Partial pole placement by feedback control with inaccessible degrees of freedom

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Abstract

Classical pole-placement theory requires that every degree of freedom shall be accessible to sensing but in physical systems there are often obstructions that make sensing at certain degrees of freedom impractical. In the classical formulation of the pole placement problem the input vector which determines the actuator gains is given and the pole placement problem is linear. If the input vector is not known and it is desired to find the gains of actuators and the gains of the measured state subject to some constraints then the problem becomes nonlinear since the unknown parameters multiply each other. It is shown that this nonlinear active vibration control problem is rendered linear by the application of a new double input control methodology implemented in conjunction with a receptance-based scheme where full pole placement is achieved while some chosen degrees of freedom are free from both actuation and sensing. A lower bound on the maximum number of degrees of freedom inaccessible to both actuation and sensing is established. A numerical example is provided to demonstrate the working of the method using the new double-input approach.

Keywords: Active vibration control; partial pole placement; inaccessible degrees of freedom; double-input control; receptance method.

1. Introduction

The eigenvalue assignment problem has many potential applications in structural dynamics, including the improvement of the stability of dynamic systems, avoidance of the damaging large-amplitude vibrations close to resonance, and adaptive changes to system behaviour. A variety of eigenvalue assignment algorithms have been developed over several decades, namely, eigenvalue assignment by state feedback [1, 2], eigenvalue assignment by output feedback [3-8], robust eigenvalue assignment problem [9-15], to name but a few. In practice, there may be a large number of eigenvalues but only a few that are undesirable. Therefore, partial pole placement, where some eigenvalues are required to be relocated and the remaining poles are rendered unchanged, is of practical value in suppressing vibration and stabilising dynamic systems. Saad [16] proposed a projection algorithm for the partial eigenvalue assignment for first-order systems. Datta et al. [17] developed a closed-form solution to the partial pole assignment problem by state feedback control in systems represented by second order differential equations. The method has been generalised for the case of multi-input control [18, 19]. Chu [15] proposed a partial pole assignment method with state feedback for second-order systems. The robust partial pole placement problem was investigated in [20-23]. The problem of optimising the control effort in partial eigenvalue assignment was addressed by Guzzardo et al. [24]. Partial pole placement with time delay was also considered [25-27].

Ram and Mottershead [28] developed a new theory known as the receptance method for eigenvalue assignment in active vibration control using experimental measurements. While in conventional pole placement methods analytical models are required, in the method of receptances, measured modal data are used instead of system matrices. Therefore, the receptance method has a wealth of advantages. There is no need to estimate or know the mass, stiffness and damping matrices, no need to estimate the unmeasured state using an observer or a Kalman filter and no need for model reduction. By virtue of partial controllability, a partial pole placement approach using measured receptances for single-input and multi-input feedback control was proposed by Tehrani et al. [29]. Very recently, Ram and Mottershead [30] developed a new theory of receptance-based partial pole placement by using partial observability. A series of experimental tests were carried out to demonstrate the capability of the receptance method in active vibration suppression [29, 31-33].

In the traditional application of active vibration control by partial pole placement with state feedback the input vectors are assumed to be given and the calculated vectors of the control gain are therefore in general fully populated. Consequently, to realize the control in practice it is required to sense the state at each degree of freedom. In applications, however, some of the degrees of freedom may not be physically accessible to actuation and sensing simultaneously. That is, there exist some inaccessible degrees of freedom.

The purpose of the paper is quite different from sparse controllability problem [34, 35] whereby optimisation is applied to ensure controllability (and separately observability) with as few variables as possible, leading the fewest total number of sensors and actuators. The present work is motivated by engineering practicality where not every degree of freedom is available for sensing, and if it is not available for sensing it is not available for actuation either. One example is the helicopter rotor blade requiring active vibration control but inaccessible to both actuation and sensing.

In the present paper the input and control-gain vectors are determined and the resulting interactions between unknown terms that would normally lead to nonlinearity are circumvented by the use of a new double input control involving position, velocity and acceleration feedback. This enables the retained modes to be separated into two sets resulting in a linear system of constrains. Further constraints are applied to assign the other modes and it is seen that the nonlinear problem of determining input vectors and the control gains for partial pole placement with inaccessible degrees of freedom is converted into a linear one. A lower bound on the maximum number of degrees of freedom completely cleared of both sensing and actuation is then established using purely linear analysis. Since the main objective is the introduction of a new concept, we address for simplicity the case involving distinct eigenvalues in both open and closed loops. Systems with repeated eigenvalues will be considered in further work beyond the scope of the present article.

Section 2 of this paper establishes the basis for the analysis that follows. In Sections 3 and 4 the necessary equations are established for partial pole placement with inaccessible actuators and sensors represented by zero terms in the input vector and the control-gain vectors. Section 5 establishes the solvability conditions that enable lower bounds on the maximum numbers of inaccessible actuators and sensors to be determined. Then in Section 6 a lower bound on the maximum number of degrees of freedom inaccessible to both actuation and sensing is achieved by equating the solutions obtained in the previous section. A numerical example is used to demonstrate the working of the proposed theory.

2. Motivation

The motion of the n degree of freedom system

$$\mathbf{M}\ddot{\mathbf{x}}+\mathbf{C}\dot{\mathbf{x}}+\mathbf{K}\mathbf{x}=\mathbf{0}\,, \tag{1}$$

where $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ are symmetric $n\times n$ matrices and where $\mathbf{M}$ is positive-definite and $\mathbf{C}$ and $\mathbf{K}$ are positive-semidefinite, may be altered by state feedback control

$$\mathbf{M}\ddot{\mathbf{x}}+\mathbf{C}\dot{\mathbf{x}}+\mathbf{K}\mathbf{x}=\mathbf{b}u\left(t\right) \tag{2}$$

where

$$u\left(t\right)=\mathbf{f}^{T}\dot{\mathbf{x}}+\mathbf{g}^{T}\mathbf{x} \tag{3}$$

and where $\mathbf{b}$, $\mathbf{f}$ and $\mathbf{g}$ are real vectors denoting force-distribution and control-gain terms.

