SIAM J. CONTROL OPTIM.
Vol. 54, No. 3, pp. 1444–1474

CONSTRAINED AND UNCONSTRAINED OPTIMAL DISCOUNTED CONTROL OF PIECEWISE DETERMINISTIC MARKOV PROCESSES[*]

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Abstract. The main goal of this paper is to study the infinite-horizon expected discounted continuous-time optimal control problem of piecewise deterministic Markov processes with the control acting continuously on the jump intensity λ and on the transition measure Q of the process but not on the deterministic flow ϕ. The contributions of the paper are for the unconstrained as well as the constrained cases. The set of admissible control strategies is assumed to be formed by policies, possibly randomized and depending on the history of the process, taking values in a set valued action space. For the unconstrained case we provide sufficient conditions based on the three local characteristics of the process ϕ, λ, Q and the semicontinuity properties of the set valued action space, to guarantee the existence and uniqueness of the integro-differential optimality equation (the so-called Bellman–Hamilton–Jacobi equation) as well as the existence of an optimal (and δ-optimal, as well) deterministic stationary control strategy for the problem. For the constrained case we show that the values of the constrained control problem and an associated infinite dimensional linear programming (LP) problem are the same, and moreover we provide sufficient conditions for the solvability of the LP problem as well as for the existence of an optimal feasible randomized stationary control strategy for the constrained problem.

Key words. unconstrained/constrained control problem, continuous control, piecewise, deterministic Markov process, continuous-time Markov decision process, discounted cost

AMS subject classifications. Primary, 90C40; Secondary, 60J25

DOI. 10.1137/140996380

1. Introduction. A general family of nondiffusion stochastic models, namely, piecewise deterministic Markov processes (PDMPs), was introduced in [9], covering an enormous variety of applications in operations research, engineering systems, and management science. These processes are determined by three local characteristics: the flow ϕ, the jump rate λ, and the transition measure Q. Roughly speaking, for the uncontrolled case, the motion of a PDMP can be described as follows: starting from x, the motion of the process follows the flow $\phi(x,t)$ until the first jump time T_1, which occurs either spontaneously in a Poisson-like fashion with rate λ or when the flow $\phi(x,t)$ hits the boundary of the state space. In either case the location of the process at the jump time T_1 is selected by the transition measure $Q(.|\phi(x,T_1))$ and the motion restarts from this new point as before. With a suitable choice of the

[*]Received by the editors November 18, 2014; accepted for publication (in revised form) April 12, 2016; published electronically June 2, 2016. This work was partially supported by FAPESP (Research Council of the State of São Paulo) grant 2013/50759-3 and by the INRIA Associate Team CDSS and by the French State, managed by the French National Research Agency (ANR) in the frame of the "Investments for the future" Programme IdEx Bordeaux–CPU (ANR-10-IDEX-03-02).
http://www.siam.org/journals/sicon/54-3/99638.html

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state space and the local characteristics ϕ, λ, and Q, a great deal of problems can be covered by these models (see, for instance, [9]).

The goal of this paper is to study the infinite-horizon expected discounted continuous-time optimal control problem of PDMPs, with the control chosen from an action set (which depends on the state variable) acting continuously on the jump intensity λ and on the transition measure Q, but not on the flow ϕ. The contributions of the paper are for the unconstrained as well as the constrained cases. For the unconstrained case the goal is to minimize the infinite-horizon total expected discounted cost, which is composed of a running cost and a boundary cost, added to the total cost each time the PDMP touches the boundary. For the constrained case there is a finite number of restrictions, also written as infinite-horizon total expected discounted costs, that has to be satisfied. The set of admissible control strategies is assumed to be formed by policies possibly randomized and depending on the history of the process.

For the unconstrained case, a common approach to tackle this problem is to characterize the value function as a solution to the Bellman–Hamilton–Jacobi (BHJ) equation associated with an embedded discrete-stage Markov decision model, with the stages defined by the jump times T_n of the process. In this case the decision is to find, at each stage, a control function that solves an embedded deterministic optimal control problem. Usually the control strategy is chosen among the set of piecewise open loop policies, that is, stochastic kernels or measurable functions that depend only on the last jump time and post jump location. We can mention [2, 3, 6, 9, 8, 13, 23, 25] as works following this technique. Another important approach for this class of problems, which we will call the infinitesimal approach, is to characterize the optimal value function as the viscosity solution of the corresponding integro-differential BHJ equation. We can mention [10, 9, 11, 12, 26] as a sample of works using this kind of approach. Our results concerning the unconstrained case, presented in section 5, follow the infinitesimal approach and provide in Theorem 5.5 sufficient conditions for the existence of a solution for an integro-differential BHJ optimality equation associated with the problem as well as conditions for the existence of an optimal or δ-optimal selector for this equation. In what follows it is shown in Proposition 5.6 that the solution of the integro-differential BHJ optimality equation is in fact unique and coincides with the optimal value for the unconstrained problem, and also solves the dual linear program (DLP). Moreover, the optimal selector (δ-optimal selector) derived in Theorem 5.5 yields an optimal (ϵ-optimal, respectively) stationary nonrandomized strategy for the unconstrained problem. When compared with the PDMP literature it should be stressed that we consider a broader class of control strategies (possibly depending on the history of the process and taking values in state-dependent action spaces) instead of the open loop policies with fixed action set considered in previous papers. Under this general class of strategies we obtain in Lemma 4.2 a discounted version of the so-called Dynkin formula associated with the controlled process, a key result for the unconstrained as well as the constrained problems. Another main novelty for the unconstrained case is that we provide sufficient conditions based on the three local characteristics of the process ϕ, λ, Q and the semicontinuity properties of the set valued action space, to guarantee the existence and uniqueness of the solution to the integro-differential BHJ equation as well as the existence of an optimal (δ-optimal) deterministic selector for this optimality equation. As far as the authors are aware, this is the first time that this kind of result is presented in the literature for discounted control problems of PDMPs considering the broader class of controls mentioned above.

The linear programming (LP) technique has proved to be a very efficient method for solving continuous-time Markov decision processes (MDPs) problems with constraints. We can mention [15, 16, 21] and the references therein as a sample of works on this subject in the context of continuous-time controlled Markov processes. On the other hand, contrary to continuous-time constrained MDPs, it should be stressed that the constrained optimal control problems of PDMPs have received much less attention. An attempt in this direction is presented in [14], where the authors study a control problem for a special class of PDMPs (with no boundary) by using an LP technique. More recently in [7] the constrained problem was studied by reducing the original continuous-time control problem into a discrete-stage Markov decision model in which the stages are the jump times T_n, similar to the approach adopted for the unconstrained case mentioned above. In this paper we follow another way, using what we called above the infinitesimal approach. The main results regarding the constrained case are presented in section 6. Initially the set of admissible finite measures are introduced in Definition 6.5, a definition that generalizes the usual definition in the continuous-time MDP case, as pointed out in Remark 6.6. Theorem 6.7 presents a key result relating the set of admissible measures and the set of occupation measures associated with any admissible control strategy for the problem. The main novelty of this section, and one of the main contributions of the paper, is presented in Theorem 6.14, which proves that the values of the constrained control problem and the LP problem are the same and provides sufficient conditions for the solvability of the LP problem as well as for the existence of an optimal feasible control strategy for the constrained problem. As mentioned above, the literature for the constrained control of PDMPs is very scarce and, as far as the authors are aware, this is the first time that the constrained control problem for PDMPs is considered under the infinitesimal approach.

The paper is organized as follows. In section 2 we introduce the notation as well as the parameters defining the model and the construction of the controlled process. In section 3 we define the infinite-horizon performance criterion we are concerned with and several different classes of admissible control strategies and introduce the main assumptions that will be considered throughout the paper. In section 4 we present some preliminary key results that will be used throughout the paper. In particular we provide conditions for the controlled process to be nonexplosive and derive the discounted version of the so-called Dynkin formula associated with the controlled process. The main results for the unconstrained case are presented in section 5, while the main results for the constrained case are presented in section 6, as discussed above.

2. The controlled PDP. The main goal of this section is to introduce the notation and the parameters defining the model and to present the construction of the controlled process. In particular a measurable space $(\Omega, \mathcal{F})$ consisting of the canonical sample paths of the multivariate point process (Θ_n, X_n) is introduced. Having defined the class of admissible strategies, we show the existence of a probability measure $\mathbb{P}^u_{x_0}$ with respect to which the controlled process (Θ_n, X_n) has the required conditional distributions.

The following notation will be used in this paper. $\mathbb{N}$ is the set of natural numbers including 0, $\mathbb{N}^* = \mathbb{N} - \{0\}$, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}_+$ denotes the set of nonnegative real numbers, $\mathbb{R}^*_+ = \mathbb{R}_+ - \{0\}$, $\overline{\mathbb{R}}_+ = \mathbb{R}_+ \cup \{+\infty\}$, and $\overline{\mathbb{R}}^*_+ = \mathbb{R}^*_+ \cup \{+\infty\}$. For any $q \in \mathbb{N}$, $\mathbb{N}_q$ is the set $\{0, 1, \ldots, q\}$ and for any $q \in \mathbb{N}^*$, $\mathbb{N}^*_q$ is the set $\{1, \ldots, q\}$. The term *measure* will always refer to a countably additive, $\overline{\mathbb{R}}_+$-valued set function. Let X be a Borel space (i.e., a Borel-measurable subset of a complete and separable metric space) and denote by $\mathcal{B}(X)$ its associated Borel σ-algebra. For any set A, I_A denotes the indicator function of the set A. The set of measures defined on $(X, \mathcal{B}(X))$

is denoted by $\mathcal{M}(X)$, $\mathcal{P}(X)$ is the set of probability measures defined on $(X, \mathcal{B}(X))$, and $\mathcal{P}(X|Y)$ is the set of stochastic kernels on X given Y, where Y denotes a Borel space. When referring to the space of measure $\mathcal{M}(X)$, it is supposed that this space is endowed with the weak topology. For any point $x \in X$, δ_x denotes the Dirac measure defined by $\delta_x(\Gamma) = I_\Gamma(x)$ for any $\Gamma \in \mathcal{B}(X)$. Suppose that X, Y, and Z are Borel spaces. If R_1 is a kernel on Y given X and R_2 is a kernel on Z given Y, the *product* of R_1 and R_2 is defined by $R_1R_2(B|x) = \int R_2(B|y)R_1(dy|x)$ for any $(x, B) \in X \times \mathcal{B}(Z)$. For a kernel R on X given X, the iterates R^n for $n \in \mathbb{N} \cup \{0\}$ are defined by setting $R^0(x, B) = \delta_x(B)$ for any $(x, B) \in X \times \mathcal{B}(X)$ and iteratively $R^n = RR^{n-1}$. Suppose that $Y = W \times Z$, where W and Z are Borel spaces. The marginal of a measure $\eta \in \mathcal{M}(Y)$ with respect to the first space W will be denoted by $\widehat{\eta}$, that is, $\widehat{\eta}(\Gamma_W) = \eta(\Gamma_W \times Z)$ for any $\Gamma_W \in \mathcal{B}(W)$. The set of bounded real-valued Borel-measurable functions defined on the Borel space X is denoted by $\mathbb{B}(X)$ and $\mathbb{C}(X)$ (respectively, $\mathbb{L}(X)$ and $\mathbb{U}(X)$) is the set of bounded real-valued continuous (respectively, lower semicontinuous and upper semicontinuous) functions defined on X. Finally, the infimum over an empty set is understood to be equal to $+\infty$, and we set $e^{-\infty} = 0$.

2.1. Parameters of the model. We will deal with a control model defined through the following elements:

- $\mathbf{X}$ is the state space, assumed to be an open subset of $\mathbb{R}^d$ ($d \in \mathbb{N}^*$), and $\partial\mathbf{X}$ denotes the boundary of $\mathbf{X}$.
- $\phi(x, t) : \mathbb{R}^d \times \mathbb{R}_+ \to \mathbb{R}^d$ is the flow associated with a given Lipschitz continuous vector field in $\mathbb{R}^d$, that is, $\phi(x, t + s) = \phi(\phi(x, s), t)$ for all $x \in \mathbb{R}^d$ and $(t, s) \in \mathbb{R}^2$.
- $\mathbf{\Xi} = \{x \in \partial\mathbf{X} : x = \phi(y, t) \text{ for some } y \in \mathbf{X} \text{ and } t \in \mathbb{R}_+^*\}$ is the so-called active boundary. Below, with some abuse of notation, $\overline{\mathbf{X}}$ denotes $\mathbf{X} \cup \mathbf{\Xi}$. For $x \in \overline{\mathbf{X}}$ we use notation $t^*(x) = \inf\{t \in \mathbb{R}_+ : \ \phi(x, t) \in \mathbf{\Xi}\}$. Actually, the flow ϕ outside the space $\overline{\mathbf{X}}$ plays no role and can be defined arbitrarily.
- $\mathbf{A}$ is the action space, assumed to be a Borel space. $\mathbf{A}^i \in \mathcal{B}(\mathbf{A})$ (respectively, $\mathbf{A}^g \in \mathcal{B}(\mathbf{A})$) is the set of impulsive (respectively, gradual) actions assumed to be nonempty and satisfying $\mathbf{A} = \mathbf{A}^i \cup \mathbf{A}^g$ with $\mathbf{A}^i \cap \mathbf{A}^g = \emptyset$.
- The set of feasible actions in state $x \in \overline{\mathbf{X}}$ is $\mathbf{A}(x)$, which is a nonempty measurable subset of $\mathbf{A}$. We assume that $\mathbf{A}(x) \subset \mathbf{A}^g$ for all $x \in \mathbf{X}$ and $\mathbf{A}(x) \subset \mathbf{A}^i$ for all $x \in \mathbf{\Xi}$. Let us introduce the following sets $\mathbf{K} = \mathbf{K}^i \cup \mathbf{K}^g$ with
$$\mathbf{K}^g = \{(x, a) \in \mathbf{X} \times \mathbf{A} : a \in \mathbf{A}(x)\} \in \mathcal{B}(\mathbf{X} \times \mathbf{A}^g),$$
$$\mathbf{K}^i = \{(x, a) \in \mathbf{\Xi} \times \mathbf{A} : a \in \mathbf{A}(x)\} \in \mathcal{B}(\mathbf{\Xi} \times \mathbf{A}^i).$$
It is assumed that $\mathbf{K}^g$ (respectively, $\mathbf{K}^i$) contains the graph of a measurable function from $\mathbf{X}$ (respectively, $\mathbf{\Xi}$) to $\mathbf{A}$.
- The transition rate (infinitesimal generator) q is a signed kernel on $\mathbf{X}$ given $\mathbf{K}^g$. This means that $\Gamma \mapsto q(\Gamma|x, a)$ is a signed measure on $(\mathbf{X}, \mathcal{B}(\mathbf{X}))$ for all $(x, a) \in \mathbf{K}^g$ and that $(x, a) \mapsto q(\Gamma|x, a)$ is measurable for all $\Gamma \in \mathcal{B}(\mathbf{X})$. It satisfies $q(\Gamma|x, a) \geq 0$ for all $\Gamma \in \mathcal{B}(\mathbf{X})$ such that $x \notin \Gamma$. It is conservative, i.e., $q(\mathbf{X}|x, a) = 0$, and stable in that $\sup_{a \in \mathbf{A}(x)} \lambda(x, a) < \infty$, where $\lambda(x, a) = -q(\{x\}|x, a) = q(\mathbf{X} \setminus \{x\}|x, a)$.
- The stochastic kernel Q on $\mathbf{X}$ given $\mathbf{K}^i$. For any $(z, b) \in \mathbf{K}^i$, $Q(\cdot|z, b)$ is the distribution of the state immediately after the jump from the boundary when an *impulsive* action $b \in \mathbf{A}(z)$ is applied. We call such jumps "forced" jumps, while the jumps governed by the generator q are called "natural" jumps.

Note that $\lambda(x,a)$, as previously introduced, is the natural jumps' intensity. For an arbitrary $\Gamma \in \mathcal{B}(\mathbf{X})$ and $x \notin \Gamma$, the ratio defined by $Q(\Gamma|x,a) = q(\Gamma|x,a)/\lambda(x,a)$ whenever $\lambda(x,a) > 0$ gives the probability that the state belongs to Γ immediately after a natural jump. In case $\lambda(x,a) = 0$, Q can be defined in an arbitrary measurable way to obtain a stochastic kernel Q on $\mathbf{X}$ given $\mathbf{K}^g$ satisfying $Q(\mathbf{X} \setminus \{x\}|x,a) = 1$ for any $(x,a) \in \mathbf{K}^g$. Now obviously,

$$(1) \qquad q(dy|x,a) = \lambda(x,a)\big[Q(dy|x,a) - \delta_x(dy)\big].$$

When adding points $(x,a) \in \mathbf{K}^i$, we obtain the stochastic kernel Q on $\mathbf{X}$ given $\mathbf{K}$.

Related to the parameters defining the process, one needs to introduce the set $\mathbb{A}(\overline{\mathbf{X}})$ of bounded measurable functions which are absolutely continuous with respect to the flow ϕ, that is, the set of functions $g \in \mathbb{B}(\overline{\mathbf{X}})$ such that for any $x \in \overline{\mathbf{X}}$, the function $g(\phi(x,\cdot))$ is absolutely continuous on $[0,t^*(x)] \cap \mathbb{R}_+$.

