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Classification of Flipped $SU(5)$ Heterotic-String Vacua

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Abstract

We extend the classification of the free fermionic heterotic–string vacua to models in which the $SO(10)$ GUT symmetry at the string scale is broken to the flipped $SU(5)$ subgroup. In our classification method, the set of basis vectors defined by the boundary conditions which are assigned to the free fermions is fixed and the enumeration of the string vacua is obtained in terms of the Generalised GSO (GGSO) projection coefficients entering the one–loop partition function. We derive algebraic expressions for the GGSO projections for all the physical states appearing in the sectors generated by the set of basis vectors. This enables the analysis of the entire string spectrum to be programmed in to a computer code therefore, we performed a statistical sampling in the space of $2^{44} \approx 10^{13}$ flipped $SU(5)$ vacua and scanned up to 10^{12} GGSO configurations. For that purpose, two independent codes were developed based on JAVA and FORTRAN95. All the results presented here are confirmed by the two independent routines. Contrary to the corresponding Pati–Salam classification, we do not find exophobic flipped $SU(5)$ vacua with an odd number of generations. We study the structure of exotic states appearing in the three generation models that additionally contain a viable Higgs spectrum. Moreover, we demonstrate the existence of models in which all the exotic states are confined by a hidden sector non–Abelian gauge symmetry as well as models that may admit the racetrack mechanism.

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1 Introduction

The LHC discovery of a Higgs–like resonance [1] lends further support to the viability of the Standard Model as the effective parameterisation of all observational subatomic data up to the GUT or Planck scales. This hypothesis is further supported by the proton lifetime, the neutrino mass suppression and the logarithmic evolution of the Standard Model parameters in the gauge and heavy generation matter sectors. The logarithmic evolution in the scalar sector is spoiled by radiative corrections from the cutoff scale. Restoration of the logarithmic running in the scalar sector suggests the existence of a new symmetry, with supersymmetry being a concrete example of contemporary interest.

Despite its enormous success in accounting for observational subatomic data, the Standard Model is unsatisfactory. It contains too many ad hoc parameters. The gauge symmetries and representations are not selected by a fundamental principle. The Standard Model* requires at least twenty–six additional parameters to account for the available data. The Standard Model gauge and flavour parameters can only be determined in a theory that unifies the gauge interactions with gravity. String theory provides a framework to study how the elementary particle's attributes may arise from a consistent theory of gauge–gravity unification. This necessitates the construction of quasi–realistic string models and the investigation of their phenomenological properties. The tools assembled for this purpose include target–space and worldsheet constructions [2].

The free fermionic formulation [3] of the heterotic–string in four space–time dimensions provides a worldsheet approach to study quasi–realistic string vacua. The models constructed in this formulation represent some of the most realistic string models corresponding to symmetric and asymmetric $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold compactifications. Early examples of quasi–realistic free fermionic models which correspond to asymmetric $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold compactifications were constructed since the late eighties. The models correspond to compactifications with $\mathcal{N} = (2, 0)$ worldsheet supersymmetry in which the observable E_8 symmetry is broken to a subgroup of $SO(10)$. The cases with $SU(5) \times U(1)$ (flipped $SU(5)$) [4, 5], $SO(6) \times SO(4)$ Pati–Salam) [6], $SU(3) \times SU(2) \times U(1)^2$ (Standard–like) [7] and $SU(3) \times SU(2)^2 \times U(1)$ (left–right symmetric) [8] was shown to give rise to quasi–realistic examples.

The early quasi–realistic free fermionic models consisted of a few examples that shared an underlying NAHE–based structure [9]. Contemporary research in string model building focuses on explorations of large classes of string vacua. Over the last decade, tools for the classification of the symmetric $\mathbb{Z}_2 \times \mathbb{Z}_2$ free fermionic orbifolds were derived in [10] for type II superstring and extended in [11, 12]. Classification of the heterotic–string vacua with unbroken $SO(10)$ and E_6 GUT groups revealed the existence of a symmetry in the space of $\mathbb{Z}_2$ and $\mathbb{Z}_2 \times \mathbb{Z}_2$ string models under the exchange of spinorial plus anti–spinorial and vectorial representations of $SO(10)$

*Including massive neutrinos.

[12, 13], which resembles mirror symmetry [14]. The classification was extended to the string vacua in which the $SO(10)$ symmetry is broken to the Pati–Salam subgroup in [15]. It revealed the existence of exophobic string vacua, in which fractionally charged states (exotics) appear in the massive spectrum, but do not exist among the massless states. A concrete three generation exophobic model was studied in [16] and was shown to accommodate qualitatively viable phenomenology. The classification method in [18], was used to fish out an exophobic model in which the E_6 symmetry is broken to the maximal $SU(6) \times SU(2)$ subgroup, which admits an additional family universal and anomaly free $U(1)$ symmetry beyond the $U(1)$ generators of the $SO(10)$ GUT group [19].

The classification methodology developed in [10, 11, 12, 15], provides a useful tool to explore the properties of large classes of string vacua. In this paper, we extend the classification to models in which the $SO(10)$ symmetry is broken to the flipped $SU(5)$ subgroup [20]. The novel aspect in the classification of this class of string models, is that the basis vectors that generate these contain boundary conditions that give rise to complex phases, whilst the models in the previous studies contained only periodic and antiperiodic boundary conditions. Extension of the classification method to the flipped $SU(5)$ case is also a necessary step towards the classification of Standard–like string vacua, which utilises both the Pati–Salam and flipped $SU(5)$ generating basis vectors. A question of particular interest in the classification is the existence of quasi–realistic exophobic three generation flipped $SU(5)$ models. We find that such a model does not exist in the space of the order of 10^{12} that we explore. Our scan shows that the exophobic flipped $SU(5)$ models exist in the string vacua with an even number of generations, but not in those with an odd number.

2 Flipped $SU(5)$ Free Fermionic Models

The quasi–realistic free fermionic models correspond to $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold compactifications with $\mathcal{N} = (2, 0)$ super–conformal worldsheet symmetry. The free fermionic formulation [3] is set at an extended symmetry point in the moduli space, where the compactified directions are represented in terms of two dimensional fermions propagating on the string worldsheet [21, 22]. Exactly marginal deformations from the free fermionic point are obtained by incorporating worldsheet Thirring interactions among the worldsheet fermions [23]. The free fermionic formulation provides a set of rules that enables straightforward derivation of the physical states and interactions plus is suited to explore the phenomenological properties of the string vacua. The matter states in the free fermionic models arise from the spinorial **16** representations of $SO(10)$, whilst the Higgs states arise from the vectorial **10** representation. The $\mathbb{Z}_2 \times \mathbb{Z}_2$ free fermionic orbifold models therefore preserve the Standard Model spectrum embedded in the $SO(10)$ GUT group. The $SO(10)$ symmetry is broken at the string scale, leading to the gauge symmetry in the low energy effective field theory being a subgroup of the $SO(10)$.

In this paper, we extend the classification method to the case of the flipped $SU(5)$ subgroup. The distinctive feature of these models is the utilisation of rational boundary conditions, whereas the $SO(10)$ and $SO(6)\times SO(4)$ that were classified previously only used periodic and anti–periodic boundary conditions. Our classification method, entails that the GGSO projections for all the states that arise in the twisted sectors are expressed in terms of algebraic equations. The equations are incorporated in a computer program that facilitates scanning a large number of models.

2.1 Free Fermionic Formulation

In the light-cone gauge, the four dimensional heterotic–string is given by 44 right–moving and 20 left–moving real worldsheet fermions in the free fermionic construction. The models are constructed by specifying the phases picked up by the fermions $f_1, \dots, f_{20}, \overline{f}_1, \dots, \overline{f}_{44}$ when parallel transported along the non-contractible loops of the vacuum to vacuum amplitude. The worldsheet fermions in the light-cone gauge in the usual notation are: $\psi^\mu, \chi^{1,\dots,6}, y^{1,\dots,6}, \omega^{1,\dots,6}$ for the left-movers and $\overline{y}^{1,\dots,6}, \overline{\omega}^{1,\dots,6}$, $\overline{\psi}^{1,\dots,5}$, $\overline{\eta}^{1,2,3}$, $\overline{\phi}^{1,\dots,8}$ for the right-movers. For a particular choice of fermion phases, each model is defined to be consistent with the modular invariance constraints, such that it can be spanned by a set of basis vectors $v_1, \dots, v_N$ given by

$$v_i = \left\{\alpha_i(f_1), \dots, \alpha_i(f_{20}) | \alpha_i(\overline{f}_1), \dots, \alpha_i(\overline{f}_{44})\right\}$$

describing the transformation properties of each worldsheet fermion

$$f_j \to -e^{i\pi\alpha_i(f_j)}\ f_j, \qquad j = 1, \dots, 64.$$

The basis vectors generate a space Ξ that consists of 2^{N+1} sectors which produce the string spectrum. Each sector is given by a linear combination of the basis vectors

$$\xi = \sum_{i=1}^{N} m_j v_i, \qquad m_j = 0, 1, \dots, N_j - 1,$$

where $N_j \cdot v_j = 0 \bmod 2$. The basis vectors induce the GGSO projections, with action on a given string state $|S_\xi\rangle$ given by

$$e^{i\pi v_i \cdot F_\xi}|S_\xi\rangle = \delta_\xi\ C\binom{\xi}{v_i}^* |S_\xi\rangle, \tag{2.1}$$

where $\delta_\xi = \pm 1$ is the space–time spin statistics index and F_ξ is the fermion number operator. Different sets of GGSO projection coefficients $c\left[{\xi \atop v_i}\right] = \pm 1; \pm i$, consistent with modular invariance produce different models. To summarize, a model is defined by a set of basis vectors $v_1, \dots, v_N$ and a set of $2^{N(N-1)/2}$ independent GGSO projection coefficients $C\binom{v_i}{v_j}$, $i > j$ defining the string spectrum.

2.2 SO(10) Models

The flipped $SU(5)$ models we consider here are generated by a set of 13 basis vectors. The first 12, consist of the same basis vectors that were used in the classification of the $SO(10)$ vacua [12], which are given by

$$
\begin{aligned}
v_1 = \mathbf{1} &= \{\psi^\mu,\ \chi^{1,\dots,6}, y^{1,\dots,6}, \omega^{1,\dots,6} | \\
&\qquad \overline{y}^{1,\dots,6}, \overline{\omega}^{1,\dots,6}, \overline{\eta}^{1,2,3}, \overline{\psi}^{1,\dots,5}, \overline{\phi}^{1,\dots,8}\}, \\
v_2 = S &= \{\psi^\mu, \chi^{1,\dots,6}\}, \\
v_{2+i} = e_i &= \{y^i, \omega^i | \overline{y}^i, \overline{\omega}^i\},\ i = 1, \dots, 6, \\
v_9 = b_1 &= \{\chi^{34}, \chi^{56}, y^{34}, y^{56} | \overline{y}^{34}, \overline{y}^{56}, \overline{\eta}^1, \overline{\psi}^{1,\dots,5}\}, \\
v_{10} = b_2 &= \{\chi^{12}, \chi^{56}, y^{12}, y^{56} | \overline{y}^{12}, \overline{y}^{56}, \overline{\eta}^2, \overline{\psi}^{1,\dots,5}\}, \\
v_{11} = z_1 &= \{\overline{\phi}^{1,\dots,4}\}, \\
v_{12} = z_2 &= \{\overline{\phi}^{5,\dots,8}\}.
\end{aligned}
\tag{2.2}
$$

The basis vectors $\mathbf{1}$ and S, generate a model with the $SO(44)$ gauge symmetry and $N = 4$ space–time supersymmetry. The vectors e_1,...,e_6, corresponding to all the possible symmetric shifts of the six internal coordinates, break the $SO(44)$ gauge group to $SO(32) \times U(1)^6$ and preserve the $N = 4$ space–time supersymmetry. The vectors b_1 and b_2 correspond to the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold twists, which break $N = 4$ to $N = 1$ supersymmetry and reduces the rank of the group as the $U(1)^6$ is broken, leaving the $SO(32)$ symmetry to decompose to the $SO(10) \times U(1)^3 \times SO(16)$ gauge group. Furthermore, the $SO(10) \times U(1)^3$ group corresponds to our observable and the $SO(16)$ group to our hidden gauge group. The remaining fermions that are not affected by the action of the previous vectors are $\overline{\phi}^{1,\dots,8}$, which correspond to the $SO(16)$ gauge group. The vectors z_1 and z_2 reduce the untwisted hidden gauge group from $SO(16)$ to $SO(8) \times SO(8)$. This choice of basis is the most general set of basis vectors with symmetric shifts for the internal fermions compatible with a $SO(10)$ GUT group. The untwisted vector bosons consistent with the GGSO projections induced by the choice of basis vectors in (2.2), generate the adjoint representation of an $SO(10) \times U(1)^3 \times SO(8)^2$ gauge group.