The quadratic eigenvalue problem corresponding to the open loop system (1) is given by

$$\left(\lambda_k^2\mathbf{M}+\lambda_k\mathbf{C}+\mathbf{K}\right)\mathbf{v}_k=\mathbf{0},\qquad k=1,2,\cdots,2n\,. \tag{4}$$

The self-conjugate set of $2n$ poles, $\{\lambda_k\}_{k=1}^{2n}$, with corresponding eigenvectors $\{\mathbf{v}_k\}_{k=1}^{2n}$ that satisfy (4) are the eigenpairs of the open-loop system.

Similarly, the eigenvalue problem of the closed loop system (2) is

$$\left(\mu_k^2\mathbf{M}+\mu_k\left(\mathbf{C}-\mathbf{bf}^T\right)+\mathbf{K}-\mathbf{bg}^T\right)\mathbf{w}_k=\mathbf{0},\qquad k=1,2,\cdots,2n\,. \tag{5}$$

with the self-conjugate set of $2n$ poles, $\{\mu_k\}_{k=1}^{2n}$, and corresponding eigenvectors $\{\mathbf{w}_k\}_{k=1}^{2n}$. The eigenvalues of the open-loop system are assumed to be distinct, as are those of the closed-loop systems, the case of repeated roots and defective systems is to be considered in further work beyond the scope of the present article.

To regulate the dynamic of the open loop system (1) it is frequently required to alter a subset of eigenvalues. Since the eigenvalues may be ordered arbitrarily, without loss of generality we may assume that the $2m \le 2n$ poles of the self-conjugate set $\{\lambda_k\}_{k=1}^{2m}$ associated with (4) are required to be changed to a predetermined self-conjugate set $\{\mu_k\}_{k=1}^{2m}$ by the applied control force. To avoid spillover it is further requested that $\{\mu_k\}_{k=2m+1}^{2n}=\{\lambda_k\}_{k=2m+1}^{2n}$. These conditions may be thus written in the form

$$\mu_k=\begin{cases}\mu_k & k=1,2,...,2m\\ \lambda_k & k=2m+1,2m+2,...,2n.\end{cases} \tag{6}$$

The classical problem of partial pole placement by state feedback control is formulated as follows.

Problem 1: *Partial pole assignment by state feedback control*

Given: $\mathbf{M}$, $\mathbf{C}$, $\mathbf{K}$, $\mathbf{b}$ and a self-conjugate set $\{\mu_k\}_{k=1}^{2m}$

Find: $\mathbf{f}$, $\mathbf{g}$ such that the poles of (5) form the closed-conjugate set (6).

Datta, Elhay and Ram [17] gave a closed form solution to Problem 1. Their solution showed for example that when we choose $\mathbf{b}=\mathbf{e}_1$, where $\mathbf{e}_k$ is the *k*-th unit vector, the solution generally leads to fully populated vectors of control gains $\mathbf{f}$ and $\mathbf{g}$. The physical meaning is that the state feedback control may be implemented in general by one actuator and n sensors measuring the complete state of the system in real time. In practice, however, some of the degrees of freedom may not be accessible to both sensing and actuation. For brevity we will refer to such degrees of freedom as the *inaccessible degrees of freedom.*

Since the degrees of freedom may be numbered arbitrarily, without loss of generality we may assume that the last p degrees of freedom are inaccessible. Let

$$\sigma_k=\left(\mathbf{e}_k^T\mathbf{b}\right)^2+\left(\mathbf{e}_k^T\mathbf{f}\right)^2+\left(\mathbf{e}_k^T\mathbf{g}\right)^2, \tag{7}$$

then the condition

$$\sum_{k=n-p+1}^{n} \sigma_k = 0\,, \tag{8}$$

implies that there is no need to sense or actuate the last p degrees of freedom since every term in (8) is non-negative and therefore $b_k = f_k = g_k = 0$ for $k = n - p + 1, n - p + 2, ..., n$.

In addressing the problem of state feedback control with inaccessible degrees of freedom we may thus attempt to modify Problem 1 to the problem of finding $\mathbf{b}$, $\mathbf{f}$ and $\mathbf{g}$ subject to the constraint (8). Problem 1, which is linear, would then be changed to a non-linear problem since the unknowns elements of $\mathbf{b}$ interact with the unknown elements of $\mathbf{f}$ and $\mathbf{g}$ nonlinearly.

It will be shown in this paper that with a new double-input controller taking form

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{C}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{b_1}u_1(t) + \mathbf{b_2}u_2(t) \tag{9}$$

where,

$$u_1(t) = \mathbf{f}^T\ddot{\mathbf{x}} + \mathbf{g}^T\dot{\mathbf{x}} \tag{10}$$

$$u_2(t) = \mathbf{f}^T\dot{\mathbf{x}} + \mathbf{g}^T\mathbf{x} \tag{11}$$

it is possible to solve in a linear fashion the partial pole placement with inaccessible degrees of freedom. In (9) the vectors $\mathbf{b}_1$ and $\mathbf{b}_2$ represent the distributions of control forces. The magnitudes of individual terms denote amplification factors to be applied to the inputs u_1 and u_2. The closed loop quadratic eigenvalue problem associated with (9) in conjunction with (10) and (11) then becomes

$$\left(\mu_k^2\mathbf{M} + \mu_k\mathbf{C} + \mathbf{K}\right)\mathbf{w}_k = \left(\mu_k^2\mathbf{b}_1\mathbf{f}^T + \mu_k\left(\mathbf{b}_1\mathbf{g}^T + \mathbf{b}_2\mathbf{f}^T\right) + \mathbf{b}_2\mathbf{g}^T\right)\mathbf{w}_k, \quad k = 1, 2, \cdots, 2n\,. \tag{12}$$

The control force on the right-hand-side of (12) may be rewritten as

$$\left(\mu_k^2\mathbf{b}_1\mathbf{f}^T + \mu_k\left(\mathbf{b}_1\mathbf{g}^T + \mathbf{b}_2\mathbf{f}^T\right) + \mathbf{b}_2\mathbf{g}^T\right)\mathbf{w}_k = \left(\mu_k\mathbf{b}_1 + \mathbf{b}_2\right)\left(\mu_k\mathbf{f}^T + \mathbf{g}^T\right)\mathbf{w}_k \tag{13}$$

so that the eigenvalue problem (12) becomes

$$\left(\mu_k^2\mathbf{M} + \mu_k\mathbf{C} + \mathbf{K}\right)\mathbf{w}_k = \left(\mu_k\mathbf{b}_1 + \mathbf{b}_2\right)\left(\mu_k\mathbf{f}^T + \mathbf{g}^T\right)\mathbf{w}_k, \quad k = 1, 2, \cdots, 2n \tag{14}$$

and the condition (8), with

$$\sigma_k = \left(\mathbf{e}_k^T\mathbf{b}_1\right)^2 + \left(\mathbf{e}_k^T\mathbf{b}_2\right)^2 + \left(\mathbf{e}_k^T\mathbf{f}\right)^2 + \left(\mathbf{e}_k^T\mathbf{g}\right)^2 \tag{15}$$

ensures that there is no need to actuate or sense the last p degrees of freedom of the controlled system (9).

