Some metric properties of spaces of stability conditions

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ABSTRACT

We show that the metric on a full component of $\mathrm{Stab}(\mathcal{C})$ for which the central charges factor through a finite rank quotient of the Grothendieck group $K(\mathcal{C})$ is complete. As an example, we compute the metric on the space of numerical stability conditions on a smooth complex projective curve of genus greater than one, and show that the quotient $\mathrm{Stab}(\mathcal{C})/\mathbb{C}$ by the natural action of the complex numbers is isometric to the upper half plane equipped with half the hyperbolic metric.

We also make two observations about the way in which the heart changes as we move through the space of stability conditions. Firstly, hearts of stability conditions in the same component of the space of stability conditions are related by finite sequences of tilts. Secondly, if each of a convergent sequence of stability conditions has the same heart then the heart of the limiting stability condition is obtained from this by a right tilt.

1. *Introduction*

The space of locally-finite stability conditions $\mathrm{Stab}(\mathcal{C})$ on a triangulated category $\mathcal{C}$ is a (possibly infinite-dimensional) complex manifold. There is a metric which induces the topology. For numerically-finite $\mathcal{C}$, for instance the coherent bounded derived category of a smooth complex projective variety or the bounded derived category of a finite-dimensional algebra, one can consider the finite-dimensional submanifold of numerical stability conditions. We show that under mild conditions this is complete in the metric and describe limiting stability conditions.

The contents are as follows: §2 contains some background material on stability conditions. Section 3 contains the proof that a full component of the space of stability conditions whose central charges factor through some finite rank quotient A of the Grothendieck group is complete. Recall that full means that the space of stability conditions is locally homeomorphic to $\mathrm{Hom}(A, \mathbb{C})$, and not to some proper subspace. The key result is Proposition 3.5 which describes the limiting Harder–Narasimhan filtrations. The assumptions are satisfied, for instance, for a full component of the space of numerical stability conditions on a numerically-finite triangulated category.

In §4 we independently verify that the metric on the space of numerical stability conditions on a smooth complex projective curve of genus ≥ 1 is complete. We compute this metric as follows. There is a natural action of the universal cover G of $GL_2^+\mathbb{R}$ on any space of stability conditions. When the phases of semistable objects are dense for a stability condition σ, the orbit through σ is free and the restriction d_G of the metric to it is independent of σ and can be explicitly described, and seen to be complete. In the case of curves of genus $g \geq 1$ this density of phases condition is satisfied and the action of G is both free and transitive, so that the space of numerical stability conditions is isometric to (G, d_G). The metric d_G is closely related to the hyperbolic metric on the upper half plane — the universal cover of the conformal linear maps forms a subgroup of G isomorphic to $\mathbb{C}$ and the quotient can be identified with the upper half-plane in such a way that the quotient metric is half the standard hyperbolic metric — see Proposition 4.1.
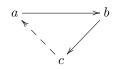
Section 5 contains some observations about relationships between the hearts of stability conditions. Corollary 5.2 states that hearts of stability conditions in the same component of $\mathrm{Stab}(\mathcal{C})$ are related by finite sequences of tilts. In the process of proving Theorem 3.6 we obtain a description of the limiting stability condition σ of a convergent sequence σ_n. As a consequence we obtain Corollary 5.3, which states that if the σ_n all have the same heart, $\mathcal{A}$ say, then the heart of σ must be a right tilt of $\mathcal{A}$.

2. Stability conditions

We fix some notation. Let $\mathcal{C}$ be an additive category. We write $c \in \mathcal{C}$ to mean c is an object of $\mathcal{C}$. We will use the term *subcategory* to mean strictly full subcategory. When $\mathcal{C}$ is triangulated with shift functor [1] exact triangles will be denoted either by $a \to b \to c \to a[1]$ or by a diagram

$$a \longrightarrow b$$
$$\nwarrow \quad \swarrow$$
$$c$$

where the dotted arrow denotes a morphism $c \to a[1]$. We will always assume that $\mathcal{C}$ is essentially small so that isomorphism classes of objects form a set. Given sets S_i of objects for $i \in I$ let $\langle S_i \mid i \in I \rangle$ denote the extension-closed subcategory generated by objects isomorphic to an element in some S_i. We use the same notation when the S_i are subcategories of $\mathcal{C}$. (We do not assume that this subcategory is closed under shifts.)

Let $\mathcal{C}$ be a triangulated category and $K(\mathcal{C})$ be its Grothendieck group. A *stability condition* $\sigma = (\mathcal{Z}_\sigma, \mathcal{P}_\sigma)$ on $\mathcal{C}$ [**3**, Definition 1.1] consists of a group homomorphism $\mathcal{Z}_\sigma : K(\mathcal{C}) \to \mathbb{C}$ and full additive subcategories $\mathcal{P}_\sigma(\varphi)$ of $\mathcal{C}$ for each $\varphi \in \mathbb{R}$ satisfying

(1) if $c \in \mathcal{P}_\sigma(\varphi)$ then $\mathcal{Z}_\sigma(c) = m(c) \exp(i\pi\varphi)$ where $m(c) \in \mathbb{R}_{>0}$;
(2) $\mathcal{P}_\sigma(\varphi + 1) = \mathcal{P}_\sigma(\varphi)[1]$ for each $\varphi \in \mathbb{R}$;
(3) if $c \in \mathcal{P}_\sigma(\varphi)$ and $c' \in \mathcal{P}_\sigma(\varphi')$ with $\varphi > \varphi'$ then $\mathrm{Hom}(c, c') = 0$;
(4) for each nonzero object $c \in \mathcal{C}$ there is a Harder–Narasimhan filtration, i.e. a finite collection of triangles

$$0 = c_0 \longrightarrow c_1 \longrightarrow \ldots \longrightarrow c_{n-1} \longrightarrow c_n = c$$
$$\nwarrow \quad \swarrow \qquad\qquad\qquad\qquad \nwarrow \quad \swarrow$$
$$s_1 \qquad\qquad\qquad\qquad\qquad s_n$$

with $s_j \in \mathcal{P}_\sigma(\varphi_j)$ where $\varphi_1 > \ldots > \varphi_n$.

