Risk management under a prudential policy*

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Abstract

In this paper, we study the structure of optimal contracts in a banking system when there is no risk of moral hazard. We consider a risk management problem under a policy that reduces the excessive risk taking behavior by making all banks bear part of the risk that they transfer to other parties in the market. First, we characterize the optimal solutions to the risk management problem, and, second, we find a necessary and sufficient condition under which the "risk of the tail events" will not be transferred. In particular, we will study the problem using two known risk measures, Value at Risk and Conditional Value at Risk, and will show that in these cases the optimal solutions are in the form of stop-loss policies.

Key words: Deposit insurance, risk measure and premium, Black-Scholes model, moral hazard, tail events, Var, CVaR, stop-loss policy

1 Introduction

In the light of the 2008 financial crisis and its aftermath, moral hazard has become an important issue for the policy makers. It is known that moral hazard is responsible for the excessive risk taking behavior in the financial markets, either taken by hedge-funds or banks. A moral hazard is a situation when some agents take excessive risk because the costs of taking risk will not be felt by them. In other words, a moral hazard occurs since some agents know the potential costs of taking further risk will be borne by other agents and/or the government. Perhaps the best example when the moral hazard happened on a large-scale was during the years preceding

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the financial crisis of 2008. In that period, two matters caused the moral hazard: first, the extensive use of deposit-insurances, which demotivated hedge-funds to be cautious about their investments and, second, the securitization of mortgage-backed securities, which demotivated banks to carefully screen the prospective mortgage holders. In both cases the problem was to shift the risk of the tail events to other parties, for example insurance companies or security buyers, without taking any responsibility about losses. For more reading on the impact of deposit-insurances on the banking system see Collins (1988), Kaufman (1988), Dowd (1996) and Freixas and Rochet (2008).

Moral hazard was first discussed by Kenneth Arrow (e.g., Arrow (1971)) as an inevitable risk caused by altering a policyholder's incentives when he/she shifts his/her risk to an insurer[1] (see also Heimer (1989)). However, as discussed above, in a banking system, moral hazard is mainly caused by a risk taking appetite in the absence of enough prudential policies. While it is widely believed that the minimum capital requirement can prevent the excessive risk taking behavior by putting banks' equity at risk, Hellmann et al. (2000) showed that it also can encourage further risk taking by harming banks' franchise values. They show that, by adding deposit-rate controls as a regulatory instrument, this problem can be partly avoided. However, after 2008, it is proven that neither these measures nor any other instruments of prudential control can prevent another crisis if the excessive risk taking behavior, and so the moral hazard, cannot be controlled. For a good reading on moral hazard during the 2008 financial crisis see Dowd (2009).[2]

One way to reduce the excessive risk taking behavior is to reduce the banks' moral hazard problem by making them pay the real cost of their risk taking. One can consider two different approaches to achieve this: an ex-ante approach by setting built-in incentives, and an ex-post approach by penalizing the bad behavior (see similar discussion for market discipline in Freixas and Rochet (2008)). In this paper, we set an ex-ante approach, where banks have to bear part of any risk that they introduce to the system. We consider a bank that wants to optimally shift part of the risk of its position to the other parties in the market (for instance by writing some insurance contracts). The regulation makes the bank mutualize both the shifted risk, which is transferred to the system, and the remaining risk, which is borne by the

[1] This also applies to the financial crisis of 2008, when the deposit-insurances were blamed as part of the problem.

[2] It is worth mentioning that moral hazard have been extensively studied in the literature of game theory in particular the principal agent problem, and also recently has received some attention in venture investment studies, which are unrelated to the risk management discussions of the present paper.

bank. By adopting a complete market model as in Merton (1977) [3], we will fully characterize the optimal contracts and we will see that, in particular cases of Value at Risk (VaR) and Conditional Value at Risk (CVaR), the contracts are in the form of stop-loss policies. This motivates banks to take care of the smaller amounts of the risk, which lie below a certain deductible level, and the greater amounts of the risk (associated with the tail events), which lie above a certain retention level. Next, we focus our attention on the risk of the tail events. Briefly, a tail event consists of all scenarios under which the value of the risk variable is above a certain number[4]. This is a risk that, in principal, all banks want to transfer to other parties in the market. We will establish a necessary and sufficient condition under which the risk of the tail events is not transferred. We will see that, if the banks use VaR or CVaR to measure their global risk, the risk of the tail events will not be transferred to the system. Remarkably, unlike Merton (1977) where the insurance contracts are considered with unbounded retention levels, because they are in the form of put options, our no-moral-hazard assumption puts a limit on the upper values of any contract. The results of this paper can properly be linked to the systemic risk. As a matterof fact deposit insurances are just financial tolls that transfer the risk of losses to other parties in the market, and therefore, cannot mitigate the risk of the system. This means that the tail events cannot be avoided unless the excessive risk taking behavior is demotivated within an ex-ante (or ex-post) policy, which will be studied in this paper.