If $g \in \mathbb{B}(\mathbf{X})$ is such that for any $x \in \mathbf{X}$, $g(\phi(x,\cdot))$ is absolutely continuous on $[0,t^*(x)[$ and $\lim_{t\to t^*(x)} g(\phi(x,t))$ exists whenever $t^*(x) < \infty$, then it can be easily seen that the domain of the mapping g can be extended to $\overline{\mathbf{X}}$ by setting $g(z) = \lim_{t\to t^*(x)} g(\phi(x,t))$, where $z = \phi(x,t^*(x))) \in \mathbf{\Xi}$. By doing so, we can consider that $g \in \mathbb{A}(\overline{\mathbf{X}})$. Let $g \in \mathbb{A}(\overline{\mathbf{X}})$. From Lemma 2.2 in [6], there exists a real-valued measurable function $\mathcal{X}g$ defined on $\mathbf{X}$ satisfying

$$(2) \qquad g(\phi(x,t)) = g(x) + \int_{[0,t]} \mathcal{X}g(\phi(x,s))ds$$

for any $t \in [0,t^*(x)[$. Observe that for any function $g \in \mathbb{A}(\overline{\mathbf{X}})$, the function $\mathcal{X}g$ satisfying (2) is not necessarily unique.

Let us introduce the following notation: $\mathcal{P}^g$ (respectively, $\mathcal{P}^i$) denotes the set of stochastic kernels π in $\mathcal{P}(\mathbf{A}^g|\mathbf{X})$ (respectively, γ in $\mathcal{P}(\mathbf{A}^i|\mathbf{X})$) satisfying $\pi(\mathbf{A}(x)|x) = 1$ for any $x \in \mathbf{X}$ (respectively, $\gamma(\mathbf{A}(z)|z) = 1$ for any $z \in \mathbf{\Xi}$).

2.2. Construction of the process. Let $\mathbf{X}_\infty = \mathbf{X} \cup \{x_\infty\}$, where x_∞ is an isolated artificial point corresponding to the case when no jumps occur in the future.

We put $\Omega_n = \mathbf{X} \times (\mathbb{R}_+^* \times \mathbf{X})^n \times (\{\infty\} \times \{x_\infty\})^\infty$. The canonical space denoted by Ω is defined as $\Omega = \bigcup_{n=0}^\infty \Omega_n \bigcup \big(\mathbf{X} \times (\mathbb{R}_+^* \times \mathbf{X})^\infty\big)$ and is endowed with its Borel σ-algebra denoted by $\mathcal{F}$. For notational convenience, $\omega \in \Omega$ will be represented as

$$\omega = (x_0, \theta_1, x_1, \theta_2, x_2, \ldots).$$

Here, $x_0 \in \mathbf{X}$ is the initial state of the controlled process ξ with values in $\mathbf{X}$, defined below. For $n \in \mathbb{N}^*$, the components $\theta_n > 0$ and x_n correspond to the time interval between two consecutive jumps and the value of the process ξ immediately after the jump. In case $\theta_n < \infty$ and $\theta_{n+1} = \infty$, the trajectory has only n jumps, and we put $\theta_m = \infty$ and $x_m = x_\infty$ (artificial point) for all $m \geq n+1$. Between jumps, the state of the process ξ moves according to the flow ϕ.

The path up to $n \in \mathbb{N}$ is denoted by $h_n = (x_0, \theta_1, x_1, \theta_2, x_2, \ldots \theta_n, x_n)$, and the collection of all such paths is denoted by $\mathbf{H}_n$. For $n \in \mathbb{N}$, introduce the mappings $X_n : \Omega \to \mathbf{X}_\infty$ by $X_n(\omega) = x_n$ and, for $n \geq 1$, the mappings $\Theta_n : \Omega \to \overline{\mathbb{R}}_+^*$ by $\Theta_n(\omega) = \theta_n$; $\Theta_0(\omega) = 0$. The sequence $(T_n)_{n\in\mathbb{N}^*}$ of $\overline{\mathbb{R}}_+^*$-valued mappings is defined on Ω by $T_n(\omega) = \sum_{i=1}^n \Theta_i(\omega) = \sum_{i=1}^n \theta_i$ and $T_\infty(\omega) = \lim_{n\to\infty} T_n(\omega)$. We denote by $H_n = (X_0, \Theta_1, X_1, \ldots, \Theta_n, X_n)$ the n-term random history taking values in $\mathbf{H}_n$ for $n \in \mathbb{N}$.

The random measure μ associated with $(\Theta_n, X_n)_{n\in\mathbb{N}}$ is a measure defined on $\mathbb{R}_+^* \times \mathbf{X}$ by

$$\mu(\omega; dt, dx) = \sum_{n\geq 1} I_{\{T_n(\omega)<\infty\}}\delta_{(T_n(\omega),X_n(\omega))}(dt, dx).$$

For notational convenience the dependence on ω will be suppressed and, instead of $\mu(\omega; dt, dx)$, it will be written $\mu(dt, dx)$. For $t \in \mathbb{R}_+$, define $\mathcal{F}_t = \sigma\{H_0\} \vee \sigma\{\mu(]0, s] \times B) : s \leq t, B \in \mathcal{B}(\mathbf{X})\}$. Finally, we define the controlled process $\{\xi_t\}_{t\in\mathbb{R}_+}$:

$$\xi_t(\omega) = \begin{cases} \phi(X_n, t - T_n) & \text{if } T_n \leq t < T_{n+1} \text{ for } n \in \mathbb{N}; \\ x_\infty, & \text{if } T_\infty \leq t. \end{cases}$$

Obviously, the process $\{\xi_t\}_{t\in\mathbb{R}_+}$ can be equivalently described by the sequence $(\Theta_n, X_n)_{n\in\mathbb{N}}$.

2.3. Admissible strategies and conditional distribution of the controlled process. An admissible control strategy is a sequence $u = (\pi_n, \gamma_n)_{n\in\mathbb{N}}$ such that, for any $n \in \mathbb{N}$, the following hold:

- π_n is a stochastic kernel on $\mathbf{A}^g$ given $\mathbf{H}_n \times \mathbb{R}_+^*$. For $h_n = (x_0, \theta_1, x_1, \dots \theta_n, x_n) \in \mathbf{H}_n$ with $x_n \neq x_\infty$, it satisfies $\pi_n(\mathbf{A}(\phi(x_n, t))|h_n, t) = 1$ for any $t \in]0, t^*(x_n)[$; in the case $x_n = x_\infty$ $\pi_n(\cdot|h_n, t)$ is an arbitrary stochastic kernel on $\mathbf{A}^g$ given $\mathbf{H}_n \times \mathbb{R}_+^*$.
- γ_n is a stochastic kernel on $\mathbf{A}^i$ given $\mathbf{H}_n$. For $h_n = (x_0, \theta_1, x_1, \dots \theta_n, x_n) \in \mathbf{H}_n$ with $x_n \neq x_\infty$ and $t^*(x_n) < \infty$, it satisfies $\gamma_n(\mathbf{A}(\phi(x_n, t^*(x_n)))|h_n) = 1$; otherwise $\gamma_n(\cdot|h_n)$ is an arbitrarily fixed kernel on $\mathbf{A}^i$ given $\mathbf{H}_n$.

The set of admissible control strategies is denoted by $\mathcal{U}$. Below, we will use the following notation. When an admissible control strategy $u = (\pi_n, \gamma_n)_{n\in\mathbb{N}}$ is considered, then we denote by π and γ the random processes with values in $\mathcal{P}(\mathbf{A}^g)$ and $\mathcal{P}(\mathbf{A}^i)$ correspondingly as

$$\pi(da|t) = \sum_{n\in\mathbb{N}} I_{\{T_n<t\leq T_{n+1}\}}\pi_n(da|H_n, t - T_n)$$

and

$$\gamma(da|t) = \sum_{n\in\mathbb{N}} I_{\{T_n<t\leq T_{n+1}\}}\gamma_n(da|H_n)$$

for $t \in \mathbb{R}_+^*$. The processes π and γ are $\{\mathcal{F}_t\}_{t\in\mathbb{R}_+}$-predictable random processes with values in $\mathcal{P}(\mathbf{A}^g)$ and $\mathcal{P}(\mathbf{A}^i)$ correspondingly.

Suppose a strategy $u = \big(\pi_n, \gamma_n\big)_{n\in\mathbb{N}} \in \mathcal{U}$ is fixed. For $h_n = (x_0, \theta_1, x_1, \dots \theta_n, x_n) \in \mathbf{H}_n$ with $x_n \neq x_\infty$, the transition rate is given by

$$\int_{\mathbf{A}^g} q(\cdot|\phi(x_n, t), a)\pi_n(da|h_n, t), \quad t \in]0, t^*(x_n)[.$$

In this connection, the intensity of the natural jumps is given by

$$\lambda_n^u(h_n, t) = \int_{\mathbf{A}^g} \lambda(\phi(x_n, t), a)\pi_n(da|h_n, t), \quad t \in]0, t^*(x_n)[,$$

and the rate of the natural jumps by

$$\Lambda_n^u(h_n, t) = \int_{]0,t[} \lambda_n^u(h_n, s)ds, \quad t \in]0, t^*(x_n)],$$

for any $n \in \mathbb{N}$. In case $x_n = x_\infty$, $\lambda_n^u(h_n,t) = \Lambda_n^u(h_n,t) = 0$ for any $t \in \mathbb{R}_+^*$. Let $\Gamma_{\mathbf{X}} \in \mathcal{B}(\mathbf{X})$. Similarly to the definition of the stochastic kernel Q, in case $\lambda_n^u(h_n,t) > 0$, we define the distribution of the state after a natural jump at time moment $\sum_{k=1}^n \theta_k + t$ for $t < t^*(x_n)$ by

$$Q_n^{g,u}(\Gamma_{\mathbf{X}}|h_n,t) = \frac{\lambda Q_n^{g,u}(\Gamma_{\mathbf{X}}|h_n,t)}{\lambda_n^u(h_n,t)},$$

where

$$\begin{aligned}\lambda Q_n^{g,u}(\Gamma_{\mathbf{X}}|h_n,t) &= \int_{\mathbf{A}^g} q(\Gamma_{\mathbf{X}} \setminus \{\phi(x_n,t)\})|\phi(x_n,t),a)\pi_n(da|h_n,t) \\ &= \int_{\mathbf{A}^g} Q(\Gamma_{\mathbf{X}}|\phi(x_n,t),a)\lambda(\phi(x_n,t),a)\pi_n(da|h_n,t).\end{aligned}$$

In case $\lambda_n^u(h_n,t) = 0$, the kernel $Q_n^{g,u}$ is fixed arbitrarily.

When $x_n \neq x_\infty$ and $t^*(x_n) < \infty$, one can introduce the distribution of the state after a forced jump at time moment $\sum_{k=1}^n \theta_k + t^*(x_n)$ by

$$Q_n^{i,u}(\Gamma_{\mathbf{X}}|h_n) = \int_{\mathbf{A}^i} Q(\Gamma_{\mathbf{X}}|\phi(x_n,t^*(x_n)),a)\gamma_n(da|h_n). \tag{3}$$

If $x_n = x_\infty$ or $t^*(x_n) = \infty$, then $Q_n^{i,u}$ is fixed arbitrarily.

Now, for any $n \in \mathbb{N}$, the stochastic kernel G_n on $\overline{\mathbb{R}}_+^* \times \mathbf{X}_\infty$ given $\mathbf{H}_n$, describing the joint distribution of the next sojourn time and state, is defined by

$$\begin{aligned}G_n(\Gamma|h_n) = &\Big[I_{\{x_n=x_\infty\}} + e^{-\Lambda_n^u(h_n,+\infty)} I_{\{x_n\in\mathbf{X}\}} I_{\{t^*(x_n)=\infty\}}\Big]\delta_{(+\infty,x_\infty)}(\Gamma) \\ &+ I_{\{x_n\in\mathbf{X}\}}\bigg[\int_{\mathbb{R}_+^*\times\mathbf{X}} I_\Gamma(t,x)\delta_{t^*(x_n)}(dt)Q_n^{i,u}(dx|h_n)e^{-\Lambda_n^u(h_n,t^*(x_n))} \\ &+ \int_{]0,t^*(x_n)[\times\mathbf{X}} I_\Gamma(t,x)Q_n^{g,u}(dx|h_n,t)\lambda_n^u(h_n,t)e^{-\Lambda_n^u(h_n,t)}dt\bigg],\end{aligned} \tag{4}$$

where $\Gamma \in \mathcal{B}(\overline{\mathbb{R}}_+^* \times \mathbf{X}_\infty)$ and $h_n = (x_0,\theta_1,x_1,\dots,\theta_n,x_n) \in \mathbf{H}_n$. Note that the kernels $Q_n^{i,u}$ and $Q_n^{g,u}$ appear in the formula for G_n only if $t^*(x_n) < \infty$ and $\lambda_n^u(h_n,t) \neq 0$, respectively.

Consider an admissible strategy $u \in \mathcal{U}$ and an initial state $x_0 \in \mathbf{X}$. From Theorem 3.6 in [18] (or Remark 3.43 on p. 87 in [19]), there exists a probability $\mathbb{P}_{x_0}^u$ on $(\Omega,\mathcal{F})$ such that the restriction of $\mathbb{P}_{x_0}^u$ to $(\Omega,\mathcal{F}_0)$ is given by

$$\mathbb{P}_{x_0}^u\big(\{X_0 = x_0\}\big) = 1 \tag{5}$$

and the positive random measure ν defined on $\mathbb{R}_+^* \times \mathbf{X}$ by

$$\nu(dt,dx) = \sum_{n\in\mathbb{N}} \frac{G_n(dt - T_n, dx|H_n)}{G_n([t-T_n,+\infty]\times\mathbf{X}_\infty|H_n)} I_{\{T_n<t\le T_{n+1}\}} \tag{6}$$

is the predictable projection of μ with respect to $\mathbb{P}_{x_0}^u$.

Remark 2.1. Observe that $\mathcal{F}_{T_n}$ is the σ-algebra generated by the random variable H_n for $n \in \mathbb{N}$. The conditional distribution of (Θ_{n+1},X_{n+1}) given $\mathcal{F}_{T_n}$ under $\mathbb{P}_{x_0}^u$ is determined by $G_n(\cdot|H_n)$ and the conditional survival function of Θ_{n+1} given $\mathcal{F}_{T_n}$ under $\mathbb{P}_{x_0}^u$ is given by $G_n([t,+\infty]\times\mathbf{X}_\infty|H_n)$.

3. Optimization problems and assumptions.

3.1. Formulation of the optimization control problems. The objective of this section is to introduce the infinite-horizon performance criterion we are concerned with for the unconstrained and constrained cases as well as several different classes of admissible strategies. First we provide in Lemma 3.1 below the decomposition of the predictable projection ν of the process in terms of two parts: one related to the natural jumps governed by the jumps intensity λ and the other to the impulsive jumps from the boundary $\mathbf{\Xi}$ entirely governed by the stochastic kernel Q.

LEMMA 3.1. *The predictable projection of the random measure μ is given by $\nu = \nu_0 + \nu_1$, where, for $\Gamma \in \mathcal{B}(\mathbb{R}_+^* \times \mathbf{X})$,*

$$\nu_0(\Gamma) = \int_\Gamma \int_{\mathbf{A}(\xi_s)} Q(dx|\xi_s, a)\lambda(\xi_s, a)\pi(da|s)ds,$$
$$\nu_1(\Gamma) = \int_\Gamma \sum_{n\in\mathbb{N}^*} I_{\{\xi_{T_n-}\in\mathbf{\Xi}\}} \int_{\mathbf{A}(\xi_{T_n-})} Q(dx|\xi_{T_n-}, a)\gamma(da|T_n-)\delta_{T_n}(ds).$$

Proof. First observe that by using the integration by parts formula, we obtain that

$$G_n([t, +\infty] \times \mathbf{X}_\infty|h_n) = \delta_{x_n}(\{x_\infty\}) + \delta_{x_n}(\mathbf{X})e^{-\Lambda_n^u(h_n,t)}I_{\{t^*(x_n)\geq t\}}.$$

Now, recalling the definition of ν (see (6)) in terms of G (see (4)), a straightforward calculation gives the result. ◻

We consider in this paper optimization problems without constraints and with p constraints for $p \in \mathbb{N}^*$. We will use the index $j = 0$ for the performance cost and $j \in \mathbb{N}_p^*$ for the constraints, with all of them given in terms of infinite-horizon discounted criteria. The cost rate C_j^g associated with a gradual action is a real-valued measurable mapping defined on $\mathbf{K}^g$ and the cost C_j^i associated with an impulsive action on the boundary $\mathbf{\Xi}$ is a real-valued measurable mapping defined on $\mathbf{K}^i$ for any $j \in \mathbb{N}_p$. The associated infinite-horizon discounted criteria corresponding to an admissible control strategy $u \in \mathcal{U}$ are defined, for $j \in \mathbb{N}_p$, by

$$\begin{aligned}\mathcal{V}_j(u, x_0) =& \mathbb{E}_{x_0}^u\left[\int_{]0,+\infty[} e^{-\alpha s}\int_{\mathbf{A}(\xi_s)} C_j^g(\xi_s, a)\pi(da|s)ds\right]\\ &+ \mathbb{E}_{x_0}^u\left[\int_{]0,+\infty[} e^{-\alpha s}I_{\{\xi_{s-}\in\mathbf{\Xi}\}}\int_{\mathbf{A}(\xi_{s-})} C_j^i(\xi_{s-}, a)\gamma(da|s)\mu(ds, \mathbf{X})\right]\end{aligned} \tag{7}$$

for any $j \in \mathbb{N}_p$. In the previous expression, $\alpha > 0$ is the discount factor, and $\mathcal{V}_j(u, x_0)$ is understood to be equal to $+\infty$ if the integrals of both the positive and negative parts of the integrand are infinite. Note that for any control strategy $u \in \mathcal{U}$, the function $\mathcal{V}_j(u, \cdot)$ is measurable.