The problem under consideration is thus

Problem 2: *Partial pole assignment with inaccessible degrees*

Given: $\mathbf{M}$, $\mathbf{C}$, $\mathbf{K}$, $\{\mu_k\}_{k=1}^{2m}$ and $0 < p < n$

Find: $\mathbf{b}_1$, $\mathbf{b}_2$, $\mathbf{f}$ and $\mathbf{g}$ such that the poles of (14) form the set (6) and the condition (8), in conjunction with (15), is satisfied.

3. Immovable and assigned eigenvalues

We begin by writing the open-loop and closed-loop eigenvalue problems, (4) and (14), as

$$\left(\lambda_k^2\mathbf{M} + \lambda_k\mathbf{C} + \mathbf{K}\right)\mathbf{v}_k = \mathbf{0}, \qquad k = 1, 2, ..., 2n \tag{16}$$

and

$$\left(\mu_k^2\mathbf{M} + \mu_k\mathbf{C} + \mathbf{K}\right)\mathbf{w}_k = \left(\mu_k\mathbf{b}_1 + \mathbf{b}_2\right)\left(\mu_k\mathbf{f}^T + \mathbf{g}^T\right)\mathbf{w}_k, \quad k = 1, 2, ..., 2n \tag{17}$$

with the understanding that

$$\mu_k = \lambda_k, \qquad k = 2m+1, 2m+2, ..., 2n \tag{18}$$

and $\{\mu_k\}_{k=1}^{2m}$ are assumed to be distinct from $\{\lambda_k\}_{k=1}^{2n}$.

It is apparent that the right-hand-side of (17) is asymmetric, so that the closed-loop eigenvalue problem may be expressed in terms of the left eigenvector $\mathbf{z}_k$,

$$\mathbf{z}_k^T\left(\mu_k^2\mathbf{M} + \mu_k\mathbf{C} + \mathbf{K}\right) = \mathbf{z}_k^T\left(\mu_k\mathbf{b}_1 + \mathbf{b}_2\right)\left(\mu_k\mathbf{f}^T + \mathbf{g}^T\right), \quad k = 1, 2, ..., 2n \tag{19}$$

It follows from (17) that $\mu_k = \lambda_k$ and $\mathbf{w}_k = \mathbf{v}_k$ whenever

$$\left(\lambda_k\mathbf{f}^T + \mathbf{g}^T\right)\mathbf{v}_k = 0 \tag{20}$$

and from (19) that $\mu_k = \lambda_k$ and $\mathbf{z}_k = \mathbf{v}_k$ when

$$\mathbf{v}_k^T\left(\lambda_k\mathbf{b}_1 + \mathbf{b}_2\right) = 0 \tag{21}$$

We now partition the set of unchanged eigenvalues $\{\mu_k\}_{k=2m+1}^{2n} = \{\lambda_k\}_{k=2m+1}^{2n}$, closed under conjugation, so that those eigenvalues with indices $k = 2m+1, \ldots, 2\tau$, rendered unchanged by virtue of (20), are separated from those with $k = 2\tau+1,\ldots,2n$, given by satisfaction of (21) and $m \le \tau \le n$. To summarise, there are $2m$ eigenvalues to be assigned arbitrarily, $2(\tau - m)$ that are unchanged due to (20) and $2(n-\tau)$ unchanged due to (21) as shown in Fig. 1

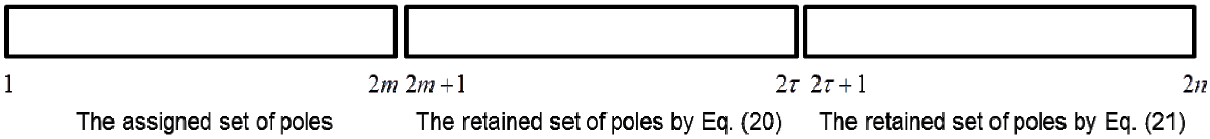

Fig. 1 Eigenvalues assigned and retained

To ensure that $\{\mu_k\}_{k=2m+1}^{2\tau} = \{\lambda_k\}_{k=2m+1}^{2\tau}$ we re-write equation (20) in the form

$$\mathbf{Q}\begin{pmatrix}\mathbf{f}\\ \mathbf{g}\end{pmatrix} = \mathbf{0}, \qquad \mathbf{Q} = \begin{bmatrix} \lambda_{2m+1}\mathbf{v}_{2m+1}^T & \mathbf{v}_{2m+1}^T \\ \lambda_{2m+2}\mathbf{v}_{2m+2}^T & \mathbf{v}_{2m+2}^T \\ \vdots & \vdots \\ \lambda_{2\tau}\mathbf{v}_{2\tau}^T & \mathbf{v}_{2\tau}^T \end{bmatrix}. \tag{22}$$

Likewise equation (21) may be recast to ensure $\{\mu_k\}_{k=2\tau+1}^{2n} = \{\lambda_k\}_{k=2\tau+1}^{2n}$

$$\mathbf{\Phi}\begin{pmatrix}\mathbf{b}_1\\ \mathbf{b}_2\end{pmatrix} = \mathbf{0}, \qquad \mathbf{\Phi} = \begin{bmatrix} \lambda_{2\tau+1}\mathbf{v}_{2\tau+1}^T & \mathbf{v}_{2\tau+1}^T \\ \lambda_{2\tau+2}\mathbf{v}_{2\tau+2}^T & \mathbf{v}_{2\tau+2}^T \\ \vdots & \vdots \\ \lambda_{2n}\mathbf{v}_{2n}^T & \mathbf{v}_{2n}^T \end{bmatrix}. \tag{23}$$

The rows of $\mathbf{Q}$ and $\mathbf{\Phi}$ are independent when the retained eigenvalues of the open-loop system are distinct.