The homomorphism $\mathcal{Z}_\sigma$ is known as the *central charge* and the objects of $\mathcal{P}_\sigma(\varphi)$ are said to be σ-*semistable of phase* φ. The objects s_j are known as the σ-*semistable factors* of c. We define $\varphi_\sigma^+(c) = \varphi_1$ and $\varphi_\sigma^-(c) = \varphi_n$. The *mass* of c is defined to be $m_\sigma(c) = \sum_{i=1}^n m(s_i)$.

LEMMA 2.1.   *If $a \to b \to c \to a[1]$ is an exact triangle, then*

$$\min\{\varphi_\sigma^-(a), \varphi_\sigma^-(c)\} \leq \varphi_\sigma^-(b) \leq \varphi_\sigma^+(b) \leq \max\{\varphi_\sigma^+(a), \varphi_\sigma^+(c)\}$$

for any stability condition σ.

Proof.   Note that $\varphi_\sigma^+(b) = \sup\{t \mid \exists b' \in \mathcal{P}_\sigma(t) \text{ with } \mathrm{Hom}(b', b) \neq 0\}$ and similarly

$$\varphi_\sigma^-(b) = \inf\{t \mid \exists b' \in \mathcal{P}_\sigma(t) \text{ with } \mathrm{Hom}(b, b') \neq 0\}.$$

Furthermore, both the extreme values are achieved. Suppose $\varphi_\sigma^+(b) > \varphi_\sigma^+(c)$. There is a semistable b' with phase $\varphi_\sigma^+(b)$ and nonzero morphism $b' \to b$. Since $\mathrm{Hom}(b', c) = 0$ for phase

reasons this morphism factors through a nonzero morphism to a. Hence $\varphi_\sigma^+(a) \geq \varphi_\sigma^+(b)$ and $\varphi_\sigma^+(b) \leq \max\{\varphi_\sigma^+(a), \varphi_\sigma^+(c)\}$. The other inequality is proved similarly. $\qquad\square$

By a σ-*filtration* of $0 \neq c \in \mathcal{C}$ we mean a filtration as above whose factors are σ-semistable. The Harder–Narasimhan filtration is one example; in general there will be others not satisfying the decreasing phase of factors condition (which uniquely characterises the Harder–Narasimhan filtration). One consequence of the previous lemma is that the Harder–Narasimhan filtration has minimal phase range amongst all σ-filtrations in the sense that if a σ-filtration of c has factors with phases φ_i then

$$\min\{\varphi_i\} \leq \varphi_\sigma^-(c) \leq \varphi_\sigma^+(c) \leq \max\{\varphi_i\}.$$

For any interval $I \subset \mathbb{R}$ let $\mathcal{P}_\sigma(I) = \langle \mathcal{P}_\sigma(\varphi) \mid \varphi \in I \rangle$ be the full extension-closed subcategory generated by σ-semistable objects with phases in I. For any $t \in \mathbb{R}$ the full subcategory $\mathcal{P}_\sigma(t, \infty)$ is a t-structure. In particular the heart

$$\mathcal{P}_\sigma(t, \infty) \cap \mathcal{P}_\sigma(t, \infty)^\perp[1] = \mathcal{P}_\sigma(t, \infty) \cap \mathcal{P}_\sigma(-\infty, t+1] = \mathcal{P}_\sigma(t, t+1]$$

is an abelian subcategory of $\mathcal{C}$. Here the (right) orthogonal subcategory $\mathcal{B}^\perp$ of a subcategory $\mathcal{B}$ of $\mathcal{C}$, is the full subcategory on $\{c \in \mathcal{C} \mid \mathrm{Hom}(b, c) = 0 \; \forall b \in \mathcal{B}\}$. Note that $Z_\sigma(a) \neq 0$ for any $0 \neq a \in \mathcal{P}_\sigma(t, t+1]$ so that the phase

$$\varphi_\sigma(a) = \frac{1}{\pi} \arg Z_\sigma(a) \in (t, t+1]$$

is well-defined. The semistable objects in $\mathcal{P}_\sigma(t, t+1]$ can be characterised as those b such that $a \rightarrowtail b$ implies $\varphi_\sigma(a) \leq \varphi_\sigma(b)$, or equivalently those for which $b \twoheadrightarrow c$ implies $\varphi_\sigma(b) \leq \varphi_\sigma(c)$. See [**3**, §5] for details. The abelian category $\mathcal{P}_\sigma(0, 1]$ is referred to as *the* heart of the stability condition σ.

When the length $|I| = \sup I - \inf I$ of the interval is < 1 the category $\mathcal{P}_\sigma(I)$ is quasi-abelian, see [**3**, Lemma 4.3]. The strict short exact sequences arise from exact triangles $a \to b \to c \to a[1]$ in $\mathcal{C}$ for which a, b and c are in $\mathcal{P}_\sigma(I)$. A stability condition σ is said to be *locally-finite* if given $t \in \mathbb{R}$ we can find $\epsilon > 0$ such that the quasi-abelian category $\mathcal{P}_\sigma(t - \epsilon, t + \epsilon)$ is both artinian and noetherian. (Such a category is called finite length in [**3**] but we prefer to avoid this terminology to avoid confusion with the *a priori* stronger notion of length category introduced in the next section.)