DEFINITION 3.2. *The optimization problem without constraint consists in minimizing the performance criterion $\mathcal{V}_0(u, x_0)$ within the class of admissible strategies $u \in \mathcal{U}$, where x_0 is the initial state.*

DEFINITION 3.3. *The optimization problem with p constraints consists in minimizing the performance criterion $\mathcal{V}_0(u, x_0)$ within the class of admissible strategies $u \in \mathcal{U}$, where x_0 is the initial state, and such that the constraint criteria $\mathcal{V}_j(u, x_0) \leq B_j$ are satisfied for any $j \in \mathbb{N}_p^*$, where $(B_j)_{j\in\mathbb{N}_p^*}$ are real numbers representing the constraint bounds.*

We introduce now several different classes of admissible strategies that will be considered throughout the paper. A control strategy $u \in \mathcal{U}$ is called

- *nonrandomized stationary* if $\pi_n(\cdot|h_n, t) = \delta_{\varphi^s(\phi(x_n,t))}(\cdot)$ and $\gamma_n(\cdot|h_n) = \delta_{\varphi^s(\phi(x_n,t))}(\cdot)$, where $\varphi^s : \overline{\mathbf{X}} \to \mathbf{A}$ is a measurable mapping satisfying $\varphi^s(y) \in \mathbf{A}(y)$ for any $y \in \overline{\mathbf{X}}$;
- *uniformly or persistently* optimal (respectively, ε-optimal for $\varepsilon > 0$) if u satisfies $\mathcal{V}_0(u, x_0) = \inf_{v \in \mathcal{U}} \mathcal{V}_0(v, x_0)$ (respectively, $\mathcal{V}_0(u, x_0) \leq \mathcal{V}_0(v, x_0) + \varepsilon$ for any $v \in \mathcal{U}$) simultaneously for all $x_0 \in \mathbf{X}$ and hence for any initial distribution;
- *stationary* if for some $(\pi, \gamma) \in \mathcal{P}^g \times \mathcal{P}^i$ the control strategy $u = (\pi_n, \gamma_n)_{n \in \mathbb{N}}$ is given by $\pi_n(da|h_n, t) = \pi(da|\phi(x_n, t))$ when $x_n \neq x_\infty$ and $\gamma_n(db|h_n) = \gamma(db|\phi(x_n, t^*(x_n)))$ when $x_n \neq x_\infty$ and $t^*(x_n) < \infty$; in case $x_n = x_\infty$ or $t^*(x_n) = \infty$, $\pi_n(\cdot|h_n, t)$ and $\gamma_n(\cdot|h_n)$ are arbitrarily fixed, by a slight abuse of notation, such strategy will be denoted by $u = (\pi, \gamma)$, and the set of stationary control strategies will be denoted by $\mathcal{U}_s$;
- *feasible* if $u \in \mathcal{U}$ and $\mathcal{V}_j(u, x_0) \leq B_j$ for any $j \in \mathbb{N}_p^*$; the set of feasible control strategies is denoted by $\mathcal{U}^f$;
- *feasible stationary* if $u \in \mathcal{U}^f \cap \mathcal{U}_s$. The set of feasible stationary control strategies will be written as $\mathcal{U}_s^f$.

3.2. Assumptions. In this subsection we present a list of assumptions that we will consider in the paper. Assumption A will be mainly used to show that the process is nonexplosive and to provide an upper bound for the sum of the expected values of $e^{-\alpha T_n}$. Assumptions B, C, and D will be mainly required to obtain the existence of an optimal or δ-optimal selectors for the problem.

Assumption A. There are constants $K \geq 0, \varepsilon_1 > 0$, and $\varepsilon_2 \in [0, 1[$ such that the following hold:

(A1) For any $(x, a) \in \mathbf{K}^g$, $\lambda(x, a) \leq K$.

(A2) For any $(z, b) \in \mathbf{K}^i$, $Q(A_{\varepsilon_1}|z, b) \geq 1 - \varepsilon_2$, where

$$A_{\varepsilon_1} = \{x \in \mathbf{X} : t^*(x) > \varepsilon_1\}.$$

(A3) There exist $b \in \mathbb{B}(\mathbf{X})$, $c > -\alpha$, and a positive function $v \in \mathbb{A}(\overline{\mathbf{X}})$ such that the following inequalities are satisfied:

$$\mathcal{X}v(x) + cv(x) - \lambda(x, a)\Big[v(x) - Qv(x, a)\Big] \leq b(x), \tag{8}$$

$$\lambda(x, a) + \frac{1}{c + \alpha} b(x) \leq v(x) \tag{9}$$

for some $\mathcal{X}v$ and for any $(x, a) \in \mathbf{K}^g$, and for any $(z, a) \in \mathbf{K}^i$

$$v(z) \geq Qv(z, a) + c + \alpha. \tag{10}$$

Assumption B.

(B1) The set $\mathbf{A}(y)$ is compact for every $y \in \overline{\mathbf{X}}$.

(B2) The kernel Q is weakly continuous (also called the weak-Feller Markov kernel).

(B3) The function λ is continuous on $\mathbf{K}^g$.

(B4) The flow ϕ is continuous on $\mathbb{R}^d \times \mathbb{R}_+$.

(B5) The function t^* is continuous on $\overline{\mathbf{X}}$.

Assumption C.

(C1) The multifunction Ψ^g from $\mathbf{X}$ to $\mathbf{A}$ defined by $\Psi^g(x) = \mathbf{A}(x)$ is upper semicontinuous. The multifunction Ψ^i from $\mathbf{\Xi}$ to $\mathbf{A}$ defined by $\Psi^i(z) = \mathbf{A}(z)$ is upper semicontinuous.

(C2) The cost function C_0^g (respectively, C_0^i) is bounded and lower semicontinuous on $\mathbf{K}^g$ (respectively, $\mathbf{K}^i$).

Assumption D.

(D1) The multifunction Ψ^g from $\mathbf{X}$ to $\mathbf{A}$ defined by $\Psi^g(x) = \mathbf{A}^g(x)$ is lower semicontinous. The multifunction Ψ^i from $\boldsymbol{\Xi}$ to $\mathbf{A}$ defined by $\Psi^i(z) = \mathbf{A}^i(z)$ is lower semicontinous.

(D2) The cost function C_0^g (respectively, C_0^i) is bounded and upper semicontinuous on $\mathbf{K}^g$ (respectively, $\mathbf{K}^i$).

Without loss of generality, we assume that in the case Assumptions (A1) and (C2) or (D2) are satisfied, inequalities $|C_0^g| \leq K$ and $|C_0^i| \leq K$ are valid, where K is the constant from (A1).

4. Preliminary results. In this section we establish some preliminary results that will be needed throughout the paper. We start with Lemma 4.1 by showing that the process is nonexplosive and provide an upper bound for the sum of the expected values of $e^{-\alpha T_n}$. A key result for the unconstrained as well as the constrained cases will be the discounted version of the so-called Dynkin formula associated with the controlled process, proved in Lemma 4.2.

LEMMA 4.1. *Suppose Assumptions* (A1) *and* (A2) *or* (A1) *and* (A3) *are satisfied. Then there exists $M < \infty$ such that for any control strategy $u \in \mathcal{U}$ and for any $x_0 \in \mathbf{X}$*

$$\mathbb{E}_{x_0}^u\Big[\sum_{n\in\mathbb{N}^*} e^{-\alpha T_n}\Big] \leq M \ \textit{and}\ \mathbb{P}_{x_0}^u(T_\infty < +\infty) = 0.$$

Proof. (a) Suppose (A1) and (A2) hold. An easy calculation gives

$$\mathbb{P}_{x_0}^u(\Theta_{n+1} > \varepsilon_1 | H_{n-1}, \Theta_n) = \int_{\mathbf{X}_\infty} \mathbb{P}_{x_0}^u(\Theta_{n+1} > \varepsilon_1 | H_{n-1}, \Theta_n, x_n)\mathbb{P}_{x_0}^u(dx_n | H_{n-1}, \Theta_n).$$

Recalling the conditional distribution of (Θ_{n+1}, X_{n+1}) given $\mathcal{F}_{T_n}$ under $\mathbb{P}_{x_0}^u$ as defined in (4), it follows that

$$\begin{aligned}
&\mathbb{P}_{x_0}^u(\Theta_{n+1} > \varepsilon_1 | H_{n-1}, \Theta_n, x_n)\\
&= \Big[I_{\{x_\infty\}}(x_n) + I_{\mathbf{X}\times\{\infty\}}(x_n, t^*(x_n))e^{-\Lambda_n^u((H_{n-1},\Theta_n,x_n),\infty)}\Big]\\
&\quad + I_{\mathbf{X}}(x_n)\Big[I_{]\varepsilon_1,\infty[}(t^*(x_n))e^{-\Lambda_n^u((H_{n-1},\Theta_n,x_n),t^*(x_n))}\\
&\quad + \int_{]0,t^*(x_n)[} I_{]\varepsilon_1,\infty[}(t)\lambda_n^u((H_{n-1},\Theta_n,x_n),t)e^{-\Lambda_n^u((H_{n-1},\Theta_n,x_n),t)}dt\Big]\\
&= I_{\{x_\infty\}}(x_n) + I_{A_{\varepsilon_1}}(x_n)e^{-\Lambda_n^u((H_{n-1},\Theta_n,x_n),\varepsilon_1)} \qquad (11)
\end{aligned}$$

for any $x_n \in \mathbf{X}_\infty$ and that the conditional distribution of X_n given $\sigma\{H_{n-1}, \Theta_n\}$ satisfies

$$\begin{aligned}
\mathbb{P}_{x_0}^u(dx_n | H_{n-1}, \Theta_n) =& I_{\{\Theta_n=\infty\}}\delta_{\{x_\infty\}}(dx_n) + I_{\{\Theta_n<\infty\}}\Big[I_{\{\Theta_n=t^*(X_{n-1})\}}Q_{n-1}^{i,u}(dx_n|H_{n-1})\\
&+ I_{\{\Theta_n<t^*(X_{n-1})\}}Q_{n-1}^{g,u}(dx_n|H_{n-1},\Theta_n)\Big].
\end{aligned}$$

Consequently,

$$\mathbb{P}^u_{x_0}(\Theta_{n+1} > \varepsilon_1 | H_{n-1}, \Theta_n) = I_{\{\Theta_n=\infty\}} + I_{\{\Theta_n<\infty\}}$$
$$\times\Bigg(I_{\{\Theta_n<t^*(X_{n-1})\}}\int_{A_{\varepsilon_1}}\int_{\mathbf{A}^g} Q(dy|\phi(X_{n-1},\Theta_n),a)\pi_{n-1}(da|H_{n-1},\Theta_n)e^{-\Lambda^u_n((H_{n-1},\Theta_n,y),\varepsilon_1)}$$
$$+ I_{\{\Theta_n=t^*(X_{n-1})\}}\int_{A_{\varepsilon_1}}\int_{\mathbf{A}^i} Q(dy|\phi(X_{n-1},\Theta_n),a)\gamma_{n-1}(da|H_{n-1})e^{-\Lambda^u_n((H_{n-1},\Theta_n,y),\varepsilon_1)}\Bigg).$$

By using Assumptions (A1) and (A2), we obtain that for any $j \in \mathbb{N}$

$$\begin{aligned}\mathbb{P}^u_{x_0}(\Theta_{j+2} > \varepsilon_1 | H_j, \Theta_{j+1}) &\geq e^{-K\varepsilon_1}(1-\varepsilon_2) I_{\{\Theta_{j+1}<\infty\}} I_{\{\Theta_{j+1}=t^*(X_j)\}} \\ &\geq e^{-K\varepsilon_1}(1-\varepsilon_2) I_{\{X_j\in\mathbf{X}\setminus A_{\varepsilon_1}\}} I_{\{\Theta_{j+1}=t^*(X_j)\}} \end{aligned} \tag{12}$$

since $\{X_j \in \mathbf{X} \setminus A_{\varepsilon_1}\} \subset \{\Theta_{j+1} < \infty\}$. Now, by using Assumption (A1)

$$\mathbb{P}^u_{x_0}(\Theta_{j+1}=t^*(X_j)|H_j) I_{\{X_j\in\mathbf{X}\setminus A_{\varepsilon_1}\}} = e^{-\Lambda^u_j(H_j,t^*(X_j))} I_{\{X_j\in\mathbf{X}\setminus A_{\varepsilon_1}\}} \geq e^{-K\varepsilon_1} I_{\{X_j\in\mathbf{X}\setminus A_{\varepsilon_1}\}}. \tag{13}$$

Combining the previous equations (12)–(13), it follows that

$$\begin{aligned}\mathbb{P}^u_{x_0}(\Theta_{j+2}+\Theta_{j+1} > \varepsilon_1 | H_j) &\geq \mathbb{P}^u_{x_0}(\{\Theta_{j+2} > \varepsilon_1\} \cap \{\Theta_{j+1} = t^*(X_j)\} | H_j) \\ &\geq e^{-K\varepsilon_1}(1-\varepsilon_2)\mathbb{P}^u_{x_0}(\Theta_{j+1} = t^*(X_j)|H_j) I_{\{X_j\in\mathbf{X}\setminus A_{\varepsilon_1}\}} \\ &\geq e^{-2K\varepsilon_1}(1-\varepsilon_2) I_{\{X_j\in\mathbf{X}\setminus A_{\varepsilon_1}\}}. \end{aligned} \tag{14}$$

Moreover, from (11) and Assumption (A1)

$$\begin{aligned}\mathbb{P}^u_{x_0}(\Theta_{j+1} > \varepsilon_1 | H_j) &\geq e^{-\Lambda^u_j(H_j,\varepsilon_1)} I_{\{X_j\in A_{\varepsilon_1}\}} \geq e^{-K\varepsilon_1} I_{\{X_j\in A_{\varepsilon_1}\}} \\ &\geq e^{-2K\varepsilon_1}(1-\varepsilon_2) I_{\{X_j\in A_{\varepsilon_1}\}}, \end{aligned}$$

implying

$$\mathbb{P}^u_{x_0}(\Theta_{j+2}+\Theta_{j+1} > \varepsilon_1 | H_j) \geq \mathbb{P}^u_{x_0}(\Theta_{j+1} > \varepsilon_1 | H_j) \geq e^{-2K\varepsilon_1}(1-\varepsilon_2) I_{\{X_j\in A_{\varepsilon_1}\}}. \tag{15}$$

It is clear that

$$\mathbb{P}^u_{x_0}(\Theta_{j+2}+\Theta_{j+1} > \varepsilon_1 | H_j) \geq I_{\{X_j=x_\infty\}} \geq e^{-2K\varepsilon_1}(1-\varepsilon_2) I_{\{X_j=x_\infty\}}. \tag{16}$$

Finally, combining (14)–(16),

$$\mathbb{P}^u_{x_0}(\Theta_{j+2}+\Theta_{j+1} > \varepsilon_1 | H_j) \geq e^{-2K\varepsilon_1}(1-\varepsilon_2). \tag{17}$$

Now, for any control strategy u, for any $x_0 \in \mathbf{X}$ we have for any $j \in \mathbb{N}$

$$\begin{aligned}&\mathbb{E}^u_{x_0}\left[e^{-\alpha(\Theta_{j+1}+\Theta_{j+2})}|H_j\right] \\ &\quad\leq \mathbb{P}^u_{x_0}(\Theta_{j+1}+\Theta_{j+2} \leq \varepsilon_1 | H_j) + e^{-\alpha\varepsilon_1}\mathbb{P}^u_{x_0}(\Theta_{j+1}+\Theta_{j+2} > \varepsilon_1 | H_j) \\ &\quad= 1 + [e^{-\alpha\varepsilon_1} - 1]\mathbb{P}^u_{x_0}(\Theta_{j+1}+\Theta_{j+2} > \varepsilon_1 | H_j) \\ &\quad\leq 1 + [e^{-\alpha\varepsilon_1} - 1][1-\varepsilon_2]e^{-2K\varepsilon_1} = \kappa < 1. \end{aligned} \tag{18}$$

For any $j \in \mathbb{N}^*$,

$$\mathbb{E}^u_{x_0}\left[e^{-\alpha T_{2j+1}}\right] = \mathbb{E}^u_{x_0}\left[e^{-\alpha T_{2j-1}}\mathbb{E}^u_{x_0}\left[e^{-\alpha(\Theta_{2j}+\Theta_{2j+1})}|H_{2j-1}\right]\right] \leq \kappa \mathbb{E}^u_{x_0}\left[e^{-\alpha T_{2j-1}}\right].$$

Therefore,

$$\mathbb{E}^u_{x_0}\left[e^{-\alpha T_{2j+1}}\right] \leq \kappa^j \mathbb{E}^u_{x_0}\left[e^{-\alpha T_1}\right] \leq \kappa^j$$

and similarly,

$$\mathbb{E}^u_{x_0}\left[e^{-\alpha T_{2j+2}}\right] \leq \kappa^j \mathbb{E}^u_{x_0}\left[e^{-\alpha T_2}\right] \leq \kappa^j$$

for any $j \in \mathbb{N}$. Therefore,

$$\mathbb{E}^u_{x_0}\Big[\sum_{n\in\mathbb{N}^*} e^{-\alpha T_n}\Big] \leq \frac{2}{1-\kappa},$$

and we obtain the statements since if $\mathbb{P}^u_{x_0}(T_\infty < +\infty) > 0$, then $\mathbb{E}^u_{x_0}\left[\sum_{n\in\mathbb{N}} e^{-\alpha T_n}\right] = \infty$.