The assignment of $2m$ eigenvalues $\{\mu_k\}_{k=1}^{2m}$ is achieved as in [30] by the satisfaction of characteristic equations arranged in the form,

$$\mathbf{P}\begin{pmatrix}\mathbf{f}\\ \mathbf{g}\end{pmatrix} = \tilde{\mathbf{e}} \tag{24}$$

where

$$\mathbf{P} = \begin{bmatrix} \mu_1\mathbf{r}_1^T & \mathbf{r}_1^T \\ \mu_2\mathbf{r}_2^T & \mathbf{r}_2^T \\ \vdots & \vdots \\ \mu_{2m}\mathbf{r}_{2m}^T & \mathbf{r}_{2m}^T \end{bmatrix}, \quad \mathbf{r}_k = \left(\mu_k^2\mathbf{M} + \mu_k\mathbf{C} + \mathbf{K}\right)^{-1}\left(\mu_k\mathbf{b}_1 + \mathbf{b}_2\right); \quad \tilde{\mathbf{e}} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \in \mathbb{R}^{2m}. \tag{25}$$

The rows of $\mathbf{P}$ are independent when the assigned eigenvalues $\{\mu_k\}_{k=1}^{2m}$ are distinct.

4. Degrees of freedom free of actuation and sensing

Let us assume that the number of inaccessible actuators is $p_1, 0 < p_1 < n$, the number of inaccessible sensors is $p_2, 0 < p_2 < n$ and zero entries are placed in the last p_1 terms of $\mathbf{b}_1$ and $\mathbf{b}_2$ and in the last p_2 terms of $\mathbf{f}$ and $\mathbf{g}$. Since there is no restriction on the choice of degrees of freedom to be assigned zero entries, we may write

$$\mathbf{E}\begin{pmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \end{pmatrix} = \mathbf{0}, \qquad \mathbf{E} = \begin{bmatrix} \mathbf{E}_{n-p_1+1} \\ \mathbf{E}_{n-p_1+2} \\ \vdots \\ \mathbf{E}_n \end{bmatrix}. \tag{26}$$

In addition,

$$\bar{\mathbf{E}}\begin{pmatrix} \mathbf{f} \\ \mathbf{g} \end{pmatrix} = \mathbf{0}, \qquad \bar{\mathbf{E}} = \begin{bmatrix} \mathbf{E}_{n-p_2+1} \\ \mathbf{E}_{n-p_2+2} \\ \vdots \\ \mathbf{E}_n \end{bmatrix}. \tag{27}$$

where $\mathbf{E}_k$ is a $2\times 2n$ matrix

$$\mathbf{E}_k = \begin{bmatrix} \mathbf{e}_k^T \\ \mathbf{e}_{k+n}^T \end{bmatrix}. \tag{28}$$

The rows of $\mathbf{E}$ and $\bar{\mathbf{E}}$ are by definition independent.

5. Lower bounds on the maximum numbers of inaccessible actuators and sensors

In this section conditions are established that determine lower bounds on the maximum numbers of inaccessible actuators and sensors. These include the existence of nontrivial solutions for the force-distribution terms $\begin{pmatrix} \mathbf{b}_1^T & \mathbf{b}_2^T \end{pmatrix}^T$ and that such solutions are always admitted when $p_1 \le (\tau - 1)$. Then the conditions under which exact solutions are admitted for the control gains $\left(\mathbf{f}^T \quad \mathbf{g}^T\right)^T$ are established. It is shown that certain identical exact solutions are available for $p_2 \le (n-\tau)$ to guarantee at least $(n-\tau)$ null terms in $\mathbf{f}$ and $\mathbf{g}$. Thus the lower bounds on the maximum numbers of inaccessible actuators and sensors are found to be $p_1 = \breve{p}_1 = (\tau - 1)$ and $p_2 = \breve{p}_2 = (n-\tau)$ respectively.

We begin by establishing the necessary systems of equations. Thus, by combining equations (23) and (26),

$$\begin{bmatrix} \mathbf{\Phi} \\ \mathbf{E} \end{bmatrix} \begin{pmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \end{pmatrix} \tag{29}$$

and also equations (22), (24) and (27),

$$\mathbf{S} \begin{pmatrix} \mathbf{f} \\ \mathbf{g} \end{pmatrix} = \begin{pmatrix} \tilde{\mathbf{e}} \\ \mathbf{0} \\ \mathbf{0} \end{pmatrix}; \quad \mathbf{S} = \begin{bmatrix} \mathbf{P} \\ \mathbf{Q} \\ \bar{\mathbf{E}} \end{bmatrix} \tag{30}$$

The inaccessible actuators and sensors are denoted by vanishing entries placed in the last p_1 terms of $\mathbf{b}_1$ and $\mathbf{b}_2$ and in the last p_2 terms of $\mathbf{f}$ and $\mathbf{g}$ respectively. Thus,

$$\begin{pmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \end{pmatrix} = \begin{pmatrix} \tilde{\mathbf{b}}_1 \\ \mathbf{0}_{(p_1 \times 1)} \\ \tilde{\mathbf{b}}_2 \\ \mathbf{0}_{(p_1 \times 1)} \end{pmatrix}; \quad \begin{pmatrix} \mathbf{f} \\ \mathbf{g} \end{pmatrix} = \begin{pmatrix} \tilde{\mathbf{f}} \\ \mathbf{0}_{(p_2 \times 1)} \\ \tilde{\mathbf{g}} \\ \mathbf{0}_{(p_2 \times 1)} \end{pmatrix} \tag{31}$$