The set of locally-finite stability conditions can be given a topology so that it is a, possibly infinite-dimensional, complex manifold $\mathrm{Stab}(\mathcal{C})$ referred to as the *space of stability conditions* on $\mathcal{C}$. For each component of $\mathrm{Stab}(\mathcal{C})$ there is a linear subspace $V \subset \mathrm{Hom}(K(\mathcal{C}), \mathbb{C})$ such that the projection $(\mathcal{Z}, \mathcal{P}) \mapsto \mathcal{Z}$ is a local homeomorphism from that component to V [**3**, Theorem 1.2].

The topology on $\mathrm{Stab}(\mathcal{C})$ arises from the generalised metric

$$d(\sigma, \tau) = \sup_{0 \neq c \in \mathcal{C}} \max \left\{ |\varphi_\sigma^-(c) - \varphi_\tau^-(c)|, |\varphi_\sigma^+(c) - \varphi_\tau^+(c)|, \left| \log \frac{m_\sigma(c)}{m_\tau(c)} \right| \right\}$$

which takes values in $[0, \infty]$, see [**3**, Proposition 8.1]. Note that $\{\tau \mid d(\sigma, \tau) < \infty\}$ is both open and closed and so is a union of components. Hence $d(\sigma, \tau) = \infty$ implies that σ and τ are in distinct components of $\mathrm{Stab}(\mathcal{C})$. For the rest of this paper we will loosely refer to d as a metric.

The group $\mathrm{Aut}(\mathcal{C})$ of automorphisms of $\mathcal{C}$ acts on the left of $\mathrm{Stab}(\mathcal{C})$ by isometries via

$$(\mathcal{Z}, \mathcal{P}) \mapsto \left(\mathcal{Z} \circ \alpha^{-1}, \alpha \circ \mathcal{P} \right).$$

There is also a right action of the universal cover G of $GL_2^+\mathbb{R}$. An element $g \in G$ corresponds to a pair (T_g, θ_g) where T_g is the projection of g to $GL_2^+\mathbb{R}$ under the covering map and $\theta_g : \mathbb{R} \to \mathbb{R}$ is an increasing map with $\theta_g(t+1) = \theta_g(t) + 1$ which induces the same map as T_g on the circle

$\mathbb{R}/2\mathbb{Z} = \mathbb{R}^2 - \{0\}/\mathbb{R}_{>0}$. In these terms the action is given by

$$\left(\mathcal{Z}, \mathcal{P}\right) \mapsto \left(T_g^{-1} \circ \mathcal{Z}, \mathcal{P} \circ \theta_g\right).$$

(Here we think of the central charge as valued in $\mathbb{R}^2$.) This action preserves the semistable objects, and also preserves the Harder–Narasimhan filtrations of all objects. The conformal linear maps in $GL_2^+\mathbb{R}$ are the rotations and radial expansions, which can be identified with multiplications by elements of $\mathbb{C}^*$ (where we identify $\mathbb{C} \cong \mathbb{R}^2$). There is a fibre product diagram

$$
\begin{array}{ccc}
\mathbb{C} & \hookrightarrow & G \\
{\scriptstyle \exp(i\pi-)}\downarrow & & \downarrow \\
\mathbb{C}^* & \hookrightarrow & GL_2^+\mathbb{R}.
\end{array}
$$

The subgroup of G consisting of pairs for which T is conformal is isomorphic to $\mathbb{C}$ with $\lambda \in \mathbb{C}$ acting via

$$(\mathcal{Z}, \mathcal{P}) \mapsto (\exp(-i\pi\lambda)\mathcal{Z}, \mathcal{P}(\varphi + \mathrm{re}\,\lambda)),$$

i.e. by rotating the phases and rescaling the masses of semistable objects. This action is free and preserves the metric. The action of $1 \in \mathbb{C}$ corresponds to the action of the shift automorphism [1], see [**2**, p9].

The distance between stability conditions σ and $\sigma\lambda$ in the same orbit of this $\mathbb{C}$ action is given by

$$d(\sigma, \sigma\lambda) = \max\{|\mathrm{re}\,\lambda|, \pi|\mathrm{im}\,\lambda|\}.$$

No sphere in the metric intersects the orbit in a smooth submanifold. Therefore d is not the path metric of a Riemannian metric. Straight lines in $\mathbb{C}$ are geodesics since they have length exactly the distance between their endpoints. By considering geodesic triangles in an orbit we see that $\mathrm{Stab}(\mathcal{C})$ can never have strictly negative curvature, although it may have non-positive curvature.

Another elementary observation is that the orbits of the $\mathbb{C}$ action are closed. Hence there is an induced quotient metric $\overline{d}$ on $\mathrm{Stab}(\mathcal{C})/\mathbb{C}$ defined by

$$\overline{d}(x\mathbb{C}, y\mathbb{C}) = \inf\{d(x, y\lambda) \mid \lambda \in \mathbb{C}\}.$$

In §4 we will see examples in which this quotient metric arises from a Riemannian metric with constant strictly negative curvature. It would be interesting to see whether the quotient metric has similar good properties in other examples.

3. *Completeness of spaces of stability conditions*

Let $\mathcal{C}$ be a triangulated category and $K(\mathcal{C}) \twoheadrightarrow A$ a finite rank quotient of its Grothendieck group. Consider the subspace of stability conditions on $\mathcal{C}$ whose central charge factors through A. Let $\mathrm{Stab}_0(\mathcal{C})$ be a component of this subspace. Suppose that $\mathrm{Stab}_0(\mathcal{C})$ is full in the sense of [**4**, Definition 4.2], i.e. that the projection $\mathrm{Stab}_0(\mathcal{C}) \to \mathrm{Hom}(A, \mathbb{C})$ is a local homeomorphism. We prove that such a component is complete in the metric. In fact this follows rather quickly from the results of [**3**, §7,8], which show that $\mathrm{Stab}_0(\mathcal{C})$ can be covered by closed balls of fixed radius in the metric, each of which is isometric to a complete metric space.[†] However, we give a different proof which has the benefit of providing a useful description of the limiting stability condition.