(b) Suppose now that (A1) and (A3) are satisfied. In this case the proof is similar to the one in Proposition 5.7 of [7]. ∎

The following lemma provides a discounted version of the so-called Dynkin formula associated with the controlled process $(\xi_t)_{t\in\mathbb{R}_+}$.

Lemma 4.2. *Let the assumptions of Lemma* 4.1 *be satisfied. Suppose a strategy* $u = (\pi_n, \gamma_n)_{n\in\mathbb{N}} \in \mathcal{U}$ *is fixed. Then we have for any* $(W, \mathcal{X}W) \in \mathbb{A}(\overline{\mathbf{X}}) \times \mathbb{B}(\mathbf{X})$ *that*

$$\begin{aligned}
0 = W(\xi_0) &+ \mathbb{E}^u_{x_0}\left[\int_{]0,\infty[} e^{-\alpha s}\big[\mathcal{X}W(\xi_s) - \alpha W(\xi_s)\big]ds\right] \\
&+ \mathbb{E}^u_{x_0}\left[\int_{]0,\infty[}\int_{\mathbf{X}} e^{-\alpha s}\Big[W(y) - W(\xi_s)\Big]\int_{\mathbf{A}(\xi_s)} Q(dy|\xi_s,a)\lambda(\xi_s,a)\pi(da|s)ds\right] \\
&+ \mathbb{E}^u_{x_0}\Bigg[\sum_{n\in\mathbb{N}^*} I_{\{\xi_{T_n-}\in\mathbf{\Xi}\}} e^{-\alpha T_n}\int_{\mathbf{X}}\Big[W(y) - W(\xi_{T_n-})\Big] \\
&\qquad \times \int_{\mathbf{A}(\xi_{T_n-})} Q(dy|\xi_{T_n-},a)\gamma(da|T_n-)\Bigg]. \qquad (19)
\end{aligned}$$

Proof. First, observe that from Lemma 4.1, we have that $\mathbb{P}^u_{x_0}(T_\infty = +\infty) = 1$. Since $W \in \mathbb{A}(\bar{\mathbf{X}})$, then any measurable function $\mathcal{X}W$ satisfies $W(\phi(x,t)) - W(x) = \int_{]0,t]} \mathcal{X}W(\phi(x,s))ds$ for any $x \in \mathbf{X}$ and $t \in [0, t^*(x)[$. Consequently, by using the product formula for functions of bounded variation (see, for example, Theorem A.4.6 in [20]) and since $\mathbb{P}^u_{x_0}(T_\infty = +\infty) = 1$, we have for any $t \in \mathbb{R}_+$

$$\begin{aligned}
e^{-\alpha t}W(\xi_t) = W(\xi_0) &+ \int_{]0,t]} e^{-\alpha s}\big[\mathcal{X}W(\xi_s) - \alpha W(\xi_s)\big]ds \\
&+ \int_{]0,t]\times\mathbf{X}} e^{-\alpha s}\Big[W(z) - W(\xi_{s-})\Big]\mu(ds,dz). \qquad (20)
\end{aligned}$$

However, it is easy to see that $\int_{]0,\infty[\times\mathbf{X}} e^{-\alpha s}\mu(ds,dz) = \sum_{n\in\mathbb{N}^*} e^{-\alpha T_n}$ and so, combining Lemma 4.1 and the bounded convergence theorem, and recalling that W is bounded, it follows that

$$\lim_{t\to\infty} \mathbb{E}^u_{x_0}\bigg[\int_{]0,t]\times\mathbf{X}} e^{-\alpha s}\Big[W(z)-W(\xi_{s-})\Big]\mu(ds,dz)\bigg]$$
$$= \mathbb{E}^u_{x_0}\bigg[\int_{]0,\infty[\times\mathbf{X}} e^{-\alpha s}\big[W(z)-W(\xi_{s-})\big]\mu(ds,dz)\bigg] < \infty.$$

Now, by using the fact that ν is the predictable projection of μ, it yields that

$$\lim_{t\to\infty} \mathbb{E}^u_{x_0}\bigg[\int_{]0,t]\times\mathbf{X}} e^{-\alpha s}\Big[W(z)-W(\xi_{s-})\Big]\mu(ds,dz)\bigg]$$
$$= \mathbb{E}^u_{x_0}\Bigg[\int_{]0,\infty[\times\mathbf{X}} e^{-\alpha s}\Big[W(z)-W(\xi_{s-})\Big]\nu(ds,dz)\Bigg].$$

Consequently, taking the expectation with respect to $\mathbb{P}^u_{x_0}$ in (20) and passing to the limit as t tends to infinity we obtain that

$$\begin{aligned}0 =&\mathbb{E}^u_{x_0}\big[W(\xi_0)\big] + \mathbb{E}^u_{x_0}\bigg[\int_{]0,\infty[} e^{-\alpha s}\big[\mathcal{X}W(\xi_s)-\alpha W(\xi_s)\big]ds\bigg]\\ &+ \mathbb{E}^u_{x_0}\bigg[\int_{]0,\infty[\times\mathbf{X}} e^{-\alpha s}\Big[W(z)-W(\xi_{s-})\Big]\nu(ds,dz)\bigg]\end{aligned}$$

since W and $\mathcal{X}W$ are bounded. We obtain the result by using Lemma 3.1. ∎

The following corollary will be useful in the proof of Proposition 5.6.

COROLLARY 4.3. *Let assumptions of Lemma* 4.1 *be satisfied and a strategy* $u = (\pi_n, \gamma_n) \in \mathcal{U}$ *be fixed. Suppose the cost functions* C^g_0 *and* C^i_0 *are bounded (below or above). Then we have for any* $(W, \mathcal{X}W) \in \mathbb{A}(\overline{\mathbf{X}}) \times \mathbb{B}(\mathbf{X})$ *that*

$$\begin{aligned}\mathcal{V}_0(u,x_0) &= W(x_0) + \mathbb{E}^u_{x_0}\bigg[\int_{]0,+\infty[} e^{-\alpha s}\big[\mathcal{X}W(\xi_s)-\alpha W(\xi_s)\big]ds\bigg]\\ &+ \mathbb{E}^u_{x_0}\bigg[\int_{]0,+\infty[} e^{-\alpha s}\int_{\mathbf{A}^g}\{C^g_0(\xi_s,a) + \int_{\mathbf{X}} W(y)Q(dy|\xi_s,a)\lambda(\xi_s,a)\\ &\qquad - W(\xi_s)\lambda(\xi_s,a)\}\pi(da|s)]ds\bigg]\\ &+ \mathbb{E}^u_{x_0}\bigg[\sum_{n\in\mathbb{N}^*} I_{\{\xi_{T_n-}\in\mathbf{\Xi}\}}e^{-\alpha T_n}\bigg[\int_{\mathbf{A}^i}\{C^i_0(\xi_{T_n-},a)\\ &\qquad + \int_{\mathbf{X}} W(y)Q(dy|\xi_{T_n-},a)\}\gamma(da|T_n-) - W(\xi_{T_n-})\bigg]\bigg].\end{aligned} \tag{21}$$

Proof. This is a straightforward consequence of Lemma 4.2 and the definition of the cost function $\mathcal{V}_0(u,x_0)$ as in (7). ∎

5. The unconstrained problem and the dynamic programming approach. In this section we present our main results concerned with the unconstrained case. We provide in Theorem 5.5 sufficient conditions based on the three local characteristics of the process ϕ, λ, Q, and the semicontinuity properties of the set valued action space, for the existence of a solution for an integro-differential BHJ optimality equation associated with the problem as well as conditions for the existence of an optimal selector or

δ-optimal selector for this equation. In what follows this result is used in Proposition 5.6 to show that the solution of the integro-differential BHJ optimality equation is in fact unique and coincides with the optimal value for the unconstrained problem and also solves the DLP. Moreover, the optimal selector (respectively, δ-optimal selector) derived in Theorem 5.5 yields an optimal (respectively, ϵ-optimal) stationary nonrandomized strategy for the unconstrained problem. But before showing these results we need four auxiliary results presented in Lemmas 5.1, 5.2, 5.3, and 5.4.

LEMMA 5.1. *Consider a bounded* $\mathbb{R}$*-valued measurable function* F *(respectively,* G*) defined on* $\mathbf{X}$ *(respectively,* Ξ*) and a real number* $\beta > 0$*. Then the mapping* V *defined on* $\overline{\mathbf{X}}$ *by*

$$V(x) = \int_{[0,t^*(x)[} e^{-\beta s} F(\phi(x,s))ds + e^{-\beta t^*(x)} G(\phi(x,t^*(x)))$$

belongs to $\mathbb{A}(\overline{\mathbf{X}})$*. Moreover, there exists a bounded function* $\mathcal{X}V$ *satisfying*

$$-\beta V(x) + \mathcal{X}V(x) = -F(x)$$

for any $x \in \mathbf{X}$*. Furthermore,* $V(z) = G(z)$ *for any* $z \in \Xi$*.*

Proof. Observe that for any $x \in \mathbf{X}$, $t^*(\phi(x,t)) = t^*(x) - t$, $\phi(\phi(x,t), t^*(\phi(x,t))) = \phi(x,t^*(x))$ and $\phi(\phi(x,t),s) = \phi(x,t+s)$ for any $(t,s) \in \mathbb{R}_+^2$ with $t+s \leq t^*(x)$. Then, a straightforward calculation shows that for any $x \in \mathbf{X}$ and $t \in [0,t^*(x)[$,

$$V(\phi(x,t)) = e^{\beta t} \int_{[t,t^*(x)[} e^{-\beta s} F(\phi(x,s))ds + e^{\beta t} e^{-\beta t^*(x)} G(\phi(x,t^*(x))). \tag{22}$$

Consequently, the function $V(\phi(x,\cdot))$ is absolutely continuous on $[0,t^*(x)] \cap \mathbb{R}_+$ and so, $V \in \mathbb{A}(\overline{\mathbf{X}})$. Equation (22) implies that for any $x \in \mathbf{X}$

$$\mathcal{X}V(\phi(x,t)) = \beta V(\phi(x,t)) - F(\phi(x,t))$$

almost everywhere w.r.t. the Lebesgue measure on $[0,t^*(x)[$. This implies that $-\beta V(x) + \mathcal{X}V(x) = -F(x)$ for any $x \in \mathbf{X}$. Moreover, we have $V(z) = G(z)$ for any $z \in \Xi$, showing the result. ◻

In case Assumption (A1) is satisfied, let us introduce for any $V \in \mathbb{B}(\overline{\mathbf{X}})$ the real-valued function $\mathfrak{R}V$ defined on $\mathbf{X}$ by

$$\mathfrak{R}V(x) = \inf_{a \in \mathbf{A}(x)} \Big\{ C_0^g(x,a) + qV(x,a) + KV(x) \Big\}, \tag{23}$$

where the constant K has been introduced in Assumption (A1) and q in (1). For notational convenience, let use denote the real-valued function $\mathfrak{T}V$ defined on Ξ by

$$\mathfrak{T}V(z) = \inf_{b \in \mathbf{A}(z)} \Big\{ C_0^i(z,b) + QV(z,b) \Big\} \tag{24}$$

for any $V \in \mathbb{B}(\overline{\mathbf{X}})$.

LEMMA 5.2. *Suppose Assumptions* (A1) *and* B–C *are satisfied. If* $V \in \mathbb{L}(\overline{\mathbf{X}})$*, then* $\mathfrak{R}V \in \mathbb{L}(\mathbf{X})$ *and* $\mathfrak{T}V \in \mathbb{L}(\Xi)$*. Moreover, the function* $\mathfrak{B}V$ *defined on* $\overline{\mathbf{X}}$ *by*

$$\mathfrak{B}V(y) = \int_{[0,t^*(y)[} e^{-(K+\alpha)t} \mathfrak{R}V(\phi(y,t))dt + e^{-(K+\alpha)t^*(y)} \mathfrak{T}V(\phi(y,t^*(y))) \tag{25}$$

belongs to $\mathbb{L}(\overline{\mathbf{X}})$*.*

Proof. Consider $V \in \mathbb{L}(\overline{\mathbf{X}})$. By using hypotheses (B2)–(B3) and the fact that λ is bounded by K on $\mathbf{K}^g$, we obtain that $qV + KV \in \mathbb{L}(\mathbf{K}^g)$, and so $C_0^g + qV + KV \in \mathbb{L}(\mathbf{K}^g)$ by Assumption (C2). Therefore, combining Lemma 17.30 in [1] with Assumptions (B1) and (C1), it yields that $\mathfrak{R}V \in \mathbb{L}(\mathbf{X})$. By using the same arguments, it can be shown that $\mathfrak{T}V \in \mathbb{L}(\mathbf{\Xi})$.

Now consider $y \in \overline{\mathbf{X}}$ and a sequence $\{y_n\}_{n\in\mathbb{N}}$ in $\overline{\mathbf{X}}$ converging to y. By a slight abuse of notation, for any $y \in \mathbf{X}$, $I_{[0,t^*(y)[}(t)\ e^{-(K+\alpha)t}\ \mathfrak{R}V(\phi(y,t))$ denotes the function defined on $\mathbb{R}_+$ which is equal to $e^{-(K+\alpha)t}\mathfrak{R}V(\phi(y,t))$ on $[0, t^*(y)[$ and zero elsewhere. It can be shown easily by using the lower semicontinuity of the function $\mathfrak{R}V$ and the continuity of the flow ϕ that $\underline{\lim}_{n\to\infty} I_{[0,t^*(y_n)[}(t)\ e^{-(K+\alpha)t}\mathfrak{R}V(\phi(y_n,t)) \geq I_{[0,t^*(y)[}(t)\ e^{-(K+\alpha)t}\mathfrak{R}V(\phi(y,t))$ for any $t \in [0, t^*(y)[$. An application of Fatou's Lemma gives that

$$\underline{\lim}_{n\to\infty} \int_{[0,t^*(y_n)[} e^{-(K+\alpha)t}\mathfrak{R}V(\phi(y_n,t))dt \geq \int_{[0,t^*(y)[} e^{-(K+\alpha)t}\mathfrak{R}V(\phi(y,t))dt.$$

The case $t^*(y) = \infty$ is trivial. Now, if $t^*(y) < \infty$, then combining the lower semicontinuity of the function $\mathfrak{T}V$ with the continuity of the flow ϕ and t^* (see Assumptions (B4)–(B5)), it gives easily that

$$\underline{\lim}_{n\to\infty} e^{-(K+\alpha)t^*(y_n)}\mathfrak{T}V(\phi(y_n,t^*(y_n))) \geq e^{-(K+\alpha)t^*(y)}\mathfrak{T}V(\phi(y,t^*(y))),$$

showing the result. □

LEMMA 5.3. *Suppose Assumptions* (A1), (B2)–(B5), *and* D *hold. If* $V \in \mathbb{U}(\overline{\mathbf{X}})$, *then* $\mathfrak{R}V \in \mathbb{U}(\mathbf{X})$ *and* $\mathfrak{T}V \in \mathbb{U}(\mathbf{\Xi})$. *Moreover, the function* $\mathfrak{B}V$ *defined on* $\overline{\mathbf{X}}$ *in* (25) *belongs to* $\mathbb{U}(\overline{\mathbf{X}})$.

Proof. The proof follows exactly the same line as in the proof of Lemma 5.2 except that in order to show that $\mathfrak{R}V \in \mathbb{U}(\mathbf{X})$ and $\mathfrak{T}V \in \mathbb{U}(\mathbf{\Xi})$ for $V \in \mathbb{L}(\overline{\mathbf{X}})$, one needs to use Lemma 17.29 in [1] and we do not need Assumption (B1), that is, the compactness hypothesis of the actions sets $\big(\mathbf{A}(y)\big)_{y\in\overline{\mathbf{X}}}$. □

In case Assumptions (A1) and (A2) are satisfied, let us introduce the constants K_A and K_B satisfying

$$K_B \geq \frac{K}{1-\varepsilon_2},$$
$$K_A \geq \frac{K(1+K_B)(1-e^{-(K+\alpha)\varepsilon_1}) + (K+\alpha)(K+K_B\varepsilon_2)e^{-(K+\alpha)\varepsilon_1}}{\alpha(1-e^{-(K+\alpha)\varepsilon_1})}.$$

LEMMA 5.4. *Suppose that Assumption* (A2) *holds and that either the assumptions of Lemma* 5.2 *are satisfied and* $V \in \mathbb{L}(\overline{\mathbf{X}})$ *or the assumptions of Lemma* 5.3 *are satisfied and* $V \in \mathbb{U}(\overline{\mathbf{X}})$. *If* $|V(y)| \leq K_A I_{A_{\varepsilon_1}}(y) + (K_A + K_B) I_{A^c_{\varepsilon_1}}(y)$ *for any* $y \in \overline{\mathbf{X}}$, *then* $\mathfrak{B}V \in \mathbb{A}(\overline{\mathbf{X}})$ *and* $|\mathfrak{B}V(y)| \leq K_A I_{A_{\varepsilon_1}}(y) + (K_A + K_B) I_{A^c_{\varepsilon_1}}(y)$.

