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Flows between dualities for 3d Chern-Simons theories

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We study Seiberg-like dualities for three-dimensional $\mathcal{N} = 2$ theories with flavors in fundamental and adjoint representations. The recent results of Intriligator and Seiberg provide a derivation of an Aharony duality from a Giveon–Kutasov duality. We extend their result to the case of more general theories involving various masses for fundamental quarks and adjoint fields. By fine-tuning the vacuum expectation value of a scalar field and using various identifications between gauge groups and their singlet duals, we derive several examples of Aharony dualities. For theories with an adjoint field, we discuss the connection between the Aharony dualities proposed by Kim and Park for theories with multiple Coulomb branches and Giveon–Kutasov–Niarchos dualities.

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I. INTRODUCTION

The breaking of supersymmetry (SUSY) from $\mathcal{N} = 2$ to $\mathcal{N} = 1$ in four-dimensional theories with flavors has been the subject of intense study in both field theory or string theory. It is achieved by turning on superpotentials for the adjoint fields and the matter fields. In string theory, one can either consider brane configurations with suspended D4-branes [1], geometries with wrapped D5-branes [2], or matrix models [3]. These distinct approaches are related to each other by T dualities [4]. String theory provides insights into strongly coupling regimes, Seiberg dualities, metastable vacua, etc.

One immediate question is whether one can translate this knowledge to three-dimensional (3d) theories, where the corresponding deformation would be $\mathcal{N} = 4 \rightarrow \mathcal{N} = 2$. A partial answer to this question was obtained some time ago when certain aspects of mirror symmetry for 3d $\mathcal{N} = 2$ theories were uncovered [5]. In recent years there has been a renewal of interest in 3d gauge theories along various exciting research avenues. On one hand 3-algebras were shown to describe multiple, parallel membranes in M theory (see Ref. [6] for a review and a complete list of references). On the other hand, a new AdS/CFT duality was proposed in Ref. [7] (see again Ref. [6] for details and references).

A third recent direction, built up on the developments of Ref. [5], has already provided important fresh tools for exploring Seiberg dualities as the computation of the partition functions on 3-spheres and the corresponding superconformal index [8–23]. Various connections between 3d dualities and usual four-dimensional (4d) dualities were discussed in Refs. [24,25].

In three dimensions there are two types of Seiberg-like dualities:

(i) Giveon–Kutasov duality, which applies for theories with any Chern-Simons level. This is reminiscent of a 4d Seiberg duality for theories with fundamental matter [26] or an adjoint field [27]. The Giveon–Kutasov duality is well understood from brane configurations with D3-branes suspended between Neveu-Schwarz (NS) branes.

(ii) Aharony duality for three dimensions, which has a peculiar coupling between electric and magnetic monopoles [28]. This does not have a clear brane picture.

The two Seiberg-like dualities can be connected by starting with the Aharony duality, adding masses and generating Chern-Simons terms, resulting in the Giveon–Kutasov duality. It is less clear how to proceed the other way because a coupling between electric and magnetic monopoles has to be generated. Very recently, Ref. [29] discussed a reverse renormalization group flow from Giveon–Kutasov duality in the UV to Aharony duality in the IR. This flow provided a derivation of the monopole operator in the Aharony duality. Intriligator and Seiberg considered the case of one massive fundamental flavor and performed a fine-tuning of the expectation value for the extra adjoint scalar coming from the 4d $\mathcal{N} = 2$ vector multiplet. The result was an extra set of $U(1)$ groups with Chern-Simons level $k = -\frac{1}{2}$, which are dual to some singlets [30,31], subsequently identified with electric monopoles in the magnetic theory.

Our work extends the discussion of Ref. [29] in two directions. First, we consider the case in which more than one flavor is massive (we consider the case of two and four massive flavors). The choices of mass and the tuning of the corresponding vacuum expectation value (vev) for the adjoint scalar field lead to various electric theories and their corresponding Giveon–Kutasov duals. For particular values of the Chern-Simons level, some unbroken groups are of type $U(k)_{-\frac{k}{2}}$ and have dual descriptions in terms of singlets [31], which are identified with the Aharony dual monopoles. The second direction that we consider involves

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theories with an extra adjoint field where the Aharony duality has multidimensional Coulomb branches and multiple monopole operators are needed [32]. The Giveon–Kutasov duality for Chern-Simons theories with adjoint matter was proposed in Ref. [27]. We explain how masses for fundamental fields provide unbroken groups with singlet duals providing pairs of monopoles and connecting the two duality pairs. We use the continuous deformation between the adjoint field potential $\mathrm{Tr}\Phi^n$ and the one with lower powers in Φ in order to understand how pairs of monopoles overlap.

II. FIELD THEORY RESULTS

We start with $\mathcal{N}=4$ theory in three dimensions obtained by suspending N D3-branes between two parallel NS-branes with extra N_f flavor D5-branes.

In $\mathcal{N}=2$ language, the hypermultiplets can be written as two chiral superfields Q and $\tilde{Q}$ with charges 1 and -1. The $\mathcal{N}=4$ vector multiplet contains an $\mathcal{N}=2$ real vector multiplet V with a real scalar σ as the lowest component and a chiral multiplet Φ with a complex scalar as the lowest component. Another type of $\mathcal{N}=4$ multiplet is the linear multiplet, which contains an $\mathcal{N}=2$ linear multiplet Σ (which is real and satisfies $D^2\Sigma=\bar{D}^2\Sigma=0$).

The kinetic part of the Lagrangian for an $\mathcal{N}=4$ theory is

$$L_{\mathrm{kin}}=\int d^4\theta(Q^\dagger e^{2V}Q+\tilde{Q}^\dagger e^{-2V}\tilde{Q}) + \left[\int d^2\theta\sqrt{2}\Phi Q\tilde{Q}+\mathrm{c.c.}\right]. \quad (1)$$

The hypermultiplets can have a complex mass m_c coming form their coupling to the multiplet Φ or a real mass m_r:

$$L_{\mathrm{mass}}=\int d^4\theta(Q^\dagger e^{2im_r\theta\bar{\theta}}Q+\tilde{Q}^\dagger e^{-2im_r\theta\bar{\theta}}\tilde{Q}) + \left[\int d^2\theta m_c Q\tilde{Q}+\mathrm{c.c.}\right]. \quad (2)$$

There are also Fayet–Iliopoulos terms,

$$\xi_V\int d^4\theta V+\xi_\Phi\int d^2\theta\Phi, \quad (3)$$

and Chern-Simons terms,

$$\int d^4\theta\frac{k}{4\pi}\Sigma V. \quad (4)$$

The $\mathcal{N}=4$ theory is deformed to an $\mathcal{N}=2$ one by adding a mass term for the field Φ,[1]

$$\int d^2\theta\frac{1}{2}\mu^2\Phi^2. \quad (5)$$

Integrating out the massive fields, we obtain $\mathcal{N}=2$, $U(N)$ theory with N_f flavor fields.