It is worth mentioning that the relation between moral hazard and the risk of the tail events has previously been studied in the literature of insurance. For instance, Froot (2001) shows that the insurance companies are reluctant to buy reinsurance contracts for catastrophe risks, and Bernard and Bernard and Tian (2009) give some theoretical justification for this behavior.

The rest of the paper is organized as follows: Section 2 introduces the mathematical notions and notations that we use in this paper. In Section 3 the general set-up of our risk management problem will be presented. In Section 4 we will present the solution to the risk management problem under our no-moral-hazard prudential policies. We also provide criteria with which the regulator can make sure that the risk of the tail events is not transferred.

[3]In Merton (1977) the author prices a deposit-insurance like an option

[4]The concept of the tail events (not exactly with the same name) has been considered in Adrian and Brunnermeier (2011) for studying the systemic risk.

2 Preliminaries and Notations

Let $(\Omega, P, \mathcal{F})$ be a probability space, where Ω is the "states of the nature", P is the physical probability measure and $\mathcal{F}$ is the σ- field of measurable subsets of Ω. Let $p, q \in [1, \infty]$ be two numbers such that $1/p + 1/q = 1$. We denote the space of all random variables with p'th finite moment with L^p, i.e., $L^p = \{X : \Omega \to \mathbb{R} : E\left(|X|^p\right) < \infty\}$, where E denotes the expectation w.r.t P.

In this paper, we consider only two period of time, 0 and T, where 0 represents the beginning of the year, when a contract is written, and T represents the end of the year, when liabilities are settled. Every random variable represents losses at time T. For any $X \in L^P$, the cumulative distribution function associated with X is denoted by F_X.

2.1 Distortion Risk Measures

Let g be a non-decreasing real function from $[0, 1]$ to $[0, 1]$ such that $g(0) = 1 - g(1) = 0$. A distortion risk measure ϱ (see for example Wang et al. (1997) and Sereda et al. (2010)) is a mapping from L^p to $\mathbb{R} \cup \{\infty\}$ and is introduced as[5]

$$\varrho(X) = \int_{-\infty}^{0} \left(g(S_X(t)) - 1\right) dt + \int_{0}^{\infty} g(S_X(t))dt, \tag{1}$$

where $S_X = 1 - F_X$ is the survival function associated with X. In the literature, g is known as the distortion function. A more convenient representation for a distortion risk measure can be found in terms of Value at Risk. Let $\Pi(x) = 1 - g(1 - x)$. By a simple change of variable, one can see that the distortion form (1) can be represented as

$$\varrho(X) = \int_0^1 \mathrm{VaR}_t(X) d\Pi(t), \tag{2}$$

where

$$\mathrm{VaR}_\alpha(X) = \inf\{x \in \mathbb{R} | P(X > x) \leq 1 - \alpha\}.$$

A popular example of a distortion risk measure is Value at Risk (VaR), where $\Pi(t) = 1_{[\alpha,1]}(t)$, for a confidence level $1 - \alpha$. A Conditional Value at Risk (CVaR) is a

[5]This form is a bit more general than the usual definition of a distortion risk measure, since we assumed for the moment that X could be any member of L^p. Actually, this is a particular form of the Choquet integral, when the capacity v is given by $v(S) = g(P(S))$, for any measurable set $S \in \mathcal{F}$.

distortion risk measure whose distortion function is given by $\Pi(t) = \frac{t-\alpha}{1-\alpha}1_{[\alpha,1]}(t)$, and can be represented as

$$\mathrm{CVaR}_\alpha(X) = \frac{1}{1-\alpha}\int_\alpha^1 \mathrm{VaR}_t(X)dt. \tag{3}$$

In the sequel, we denote sometimes the risk measure ϱ in (2) with ϱ_Π to show its connection with Π.

A popular example of a risk premium is a Wang's premium introduced by the following distortion function known as Wang's transformation

$$g_\beta(x) = \Phi\left(\Phi^{-1}(x) + \beta\right), \tag{4}$$

where $\beta \in \mathbb{R}$ is a real number and Φ is the CDF of the normal distribution with the mean equal to zero and the standard deviation equal to one. Actually, the Wang's premium is used to price a risk variable, however, it is also an appropriate risk measure if risk is regarded as the price of the risky position. The parameter β represents the level of risk aversion of an agent, and also represents the agents expectations from the market risk-premium. That is why we call it the "expected market risk-premium". It is not very difficult to see that $\Pi_\beta(x) = 1 - g_\beta(1-x) = \Phi\left(\Phi^{-1}(x) - \beta\right)$.

The family of spectral risk measures which was introduced first in Acerbi (2002), has the same representation as (2), where Π is also convex. One can readily see that ϱ_Π is positive homogenous, translation invariant, monotone, law invariant and co-monotonic additive. It can be shown that all law-invariant co-monotone additive coherent risk measures can be represented as (2); see Kusuoka (2001). A risk measure in the form (2) is important from different perspectives. First of all it makes a link between the risk measures theory and the behavioral finance as the form (2) is a particular form of Choquet utility. Second, (2) contains a family of risk measures which are statistically robust. In Cont et al. (2010) it is shown that a risk measure $\varrho(x) = \int_0^1 \mathrm{VaR}_t(x)d\Pi(t)$ is robust if and only if the support of Π is away from zero and one. For example Value at Risk is a risk measure with this property.