[†]This argument was pointed out to me by Arend Bayer.

By assumption stability conditions in $\mathrm{Stab}_0(\mathcal{C})$ are locally-finite. We sharpen this slightly in Lemma 3.2 below by showing that they are *locally-length* in the sense that given $t \in \mathbb{R}$ we can find $\epsilon > 0$ such that the quasi-abelian category $\mathcal{P}_\sigma(t - \epsilon, t + \epsilon)$ is a quasi-abelian length category, or *length category* for short. By this we mean a quasi-abelian category in which every object $a \neq 0$ has a length $l(a) \in \mathbb{N}$ such that any filtration

$$\ldots \rightarrowtail a_i \rightarrowtail a_{i+1} \rightarrowtail \ldots \rightarrowtail a$$

by strict subobjects has length $\leq l(a)$ with equality for some filtration. (See [7] for more on quasi-abelian categories.) Clearly a length category is both noetherian and artinian, so that a locally-length stability condition is locally-finite. However, the locally-length condition is *a priori* stronger in that it requires the existence of an upper bound on the length of any filtration, not merely that any filtration have finite length.

REMARK 1. If $\mathcal{A}$ is a noetherian and artinian abelian category the Jordan–Hölder theorem asserts that any $0 \neq a \in \mathcal{A}$ has a finite composition series with simple factors, and that all such composition series have the same length. So this definition of length category agrees with the usual one in case $\mathcal{A}$ is abelian, and any noetherian and artinian abelian category is a length category. Furthermore, for an abelian length category the length descends to a homomorphism $l : K(\mathcal{A}) \to \mathbb{Z}$ defined on the Grothendieck group. However, it is not clear that the analogues hold in the quasi-abelian setting.

LEMMA 3.1. *Suppose σ and τ are stability conditions such that $\mathcal{P}_\sigma(I) \subset \mathcal{P}_\tau(J)$ for intervals I and J of length < 1. Then $\mathcal{P}_\sigma(I)$ is a quasi-abelian length category whenever $\mathcal{P}_\tau(J)$ is. Moreover the length of a in $\mathcal{P}_\sigma(I)$ is bounded above by its length in $\mathcal{P}_\tau(J)$.*

Proof. A strict monomorphism $\imath : a \rightarrowtail a'$ in $\mathcal{P}_\sigma(I)$ fits into a triangle $a \xrightarrow{\imath} a' \longrightarrow a'' \longrightarrow a[1]$ with $a, a', a'' \in \mathcal{P}_\sigma(I)$ and therefore also in $\mathcal{P}_\tau(J)$. Hence $\imath$ is a strict monomorphism in $\mathcal{P}_\tau(J)$ too. The result follows. $\qquad\square$

The next result is a slightly stronger version of [4, Lemma 4.4]; the idea of the proof is the same.

LEMMA 3.2. *Any $\sigma \in \mathrm{Stab}_0(\mathcal{C})$ is locally-length, indeed $\mathcal{P}_\sigma(I)$ is a quasi-abelian length category whenever I is a closed interval of length $|I| < 1$.*

Proof. Suppose $\sigma \in \mathrm{Stab}_0(\mathcal{C})$ and that the central charge Z_σ is rational, i.e. has image in $\mathbb{Q}[i]$. Then the image is a discrete subgroup of $\mathbb{C}$. Assume, without loss of generality, that $I \subset (0, 1)$. Fix $0 \neq a \in \mathcal{P}_\sigma(I)$. Then

$$K = \left\{ z \in \mathbb{C} \ \middle| \ \frac{1}{\pi} \arg z \in I \text{ and } \mathrm{im}\, z \leq \mathrm{im}\, Z_\sigma(a) \right\}$$

is compact, and so contains only finitely many points, say L, in the image of Z_σ. Any filtration of a in $\mathcal{P}_\sigma(I)$ is mapped under Z_σ to a sequence in K. Since $Z_\sigma(b) \neq 0$ for $0 \neq b \in \mathcal{P}_\sigma(I)$ the images of the subobjects in the filtration are distinct, and hence the filtration cannot have length greater than L.

In the general case, since $\mathrm{Stab}_0(\mathcal{C})$ is full we can approximate σ by a stability condition τ with rational central charge, see the proof of [4, Lemma 4.5], and such that $\mathcal{P}_\sigma(I) \subset \mathcal{P}_\tau(J)$ for some closed interval J of length < 1. The result follows from Lemma 3.1. $\qquad\square$

Suppose that σ_n is a Cauchy sequence in this component. For ease of reading we write $\mathcal{P}_n, Z_n, \varphi_n, m_n$ for $\mathcal{P}_{\sigma_n}, Z_{\sigma_n}, \varphi_{\sigma_n}$ and m_{σ_n}.