2.3 Flipped $SU(5)$ Construction

The $SO(10)$ GUT models generated by (2.2) are broken to the flipped $SU(5)$ subgroup, by the rational boundary condition assignment of the complex right-moving fermions $\overline{\psi}^{1,\dots,5} = \pm\frac{1}{2}$. This is achieved with the addition of the basis vector $v_{13} = \alpha$. The case of the $SO(6) \times SO(4)$ models, which were classified in [15], utilize solely periodic and anti–periodic boundary conditions. In this case, the choice of α compatible with the set (2.2) is unique and is given by $\alpha = \{\overline{\psi}^{4,5}, \overline{\phi}^{1,2}\}$. All other possible assignments that reduce the $SO(10)$ symmetry to the $SO(6) \times SO(4)$ are equivalent. As in the cases of other free fermionic flipped $SU(5)$ models constructed to date [4, 5], we restrict the assignment of $\overline{\psi}^{1,\dots,5}$ to the case of positive 1/2 boundary conditions. Furthermore, unlike the case of the $SO(6)\times SO(4)$ models, the choice of the basis vector α that breaks the $SO(10)$ symmetry to $SU(5)\times U(1)$ is not unique. Also, the assignment of the three complex worldsheet fermions $\overline{\eta}^{1,2,3}=1/2$ is fixed by the modular invariance constraint $b_j\cdot\alpha=0 \bmod 1$. Consequently, it follows that the assignment of the boundary conditions of the eight worldsheet complex fermions $\overline{\psi}^{1,\dots,5},\overline{\eta}^{1,2,3}$ is unique and the variation is in the boundary conditions of the worldsheet fermions $\overline{\phi}^{1,\dots,8}$. Modular invariance constraints, restrict the possibilities to assigning 1/2 boundary conditions of $\overline{\phi}^{1,\dots,8}$ worldsheet fermions to 0, 4 or 8. The null case been given by

$$\alpha=\{\overline{\psi}^{1,\dots,5}=\tfrac{1}{2},\,\overline{\eta}^{1,2,3}=\tfrac{1}{2},\,\overline{\phi}^{1,2}=1,\,\overline{\phi}^{3,4}=1,\,\overline{\phi}^{5}=0,\,\overline{\phi}^{6,7}=0,\,\overline{\phi}^{8}=0\}.$$

is automatically excluded because the sector $x=2\alpha=\{\overline{\psi}^{1,\dots,5},\overline{\eta}^{1,2,3}\}$ enhances the $SU(5)\times U(1)$ gauge group back to the $SO(10)$ symmetry. The condition $z_{1,2}\cdot\alpha=0 \bmod 1$, imposes the assignment of 1/2 boundary conditions to 0, 2 or 4 of each of the groups of worldsheet fermions $\overline{\phi}^{1,\dots,4}$ and $\overline{\phi}^{5,\dots,8}$. The possible choices of v_{13} are then given by

$$\begin{aligned}
\alpha_1 &= \{\overline{\psi}^{1,\dots,5}=\tfrac{1}{2},\,\overline{\eta}^{1,2,3}=\tfrac{1}{2},\,\overline{\phi}^{1,2}=\tfrac{1}{2},\,\overline{\phi}^{3,4}=\tfrac{1}{2},\,\overline{\phi}^{5}=1,\,\overline{\phi}^{6,7}=0,\,\overline{\phi}^{8}=0\,\},\\
\alpha_2 &= \{\overline{\psi}^{1,\dots,5}=\tfrac{1}{2},\,\overline{\eta}^{1,2,3}=\tfrac{1}{2},\,\overline{\phi}^{1,2}=\tfrac{1}{2},\,\overline{\phi}^{3,4}=\tfrac{1}{2},\,\overline{\phi}^{5}=\tfrac{1}{2},\,\overline{\phi}^{6,7}=\tfrac{1}{2},\,\overline{\phi}^{8}=\tfrac{1}{2}\},\\
\alpha_3 &= \{\overline{\psi}^{1,\dots,5}=\tfrac{1}{2},\,\overline{\eta}^{1,2,3}=\tfrac{1}{2},\,\overline{\phi}^{1,2}=\tfrac{1}{2},\,\overline{\phi}^{3,4}=0,\,\overline{\phi}^{5}=1,\,\overline{\phi}^{6,7}=\tfrac{1}{2},\,\overline{\phi}^{8}=0\}.
\end{aligned}\tag{2.3}$$

The α's above require that the sets of basis vectors are linearly independent. This does not hold for the cases with α_1 and α_2, since in these cases we obtain

$$\begin{aligned}
1 &= S+\sum_{i=0}^{6}e_i+2\alpha_1+z_2,\\
1 &= S+\sum_{i=0}^{6}e_i+2\alpha_2+z_1+z_2\ .
\end{aligned}$$

In order to keep the set of basis vectors in (2.2), in addition to α_1 or α_2 being linearly independent, we choose to remove the basis vector $\mathbf{1}$ leaving the set of 12 vectors $\{S,e_1,e_2,e_3,e_4,e_5,e_6,b_1,b_2,z_1,z_2,\alpha_i\}$, where i = 1 or 2. In the case with α_3, the set in (2.2) is linearly independent giving us the set $\{1,S,e_1,e_2,e_3,e_4,e_5,e_6,b_1,b_2,z_1,z_2,\alpha_3\}$.

The plan for the remainder of the paper, is to give a comprehensive view of the methodology and an insight into the classification of the $SU(5)\times U(1)$ models with the inclusion of α_1 in the basis. The classification was carried out by using two independent codes, the first being the JAVA and the second being the FORTRAN95 code. It was then also carried out in the case of α_2. The details of the formulae needed for this classification can be obtained from the authors and will be published in a separate publication [24]. The classification using α_3 will be reported in future work.

2.4 GGSO projections

In order to define the string vacua, the GGSO projection coefficients appearing in the one-loop partition function $c\binom{v_i}{v_j}$ need to be specified. Taking the coefficients to span a 12×12 matrix, only the elements $i \geq j$ are independent. Modular invariance dictates that the 66 lower triangle elements of the matrix are fixed by the corresponding 66 upper triangle elements. Adding the remaining 12 diagonal terms, we are left with 78 independent coefficients corresponding to $2^{78} \approx 3 \times 10^{23}$ different string vacua. Moreover, requiring that the models possess $N = 1$ space–time supersymmetry, we fix eleven of the coefficients. Without loss of generality we set the associated GGSO projection coefficients

$$C\binom{S}{S} = C\binom{S}{e_i} = C\binom{S}{b_k} = C\binom{S}{z_1} = C\binom{S}{\alpha} = -1, \qquad i = 1, ..., 6,\ k = 1, 2. \tag{2.4}$$

Modular invariance imposes additional constraints on the diagonal terms. In our case, where the vector $\mathbf{1}$ is composite, they are given by

$$\begin{aligned}
C\binom{S}{z_2} &= -\prod_{i=1}^{6} C\binom{S}{e_i}, \\
C\binom{e_k}{z_2} &= \prod_{\substack{i=1\\ i\neq k}}^{6} C\binom{e_k}{e_i}, \quad k = 1\dots 6, \\
C\binom{b_k}{b_k} &= -\prod_{i=1}^{6} C\binom{b_k}{e_i} C\binom{b_k}{z_2}, \quad k = 1, 2, \\
C\binom{z_1}{z_1} &= -\prod_{i=1}^{6} C\binom{z_1}{e_i} C\binom{z_1}{z_2}, \\
C\binom{\alpha}{\alpha} &= -\prod_{i=1}^{6} C\binom{\alpha}{e_i} C\binom{\alpha}{z_2},
\end{aligned} \tag{2.5}$$

where $C\binom{z_2}{z_2}$ is independent of any term. Further analysis of the GGSO projections of interest, shows that there are additional phases that do not effect the properties of the string spectrum. As a result, the following coefficients are fixed in the ensuing analysis

$$C\binom{e_i}{e_i} = C\binom{e_3}{b_1} = C\binom{e_4}{b_1} = C\binom{e_1}{b_2} = C\binom{e_2}{b_2} = C\binom{b_1}{b_2} = C\binom{z_2}{z_2} = 1, \quad (2.6)$$

where $i = 1, ..., 6$. Taking the equations (2.4), (2.5) and (2.6) we are left with 44 independent coefficients which can take two discrete values ± 1, except in the cases $C\binom{\alpha}{b_1}$, $C\binom{\alpha}{b_2}$ and $C\binom{\alpha}{z_2}$, where they take the values $\pm i$ since $\alpha \cdot b_1 = -3$ (odd), $\alpha \cdot b_2 = -3$ (odd) and $\alpha \cdot z_2 = -1$ (odd). Furthermore, a simple counting gives $2^{44} \approx 1.76 \times 10^{13}$ vacua in this class of superstring models. We note that there may still exist some degeneracies in this space of vacua with regard to the characteristics of the low energy effective field theory, and in particular with respect to the observable massless states. For instance, the three twisted sectors of $\mathbb{Z}_2 \times \mathbb{Z}_2$ toroidal orbifolds possess a cyclic permutation symmetry. Nevertheless, some of the vacua that may seem identical in the low energy effective field theory limit of the observable sector, differ by other properties, such as the massive spectrum, superpotential couplings, hidden sector matter states and are therefore distinct.