Proof. Under the assumptions, it is clear from Lemmas 5.2 and 5.3 that $\mathfrak{B}V$ as an integral of the measurable function $\mathfrak{R}V$ along the flow (see (25)) is well defined. Now, applying Lemma 5.1, it follows that $\mathfrak{B}V \in \mathbb{A}(\overline{\mathbf{X}})$ and $\mathfrak{B}V(z) = \mathfrak{T}V(z)$ for any $z \in \mathbf{\Xi}$. Observe that

$$\sup_{x\in\mathbf{X}} \big|\mathfrak{R}V(x)\big| \leq K(1+K_A+K_B). \tag{26}$$

Moreover, for $(z,b) \in \mathbf{K}^i$, we have

$$\big|QV(z,b)\big| \leq \int_{\mathbf{X}} I_{A_{\varepsilon_1}}(y)\big|V(y)\big|Q(dy|z,b) + \int_{\mathbf{X}} I_{A^c_{\varepsilon_1}}(y)\big|V(y)\big|Q(dy|z,b) \leq K_A + K_B\varepsilon_2,$$

and so

$$\sup_{z\in\mathbf{\Xi}} \big|\mathfrak{B}V(z)\big| = \sup_{z\in\mathbf{\Xi}} \big|\mathfrak{T}V(z)\big| \leq K + K_A + K_B\varepsilon_2 \leq K_A + K_B,$$

where for the last inequality we have used the definition of K_B. We study now three different cases:

(a) For $x \in A^c_{\varepsilon_1} \cap \mathbf{X}$, we have

$$\begin{aligned} |\mathfrak{B}V(x)| \leq{}& \frac{K(1+K_A+K_B)}{K+\alpha}\left(1-e^{-(K+\alpha)t^*(x)}\right) \\ &+\Big[K+K_A+K_B\varepsilon_2\Big]e^{-(K+\alpha)t^*(x)}. \end{aligned} \tag{27}$$

By the definition of K_A we have $K_A\alpha \geq K(1+K_B)$ and so, $\frac{K_A\alpha}{K+\alpha} > \frac{K(1+K_B)}{K+\alpha} - K - K_B\varepsilon_2$, implying

$$K + K_A + K_B\varepsilon_2 - \frac{K(1+K_A+K_B)}{K+\alpha} > 0. \tag{28}$$

Therefore, (27) gives

$$\begin{aligned} |\mathfrak{B}V(x)| &\leq K + K_A + K_B\varepsilon_2 + \Big[\frac{K(1+K_A+K_B)}{K+\alpha} \\ &\quad - \big(K+K_A+K_B\varepsilon_2\big)\Big]\left(1-e^{-(K+\alpha)t^*(x)}\right) \\ &\leq K + K_A + K_B\varepsilon_2 \leq K_A + K_B. \end{aligned}$$

(b) Suppose $x \in A_{\varepsilon_1}$ and $t^*(x) < \infty$. Then

$$\begin{aligned} \mathfrak{B}V(x) ={}& \int_{[0,t^*(x)-\varepsilon_1[} e^{-(K+\alpha)t}\mathfrak{R}V(\phi(x,t))dt \\ &+ \int_{[t^*(x)-\varepsilon_1,t^*(x)[} e^{-(K+\alpha)t}\mathfrak{R}V(\phi(x,t))dt + e^{-(K+\alpha)t^*(x)}\mathfrak{T}V(\phi(x,t^*(x))) \\ ={}& \int_{[0,t^*(x)-\varepsilon_1[} e^{-(K+\alpha)t}\mathfrak{R}V(\phi(x,t))dt + e^{-(K+\alpha)(t^*(x)-\varepsilon_1)}\mathfrak{B}V(x^*), \end{aligned} \tag{29}$$

since $t^*(x^*) = \varepsilon_1$ with $x^* = \phi(x, t^*(x)-\varepsilon_1) \in A^c_{\varepsilon_1} \cap \mathbf{X}$. Now, recalling (27) it follows that $|\mathfrak{B}V(x^*)| \leq M$ with

$$M = \frac{K(1+K_A+K_B)}{K+\alpha}\left(1-e^{-(K+\alpha)\varepsilon_1}\right) + (K+K_A+K_B\varepsilon_2)e^{-(K+\alpha)\varepsilon_1}.$$

Therefore, (29) gives

$$\begin{aligned} |\mathfrak{B}V(x)| &\leq \frac{K(1+K_A+K_B)}{K+\alpha}\left(1-e^{-(K+\alpha)(t^*(x)-\varepsilon_1)}\right) + Me^{-(K+\alpha)(t^*(x)-\varepsilon_1)} \\ &= \frac{K(1+K_A+K_B)}{K+\alpha} + e^{-(K+\alpha)t^*(x)}\Big[K+K_A+K_B\varepsilon_2 - \frac{K(1+K_A+K_B)}{K+\alpha}\Big]. \end{aligned}$$

From inequality (28) and using $t^*(x) > \varepsilon_1$, we conclude that $|\mathfrak{B}V(x)| \leq M$. However, a straightforward calculation shows that $M \leq K_A$ based on the definitions of K_A and M. Consequently, it yields $|\mathfrak{B}V(x)| \leq K_A$.

(c) Suppose $x \in A_{\varepsilon_1}$ and $t^*(x) = \infty$. Then we have

$$|\mathfrak{B}V(x)| \leq \frac{K(1 + K_A + K_B)}{K + \alpha} \leq M \leq K_A,$$

giving the result. ◻

The next theorem provides sufficient conditions for the existence of a solution for the BHJ equation associated with the optimization problem as well as conditions for the existence of an optimal selector or δ-optimal selector for this equation.

THEOREM 5.5. *Suppose the assumptions of Lemma* 4.1 *are satisfied. If either Assumptions* B *and* C *hold, or Assumptions* (B2)–(B5) *and* D *hold, then there exist* $W \in \mathbb{A}(\overline{\mathbf{X}})$ *and* $\mathcal{X}W \in \mathbb{B}(\mathbf{X})$ *satisfying*

$$-\alpha W(x) + \mathcal{X}W(x) + \inf_{a \in A^g(x)} \Big\{ C_0^g(x, a) + qW(x, a) \Big\} = 0 \tag{30}$$

for any $x \in \mathbf{X}$ *and*

$$W(z) = \inf_{b \in A^i(z)} \Big\{ C_0^i(z, b) + QW(z, b) \Big\} \tag{31}$$

for any $z \in \mathbf{\Xi}$. *Moreover the following assertions hold:*

(i) *If Assumptions* B *and* C *are satisfied, then there is a measurable mapping* $\widehat{\varphi} : \overline{\mathbf{X}} \to \mathbf{A}$ *such that* $\widehat{\varphi}(y) \in \mathbf{A}(y)$ *for any* $y \in \overline{\mathbf{X}}$ *and satisfying*

$$C_0^g(x, \widehat{\varphi}(x)) + qW(x, \widehat{\varphi}(x)) = \inf_{a \in \mathbf{A}(x)} \Big\{ C_0^g(x, a) + qW(x, a) \Big\} \tag{32}$$

for any $x \in \mathbf{X}$ *and*

$$C_0^i(z, \widehat{\varphi}(z)) + QW(z, \widehat{\varphi}(z)) = \inf_{b \in \mathbf{A}(z)} \Big\{ C_0^i(z, b) + QW(z, b) \Big\} \tag{33}$$

for any $z \in \mathbf{\Xi}$.

(ii) *If Assumptions* (B2)–(B5) *and* D *are satisfied then, for any* $\delta > 0$, *there is a measurable mapping* $\widehat{\varphi}_\delta : \overline{\mathbf{X}} \to \mathbf{A}$ *such that* $\widehat{\varphi}_\delta(y) \in \mathbf{A}(y)$ *for any* $y \in \overline{\mathbf{X}}$ *and, for all* $x \in \mathbf{X}$,

$$C_0^g(x, \widehat{\varphi}_\delta(x)) + qW(x, \widehat{\varphi}_\delta(x)) \leq \inf_{a \in \mathbf{A}(x)} \Big\{ C_0^g(x, a) + qW(x, a) \Big\} + \delta \tag{34}$$

and, for all $z \in \mathbf{\Xi}$,

$$C_0^i(z, \widehat{\varphi}_\delta(z)) + QW(z, \widehat{\varphi}_\delta(z)) \leq \inf_{b \in \mathbf{A}(z)} \Big\{ C_0^i(z, b) + QW(z, b) \Big\} + \delta. \tag{35}$$

Proof. Suppose first that Assumptions B and C hold. By Lemma 5.2, one can define recursively the sequence of functions $\big\{W_i\big\}_{i \in \mathbb{N}}$ in $\mathbb{L}(\overline{\mathbf{X}})$ as follows: $W_{i+1}(y) = \mathfrak{B}W_i(y)$ for $i \in \mathbb{N}$ and $W_0(y) = -K_A I_{A_{\varepsilon_1}}(y) - (K_A + K_B) I_{A_{\varepsilon_1}^c}(y)$ for any $y \in \overline{\mathbf{X}}$. By using Lemma 5.4 and the definition of W_0, we obtain that $W_1(y) \geq W_0(y)$ for any $y \in \overline{\mathbf{X}}$. Now, note that the operator $\mathfrak{B}$ is monotone, that is, $V_1 \leq V_2$ implies

$\mathfrak{B}V_1 \leq \mathfrak{B}V_2$. Consequently, it can be shown by induction that the sequence $\{W_i\}_{i\in\mathbb{N}}$ is increasing. Using again Lemma 5.4, this sequence is uniformly bounded, that is, $\sup_{y\in\overline{\mathbf{X}}} |W_i(y)| \leq K_A + K_B$ for any $i \in \mathbb{N}$. As a result, $\{W_i\}_{i\in\mathbb{N}}$ converges to a mapping $W \in \mathbb{B}(\overline{\mathbf{X}})$. Since $\{W_i\}_{i\in\mathbb{N}}$ is an increasing sequence of lower semicontinuous functions, $W \in \mathbb{L}(\overline{\mathbf{X}})$, $KW_i + qW_i \in \mathbb{L}(\mathbf{K}^g)$, and so, $C_0^g + KW_i + qW_i \in \mathbb{L}(\mathbf{K}^g)$ by Assumption (C2). Therefore, combining Assumptions (B1) and (C1) and Lemma 2.1 in [22], it follows that $\lim_{i\to\infty} \mathfrak{R}W_i(x) = \mathfrak{R}W(x)$ for any $x \in \mathbf{X}$ and $\lim_{i\to\infty} \mathfrak{T}W_i(z) = \mathfrak{T}W(z)$ for any $z \in \mathbf{\Xi}$. By using the bounded convergence theorem, it implies that the mapping W satisfies the following equations:

$$
\begin{aligned}
W(y) &= \mathfrak{B}W(y) \\
&= \int_{[0,t^*(y)[} e^{-(K+\alpha)t} \mathfrak{R}W(\phi(y,t))dt + e^{-(K+\alpha)t^*(y)} \mathfrak{T}W(\phi(y,t^*(y))),
\end{aligned} \tag{36}
$$

where $y \in \overline{\mathbf{X}}$. Applying Lemma 5.1 to the mapping W where the function F (respectively, G) is given by $\mathfrak{R}W$ (respectively, $\mathfrak{T}W$), it yields that the function $W \in \mathbb{A}(\overline{\mathbf{X}})$ and satisfies

$$
-(\alpha + K)W(x) + \mathcal{X}W(x) = - \inf_{a\in A^g(x)} \Big\{ C_0^g(x,a) + qW(x,a) - KW(x) \Big\}
$$

for any $x \in \mathbf{X}$ and

$$
W(z) = \inf_{b\in A^i(z)} \Big\{ C_0^i(z,b) + QW(z,b) \Big\}
$$

for any $z \in \mathbf{\Xi}$.

Now, if Assumptions (B2)–(B5) and D hold, then by using Lemma 5.3, one can define recursively the sequence of functions $\{V_i\}_{i\in\mathbb{N}}$ in $\mathbb{U}(\overline{\mathbf{X}})$ as follows: $V_{i+1}(y) = \mathfrak{B}V_i(y)$ for $i \in \mathbb{N}$ and $V_0(y) = K_A I_{A_{\varepsilon_1}}(y) + (K_A + K_B) I_{A^c_{\varepsilon_1}}(y)$ for any $y \in \overline{\mathbf{X}}$. By using Lemma 5.4 and the definition of V_0, we obtain that $V_1(y) \leq V_0(y)$ for any $y \in \overline{\mathbf{X}}$. Consequently, it can be shown by induction that $\{V_i\}_{i\in\mathbb{N}}$ is a decreasing sequence of upper semicontinuous functions converging to a mapping $V \in \mathbb{U}(\overline{\mathbf{X}})$. At this point, the same arguments can be used to show that the mapping V satisfies (36) and again applying Lemma 5.1 we get the result.

Now, under Assumptions B and C, for any $x \in \mathbf{X}$ the mapping defined on $\mathbf{A}(x)$ by

$$
a \to C_0^g(x,a) + \lambda(x,a)\big[QW(x,a) - W(x)\big]
$$

is lower semicontinuous and since Ψ^g is upper semicontinuous, it follows from Proposition D.5 in [17] that there exists a measurable mapping $\varphi^g : \mathbf{X} \to \mathbf{A}^g$ such that for all $x \in \mathbf{X}$ $\varphi^g(x) \in \mathbf{A}(x)$ and (32) holds. Similar arguments can be used to show the existence of a measurable mapping $\varphi^i : \mathbf{\Xi} \to \mathbf{A}^i$ satisfying $\varphi^i(z) \in \mathbf{A}(z)$ for any $z \in \mathbf{\Xi}$ and (33). Therefore, the measurable mapping $\widehat{\varphi}$ defined by $\widehat{\varphi}(x) = \varphi^i(x)$ for any $x \in \mathbf{X}$ and $\widehat{\varphi}(z) = \varphi^i(z)$ for any $z \in \mathbf{\Xi}$ satisfies the claim.

To prove the last part, observe that under Assumptions (B2)–(B5) and D the mapping defined on $\mathbf{K}^g$ by

$$
(x,a) \to C_0^g(x,a) + \lambda(x,a)\big[QW(x,a) - W(x)\big] + KW(x)
$$

and the function defined on $\mathbf{K}^i$ by

$$
(z,b) \to C_0^i(z,b) + QW(z,b)
$$

are upper semicontinuous. Now applying Proposition 7.34 in [4], we get the result. $\square$

The following proposition shows that the solution of the BHJ equation is in fact unique and coincides with the optimal value for the unconstrained problem and also solves the DLP. Moreover this result provides the existence of an optimal (respectively, ϵ-optimal) stationary nonrandomized strategy for the unconstrained problem.

PROPOSITION 5.6. *Suppose all the conditions of Theorem* 5.5 *are satisfied. Under Assumptions* B *and* C*, the stationary nonrandomized strategy* $\widehat{\varphi}$ *as defined in item* (i) *of Theorem* 5.5 *is uniformly optimal. Under Assumptions* (B2)–(B5) *and* D*, for any* $\varepsilon > 0$ *there is* $\delta > 0$ *such that the stationary nonrandomized strategy* $\widehat{\varphi}_\delta$ *as defined in item* (ii) *of Theorem* 5.5 *is uniformly* ε*-optimal.*

Moreover the function $W \in \mathbb{A}(\overline{\mathbf{X}})$*, solution of* (30)–(31)*, is unique and coincides with* $\inf_{u\in\mathcal{U}} \mathcal{V}_0(u, x)$*. Under a fixed initial condition* $x_0 \in \mathbf{X}$*, it solves the so-called dual linear program (DLP)*

$$\sup_{V\in\left\{S\in\mathbb{A}(\overline{\mathbf{X}}):\exists \mathcal{X}S\in\mathbb{B}(\mathbf{X})\right\}} V(x_0) \tag{37}$$

subject to

$$\begin{aligned} &\mathcal{X}V(x) - \alpha V(x) + C_0^g(x,a) + qV(x,a) \geq 0,\\ &C_0^i(z,b) + QV(z,b) \geq V(z) \end{aligned} \tag{38}$$

for $(x, a) \in \mathbf{K}^g$ *and* $(z, b) \in \mathbf{K}^i$*.*

Proof. Let us denote by W a mapping in $\mathbb{A}(\overline{\mathbf{X}})$ satisfying (30)–(31). According to Corollary 4.3, for an arbitrary control strategy $u \in \mathcal{U}$ we have $\mathcal{V}_0(u, x) \geq W(x)$ for any $x \in \mathbf{X}$.

In case Assumptions B and C are satisfied, the stationary nonrandomized strategy $\widehat{\varphi}$ as defined in item (i) of Theorem 5.5 is uniformly optimal since $\mathcal{V}_0(\widehat{\varphi}, x) = W(x)$ for any $x \in \mathbf{X}$.

Suppose Assumptions (B2)–(B5) and D are satisfied. Then the stationary nonrandomized strategy $\widehat{\varphi}_\delta$ as defined in item (ii) of Theorem 5.5 satisfies, for any x and any $u \in \mathcal{U}$, the inequality

$$\mathcal{V}_0(\widehat{\varphi}_\delta, x) \leq \mathcal{V}_0(u, x) + \delta \int_{[0,\infty[} e^{-\alpha s} ds + \delta \mathbb{E}_x^u \Big[\sum_{n\in\mathbb{N}^*} I_{\{\xi_{T_n -}\}} e^{-\alpha T_n^i} \Big] \tag{39}$$

due to (21). From Lemma 4.1, it implies that $\mathcal{V}_0(\widehat{\varphi}_\delta, x) \leq \mathcal{V}_0(u, x) + \delta\big[\frac{1}{\alpha} + M\big]$ for any x and any $u \in \mathcal{U}$. By choosing δ such $\delta\big[\frac{1}{\alpha} + M\big] \leq \varepsilon$, the strategy $\widehat{\varphi}_\delta$ is uniformly ε-optimal.

Consequently, if the assumptions of Theorem 5.5 are satisfied, then $W(x) = \inf_{u\in\mathcal{U}} \mathcal{V}_0(u, x)$ is unique.