PROPOSITION 3.3.   *Let $0 \neq c \in \mathcal{C}$. The lengths of the Harder–Narasimhan σ_n-filtrations of c are bounded.*

Proof.   Fix $0 < \epsilon < 1/8$ and $\theta \in \mathbb{R}$. Choose M such that $d(\sigma_m, \sigma_n) < \epsilon$ whenever $m, n \geq M$. In particular, for any $n \geq M$,

$$\mathcal{P}_n(-\infty, \theta + \epsilon) \subset \mathcal{P}_M(-\infty, \theta + 2\epsilon) \quad \text{and} \quad \mathcal{P}_n(\theta - \epsilon, \infty) \subset \mathcal{P}_M(\theta - 2\epsilon, \infty).$$

Let $\tau_n^{<t}$ be the truncation functor right adjoint to the inclusion of $\mathcal{P}_n(-\infty, t)$ in $\mathcal{C}$, and so on. Then

$$\begin{aligned}
\tau_n^{<\theta+\epsilon} \tau_n^{>\theta-\epsilon} c &= \tau_n^{<\theta+\epsilon} \tau_n^{>\theta-\epsilon} \tau_M^{>\theta-2\epsilon} c \\
&= \tau_n^{>\theta-\epsilon} \tau_n^{<\theta+\epsilon} \tau_M^{>\theta-2\epsilon} c \\
&= \tau_n^{>\theta-\epsilon} \tau_n^{<\theta+\epsilon} \tau_M^{<\theta+2\epsilon} \tau_M^{>\theta-2\epsilon} c.
\end{aligned}$$

Hence the number of σ_n-semistable factors of c with phase in $(\theta - \epsilon, \theta + \epsilon)$ is the same as the number of σ_n-semistable factors of $\tau_M^{<\theta+2\epsilon} \tau_M^{>\theta-2\epsilon} c$ with phase in $(\theta - \epsilon, \theta + \epsilon)$. However, the Harder–Narasimhan σ_n-filtration of $\tau_M^{<\theta+2\epsilon} \tau_M^{>\theta-2\epsilon} c$ is contained within

$$\mathcal{P}_n(\theta - 3\epsilon, \theta + 3\epsilon) \subset \mathcal{P}_M(\theta - 4\epsilon, \theta + 4\epsilon)$$

and the latter is a length category by Lemma 3.2. Therefore there is a uniform (in $n \geq M$) bound on the number of σ_n-semistable factors of c with phase in $(\theta - \epsilon, \theta + \epsilon)$. Note that M depends only on ϵ, and not on θ. Therefore, covering $[\varphi_M^-(c) - \epsilon, \varphi_M^+(c) + \epsilon]$ by finitely many intervals of the form $(\theta - \epsilon, \theta + \epsilon)$ we obtain a uniform bound on the length of the Harder–Narasimhan σ_n-filtration of c whenever $n \geq M$.   □

For $\theta \in \mathbb{R}$ we define the *limiting semistable objects* of phase θ to be

$$\mathcal{P}(\theta) = \langle 0 \neq c \in \mathcal{C} \mid \varphi_n^\pm(c) \to \theta \rangle.$$

The next lemma shows that limiting semistable objects have non-zero limiting central charge.

LEMMA 3.4.   *Let $0 \neq c \in \mathcal{C}$. If $\varphi_n^+(c) - \varphi_n^-(c) \to 0$ as $n \to \infty$ then $Z_n(c) \not\to 0$.*

Proof.   If $\varphi_n^+(c) - \varphi_n^-(c) \to 0$ then $m_n(c) - |Z_n(c)| \to 0$. Therefore $Z_n(c) \to 0$ implies $m_n(c) \to 0$, contradicting the assumption that σ_n is Cauchy.   □

PROPOSITION 3.5.   *Each $0 \neq c \in \mathcal{C}$ has a filtration by limiting semistable objects:*

where the factors $b_i \in \mathcal{P}(\theta_i)$ with $\theta_1 > \ldots > \theta_r$.

Proof.   By Proposition 3.3 there is some $L \in \mathbb{N}$ such that the Harder–Narasimhan σ_n-filtration of c has length $\leq L$ for all sufficiently large n. We prove the result by induction

on L. The case $L = 1$ is easy, for in this case c is σ_n-semistable for all sufficiently large n and so $\varphi_n^-(c) - \varphi_n^+(c) \to 0$ (is in fact 0 for sufficiently large n) and c is itself a limiting semistable. Now assume that the result is true for any object whose Harder–Narasimhan σ_n-filtrations are of length $< L$ for all sufficiently large n.

Since σ_n is a Cauchy sequence both $\varphi_n^+(c)$ and $\varphi_n^-(c)$ converge. Suppose that $\varphi_n^+(c) - \varphi_n^-(c) \to \alpha$. If $\alpha = 0$ then c is a limiting semistable and the result holds. So we may assume $\alpha > 0$. Choose $0 < \epsilon < \alpha/2(L + 1)$. Choose M so that $d(\sigma_m, \sigma_n) < \epsilon$ whenever $m, n \geq M$. For any $n \geq M$ we can find successive semistable factors in the Harder–Narasimhan σ_n-filtration of c whose phases differ by more than

$$\frac{\alpha - 2\epsilon}{L} > \frac{2(L + 1)\epsilon - 2\epsilon}{L} = 2\epsilon.$$

Splitting the Harder–Narasimhan σ_n-filtration of c between the above-mentioned factors one obtains a triangle $c' \to c \to c'' \to c'[1]$ with $c', c'' \neq 0$ and such that $\varphi_n^-(c') > \varphi_n^+(c'')$ whenever $n \geq M$. Thus the Harder–Narasimhan σ_n-filtration of c for any $n \geq M$ is obtained by concatenating those of c' and c''. It follows that the Harder–Narasimhan σ_n-filtrations of c' and c'' must have length $< L$ for any $n \geq M$. Hence, by the inductive hypothesis, both c' and c'' have filtrations by limiting semistable objects of decreasing phase. Moreover, the minimal phase of a factor of c' is at least as large as the maximal phase of a factor of c''. The result follows by concatenating these limiting Harder–Narasimhan filtrations of c' and c''.   □

THEOREM 3.6.   *Let* $\mathrm{Stab}_0(\mathcal{C})$ *be a full component of the space of locally-finite stability conditions on* $\mathcal{C}$ *whose central charges factor through a given finite rank quotient of the Grothendieck group. Then* $\mathrm{Stab}_0(\mathcal{C})$ *is complete in the natural metric.*

