Periodically fighting shake, rattle and roll

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How easy is it to suppress shake, rattle and roll in a long bridge or a skyscraper? Most practical structures are designed so that long wave resonance vibrations can be avoided. However, there are recent examples, such as the Millennium Bridge in London or the Volga Bridge in Volgograd, which show that unexpected external forces may result in large scale unwanted shake and rattle. Full scale alteration of a bridge (or a skyscraper) would not be considered as an acceptable option, unless the structure has collapsed.

Can we fix this by examining a representative part of the structure only and making small lightweight changes? We will do it here and illustrate an idea linking the engineering analysis to elastic waveguides.

Background of the problem. Figs. 1A and 1B show the troubled automobile bridge across the river Volga in Volgograd. The large deformation of the upper deck was completely unexpected. As described in the Daily Mail (see [1]) and in IMechanica professional discussion forum [2], large vibrations have been invoked by relatively small external forces. The bridge was inaugurated in October 2009, and in May 2010 a long wave resonance vibration caused sections of the bridge to bend as shown in Figs. 1B and 1C. A fundamental alteration of the bridge on its foundation appears to be problematic and no efficient technical solutions have been proposed to date. The model illustrated here gives a solution of this challenging problem and, being generic, also leads to design methodologies for vibrating slender structures such as skyscrapers and earthquake-resistant systems. The analysis complements the models of lateral vibration induced by "crowd-synchronization" and "balancing pedestrians" for the Millennium Bridge discussed in [3, 4, 5].

Although the general theory of waves in periodic structures is a classical topic described in many textbooks (see, for example, [6] and [7]), until recently the bulk of the work was related to problems of electromagnetism and acoustics as confirmed by the extensive bibliography on photonic band gap structures [8] compiled in 2008 by J.P. Dowling.

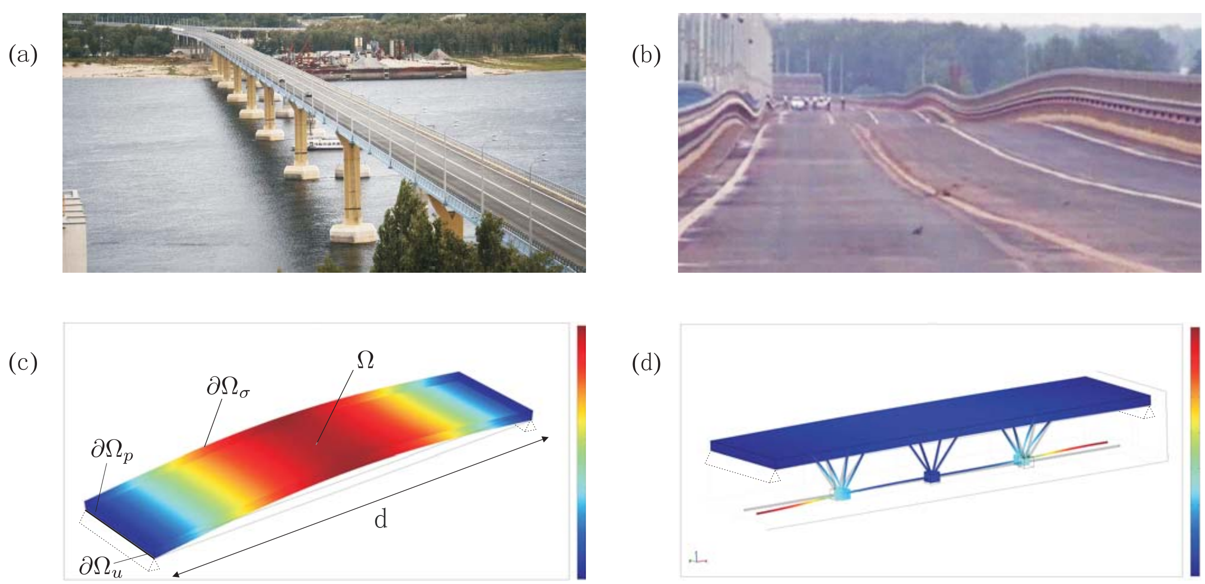

Figure 1: Volga bridge, ((A), (B) HTB Volgograd News). Flexural deformation of the main upper body of the bridge: (B) real structure, (C) simplified numerical model, (D) modified 3D structured wave guide with lightweight resonators.