According to (21), for any function $V \in \mathbb{A}(\overline{\mathbf{X}})$ satisfying inequalities (38) we have $\mathcal{V}_0(u, x_0) \geq V(x_0)$ for any $u \in \mathcal{U}$ and for any initial value x_0, implying $\inf_{u\in\mathcal{U}} \mathcal{V}_0(u, x_0) \geq V(x_0)$. On the other hand, the function W satisfies (38) and $\inf_{u\in\mathcal{U}} \mathcal{V}_0(u, x_0) = W(x_0)$. Therefore, the function W is a solution of the DLP described by (37) and (38). ∎

Remark 5.7. (a) Note that for a fixed $x_0 \in \mathbf{X}$, the DLP can have solutions different from the function W satisfying (30)–(31) in Theorem 5.5. However, the value of all these solutions at point x_0 is the same and equal to $W(x_0)$.

(b) Suppose the assumptions of Lemma 4.1 are satisfied and function $W \in \mathbb{A}(\overline{\mathbf{X}})$ satisfies (30), (31). Moreover, assume that there is a measurable mapping $\widehat{\varphi}$ (respectively, $\widehat{\varphi}_\delta$) satisfying equalities (32), (33) (respectively, inequalities (34), (35) for any fixed $\delta > 0$). Then the stationary nonrandomized strategy $\widehat{\varphi}$ (respectively, $\widehat{\varphi}_\delta$) is uniformly optimal (respectively, uniformly ε-optimal for $\varepsilon \geq \delta(\frac{1}{\alpha}+M)$). All the other assertions of Proposition 5.6 hold true as well.

6. The constrained problem and the LP approach. In this section, we present the main results regarding the constrained case. In order to do that we need to introduce in Definition 6.5 the set of admissible finite measures, a definition that generalizes the usual definition in the continuous-time MDP case, as pointed out in Remark 6.6. A key result in this section is presented in Theorem 6.7, relating the set of admissible measures and the set of occupation measures associated with any admissible control strategy for the problem introduced in Definition 6.4. Theorem 6.14 provides the main result of this section, showing that the values of the constrained control problem and of an infinite dimensional LP problem are the same. Furthermore it gives sufficient conditions for the solvability of the LP problem and the existence of an optimal feasible control strategy for the constrained problem. A simple illustrative case of the calculation of the occupation measure by using Theorem 6.7 is presented at the end of the section. But first we need the following auxiliary result.

PROPOSITION 6.1. *Suppose the assumptions of Lemma* 4.1 *are satisfied. Consider* $v \in \mathbb{B}(\mathbf{K}^g)$ *and* $w \in \mathbb{B}(\mathbf{K}^i)$ *and a stationary control strategy* $u = (\pi, \gamma)$, *where* $(\pi,\gamma) \in \mathcal{P}^g \times \mathcal{P}^i$. *Under these conditions the function* U *defined on* $\mathbf{X}$ *by*

$$\begin{aligned} U(x) = {} & \mathbb{E}_x^u\bigg[\int_{]0,\infty[} e^{-\alpha s}\int_{\mathbf{A}^g} v(\xi_s,a)\pi(da|\xi_s)ds\bigg] \\ & + \mathbb{E}_x^u\bigg[\int_{]0,\infty[} e^{-\alpha s}\int_{\mathbf{A}^i} w(\xi_{s-},b)\gamma(db|\xi_{s-})I_{\mathbf{\Xi}}(\xi_{s-})\mu(ds,\mathbf{X})\bigg] \end{aligned} \tag{40}$$

is absolutely continuous along the flow ϕ, *that is, for any* $x \in \mathbf{X}$, $U(\phi(x,\cdot))$ *is absolutely continuous on* $[0,t^*(x)[$ *and when* $t^*(x)<\infty$, $\lim_{t\to t^*(x)} U(\phi(x,t))$ *exists (in this case,* $\lim_{t\to t^*(x)} U(\phi(x,t))$ *will be denoted by* $U(\phi(x,t^*(x)))$*). Moreover, there exists* $\mathcal{X}U \in \mathbb{B}(\mathbf{X})$ *satisfying for any* $x \in \mathbf{X}$

$$\mathcal{X}U(x) - \alpha U(x) + \int_{\mathbf{X}\times\mathbf{A}(x)} U(y)q(dy|x,a)\pi(da|x) + \int_{\mathbf{A}(x)} v(x,a)\pi(da|x) = 0, \tag{41}$$

where the signed kernel q *has been defined in* (1), *and for any* $z \in \mathbf{\Xi}$,

$$\int_{\mathbf{A}(z)} w(z,b)\gamma(db|z) + \int_{\mathbf{X}\times\mathbf{A}(z)} U(y)Q(dy|z,b)\gamma(db|z) - U(z) = 0. \tag{42}$$

Proof. This proof is similar to the one in Theorem 32.2 of [9]. ∎

Remark 6.2. According to Proposition 6.1, it can be easily seen that the domain of definition of the mapping U can be extended to $\overline{\mathbf{X}}$ by setting $U(z) = \lim_{t\to t^*(x)} U(\phi(x,t))$, where $z = \phi(x,t^*(x))) \in \mathbf{\Xi}$. By doing so, we have that $U \in \mathbb{A}(\overline{\mathbf{X}})$.

DEFINITION 6.3. *For any* $\eta \in \mathcal{M}(\mathbf{K})$ *we define* $\eta^g \in \mathcal{M}(\mathbf{K}^g)$ *and* $\eta^i \in \mathcal{M}(\mathbf{K}^i)$ *as follows:*

$$\eta^g(\Gamma_1) = \eta(\Gamma_1 \cap \mathbf{K}^g), \tag{43}$$

$$\eta^i(\Gamma_2) = \eta(\Gamma_2 \cap \mathbf{K}^i) \tag{44}$$

for any $\Gamma_1 \in \mathcal{B}(\mathbf{K}^g)$ *and* $\Gamma_2 \in \mathcal{B}(\mathbf{K}^i)$. *Since* $\mathbf{K}^g \cap \mathbf{K}^i = \emptyset$ *we have that*

$$\eta(\Gamma) = \eta^g(\Gamma \cap \mathbf{K}^g) + \eta^i(\Gamma \cap \mathbf{K}^i)$$

for any $\Gamma \in \mathcal{B}(\mathbf{K})$.

Definition 6.4. *For any admissible control strategy* $u \in \mathcal{U}$, *we introduce the occupation measure* $\eta_u \in \mathcal{M}(\mathbf{K})$ *associated with* u, *as follows:*

$$\begin{aligned}\eta_u(\Gamma) =& \mathbb{E}^u_{x_0}\left[\int_{\Gamma\cap\mathbf{K}^g}\int_{]0,\infty[} e^{-\alpha s}\delta_{\xi_s}(dx)\pi(da|s)ds\right]\\ &+ \mathbb{E}^u_{x_0}\left[\int_{\Gamma\cap\mathbf{K}^i}\sum_{n\in\mathbb{N}^*} e^{-\alpha T_n}\delta_{\xi_{T_n-}}(dz)\gamma(db|T_n-)\right]\end{aligned} \tag{45}$$

for any $\Gamma \in \mathcal{B}(\mathbf{K})$.

From Definition 6.3 and (45) it follows that

$$\eta^g_u(\Gamma_{\mathbf{K}^g}) = \mathbb{E}^u_{x_0}\left[\int_{\Gamma_{\mathbf{K}^g}}\int_{]0,\infty[} e^{-\alpha s}\delta_{\xi_s}(dx)\pi(da|s)ds\right], \tag{46}$$

$$\eta^i_u(\Gamma_{\mathbf{K}^i}) = \mathbb{E}^u_{x_0}\left[\int_{\Gamma_{\mathbf{K}^i}}\sum_{n\in\mathbb{N}^*} e^{-\alpha T_n}\delta_{\xi_{T_n-}}(dz)\gamma(db|T_n-)\right] \tag{47}$$

for any $\Gamma_{\mathbf{K}^g} \in \mathcal{B}(\mathbf{K}^g)$ and $\Gamma_{\mathbf{K}^i} \in \mathcal{B}(\mathbf{K}^i)$. Notice that $\eta^g_u(\mathbf{K}^g) = \frac{1}{\alpha}$. Moreover if the conditions of Lemma 4.1 are satisfied, then η^i is finite and, furthermore, if Assumption (C2) or (D2) holds and the functions C^g_j and C^i_j are bounded from below (or above), then $\mathcal{V}_j(u, x_0) = \eta^g_u(C^g_j) + \eta^i_u(C^i_j)$ for any $u \in \mathcal{U}$ and $j \in \mathbb{N}_p$.

Definition 6.5. *A finite measure* $\eta \in \mathcal{M}(\mathbf{K})$ *is called admissible if the following equality holds:*

$$\begin{aligned}&\int_{\mathbf{X}}\big[\alpha W(x) - \mathcal{X}W(x)\big]\widehat{\eta}^g(dx) + \int_{\mathbf{\Xi}} W(z)\widehat{\eta}^i(dz)\\ &\qquad = W(x_0) + \int_{\mathbf{K}^g} qW(x,a)\eta^g(dx,da) + \int_{\mathbf{K}^i} QW(z,b)\eta^i(dz,db)\end{aligned} \tag{48}$$

for any $(W, \mathcal{X}W) \in \mathbb{A}(\overline{\mathbf{X}}) \times \mathbb{B}(\mathbf{X})$.

Remark 6.6. Notice that for the case in which there is no flow, that is, $\phi(x,t) = x$ for all $t \in \mathbb{R}_+$ and therefore the boundary is empty, then $\mathbb{A}(\overline{\mathbf{X}}) = \mathbb{B}(\mathbf{X})$ since $\mathcal{X}W(x) = 0$ for every $W \in \mathbb{B}(\mathbf{X})$. Thus it is easy to see that in this case it is enough to consider the indicator functions $W(x) = I_{\Gamma_{\mathbf{X}}}(x)$ for $\Gamma_{\mathbf{X}} \in \mathcal{B}(\mathbf{X})$. Thus, noticing that $\eta^g = \eta$, and defining $\rho = \alpha\eta$, we have that (48) becomes

$$\widehat{\rho}(\Gamma_{\mathbf{X}}) = \delta_{x_0}(\Gamma_{\mathbf{X}}) + \frac{1}{\alpha}\rho q(\Gamma_{\mathbf{X}}), \tag{49}$$

which is the usual admissibility condition known in the literature on continuous-time MDPs (see, for instance, [21, (3.1)]).

The next important result shows the link between the set of admissible measures and the set of occupation measures.

THEOREM 6.7. *Let the conditions of Lemma* 4.1 *be satisfied. Then the following assertions hold:*

(i) *For any control strategy $u \in \mathcal{U}$, the occupation measure η_u as introduced in Definition* 6.4 *is admissible.*

(ii) *Suppose that the measure η is admissible according to Definition* 6.5 *and consider the measures η^g and η^i as defined in* (43) *and* (44). *Then there exist stochastic kernels $\pi \in \mathcal{P}^g$ and $\gamma \in \mathcal{P}^i$ satisfying*

$$\eta^g(\Gamma_{\mathbf{K}^g}) = \int_{\Gamma_{\mathbf{K}^g}} \pi(da|x)\widehat{\eta}^g(dx), \tag{50}$$

$$\eta^i(\Gamma_{\mathbf{K}^i}) = \int_{\Gamma_{\mathbf{K}^i}} \gamma(db|z)\widehat{\eta}^i(dz) \tag{51}$$

for any $\Gamma_{\mathbf{K}^g} \in \mathcal{B}(\mathbf{K}^g)$ and $\Gamma_{\mathbf{K}^i} \in \mathcal{B}(\mathbf{K}^i)$.

The stationary control strategy $u = (\pi, \gamma) \in \mathcal{U}_s$ is such that $\eta = \eta_u$, where η_u is the occupation measure associated with u according to the Definition 6.4.

Proof. For item (i), consider $W \in \mathbb{A}(\mathbf{X})$ with $\mathcal{X}W \in \mathbb{B}(\mathbf{X})$. According to (19)

$$\begin{aligned} 0 = W(x_0) &+ \mathbb{E}^u_{x_0}\left[\int_{]0,\infty[} e^{-\alpha s}\big[\mathcal{X}W(\xi_s) - \alpha W(\xi_s)\big]ds\right] \\ &+ \mathbb{E}^u_{x_0}\left[\int_{]0,\infty[} e^{-\alpha s} qW(\xi_s,a)\pi(da|s)ds\right] \\ &+ \mathbb{E}^u_{x_0}\Bigg[\sum_{n\in\mathbb{N}^*} I_{\{\xi_{T_n-}\in\mathbf{\Xi}\}} e^{-\alpha T_n} \\ &\qquad \times \int_{\mathbf{X}} \big[W(y) - W(\xi_{T_n-})\big] \int_{\mathbf{A}(\xi_{T_n-})} Q(dy|\xi_{T_n-},a)\gamma(da|T_n-)\Bigg]. \end{aligned} \tag{52}$$

Recalling the definitions of η^g_u and η^i_u (see (46) and (47)), we have that

$$\mathbb{E}^u_{x_0}\left[\int_{]0,\infty[} e^{-\alpha s}\big[\mathcal{X}W(\xi_s) - \alpha W(\xi_s)\big]ds\right] = \int_{\mathbf{X}} \big[\mathcal{X}W(x) - \alpha W(x)\big]\widehat{\eta}^g_u(dx), \tag{53}$$

$$\mathbb{E}^u_{x_0}\left[\int_{]0,\infty[} e^{-\alpha s} qW(\xi_s,a)\pi(da|s)ds\right] = \int_{\mathbf{K}^g} qW(x,a)\eta^g_u(dx,da), \tag{54}$$

$$\begin{aligned} &\mathbb{E}^u_{x_0}\left[\sum_{n\in\mathbb{N}^*} I_{\{\xi_{T_n-}\in\mathbf{\Xi}\}} e^{-\alpha T_n} \int_{\mathbf{A}(\xi_{T_n-})}\int_{\mathbf{X}} W(y)Q(dy|\xi_{T_n-},a)\gamma(da|T_n-)\right] \\ &\quad = \int_{\mathbf{K}^i} QW(z,b)\eta^i_u(dz,db), \end{aligned} \tag{55}$$

and

$$\begin{aligned} &\mathbb{E}^u_{x_0}\left[\sum_{n\in\mathbb{N}^*} I_{\{\xi_{T_n-}\in\mathbf{\Xi}\}} e^{-\alpha T_n} \int_{\mathbf{X}} W(\xi_{T_n-})\int_{\mathbf{A}(\xi_{T_n-})} Q(dy|\xi_{T_n-},a)\gamma(da|T_n-)\right] \\ &= \mathbb{E}^u_{x_0}\left[\sum_{n\in\mathbb{N}^*} I_{\{\xi_{T_n-}\in\mathbf{\Xi}\}} e^{-\alpha T_n} W(\xi_{T_n-})\right] = \int_{\mathbf{\Xi}} W(z)\widehat{\eta}^i_u(dz). \end{aligned} \tag{56}$$

Combining (52)–(56), it follows that

$$W(x_0) + \int_{\mathbf{X}} \big[\mathcal{X}W(x) - \alpha W(x)\big]\widehat{\eta}_u^g(dx) + \int_{\mathbf{K}^g} qW(x,a)\eta_u^g(dx,da) \\ + \int_{\mathbf{K}^i} QW(z,b)\eta_u^i(dz,db) - \int_{\mathbf{\Xi}} W(z)\widehat{\eta}_u^i(dz) = 0, \tag{57}$$

showing item (i).

For item (ii) suppose that η is admissible. The existence of stochastic kernels π and γ follows from Proposition D.8 in [17]. Let η_u be the occupation measure on $\mathbf{K}$ associated with u as introduced in Definition 6.4. If we show that $\eta(g) = \eta_u(g)$ for every $g \in \mathbb{B}(\mathbf{K})$, then the result follows. Let us denote by v (respectively, w) the restriction of g to $\mathbf{K}^g$ (respectively, $\mathbf{K}^i$). Considering U as in (40) with v and w previously defined, we have from Proposition 6.1 and Remark 6.2 that $U \in \mathbb{A}(\overline{\mathbf{X}})$ satisfies (41), (42), and $\mathcal{X}U \in \mathbb{B}(\mathbf{X})$. From (40) and (45) it follows that

$$\eta_u(g) = U(x_0). \tag{58}$$

Since η is admissible and $U \in \mathbb{A}(\overline{\mathbf{X}})$, $\mathcal{X}U \in \mathbb{B}(\mathbf{X})$, we have from (48) that

$$\int_{\mathbf{X}} \big[\alpha U(x) - \mathcal{X}U(x)\big]\widehat{\eta}^g(dx) + \int_{\mathbf{\Xi}} U(z)\widehat{\eta}^i(dz) \\ = U(x_0) + \int_{\mathbf{K}^g} qU(x,a)\eta^g(dx,da) + \int_{\mathbf{K}^i} QU(z,b)\eta^i(dz,db). \tag{59}$$

On the other hand, from the definitions of π and γ (see (50) and (51), respectively)

$$\eta(g) = \int_{\mathbf{K}^g} v(x,a)\pi(da|x)\widehat{\eta}^g(dx) + \int_{\mathbf{K}^i} w(z,b)\gamma(db|z)\widehat{\eta}^i(dz).$$

Now, (41) and (42) imply that

$$\eta(g) = \int_{\mathbf{X}} \big[\alpha U(x) - \mathcal{X}U(x)\big]\widehat{\eta}^g(dx) - \int_{\mathbf{K}^g} qU(x,a)\pi(da|x)\widehat{\eta}^g(dx) \\ + \int_{\mathbf{\Xi}} U(z)\widehat{\eta}^i(dz) - \int_{\mathbf{K}^i} QU(z,b)\gamma(db|z)\widehat{\eta}^i(dz)$$

and with (59) it follows that

$$\eta(g) = U(x_0). \tag{60}$$

From (58) and (60) we get that $\eta_u(g) = \eta(g) = U(x_0)$, completing the proof. ◻

It is worth noticing, from Theorem 6.7, that if the conditions of Lemma 4.1 and Assumptions (C2) or (D2) hold, and the cost functions C_j^g and C_j^i are bounded from below (or above), then any cost $\mathcal{V}_j(u, x_0)$ for $j \in \mathbb{N}_p$ corresponding to an arbitrary admissible control strategy $u \in \mathcal{U}$ can be attained by a stationary control strategy in $\mathcal{U}_s$.