Proof.   Let σ_n be a Cauchy sequence. Then the central charges Z_n converge, say to $Z : K(\mathcal{C}) \to \mathbb{C}$. We claim that $\sigma = (Z, \mathcal{P})$ is a locally-length stability condition with $\sigma_n \to \sigma$. It is immediate from the definition of limiting semistable objects that $\mathcal{P}(\theta + 1) = \mathcal{P}(\theta)[1]$. By Lemma 3.4 if $0 \neq c \in \mathcal{P}(\theta)$ then $Z(c) = m \exp(i\pi\theta)$ for some $m > 0$. If $c \in \mathcal{P}(\theta)$ and $c' \in \mathcal{P}(\theta')$ with $\theta > \theta'$ then, for sufficiently large n, we have $\varphi_n^-(c) > \varphi_n^+(c')$ so that $\mathrm{Hom}(c, c') = 0$. The existence of Harder–Narasimhan filtrations is guaranteed by Proposition 3.5. Hence σ is a stability condition. If the kernel of $K(\mathcal{C}) \twoheadrightarrow A$ is annihilated by each of the Z_n then it is also annihilated by Z, i.e. the central charge of σ factors through A.

Given $\epsilon > 0$ we can choose $M \in \mathbb{N}$ so that $d(\sigma_m, \sigma_n) < \epsilon$ for $m, n \geq M$. Then $\mathcal{P}(t - \epsilon, t + \epsilon) \subset \mathcal{P}_m(t - 2\epsilon, t + 2\epsilon)$ whenever $m \geq M$. Hence σ is locally-length by Lemma 3.1, in particular σ is locally-finite. By construction $\sigma_n \to \sigma$, so that $\sigma \in \mathrm{Stab}_0(\mathcal{C})$ and the component is complete.   □

REMARK 2.   The condition that $\mathrm{Stab}_0(\mathcal{C})$ is a full component of the space of locally-finite stability conditions on $\mathcal{C}$ whose central charges factor through a given finite rank quotient of the Grothendieck group enters only in the proof that the lengths of the Harder–Narasimhan σ_n-filtrations of a fixed object are bounded as $n \to \infty$. Hence Theorem 3.6 shows that any Cauchy sequence in the space of locally-finite stability conditions for which this is true converges.

4. *Examples*

In general the metric d, and the induced quotient metric on $\mathrm{Stab}(\mathcal{C})/\mathbb{C}$, are hard to compute. In this section, under the assumption that the phases of σ-semistables are dense in $\mathbb{R}$, we compute the restricted metric on the orbit σG, where G is the universal cover of $GL_2^+\mathbb{R}$, and

show that it is independent of σ. In this case the induced metric on $\sigma G/\mathbb{C} \cong \mathbb{H}$ is half the standard hyperbolic metric on the upper half-plane. This allows us to compute the metric on $\mathrm{Stab}(X)$ whenever X is a smooth complex projective curve of genus ≥ 1, and verify directly that the metrics are complete in these cases.

PROPOSITION 4.1. *Suppose $\sigma \in \mathrm{Stab}(\mathcal{C})$ is a stability condition for which the phases of semistable objects are dense in $\mathbb{R}$. Then the G orbit through σ is free, the restriction of the metric d to it is independent of σ and thus gives a well-defined metric d_G on G. This metric is invariant under the action of G by left multiplication and is complete. The induced metric on $\sigma G/\mathbb{C} \cong \mathbb{H}$ is half the hyperbolic metric d_{hyp} on the upper half-plane.*

Proof. Let $g \in G$ correspond to the pair (T_g, θ_g) where $T_g \in GL_2^+\mathbb{R}$ and $\theta_g : \mathbb{R} \to \mathbb{R}$ is increasing with $\theta_g(t+1) = \theta_g(t) + 1$ and induces the same map as T_g on the circle $\mathbb{R}/2\mathbb{Z} = \mathbb{R}^2 - \{0\}/\mathbb{R}_{>0}$. Since σ and σg have the same semistable objects and Harder–Narasimhan filtrations

$$d(\sigma g, \sigma) = \sup_{\substack{\text{semistable } c}} \max\left\{ |\varphi_{\sigma g}(c) - \varphi_\sigma(c)|, \left|\log \frac{m_{\sigma g}(c)}{m_\sigma(c)}\right| \right\}$$

$$= \sup_{\substack{\text{semistable } c}} \max\left\{ |\theta_g(\varphi_\sigma(c)) - \varphi_\sigma(c)|, \left|\log \frac{|T_g Z_\sigma(c)|}{|Z_\sigma(c)|}\right| \right\}$$

$$= \sup_{\substack{\text{semistable } c}} \max\left\{ |\theta_g(\varphi_\sigma(c)) - \varphi_\sigma(c)|, |\log(|T_g v|)| \right\}$$

where $v = Z_\sigma(c)/|Z_\sigma(c)|$ is the unit vector in the direction of $Z_\sigma(c) \in \mathbb{R}^2$. Under the assumption that phases of semistables are dense we thus have

$$d(\sigma g, \sigma) = \max\left\{ \sup_t |\theta_g(t) - t|, \sup_{|v|=1} |\log(|T_g v|)| \right\}$$

$$= \max\left\{ ||\theta_g - \mathrm{id}||, \log||T_g||, \log||T_g^{-1}|| \right\}$$

which depends only on g, and not on σ. Set $\Delta(g) = d(\sigma g, \sigma)$. It is easy to check that $\Delta(g) = 0$ if and only if $g = 1$, so that

$$d_G(g, h) = d(\sigma g, \sigma h) = d((\sigma h)h^{-1}g, \sigma h) = \Delta(h^{-1}g)$$

is a metric on G, and the orbit σG is free. Note that

$$d_G(fg, fh) = \Delta((fh)^{-1}fg) = \Delta(h^{-1}g) = d_G(g, h)$$

so that d_G is invariant under left multiplication. The explicit description shows that d_G is complete.