In the recent years, the theory of elastic waves in periodic systems has received increasing attention, with applications ranging from the design of elastic filters and polarisers to the modelling of earthquake-resistant structures. Analytical models for phononic crystals, having periodic structures, have been developed in [9, 10, 11, 12, 13]. Transmission problems for arrays of elastic structured stacks were studied in [14, 15, 16]. These incorporate the comparative analysis of the filtering properties of elastic waves in doubly periodic media and the transmission properties for the corresponding singly periodic stack structure. Propagation and dispersion of waves in plate structures have been analysed in [17, 18, 19, 20]; in particular, Green's functions and localised vibration modes within plate structures are included in [18]. Heterogeneous lattices were used in [30, 31] to build models of structured interfaces, which possess specially designed transmission properties. These models also included the means of controlling dispersion properties. Effects of disorder on propagation of waves in structured media as well as localisation have been addressed in [16, 18], where the multipole Rayleigh method and a novel recurrence algorithm were used to obtain the transmission characteristics of a composite stack incorporating multiple arrays of circular inclusions/voids.

Smart composite structures have been designed and built for a range of applications in optics and problems of electromagnetism to encompass phenomena such as negative refraction and the cloaking of finite sized objects. Complex models have been constructed in [21, 22, 23]. The theoretical findings have also been verified experimentally [24, 25, 26, 27]. Specially designed composite structures, referred to as metamaterials, may channel waves around finite sized obstacles (cloaking) or create the effect of negative refraction which can be further used in focussing of electromagnetic waves by flat interfaces. Analysis of the geometrical transforms used in modelling of such materials is presented in [28]. Modelling of elastic waves brings new challenging questions in the design of metamaterials. In particular, a geometric transform was used in [29] to model an "invisibility cloak" for elastic waves.

In this paper, we consider a problem where elastic waves have to be channeled around some part of an engineering structure, for example the main deck of a bridge. The background work of the authors, which addresses analysis of waves interacting with structured interfaces, is included in [29, 20, 30]. Although, the problem is very different from the formulations of electromagnetism, there is a direct link with the design of metamaterial structures in [27]. Physically, the interpretation of the model can be thought of in terms of a "by-pass" for the undesired elastic waves. The method of design is non-trivial, and the analysis presented here is original. Our approach provides accurate estimates for frequencies of standing waves within the elastic structure and the design for the by-pass of elastic waves, so that shake, rattle and roll is no more.

Physical model for a periodic waveguide. Instead of looking into resonance modes of a finite slender body such as a bridge or a tall building, we consider a periodic waveguide and then analyse dispersion properties of Bloch-Floquet elastic waves. A typical span of the bridge between two neighboring pillars is shown in Fig. 1C and will be used as an elementary cell of the periodic structure.

The geometrical notations are introduced as follows. We denote by Ω the domain occupied by the elementary cell of the periodic waveguide. The boundary of Ω is divided into the constrained part $\partial\Omega_u$, which is in contact with the supporting pillars, the traction free part $\partial\Omega_\sigma$ and the contact region between consecutive unit cells $\partial\Omega_p$. These notations are shown in Fig. 1C, Fig. 2A and Fig. 6H.

The elastic displacement is assumed to be time-harmonic of angular frequency ω, and its amplitude $\mathbf{u}$ satisfies the Lamé equation

$$\mu\Delta\mathbf{u} + (\lambda+\mu)\nabla\nabla\cdot\mathbf{u} + \rho\omega^2\mathbf{u} = 0 \quad \text{in } \Omega, \tag{1}$$

where μ and λ are the Lamé elastic moduli and ρ is the mass density. The boundary conditions are

$$\mathbf{u} = 0, \quad \text{on} \quad \partial\Omega_u, \tag{2}$$

representing the fixed part of the boundary, and on the remaining boundary subjected to the traction boundary conditions we set

$$\mathbf{t}^{(n)}(\mathbf{u}) = 0, \quad \text{on} \quad \partial\Omega_\sigma, \tag{3}$$

where $\mathbf{t}^{(n)}(\mathbf{u})$ is the vector of tractions. A Bloch-Floquet wave, characterised by the wave number k, would also satisfy the quasi-periodicity condition within the elementary cell

$$\mathbf{u}(\mathbf{x} + d\mathbf{e}^{(1)}) = \mathbf{u}(\mathbf{x})e^{ikd}, \tag{4}$$

where d is the period of the structure. Subject to modification of boundary conditions, a similar formulation can be used to describe a time-harmonic vibration of a tall building. Analysis of dispersion properties of elastic waves characterizes the group velocity for different values of the frequency ω. In the low frequency range, this analysis also provides information about stop bands, defined as intervals of frequencies where no waves can propagate through the structured waveguide. One would also learn about standing waves, which may exist within the slender engineering structure and, in turn, one would be able to suppress undesired vibrations or channel waves around some parts of the structure such as the upper deck of a long bridge.