3 String Spectrum

The vector bosons from the untwisted sector generate the

$$SU(5) \times U(1) \times U(1)^3 \times SU(4) \times U(1) \times U(1) \times SO(6)$$

gauge symmetry. Depending on the choices of the GGSO projection coefficients, extra space–time vector bosons may be obtained from the following twelve sectors

$$\mathbf{G} = \left\{ \begin{array}{cccc} z_1, & z_2, & z_1 + z_2, & z_1 + 2\alpha, \\ \alpha, & z_1 + \alpha, & z_2 + \alpha, & z_1 + z_2 + \alpha, \\ 3\alpha, & z_1 + 3\alpha, & z_2 + 3\alpha, & z_1 + z_2 + 3\alpha \end{array} \right\}. \quad (3.1)$$

The projections on the sectors 3α, $z_1 + 3\alpha$, $z_2 + 3\alpha$, $z_1 + z_2 + 3\alpha$ can be inferred from the projections on the sectors α, $z_1 + \alpha$, $z_2 + \alpha$, $z_1 + z_2 + \alpha$ respectively. Therefore, we will not discuss them in detail. The gauge bosons that are obtained from the sectors in (3.1) enhance the untwisted gauge symmetry. We impose the restriction that the only gauge bosons that remain in the spectrum are those that are obtained from the untwisted sector. The gauge groups in these models are therefore

$$\begin{array}{rcl} \text{Observable} & : & SU(5)_{obs} \times U(1)_5 \times U(1)_1 \times U(1)_2 \times U(1)_3 \\ \text{Hidden} & : & SU(4)_{hid} \times U(1)_4 \times U(1)_{hid} \times SO(6)_{hid} \end{array}$$

The NS sector matter spectrum is common in these models and consists of three pairs of $\mathbf{5}$ and $\overline{\mathbf{5}}$ representations of the observable $SU(5) \times U(1)_5$ gauge group and twelve that are singlets under the non–Abelian gauge symmetries.

3.1 Observable Matter Spectrum

The chiral matter spectrum arises from the twisted sectors. The method of classification enables a straightforward enumeration of all the twisted sectors that produce massless states and the GGSO projection that operate on them. We provide below the details of the method in the case of α_1 in (2.3). The chiral spinorial representations of the observable $SU(5)\times U(1)_5$ arise from the sectors

$$\begin{aligned}
B^{(1)}_{pqrs} &= S+b_1+pe_3+qe_4+re_5+se_6 \\
&= \{\psi^\mu,\chi^{12},(1-p)y^3\overline{y}^3,p\omega^3\overline{\omega}^3,(1-q)y^4\overline{y}^4,q\omega^4\overline{\omega}^4, \\
&\qquad (1-r)y^5\overline{y}^5,r\omega^5\overline{\omega}^5,(1-s)y^6\overline{y}^6,s\omega^6\overline{\omega}^6,\overline{\eta}^1,\overline{\psi}^{1,\dots,5}\}, \qquad (3.2)\\
B^{(2)}_{pqrs} &= S+b_2+pe_1+qe_2+re_5+se_6, \\
B^{(3)}_{pqrs} &= S+b_3+pe_1+qe_2+re_3+se_4,
\end{aligned}$$

where $p,q,r,s=0,1$ and $b_3=b_1+b_2+2\alpha+z_1$. These 48 sectors give rise to $\mathbf{16}$ and $\overline{\mathbf{16}}$ multiplets of $SO(10)$ decomposed under $SU(5)\times U(1)$, which are given by

$$\begin{aligned}
\mathbf{16} &= \left(\overline{\mathbf{5}},-\tfrac{3}{2}\right)+\left(\mathbf{10},+\tfrac{1}{2}\right)+\left(\mathbf{1},+\tfrac{5}{2}\right), \\
\overline{\mathbf{16}} &= \left(\mathbf{5},+\tfrac{3}{2}\right)+\left(\overline{\mathbf{10}},-\tfrac{\mathbf{1}}{\mathbf{2}}\right)+\left(\mathbf{1},-\tfrac{5}{2}\right).
\end{aligned}$$

Additionally, vector–like representations of the observable $SU(5)\times U(1)_5$ gauge group arise from the sectors

$$\begin{aligned}
B^{(4)}_{pqrs} &= B^{(1)}_{pqrs}+z_1+2\alpha \\
&= \{\psi^\mu,\chi^{12},(1-p)y^3\overline{y}^3,p\omega^3\overline{\omega}^3,(1-q)y^4\overline{y}^4,q\omega^4\overline{\omega}^4, \\
&\qquad (1-r)y^5\overline{y}^5,r\omega^5\overline{\omega}^5,(1-s)y^6\overline{y}^6,s\omega^6\overline{\omega}^6,\overline{\eta}^{2,3}\}, \qquad (3.3)\\
B^{(5,6)}_{pqrs} &= B^{(2,3)}_{pqrs}+z_1+2\alpha.
\end{aligned}$$

These sectors contain four periodic worldsheet right–moving complex fermions. The massless states are obtained by acting on the vacuum with a Neveu–Schwarz right–moving fermionic oscillator. Furthermore, if the oscillator is given by $\{\overline{\psi}^{1,\dots,5}\}$ or $\{\overline{\psi}^{*1,\dots,5}\}$ then some of the 48 twisted sectors can give rise to the vectorial $\mathbf{10}$ representation of $SO(10)$ decomposed under $SU(5)\times U(1)$, which is given by

$$\mathbf{10} = \left(\overline{\mathbf{5}},+1\right)+\left(\mathbf{5},-1\right).$$

These states are identified with light Higgs representations that are used to break the Standard Model gauge symmetry to $SU(3)\times U(1)_{\text{e.m.}}$. Additional states which are singlets under the observable $SU(5)\times U(1)_5$ might also arise from any of the 48 sectors in (3.3), given by the following representations:

- $\{\overline{\eta}^i\}|R\rangle^{(4,5,6)}_{pqrs}$ or $\{\overline{\eta}^{*i}\}|R\rangle^{(4,5,6)}_{pqrs}$, $i=1,2,3$, where $|R\rangle^{(4,5,6)}_{pqrs}$ is the degenerated Ramond vacuum of the $B^{(4,5,6)}_{pqrs}$ sector. These states transform as a vector–like representations under the $U(1)_i$'s.

- $\{\overline{\phi}^{1,\dots,4}\}|R\rangle^{(4,5,6)}_{pqrs}$ or $\{\overline{\phi}^{*1,\dots,4}\}|R\rangle^{(4,5,6)}_{pqrs}$. These states transform as a vector–like representations of $SU(4)\times U(1)_4$.

- $\{\overline{\phi}^{5}\}|R\rangle^{(4,5,6)}_{pqrs}$ or $\{\overline{\phi}^{*5}\}|R\rangle^{(4,5,6)}_{pqrs}$. These states transform as a vector–like representations under the $U(1)_5$'s.

- $\{\overline{\phi}^{6,7,8}\}|R\rangle^{(4,5,6)}_{pqrs}$ or $\{\overline{\phi}^{*6,7,8}\}|R\rangle^{(4,5,6)}_{pqrs}$. These states transform as a vectorial representation of $SO(6)$.

3.2 Hidden Matter Spectrum

The sectors which produce states that transform under representations of the hidden gauge group are singlets of the observable $SO(10)$ GUT gauge group. These states are hidden matter states that are obtained in the string model, but are not exotic with respect to the Standard Model gauge charges. The 48 sectors in $B^{1,2,3}_{pqrs}+2\alpha$ produce states that transforms under the $(\overline{\mathbf{4}},+1)$, $(\mathbf{4},-1)$, $(\mathbf{6},0)$, $(\mathbf{1},+2)$ and $(\mathbf{1},-2)$ representations of the $SU(4)\times U(1)$ hidden gauge group and are given by

$$
\begin{aligned}
B^{(7)}_{pqrs} &= B^{(1)}_{pqrs}+2\alpha \\
&= \{\psi^\mu,\chi^{12},(1-p)y^3\overline{y}^3,p\omega^3\overline{\omega}^3,(1-q)y^4\overline{y}^4,q\omega^4\overline{\omega}^4, \\
&\qquad (1-r)y^5\overline{y}^5,r\omega^5\overline{\omega}^5,(1-s)y^6\overline{y}^6,s\omega^6\overline{\omega}^6,\overline{\eta}^{2,3},\overline{\phi}^{1,\dots,4}\}, \qquad (3.4)\\
B^{(8,9)}_{pqrs} &= B^{(2,3)}_{pqrs}+2\alpha.
\end{aligned}
$$

In addition we have the following 48 sectors

$$
\begin{aligned}
B^{(10)}_{pqrs} &= B^{(1)}_{pqrs}+z_1+z_2+2\alpha \\
&= \{\psi^\mu,\chi^{12},(1-p)y^3\overline{y}^3,p\omega^3\overline{\omega}^3,(1-q)y^4\overline{y}^4,q\omega^4\overline{\omega}^4, \\
&\qquad (1-r)y^5\overline{y}^5,r\omega^5\overline{\omega}^5,(1-s)y^6\overline{y}^6,s\omega^6\overline{\omega}^6,\overline{\eta}^{2,3},\overline{\phi}^{5,\dots,8}\}, \qquad (3.5)\\
B^{(11,12)}_{pqrs} &= B^{(2,3)}_{pqrs}+z_1+z_2+2\alpha,
\end{aligned}
$$

which produce states in the $\mathbf{4}$ and $\overline{\mathbf{4}}$ spinorial representations of the hidden $SO(6)$ gauge group.

3.3 Exotic Matter Spectrum

In the string spectrum, additional sectors exist which produce fractionally charged states under the $SU(5)\times U(1)$ symmetry. These sectors arise when we have massless states, which are produced from a linear combination of basis vectors that include the vector α, resulting in the breaking of $SO(10)$ symmetry. Moreover, these sectors produce states that do not fall into representations of the underlying $SO(10)$ GUT symmetry. Specifically, they possess fractionally charged assignments with respect to the $U(1)$ symmetry in the decomposition $SO(10)\longrightarrow SU(5)\times U(1)$. Consequently, provided that the weak hypercharge has the canonical $SO(10)$ GUT embedding and the canonical GUT prediction $\sin^2\theta_w = 3/8$, these sectors produce states that carry fractional electric charges. This is a generic feature of string compactifications [25, 26], that may have interesting phenomenological implications [27], as electric charge conservation implies that the lightest of those exotic states is necessarily stable. Many experimental searches for fractionally charged matter have been conducted [28]. However, no reported observation of any such particles has ever been confirmed and there are strong upper bounds on their abundance [28]. This implies that such exotic states in string models should be either confined into integrally charged states [4], or be sufficiently heavy and diluted in the cosmological evolution of the universe [27]. The first of these solutions is problematic, due to the effect of the charged states on the renormalisation group running of the weak–hypercharge and gauge coupling unification. The preferred solution is therefore for the fractionally charged states to become sufficiently massive, *i.e.* with a mass which is larger than the GUT scale. In this case the fractionally charged states can be diluted by the inflationary evolution of the universe. Due to their heavy mass they will not be reproduced during reheating and the experimental constraints can be evaded. Three generation Pati–Salam heterotic–string models in which the fractionally charged states arise in the massive string spectrum but not as massless states, that were constructed in [15], which are dubbed as the quasi–realistic exophobic Pati–Salam string models. A particular question of interest in the current work is the existence of quasi–realistic flipped $SU(5)$ heterotic–string models. Also, it should be noted that the sectors appearing in (3.4) and (3.5) contain the combination 2α and do not break the $SO(10)$ symmetry. Therefore, these sectors do not produce exotic states under the $SU(5)\times U(1)$ gauge symmetry.