Then equations (22), (23) and (24) may be recast in the form,

$$\begin{bmatrix} \mathbf{\Phi}_1 & \mathbf{\Phi}_2 \end{bmatrix} \begin{pmatrix} \tilde{\mathbf{b}}_1 \\ \tilde{\mathbf{b}}_2 \end{pmatrix} = \mathbf{0}; \quad \mathbf{\Phi}_1 = \mathbf{\Phi}_{(\,:\,,1:n-p_1)}; \quad \mathbf{\Phi}_2 = \mathbf{\Phi}_{(\,:\,,n+1:2n-p_1)} \tag{32}$$

$$\begin{bmatrix} \mathbf{P}_1 & \mathbf{P}_2 \\ \mathbf{Q}_1 & \mathbf{Q}_2 \end{bmatrix} \begin{pmatrix} \tilde{\mathbf{f}} \\ \tilde{\mathbf{g}} \end{pmatrix} = \begin{pmatrix} \tilde{\mathbf{e}} \\ \mathbf{0} \end{pmatrix}; \quad \begin{matrix} \mathbf{P}_1 = \mathbf{P}_{(\,:\,,1:n-p_2)}; & \mathbf{P}_2 = \mathbf{P}_{(\,:\,,n+1:2n-p_2)} \\ \mathbf{Q}_1 = \mathbf{Q}_{(\,:\,,1:n-p_2)}; & \mathbf{Q}_2 = \mathbf{Q}_{(\,:\,,n+1:2n-p_2)} \end{matrix} \tag{33}$$

where,

$$\dim\begin{bmatrix} \mathbf{\Phi}_1 & \mathbf{\Phi}_2 \end{bmatrix} = 2(n-\tau) \times 2(n-p_1); \quad \dim\begin{bmatrix} \mathbf{P}_1 & \mathbf{P}_2 \\ \mathbf{Q}_1 & \mathbf{Q}_2 \end{bmatrix} = 2\tau \times 2(n-p_2) \tag{34}$$

Lemma 1: There exists a non-trivial solution $\left(\mathbf{b}_1^T \quad \mathbf{b}_2^T\right)^T$ to equation (29) if and only if $\operatorname{rank}\begin{bmatrix} \mathbf{\Phi}_1 & \mathbf{\Phi}_2 \end{bmatrix} < 2(n-p_1)$.

Proof: The necessary and sufficient condition for the existence of a nontrivial solution to a homogeneous system of linear equations is that the rank of the coefficient matrix is smaller than the number of unknowns.

□

Corollary 1: There always exists a non-trivial solution $\left(\mathbf{b}_1^T \quad \mathbf{b}_2^T\right)^T$ to equation (29) when $p_1 \leq \tau - 1$.

Proof: From (34), if $p_1 \leq \tau - 1$ then $\text{nullity}\left(\left[\boldsymbol{\Phi}_1 \quad \boldsymbol{\Phi}_2\right]\right) \geq 2$.

Therefore, a nontrivial solution is available from,

$$\begin{pmatrix} \tilde{\mathbf{b}}_1 \\ \tilde{\mathbf{b}}_2 \end{pmatrix} = \mathbf{N}\boldsymbol{\alpha}; \quad \mathbf{N} = \text{null}\left[\boldsymbol{\Phi}_1 \quad \boldsymbol{\Phi}_2\right] \tag{35}$$

and $p_1 \leq \tau - 1$ denotes the number of null entries in $\left(\mathbf{b}_1^T \quad \mathbf{b}_2^T\right)^T$.

□

Corollary 2: The lower bound on the maximum number of inaccessible actuators is given by $\breve{p}_1 = \tau - 1$.

Proof: If $p_1 = \tau - 1$ and $2h$ of the $2(n-\tau)$ rows of $\left[\boldsymbol{\Phi}_1 \quad \boldsymbol{\Phi}_2\right]$ are redundant then $\text{nullity}\left(\left[\boldsymbol{\Phi}_1 \quad \boldsymbol{\Phi}_2\right]\right) = 2 + 2h$ and a further h inaccessible actuators may be admitted while still ensuring that $2(n-\tau)$ eigenvalues remain unchanged. Therefore the lower bound on the maximum number of inaccessible actuators is given when $h = 0$ so that $p_1 = \breve{p}_1 = \tau - 1$.

□

Lemma 2: There always exists one or more identical exact solutions $\left(\mathbf{f}^T \quad \mathbf{g}^T\right)^T$ to equation (30) for different $p_2 \leq (n-\tau)$ when $\text{rank}\begin{bmatrix} \mathbf{P}_1 & \mathbf{P}_2 \\ \mathbf{Q}_1 & \mathbf{Q}_2 \end{bmatrix}_{p_2 = n-\tau} = \text{rank}\begin{bmatrix} \mathbf{P}_1 & \mathbf{P}_2 & \tilde{\mathbf{e}} \\ \mathbf{Q}_1 & \mathbf{Q}_2 & \mathbf{0} \end{bmatrix}_{p_2 = n-\tau}$, and any other solution requires a greater number of sensors.

Proof: One or more exact solutions exist when $\text{rank}\begin{bmatrix} \mathbf{P}_1 & \mathbf{P}_2 \\ \mathbf{Q}_1 & \mathbf{Q}_2 \end{bmatrix}_{p_2 \leq n-\tau} = \text{rank}\begin{bmatrix} \mathbf{P}_1 & \mathbf{P}_2 & \tilde{\mathbf{e}} \\ \mathbf{Q}_1 & \mathbf{Q}_2 & \mathbf{0} \end{bmatrix}_{p_2 \leq n-\tau}$ so the right-hand-side of equation (33) is given by a linear combination of the columns of $\begin{bmatrix} \mathbf{P}_1 & \mathbf{P}_2 \\ \mathbf{Q}_1 & \mathbf{Q}_2 \end{bmatrix}_{p_2 \leq n-\tau}$. However, any exact solution when $p_2 = (n-\tau)$ is also a solution when $p_2 < (n-\tau)$ because the columns of $\begin{bmatrix} \mathbf{P}_1 & \mathbf{P}_2 \\ \mathbf{Q}_1 & \mathbf{Q}_2 \end{bmatrix}_{p_2=n-\tau}$ are included in $\begin{bmatrix} \mathbf{P}_1 & \mathbf{P}_2 \\ \mathbf{Q}_1 & \mathbf{Q}_2 \end{bmatrix}_{p_2 \le n-\tau}$. Other solutions exists when $p_2 < (n-\tau)$ but are given by the linear combination of a greater number of columns, therefore requiring a greater number of sensors.