Two-dimensional implementation. Firstly, we provide an illustration of a periodic waveguide for a two-dimensional analogue of the model of the upper deck shown in Fig. 1C, which is given as a periodically constrained slender elastic solid, that can be approximated in the low frequency range as an elastic beam. We refer to [32] for technical details of the asymptotic analysis, which leads to the lower-dimensional *beam type* approximations of solutions of the problem (1)–(4) for the Lamé system in the desired range of frequencies. Within the elementary cell, the beam is simply supported and such a system possesses low frequency flexural modes. We show with a simple representative example (see Fig. 2), that such flexural modes can be suppressed by the addition of a lightweight periodic system of resonators. The analysis of dispersion properties of elastic waves enables one to tune the system accurately and to choose the correct design.

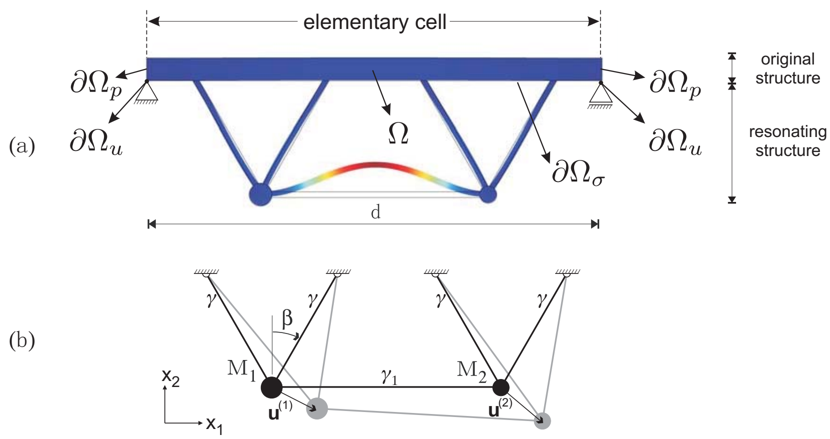

Figure 2: Two-dimensional elementary cell with the system of resonators. (A) The first eigenmode. (B) Truss model used for the analytical estimates of the eigenfrequencies: each mass M_i undergoes the horizontal and vertical displacements $u_1^{(i)}$ and $u_2^{(i)}$, respectively, with $i = 1, 2$. Parameter values are: $d = 4$ m; the thickness $s = 0.2$ m; the radii of the disks are 0.1 m and 0.075 m; $h_1 = 2$ m, $h_2 = 1$ m, $\beta = \pi/6$; the longitudinal stiffness coefficients are $\gamma = 0.14$ GPa, $\gamma_1 = 0.018$ GPa; the main plate has mass density $\rho = 7850$ kg/m^3 and shear modulus $\mu = 80$ GPa; the disks and the elastic links have mass density $\rho_M = 7850$ kg/m^3 and $\rho_\gamma = 200$ kg/m^3, respectively.

For flexural vibrations of the periodically constrained beam, the dispersion diagram, presenting the normalised frequency $\mathcal{F} = fd/v$ as a function of kd, is shown in Fig. 3A (Here $f = \omega/2\pi$ is the frequency, d is the period, v is the speed of the shear wave in the upper deck of the bridge).

The flexural mode, that is similar to the one shown in Fig. 1C, corresponds to $\mathcal{F} = 0.0358$. Our aim is to use the data from the dispersion diagram to tune the elastic system so that the above flexural mode is suppressed.

Now, we introduce a lightweight modification of the original structure which would alter