The terms containing Φ are

$$\int d^2\theta\left(\sqrt{2}\Phi Q\tilde{Q}+\frac{1}{2}\mu^2\Phi^2\right). \quad (6)$$

The equation of motion for Φ is

$$\sqrt{2}Q\tilde{Q}+\mu\Phi=0, \quad (7)$$

with the simple solution $Q\tilde{Q}=\Phi=0$, suggesting either $Q=\Phi=0$ or $\tilde{Q}=\Phi=0$.

In terms of brane configurations, we consider that one NS-brane rotates by an angle $\tan(\theta)=\mu$, which goes to ∞ for $\mathcal{N}=2$ theory. In this limit, the equations of motion for the remaining fields in the $\mathcal{N}=2$ theory are [29]:

$$\int d^4\theta(V+m_r)Q=0; \qquad \int d^4\theta(-V+m_r)\tilde{Q}=0, \quad (8)$$

which for the scalar components of Q and $\tilde{Q}$ are [29]

$$(\sigma+m_r)Q=0; \qquad (-\sigma+m_r)\tilde{Q}=0. \quad (9)$$

We have the following possibilities:

(1) $\sigma=0$, $Q=0$, $\tilde{Q}=0$,
(2) $Q=0$, $\sigma=m_r$,
(3) $\tilde{Q}=0$, $\sigma=-m_r$.

We also impose the D-term equation, which becomes

$$-\frac{k}{2\pi}\sigma-\frac{\xi}{2\pi}+Q^\dagger Q-\tilde{Q}^\dagger\tilde{Q}=0. \quad (10)$$

III. REVIEW OF INTRILIGATOR–SEIBERG RESULT

We are reviewing the Seiberg duality flow discussion for one massive flavor described in Ref. [29].

Consider first the case in which only one flavor gets massive and we have zero σ, Q^{N_f}, $\tilde{Q}^{N_f}$. In brane configurations we have a stack of N D3-branes at $\sigma=0$. On the other hand, integrating out the massive flavor causes the Chern-Simons level to increase by 1. The flavor D5-branes split into (1,1)-branes [33], and the gauge group has a bigger Chern-Simons level, becoming $U(N)_1$. If the initial level was k, a $(1,k)$-brane splits into $(1,k+1)$-branes, and the gauge group becomes $U(N)_{k+1}$.

When $\sigma=\pm m_r$, some of the D3-branes move to the points $\pm m_r$. For one massive flavor, the gauge group is broken to $U(N_c-1)_{k+1}\times U(1)_{k+\frac{1}{2}}$. The $U(N_c-1)$ group has N_f-1 massless flavors and one massive flavor Q^{N_f}, $\tilde{Q}^{N_f}$. $U(1)$ only has either Q^{N_f} or $\tilde{Q}^{N_f}$ as massive flavors. After integrating out the massive flavor, the level of the $U(N_c-1)$ gauge group increases by 1, whereas the level of the $U(1)$ gauge group changes by $+\frac{1}{2}$. In brane configuration language, the $U(N_c-1)$ group lives on the N_c-1 D3-branes, which are not moved, whereas $U(1)$ lives on the single D3-brane moved and suspended between an

[1]We will consider more general deformations in a later section.

NS-brane and the intersection point of $(1, k)$ 5-brane, D5-brane, and $(1, k+1)$ 5-branes.

For $\sigma = \pm m_r$ we also need to consider the D-term equation (10). When $Q^\dagger Q$ and $\tilde{Q}^\dagger \tilde{Q}$ are zero, the value for the Fayet–Iliopoulos parameter is $\xi = m$. In brane configurations, the Fayet–Iliopoulos parameter is the distance between the two NS-branes. It is generally associated to a supersymmetry breaking. Nevertheless, in our present case, the Fayet–Iliopoulos term balances the nonzero expectation value for the field σ and does not break supersymmetry.

The $U(N_c - 1)_{k+1} \times U(1)_{k+\frac{1}{2}}$-brane configuration contains:

(i) one $(1, k+1)$ 5-brane between two junctions of D5 intersecting $(1, k)$-branes,

(ii) three rotated NS-branes: NS_0 and $NS_\pm$,

(iii) $N_c - 1$ D3-branes between NS and NS_0,

(iv) one D3-brane between NS and either NS_+ or NS_-.

A. Giveon–Kutasov to Aharony duality for one massive flavor

Reference [29] considered the flow between Giveon–Kutasov [26] and Aharony dualities [28].

The Giveon–Kutasov duality maps $\mathcal{N} = 2$, $U(N_c)$ Chern-Simons theory with N_f flavors and level k to a dual theory $U(N_f + |k| - N_c)_{-k}$ with a superpotential $Mq\tilde{q}$, where q are the dual quarks and M is a singlet. On the other hand, the Aharony duality maps $\mathcal{N} = 2$, $U(N_c)$ theory with N_f flavors and level 0 to $U(N_f - N_c)$ with an unusual superpotential coupling the electric $X_\pm$ and magnetic monopoles $\tilde{X}_\pm$:

$$Mq\tilde{q} + X_+\tilde{X}_- + X_-\tilde{X}_+. \tag{11}$$

One can flow from Aharony duality to Giveon–Kutasov duality by turning on real masses for the electric flavors. It is less clear how to flow from Giveon–Kutasov duality to Aharony duality, and the solution was offered in Ref. [29]:

(i) Start with Giveon–Kutasov electric theory $U(N_c)_k$ with N_f quark flavors, and add a real mass for one of the flavors Q^{N_f} and $\tilde{Q}_{N_f}$.

(ii) For $\sigma = 0$, $Q = 0$, $\tilde{Q} = 0$, the Giveon–Kutasov electric theory modifies into an Aharony electric theory $U(N_c)_{k+1}$ with $N_f - 1$ flavors, when $k = -1$.

(iii) For $\sigma = \pm m_r$, the Giveon–Kutasov magnetic theory becomes

$$U(N_f - N_c - 1)_{k+1} \times U(1)_{k+\frac{1}{2}} \times U(1)_{k+\frac{1}{2}}. \tag{12}$$

This is to be identified with the Aharony magnetic theory when $k = -1$. The identification holds for $k = -1$, and the proposed Aharony duality takes $U(N_c)_0$ with $N_f - 1$ flavors into $U(N_f - N_c - 1)_0 \times U(1)_{-\frac{1}{2}} \times U(1)_{-\frac{1}{2}}$ with $N_f - 1$ flavors. The product $U(1)_{-\frac{1}{2}} \times U(1)_{-\frac{1}{2}}$ is dual to a pair of single chiral superfields $X_\pm$ [30,31]. The single superfields couple to the magnetic monopoles $\tilde{X}_\pm$ associated to $U(1)$ inside $U(N_c - 1)$ as $X_-\tilde{X}_+ + X_+\tilde{X}_-$, as required for Aharony duality.