In the free fermionic construction, we classify the sectors that produce exotic states according to the product $\xi_R \cdot \xi_R = 4$, 6, or 8. In the first case, massless states are obtained by acting on the vacuum with a Neveu–Schwarz fermion or with two oscillators with 1/4 frequencies. In the second case, oscillators with 1/4 frequency are needed to produce massless states, whereas in the third case no oscillators are used to produce massless states. Furthermore, in the third case with no oscillators, we have the following 96 sectors

$$\begin{aligned}
B^{(13)}_{pqrs} &= B^{(1)}_{pqrs} + z_2 + \alpha \\
&= \{\psi^\mu, \chi^{12}, (1-p)y^3\overline{y}^3, p\omega^3\overline{\omega}^3, (1-q)y^4\overline{y}^4, q\omega^4\overline{\omega}^4, \\
&\qquad (1-r)y^5\overline{y}^5, r\omega^5\overline{\omega}^5, (1-s)y^6\overline{y}^6, s\omega^6\overline{\omega}^6, \overline{\eta}^1 = -\tfrac{1}{2}, \\
&\qquad\qquad \overline{\eta}^{2,3} = \tfrac{1}{2}, \overline{\psi}^{1,\dots,5} = -\tfrac{1}{2}, \overline{\phi}^{1,\dots,4} = \tfrac{1}{2}, \overline{\phi}^{6,7,8}\}, \qquad (3.6) \\
B^{(14,15)}_{pqrs} &= B^{(2,3)}_{pqrs} + z_2 + \alpha,
\end{aligned}$$

$$
\begin{aligned}
B_{pqrs}^{(16)} &= B_{pqrs}^{(1)} + z_1 + z_2 + \alpha \\
&= \{\psi^\mu, \chi^{12}, (1-p)y^3\overline{y}^3, p\omega^3\overline{\omega}^3, (1-q)y^4\overline{y}^4, q\omega^4\overline{\omega}^4, \\
&\quad r(1-r)y^5\overline{y}^5, \omega^5\overline{\omega}^5, (1-s)y^6\overline{y}^6, s\omega^6\overline{\omega}^6, \overline{\eta}^1 = -\tfrac{1}{2}, \\
&\quad \overline{\eta}^{2,3} = \tfrac{1}{2}, \overline{\psi}^{1,\dots,5} = -\tfrac{1}{2}, \overline{\phi}^{1,\dots,4} = -\tfrac{1}{2}, \overline{\phi}^{6,7,8}\}, \\
B_{pqrs}^{(17,18)} &= B_{pqrs}^{(2,3)} + z_1 + z_2 + \alpha.
\end{aligned} \tag{3.7}
$$

These produce states that are singlets under the observable $SU(5)$ but are charged under the $U(1)_5$ and are given by $\left(\mathbf{1}, -\frac{5}{4}\right)$ and $\left(\mathbf{1}, +\frac{5}{4}\right)$. We now move on to the second case that consists of oscillators with one $1/4$ frequency giving rise to additional massless vector–like states given by the following 48 sectors

$$
\begin{aligned}
B_{pqrs}^{(19)} &= B_{pqrs}^{(1)} + \alpha \\
&= \{\psi^\mu, \chi^{12}, (1-p)y^3\overline{y}^3, p\omega^3\overline{\omega}^3, (1-q)y^4\overline{y}^4, q\omega^4\overline{\omega}^4, \\
&\quad r(1-r)y^5\overline{y}^5, \omega^5\overline{\omega}^5, (1-s)y^6\overline{y}^6, s\omega^6\overline{\omega}^6, \overline{\eta}^1 = -\tfrac{1}{2}, \\
&\quad \overline{\eta}^{2,3} = \tfrac{1}{2}, \overline{\psi}^{1,\dots,5} = -\tfrac{1}{2}, \overline{\phi}^{1,\dots,4} = \tfrac{1}{2}, \overline{\phi}^{5}\}, \\
B_{pqrs}^{(20,21)} &= B_{pqrs}^{(2,3)} + \alpha.
\end{aligned} \tag{3.8}
$$

As an example, the sectors in $B_{pqrs}^{(19)}$ produce the following states:

- $\{\overline{\eta}^1\}|R\rangle_{pqrs}^{(19)}$, where $|R\rangle_{pqrs}^{(19)}$ is the degenerate Ramond vacuum of the $B_{pqrs}^{(19)}$ sector. These states transform as vector–like representations under the $U(1)_1$.

- $\{\overline{\eta}^{*2}\}|R\rangle_{pqrs}^{(19)}$ and $\{\overline{\eta}^{*3}\}|R\rangle_{pqrs}^{(19)}$. These states transform as vector–like representations under the $U(1)_{2/3}$.

- $\{\overline{\psi}^{1,\dots,5}\}|R\rangle_{pqrs}^{(19)}$. These states transform as $\left(\overline{\mathbf{5}}, +\frac{1}{4}\right)$ and $\left(\mathbf{5}, -\frac{1}{4}\right)$ representations of $SU(5) \times U(1)$.

- $\{\overline{\phi}^{*1,\dots,4}\}|R\rangle_{pqrs}^{(19)}$. These states transform as vector–like representations of $SU(4) \times U(1)$.

Similarly the sectors in $B_{pqrs}^{(20)}$ and $B_{pqrs}^{(21)}$ produce the states above. What is more, similar states appear in the following 48 sectors

$$
\begin{aligned}
B_{pqrs}^{(22)} &= B_{pqrs}^{(1)} + z_1 + \alpha \\
&= \{\psi^\mu, \chi^{12}, (1-p)y^3\overline{y}^3, p\omega^3\overline{\omega}^3, (1-q)y^4\overline{y}^4, q\omega^4\overline{\omega}^4, \\
&\quad r(1-r)y^5\overline{y}^5, \omega^5\overline{\omega}^5, (1-s)y^6\overline{y}^6, s\omega^6\overline{\omega}^6, \overline{\eta}^1 = -\tfrac{1}{2}, \\
&\quad \overline{\eta}^{2,3} = \tfrac{1}{2}, \overline{\psi}^{1,\dots,5} = -\tfrac{1}{2}, \overline{\phi}^{1,\dots,4} = -\tfrac{1}{2}, \overline{\phi}^{5}\}, \\
B_{pqrs}^{(23,24)} &= B_{pqrs}^{(2,3)} + z_1 + \alpha.
\end{aligned} \tag{3.9}
$$

The only difference between the sectors in (3.8) and (3.9) is the sign of the 1/2 boundary condition of the worldsheet fermion $\overline{\phi}^{1,\dots,4}$. This changes some of the $U(1)$ charges arising in (3.8) compared to those arising in (3.9), but the structure and type of states are similar to those listed above. Finally, the first case of exotic states arise in the sectors α and $z_1 + \alpha$. These exotic states can be eliminated by the same conditions that eliminate the space–time vector bosons arising in these sectors which will be discussed in section 5.

4 Twisted matter spectrum

The counting of spinorial and vector–like representations in the given string vacua is realised by utilising the so called projectors. Each sector B^i_{pqrs}, corresponds to a projector, $P^i_{pqrs} = 0, 1$, which is expressed in terms of GGSO coefficients and determines whether a given sector survives the GGSO projections. It is noted with the basis vectors given in (2.2), each fixed point of the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold corresponds to a distinct sector ξ in the additive group. In this method, the states arising from each fixed point are, therefore, controlled individually. Furthermore, the computational analysis is facilitated by rewriting the projectors in an analytic form. These are written as algebraic conditions, for the individual states arising in the string spectrum, in terms of the GGSO phases of the basis vectors. The algebraic expressions are inserted into the computer code, which enables the scan of the large space of models spanned by the basis GGSO phases.

4.1 Observable spinorial states

In order to get the particle content for the representations for the sectors in (3.2), we used the following normalisations for the hypercharge and the electromagnetic charge:

$$
\begin{aligned}
Y &= \frac{1}{3}(Q_1 + Q_2 + Q_3) + \frac{1}{2}(Q_4 + Q_5), \\
Q_{em} &= Y + \frac{1}{2}(Q_4 - Q_5).
\end{aligned}
$$

Where the Q_i charges of a state, arise due to ψ^i for $i = 1, ..., 5$. The following table summarises the charges of the colour $SU(3)$ and electroweak $SU(2) \times U(1)$ Cartan generators, of the states which form the $SU(5) \times U(1)$ matter representations

Representation	$\overline{\psi}^{1,2,3}$	$\overline{\psi}^{4,5}$	Y	Q_{em}
$\left(\mathbf{5}\,,+\frac{3}{2}\right)$	$(+,+,+)$	$(+,-)$	1/2	1,0
	$(+,+,-)$	$(+,+)$	2/3	2/3
$\left(\overline{\mathbf{5}}\,,-\frac{3}{2}\right)$	$(+,-,-)$	$(-,-)$	-2/3	-2/3
	$(-,-,-)$	$(+,-)$	-1/2	-1,0
$\left(\mathbf{10},+\frac{1}{2}\right)$	$(+,+,+)$	$(-,-)$	0	0
	$(+,-,-)$	$(+,+)$	1/3	1/3
	$(+,+,-)$	$(+,-)$	1/6	-1/3,2/3
$\left(\overline{\mathbf{10}},-\frac{1}{2}\right)$	$(+,+,-)$	$(-,-)$	-1/3	-1/3
	$(+,-,-)$	$(+,-)$	-1/6	1/3,-2/3
	$(-,-,-)$	$(+,+)$	0	0
$\left(\mathbf{1}\,,+\frac{5}{2}\right)$	$(+,+,+)$	$(+,+)$	1	1
$\left(\mathbf{1}\,,-\frac{5}{2}\right)$	$(-,-,-)$	$(-,-)$	-1	-1

Here " + ", and " − ", label the contribution of an oscillator with fermion number $F=0$, or $F=-1$, to the degenerate vacuum. For example $(+,+,-)$ under $\overline{\psi}^{1,2,3}$ corresponds to a part of the Ramond vacuum formed by two oscillators with fermion number $F=0$ and one oscillator with fermion numbers $F=-1$. These states correspond to particles of the Standard Model. More precisely we can decompose these representations under $SU(3)\times SU(2)\times U(1)$:

$$\begin{aligned}
\left(\overline{\mathbf{5}}\,,-\frac{3}{2}\right) &= \left(\overline{\mathbf{3}},\mathbf{1},-\frac{2}{3}\right)_{u^c} + \left(\mathbf{1},\mathbf{2},-\frac{1}{2}\right)_{L}\,, \\
\left(\mathbf{10},+\frac{1}{2}\right) &= \left(\mathbf{3},\mathbf{2},+\frac{1}{6}\right)_{Q} + \left(\overline{\mathbf{3}},\mathbf{1},+\frac{1}{3}\right)_{d^c} + (\mathbf{1},\mathbf{1},0)_{\nu^c}\,, \\
\left(\mathbf{1}\,,+\frac{5}{2}\right) &= (\mathbf{1},\mathbf{1},+1\,)_{e^c}\,,
\end{aligned}$$

where L is the lepton–doublet; Q is the quark–doublet; d^c, u^c, e^c and ν^c are the quark and lepton singlets. Because of the α–projection, which projects on incomplete **16** and $\overline{\mathbf{16}}$ representations, complete families and anti–families are formed by combining states from different sectors.