For more reading on distortion risk measures one can see Sereda et al. (2010), Wu and Zhou (2006), Balbás et al. (2009) and Wang et al. (1997).

3 Model and Problem Set-up

Let $X_0 \geq 0$ denote a loss variable for a bank's business. The bank aims to outsource part of the risk, denoted by X with values in $[0, X_0]$, through a contract. The contract's market-value is $E(\varphi X)$, where φ is a stochastic discount factor evaluating risky positions in the market. Therefore, the bank's global position is composed of a part whose risk is not transferred, $X_0 - X$, and an amount that is payed for the contract, $E(\varphi X)$. Hence the bank's global position is $X_0 - X + E(\varphi X)$. Bank's optimal contract will be found by minimizing the global risk

$$\min \varrho\left(E(\varphi X) + X_0 - X\right), \text{s.t. } 0 \le X \le X_0.$$

By cash-invariance property of ϱ this is equivalent to

$$\min E(\varphi X) + \varrho(X_0 - X), \text{s.t. } 0 \le X \le X_0. \tag{5}$$

Let us now focus our attention on the market evaluation of risk and the stochastic discount factor. Inspired by Merton (1977), we assume the market is complete and set up a Black-Scholes model. Assume the following geometric Brownian motion

$$dS_t = \mu S_t dt + \sigma S_t dW_t,$$

where W_t is the Wiener process, and μ and σ are the drift and volatility, respectively. One can solve this SDE and find S_t however, in this paper we will be working with the discounted process $\tilde{S}_t = \exp(-rt) S_t$ which can be given by

$$\tilde{S}_t = S_0 \exp\left(\left(m - \frac{\sigma^2}{2}\right) t + \sigma W_t\right)$$

where $m = \mu - r$ and r is the risk-free interest rate. In that case, it is known that the market is complete and the unique martingale measure $\mathbb{Q}$ has the density

$$\varphi = \frac{d\mathbb{Q}}{d\mathbb{P}} = \exp\left(-\frac{m}{\sigma} W_T - \frac{1}{2}\left(\frac{m}{\sigma}\right)^2 T\right).$$

where T is the time horizon. We will be using the following relation between φ and the price process of the stock, cf Nakano (2004).

$$\varphi = \exp\left(\left(\frac{1}{2} m^2/\sigma^2 - m/2\right) T\right) \left(\frac{\tilde{S}_T}{S_0}\right)^{-m/\sigma^2} \tag{6}$$

It is then easily seen that φ is a decreasing function of the underlying asset $\tilde{S}_T$.

After introducing the model, we go back to the loss variable. We assume that the bank's loss is given by $X_0 = L\left(\tilde{S}_T\right)$, where $L : \mathbb{R} \to \mathbb{R}_+$ is a non-increasing function.

A natural example is $L(x) = S_0 \exp\left((\log(x/S_0))_-\right)$, where $(x)_- = \min\{x, 0\}$. This loss function focuses on the scenarios when the return is negative.

Given that φ is a decreasing function of the underlying asset $\tilde{S}_T$, the Hardy-Littlewood Inequality implies that

$$E(\varphi X_0) = \int_0^1 \mathrm{VaR}_\alpha(\varphi)\mathrm{VaR}_\alpha(X_0)d\alpha = \int_0^1 \mathrm{VaR}_\alpha(X_0)d\Gamma(\alpha),$$

where $\Gamma(x) = \int_0^x \mathrm{VaR}_\alpha(\varphi)d\alpha$. On the other hand,

$$\begin{aligned} P(\varphi \leq x) &= P\left(\exp\left(-\frac{m}{\sigma}W_T - \frac{1}{2}\left(\frac{m}{\sigma}\right)^2 T\right) \leq x\right) \\ &= P\left(-W_T \leq \frac{\sigma}{m}\left(\log x + \frac{1}{2}\left(\frac{m}{\sigma}\right)^2 T\right)\right) = \Phi\left(\frac{\sigma}{m\sqrt{T}}\left(\log x + \frac{1}{2}\left(\frac{m}{\sigma}\right)^2 T\right)\right), \end{aligned}$$

where Φ is the cumulative distribution function of a normal random variable with mean zero and variance one. Therefore,

$$\mathrm{VaR}_\alpha(\varphi) = \exp\left(\frac{m\sqrt{T}}{\sigma}\Phi^{-1}(\alpha) - \frac{1}{2}\left(\frac{m}{\sigma}\right)^2 T\right).$$

Given this, one can show further that

$$\Gamma(x) = \Phi\left(\Phi^{-1}(x) - \frac{m\sqrt{T}}{\sigma}\right),$$

which clearly gives a distortion function associated with a Wang's premium when $\beta = \frac{m\sqrt{T}}{\sigma}$ (see Wang (2000)).