IV. FLOW BETWEEN DUALITIES FOR TWO MASSIVE FLAVORS

We now generalize the results of Ref. [29] to the case of more massive flavors. We start with two massive flavors Q^{N_f}, Q^{N_f-1} and $\tilde{Q}_{N_f}$, $\tilde{Q}_{N_f-1}$ with masses

$$\begin{aligned} m_{1}{}^{f'}_{f} &= m_1 \delta^{f' N_f}_{f N_f}; & m_{2}{}^{f'}_{f} &= m_2 \delta^{f' N_f - 1}_{f N_f - 1}, \\ \tilde{m}_{1}{}^{f'}_{f} &= m_1 \delta^{\tilde{f}' N_f}_{\tilde{f} N_f}; & \tilde{m}_{2}{}^{f'}_{f} &= m_2 \delta^{\tilde{f}' N_f - 1}_{\tilde{f} N_f - 1}. \end{aligned} \tag{13}$$

We start with the case of equal masses $m_1 = m_2 = m$.

A. Equal masses $m_1 = m_2$

If the masses are equal $m_1 = m_2 = m$, we can have one of the following (all entries of Q, $\tilde{Q}$ are always zero):

Case 1: All components of σ are 0.

After integrating out the massive flavors, the gauge group becomes $U(N_c)_{k+2}$. This group lives on N_c D3-branes suspended between an NS-brane and a $(1, k+2)$ 5-brane.

Case 2: $\sigma^{N_c}_{N_c} = \sigma^{N_c-1}_{N_c-1}$ are $-m$. The other components are 0.

The gauge group becomes the $U(N_c - 2)_{k+2} \times U(2)_{k+1}$ group. The unbroken $U(2)_{k+1}$ is a decoupled factor with two $\tilde{Q}$ fields with mass $2m$ and two Q fields with mass 0. By integrating out the massive flavors $\tilde{Q}$, we get a modification in the level and Fayet–Iliopoulos term:

$$k \to k + 1, \qquad \xi \to \xi + 2m, \tag{14}$$

which, after a shift, relates the Fayet-Iliopoulos (FI) term and the mass m as $\xi = m(k+2)$.

We analyze the low-energy $U(2)_{k+1}$ theory. The Witten index for the theory with two massless Q fields and level $k+1$ is [29]

$$\mathrm{Tr}(-1)^F = |k+1| + 1, \tag{15}$$

which is nonzero for any value of k. The theory has SUSY vacua, but they can be topological. For $|k+1| = 1$, if one adds massive matter and flows down, the theory at the origin is an IR-free theory consisting of one chiral superfield with no superpotential. This superfield is to be identified with the electric monopole X_-.

One special case is $k = -2$, in which the broken gauge group becomes $U(N_c - 2)_0 \times U(2)_{-1}$. As in Ref. [31], $U(2)_{-1}$ with two Q fields has a magnetic dual with only one singlet X_-, which couples to the operator $\tilde{X}_+$ associated to the $U(1)$ inside $U(N_c - 2)$. X_- is exactly the chiral superfield describing the vacuum of $U(2)_{-1}$.

The Chern-Simons level of the field theory on the D3-branes is

(i) k for D3-branes between the NS-brane and the $(1, k)$-brane,

(ii) $k+2$ for D3-branes between the NS-brane and the $(1, k+2)$-brane,

(iii) $k+1$ for D3-branes between the NS-brane and the junction point of $(1, k)$, $(1, k+2)$, and 2 D5-branes

Case 3: $\sigma_{N_c}^{N_c} = \sigma_{N_c-1}^{N_c-1} = +m$ and $\sigma_{N_c-2}^{N_c-2} = \sigma_{N_c-3}^{N_c-3} = -m$. The gauge group becomes $U(N_c-4)_{k+2} \times U(2)_{k+1} \times U(2)_{k+1}$. For the case $k=-2$, the duals for both $U(2)_{-1}$ provide two chiral superfields X_-, X_+.

Case 4: $\sigma_{N_c}^{N_c} = \pm m$ and $\sigma_{N_c-1}^{N_c-1} = 0$. The two massive flavors are integrated out, and one obtains the product between $U(N_c-1)_{k+2}$ with N_f-2 flavors and $U(1)_{k+1}$ with two massless Q fields. The Witten index for $U(1)_{k+1}$ is $|k+1|+1$, which is always positive so the supersymmetry is preserved. Nevertheless, for $k=-2$, the Abelian group becomes $U(1)_{-1}$, and no candidate for magnetic monopoles exists.

Case 5: $\sigma_{N_c}^{N_c} = \pm m$, $\sigma_{N_c-1}^{N_c-1} = \mp m$ and $\sigma_{N_c-2}^{N_c-2} = 0$. After integrating out the two massive flavors, the remaining gauge group is $U(N_c-2)_{k+2} \times U(1)_{k+1} \times U(1)_{k+1}$.

B. Different masses for the two flavors

We now consider the case when the mass for the Q^{N_f} flavor is bigger than the one for Q^{N_f-1}.

Case 1 remains the same as for equal masses.

For case 2 we choose two nonzero entries for σ as $\sigma_{N_c}^{N_c} = -m_1$, $\sigma_{N_c-1}^{N_c-1} = -m_2$. The unbroken gauge group is $U(N_c-2)_{k+2} \times U(1)_{k_1} \times U(1)_{k_2}$, where $k_{1,2}$ need to be determined.

The $U(1)_{k_1}$ on the D3-brane displaced to $\sigma_{N_c}^{N_c} = -m_1$ has two massive $\tilde{Q}$ fields with positive masses $2m_1$ and m_1+m_2, respectively. The effective real masses for the fields Q are

$$m_i(\sigma) = m_i + n_i\sigma, \quad \text{where } n_i \text{ is the charge under } U(1), \tag{16}$$

so the effective mass for Q^{N_f} is 0 and the one for Q^{N_f-1} is $m_2 - m_1 < 0$. The $U(1)_{k_2}$ on the D3-brane displaced to $\sigma_{N_c-1}^{N_c-1} = -m_2$ contains two massive $\tilde{Q}$ fields with positive masses $2m_2$ and m_1+m_2, respectively, and one field Q with positive mass. The effective mass for the fields Q is $m_1 - m_2 > 0$ for Q^{N_f} and zero for Q^{N_f-1}. The effective Chern-Simons level

$$k_{\text{eff}} = k + \frac{1}{2}\sum_i n_i^2 \text{sign}(m_i(\sigma)) \tag{17}$$

implies that $k_1 = k + \frac{1}{2}$, $k_2 = k + \frac{3}{2}$, and the unbroken gauge group becomes

$$U(N_c-2)_{k+2} \times U(1)_{k+\frac{1}{2}} \times U(1)_{k+\frac{3}{2}}. \tag{18}$$

To connect to Aharony duality, we consider $k=-2$, and the unbroken gauge group becomes $U(N_c-2)_0 \times U(1)_{-\frac{3}{2}} \times U(1)_{-\frac{1}{2}}$. $U(1)_{-\frac{3}{2}}$ and $U(1)_{-\frac{1}{2}}$ have two $\tilde{Q}$ fields and one Q field. $U(1)_{-\frac{1}{2}}$ has a dual with a free singlet field X_+. What about the group $U(1)_{-\frac{3}{2}}$? The effective FI term is $k_{\text{eff}}\sigma + \xi_{\text{eff}}$, which is negative and so would signal no SUSY vacuum. Nevertheless, as $U(1)_{-\frac{3}{2}}$ does not contribute to the Aharony duality, it can decouple, and the theory would remain SUSY. It would be interesting to discuss this decoupling in detail.