We have identifications $G/\mathbb{C} \cong GL_2^+\mathbb{R}/\mathbb{C}^* \cong SL_2\mathbb{R}/SO(2) \cong \mathbb{H}$ where the last isomorphism is given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \frac{ai+b}{ci+d}.$$

Under this identification the left multiplication action of $SL_2\mathbb{R}$ corresponds to its action by Möbius transformations on $\mathbb{H}$. Hence the induced quotient metric on $\sigma G/\mathbb{C} \cong \mathbb{H}$ is invariant under the Möbius action of $SL_2\mathbb{R}$. Under this action any pair of points can be mapped respectively to i and ai for some real $a \geq 1$. Therefore to pin down the metric it suffices to compute the distance between i and ai. Under the map $G \to G/\mathbb{C} \cong \mathbb{H}$ the point ai is the image of $g_a = (T_{g_a}, \theta_{g_a})$ where

$$T_{g_a} = \begin{pmatrix} \sqrt{a} & 0 \\ 0 & 1/\sqrt{a} \end{pmatrix} \text{ and } \theta_{g_a}(t) = \frac{1}{\pi} \tan^{-1}\left(\frac{\tan(\pi t)}{a}\right).$$

By definition the distance from i to ai is $\inf_{\lambda \in \mathbb{C}} \Delta(g_a \lambda)$. To compute this we note that

$$\log \|T_{g_a} \lambda\| = \log(\sqrt{a}) - \pi \operatorname{im} \lambda = \log(a)/2 - \pi \operatorname{im} \lambda$$

and similarly $\log \|T_{g_a}^{-1} \lambda\| = \log(a)/2 + \pi \operatorname{im} \lambda$. To maximise $|\theta_{g_a} \lambda(t) - t| = |\operatorname{re} \lambda + \theta_{g_a}(t) - t|$ we differentiate; the critical points are at $t = \pm \tan^{-1}(\sqrt{a})/\pi$ where the values are respectively

$$\left| \operatorname{re} \lambda \pm \frac{1}{\pi} \left(\tan^{-1} \left(\frac{1}{\sqrt{a}} \right) - \tan^{-1} \left(\sqrt{a} \right) \right) \right|.$$

It follows from these calculations that the infimum occurs when $\lambda = 0$. Moreover, one can check that $\log(a)/2 \geq \left(\tan^{-1}(\sqrt{a}) - \tan^{-1}(1/\sqrt{a}) \right)/\pi \geq 0$ for $a \geq 1$ so that

$$\inf_{\lambda \in \mathbb{C}} \Delta(g_a \lambda) = \frac{1}{2} \log(a) = \frac{1}{2} d_{\mathrm{hyp}}(i, ai),$$

is half the hyperbolic distance between i and ai. $\qquad \square$

If X is a smooth complex projective curve of genus ≥ 1 then $\mathrm{Stab}(X) \cong G$ is a single orbit [6]. Furthermore, the phases of semistables are dense since, for the standard stability condition, there are semistable sheaves of any rational slope. (Both of these facts are false for the genus 0 case.) Therefore the spaces of stability conditions are isometric to (G, d_G) for any such X, and in particular are complete. Furthermore, $\mathrm{Stab}(X)/\mathbb{C}$ is isometric to the upper half-plane with metric $\frac{1}{2} d_{\mathrm{hyp}}$.

5. Tilting and hearts of stability conditions

A *torsion theory* in an abelian category $\mathcal{A}$ is a pair $(\mathcal{T}, \mathcal{F})$ of subcategories such that $\mathcal{F} \subset \mathcal{T}^\perp$ and every $a \in \mathcal{A}$ sits in a short exact sequence $0 \to t \to a \to f \to 0$ with $t \in \mathcal{T}$ and $f \in \mathcal{F}$, see for example [1, Definition 1.1]. In fact $\mathcal{F} = \mathcal{T}^\perp$ and $\mathcal{T} = {}^\perp\mathcal{F}$ so that it is only necessary to specify either $\mathcal{T}$ or $\mathcal{F}$.

Let $\mathcal{A}$ be the heart of a t-structure $\mathcal{D}$. A torsion theory $\mathcal{T}$ in $\mathcal{A}$ determines a new t-structure $\langle \mathcal{D}, \mathcal{T}[-1] \rangle$, see [5, Proposition 2.1]: we say this new t-structure is obtained from the original by *left tilting* at $\mathcal{T}$ and denote its heart by $L_{\mathcal{T}} \mathcal{A}$. Explicitly $L_{\mathcal{T}} \mathcal{A} = \langle \mathcal{F}, \mathcal{T}[-1] \rangle$. A torsion theory also determines another t-structure $\langle \mathcal{D}^\perp, \mathcal{F}[1] \rangle^\perp$ by a 'double dual' construction. This is the shift by [1] of the other: we say it is obtained by *right tilting* at $\mathcal{T}$ and denote the new heart $\langle \mathcal{F}[1], \mathcal{T} \rangle$ by $R_{\mathcal{T}} \mathcal{A}$. Left and right tilting are inverse to one another: $\mathcal{F}$ is a torsion theory in $L_{\mathcal{T}} \mathcal{A}$ and right tilting with respect to this we recover the original heart $\mathcal{A}$. In the opposite direction, given a t-structure $\mathcal{E}$ with $\mathcal{D} \subset \mathcal{E} \subset \mathcal{D}[-1]$ Beligiannis and Reiten [1, Theorem 3.1] show that

$$\mathcal{T} = (\mathcal{E} \cap \mathcal{D}^\perp)[1] = \langle a \in \mathcal{A} \mid a = H^1 e \text{ for some } e \in \mathcal{E} \rangle$$

determines a torsion theory in $\mathcal{A}$ such that $L_{\mathcal{T}} \mathcal{A}$ is the heart of $\mathcal{E}$.