As it was discussed earlier, we assume that the market regulation is toward avoiding moral hazard. To this end, we set up for an ex-ante policy under which all contracts in the market have to be issued in a way that the losses can be felt by both the issuer and the holder. Therefore, the contracts have to move co-monotonically with the total risk. Assuming that a contract is a function of the risk X_0, denoted by f, the contract $X = f(X_0)$ is appropriate if the bank bears the risk generated by the risk variable X_0. Therefore, the un-shifted risk, $X_0 - f(X_0)$, which is borne by the bank, has to be non-decreasing in X_0. On the other hand, since the banks are rational, they also want to outsource the losses; that is why they consider contracts are non-decreasing in X_0. As a result, both the functions $x \mapsto f(x)$ and

$x \mapsto x - f(x)$ have to be non-decreasing. To set this economic assumption on a sound mathematical basis, we introduce the following set of admissible contracts as

$$\mathcal{A} = \{f(X_0) | f \in C\},$$

where

$$C = \{f \geq 0 | f(x) \text{ and } x - f(x) \text{ are non-decreasing}\}.$$

Note that all members in C are Lipschitz continuous since for any $0 \leq x_1 \leq x_2$ we have $x_1 - f(x_1) \leq x_2 - f(x_2)$ and $f(x_1) \leq f(x_2)$ which result in $0 \leq f(x_2) - f(x_1) \leq x_2 - x_1$. In particular we have that $0 \leq f(x) \leq x$, $\forall f \in C, x \geq 0$.

Now , we rewrite the problem (5) as follows

$$\min E(\varphi f(X_0)) + \varrho(X_0 - f(X_0)), \text{s.t. } f \in C. \tag{7}$$

It is known that if a function f is Lipschitz continuous, it is almost everywhere differentiable and its derivative is essentially bounded by its Lipschitz constant. As a result, C can be represented as

$$C = \left\{ f \geq 0 \middle| f(x) = \int_0^x h(t)dt, 0 \leq h \leq 1 \right\}.$$

We introduce the set of *marginal contracts* as

$$D = \left\{ h : \mathbb{R}_+ \to \mathbb{R}_+ \middle| 0 \leq h \leq 1 \right\}.$$

Definition 1. For any indemnification function $f \in C$, the associated marginal contract is a function $h \in D$ such that

$$f(x) = \int_0^x h(t)dt, x \geq 0.$$

The interpretation of a marginal contract is as follows: if $f(x) = \int_0^x h(t)dt$ is a contract, then at each value $X_0 = x$, a marginal change δ to the value of the total risk will result in marginal change of the contract in the size $\delta h(x)$. We will see in the following that in our framework the marginal change of an optimal contract is either 0 or δ, i.e., $h = 0$ or 1.

4 Optimal Solutions and Risk of Tail Events

4.1 Main Results

In this section, we will characterize the optimal solutions to the risk management problem (7) and discuss the level of the tail risk that can be transferred to the system. As a tail event we mean $\{X_0 \geq B\}$, for some positive number $B \in \mathbb{R}$. Indeed, a tail event is associated with large losses of the position S_T. This is the riskiest part of the position S_T and as a matter of fact, this is the part that all banks, under a poor regulation, would like to get rid of. We say that the risk of the tail events cannot be transferred if there is $B > 0$, such that the bank's optimal contract $X = f(X_0)$ is bounded by B. In the following, we will discuss that transferring the risk of the tail events is impossible if the risk measures are not too sensitive about big losses. This will give the regulator a benchmark to avoid risk of moral hazard. We will see that this is usually the case for the known risk measures such as VaR and CVaR. We also give examples that the risk of the tail events would be transferred as a result of optimal decision-making process.

Here we have the optimal solution

Theorem 1. *The solution to the general optimization problem* (7) *is given by* $f^*(x) = \int_0^x h^*(t)dt$ *and*

$$h^*(t) = \begin{cases} 1 & \Gamma\left(F_{X_0}(t)\right) > \Pi\left(F_{X_0}(t)\right) \\ 0 & \Gamma\left(F_{X_0}(t)\right) < \Pi\left(F_{X_0}(t)\right) \\ \tilde{h}(t) & \text{otherwise} \end{cases}, \tag{8}$$

where $\tilde{h}$ *could be any function between* 0 *and* 1 *on* $\Gamma\left(F_{X_0}(t)\right) = \Pi\left(F_{X_0}(t)\right)$.