In case 3 all entries of Q are 0, and two nonzero entries of σ are $\sigma_{N_c}^{N_c} = +m_1$, $\sigma_{N_c-1}^{N_c-1} = +m_2$. The unbroken gauge group is $U(N_c-2)_{k+2} \times U(1)_{k_3} \times U(1)_{k_4}$.

The $U(1)_{k_3}$ on D3-branes displaced to $\sigma_{N_c}^{N_c} = m_1$ contains two massive Q fields with positive masses $2m_1$ and m_1+m_2, respectively, together with a $\tilde{Q}$ field with negative mass m_2-m_1. The value for k_3 is $k+\frac{1}{2}$. The $U(1)_{k_4}$ on D3-branes located at $\sigma_{N_c}^{N_c} = m_2$ has two massive Q fields with positive mass $2m_2$ and m_1+m_2, respectively, plus a $\tilde{Q}$ field with positive mass m_1-m_2. The value for k_4 is then $k+\frac{3}{2}$ so the unbroken gauge group is $U(N_c-2)_{k+2} \times U(1)_{k+\frac{1}{2}} \times U(1)_{k+\frac{3}{2}}$. For $k=-2$, the group becomes $U(N_c-2)_0 \times U(1)_{-\frac{3}{2}} \times U(1)_{-\frac{1}{2}}$.

For case 4, the nonzero entries of σ are $\sigma_{N_c}^{N_c} = m_1$, $\sigma_{N_c-1}^{N_c-1} = m_2$, $\sigma_{N_c-2}^{N_c-2} = -m_2$, $\sigma_{N_c-3}^{N_c-3} = -m_1$. The unbroken gauge group is $U(N_c-2)_{k+2} \times U(1)_{k_5} \times U(1)_{k_6} \times U(1)_{k_7} \times U(1)_{k_8}$. The field content for all $U(1)$ groups implies that

$$k_5 = k_8 = k + \frac{1}{2}; \qquad k_6 = k_7 = k + \frac{3}{2}. \tag{19}$$

For $k=-2$, the unbroken group becomes

$$U(N_c-4)_0 \times U(1)_{-\frac{3}{2}} \times U(1)_{-\frac{1}{2}} \times U(1)_{-\frac{1}{2}} \times U(1)_{-\frac{3}{2}}. \tag{20}$$

The two $U(1)_{-\frac{1}{2}}$ groups have duals represented by the scalars $X_\pm$.

In case 5 other interesting cases are $\sigma_{N_c}^{N_c} = 0$, $\sigma_{N_c-1}^{N_c-1} = -m_1$ or $\sigma_{N_c}^{N_c} = 0$, $\sigma_{N_c-1}^{N_c-1} = -m_2$, in which case the unbroken gauge group is $U(N_c-1)_{k+2} \times U(1)_{k+\frac{1}{2}}$ or $U(N_c-1)_{k+2} \times U(1)_{k+\frac{3}{2}}$. For $k=-2$, this becomes $U(N_c-1)_0 \times U(1)_{-\frac{3}{2}}$ or $U(N_c-1)_0 \times U(1)_{-\frac{1}{2}}$.

For all cases 3, 4, and 5, we need to decouple the $U(1)_{-\frac{3}{2}}$ in order to save the SUSY.

C. Giveon–Kutasov to Aharony duality for two massive flavors

We start with Giveon–Kutasov electric theory with gauge group $U(N_c)_{-2}$ with N_f quark flavors. The N_f and

$N_f - 1$ flavors have a real mass. After integrating out the massive flavors and considering a vacuum with $\sigma = 0$, the electric theory becomes an Aharony electric theory with $U(N_c)_0$ with $N_f - 2$ massless flavors.

One the other hand, the Giveon–Kutasov magnetic dual is $U(N_f + 2 - N_c)_{-2}$. How can we reach the Aharony magnetic dual? The answer depends on the choice for the masses:

(i) If the two masses are equal, the corresponding case 3 provides an Aharony magnetic dual with a gauge group $U(N_f - 2 - N_c)_0 \times U(2)_{-1} \times U(2)_{-1}$ with $N_f - 2$ flavors. Each $U(2)_{-1}$ is dual to a singlet $X_\pm$, which contributes to the mysterious superpotential. Therefore, the corresponding case 1 for electric theory with equal masses is dual to case 3 for magnetic theory with equal masses. This is the Aharony duality for equal flavor masses.

(ii) If the two masses are equal and we consider case 4, the electric theory would be $U(N_c - 1)_0 \times U(1)_{-1}$ and the magnetic one $U(N_f + 1 - N_c)_0 \times U(1)_{-1}$. This now looks like an Aharony duality.

(iii) If the masses are not equal, we consider the case 3 for nonequal masses and obtain a theory with a gauge group (20). Both $U(1)_{-\frac{1}{2}}$ groups can be dualized into singlets, and we obtain $U(N_f - 2 - N_c)$ with singlets and a superpotential, together with an extra product of two $U(1)_{-\frac{3}{2}}$. The Aharony duality is valid only if the two masses are very large and almost equal. The product of two $U(1)_{-\frac{3}{2}}$ can decouple to preserve SUSY.

(iv) If we are in case 5 of nonequal masses, the electric theory is $U(N_c - 1)_0 \times U(1)_{-\frac{3}{2}}$ or $U(N_c - 1)_0 \times U(1)_{-\frac{1}{2}}$, and the magnetic theory is $U(N_f + 1 - N_c)_0 \times U(1)_{-\frac{3}{2}}$ or $U(N_f + 1 - N_c)_0 \times U(1)_{-\frac{1}{2}}$. We have Giveon–Kutasov duality between $U(N_c - 1)_0 \times U(1)_{-\frac{1}{2}}$ and $U(N_f + 1 - N_c)_0 \times U(1)_{-\frac{1}{2}}$ if both $U(1)_{-\frac{1}{2}}$ are dualized into singlets. For $U(N_c - 1)_0 \times U(1)_{-\frac{3}{2}}$ and $U(N_f + 1 - N_c)_0 \times U(1)_{-\frac{3}{2}}$, we have a Giveon–Kutasov duality if the $U(1)_{-\frac{3}{2}}$ groups decouple to save SUSY.

The conclusion is that by starting from a Giveon–Kutasov duality, we can reach a multitude of dualities, depending on the choice of the vevs for σ.