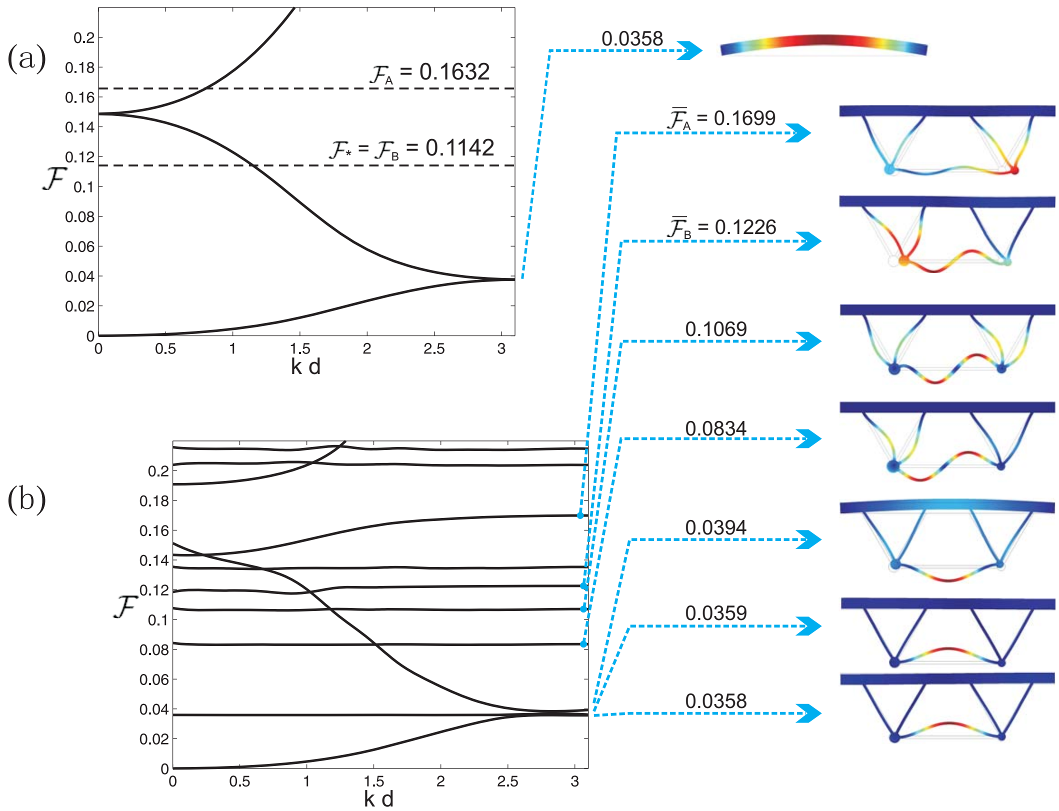

Figure 3: Dispersion diagrams and corresponding eigenmodes for the 2D structured wave guide (finite element computations). (A) The original beam structure and the eigenmode to suppress. $\mathcal{F}_A$ and $\mathcal{F}_B$ correspond to the analytical values given in eqn. (5)a. (B) The modified structure and the first several eigenmodes with corresponding normalized frequencies. $\overline{\mathcal{F}}_A$ and $\overline{\mathcal{F}}_B$ are finite element results for the analytical frequencies $\mathcal{F}_A$ and $\mathcal{F}_B$.

the long-wave flexural oscillations. Such a modification does not involve any major alteration of the structure such as a change of the boundary conditions in the contact region between the deck of the bridge and the supporting pillars or variation of the bending stiffness of the upper deck.

For the two-dimensional configuration, the proposed design is shown in Fig. 2, where a connected system of lightweight resonators is attached to the main body of the bridge. The overall mass of a single resonator is much smaller compared to the total mass of the elementary cell of the structure. Also, this system has a relatively low stiffness (γ, γ_1) of additional elastic links compared to the stiffness of the main deck of the bridge. In turn, the eigenfrequencies of such a resonator system can be placed sufficiently close to the low eigenfrequencies of the original macrostructure.

The new set of dispersion curves is shown in Fig. 3B, which contains stop bands, i.e. the frequency intervals where no propagating elastic waves exist. We note a narrow band gap around the normalised frequency $\mathcal{F} = 0.0359$ and a wider band gap at normalised frequencies around $\mathcal{F} = 0.1699$.

It is vital to pay attention to the vibration modes near the boundaries of the band gaps mentioned and emphasise that they do not involve large flexural vibrations of the main body of the bridge, as required in the proposed design.

It is important to be able to tune the system correctly to match the band gap frequency with the frequency of an unwanted vibration of the upper deck of the bridge. This task can be achieved by using the analytical approximations described below.

Analytical tuning of the vibrating elastic system. Analytical estimates of the eigenfrequencies for a class of standing waves provide a convenient design tool for the resonating structure and are accompanied by a numerical finite element simulation for an elementary cell comprising a computation of the eigenmodes and providing a dispersion diagram for elastic waves within the structured waveguide.

The approximation scheme is illustrated for the configuration displayed in Fig. 2B, where the system of resonators is represented as a low-frequency truss structure with two masses M_1 and M_2 and linear elastic links with longitudinal stiffness values γ for the diagonal connections and γ_1 for the horizontal one. Assuming that the upper deck is fixed, the low eigenfrequencies of vibration involving translational motion of masses M_1 and M_2 are given in the simple analytical form

$$f_{A,B}^2 = \frac{1}{8\pi^2}\left(\frac{1}{M_1}+\frac{1}{M_2}\right)\left[\gamma_1 + 2\gamma\sin^2\beta \pm \sqrt{\gamma_1^2 + 4\left(\frac{M_1-M_2}{M_1+M_2}\right)^2\gamma\sin^2\beta(\gamma_1+\gamma\sin^2\beta)}\,\right] \quad ,$$
$$f_C^2 = \frac{\gamma}{2\,M_1\pi^2}\cos^2\beta, \qquad f_D^2 = \frac{\gamma}{2\,M_2\pi^2}\cos^2\beta, \tag{5}$$