Before establishing our main result of this section, we need four auxiliary results, presented in Proposition 6.8, Lemma 6.9, Proposition 6.10, and Lemma 6.11. Recalling the definition of product of kernels as introduced at the beginning of section 2, we obtain the following technical result.

PROPOSITION 6.8. *Let the conditions of Lemma* 4.1 *be satisfied. Then the occupation measure* η_u *generated by the stationary control strategy* $u = (\pi, \gamma) \in \mathcal{U}$ *where* $\pi \in \mathcal{P}^g$ *and* $\gamma \in \mathcal{P}^i$ *satisfies*

$$\eta_u^g(dx, da) = \sum_{k=0}^{\infty} H_u^k J_u(dx, da|x_0), \tag{61}$$

$$\eta_u^i(dx, da) = \sum_{k=0}^{\infty} H_u^k K_u(dz, db|x_0), \tag{62}$$

where H_u *(respectively,* J_u *and* K_u*) is the kernel on* $\mathbf{X}$ *(respectively,* $\mathbf{K}^g$ *and* $\mathbf{K}^i$*) given* $\mathbf{X}$ *defined by*

$$\begin{aligned} H_u(dy|x) = {} & e^{-\alpha t^*(x) - \int_{]0,t^*(x)[} \lambda(\phi(x,s),a)\pi(da|\phi(x,s))ds} \\ & \times \int_{\mathbf{A}(\phi(x,t^*(x)))} Q(dy|\phi(x,t^*(x)),b)\gamma(db|\phi(x,t^*(x))) \\ & + \int_{]0,t^*(x)[} \int_{\mathbf{A}(\phi(x,t))} Q(dy|\phi(x,t),a)\lambda(\phi(x,t),a)\pi(da|\phi(x,t)) \\ & \times e^{-\alpha t - \int_{]0,t[} \lambda(\phi(x,s),a)\pi(da|\phi(x,s))ds} dt, \end{aligned}$$

$$J_u(dy, da|x) = \int_{]0,t^*(x)[} e^{-\alpha t - \int_{]0,t[} \lambda(\phi(x,s),a)\pi(da|\phi(x,s))ds} \delta_{\phi(x,t)}(dy)\pi(da|\phi(x,t))dt,$$

$$K_u(dz, db|x) = e^{-\alpha t^*(x) - \int_{]0,t^*(x)[} \lambda(\phi(x,s),a)\pi(da|\phi(x,s))ds} \delta_{\phi(x,t^*(x))}(dz)\gamma(db|\phi(x,t^*(x))).$$

Proof. According to the conditional distribution of (Θ_k, X_k) given $\mathcal{F}_{T_{k-1}}$ under $\mathbb{P}_x^u$ (see (4)), it follows that

$$\begin{aligned} & \mathbb{E}_{x_0}^u\left[\int_{\Gamma_{\mathbf{K}^g}} \int_{]0,\Theta_k[} e^{-\alpha s}\delta_{\phi(X_{k-1},s)}(dx)\pi(db|\phi(X_{k-1},s))ds \Big| \mathcal{F}_{T_{k-1}}\right] \\ & \qquad = J_u(\Gamma_{\mathbf{K}^g}|X_{k-1}), \end{aligned} \tag{63}$$

$$\mathbb{E}_{x_0}^u\left[I_{\{\Theta_k = t^*(X_{k-1})\}}\Big|\mathcal{F}_{T_{k-1}}\right] = e^{-\int_{]0,t^*(X_{k-1})[} \lambda(\phi(X_{k-1},s),a)\pi(da|\phi(X_{k-1},s))ds}, \tag{64}$$

and

$$\mathbb{E}_{x_0}^u\left[e^{-\alpha\Theta_k} I_{\Gamma_{\mathbf{X}}}(X_k)\Big|\mathcal{F}_{T_{k-1}}\right] = H_u(\Gamma_{\mathbf{X}}|X_{k-1}) \tag{65}$$

for any $\Gamma_{\mathbf{K}^g} \in \mathcal{B}(\mathbf{K}^g)$ and $\Gamma_{\mathbf{X}} \in \mathcal{B}(\mathbf{X})$. Recalling the definition of η_u^g as introduced in (46), we have for any $\Gamma_{\mathbf{K}^g} \in \mathcal{B}(\mathbf{K}^g)$

$$\begin{aligned} \eta_u^g(\Gamma_{\mathbf{K}^g}) = {} & \sum_{k=1}^{\infty} \mathbb{E}_{x_0}^u\left[e^{-\alpha T_{k-1}} \int_{\Gamma_{\mathbf{K}^g}} \int_{]0,\Theta_k[} e^{-\alpha s}\delta_{\phi(X_{k-1},s)}(dx)\pi(da|\phi(X_{k-1},s))ds\right] \\ = {} & \sum_{k=1}^{\infty} \mathbb{E}_{x_0}^u\left[e^{-\alpha T_{k-1}}\mathbb{E}_{x_0}^u\left[\int_{\Gamma_{\mathbf{K}^g}} \int_{]0,\Theta_k[} e^{-\alpha s}\delta_{\phi(X_{k-1},s)}(dx)\pi(da|\phi(X_{k-1},s))ds\Big|\mathcal{F}_{T_{k-1}}\right]\right], \end{aligned}$$

and so using (63),

$$\eta_u^g(\Gamma_{\mathbf{K}^g}) = \sum_{k=1}^{\infty} \mathbb{E}_{x_0}^u \Big[e^{-\alpha T_{k-1}} J_u(\Gamma_{\mathbf{K}^g} \big| X_{k-1}) \Big]. \tag{66}$$

Now, from (65) it is easy to show by iteration that for any $k \in \mathbb{N}^*$

$$\mathbb{E}_{x_0}^u \Big[e^{-\alpha T_{k-1}} J_u(\Gamma_{\mathbf{K}^g} \big| X_{k-1}) \Big] = H_u^{k-1} J_u(\Gamma_{\mathbf{K}^g} | x_0). \tag{67}$$

Combining (66) and (67) we obtain the first part of the result (61).

Finally, using the definition of η_u^i (see (47)) we obtain that for any $\Gamma_{\mathbf{K}^i} \in \mathcal{B}(\mathbf{K}^i)$

$$\eta_u^i(\Gamma_{\mathbf{K}^i}) = \sum_{k \in \mathbb{N}^*} \mathbb{E}_{x_0}^u \Bigg[\int_{\Gamma_{\mathbf{K}^i}} e^{-\alpha T_{k-1}} e^{-\alpha t^*(X_{k-1})} \delta_{\phi(X_{k-1}, t^*(X_{k-1}))}(dz) \\ \times \gamma(db | \phi(X_{k-1}, t^*(X_{k-1}))) \mathbb{E}_{x_0}^u \Big[I_{\{\Theta_k = t^*(X_{k-1})\}} \Big| \mathcal{F}_{T_{k-1}} \Big] \Bigg].$$

Now, combining (64) and the definition of K_u we have that

$$\eta_u^i(\Gamma_{\mathbf{K}^i}) = \sum_{k \in \mathbb{N}^*} \mathbb{E}_{x_0}^u \Big[e^{-\alpha T_{k-1}} K_u(\Gamma_{\mathbf{K}^i} | X_{k-1}) \Big].$$

However, for any $k \in \mathbb{N}^*$, $\mathbb{E}_{x_0}^u \big[e^{-\alpha T_{k-1}} K_u(\Gamma_{\mathbf{K}^i} | X_{k-1}) \big] = H_u^{k-1} K_u(\Gamma_{\mathbf{K}^i} | x_0)$, showing (62) and completing the proof. ∎

LEMMA 6.9. *Suppose Assumptions* (B1)–(B2), (B4)–(B5), *and* (C1) *hold. Then for any* $f \in \mathbb{U}(\mathbf{K}^g)$, $g \in \mathbb{U}(\mathbf{X})$ *and* $h \in \mathbb{U}(\mathbf{K}^i)$, *there exist* $\widetilde{f} \in \mathbb{U}(\mathbf{X})$, $\widetilde{g} \in \mathbb{U}(\mathbf{X})$, *and* $\widetilde{h} \in \mathbb{U}(\mathbf{X})$ *defined by*

$$\widetilde{f}(x) = \int_{]0, t^*(x)[} e^{-\alpha t} \sup_{a \in \mathbf{A}(\phi(x,t))} f(\phi(x,t), a) dt,$$

$$\widetilde{g}(x) = e^{-\alpha t^*(x)} \sup_{b \in \mathbf{A}(\phi(x, t^*(x)))} Qg(\phi(x, t^*(x)), b) \\ + K \int_{]0, t^*(x)[} e^{-\alpha t} \sup_{a \in \mathbf{A}(\phi(x,t))} Qg(\phi(x,t), a) dt,$$

and

$$\widetilde{h}(x) = e^{-\alpha t^*(x)} \sup_{b \in \mathbf{A}(\phi(x, t^*(x)))} h(\phi(x, t^*(x)), b)$$

satisfying

$$\sup_{u \in \mathcal{U}_s} J_u f(x) \leq \widetilde{f}(x), \quad \sup_{u \in \mathcal{U}_s} H_u g(x) \leq \widetilde{g}(x), \ \text{and} \ \sup_{u \in \mathcal{U}_s} K_u h(x) \leq \widetilde{h}(x) \tag{68}$$

for any $x \in \mathbf{X}$. *Moreover, if* $\{f_p\}_{p \in \mathbb{N}}$ *(respectively,* $\{g_p\}_{p \in \mathbb{N}}$ *and* $\{h_p\}_{p \in \mathbb{N}}$*) is a sequence of nonnegative functions in* $\mathbb{U}(\mathbf{K}^g)$ *(respectively,* $\mathbb{U}(\mathbf{X})$ *and* $\mathbb{U}(\mathbf{K}^i)$*) such that* $f_p \downarrow 0$ *(respectively,* $g_p \downarrow 0$ *and* $h_p \downarrow 0$*) as* $p \to \infty$, *then* $\widetilde{f}_p \downarrow 0$ *(respectively,* $\widetilde{g}_p \downarrow 0$ *and* $\widetilde{h}_p \downarrow 0$*) as* $p \to \infty$.

Proof. Clearly, for any $f \in \mathbb{U}(\mathbf{K}^g)$, we have that

$$J_u f(x) = \int_{]0,t^*(x)[} e^{-\alpha t} \int_{\mathbf{A}(x)} f(\phi(x,t),a)\pi(da|\phi(x,t))dt \leq \widetilde{f}(x).$$

For $t \in \mathbb{R}_+^*$ fixed, we have that the mapping $x \to \sup_{a\in\mathbf{A}(\phi(x,t))} f(\phi(x,t),a)$ is upper semicontinuous on $\mathbf{X}$ by using Lemma 17.30 in [1] and Assumptions (B1), (B4), and (C1). Now, for any $x \in \mathbf{X}$, $t \in \mathbb{R}_+ - \{t^*(x)\}$ and any sequence $\{x_n\}_{n\in\mathbb{N}}$ in $\mathbf{X}$ converging to x, we have from Assumption (B5) that

$$\overline{\lim_{n\to\infty}} I_{]0,t^*(x_n)[}(t) \sup_{a\in\mathbf{A}(\phi(x_n,t))} f(\phi(x_n,t),a) \leq I_{]0,t^*(x)[}(t) \sup_{a\in\mathbf{A}(\phi(x,t))} f(\phi(x,t),a).$$

Consequently, by using Fatou's lemma we obtain that

$$\begin{aligned}\overline{\lim_{n\to\infty}} \widetilde{f}(x_n) &= \overline{\lim_{n\to\infty}} \int_{\mathbb{R}_+^*} I_{]0,t^*(x_n)[}(t)e^{-\alpha t} \sup_{a\in\mathbf{A}(\phi(x_n,t))} f(\phi(x_n,t),a)dt \\ &\leq \int_{\mathbb{R}_+^*} I_{]0,t^*(x)[}(t)e^{-\alpha t} \sup_{a\in\mathbf{A}(\phi(x,t))} f(\phi(x,t),a)dt = \widetilde{f}(x),\end{aligned}$$

showing that $\widetilde{f} \in \mathbb{U}(\mathbf{X})$. Moreover, let $\{f_p\}_{p\in\mathbb{N}}$ be a sequence of nonnegative functions in $\mathbb{U}(\mathbf{K}^g)$ such that $f_p \downarrow 0$. Then, it follows easily by definition that $\{\widetilde{f}_p\}_{p\in\mathbb{N}}$ is a decreasing sequence of nonnegative functions. By using the bounded convergence theorem,

$$\lim_{p\to\infty} \widetilde{f}_p(x) = \int_{]0,t^*(x)[} e^{-\alpha t} \lim_{p\to\infty} \sup_{a\in\mathbf{A}(\phi(x,t))} f_p(\phi(x,t),a)dt.$$

However, from Lemma 2.1 in [22], $\lim_{p\to\infty} \sup_{a\in\mathbf{A}(\phi(x,t))} f_p(\phi(x,t),a) = \sup_{a\in\mathbf{A}(\phi(x,t))} \lim_{p\to\infty} f_p(\phi(x,t),a) = 0$ for any $x \in \mathbf{X}$, $t \in \mathbb{R}_+^*$ and so, $\lim_{p\to\infty} \widetilde{f}_p(x) = 0$ showing the first part of the result. The proof of the other claims uses exactly the same arguments and is, therefore, omitted. ∎

PROPOSITION 6.10. *Suppose Assumptions* (A1)–(A2), (B1)–(B2), (B4)–(B5), *and* (C1) *hold. Then the set* $\{(\eta_u^g, \eta_u^i) \in \mathcal{M}(\mathbf{K}^g) \times \mathcal{M}(\mathbf{K}^i) : u \in \mathcal{U}_s\}$ *is relatively compact with respect to the product topology.*

Proof. Let $\{f_p\}_{p\in\mathbb{N}}$ (respectively, $\{h_p\}_{p\in\mathbb{N}}$) be a sequence of nonnegative functions in $\mathbb{U}(\mathbf{K}^g)$ (respectively, $\mathbb{U}(\mathbf{K}^i)$) such that $f_p \downarrow 0$ (respectively, $h_p \downarrow 0$) as $p \to \infty$. From Proposition 6.8, it follows that

$$\eta_u^g(f_p) + \eta_u^i(h_p) = \sum_{k=0}^{\infty} H_u^k J_u f_p(x_0) + \sum_{k=0}^{\infty} H_u^k K_u h_p(x_0).$$

Observe that $H_u^2(\mathbf{X}|x) = \mathbb{E}_{x_0}^u\big[e^{-\alpha(\Theta_2+\Theta_1)}\big] \leq \kappa < 1$ for any $x \in \mathbf{X}$ by using (18), where $\kappa = 1 + [e^{-\alpha\varepsilon_1} - 1][1-\varepsilon_2]e^{-2K\varepsilon_1} < 1$. Moreover, $\|J_u f_p\| \leq \frac{\|f_p\|}{\alpha} \leq \frac{\|f_0\|}{\alpha}$ and $\|K_u h_p\| \leq \|h_p\| \leq \|h_0\|$. Consequently,

$$\eta_u^g(f_p) + \eta_u^i(h_p) \leq \sum_{k=0}^{2n-1} H_u^k J_u f_p(x_0) + \sum_{k=0}^{2n-1} H_u^k K_u h_p(x_0) + \frac{2\kappa^n}{1-\kappa}\left[\frac{\|f_0\|}{\alpha} + \|h_0\|\right].$$

Applying recursively Lemma 6.9, we obtain easily that

$$\lim_{p\to\infty}\sup_{u\in\mathcal{U}_s} H_u^k J_u f_p(x_0) = 0 \text{ and } \lim_{p\to\infty}\sup_{u\in\mathcal{U}_s} H_u^k K_u h_p(x_0) = 0$$

for any $k \in \mathbb{N}_n$ and so, we obtain that $\lim_{p\to\infty}\sup_{u\in\mathcal{U}_s} \eta_u^g(f_p) = \lim_{p\to\infty}\sup_{u\in\mathcal{U}_s} \eta_u^i(h_p) = 0$. By Theorem 8.6.11 in [5, p. 207] (or Theorem 25 in [24, p. 200]), it gives that $\{\eta_u^g \in \mathcal{M}(\mathbf{K}^g) : u \in \mathcal{U}_s\}$ and $\{\eta_u^i \in \mathcal{M}(\mathbf{K}^i) : u \in \mathcal{U}_s\}$ are relatively compact, completing the proof. ∎

LEMMA 6.11. *A finite measure $\eta \in \mathcal{M}(\mathbf{K})$ is admissible if and only if η^g and η^i satisfy the equation*

$$\begin{aligned}\widehat{\eta}^g(\Gamma_{\mathbf{X}}) + \widehat{\eta}^i(\Gamma_{\mathbf{\Xi}}) =& T(\Gamma_{\mathbf{X}} \cup \Gamma_{\mathbf{\Xi}}|x_0) + \int_{\mathbf{K}^g}\int_{\mathbf{X}} T(\Gamma_{\mathbf{X}} \cup \Gamma_{\mathbf{\Xi}}|y) q(dy|x,a)\eta^g(dx,da) \\ &+ \int_{\mathbf{K}^i}\int_{\mathbf{X}} T(\Gamma_{\mathbf{X}} \cup \Gamma_{\mathbf{\Xi}}|y) Q(dy|z,b)\eta^i(dz,db),\end{aligned} \tag{69}$$

where T is the kernel on $\overline{\mathbf{X}}$ given $\overline{\mathbf{X}}$ defined by

$$T(\Gamma_{\mathbf{X}} \cup \Gamma_{\mathbf{\Xi}}|y) = \int_{[0,t^*(y)[} e^{-\alpha t} I_{\Gamma_{\mathbf{X}}}(\phi(y,t))dt + e^{-\alpha t^*(y)} I_{\Gamma_{\mathbf{\Xi}}}(\phi(y,t^*(y))) \tag{70}$$

for any $\Gamma_{\mathbf{X}} \in \mathcal{B}(\mathbf{X})$, $\Gamma_{\mathbf{\Xi}} \in \mathcal{B}(\mathbf{\Xi})$, and $y \in \overline{\mathbf{X}}$.