Proof. Let $f \in C$. Since $f(x)$ and $x - f(x)$ are both non-decreasing, and also since VaR_t commutes with monotone functions, we get

$$\begin{aligned} &E(\varphi f(X_0)) + \varrho(X_0 - f(X_0)) \\ &\qquad = \int_0^1 \mathrm{VaR}_t(f(X_0))d\Gamma(t) + \int_0^1 \mathrm{VaR}_t(X_0 - f(X_0))d\Pi(t) \\ &\qquad = \int_0^1 f(\mathrm{VaR}_t(X_0))d\Gamma(t) + \int_0^1 (\mathrm{VaR}_t(X_0) - f(\mathrm{VaR}_t(X_0)))d\Pi(t). \end{aligned} \tag{9}$$

Let $h \in D$ be the derivative of f, i.e., $f(x) = \int_0^x h(t)dt$. Therefore, we have

$$E(\varphi f(X_0)) + \varrho(X_0 - f(X_0)) \\ = \int_0^1 \left(\int_0^{\mathrm{VaR}_t(X_0)} h(s)ds \right) d\Gamma(t) + \int_0^1 \left(\mathrm{VaR}_t(X_0) - \int_0^{\mathrm{VaR}_t(X_0)} h(s)ds \right) d\Pi(t). \tag{10}$$

First, we assume X_0 is bounded. By Fubini's Theorem, (10) gives

$$\begin{aligned} &E(\varphi f(X_0)) + \varrho(X_0 - f(X_0)) \\ &\quad = \int_0^1 \mathrm{VaR}_t(X_0) d\Pi(t) + \int_0^\infty \left(\int_{F_{X_0}(t)}^1 d\Gamma(s) - \int_{F_{X_0}(t)}^1 d\Pi(s) \right) h(t)dt \\ &\quad = \int_0^1 \mathrm{VaR}_t(X_0) d\Pi(t) + \int_0^\infty \left((1 - \Gamma(F_{X_0}(t))) - (1 - \Pi(F_{X_0}(t))) \right) h(t)dt \\ &\quad = \int_0^1 \mathrm{VaR}_t(X_0) d\Pi(t) + \int_0^\infty \left(\Pi(F_{X_0}(t)) - \Gamma(F_{X_0}(t)) \right) h(t)dt, \end{aligned} \tag{11}$$

Now, it is clear that h^* minimizes (11). It is also clear that we are free to choose the values of h^* on $\Pi(F_{X_0}(t)) = \Gamma(F_{X_0}(t))$. The value of the minimum also is equal to

$$\int_0^1 \mathrm{VaR}_t(X_0) d\tilde{\Pi}(t),$$

where $\tilde{\Pi}(x) = \min\{\Gamma(x), \Pi(x)\}$.

Now, let us in general assume that X_0 is not bounded. It is clear that at each point t, $\{\Pi \circ F_{X_0 \wedge n}(t)\}_{n=1,2,\ldots}$ is non-increasing with respect to n. On the other hand, for any t, there exist n_t such that if $n > n_t$ then $F_{X_0 \wedge n}(t) = F_{X_0}(t)$. Therefore, for any t, we have that $\Pi(F_{X_0 \wedge n}(t)) \downarrow \Pi(F_{X_0}(t))$. Now by Monotone Convergence Theorem we have that

$$\lim_{n \to \infty} \int_0^\infty \Pi(F_{X_0 \wedge n}(t)) h(t)dt = \int_0^\infty \Pi(F_{X_0}(t)) h(t)dt,$$

for any function $h \in D$. The same is true if we replace Π by Γ. Using this fact and that f is non-decreasing, we have

$$E(\varphi f(X_0)) + \varrho(X_0 - f(X_0)) \\ = \int_0^1 \mathrm{VaR}_t(X_0) d\Pi(t) + \int_0^\infty \left(\Pi(F_{X_0}(t)) - \Gamma(F_{X_0}(t)) \right) h(t)dt$$

The rest of the proof follows the same lines after (10). $\square$

Remark 1. As oppose to Merton (1977), where it is assumed that an insurance contract is similar to an option, our assumptions give raise to the contracts that are bounded from above. This will be better observe when we study particular cases of VaR and CVaR in the following section.

Now, we discuss the risk of the tail events. First, we have the following lemma from discussions in Section 4

Lemma 1. *Following all notation above,* $\Gamma'(\alpha) = \exp\left(\frac{m\sqrt{T}}{\sigma}\Phi^{-1}(\alpha) - \frac{1}{2}\left(\frac{m}{\sigma}\right)^2 T\right)$. *In particular* $\Gamma'(1) = +\infty$.

The following theorem is an implication of the previous lemma

Theorem 2. *Assume* Π *is differentiable around 1 and that* X_0 *is unbounded above. If* $m \neq 0$*, the risk of the tail events cannot be transferred iff*

$$\limsup_{y\to+\infty} \frac{\log\left(\Pi'\left(\Phi\left(y\right)\right)\right)}{y} < \frac{m\sqrt{T}}{\sigma}.$$

If $m = 0$*, the tail events cannot be transferred if and only if* $\liminf_{\alpha\to 1} \Pi'(\alpha) < 1$.