V. FLOW BETWEEN DUALITIES FOR FOUR MASSIVE FLAVORS

We now move to a more involved case with four massive flavors $Q^{N_f}, Q^{N_f-1}, Q^{N_f-2}, Q^{N_f-3}$ and $\tilde{Q}_{N_f}, \tilde{Q}_{N_f-1}, \tilde{Q}^{N_f-2}, \tilde{Q}^{N_f-3}$ as

$$\begin{aligned} m_{1\,f}^{f'} &= m_1 \delta_{fN_f}^{f'N_f}, & m_{2\,f}^{f'} &= m_2 \delta_{fN_f-1}^{f'N_f-1}, \\ m_{3\,f}^{f'} &= m_3 \delta_{fN_f-2}^{f'N_f-2}, & m_{4\,f}^{f'} &= m_4 \delta_{fN_f-3}^{f'N_f-3} \end{aligned} \tag{21}$$

$$\begin{aligned} \tilde{m}_{1\,f}^{f'} &= m_1 \delta_{\tilde{f}N_f}^{\tilde{f}'N_f}, & \tilde{m}_{2\,f}^{f'} &= m_2 \delta_{\tilde{f}N_f-1}^{\tilde{f}'N_f-1}, \\ \tilde{m}_{3\,f}^{f'} &= m_3 \delta_{\tilde{f}N_f-2}^{\tilde{f}'N_f-2}, & \tilde{m}_{4\,f}^{f'} &= m_4 \delta_{fN_f-3}^{f'N_f-3}. \end{aligned} \tag{22}$$

We have several possibilities, among which we consider the following:

(i) Case a: all the masses are equal $m_1 = m_2 = m_3 = m_4 = m$.

(ii) Case b: three masses are equal.

(iii) Case c: two masses are equal.

(iv) Case d: all the masses are different and choose $m_1 > m_2 > m_3 > m_4$.

A. Case a: All four masses are equal

(1) All components of σ are 0, and all entries of Q, $\tilde{Q}$ are 0.
After integrating out the massive flavors, the gauge group changes from $U(N_c)_k$ with N_f flavors to $U(N_c)_{k+4}$ with $N_f - 4$ massless flavors. One special case is $k = -4$, where the level becomes 0.

(2) All entries of Q are 0, and the σ nonzero entries are

$$\sigma_{N_c}^{N_c} = \sigma_{N_c-1}^{N_c-1} = \sigma_{N_c-2}^{N_c-2} = \sigma_{N_c-3}^{N_c-3} = -m. \tag{23}$$

The gauge group is $U(N_c - 4)_{k+4} \times U(4)_{k+2}$. The unbroken $U(4)_{k+2}$ has four $\tilde{Q}$ fields with mass $2m$ and four Q fields with mass 0, which implies an effective Chern-Simons level $k + 2$ and an effective shifted FI term $\xi = m(k + 4)$.
One special case is $k = -4$, in which the gauge group becomes $U(N_c - 4)_0 \times U(4)_{-2}$. $U(4)_{-2}$ theory with four Q fields has a dual with a singlet X_-. It would be very interesting to check that the Witten index is indeed nonzero for this case, and we expect that the IR description is obtained by adding massive matter and flowing down as in Ref. [29] for $U(2)$ groups.
A slight generalization is written as

$$\begin{aligned} \sigma_{N_c}^{N_c} &= \sigma_{N_c-1}^{N_c-1} = \sigma_{N_c-2}^{N_c-2} = \sigma_{N_c-3}^{N_c-3} = m, \\ \sigma_{N_c-4}^{N_c-4} &= \sigma_{N_c-5}^{N_c-5} = \sigma_{N_c-6}^{N_c-6} = \sigma_{N_c-7}^{N_c-7} = -m, \end{aligned} \tag{24}$$

when the gauge group becomes $U(N_c - 8)_{k+4} \times U(4)_{k+2} \times U(4)_{k+2}$. For $k = -4$ this is $U(N_c - 8)_0 \times U(4)_{-2} \times U(4)_{-2}$, and both $U(4)_{-2}$ groups have duals with singlets $X_\pm$.

(3) We consider some special values for $\sigma_{N_c}^{N_c}$, $\sigma_{N_c-1}^{N_c-1}$, $\sigma_{N_c-2}^{N_c-2}$, $\sigma_{N_c-3}^{N_c-3}$ for which the unbroken gauge group is the following:

(a) for $\sigma_{N_c}^{N_c} = -m$, $\sigma_{N_c-1}^{N_c-1} = \sigma_{N_c-2}^{N_c-2} = \sigma_{N_c-3}^{N_c-3} = 0$: $U(N_c - 1)_{k+4} \times U(1)_{k+2}$

(b) for $\sigma_{N_c}^{N_c} = \sigma_{N_c-1}^{N_c-1} = -m$, $\sigma_{N_c-2}^{N_c-2} = \sigma_{N_c-3}^{N_c-3} = 0$: $U(N_c - 2)_{k+4} \times U(2)_{k+2}$

(c) for $\sigma_{N_c}^{N_c} = \sigma_{N_c-1}^{N_c-1} = \sigma_{N_c-2}^{N_c-2} = -m$, $\sigma_{N_c-3}^{N_c-3} = 0$: $U(N_c - 3)_{k+4} \times U(3)_{k+2}$

The cases with eigenvalues equal to m or mixed m and $-m$ are similar.

B. Case b: Three equal masses

(1) $m_1 = m_2 = m_3 > m_4$
Choose $\sigma_{N_c}^{N_c} = \sigma_{N_c-1}^{N_c-1} = \sigma_{N_c-2}^{N_c-2} = -m_1$, $\sigma_{N_c-3}^{N_c-3} = -m_4$. In this case, for the first set of vevs of σ, one gets a $U(3)$ theory with four $\tilde{Q}$ fields with positive mass and one Q field with negative mass, which are integrated out. The effective level is $k + \frac{3}{2}$. For $\sigma_{N_c-3}^{N_c-3} = -m_4$, we get $U(1)$ with four $\tilde{Q}$ with positive masses and three Q with positive masses, so the effective level is $k + \frac{7}{2}$. The gauge group becomes

$$U(N_c - 4)_{k+4} \times U(3)_{k+\frac{3}{2}} \times U(1)_{k+\frac{7}{2}}. \tag{25}$$

For $k = -4$ this becomes

$$U(N_c - 4)_0 \times U(3)_{-\frac{5}{2}} \times U(1)_{-\frac{1}{2}}. \tag{26}$$

To preserve SUSY, we expect that $U(3)_{-\frac{5}{2}}$ decouples.

There are also other special cases:

(a) $\sigma_{N_c}^{N_c} = \sigma_{N_c-1}^{N_c-1} = \sigma_{N_c-2}^{N_c-2} = -m_1$, $\quad \sigma_{N_c-3}^{N_c-3} = 0$ with a gauge group

$$U(N_c - 3)_{k+4} \times U(3)_{k+\frac{3}{2}} \tag{27}$$

(b) $\sigma_{N_c}^{N_c} = \sigma_{N_c-1}^{N_c-1} = \sigma_{N_c-2}^{N_c-2} = 0$, $\quad \sigma_{N_c-3}^{N_c-3} = -m_4$ with a gauge group

$$U(N_c - 1)_{k+4} \times U(1)_{k+\frac{7}{2}} \tag{28}$$

(c) $\sigma_{N_c}^{N_c} = \sigma_{N_c-1}^{N_c-1} = -m_1$, $\quad \sigma_{N_c-2}^{N_c-2} = \sigma_{N_c-3}^{N_c-3} = 0$ with a gauge group

$$U(N_c - 2)_{k+4} \times U(2)_{k+\frac{3}{2}} \tag{29}$$

(d) $\sigma_{N_c}^{N_c} = -m_1$, $\quad \sigma_{N_c-1}^{N_c-1} = \sigma_{N_c-2}^{N_c-2} = \sigma_{N_c-3}^{N_c-3} = 0$ with a gauge group

$$U(N_c - 1)_{k+4} \times U(1)_{k+\frac{3}{2}} \tag{30}$$

(e) $\sigma_{N_c}^{N_c} = -m_1$, $\sigma_{N_c-1}^{N_c-1} = \sigma_{N_c-2}^{N_c-2} = \sigma_{N_c-3}^{N_c-3} = -m_4$ with a gauge group

$$U(N_c - 2)_{k+4} \times U(1)_{k+\frac{3}{2}} \times U(1)_{k+\frac{3}{2}} \tag{31}$$