The frequencies $f_{A,B}$ $(f_B < f_A)$ correspond to horizontal translational motions of the interconnected masses M_1 and M_2, whereas the frequencies f_C, f_D describe the vertical motion of the masses M_1 and M_2, respectively. We use the notation $f_* = \min\{f_B, f_C, f_D\}$, representing the quantity which serves as an upper bound for a cluster of frequencies corresponding to standing waves associated with rotational motion of the resonators and flexural vibration of the inertial links. Some of these standing waves are shown to correspond to flat bands on the dispersion diagram in Fig. 3B. For our particular choice of material parameters, we have $f_* = f_B$. We also note that f_A gives an analytical estimate for the position of the second band gap displayed in Fig. 3B.

A supplementary finite element computation for an elementary cell of the structure, addressed in Fig. 3B, allows for a full interaction between the elastic upper deck of the bridge and the built-in resonator structure, which acts as the energy localizer and hence reduces significantly the amplitude of vibration of the upper deck of the bridge, as shown in Fig. 2A.

Comparative analysis of the dispersion diagrams in Figs. 3A and 3B shows formation of band gaps and a cluster of standing waves for the structure containing lightweight resonators. It appears that some of the waves of sufficiently low group velocity, illustrated in Fig. 3B, correspond to vibrations of low frequency near the boundaries of the band gaps. The change of the inertial and stiffness properties of the high-contrast lightweight resonator structure would alter the position of the stop bands and hence can be used as the means of control of propagating waves of other frequencies within the periodic elastic system.

Further fine tuning of the system may involve, for example, a very slight change of stiffness

γ_1 of the horizontal bar. This would lead to an accurate match of a band gap frequency with the frequency of vibration of the upper deck of the unaltered bridge. In the proposed design, vibrations of the upper deck of the bridge become negligibly small compared to the vibration of the additional resonator structure, as shown in Fig. 2A, and this has been achieved without any alterations of junction conditions between the bridge deck and supporting pillars or any change in the stiffness of the main deck. A full three-dimensional simulation is discussed in the text below.

Three-dimensional simulation. We use the ideas described above to give the required design solution for the "Volga bridge problem", discussed in [1, 2]. In the three-dimensional computational model, we consider two cases: the original bridge structure, which possesses a standing flexural wave as shown in Fig. 1C, and the new design including a system of lightweight resonators illustrated in Fig. 1D, where the flexural wave in question has been suppressed.

As above, instead of referring to a full scale finite length bridge, we consider a waveguide, with emphasis on the dispersion properties of elastic waves within a periodic structure.

A simple analysis of the resonator structure on its own (as shown in Fig. 4), with the assumption that the surfaces of contact with the bridge deck are fixed, gives an accurate estimate of frequencies for a class of standing waves within the modified bridge after installation of the periodic system of resonators. Fig. 4 also shows the four eigenmodes within the frequency interval $0 < \mathcal{F} < 0.0303$. The physical and stiffness parameters of the resonators are chosen so that one of the eigenfrequencies of the lightweight resonator structure of Fig. 4 matches the frequency $\mathcal{F} = 0.0182$ of the standing wave of the unaltered bridge. When the two frequencies in question are sufficiently close to each other the combined structure will change its dynamic response within the required low frequency range. Embedding of a periodic system of low-frequency resonators leads to the formation of the required cluster of standing waves, where the