□

Corollary 3: If $\operatorname{rank}\begin{bmatrix} \mathbf{P}_1 & \mathbf{P}_2 \\ \mathbf{Q}_1 & \mathbf{Q}_2 \end{bmatrix}_{p_2=n-\tau} = \operatorname{rank}\begin{bmatrix} \mathbf{P}_1 & \mathbf{P}_2 & \tilde{\mathbf{e}} \\ \mathbf{Q}_1 & \mathbf{Q}_2 & \mathbf{0} \end{bmatrix}_{p_2=n-\tau}$ then the lower bound on the maximum number of inaccessible sensors is given by $\breve{p}_2 = n - \tau$.

Proof: If 2ℓ of the 2τ rows of $\begin{bmatrix} \mathbf{P}_1 & \mathbf{P}_2 \\ \mathbf{Q}_1 & \mathbf{Q}_2 \end{bmatrix}_{p_2=n-\tau}$ are redundant then $\operatorname{nullity}\left(\begin{bmatrix} \mathbf{P}_1 & \mathbf{P}_2 \\ \mathbf{Q}_1 & \mathbf{Q}_2 \end{bmatrix}_{p_2=n-\tau}\right) = 2\ell$ and a further ℓ inaccessible sensors may be admitted while still ensuring that $2(\tau - m)$ eigenvalues remain unchanged and $2m$ eigenvalues are assigned. Therefore the lower bound on the maximum number of inaccessible sensors is given when $\ell = 0$ so that $p_2 = \breve{p}_2 = n - \tau$.

□

The solution of equation (30) is dependent upon the solution of (29) in that the eigenvalues to be assigned must be controllable. This imposes a condition on the solution of (29) that,

$$\begin{bmatrix} \lambda_k \mathbf{v}_k^T & \mathbf{v}_k^T \end{bmatrix} \begin{pmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \end{pmatrix} \neq 0; \quad k = 1, 2, \cdots, 2m \tag{36}$$

or,

$$\begin{bmatrix} \lambda_k \tilde{\mathbf{v}}_k^T & \tilde{\mathbf{v}}_k^T \end{bmatrix} \begin{pmatrix} \tilde{\mathbf{b}}_1 \\ \tilde{\mathbf{b}}_2 \end{pmatrix} \neq 0; \quad \tilde{\mathbf{v}}_k = (\mathbf{v}_k)_{(1:n-p_1)} \tag{37}$$

Lemma 3: The eigenvalues to be assigned in (30) are controllable when the $\boldsymbol{\alpha}$ is chosen so that

$$\begin{bmatrix} \lambda_k \tilde{\mathbf{v}}_k^T & \tilde{\mathbf{v}}_k^T \end{bmatrix} \mathbf{N}\boldsymbol{\alpha} \neq 0; \quad k = 1, 2, \cdots, 2m \tag{38}$$

Proof:

Equation (37) may be obtained by the combination of equation (38) with (35).

□

6. Lower bound on the maximum number of inaccessible degrees of freedom.

The numbers of degrees of freedom inaccessible to actuation and sensing are p_1 and p_2 respectively. Our objective is to have equal values for p_1 and p_2 so that the number of inaccessible degrees of freedom is maximised. It was already shown that the lower bound on the maximum numbers of inaccessible actuators is $\breve{p}_1 = \tau - 1$ and if equation (30) is satisfied for $p_2 = \breve{p}_2 = n - \tau$, the lower bound on the maximum numbers of inaccessible sensors is $\breve{p}_2 = n - \tau$. Therefore, under the solvability condition, a lower bound on the maximum inaccessible degrees of freedom may be achieved by equating $\breve{p}_1 = \tau - 1$ and $\breve{p}_2 = n - \tau$.

We have already established that the eigenvalues can be separated into three groups:

- $2m$ eigenvalues to be assigned
- $2(\tau - m)$ eigenvalues to be unchanged due to equation (29) and
- $2(n - \tau)$ eigenvalues to be unchanged due to equation (30).

where $n \geq \tau \geq m$.

Equal numbers of degrees of freedom without sensing and actuation can be achieved when $\breve{p}_1 = \breve{p}_2$, so that $\tau - 1 = n - \tau$ and is possible when n is odd and $m \leq \frac{n+1}{2}$, in which case $\tau = \frac{n+1}{2}$. This case is illustrated in Fig. 2 where it is seen that it corresponds to an optimal maximum solution $p = \breve{p}_1 = \breve{p}_2 = \frac{n-1}{2}$ of equations (29) and (30).

When n is even and $m \leq \frac{n+1}{2}$ a sub-optimal solution is obtained as shown in Fig. 3. This results in two solutions $\tau = \frac{n}{2}$ or $\tau = \frac{n}{2} + 1$ corresponding to $p = \frac{n}{2} - 1$. In practice, $\tau = \frac{n}{2} + 1$ is preferable because it requires fewer actuators than sensors. When $m > \frac{n+1}{2}$ and $\tau \geq m$ then the only solution available is that denoted by the thick line in Fig. 4. We are free to choose any value of $p = n - \tau$ on the thick line and the best available solution is $\tau = m$. This solution results in fewer degrees of freedom free of sensing than those free of actuation and, as such, is a practical solution because fewer actuators are required than sensors.

To summarise:

Case 1 - n is an odd number and $m \leq \frac{n+1}{2}$:

$$\tau = \frac{n+1}{2}; \quad p = \frac{n-1}{2} \tag{39}$$

Case 2 - n is an even number and $m \le \frac{n+1}{2}$:

$$\tau = \frac{n}{2} \quad \text{or} \quad \tau = \frac{n}{2}+1; \quad p = \frac{n}{2}-1 \tag{40}$$