(2) $m_1 = m_2 = m_3 < m_4$
Choose $\sigma_{N_c}^{N_c} = \sigma_{N_c-1}^{N_c-1} = \sigma_{N_c-2}^{N_c-2} = -m_1$, $\sigma_{N_c-3}^{N_c-3} = -m_4$. For the first set of vevs of σ, one gets a $U(3)$ theory in which four $\tilde{Q}$ with positive masses and one Q with positive mass have been integrated out. The effective level is then $k + \frac{5}{2}$.

If $\sigma_{N_c-2}^{N_c-2} = -m_4$ we get $U(1)$ with four $\tilde{Q}$ with positive mass and three Q with negative mass, and the effective level is $k + \frac{1}{2}$. The gauge group is

$$U(N_c - 4)_{k+4} \times U(3)_{k+\frac{5}{2}} \times U(1)_{k+\frac{1}{2}} \tag{32}$$

or in the $k = -4$ case

$$U(N_c - 4)_0 \times U(3)_{-\frac{3}{2}} \times U(1)_{-\frac{7}{2}}. \tag{33}$$

Among the cases with some zero values for the above σ, we mention the one with $\sigma_{N_c}^{N_c} = \sigma_{N_c-1}^{N_c-1} = \sigma_{N_c-2}^{N_c-2} = -m_1$, $\sigma_{N_c-2}^{N_c-2} = 0$, which provides the unbroken gauge group

$$U(N_c - 3)_0 \times U(3)_{-\frac{3}{2}}. \tag{34}$$

This is part of a candidate for Aharony duality because $U(3)_{-\frac{3}{2}}$ has a singlet dual.

C. Case c: Two masses are equal

We only consider the choice $m_1 = m_2 > m_3 > m_4$. Take the vacuum

$$\sigma_{N_c}^{N_c} = \sigma_{N_c-1}^{N_c-1} = -m_1, \quad \sigma_{N_c-2}^{N_c-2} = -m_3, \quad \sigma_{N_c-3}^{N_c-3} = -m_4, \tag{35}$$

corresponding to the gauge group

$$U(N_c - 4)_{k+4} \times U(2)_{k+1} \times U(1)_{k+\frac{5}{2}} \times U(1)_{k+\frac{7}{2}}. \tag{36}$$

For $k = -4$ this becomes

$$U(N_c - 4)_0 \times U(2)_{-3} \times U(1)_{-\frac{3}{2}} \times U(1)_{-\frac{1}{2}}. \tag{37}$$

As in Ref. [29], $U(2)_{-3}$ with two massless flavors is an superconformal field theory and should decouple in order to obtain an Aharony duality with singlets coming from $U(1)_{-\frac{1}{2}}$.

One other important choice for two equal masses is $\sigma_{N_c}^{N_c} = \sigma_{N_c-1}^{N_c-1} = \sigma_{N_c-2}^{N_c-2} = 0$, $\sigma_{N_c-3}^{N_c-3} = -m_4$, which provides an unbroken gauge group,

$$U(N_c - 1)_0 \times U(1)_{-\frac{1}{2}}, \tag{38}$$

containing a potential singlet coming from the dual of $U(1)_{-\frac{1}{2}}$.

D. Case d: All masses are different

We consider $m_1 > m_2 > m_3 > m_4$ and the following vacuum:

$$\sigma_{N_c-i}^{N_c-i} = -m_{i+1}, \quad i = 0, 1, 2, 3. \tag{39}$$

Each branch $\sigma_{N_c-i}^{N_c-i} = -m_{i+1}$ has 4 $\tilde{Q}$ with positive masses, but the masses for Q fields are positive or negative, so the unbroken gauge group is

$$U(N_c-4)_{k+4}\times U(1)_{k+\frac{1}{2}}\times U(1)_{k+\frac{5}{2}}\times U(1)_{k+\frac{7}{2}}, \quad (40)$$

becoming for $k=-4$

$$U(N_c-4)_0\times U(1)_{-\frac{7}{2}}\times U(1)_{-\frac{3}{2}}\times U(1)_{-\frac{1}{2}}. \quad (41)$$

One other important solution is $\sigma_{N_c}^{N_c}=\sigma_{N_c-1}^{N_c-1}=\sigma_{N_c-2}^{N_c-2}=0$, $\sigma_{N_c-3}^{N_c-3}=-m_4$, which contains a singlet coming from $U(1)_{-\frac{1}{2}}$. The $U(1)_{-\frac{7}{2}}\times U(1)_{-\frac{3}{2}}$ groups should decouple in order to preserve SUSY.

E. From Giveon–Kutasov to Aharony duality for four massive flavors

We consider a Giveon–Kutasov electric theory with gauge group $U(N_c)_{-4}$ with N_f quark flavors with real masses for the N_f, N_f-1, N_f-2, N_f-3 flavors. The Giveon–Kutasov magnetic dual is $U(N_f+4-N_c)_{-4}$. We integrate out the four massive quarks, choose the $\sigma=0$ vacuum, and obtain the Aharony electric theory $U(N_c)_0$ with N_f-4 massless flavors.

The modification of the Giveon–Kutasov magnetic picture depends on how we choose the masses for the four quarks, as discussed in the previous subsections:

(i) If all four masses are equal, we consider case 2a and obtain the Aharony dual with a gauge group $U(N_f-4-N_c)_0\times U(4)_{-2}\times U(4)_{-2}$ w.

(ii) If only three masses are equal, we consider case b and obtain the Aharony dual with a gauge group which is either

$$U(N_f-4-N_c)_0\times U(3)_{-\frac{5}{2}}\times U(3)_{-\frac{5}{2}}\times U(1)_{-\frac{1}{2}}\times U(1)_{-\frac{1}{2}} \quad (42)$$

or

$$U(N_f-4-N_c)_0\times U(3)_{-\frac{3}{2}}\times U(3)_{-\frac{3}{2}}\times U(1)_{-\frac{7}{2}}\times U(1)_{-\frac{7}{2}}. \quad (43)$$

For the case (42), we get the two singlets $X_\pm$ from dualizing $U(1)_{-\frac{1}{2}}\times U(1)_{-\frac{1}{2}}$ and then decouple $U(3)_{-\frac{5}{2}}\times U(3)_{-\frac{5}{2}}$ by sending the masses to infinity. For the case (43), we get the singlets $X_\pm$ from dualizing $U(3)_{-\frac{3}{2}}\times U(3)_{-\frac{3}{2}}$ and then decoupling $U(1)_{-\frac{7}{2}}\times U(1)_{-\frac{7}{2}}$.