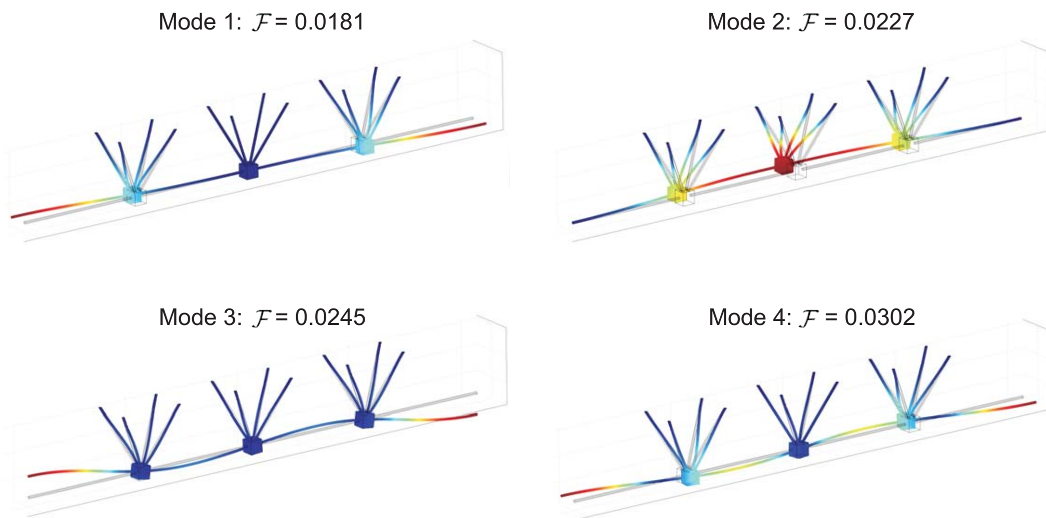

Figure 4: First four vibration modes of the resonant structure. The first mode, at normalised frequency $\mathcal{F} = fd/v = 0.0181$, is designed to eliminate the flexural vibration of the deck of the bridge. In the computation $d = 4$ m and $v = \sqrt{\mu/\rho} = 3194$ m/s is the shear wave speed in the upper deck.

amplitude of vibration of the bridge deck becomes negligibly small. The proposed mechanical system can be tuned to filter waves of desired frequencies and hence to eliminate vibration of the upper deck of the bridge within the specified frequency range.

The computation shows that for the unaltered bridge, the fundamental flexural mode corresponds to the flexural vibrations of the upper deck of the bridge at a low normalised frequency $\mathcal{F} = 0.0182$. After embedding of the appropriately tuned lightweight structure of the resonators the eigenfrequency in question has been slightly shifted, and the eigenmodes of vibration have been dramatically changed.

In Fig. 5 we show a finite structure representing a sufficiently large section of the modified bridge, where the left and the right boundaries of the deck of the bridge are simply supported. By analysing the vibration mode of the modified structure, we confirm that the motion of the main body of the bridge is negligibly small compared to the motion within the system of res-

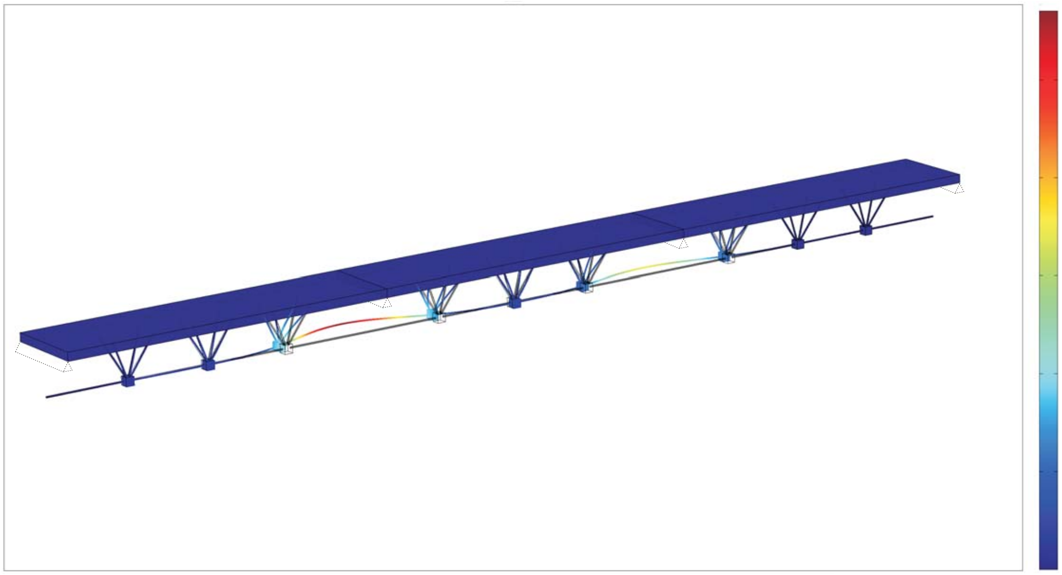

Figure 5: Modified bridge: the flexural vibration of the main body has been suppressed. The built-in resonators have taken on the vibrational motion.

onators. This gives the desired design, which eliminates the flexural vibration of the upper deck of the bridge without any major alteration of the structure in terms of mass, stiffness or boundary conditions.

Suppression of lateral vibrations of a skyscraper. The generic method based on the analysis of waves in a periodic waveguide is now applied to such a system as a tall building, or 'skyscraper', composed of several copies of an elementary cell, as shown in Fig. 6 panel (2). The displacement vector is assumed to be time-harmonic, with the radian frequency ω and the amplitude $\mathbf{u}$, which satisfies the Lamé system (1) and the boundary conditions (2) and (3) on the fixed foundation and on the boundary, which is free of tractions, respectively.