4.2 Chirality Operators

A phenomenologically viable model consists of 3 families of chiral **16** representations of $SO(10)$ decomposed under $SU(5)\times U(1)$. Therefore, we have to count the number of **16**s and $\overline{\mathbf{16}}$s. The choice of GGSO coefficients determine the model we consider and therefore the number of families. In order to be able to distinguish between **16** and $\overline{\mathbf{16}}$, one has to define operators that determine the representations in which the states of each observable sector fall into. The operators $X_{pqrs}^{(1,2,3)_{SO(10)}}=\pm 1$, defines the $SO(10)$ chirality ($\mathbf{16}$ or $\overline{\mathbf{16}}$) for B^1_{pqrs}, B^2_{pqrs} and B^3_{pqrs}, which are given by

$$\begin{aligned} X^{(1)_{SO(10)}}_{pqrs} &= C\binom{B^{(1)}_{pqrs}}{b_2+(1-r)e_5+(1-s)e_6}, \\ X^{(2)_{SO(10)}}_{pqrs} &= C\binom{B^{(2)}_{pqrs}}{b_1+(1-r)e_5+(1-s)e_6}, \\ X^{(3)_{SO(10)}}_{pqrs} &= C\binom{B^{(3)}_{pqrs}}{b_1+(1-r)e_3+(1-s)e_4}. \end{aligned} \tag{4.1}$$

This is in contrast to the case in the Pati–Salam heterotic–string models, where one needs to determine the chirality of the $SO(6)$ and $SO(4)$ representations separately [15]. Additionally, we determine which components in the $\mathbf{16}$ and $\overline{\mathbf{16}}$ survive the α projection, which breaks $SO(10)$ to $SU(5)\times U(1)$. In this respect, we note that the α projection operates identically on the $\mathbf{1}\equiv(\mathbf{1},+5/2)$ and $\overline{\mathbf{5}}\equiv(\overline{\mathbf{5}},-3/2)$ states and similarly on the conjugate representations $\overline{\mathbf{1}}\equiv(\mathbf{1},-5/2)$ and $\mathbf{5}\equiv(\mathbf{5},+3/2)$. The surviving components are determined by defining the operators $X^{(1,2,3)_{SU(5)}}_{pqrs}=\pm1$, where $X^{(i)_{SU(5)}}_{pqrs}=1$ indicates survival of the $(\mathbf{1},+5/2)$ and $(\overline{\mathbf{5}},-3/2)$ pair and $X^{(1,2,3)_{SU(5)}}_{pqrs}=-1$ indicates survival of the $(\mathbf{10},+1/2)$ states. The operator $X^{(i)_{SU(5)}}_{pqrs}$ acts similarly on the $\overline{\mathbf{16}}$ of $SO(10)$. These conditions are expressed as

$$\begin{aligned} X^{(1)_{SU(5)}}_{pqrs} &= C\binom{B^{(1)}_{pqrs}}{\alpha}, \\ X^{(2)_{SU(5)}}_{pqrs} &= C\binom{B^{(2)}_{pqrs}}{\alpha}, \\ X^{(3)_{SU(5)}}_{pqrs} &= C\binom{B^{(3)}_{pqrs}}{\alpha}. \end{aligned} \tag{4.2}$$

4.3 Projectors

The states in the sectors in $B^{(1)}_{pqrs}$, as given in (3.2), can be projected in or out of the string spectrum depending on the GGSO projections of the vectors e_1, e_2, z_1 and z_2. Likewise for $B^{(2)}_{pqrs}$ and $B^{(3)}_{pqrs}$, we define a projector P such that the states survive when $P=1$ and are projected out when $P=0$, which are given as:

$$\begin{aligned} P^{(1)}_{pqrs} &= \frac{1}{16}\left(1-C\binom{e_1}{B^{(1)}_{pqrs}}\right)\cdot\left(1-C\binom{e_2}{B^{(1)}_{pqrs}}\right)\cdot\left(1-C\binom{z_1}{B^{(1)}_{pqrs}}\right)\cdot\left(1-C\binom{z_2}{B^{(1)}_{pqrs}}\right), \\ P^{(2)}_{pqrs} &= \frac{1}{16}\left(1-C\binom{e_3}{B^{(2)}_{pqrs}}\right)\cdot\left(1-C\binom{e_4}{B^{(2)}_{pqrs}}\right)\cdot\left(1-C\binom{z_1}{B^{(2)}_{pqrs}}\right)\cdot\left(1-C\binom{z_2}{B^{(2)}_{pqrs}}\right), \\ P^{(3)}_{pqrs} &= \frac{1}{16}\left(1-C\binom{e_5}{B^{(3)}_{pqrs}}\right)\cdot\left(1-C\binom{e_6}{B^{(3)}_{pqrs}}\right)\cdot\left(1-C\binom{z_1}{B^{(3)}_{pqrs}}\right)\cdot\left(1-C\binom{z_2}{B^{(3)}_{pqrs}}\right). \end{aligned} \tag{4.3}$$

These projectors can be expressed as a system of linear equations with p, q, r and s as unknowns. The solutions of such a system of equations yield the different combinations of p, q, r and s for which sectors survive the GGSO projections. The analytic expressions for each of the different projectors $P^{1,2,3}_{pqrs}$ are given in a matrix form $\Delta^i W^i = Y^i$, where

$$
\begin{aligned}
&\begin{pmatrix}
(e_1|e_3) & (e_1|e_4) & (e_1|e_5) & (e_1|e_6)\\
(e_2|e_3) & (e_2|e_4) & (e_2|e_5) & (e_2|e_6)\\
(z_1|e_3) & (z_1|e_4) & (z_1|e_5) & (z_1|e_6)\\
(z_2|e_3) & (z_2|e_4) & (z_2|e_5) & (z_2|e_6)
\end{pmatrix}
\begin{pmatrix} p\\ q\\ r\\ s \end{pmatrix}
=
\begin{pmatrix} (e_1|b_1)\\ (e_2|b_1)\\ (z_1|b_1)\\ (z_2|b_1) \end{pmatrix}
= Y^1,\\
&\begin{pmatrix}
(e_3|e_1) & (e_3|e_2) & (e_3|e_5) & (e_3|e_6)\\
(e_4|e_1) & (e_4|e_2) & (e_4|e_5) & (e_4|e_6)\\
(z_1|e_1) & (z_1|e_2) & (z_1|e_5) & (z_1|e_6)\\
(z_2|e_1) & (z_2|e_2) & (z_2|e_5) & (z_2|e_6)
\end{pmatrix}
\begin{pmatrix} p\\ q\\ r\\ s \end{pmatrix}
=
\begin{pmatrix} (e_3|b_2)\\ (e_4|b_2)\\ (z_1|b_2)\\ (z_2|b_2) \end{pmatrix}
= Y^2,\\
&\begin{pmatrix}
(e_5|e_1) & (e_5|e_2) & (e_5|e_3) & (e_5|e_4)\\
(e_6|e_1) & (e_6|e_2) & (e_6|e_3) & (e_6|e_4)\\
(z_1|e_1) & (z_1|e_2) & (z_1|e_3) & (z_1|e_4)\\
(z_2|e_1) & (z_2|e_2) & (z_2|e_3) & (z_2|e_4)
\end{pmatrix}
\begin{pmatrix} p\\ q\\ r\\ s \end{pmatrix}
=
\begin{pmatrix} (e_5|b_3)\\ (e_6|b_3)\\ (z_1|b_3)\\ (z_2|b_3) \end{pmatrix}
= Y^3.
\end{aligned}
\tag{4.4}
$$

Here the GGSO phases are defined as

$$
C\binom{v_i}{v_j} = e^{i\pi(v_i|v_j)} \ ,
$$

where v_i and v_j refer to the basis vectors and the GGSO projections are defined as in (2.1). The corresponding algebraic expressions for the states from the remaining sectors above are enumerated in the appendix, as well as the states in the hidden sector which are not classified here, may play a substantial role in the string phenomenology, such as in the case of the SUSY breaking. Furthermore, the projectors presented in the appendix determine the number of surviving observable, hidden and exotic states in each model.

5 Gauge Group Enhancements

The $SU(5)\times U(1)$ gauge symmetry generated by the untwisted space–time vector bosons, may be enhanced by the vector bosons that arise from the sectors listed in (3.1). We impose that all the additional space–time vector bosons are projected out. The gauge symmetry is therefore identical in all the models that we scan, though the occurrence of models with enhancements is approximately about 23.8% of the total models. The string models in our classification differ by the string spectrum that arises from the twisted sectors. In our classification method, we encode the GGSO projections coefficients in terms of algebraic equations, which are applied to all the sectors listed in section 3.

The gauge bosons of any given sector in (3.1) transform under a subgroup of the Neveu–Schwarz gauge group. If they survive the GGSO projections, then the NS gauge group is enhanced. We restrict our classification here to the cases without enhancement, by identifying when the gauge bosons survive the GGSO projections and generalize the formulae to eliminate them. We remark that models with enhanced gauge symmetry in the observable or hidden sectors may be of interest for various phenomenological reasons, as, for example, the $SU(6) \times SU(2)$ string models presented in [18]. Below, we present the different types of enhancements that can occur within the string spectrum from the sectors given in (3.1). In addition, we assume that only one set of conditions is satisfied from any one given sector in (3.1).

5.1 Observable gauge group enhancement

There is one sector contributing only to the enhancement of the observable gauge group i.e. $SU(5)_{obs} \times U(1)_5 \times U(1)_1 \times U(1)_2 \times U(1)_3$. This is the sector $z_1 + 2\alpha$, given by the conditions:

- $z_1 + 2\alpha = \{\overline{\psi}^{1,\dots,5}, \overline{\eta}^{1,2,3}\}$

Sector Condition
$(z_1 + 2\alpha \vert e_i) = (z_1 + 2\alpha \vert z_k) = 0$

Enhancement Condition	**Resulting Enhancement**
$(z_1 + 2\alpha \vert \alpha) = (z_1 + 2\alpha \vert b_2)$	$SU(5)_{obs} \times U(1)_5 \times U(1)_1 \times U(1)_2 \times U(1)_3 \longrightarrow SU(6) \times SU(2) \times U(1)^2$
$(z_1 + 2\alpha \vert \alpha) \neq (z_1 + 2\alpha \vert b_2)$	$SU(5)_{obs} \times U(1)_5 \times U(1)_1 \times U(1)_2 \times U(1)_3 \longrightarrow SO(10) \times U(1)^3$

where $i = 1, \dots, 6$ and $k = 1, 2$.

5.2 Hidden gauge group enhancement

The vector bosons arising from the untwisted sector produce the hidden gauge symmetry, which is given as $SU(4)_{hid} \times U(1)_4 \times SO(6)_{hid} \times U(1)_{hid}$. Similar to the observable sector, there is one sector that enhances only the untwisted hidden sector gauge symmetry and is given by the sector $z_1 + z_2$, where the conditions are given by:

- $z_1 + z_2 = \{\overline{\phi}^{1,\ldots,8}\}$

Sector Condition
$(z_1 + z_2 \vert e_i) = (z_1 + z_2 \vert b_k) = 0$

Enhancement Condition	**Resulting Enhancement**
$(z_1 + z_2 \vert z_1) = 1$	$SU(4)_{hid} \times U(1)_4 \times SO(6)_{hid} \times U(1)_{hid}$ $\longrightarrow SU(8) \times U(1)$

where $i = 1, \ldots, 6$ and $k = 1, 2$.