Not all the gauge groups breaking give rise to Aharony duals. One example when we get the Aharony dual is $\sigma_{N_c}^{N_c}=\sigma_{N_c-1}^{N_c-1}=\sigma_{N_c-2}^{N_c-2}=0$, $\sigma_{N_c-2}^{N_c-2}=-m_4$ when the breaking is as in Eq. (28) $U(N_c-1)_{k+4}\times U(1)_{k+\frac{7}{2}}$. This becomes $U(N_c-1)_0\times U(1)_{-\frac{1}{2}}$ when $k=-4$. We recover the Aharony magnetic dual $U(N_f+1-N_c)_0\times U(1)_{-\frac{1}{2}}$.

(iii) If two masses are equal and we choose $m_1=m_2>m_3>m_4$, case c discussed above implies that the Aharony dual is

$$U(N_f-4-N_c)_0\times U(2)^2_{-3}\times U(1)^2_{-\frac{3}{2}}\times U(1)^2_{-\frac{1}{2}}, \quad (44)$$

with the two singlets $X_\pm$ being given by the duals of $U(1)^2_{-\frac{1}{2}}$.

For $\sigma_{N_c}^{N_c}=\sigma_{N_c-1}^{N_c-1}=\sigma_{N_c-2}^{N_c-2}=0$, $\sigma_{N_c-3}^{N_c-3}=-m_4$, we have an unbroken electric gauge group $U(N_c-1)_0\times U(1)_{-\frac{1}{2}}$ and unbroken magnetic gauge group $U(N_f+1-N_c)_0\times U(1)_{-\frac{1}{2}}$, which are indeed Aharony dual to each other.

(iv) If all masses are different and we choose $m_1>m_2>m_3>m_4$, the above case d implies that the Aharony magnetic dual is

$$U(N_f-4-N_c)_0\times U(1)^2_{-\frac{7}{2}}\times U(1)^2_{-\frac{3}{2}}\times U(1)^2_{-\frac{1}{2}}, \quad (45)$$

with the two singlets $X_\pm$ being again the duals of $U(1)^2_{-\frac{1}{2}}$. The group $U(1)^2_{-\frac{7}{2}}\times U(1)^2_{-\frac{3}{2}}$ needs to decouple to preserve SUSY.

VI. THREE-DIMENSIONAL THEORIES WITH ADJOINT MATTER

We consider the $\mathcal{N}=4\to\mathcal{N}=2$ deformation for a general potential for the field Φ [1,34],

$$\int d^2\theta\sum_{i=0}^{n}\frac{c_i}{n+1-i}\Phi^{n+1-i}, \quad (46)$$

which implies that the superpotential has n distinct minima $x=a_i$:

$$W'(x)=\sum_{j=0}^{n}c_ix^{n-i}=c_0\prod_{i=1}^{n}(x-a_i). \quad (47)$$

The vacua are labeled by integers $(r_1,\ldots,r_n)$. When all the values of a_i are distinct, the gauge group is Higgsed,

$$U(N_c)\to U(r_1)\times U(r_2)\times\cdots\times U(r_n), \quad (48)$$

and we get n decoupled copies of the $\mathcal{N}=2$ theories discussed in the previous section.

If we add N_f flavor fields, the full superpotential is

$$\int d^2\theta\sqrt{2}\Phi Q\tilde{Q}+\int d^2\theta\sum_{i=0}^{n}\frac{c_i}{n+1-i}\Phi^{n+1-i}. \quad (49)$$

The equation of motion for Φ is

$$\int d^2\theta\Big(\sqrt{2}Q\tilde{Q}+\sum_{j=0}^{n}c_i\Phi^{n-i}\Big), \quad (50)$$

and we take the simple solution $Q\tilde{Q}=W'(x)=0$.

The Giveon–Kutasov duality [26] was generalized to the case with an adjoint field in Ref. [27]. The magnetic dual is an $\mathcal{N}=2$, $U(nN_f + n|k| - N_c)$ theory with N_f pairs of chiral multiplets $q\tilde{q}$, an adjoint field Ψ, and n magnetic mesons $M_i (i = 1, \ldots, n)$. On the other hand, the Aharony duality [28] was generalized to the case with an adjoint field in Ref. [32]. The proposed magnetic dual is $U(nN_f - N_c)$ with N_f pairs of chiral multiplets q, $\tilde{q}$, an adjoint field Y, n magnetic mesons $M_i (i = 1, \ldots, n)$, and $2n$ singlet superfields $X_{\pm,1}, \ldots, X_{\pm,n}$. The magnetic superpotential is

$$W_m = \mathrm{Tr} Y^{n+1} + \sum_{j=1}^{n-1} M_j \tilde{q} Y^{n-1-j} q + \sum_{i=1}^{n} (X_{+,i}\tilde{X}_{-,n+1-i} + X_{-,i}\tilde{X}_{+,n+1-i}). \quad (51)$$

$X_{1,\pm}$ and $\tilde{X}_{1,\pm}$ are the monopoles of the electric and magnetic theory. The coupling between the monopoles is dictated by the requirement of having the right R symmetry charge together with the superconformal index matching requirement [32].

We now want to flow from a Giveon–Kutasov duality with adjoint matter to an Aharony duality with adjoint matter. In order to do so, we start by generalizing the results of Ref. [29] to the case of theories with adjoint matter.

A. Symmetry breaking for theories with adjoint matter

Let us consider the deformation $\mathcal{N}=4 \rightarrow \mathcal{N}=2$ for the three-dimensional theory given by the superpotential (46). Each of the $U(r_i)$ groups in Eq. (48) has N_f fundamental flavor, and we consider that each group has the same Chern-Simons level k.

Fixing the expectation value for Φ does not imply that σ is also fixed, and, for each value $\Phi = a_i$, we have a classical vacua encoded by σ_i, $Q^{N_f;i}$, $\tilde{Q}_{N_f;i}$, $i = 1, \ldots, n$ obeying Eq. (9) together with the D-term equation for each group $U(r_i)$.

We now give a real mass to one or more flavors. If we give mass to only one flavor Q^{N_f} and $\tilde{Q}_{N_f}$, we have three cases:

(i) σ_i, $Q^{N_f;i} = \tilde{Q}_{N_f;i}$, $i = 1, \ldots, n$ are all zero.
By integrating out the massive fundamental flavors, each $U(r_i)_k$ group becomes $U(r_i)_{k+1}$ with $N_f - 1$ flavors. It is now easy to see what happens when the coefficients c_i, $i \neq 1$ in Eq. (46) are zero, in which case the superpotential becomes Φ^{n+1}. All a_i become zero, and the gauge group enlarges to $U(N_c)_{k+1}$ with $N_f - 1$ flavors.