We introduce a multi-scale resonator within every elementary cell, as shown in Fig. 6H, and make use of the analysis of elastic waves within a periodic plain strain structure. The resonator consists of a mass connected to the main frame of the elementary cell by a set of parallel thin ligaments, whose overall flexural stiffness is sufficiently small.

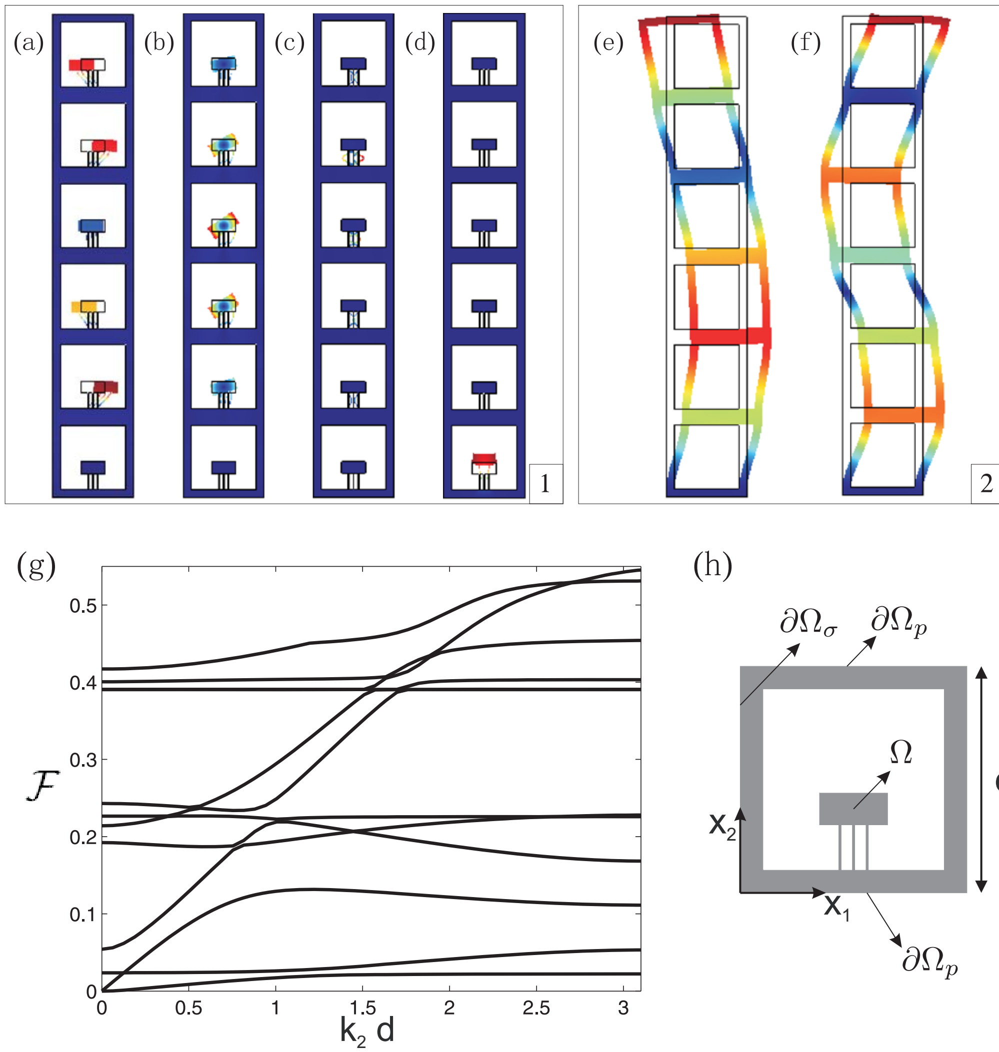

Figure 6: Eigenmodes of the multi-storey buildings. The horizontal foundation of the structure is fixed, whereas all the remaining boundary is traction free. Panel 1: first four eigenmodes of the multi-structure, each showing energy absorption by the resonators. Normalised eigenfrequencies $\mathcal{F} = fd/v$: (A) 0.0225; (B) 0.2254; (C) 0.3936; (D) 0.4446. Panel 2: first two eigenmodes of the frame structure. Normalised eigenfrequencies: (E) 0.0144; (F) 0.0260. (G) The dispersion diagram showing normalised frequency $\mathcal{F}$ versus $k_2\,d$. (H) Square elementary cell with dimension $d = 1$ m. The external structure has thickness $s = 0.1$ m, the resonator is composed of a rectangular solid with dimensions 0.3 m $\times$ 0.14 m and three thin ligaments with dimensions 0.2 m $\times$ 0.01 m. Material parameters are: shear modulus $\mu = 80$ GPa, Poisson ratio $\nu = 0.28$, mass density $\rho = 7850$ Kg/m^3.