Proof. By definition, the risk of the tail events cannot be transferred to the market if there is $B > 0$ such that $X \leq B$, $P-$a.s. According to Theorem 1, this means $\left\{X = f^*(X_0) = \int_0^{X_0} h^*(t)dt \leq B\right\}$. Since X_0 is unbounded above this is equivalent to the fact that $h^*(\beta_n) = 0$ for a sequence β_n converging to 1. By (8), this is equivalent to the fact that $\Gamma(\beta_n) < \Pi(\beta_n)$. Since Π is differentiable around 1 and that $\Gamma(1) = \Pi(1) = 1$, it is easy to see that the last inequality is equivalent to the existence of a sequence $\alpha_n \to 1$, such that $\Gamma'(\alpha_n) > \Pi(\alpha_n)$. On its own this is equivalent to $\limsup_{\alpha\to 1} \Gamma'(\alpha)/\Pi'(\alpha) > 1$. Using Lemma 1, this inequality is equivalent to

$$\limsup_{\alpha\to 1} \exp\left(\frac{m\sqrt{T}}{\sigma}\Phi^{-1}(\alpha) - \frac{1}{2}\left(\frac{m}{\sigma}\right)^2 T\right)/\Pi'(\alpha) > 1. \tag{12}$$

Given that the logarithm function is an increasing function, one can see that the inequality (12) is equivalent to $\limsup_{\alpha\to 1} \frac{m\sqrt{T}}{\sigma}\Phi^{-1}(\alpha) - \frac{1}{2}\left(\frac{m}{\sigma}\right)^2 T - \log \Pi'(\alpha) > 0$. This implies that for a small positive number δ, $\frac{m\sqrt{T}}{\sigma}\Phi^{-1}(\alpha) - \frac{1}{2}\left(\frac{m}{\sigma}\right)^2 T - \log \Pi'(\alpha) > 0, \forall \alpha \in (1-\delta, 1)$. If we divide the both sides of this inequality by $\Phi^{-1}(\alpha)$, it is equivalent to

$$\frac{m\sqrt{T}}{\sigma} - \frac{\frac{1}{2}\left(\frac{m}{\sigma}\right)^2 T}{\Phi^{-1}(\alpha)} - \log \Pi'(\alpha) > 0, \forall \alpha \in (1-\delta, 1). \tag{13}$$

Since $-\frac{\frac{1}{2}\left(\frac{m}{\sigma}\right)^2 T}{\Phi^{-1}(\alpha)} \uparrow 0$ as $\alpha \to 1$, the inequality (13) is equivalent to $\limsup_{\alpha\to 1} \frac{m\sqrt{T}}{\sigma} - \frac{\log \Pi'(\alpha)}{\Phi^{-1}(\alpha)} > 0$. If we put $y = \Phi^{-1}(\alpha)$, then the theorem is proven for $m \neq 0$.

If $m = 0$ then $\Gamma'(\alpha) = 1, \forall \alpha \in (0,1)$. Therefore, $\limsup_{\alpha\to 1} \Gamma'(\alpha)/\Pi'(\alpha) > 1$, is equivalent to $\liminf_{\alpha\to 1} \Pi'(\alpha) < 1$. $\square$

This theorem provides the regulator with the degree of the risk measure sensitivity with respect to the large losses, in order to avoid outsourcing the risk of the tail events.

Remark 2. It can be easily seen that if the risk measure ϱ is a distortion risk measure given by distortion function g_β as in (4), then $\limsup_{y\to+\infty} \frac{\log(\Pi'(\Phi(y)))}{y} = \beta$, which is the expected market risk premium. That is why one can consider $\limsup_{y\to+\infty} \frac{\log(\Pi'(\Phi(y)))}{y}$ as the generalized excepted market risk premium. With this in mind, then one can say that the risk of the tail events will not be transferred if an only if the generalized expected market risk premium is less than the real one. This makes sense if we look at the problem from an investor point of view: if the expected market risk premium is higher than the real one, one could tolerate the risk of the tail event in a hope to receive a higher (spread) premium.

4.2 Corollaries and Examples

Corollary 1. *If $\varrho = \mathrm{VaR}_\alpha$, then the optimal solution can be described as*

$$X = L\left(S_0 \exp\left(\left(m - \frac{\sigma^2}{2}\right)T\right)\exp\left(\sigma \max\left\{W_T, \sqrt{T}\Phi^{-1}(1-\alpha)\right\}\right)\right).$$

Proof. We know that since $\varrho = \mathrm{VaR}_\alpha$, $\Pi(t) = 1_{[\alpha,1]}(t)$. First of all since $\Gamma'(1) = +\infty$ then Π is greater than Γ around 1. Since Γ is a strictly increasing function between zero and one, then $\Gamma\left(F_{X_0}(t)\right) > \Pi\left(F_{X_0}(t)\right)$ if and only if $F_{X_0}(t) < \alpha$. Therefore,

$$h^*(t) = \begin{cases} 1 & F_{X_0}(t) < \alpha \\ 0 & F_{X_0}(t) \geq \alpha \end{cases}.$$