Proof. Consider $\Gamma_{\mathbf{X}} \in \mathcal{B}(\mathbf{X})$ and $\Gamma_{\mathbf{\Xi}} \in \mathcal{B}(\mathbf{\Xi})$ fixed. For notational convenience, let us denote by W the mapping defined on $\overline{\mathbf{X}}$ by $W(y) = T(\Gamma_{\mathbf{X}} \cup \Gamma_{\mathbf{\Xi}}|y)$. According to Lemma 5.1, $W \in \mathbb{A}(\overline{\mathbf{X}})$ and satisfies $\alpha W(x) - \mathcal{X}W(x) = I_{\Gamma_{\mathbf{X}}}(x)$ for any $x \in \mathbf{X}$ and so (48) gives (69).

Conversely, consider any function $W \in \mathbb{A}(\overline{\mathbf{X}})$ with $\mathcal{X}W \in \mathbb{B}(\mathbf{X})$. Introduce the function V defined on $\overline{\mathbf{X}}$ by $V(x) = \alpha W(x) - \mathcal{X}W(x)$ for $x \in \mathbf{X}$ and $V(z) = W_{\mathbf{\Xi}}(z)$ for $z \in \mathbf{\Xi}$. Then, for any $x \in \mathbf{X}$,

$$\begin{aligned}TV(x) &= \int_{[0,t^*(x)[} e^{-\alpha t} V(\phi(x,t))dt + e^{-\alpha t^*(x)} V(\phi(x,t^*(x))) \\ &= \int_{[0,t^*(x)[} \alpha e^{-\alpha t} W(\phi(x,t))dt - \int_{[0,t^*(x)[} e^{-\alpha t}\mathcal{X}W(\phi(x,t))dt \\ &\quad + e^{-\alpha t^*(x)} V(\phi(x,t^*(x))).\end{aligned}$$

Integrating by parts the first term on the right-hand side of the previous equation, it gives

$$TV(y) = W(y)$$

for any $y \in \overline{\mathbf{X}}$. Consequently, from (69) it follows that

$$\begin{aligned}\int_{\mathbf{X}} V(x)\widehat{\eta}^g(dx) + \int_{\mathbf{\Xi}} V(z)\widehat{\eta}^i(dz) = TV(x_0) &+ \int_{\mathbf{K}^g}\int_{\mathbf{X}} TV(y) q(dy|x,a)\eta^g(dx,da) \\ &+ \int_{\mathbf{K}^g}\int_{\mathbf{X}} TV(y) Q(dy|z,b)\eta^i(dz,db)\end{aligned}$$

implying that

$$\int_{\mathbf{X}} \big[\alpha W(x) - \mathcal{X} W(x)\big] \widehat{\eta}^g(dx) + \int_{\mathbf{\Xi}} W(z) \widehat{\eta}^i(dz)$$
$$= W(x_0) + \int_{\mathbf{K}^g} \int_{\mathbf{X}} W(y) q(dy|x,a) \eta^g(dx,da)$$
$$+ \int_{\mathbf{K}^i} \int_{\mathbf{X}} W(y) Q(dy|z,b) \eta^i(dz,db),$$

showing the result. □

The next theorem provides sufficient conditions for the compactness of the space of measures that we will work with.

THEOREM 6.12. *Suppose Assumptions* (A1)–(A2), (B1)–(B5)*, and* (C1) *hold. The set*

$$\big\{(\eta_u^g, \eta_u^i) \in \mathcal{M}(\mathbf{K}^g) \times \mathcal{M}(\mathbf{K}^i) : u \in \mathcal{U}_s\big\}$$

is compact with respect to the product topology.

Proof. Let us show that $\big\{(\eta_u^g, \eta_u^i) \in \mathcal{M}(\mathbf{K}^g) \times \mathcal{M}(\mathbf{K}^i) : u \in \mathcal{U}_s\big\}$ is closed and from Proposition 6.10 we will get the result. Consider $\{u_n\}_{n\in\mathbb{N}}$ a sequence in $\mathcal{U}_s$ and any bounded real-valued function f defined on $\overline{\mathbf{X}}$ such that the restriction of f to $\mathbf{X}$ (respectively, $\mathbf{\Xi}$) is continuous. Then Lemma 6.11 gives

$$\int_{\mathbf{X}} f(x) \widehat{\eta}_{u_n}^g(dx) + \int_{\mathbf{\Xi}} f(z) \widehat{\eta}_{u_n}^i(dz) = Tf(x_0)$$
$$+ \int_{\mathbf{K}^g} \int_{\mathbf{X}} Tf(y) q(dy|x,a) \eta_{u_n}^g(dx,da) + \int_{\mathbf{K}^i} \int_{\mathbf{X}} Tf(y) Q(dy|z,b) \eta_{u_n}^i(dz,db). \tag{71}$$

Now, for any $y \in \overline{\mathbf{X}}$, $t \in \mathbb{R}_+ - \{t^*(y)\}$ and any sequence $\{y_n\}_{n\in\mathbb{N}}$ in $\overline{\mathbf{X}}$ converging to y, we have from Assumptions (B4)–(B5) that

$$\lim_{n\to\infty} I_{]0,t^*(y_n)[}(t) f(\phi(y_n,t)) = I_{]0,t^*(y)[}(t) f(\phi(y,t))$$

and

$$\lim_{n\to\infty} e^{-\alpha t^*(y_n)} f(\phi(y_n, t^*(y_n))) = e^{-\alpha t^*(y)} f(\phi(y, t^*(y))),$$

implying with the bounded convergence theorem that

$$Tf(y_n) = \lim_{n\to\infty} \int_{]0,t^*(y_n)[} e^{-\alpha t} f(\phi(y_n,t)) dt + e^{-\alpha t^*(y_n)} f(\phi(y_n, t^*(y_n)))$$
$$= \int_{]0,t^*(y)[} e^{-\alpha t} f(\phi(y,t)) dt + e^{-\alpha t^*(y)} f(\phi(y, t^*(y))) = Tf(y).$$

By using Assumptions (B2)–(B3), it follows that qTf and QTf are continuous on $\mathbf{K}$.

Assume that the sequence $\big\{(\eta_{u_n}^g, \eta_{u_n}^i)\big\}_{n\in\mathbb{N}}$ converges to $(\eta_1, \eta_2) \in \mathcal{M}(\mathbf{K}^g) \times \mathcal{M}(\mathbf{K}^i)$. Then, from (71)

$$\int_{\mathbf{X}} f(x) \widehat{\eta}_1(dx) + \int_{\mathbf{\Xi}} f(z) \widehat{\eta}_2(dz) = Tf(x_0) + \int_{\mathbf{K}^g} \int_{\mathbf{X}} Tf(y) q(dy|x,a) \eta_1(dx,da)$$
$$+ \int_{\mathbf{K}^i} \int_{\mathbf{X}} Tf(y) Q(dy|z,b) \eta_2(dz,db). \tag{72}$$

By using Lemma 6.11, it follows that the measure η defined by $\eta(\Gamma) = \eta_1(\Gamma \cap \mathbf{K}^g) + \eta_2(\Gamma \cap \mathbf{K}^i)$ for any $\Gamma \in \mathcal{B}(\mathbf{K})$ is admissible and so from Theorem 6.7, there exists $u \in \mathcal{U}_s$ such that $\eta_1 = \eta_u^g$ $\eta_2 = \eta_u^i$, implying that $\big\{(\eta_u^g, \eta_u^i) \in \mathcal{M}(\mathbf{K}^g) \times \mathcal{M}(\mathbf{K}^i) : u \in \mathcal{U}_s\big\}$ is closed. ∎

DEFINITION 6.13. *The constrained linear program, labeled* $\mathbb{LP}$, *is defined as*

$$\inf_{(\eta^g,\eta^i)\in\mathbb{M}} \eta^g(C_0^g) + \eta^i(C_0^i), \tag{73}$$

where $\mathbb{M}$ *is the set of measures* (η^g, η^i) *in* $\mathcal{M}(\mathbf{K}^i) \times \mathcal{M}(\mathbf{K}^g)$ *satisfying* (48) and

$$\eta^g(C_j^g) + \eta^i(C_j^i) \le B_j \tag{74}$$

for any $j \in \mathbb{N}_p^*$.

The real number $\inf_{(\eta^g,\eta^i)\in\mathbb{M}} \eta^g(C_0^g) + \eta^i(C_0^i)$ is called the value of the constrained linear program $\mathbb{LP}$. We say that $\mathbb{LP}$ is solvable if $\inf_{(\eta^g,\eta^i)\in\mathbb{M}} \eta^g(C_0^g) + \eta^i(C_0^i) = \widetilde{\eta}^g(C_0^g) + \widetilde{\eta}^i(C_0^i)$ for some $(\widetilde{\eta}^g, \widetilde{\eta}^i) \in \mathbb{M}$.

The next theorem is the main result of this section. It shows that the values of the constrained control problem and the constrained linear program $\mathbb{LP}$ are the same. Moreover, it provides sufficient conditions for the solvability of problem $\mathbb{LP}$ and the existence of an optimal feasible control strategy for the constrained problem.

THEOREM 6.14. *The following assertions hold:*

(i) *Suppose Assumptions* (A1) *and* (A2) *or* (A1) *and* (A3) *are satisfied and the cost functions* C_j^g *and* C_j^i *are bounded from below (or above) for any* $j \in \mathbb{N}_p$. *Then the values of the constrained control problem and the linear program* $\mathbb{LP}$ *are equivalent:*

$$\inf_{(\eta^g,\eta^i)\in\mathbb{M}} \eta^g(C_0^g) + \eta^i(C_0^i) = \inf_{u\in\mathcal{U}^f} \mathcal{V}_0(u, x_0).$$

(ii) *Suppose Assumptions* (A1)–(A2), (B1)–(B5), (C1) *are satisfied and the cost functions* C_j^g *(respectively,* C_j^i*) are bounded from below and lower semicontinuous on* $\mathbf{K}^g$ *(respectively,* $\mathbf{K}^i$*) for any* $j \in \mathbb{N}_p$. *If* $\mathcal{U}^f \neq \emptyset$, *then problem* $\mathbb{LP}$ *is solvable and there exists* $\widehat{u} \in \mathcal{U}_s^f$ *such that*

$$\begin{aligned}\eta_{\widehat{u}}^g(C_0^g) + \eta_{\widehat{u}}^i(C_0^i) &= \inf_{(\eta^g,\eta^i)\in\mathbb{M}} \eta^g(C_0^g) + \eta^i(C_0^i)\\ &= \inf_{u\in\mathcal{U}^f} \mathcal{V}_0(u, x_0) = \mathcal{V}_0(\widehat{u}, x_0).\end{aligned}$$

Proof of item (i). Since the cost functions C_j^g and C_j^i are bounded from below (or above), then we have $\mathcal{V}_j(u, x_0) = \eta_u^g(C_j^g) + \eta_u^i(C_j^i)$ for any $u \in \mathcal{U}$ and $j \in \mathbb{N}_p$. The statement then follows easily from Theorem 6.7.

Proof of item (ii). By assumption $\mathcal{U}^f \neq \emptyset$ and so $\mathbb{M} \neq \emptyset$. We have that the real-valued mapping defined on $\mathcal{M}(\mathbf{K}^g) \times \mathcal{M}(\mathbf{K}^i)$ by $(\eta^g, \eta^i) \to \eta^g(C_j^g) + \eta^i(C_j^i)$ is lower semicontinuous for any $j \in \mathbb{N}_p^*$ since the cost function C_j^g (respectively, C_j^i) is bounded from below and lower semicontinuous on $\mathbf{K}^g$ (respectively, $\mathbf{K}^i$) for any $j \in \mathbb{N}_p$. Consequently, the set

$$\big\{(\eta^g, \eta^i) \in \mathcal{M}(\mathbf{K}^g) \times \mathcal{M}(\mathbf{K}^i) : \eta^g(C_j^g) + \eta^i(C_j^i) \le B_j\big\}$$

is closed for any $j \in \mathbb{N}_p^*$. Now, by using Theorem 6.7, it follows that

$$\begin{aligned}&\big\{(\eta^g, \eta^i) \in \mathcal{M}(\mathbf{K}^g) \times \mathcal{M}(\mathbf{K}^i) : (\eta^g, \eta^i) \text{ satisfies } (48)\big\}\\ &\qquad = \big\{(\eta_u^g, \eta_u^i) \in \mathcal{M}(\mathbf{K}^g) \times \mathcal{M}(\mathbf{K}^i) : u \in \mathcal{U}\big\}\\ &\qquad = \big\{(\eta_u^g, \eta_u^i) \in \mathcal{M}(\mathbf{K}^g) \times \mathcal{M}(\mathbf{K}^i) : u \in \mathcal{U}_s\big\}.\end{aligned}$$

Therefore, from Theorem 6.12 we obtain that the set $\mathbb{M}$ is compact and satisfies

$$\mathbb{M} \subset \left\{(\eta_u^g, \eta_u^i) \in \mathcal{M}(\mathbf{K}^g) \times \mathcal{M}(\mathbf{K}^i) : u \in \mathcal{U}_s\right\},$$

and we obtain the result since the mapping defined on $\mathbb{M}$ by $(\eta^g, \eta^i) \to \eta^g(C_0^g) + \eta^i(C_0^i)$ is lower semicontinuous. □

Example. The goal of this example is to illustrate, through a simple case without control, the calculation of the occupation measure by using Theorem 6.7. For a number $\tau > 0$ consider the state space $\mathbf{X} = [0, \tau)$ and the active boundary $\boldsymbol{\Xi} = \{\tau\}$. As mentioned, in this example we assume that there is no control. The flow is given by $\phi(x,t) = x + t$, the jump rate is constant and equals to λ, and, after a jump (natural or from the boundary), the system always goes to 0, that is, $Q(\{0\}|x) = 1$ for every $0 \leq x \leq \tau$. The initial distribution is assumed to be $\nu_0 = \delta_0$, that is, the process starts from 0. In this case it is easy to see that $qW(x) = \lambda(W(0) - W(x))$. From Theorem 6.7, if we can find an admissible measure η, then it will be the occupation measure. For this we will try a measure η such that η^g has a density function $f(x)$ with respect the Lebesgue measure for $0 \leq x < \tau$ and η^i has a mass function on τ with value $g(\tau)$, in other words, $\eta^i(\tau) = g(\tau)\delta_\tau$. Note that $\int_0^\tau f(x)dx = \frac{1}{\alpha}$. For $W \in \mathbb{A}(\overline{\mathbf{X}})$, writing for simplicity $\mathcal{X}W(x) = W'(x)$, we have that (48) becomes $\alpha \int_0^\tau W(x)f(x)dx + g(\tau)W(\tau) = W(0) + \int_0^\tau W'(x)f(x)dx + \int_0^\tau \lambda(W(0) - W(x))f(x)dx + W(0)g(\tau)$. Assuming that the derivative of $f(x)$ exists, and denoting it by $f'(x)$, we can write, by integration by parts, that $\int_0^\tau W'(x)f(x)dx = W(\tau)f(\tau) - W(0)f(0) - \int_0^\tau W(x)f'(x)dx$. Reordering the terms we get that $\int_0^\tau W(x)((\lambda+\alpha)f(x) + f'(x))dx = W(0)(1 + \frac{\lambda}{\alpha} + g(\tau) - f(0)) + W(\tau)(f(\tau) - g(\tau))$. Since this equation must hold for every $W \in \mathbb{A}(\overline{\mathbf{X}})$, we make $(\lambda + \alpha)f(x) + f'(x) = 0$, $0 \leq x < \tau$, $g(\tau) = f(\tau)$, $f(0) - g(\tau) = \frac{\lambda+\alpha}{\alpha}$. Solving the above system we get that $f(x) = \left(\frac{\lambda+\alpha}{\alpha}\right)\left(\frac{1}{1-e^{-(\lambda+\alpha)\tau}}\right)e^{-(\lambda+\alpha)x}$, $0 \leq x < \tau$, $g(\tau) = f(\tau) = \left(\frac{\lambda+\alpha}{\alpha}\right)\left(\frac{1}{1-e^{-(\lambda+\alpha)\tau}}\right)e^{-(\lambda+\alpha)\tau}$, and the occupation measure is given by $\eta^g(dx) = f(x)dx$ on $[0, \tau)$ and $\eta^i(\tau) = g(\tau)\delta_\tau$. Notice that for the case in which $\tau = \infty$ (no boundaries) we get that $\alpha\eta$ is the exponential distribution with parameter $\lambda + \alpha$.

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