As a design target, we would like to alter the flexural vibration modes of the original structure, within a predefined range of frequencies, to provide a set of vibration modes which do not involve a large amplitude motion of the walls of the building. Moreover, the "thin-legged resonators" may move and hence absorb the energy of the system. This is clearly a task of high practical importance, which may appear in problems such as the design of earthquake-resistant structures.

The lowest translational and rotational eigenfrequencies of the thin-legged resonators can be estimated analytically [20, 32] and the design of the resonators can be chosen to place these eigenfrequencies within the required interval corresponding to the modes of the original structure in panel (2) of Fig. 6.

The dispersion diagram of Fig. 6G has been computed for the periodic system with the unit cell given in Fig. 6H. This includes a group of flat bands corresponding to standing waves related to vibration of resonators within the periodic structure. We emphasize the connection between the standing waves associated with vibrations of the thin-legged resonator and groups or clusters of eigenmodes of the finite structure, some of which are shown in Fig. 6, panel (1). For such frequencies, the motion of the sides of the skyscraper is negligibly small. We note that an ad-hoc estimate of such frequencies is difficult for any realistic structure, and hence the appropriate spectral analysis on a single unit cell is desirable, in particular, in the design process.

The two low frequency modes of vibration of the "multi-storey" structure without built-in resonators in Fig. 6 panel (2) show that the whole macro-structure behaves like a homogenized beam. It is noted that the building, without the system of resonators, will experience lateral vibrations of significant amplitude and, in practical configurations, this may lead to an overall structural failure. In Fig. 6 panel (2) we show the structure vibrating at normalised frequencies $\mathcal{F} = 0.0144, 0.0260$. These values represent two neighboring frequencies at the ends of the interval containing $\mathcal{F} = 0.0225$, which is the first eigenfrequency of the thin-legged resonator, as shown in Fig. 6 panel (1).

The alteration of the vibration modes of panel (2) of Fig. 6 is achieved by introduction of the resonators, as shown in Fig. 6 panel (1). This is fully consistent with the idea of the modified design for the Volga Bridge discussed earlier.

We pay particular attention to clusters of normalised eigenfrequencies around the values 0.0225, 0.2254, 0.3936 and 0.4446, corresponding to standing waves. For such waves, the exterior boundary of the elementary cell moved with very small amplitude compared to the amplitude of vibrations of the thin-legged resonators. The four frequencies outlined above correspond to lateral motion of the resonator, rotational mode of the resonator, localised vibrations of thin legs within the resonator and finally, the longitudinal translational mode of the resonator, respectively. The associated eigenmodes are shown in Fig. 6 panel (1) where the color map corresponds to the total displacement. In Fig. 6A we have a mode dominated by the lateral translational motion of the resonators; in Fig. 6B we observe the rotational motion of the resonators; Fig. 6C gives an example from a large cluster of eigenmodes corresponding to the case of localised vibrations of thin legs within the resonators. Finally, in Fig. 6D we observe a localisation near the foundation of the structure (namely only one resonator located at the ground level is moving, whereas the motion of the remaining structure has a negligibly small amplitude). This last example shows an exponential localisation within a structured waveguide; in this case, the skyscraper is treated as a waveguide and the embedded structure of resonators creates a stop band preventing the waves of certain frequencies from propagating along such a waveguide. In all of these computations, the amplitude of vibration of the sides of the large structure is very small compared to that of the individual resonators. Hence, if the solid is subjected to an external impact, such resonance modes can be initiated and the energy can be absorbed into the resonators and then dissipated via the desired mechanism, for example damage of the thin legs. Even though you may not be able to eat your lunch off the resonating tables the building will survive the earthquake.

Dissipation of energy and final remarks In the re-designed bridge structure shown in Fig. 5 the waves have been channeled through the system of resonators, away from the upper deck of the bridge. For the case of a skyscraper, inclusion of a system of resonators has led to suppression of low-frequency lateral vibrations. Although the models discussed above did not incorporate energy dissipation, additional viscous dampers attached to the resonators can be considered as a feasible development in a practical implementation.

The main emphasis of the paper is on the analysis of waves in a periodic system, rather than resonance modes of a finite solid. It is shown that the analysis of the elementary cell of the periodic waveguide provides insight and required numerical data for the design of the full-scale engineering system. The ideas presented in this paper are generic, and the bridge problem in addition to the structural modification of a skyscraper, can be considered as possible practical implementations. The proposed methods of analysis would equally apply to other slender structures in civil engineering and problems of structural design.

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