Case 3 - $m > \frac{n+1}{2}$:

$$\tau = m; \quad p = n-m \tag{41}$$

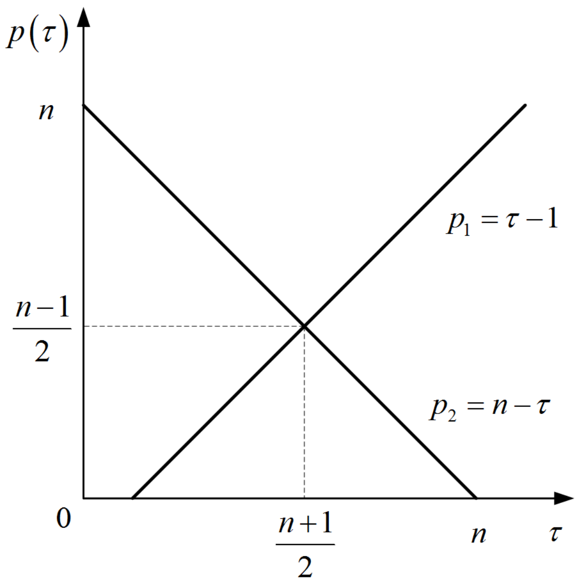

Fig. 2 Number of inaccessible degrees of freedom (Case 1)

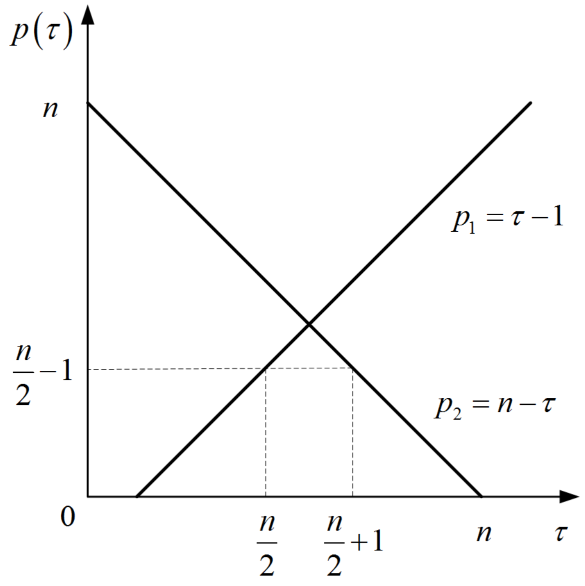

Fig. 3 Number of inaccessible degrees of freedom (Case 2)

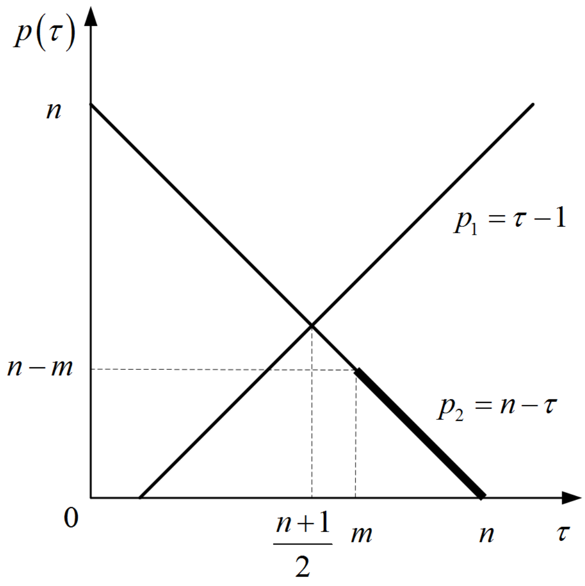

Fig. 4 Number of inaccessible degrees of freedom (Case 3)

The procedure for partial pole placement with inaccessible degrees of freedom may be summarised as follows:

1. Determine τ such that the lower bound on the maximum number of inaccessible degrees of freedom is achieved;

2. Choose $p_1 = \tau - 1$ and $p_2 = n - \tau$ and check the solvability of equation (30);

3. Solve equations (29) and (30).

Sufficient conditions for achieving the lower bound of the maximum number of inaccessible degrees of freedom are:

1. The force distribution vector should not be orthogonal to the first $2m$ modes (by choice of vector $\boldsymbol{\alpha}$);
2. $\operatorname{rank}\begin{bmatrix} \mathbf{P}_1 & \mathbf{P}_2 \\ \mathbf{Q}_1 & \mathbf{Q}_2 \end{bmatrix}_{p_2=n-\tau} = \operatorname{rank}\begin{bmatrix} \mathbf{P}_1 & \mathbf{P}_2 & \tilde{\mathbf{e}} \\ \mathbf{Q}_1 & \mathbf{Q}_2 & \mathbf{0} \end{bmatrix}_{p_2=n-\tau}$.

Example 1: Partial pole placement with inaccessible degrees of freedom

Consider the open loop system with

$$\mathbf{M}=\begin{bmatrix} 3 & & & \\ & 10 & & \\ & & 20 & \\ & & & 12 \end{bmatrix},\ \mathbf{C}=\begin{bmatrix} 2.3 & -1 & & \\ -1 & 2.2 & -1.2 & \\ & -1.2 & 2.7 & -1.5 \\ & & -1.5 & 1.5 \end{bmatrix},\ \mathbf{K}=\begin{bmatrix} 40 & -30 & & \\ -30 & 60 & -30 & \\ & -30 & 90 & -30 \\ & & -30 & 30 \end{bmatrix}.$$

The open-loop eigenvalues are

$$\begin{aligned} \lambda_{1,2} &= -0.0108 \pm 0.8736\mathrm{i} \\ \lambda_{3,4} &= -0.0809 \pm 1.6766\mathrm{i} \\ \lambda_{5,6} &= -0.1336 \pm 2.5280\mathrm{i} \\ \lambda_{7,8} &= -0.3980 \pm 4.0208\mathrm{i}. \end{aligned}$$

We wish to assign the first two pairs of eigenvalues while the remaining eigenvalues are unchanged

$$\begin{aligned} \mu_{1,2} &= -0.03 \pm 1\mathrm{i} \\ \mu_{3,4} &= -0.1 \pm 2\mathrm{i}. \end{aligned}$$

