\theta_1^*(t), \theta_2^*(t))$ that minimizes the maximum entropy
396 between an EMM and the real world probability measure over different states. The idea is
397 from [22].

398 Define first the entropy between $\mathbb{Q}_{\theta^*}$ and $\mathbb{P}$ conditional on $X(0) \in \{\mathbf{e}_1, \mathbf{e}_2\}$ as follows.

$$
\begin{aligned}
I(\mathbb{Q}_{\theta^*}, \mathbb{P}) :&= E^{\mathbb{P}}\left[\frac{\mathrm{d}\mathbb{Q}_{\theta^*}}{\mathrm{d}\mathbb{P}}\ln\left(\frac{\mathrm{d}\mathbb{Q}_{\theta^*}}{\mathrm{d}\mathbb{P}}\right)\bigg|X(0) = \mathbf{e}_i\right] = E^{\mathbb{P}}\left[\Lambda_T^{\theta^*}\ln\Lambda_T^{\theta^*}\bigg|X(0) = \mathbf{e}_i\right] \\
&= \frac{E^{\mathbb{P}}\left[-\int_0^T \theta(s)\,\mathrm{d}Y(s)e^{-\int_0^T \theta(s)\,\mathrm{d}Y(s)}\Big|X(0) = \mathbf{e}_i\right]}{E^{\mathbb{P}}\left[e^{-\int_0^T \theta(s)\,\mathrm{d}Y(s)}\Big|X(0) = \mathbf{e}_i\right]} \\
&\quad - \ln E^{\mathbb{P}}\left[e^{-\int_0^T \theta(s)\,\mathrm{d}Y(s)}\Big|X(0) = \mathbf{e}_i\right].
\end{aligned}
\tag{A.1}
$$

399 Let $\Gamma := \{\theta^* \in \mathbb{R}^2 | \theta^*$ satisfies (2.38) and (2.39)$\}$ and denote by $I_M(\mathbb{Q}_{\theta^*}, \mathbb{P})$ the maximum
400 entropy between $\mathbb{Q}_{\theta^*}$ and $\mathbb{P}$ over the different values of $X(0)$, i.e.,

$$
I_M(\mathbb{Q}_{\theta^*}, \mathbb{P}) := \max_{i=1,2} I(\mathbb{Q}_{\theta^*}, \mathbb{P}|X(0) = \mathbf{e}_i).
\tag{A.2}
$$

401 One can show as in [22] that

$$
\begin{aligned}
I(\mathbb{Q}_{\theta^*}, \mathbb{P}|X(0) = \mathbf{e}_i) :&= E^{\mathbb{P}}\left[\frac{\mathrm{d}\mathbb{Q}_{\theta^*}}{\mathrm{d}\mathbb{P}}\ln\left(\frac{\mathrm{d}\mathbb{Q}_{\theta^*}}{\mathrm{d}\mathbb{P}}\right)\bigg|X(0) = \mathbf{e}_i\right] \\
&= \frac{\langle e^{\int_0^T (A+diag(\xi_i^k(\theta_i^*(t))))\,\mathrm{d}t}X(0), \mathbf{1}_2\rangle}{\langle e^{\int_0^T (A+diag(\xi_i(\theta_i^*(t))))\,\mathrm{d}t}X(0), \mathbf{1}_2\rangle} - \ln\langle e^{\int_0^T (A+diag(\xi_i(\theta_i^*(t))))\,\mathrm{d}t}X(0), \mathbf{1}_2\rangle.
\end{aligned}
\tag{A.3}
$$

402 The selected $(\theta_1^*(t), \theta_2^*(t))$ shall be solution to the following problem: Find $(\hat{\theta}_1^*(t), \hat{\theta}_2^*(t)) \in \Gamma$
403 such that

$$
I_M(\mathbb{Q}_{\hat{\theta}^*}, \mathbb{P}) = \min_{\theta^* \in \Gamma} I_M(\mathbb{Q}_{\theta^*}, \mathbb{P}),
\tag{A.4}
$$

404 with $\Gamma := \{\theta^* \in \mathbb{R}^2 | \theta^*$ satisfies (2.38) and (2.39)$\}$

Appendix B. Simulation procedure

In this section, we discuss the simulation procedure. We adopt a straight forward Monte-Carlo procedure in order to obtain simulation approximations for the European call price. Suppose we want to evaluate the price of a European call option at the current time $t = 0$ with maturity T and strike price K. We note that the call option $C(0, S(0), X(0))$ can be evaluated as follows:

$$
\begin{aligned}
C(0, X(0), S(0)) &= E^{\theta^*}\Big[\exp\Big(-\int_0^t r(u)\,\mathrm{d}u\Big)(S(T) - K)^+\Big] \\
&= E^{\mathbb{P}}\Big[\frac{\mathrm{d}\mathbb{Q}_{\theta^*}}{\mathrm{d}\mathbb{P}}\exp\Big(-\int_0^t r(u)\,\mathrm{d}u\Big)(S(T) - K)^+\Big|S(0), X(0)\Big].
\end{aligned}
\tag{B.1}
$$

We assume that the process S is simulated over a discrete grid. To achieve this, we divide the time horizon $[0, T]$ into J subintervals $[t_j, t_{j+1}]$ for $j = 0, 1, \ldots, J - 1$ of equal length $\Delta = \frac{T}{J}$ where $t_0 = 0$ and $t_J = T$.
For the discrete-time version of the Markov chain X, we suppose that the transition probability matrix in a subinterval is $I + \mathbf{A}\Delta$ given $X(0)$.
Given the simulated path of X, the sample paths of the processes $\{\mu(t_j)\}_{j=1}^J$, $\{\sigma(t_j)\}_{j=1}^J$, $\{\theta(t_j)\}_{j=1}^J$ and $\{r(t_j)\}_{j=1}^J$ are identified. Then, we can now use these to construct a Euler forward discretization scheme to discritize the log return process Y as follows

$$
\begin{aligned}
Y(t_{j+1}) &= Y(t_j) + \Delta * (\mu(t_j) - \frac{1}{2}\sigma^2(t_j)) + \Delta * \int_{\mathbb{R}}(e^z - 1 - z)\rho^{X(t_j)}(dz) \\
&\quad + \sigma(t_j) * \xi * \sqrt{\Delta} + \tilde{J}_j^X(t_{j+1}) - \tilde{J}_j^X(t_j).
\end{aligned}
\tag{B.2}
$$

where $\xi \sim N(0,1)$ and

$$
\tilde{J}_j^X(t) = \int_{\mathbb{R}} z J_j^X(t;\,\mathrm{d}z) - \int_0^t \int_{\mathbb{R}} z \rho^{X(t_j)}(\,\mathrm{d}z)\,\mathrm{d}t.
\tag{B.3}
$$

Given $\{X(t_j)\}_{j=1}^J$ and $Y(0) = 0$, we then sample $\{Y(t_j)\}_{j=1}^J$ using (B.2) recursively. The Monte Carlo simulation procedure can be found in [22].

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