LEMMA 5.1. *If $d(\sigma, \tau) < 1/2$ then $\mathcal{A}_\tau$ can be obtained from $\mathcal{A}_\sigma$ by performing a left and then a right tilt. (Of course one or both of these tilts may be trivial.)*

Proof. Recall that $\mathcal{A}_\sigma = \mathcal{P}_\sigma(0,1]$ is the heart of the t-structure $\mathcal{P}_\sigma(0,\infty)$, and analogously for $\mathcal{A}_\tau$. Since $d(\sigma,\tau) < 1/2$ there are inclusions of t-structures

$$\mathcal{P}_\tau(0,\infty)$$

$$\downarrow$$

$$\mathcal{P}_\sigma(0,\infty) \longrightarrow \mathcal{P}_\tau\left(-\tfrac{1}{2},\infty\right) \longrightarrow \mathcal{P}_\sigma(-1,\infty)$$

$$\downarrow$$

$$\mathcal{P}_\tau(-1,\infty)$$

So $\mathcal{P}_\tau(-1/2,\infty)$ determines torsion theories $\mathcal{T}$ in $\mathcal{A}_\sigma$ and $\mathcal{T}'$ in $\mathcal{A}_\tau$ with

$$L_\mathcal{T}\mathcal{A}_\sigma = \mathcal{P}_\tau(-1/2,1/2] = L_{\mathcal{T}'}\mathcal{A}_\tau.$$

Hence $\mathcal{A}_\tau = R_{\mathcal{F}'}L_\mathcal{T}\mathcal{A}_\sigma$. $\qquad\square$

COROLLARY 5.2. *If σ and τ are in the same component of $\mathrm{Stab}(\mathcal{C})$ then the hearts $\mathcal{A}_\sigma$ and $\mathcal{A}_\tau$ are related by a finite sequence of left and right tilts.*

Proof. The connected components of $\mathrm{Stab}(\mathcal{C})$ are the path components. Choose a path from σ to τ, cover it with finitely many balls of diameter $< 1/2$ and apply Lemma 5.1. $\qquad\square$

For a heart $\mathcal{A}$ let $U(\mathcal{A}) = \{\sigma \in \mathrm{Stab}(\mathcal{C}) \mid \mathcal{A}_\sigma = \mathcal{A}\}$. Note that $U(\mathcal{A})$ may be empty and need not be either open or closed. The next result describes the hearts of stability conditions in the closure of $U(\mathcal{A})$.

COROLLARY 5.3. *Suppose σ_n is a sequence in $U(\mathcal{A})$ with $\sigma_n \to \sigma$. Then $\mathcal{A}_\sigma = R_\mathcal{T}\mathcal{A}$ for the torsion theory with*

$$\mathcal{T} = \langle 0 \neq a \in \mathcal{A} \mid \varphi_n^-(a) \not\to 0\rangle$$

and $\mathcal{F} = \langle 0 \neq a \in \mathcal{A} \mid \varphi_n^+(a) \to 0\rangle$.

Proof. It is easy to check that $\mathcal{T}$ is a torsion theory; the required short exact sequences arise from the description of the Harder–Narasimhan σ-filtration as a limiting filtration. The description of the limiting semistable objects shows that $R_\mathcal{T}\mathcal{A} \subset \mathcal{A}_\sigma$. Hence they are equal as hearts of distinct bounded t-structures cannot be nested. (To see this, suppose $\mathcal{H} \subset \mathcal{K}$ are nested hearts of bounded t-structures $\mathcal{D}_\mathcal{H}$ and $\mathcal{D}_\mathcal{K}$ respectively. Then

$$\mathcal{D}_\mathcal{H} = \langle \mathcal{H}, \mathcal{H}[1],\ldots\rangle \subset \langle \mathcal{K}, \mathcal{K}[1],\ldots\rangle = \mathcal{D}_\mathcal{K}$$

and $\mathcal{D}_\mathcal{H}^\perp = \langle \mathcal{H}[-1], \mathcal{H}[-2],\ldots\rangle \subset \langle \mathcal{K}[-1], \mathcal{K}[-2],\ldots\rangle = \mathcal{D}_\mathcal{K}^\perp$. Together these imply $\mathcal{D}_\mathcal{H} = \mathcal{D}_\mathcal{K}$ as claimed.) $\qquad\square$

It is not always the case that a torsion theory has the form $\langle 0 \neq a \in \mathcal{A} \mid \varphi_n^-(a) \not\to 0\rangle$ for some Cauchy sequence σ_n of stability conditions, and it is not true that $\mathcal{B} = R_\mathcal{T}\mathcal{A}$ implies $\overline{U(\mathcal{A})} \cap U(\mathcal{B}) \neq \emptyset$. For example, let $\mathcal{A}$ be the category of representations of the Kronecker quiver. This is well-known to be derived equivalent to the coherent derived category of $\mathbb{P}^1$ and (after fixing an appropriate equivalence) $\mathcal{A} = L_\mathcal{T}\mathrm{Coh}(\mathbb{P}^1)$ where $\mathcal{T} = {}^\perp\langle \mathcal{O}(d) \mid d < 0\rangle$. However there is no sequence σ_n in $U(\mathcal{A})$ with $\sigma_n \to \sigma$ where $\mathcal{A}_\sigma = \mathrm{Coh}(\mathbb{P}^1)$; see [8, §3.2] for details.

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