This results in

$$X = \min\left\{X_0, \mathrm{VaR}_\alpha(X_0)\right\}.$$

Since $X_0 = L(\tilde{S}_T)$, and since VaR and decreasing functions can commute, we have

$$X = \min\left\{L\left(\tilde{S}_T\right), L\left(\mathrm{VaR}_\alpha(\tilde{S}_T)\right)\right\} = L\left(\max\left\{\tilde{S}_T, \mathrm{VaR}_{1-\alpha}(\tilde{S}_T)\right\}\right).$$

On the other hand, since $W_T \sim N(0,T)$,

$$\mathrm{VaR}_{1-\alpha}(\tilde{S}_T) = S_0 \exp\left((m - \frac{\sigma^2}{2})T + \sigma\sqrt{T}\Phi^{-1}(1-\alpha))\right),$$

which completes the proof. $\square$

Example 1. Let us assume that $L(x) = S_0 \exp\left((\log(x/S_0))_-\right)$. In this case we have

$$X = S_0 \exp\left(\left(m - \frac{\sigma^2}{2}\right)T + \sigma Y(\alpha)\right),$$

where

$$Y(\alpha) = \begin{cases} \sqrt{T}\Phi^{-1}(1-\alpha) & W_T \leq \sqrt{T}\Phi^{-1}(1-\alpha), \\ W_T & \sqrt{T}\Phi^{-1}(1-\alpha) < W_T < 0, \\ 0 & W_T \geq 0. \end{cases} \tag{14}$$

Corollary 2. *If $\varrho = \mathrm{CVaR}_\alpha$ then the optimal solution can be described as*

$$X = L\left(S_0 \exp\left((m - \frac{\sigma^2}{2})T\right)\exp\left(\sigma \max\left\{W_T, \sqrt{T}\Phi^{-1}(1-\alpha^*)\right\}\right)\right).$$

where α^* is a solution to $(1-\alpha)\Gamma(x) - x + \alpha = 0$.

Proof. In this case $\Pi(x) = \frac{x-\alpha}{1-\alpha}1_{[\alpha,1]}$. First of all since $\Gamma'(1) = +\infty$ then Π is greater than Γ around 1. Note that since the function Γ is strictly increasing and also because Π is linear between α and 1, the solution to the equation $(1-\alpha)\Gamma(x) - x + \alpha = 0$ is unique. Now, it is clear that $\Gamma(F_{X_0}(t)) > \Pi(F_{X_0}(t))$ if $F_{X_0}(t) < \alpha$. On the other hand, if $F_{X_0}(t) \geq \alpha$ then $\Gamma(F_{X_0}(t)) > \Pi(F_{X_0}(t)) = \frac{F_{X_0}(t)-\alpha}{1-\alpha}$, as long as $F_{X_0}(t) < \alpha^*$. Therefore, h^* equals to one as long as $F_{X_0} < \alpha^*$. The rest of the proof is similar to the previous corollary.

As one can see $\alpha^*$is always greater than α. This means that CVaR allows for more risk to be transferred rather than VaR. In the following figure we have depicted the differences between α and α^*.