5.3 Mixed gauge group enhancements

The additional sectors in (3.1), produce vector bosons coming from the mixture of the observable and hidden sector gauge groups. The mixed gauge group enhancements are formed from the untwisted symmetries of the observable and hidden gauge group. These are given from the sectors z_1, z_2, α, $z_1 + \alpha$, $z_2 + \alpha$ and $z_1 + z_2 + \alpha$. The conditions are as follows:

- $z_2 = \{\overline{\phi}^{5,\ldots,8}\}$

Sector Condition
$(z_2 \vert e_i) = 0$

Enhancement Condition	**Resulting Enhancement**
$(z_2 \vert z_1) = 1$ $(z_2 \vert b_k) = 0$	$SU(4)_{hid} \times U(1)_4 \times SO(6)_{hid} \times U(1)_{hid}$ $\longrightarrow SU(8) \times U(1)$
$(z_2 \vert z_1) = 0$ $(z_2 \vert b_k) \neq 1$	$U(1)_{1/2/3} \times SO(6)_{hid} \times U(1)_{hid}$ $\longrightarrow SU(5) \times U(1)$
$(z_2 \vert z_1) = 0$ $(z_2 \vert b_k) = 1$	$SU(5)_{obs} \times U(1)_5 \times SO(6)_{hid} \times U(1)_5$ $\longrightarrow SU(9) \times U(1)$

where $i = 1, \ldots, 6$ and $k = 1, 2$.

- $z_1 = \{\overline{\phi}^{1,\dots,4}\}$

Enhancement Condition	**Resulting Enhancement**
$(z_1\|e_i) = (z_1\|b_k) = (z_1\|z_1) = (z_1\|\alpha) = 0$ $(z_1\|z_2) = 1$	$SU(4)_{hid} \times U(1)_4 \times U(1)_{hid} \times SO(6)_{hid}$ $\longrightarrow SO(10) \times SU(3) \times SU(2)$
$(z_1\|e_i) = (z_1\|b_k) = (z_1\|z_1) = 0$ $(z_1\|z_2) = (z_1\|\alpha) = 1$	$SU(4)_{hid} \times U(1)_4 \times U(1)_{hid} \times SO(6)_{hid}$ $\longrightarrow SO(12) \times SO(4)$
$(z_1\|e_i) = (z_1\|z_2) = 0$ $(z_1\|b_k) \neq 1$ $(z_1\|z_1) = 1$	$U(1)_{1/2/3} \times SU(4)_{hid} \times U(1)_4$ $\longrightarrow SU(5) \times U(1)$
$(z_1\|e_i) = (z_1\|z_2) = 0$ $(z_1\|b_k) = (z_1\|z_1) = 1$	$SU(5)_{obs} \times U(1)_5 \times SU(4)_{hid} \times U(1)_4$ $\longrightarrow SU(9) \times U(1)$
$(z_1\|e_j) = (z_1\|z_1) = (z_1\|z_2) = (z_1\|\alpha) = 0$ $(z_1\|e_i) = 1$ AND $(z_1\|b_1) = 0,\ i = 1, 2$ or $(z_1\|b_2) = 0,\ i = 3, 4$ or $(z_1\|b_1) = (z_1\|b_2),\ i = 5, 6$	$SU(4)_{hid} \times U(1)_4 \longrightarrow SO(5) \times SO(5)$
$(z_1\|e_j) = (z_1\|z_1) = (z_1\|z_2) = 0$ $(z_1\|e_i) = (z_1\|\alpha) = 1$ AND $(z_1\|b_1) = 0,\ i = 1, 2$ or $(z_1\|b_2) = 0,\ i = 3, 4$ or $(z_1\|b_1) = (z_1\|b_2),\ i = 5, 6$	$SU(4)_{hid} \times U(1)_4 \longrightarrow SO(8)$

where $i, j = 1, \dots, 6$, $i \neq j$ and $k = 1, 2$.

- $\alpha = \{\overline{\psi}^{1,\ldots,5} = \frac{1}{2}, \overline{\eta}^{1,2,3} = \frac{1}{2}, \overline{\phi}^{1,\ldots,4} = \frac{1}{2}, \overline{\phi}^{5}\}$

  The states that give two $\frac{1}{4}$ oscillators are produced from the following conditions:

Sector Condition
$(\alpha\|e_i) = 0$ $(\alpha\|z_2) \neq (\alpha\|\alpha)$

Enhancement Condition	**Resulting Enhancement**
$(\alpha\|z_1) = (\alpha\|b_k) = 0$	$SU(5)_{obs} \times U(1)_5 \times U(1)_1 \times U(1)_2 \times U(1)_3 \times SU(4)_{hid} \times U(1)_4 \times U(1)_{hid}$ $\longrightarrow SO(10) \times SO(4) \times SO(4) \times U(1)^4$
$(\alpha\|z_1) = 0$ $(\alpha\|b_k) \neq 0$	$SU(5)_{obs} \times U(1)_5 \times U(1)_1 \times U(1)_2 \times U(1)_3 \times U(1)_{hid}$ $\longrightarrow SU(5) \times SU(3) \times U(1)^3$
$(\alpha\|z_1) = 1$ $(\alpha\|b_k) \neq 1$	$U(1)_{1/2/3} \times SU(4)_{hid} \times U(1)_4 \times U(1)_{hid}$ $\longrightarrow SO(6) \times SO(4) \times U(1)$
$(\alpha\|z_1) = (\alpha\|b_k) = 1$	$SU(5)_{obs} \times U(1)_5 \times SU(4)_{hid} \times U(1)_4 \times U(1)_{hid}$ $\longrightarrow SU(7) \times SO(4) \times SU(3)$

where $i = 1, \ldots, 6$ and $k = 1, 2$. Additionally, the states that give one $\frac{1}{2}$ oscillators are produced from the following conditions:

Sector Condition
$(\alpha\|z_1) = 0$

Enhancement Condition	**Resulting Enhancement**
$(\alpha\|e_i) = (\alpha\|b_k) = 0$ $(\alpha\|z_2) \neq (\alpha\|\alpha)$	$SO(6)_{hid} \times U(1)_{hid} \longrightarrow SO(7) \times U(1)$
$(\alpha\|z_2) = (\alpha\|\alpha)$ $(\alpha\|e_j) = 0$ $(\alpha\|e_i) = 1$ AND $(\alpha\|b_1) = 0,\ i = 1, 2$ or $(\alpha\|b_2) = 0,\ i = 3, 4$ or $(\alpha\|b_1) = (\alpha\|b_2),\ i = 5, 6$	$U(1)_{hid} \longrightarrow SU(2)$

where $i, j = 1, \ldots, 6$, $i \neq j$ and $k = 1, 2$.

- $z_1 + \alpha = \{\overline{\psi}^{1,\dots,5} = \frac{1}{2}, \overline{\eta}^{1,2,3} = \frac{1}{2}, \overline{\phi}^{1,\dots,4} = -\frac{1}{2}, \overline{\phi}^{5}\}$

  The states that give two $\frac{1}{4}$ oscillators are produced from the following conditions:

Sector Condition	
$(z_1 + \alpha \vert e_i) = 0$	
Enhancement Condition	**Resulting Enhancement**
$(z_1 + \alpha \vert z_1) = (z_1 + \alpha \vert b_k) = 0$ $(z_1 + \alpha \vert z_2) \neq (z_1 + \alpha \vert \alpha)$	$SU(5)_{obs} \times U(1)_5 \times U(1)_1 \times U(1)_2 \times U(1)_3 \times SU(4)_{hid} \times U(1)_4 \times U(1)_{hid}$ $\longrightarrow SO(10) \times SO(4) \times SO(4) \times U(1)^4$
$(z_1 + \alpha \vert z_1) = 0$ $(z_1 + \alpha \vert b_k) \neq 0$ $(z_1 + \alpha \vert z_2) \neq (z_1 + \alpha \vert \alpha)$	$SU(5)_{obs} \times U(1)_5 \times U(1)_1 \times U(1)_2 \times U(1)_3 \times U(1)_{hid}$ $\longrightarrow SU(5) \times SU(3) \times U(1)^3$
$(z_1 + \alpha \vert z_1) = 1$ $(z_1 + \alpha \vert b_k) \neq 1$ $(z_1 + \alpha \vert z_2) = (z_1 + \alpha \vert \alpha)$	$U(1)_{1/2/3} \times SU(4)_{hid} \times U(1)_4 \times U(1)_{hid}$ $\longrightarrow SO(6) \times SO(4) \times U(1)$
$(z_1 + \alpha \vert z_1) = (z_1 + \alpha \vert b_k) = 1$ $(z_1 + \alpha \vert z_2) = (z_1 + \alpha \vert \alpha)$	$SU(5)_{obs} \times U(1)_5 \times SU(4)_{hid} \times U(1)_4 \times U(1)_{hid}$ $\longrightarrow SU(7) \times SO(4) \times SU(3)$

where $i = 1, \dots, 6$ and $k = 1, 2$. Additionally, the states that give one $\frac{1}{2}$ oscillators are produced from the following conditions:

Sector Condition	
$(z_1 + \alpha \vert z_1) = 0$	
Enhancement Condition	**Resulting Enhancement**
$(z_1 + \alpha \vert e_i) = (z_1 + \alpha \vert b_k) = 0$ $(z_1 + \alpha \vert z_2) \neq (z_1 + \alpha \vert \alpha)$	$SO(6)_{hid} \times U(1)_{hid} \longrightarrow SO(7) \times U(1)$
$(z_1 + \alpha \vert z_2) = (z_1 + \alpha \vert \alpha)$ $(z_1 + \alpha \vert e_j) = 0$ $(z_1 + \alpha \vert e_i) = 1$ AND $(z_1 + \alpha \vert b_1) = 0$, $i = 1, 2$ or $(z_1 + \alpha \vert b_2) = 0$, $i = 3, 4$ or $(z_1 + \alpha \vert b_1) = (z_1 + \alpha \vert b_2)$, $i = 5, 6$	$U(1)_{hid} \longrightarrow SU(2)$

where $i, j = 1, \dots, 6$, $i \neq j$ and $k = 1, 2$.

- $z_2 + \alpha = \{\overline{\psi}^{1,..,5} = \frac{1}{2}, \overline{\eta}^{1,2,3} = \frac{1}{2}, \overline{\phi}^{1,..,4} = \frac{1}{2}, \overline{\phi}^{6,7,8}\}$

Sector Condition
$(z_2 + \alpha \vert e_i) = 0$ $(z_2 + \alpha \vert \alpha) = \frac{1}{2}$

Enhancement Condition	**Resulting Enhancement**
$(z_2 + \alpha \vert z_1) = 0$ $(z_2 + \alpha \vert b_k) = 1$	$SU(5)_{obs} \times U(1)_5 \times SO(6)_{hid}$ $\longrightarrow SO(10) \times SO(6)$
$(z_2 + \alpha \vert z_1) = 0$ $(z_2 + \alpha \vert b_k) \neq 1$	$U(1)_{1/2/3} \times SO(6)_{hid}$ $\longrightarrow SO(5) \times SO(5)$
$(z_2 + \alpha \vert b_k) = 0$ $(z_2 + \alpha \vert z_1) = 1$	$SU(4)_{hid} \times U(1)_4 \times SO(6)_{hid}$ $\longrightarrow SO(10) \times U(1)^2$

where $i = 1, \ldots, 6$ and $k = 1, 2$.