Following the analysis given in Section 6 the system has $p=1$ degree of freedom inaccessible when either $\tau=2$ or $\tau=3$. Here we choose $\tau=2$ and then $p_1=1$ and $p_2=2$. Equation (30) is found to be solvable. The vector $\begin{pmatrix}\mathbf{b}_1^T & \mathbf{b}_2^T\end{pmatrix}^T$ is required to be orthogonal to the last two pairs of open-loop eigenvectors

$$\mathbf{\Phi}\begin{pmatrix}\mathbf{b}_1 \\ \mathbf{b}_2\end{pmatrix}=\mathbf{0}$$

where

$$\boldsymbol{\Phi} = \begin{bmatrix} \lambda_5 \mathbf{v}_5^T & \mathbf{v}_5^T \\ \lambda_6 \mathbf{v}_6^T & \mathbf{v}_6^T \\ \lambda_7 \mathbf{v}_7^T & \mathbf{v}_7^T \\ \lambda_8 \mathbf{v}_8^T & \mathbf{v}_8^T \end{bmatrix}$$

$$\mathbf{v}_5^T = \begin{bmatrix} -0.0941+0.2578\mathrm{i} & -0.0829+0.1727\mathrm{i} & 0.1056-0.2807\mathrm{i} & -0.0738+0.1775\mathrm{i} \end{bmatrix}$$
$$\mathbf{v}_7^T = \begin{bmatrix} 0.0535+0.2107\mathrm{i} & -0.0220-0.0613\mathrm{i} & 0.0033+0.0077\mathrm{i} & -0.0006-0.0014\mathrm{i} \end{bmatrix}$$

$$\mathbf{v}_6 = \mathbf{v}_5^*, \quad \mathbf{v}_8 = \mathbf{v}_7^*.$$

It is assumed that the fourth degree of freedom is inaccessible. Then

$$\mathbf{E}_4 \begin{pmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \end{pmatrix} = \mathbf{0}$$

where

$$\mathbf{E}_4 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Then

$$\begin{pmatrix} \tilde{\mathbf{b}}_1 \\ \tilde{\mathbf{b}}_2 \end{pmatrix} = \mathbf{N}\boldsymbol{\alpha}, \quad \mathbf{N} = \mathrm{null}\left(\begin{bmatrix} \boldsymbol{\Phi}_1 & \boldsymbol{\Phi}_2 \end{bmatrix}\right), \qquad \boldsymbol{\Phi}_1 = \boldsymbol{\Phi}_{(:,1:3)} \text{ and } \boldsymbol{\Phi}_2 = \boldsymbol{\Phi}_{(:,5:7)}.$$

By choosing $\boldsymbol{\alpha} = \begin{pmatrix} 0.5 & 1 \end{pmatrix}^T$, we obtain

$$\mathbf{b}_1 = \begin{pmatrix} -0.1277 \\ -0.4544 \\ -0.3831 \\ 0 \end{pmatrix}, \qquad \mathbf{b}_2 = \begin{pmatrix} 0.2199 \\ 1.0059 \\ 0.9057 \\ 0 \end{pmatrix}.$$

Also, from

$$\begin{bmatrix} \mathbf{P} \\ \mathbf{E}_3 \\ \mathbf{E}_4 \end{bmatrix} \begin{pmatrix} \mathbf{f} \\ \mathbf{g} \end{pmatrix} = \begin{pmatrix} \tilde{\mathbf{e}} \\ \mathbf{0} \\ \mathbf{0} \end{pmatrix}$$

where

$$\mathbf{P} = \begin{bmatrix} \mu_1 \mathbf{r}_1^T & \mathbf{r}_1^T \\ \mu_2 \mathbf{r}_2^T & \mathbf{r}_2^T \\ \mu_3 \mathbf{r}_3^T & \mathbf{r}_3^T \\ \mu_4 \mathbf{r}_4^T & \mathbf{r}_4^T \end{bmatrix},$$

$$\mathbf{r}_1^T = [-0.0869 + 0.0672\mathrm{i} \quad -0.1165 + 0.0848\mathrm{i} \quad -0.1399 + 0.0916\mathrm{i} \quad -0.2343 + 0.1512\mathrm{i}],$$

$$\mathbf{r}_3^T = [-0.0547 + 0.0592\mathrm{i} \quad -0.0613 + 0.0615\mathrm{i} \quad -0.0168 + 0.0162\mathrm{i} \quad 0.0278 - 0.0269\mathrm{i}],$$

$$\mathbf{r}_2 = \mathbf{r}_1^*, \qquad \mathbf{r}_4 = \mathbf{r}_3^*,$$

$$\tilde{\mathbf{e}}^T = \begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix},$$

and

$$\mathbf{E}_3 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

the control gains are found to be

$$\mathbf{f} = \begin{pmatrix} 0.4784 \\ -4.8277 \\ 0 \\ 0 \end{pmatrix}, \ \mathbf{g} = \begin{pmatrix} -17.1376 \\ 7.3027 \\ 0 \\ 0 \end{pmatrix}.$$

The last two terms of $\mathbf{g}$ and $\mathbf{f}$ are made zero so that there is one totally inaccessible degree of freedom and a further degree of freedom where there is actuation but no sensor.

Now closed-loop system becomes

$$\left(\mathbf{M} - \mathbf{b}_1 \mathbf{f}^T\right)\ddot{\mathbf{x}} + \left(\mathbf{C} - \mathbf{b}_1 \mathbf{g}^T - \mathbf{b}_2 \mathbf{f}^T\right)\dot{\mathbf{x}} + \left(\mathbf{K} - \mathbf{b}_2 \mathbf{g}^T\right)\mathbf{x} = 0$$

with roots given by

$$\begin{aligned} \mu_{1,2} &= -0.03 \pm 1\mathrm{i} \\ \mu_{3,4} &= -0.1 \pm 2\mathrm{i} \\ \mu_{5,6} &= -0.1336 \pm 2.5280\mathrm{i} \\ \mu_{7,8} &= -0.3980 \pm 4.0208\mathrm{i} \end{aligned}$$

which are exactly the prescribed eigenvalues.

7. Conclusions

A new theory for partial eigenvalue assignment by active vibration control in the presence of inaccessible degrees of freedom is proposed. A new double input control involving position, velocity and acceleration feedback is proposed. The eigenvalues of the open-loop, intended to be unchanged, are maintained in the closed-loop system by the application of orthogonality conditions on the input and feedback gain vectors, resulting in the appearance of zero terms in desired locations corresponding to degrees of freedom inaccessible to both actuation and sensing. The methodology is based entirely on linear systems of equations, thereby avoiding the need to use nonlinear optimisation routines. A lower bound on the maximum number of inaccessible degrees of freedom allowed for precise implementation of partial pole placement is given. The theory is of practical value to the vibration control of large-dimension structures with many inaccessible degrees of freedom.

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