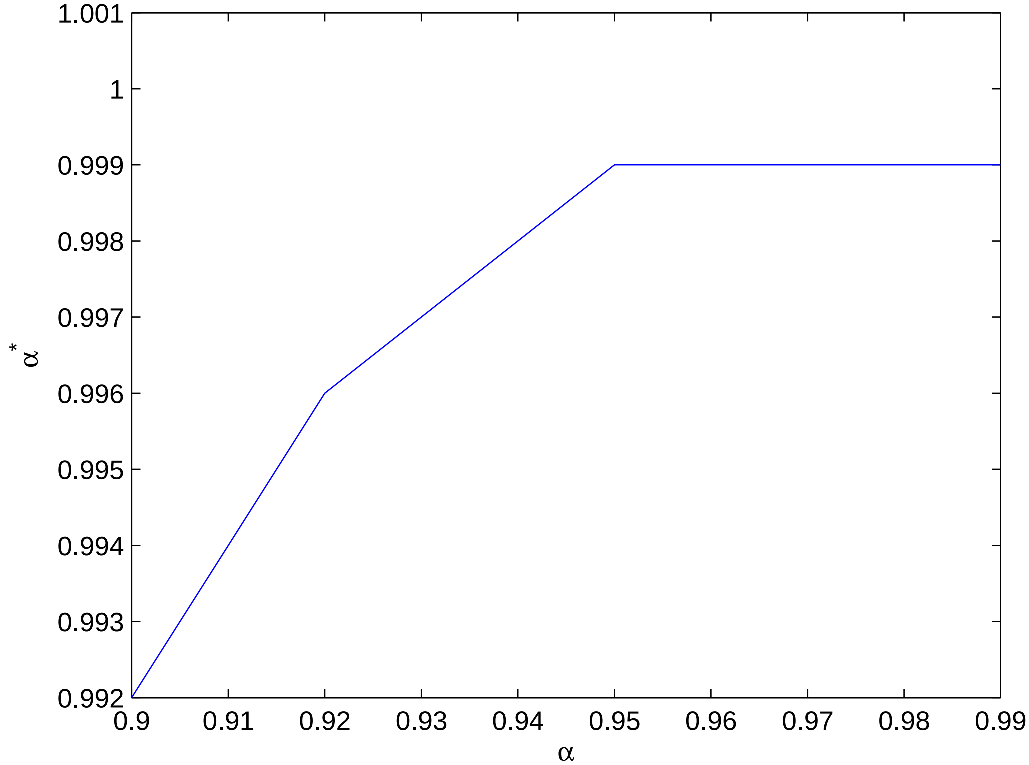

Figure 1: The figure depicts α versus α^* for the parameters $T = 1, m = 0.05$ and $\sigma = 0.05$.

$\square$

In the following we have depicted the intersection of $\Gamma$and Π for different levels of α.

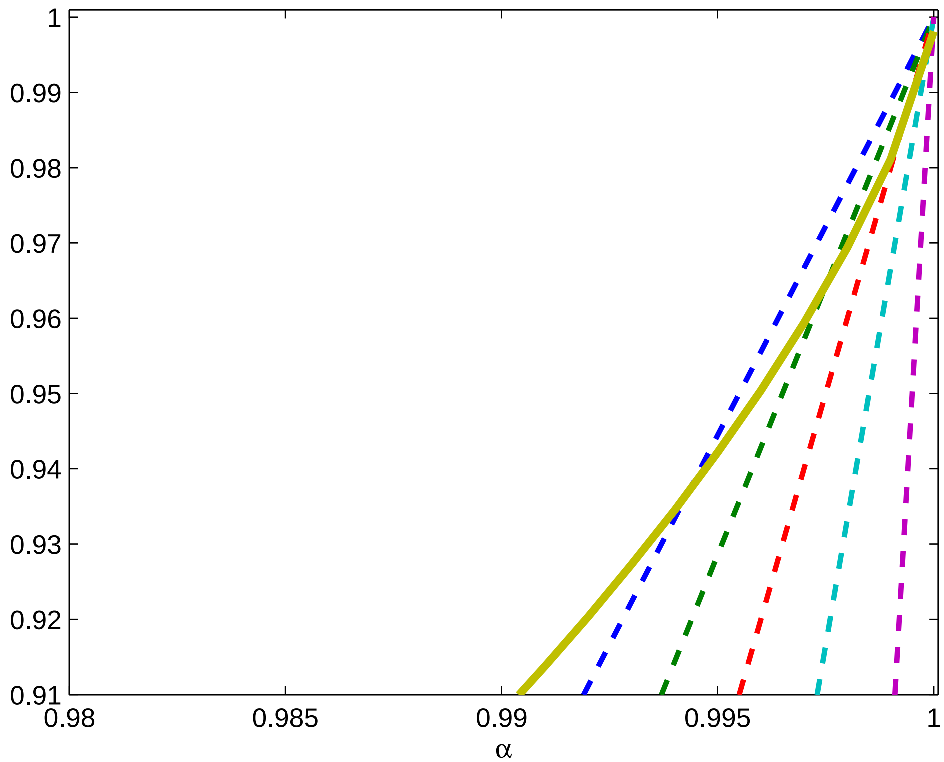

Figure 2: The intersection of $\Gamma$and Π, for the parameters $T = 1, m = 0.05$ and $\sigma = 0.05$. The curvy line is Γ and the lines in top right of the Π's for different $\alpha = 91, 93, 95, 97$ and 99%.

Example 2. Let us assume that $L(x) = \exp\left((\log(x/S_0))_-\right)$. In this case we have

$$X = S_0 \exp\left(\left(m - \frac{\sigma^2}{2}\right)T + \sigma Y(\alpha^*)\right),$$

where Y is defined by (14).

Corollary 3. *If $m \neq 0$ and $\varrho = \mathrm{VaR}_\alpha$ or $\varrho = \mathrm{CVaR}_\alpha$, the risk of the tail events are not transferred to the system. If $m = 0$ and $\varrho = \mathrm{VaR}_\alpha$ then the risk of tail events are not transferred to the system whereas the opposite holds for $m = 0$ and $\varrho = \mathrm{CVaR}_\alpha$.*

5 Final remarks

In this paper we developed a framework for studying the deposit insurances and the risk of tail events. In order to control the risk of moral hazard, inspired by the excessive risk taking behavior, deposit insurances must have a stop loss structure. We also studied the risk of tail events and the circumstances that this risk cannot be avoided. For technical reasons we have chosen to study deposit insurances for the family of distortion risk measures, including VaR and CVaR. However, for future research one can consider an extension of our setting to the family of law-invariant coherent risk measure. This extension would be possible in light of the law-invariant coherent risk measures' representation given in Kusuoka (2001). Another extension is to study this problem in an incomplete market framework. For instance, one can use the Heston model for pricing the assets in the financial market.

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