- $z_1 + z_2 + \alpha = \{\overline{\psi}^{1,..,5} = \frac{1}{2}, \overline{\eta}^{1,2,3} = \frac{1}{2}, \overline{\phi}^{1,..,4} = -\frac{1}{2}, \overline{\phi}^{6,7,8}\}$

Sector Condition
$(z_1 + z_2 + \alpha \vert e_i) = 0$

Enhancement Condition	**Resulting Enhancement**
$(z_1 + z_2 + \alpha \vert z_1) = 0$ $(z_1 + z_2 + \alpha \vert \alpha) = \frac{1}{2}$ $(z_1 + z_2 + \alpha \vert b_k) = 1$	$SU(5)_{obs} \times U(1)_5 \times SO(6)_{hid}$ $\longrightarrow SO(10) \times SO(6)$
$(z_1 + z_2 + \alpha \vert z_1) = 0$ $(z_1 + z_2 + \alpha \vert \alpha) = \frac{1}{2}$ $(z_1 + z_2 + \alpha \vert b_k) \neq 1$	$U(1)_{1/2/3} \times SO(6)_{hid}$ $\longrightarrow SO(5) \times SO(5)$
$(z_1 + z_2 + \alpha \vert b_k) = 0$ $(z_1 + z_2 + \alpha \vert \alpha) = -\frac{1}{2}$ $(z_1 + z_2 + \alpha \vert z_1) = 1$	$SU(4)_{hid} \times U(1)_4 \times SO(6)_{hid}$ $\longrightarrow SO(10) \times U(1)^2$

where $i = 1, \ldots, 6$ and $k = 1, 2$.

Finally, we remark that as noted in section 3.3, the sectors α, $z_1 + \alpha$, $z_2 + \alpha$ and $z_1 + z_2 + \alpha$ may also give rise to exotic states, when the left–moving ψ^μ oscillator is replaced by a left–moving χ^i oscillator. Moreover, we note that the GGSO projections of the basis vectors $e_{1,\ldots,6}$, $z_{1,2}$ and α do not distinguish between ψ^μ and χ^i, which can therefore be used to project both the enhancements, as well as the exotic states arising from the sectors α, $z_1 + \alpha$, $z_2 + \alpha$ and $z_1 + z_2 + \alpha$.

6 Results

By use of the algebraic expressions given in the sections previously, as well as in the appendix, we analyse the entire massless spectrum for a given choice of configuration of GGSO projection coefficients. These expressions can be seen as matrix equations which we programmed into a JAVA and FORTRAN95 code independently, that are used to scan the space of the string vacua. The number of possible configurations is $2^{44} \approx 10^{13}$, which is very large, for a classification of the entire string vacua. For this purpose, a random generation algorithm is used† and the characteristics of the models for each set of random GGSO projection coefficients are extracted. From the generated sample, a model with the desired phenomenological criteria can be fished out. This procedure was followed in [15, 16], which produced a three generation Pati–Salam heterotic–string models that do not contain any exotic massless states with fractional electric charge. In this paper, we use this methodology to classify the flipped $SU(5)$ free fermionic string models with respect to some phenomenological criteria. For example, a question of interest is the existence of viable three generation exophobic flipped $SU(5)$ vacua. The observable sector of a heterotic–string flipped $SU(5)$ model is characterised by 15 integers $(n_1, n_{\bar{1}}, n_{5s}, n_{\bar{5}s}, n_{10}, n_{\overline{10}}, n_g, n_{10H}, n_{\bar{5}v}, n_{5v}, n_{5h}, n_{1e}, n_{\bar{1}e}, n_{5e}, n_{\bar{5}e})$, which are given by:

$$\begin{aligned}
n_1 &= \#\text{ of } (\mathbf{1}, +\tfrac{5}{2}),\\
n_{\bar{1}} &= \#\text{ of } (\mathbf{1}, -\tfrac{5}{2}),\\
n_{5s} &= \#\text{ of } (\mathbf{5}, +\tfrac{3}{2}),\\
n_{\bar{5}s} &= \#\text{ of } (\overline{\mathbf{5}}, -\tfrac{3}{2}),\\
n_{10} &= \#\text{ of } (\mathbf{10}, +\tfrac{1}{2}),\\
n_{\overline{10}} &= \#\text{ of } (\overline{\mathbf{10}}, -\tfrac{1}{2}),\\
n_g &= n_{10} - n_{\overline{10}} = n_{\bar{5}} - n_5 = \#\text{ of generations},\\
n_{10H} &= n_{10} + n_{\overline{10}} = \#\text{ of non chiral heavy Higgs pairs},\\
n_{\bar{5}v} &= \#\text{ of } (\overline{\mathbf{5}}, +1),\\
n_{5v} &= \#\text{ of } (\mathbf{5}, -1),\\
n_{5h} &= n_{5v} + n_{\bar{5}v} = \#\text{ of non chiral light Higgs pairs},\\
n_{1e} &= \#\text{ of } (\mathbf{1}, -\tfrac{5}{4}) \quad \text{(exotic)},\\
n_{\bar{1}e} &= \#\text{ of } (\mathbf{1}, +\tfrac{5}{4}) \quad \text{(exotic)},\\
n_{5e} &= \#\text{ of } (\mathbf{5}, -\tfrac{1}{4}) \quad \text{(exotic)},\\
n_{\bar{5}e} &= \#\text{ of } (\overline{\mathbf{5}}, +\tfrac{1}{4}) \quad \text{(exotic)}.
\end{aligned}$$

These numbers above are all relevant for the classification of the string vacua. As noted in section 4.2, the α projection dictates that $n_{\bar{1}} = n_{5s}$ and $n_1 = n_{\bar{5}s}$. Therefore,

† We, herein acknowledge the analysis of large sets of string vacua which has been performed by other groups [29].

the counting of n_5 and $n_{\bar{5}}$ is sufficient for the number generations as shown above. Moreover, we note the distinction between the **5** and $\overline{\mathbf{5}}$ representations that arise from the spinorial **16** representation of $SO(10)$ decomposed under $SU(5) \times U(1)$, denoted by $n_{5s}, n_{\bar{5}s}$ and the **5** and $\overline{\mathbf{5}}$ that arise from its vectorial **10** representation, denoted by $n_{5v}, n_{\bar{5}v}$. While the former gives rise to the Standard Model up–type quark electroweak singlet and lepton–doublet, the later accommodates the light electroweak Higgs doublets. In the flipped $SU(5)$ models they are distinguished by their charges under the $U(1)_5$ symmetry. Using the methodology outlined in section 4, we obtained analytic formulas for all these quantities. In order to extract a string spectrum from the phenomenologically viable models of the flipped $SU(5)$, we must have:

$n_g = 3$	Three light chiral of generations,
$n_{10H} \geq 1$	At least one heavy Higgs pair to break the $SU(5) \times U(1)$ symmetry,
$n_{5h} \geq 1$	At least one pair of light Minimal SM Higgs doublets,
$n_{1e} = n_{\bar{1}e} \geq 0$	Heavy mass can be generated for vector–like exotics,
$n_{5e} = n_{\bar{5}e} \geq 0$	Heavy mass can be generated for vector–like exotics.

Here, we imposed the constraints $n_{\bar{5}h} = n_{5h}$, $n_{1e} = n_{\bar{1}e}$ and $n_{5e} = n_{\bar{5}e}$, in order to sustain anomaly free flipped $SU(5)$ models.

6.1 Minimal exophilic models

Compared to the case of the Pati–Salam classification [15], which yielded 3 generation models that are completely free of massless exotic states, no such models were found in our scan of the flipped $SU(5)$ models. We emphasise that this does not indicate that exophobic free fermionic flipped $SU(5)$ vacua do not exist, but merely that they do not exist in the space of vacua that we explored. Nevertheless, it does show that large spaces of vacua may not contain exophobic models, which is in line with related searches [30]. A model with a minimal number of exotic states that we find in our scan has $n_g = 3$, $n_{\bar{5}s} = 3$, $n_{5s} = 0$, $n_{10} = 4$, $n_{\overline{10}} = 1$, $n_{10H} = 1$, $n_{5h} = 4$, $n_{1e} = 2$ and $n_{5e} = 0$. We note that this minimal model still contains exotics which has 4 states. The minimal model is then given by the following GGSO coefficients matrix:

$$(v_i|v_j) \quad = \quad \begin{array}{c} \\ S \\ e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \\ b_1 \\ b_2 \\ z_1 \\ z_2 \\ \alpha \end{array} \!\!\! \begin{array}{c} \begin{array}{cccccccccccc} S & e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & b_1 & b_2 & z_1 & z_2 & \alpha \end{array} \\ \left(\begin{array}{cccccccccccc} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1/2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1/2 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & -1/2 \\ 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \end{array} \right) \end{array} \qquad (6.1)$$

In section 6.3 we elaborate further on the structure of the exotic states in the flipped $SU(5)$ models.

6.2 Classification

Next, we elaborate on the classification of the space of the free fermionic flipped $SU(5)$ string vacua by performing a statistical sampling, in a space of 10^{12} models out of the 2^{44} possibilities. For this purpose, we developed two independent computer codes. One being a FORTRAN95 computer program running on a single node of the Theoretical Physics Division of University of Ioannina, HPC cluster. The other being a JAVA code running on 10 nodes of the University of Liverpool, Department of Physics ULGQCD cluster that runs on AMD Opteron 6128 2GhZ CPUs. Additionally, assistance from five servers of the University of Liverpool, Department of Mathematics were also used, which totalled to 200 CPUs that ran for about 2 weeks

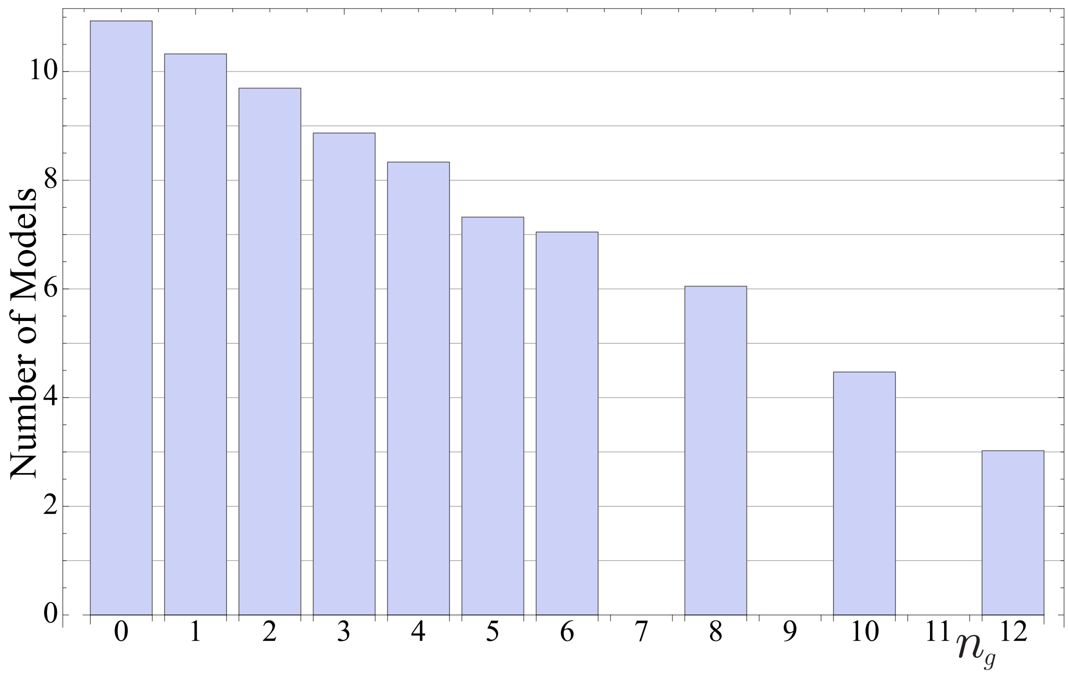

Figure 1: *Logarithm of the number of models against the number of generations (n_g) in a random sample of* 10^{12} *flipped* $SU(5)$ *configurations.*

in order to scan 10^{12} random models. Some of the results are presented in figures 1 - 3 and table 1 - 3.