(ii) $\sigma^{N_c;i}_{N_c;i} = m_i$, $i = 1, \ldots, n$ as the only nonvanishing σ. Only one flavor is massive, and the gauge group becomes

$$U(r_1 - 1)_{k+1} \times \cdots \times U(r_n - 1)_{k+1} \quad (52)$$

with $N_f - 1$ flavors each, together with n decoupled $U(1)_{k+\frac{1}{2}}$ factors, each having a single light field of charge $+1$ and a FI parameter $\xi = m_i(k+1)$.
If we now take c_i, $i \neq 1$ in Eq. (46) to zero, the $U(r_i - 1)_{k+1}$ factors in Eq. (52) add up to $U(N_c - n)_{k+1}$ with $N_f - 1$ massless flavors together with a product of n $U(1)_{k+\frac{1}{2}}$ groups, each having a FI parameter $\xi = m_i(k+1)$.

(iii) $\sigma^{N_i}_{N_c;i} = m_i$, $\sigma^{N_c-1;i}_{N_c-1;i} = -m_i$, $i = 1, \ldots, n$ break the group to

$$U(r_1 - 2)_{k+1} \times \cdots \times U(r_n - 2)_{k+1} \quad (53)$$

with $N_f - 1$ massless flavors, each together with a product of n $U(1)_{k+\frac{1}{2}} \times U(1)_{k+\frac{1}{2}}$ groups, each $U(1)_{k+\frac{1}{2}}$ having a single charged field and a FI parameter $\xi = \pm m_i(k+1)$.
If c_i, $i \neq 1$ are zero, all $U(r_i - 1)_{k+1}$ factors in Eq. (53) overlap and provide $U(N_c - 2n)_{k+1}$ with $N_f - 1$ massless flavors. The n $U(1)_{k+\frac{1}{2}} \times U(1)_{k+\frac{1}{2}}$ factors are distinguished by the value of the FI parameter $\xi = \pm m_i(k+1)$.

For $k = -1$, the above three cases become in the limit c_i, $i \neq 1$ being zero, the following:

(1) $U(N_c)_0$ with $N_f - 1$ flavors
(2) $U(N_c - n)_0$ together with n $U(1)_{-\frac{1}{2}}$ groups
(3) $U(N_c - 2n)_0$ together with n $U(1)_{-\frac{1}{2}} \times U(1)_{-\frac{1}{2}}$ groups

We can apply the methods of the previous section to generalize this case in the presence of more massive fundamental flavors.

B. Coupling between monopoles

Before moving to dualities, we need to address the question of the coupling between electric and magnetic monopoles in the potential dual theory. As in Ref. [29], the electric monopoles come from dualizing the $U(1)_{-\frac{1}{2}}$ into $X_{\pm}$ chiral superfields, whereas the magnetic monopoles are associated with $U(1)$ subgroups of the gauge groups. For the second case above, we can dualize the $U(1)_{-\frac{1}{2}}$ factors into $X_{+;i}$, $i = 1, \ldots n$ chiral superfields, each one coupling to operators $\tilde{X}_{-;i}$ associated to $U(1)_i$ subgroups of $U(r_i - 1)_0$, and a superpotential is generated as

$$W = \sum_{i=1}^{n} X_{+;i}\tilde{X}_{-;i}. \quad (54)$$

For the third case above, we dualize one group of $U(1)_{-\frac{1}{2}}$ factors into $X_{+;i}$, $i = 1, \ldots n$ and the other into $X_{-;i}$, $i = 1, \ldots n$, which couple to the magnetic monopoles as

$$W = \sum_{i=1}^{n} (X_{+;i}\tilde{X}_{-;i} + X_{-;i}\tilde{X}_{+;i}). \quad (55)$$

What happens if all c_i, $i \neq 1$ in Eq. (46) are taken to zero and all the masses m_i are equal? In this case all the monopoles $X_{\pm;i}$, $\tilde{X}_{\pm;i}$, $i = 1, \ldots, n$ for the cases 2 and 3 overlap, and we need to reconsider their couplings. A brane configuration discussion cannot provide an answer regarding this coupling, and we would need to lift our theory to an M-theory/F-theory picture. Nevertheless, we can use the arguments of Ref. [32] concerning the matching of the chiral rings between magnetic and electric theories, together with considerations of the superconformal index for $U(N)$ theory with an adjoint.

For a cubic superpotential in Φ ($n = 2$), this would mean that $X_{\pm;1}$ are identified with the electric monopoles and $\tilde{X}_{\pm;1}$ with the magnetic monopoles. From Table 2 in Ref. [32], we see that on one hand the state $X_{\pm;1}$ $\mathrm{Tr}\Phi$ contributes to the index, and this is represented by a contribution $X_{+;1}\tilde{X}_{-;2} + X_{-;1}\tilde{X}_{+;2}$ to the superpotential. On the other hand, $X_{\pm;2}$ and the $N_f \times N_f$ singlet M_1 would appear as singlets in the magnetic theory with no correspondent in the electric theory. They must be paired with monopole operators and disappear. The M_1 operator is canceled by $X_{+;2}\tilde{X}_{-;1} + X_{-;2}\tilde{X}_{+;1}$. The total contribution to the superpotential is then

$$X_{+;1}\tilde{X}_{-;2} + X_{-;1}\tilde{X}_{+;2} + X_{+;2}\tilde{X}_{-;1} + X_{-;2}\tilde{X}_{+;1}. \quad (56)$$

For a quartic superpotential in $\Phi(n = 3)$, one has three electric monopoles and three magnetic monopoles. To see their coupling, one can again consult Table 2 in Ref. [32] to see that $X_{\pm;1}\tilde{X}_{\mp;3}$ and $X_{\pm;2}\tilde{X}_{\mp;2}$ contribute to the superconformal index, whereas $X_{\pm;3}\tilde{X}_{\mp;1}$ is needed to cancel unwanted singlets. For general n we recover formula (51).

C. From Giveon–Kutasov duality to Aharony duality for adjoint matter

We start with a Giveon–Kutasov electric theory $U(N_c)_{-1}$ with N_f fundamental flavors and an adjoint field with a superpotential $\mathrm{Tr}\Phi^{n+1}$. By giving mass to one fundamental flavor and choosing $\sigma = 0$, we get the Aharony electric theory $U(N_c)_0$ with $N_f - 1$ fundamental flavors and an adjoint field.

To get to the Aharony magnetic dual, we start by considering a Giveon–Kutasov duality, which takes the $U(N_c)_{-1}$ with N_f fundamental flavors and an adjoint field to $U(nN_f + n - N_c)_{-1}$ with flavors and an adjoint dual as in Ref. [27]. The Giveon–Kutasov magnetic theory is then deformed by a nonzero vev for σ as in the third case above. The theory flows to one in which the rank of the non-Abelian dual group decreases by $2n$, and it becomes $U(nN_f - n - N_c)_{-1}$ with $N_f - 1$ fundamental flavors and an adjoint field. We also have the n products of $U(1)_{-\frac{1}{2}} \times U(1)_{-\frac{1}{2}}$ groups, which are to be dualized into electric monopoles that couple to magnetic monopoles as above:

$$X_{+,1}\tilde{X}_{-,n} + X_{-,1}\tilde{X}_{+,n} + \cdots + X_{+,n}\tilde{X}_{-,1} + X_{-,n}\tilde{X}_{+,1}. \quad (57)$$

We therefore found a pair of Aharony duals that are the same as in Ref. [32].

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