In figure 1, the number of models against the number of generations is displayed. This is in agreement with the results of [11, 12, 15], where the number of models peaks for 0 generation and decreases as the number of generations increases. Also in this figure, we note the absence of models with 7, 9, 11 and greater than 12 generations. These result can be understood in light of the corresponding results in the $SO(10)$ classification [12]. We recall that the α projection which breaks the $SO(10)$ symmetry to the flipped $SU(5)$, truncates the number of generations by two. Examining the corresponding figure in the $SO(10)$ classification, we observe the absence of the models with double the number of generations, *i.e.* with 14, 18, 22 and more than 24 generations. We remark that this result is applicable to the case in which all gauge group enhancements are projected out, as discussed in section 5. Therefore, models with the excluded number of generations may occur when the hidden gauge group is enhanced. However, we compare figure 1 to the corresponding figure in [15] and notice the existence of models with 16 generations, which seems to contradict our argument. This apparent contradiction is resolved by noting that the α projection in the Pati-Salam case projects out some of the gauge group enhancements, whereas from section 5 we note that this is not the case for the basis vector α_1 we used here. Therefore, some of these models descend from $SO(10)$

n_{5h}/n_{10H}	0	1	2	3
0	281477	28518	0	0
1	3626622	275967	8197	651
2	630727	61910	2092	0
3	23924485	63774	5901	0
4	78959	67900	0	0
5	139642	12380	0	0

Table 1: Number of three generation models as a function of the flipped $SU(5)$ breaking Higgs pairs (n_{10H}) and SM breaking Higgs pairs (n_{5h}) in a random sample of 10^{10} models. Models with $n_{10H} = 0$ or $n_{5h} = 0$ are not SM compatible, the former because $SU(5)$ cannot be broken to the SM via the Higgs mechanism and the latter due to the lack of SM breaking Higgs scalars.

GUTs with enhanced gauge group, which do not arise in the case of the flipped $SU(5)$ models studied here.

In table 1, we display the number of three generation models against the number of pairs of light and heavy Higgs representations appearing in the models, with the light and heavy pairs being $\mathbf{5}+\overline{\mathbf{5}}$ and $\mathbf{10}+\overline{\mathbf{10}}$ representations of $SU(5)$, respectively. Clearly, the null cases are not viable phenomenologically and the minimal cases are models with one pair of each. In models with a larger number of light Higgs pairs it may be easier to accommodate the Standard Model fermion mass textures, whereas models with a larger number of heavy Higgs pairs may facilitate gauge coupling unification at the string scale [5, 33].

As seen in section 3.3, some of the exotic matter states in the models transform in vector–like representations of the hidden sector non–Abelian group factors. They carry fractional electric charge and must be sufficiently massive or confined. These exotic states may nevertheless have interesting phenomenological implications. In table 2, we explore the structure of the exotic states arising in the models, which are labelled by four integers, $(\mathbf{n}^e_5, \mathbf{n}^e_1, \mathbf{n}^e_4, \mathbf{n}^e_{4'})$, where $\mathbf{n}^e_5 = n_{5e} + n_{\overline{5}e}$ is the number of exotic states that transform as $\mathbf{5}$ and $\overline{\mathbf{5}}$ of the observable $SU(5)$; $\mathbf{n}^e_1 = n_{1e} + n_{\overline{1}e}$ is the number of exotic states that transform as singlets of all non–Abelian group factors; $\mathbf{n}^e_4 = n_{4e} + n_{\overline{4}e}$ is the number of exotic states that transform as $\mathbf{4}$ and $\overline{\mathbf{4}}$ of the hidden $SU(4)$; $\mathbf{n}^e_{4'} = n_{4'e} + n_{\overline{4}'e}$ is the number of exotic states that transform as $\mathbf{4}$ and $\overline{\mathbf{4}}$ of the hidden $SO(6)$ gauge group.

In figure 2, we display the number of exophobic models against the number of generations. The striking feature in this figure is the absence of models with three chiral generations. This is in contrast to the case of the Pati–Salam models that yielded numerous three generation exophobic models. Figure 2 also reveals the absence of any exophobic models with 1, 3, 5, 7, 9, 11 and any exophobic models with more than 12 generation, whereas exophobic models arise for even number of generations, up to 12. As a result, exophobic models in this class arise in configurations with even

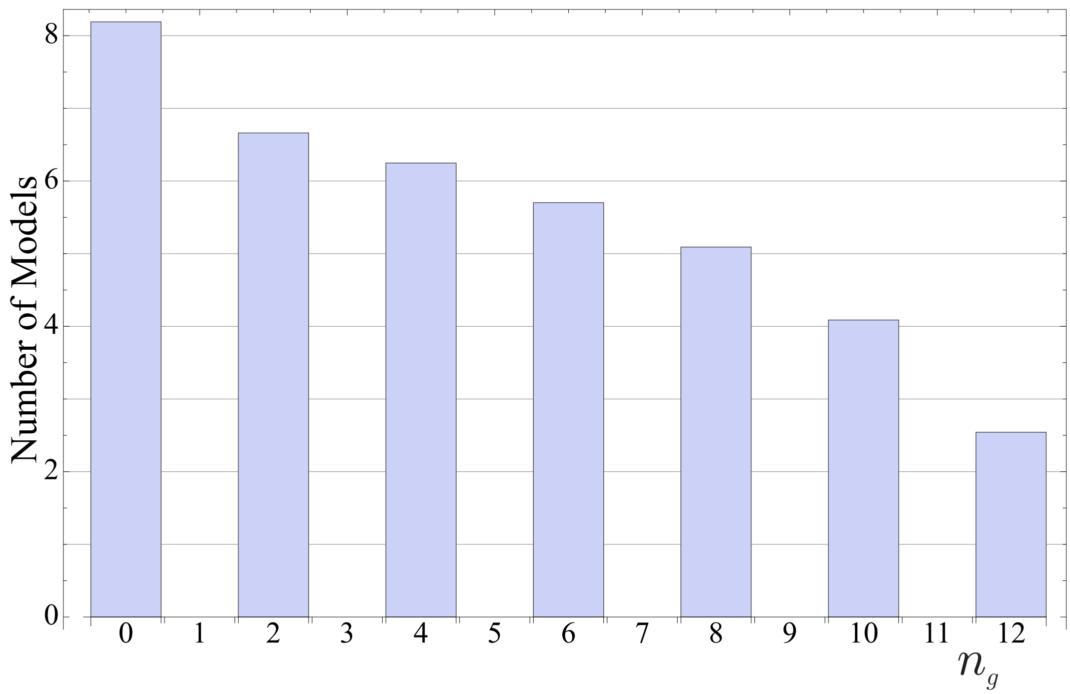

Figure 2: *Logarithm of the number of exophobic models against the number of generations (n_g) in a random sample of* 10^{12} *flipped* $SU(5)$ *configurations.*

number of generations but not in models with an odd number of generations. We emphasize that these results hold in the space of models that we explore here and does indicate absence of three generation exophobic flipped $SU(5)$ models. In figure 3, we display the number of three generation models against the number of exotic multiplets. We note from the figure that the minimal number of exotic multiplets is 4.

6.3 The structure of the exotic states

One of the main highlights of our classification method in the case of the Pati–Salam heterotic–string models has been the discovery of the exophobic heterotic–string models, in which all exotic states are limited to the massive spectrum and do not appear among the massless states. As shown in figures 2 and 3, in the class of 10^{12} flipped $SU(5)$ models that we analysed here, there are no exophobic 3 generation vacua with statistical a frequency larger than $1 : 10^{12}$. The structure of the exotic states arising in the models is analysed further in table 2. All the models given in the table 2 contain three chiral generations of which at least one–pair is the light Higgs states and at least one pair is the heavy Higgs states. Thus, in all these models the gauge symmetry can be broken to the Standard Model in the effective field theory limit and contain all the fields required for viable Standard Model phenomenology.

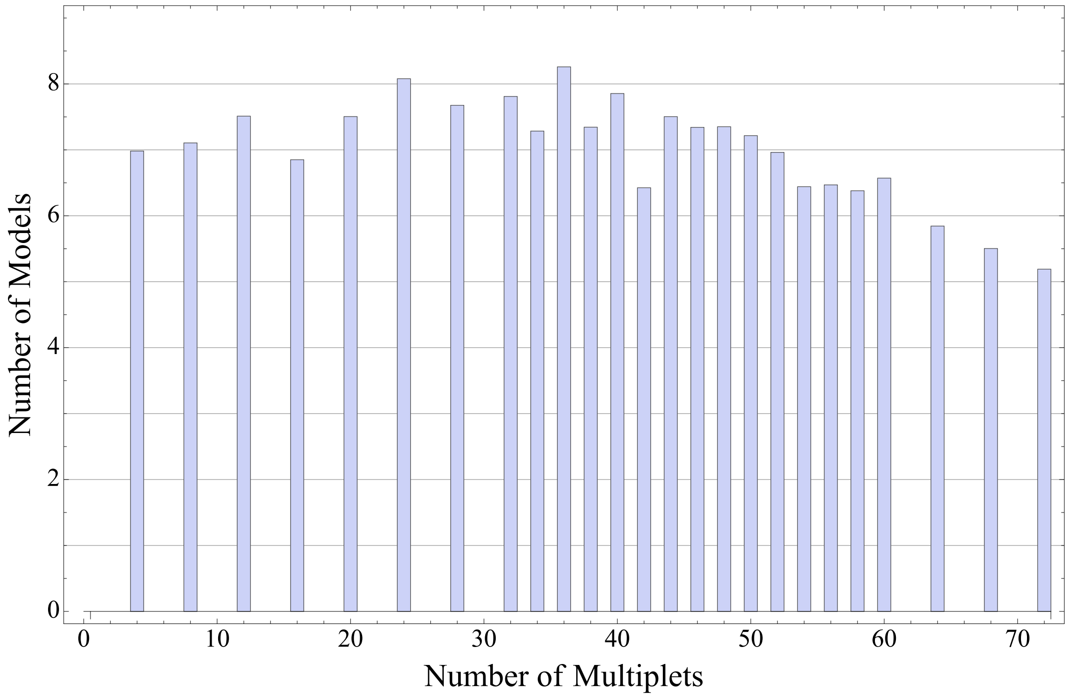

Figure 3: *Logarithm of the number 3 generation models against the number of exotic multiplets ($n_e = n_{1e} + n_{\bar{1}e} + n_{5e} + n_{\bar{5}e}$) in a random sample of 10^{12} flipped $SU(5)$ configurations.*

From table 2, we note the occurrence of models in which all exotic states transform in representations of an hidden non–Abelian gauge group. In this case, the exotic states are confined into integrally charged states and produce the so–called cryptons [31]. We further note from this table, the existence of models with equal number of $\mathbf{4}$ and $\mathbf{4'}$ states. This suggests the possible existence of the free fermionic models that admit the race–track mechanism to stabilise the vacuum expectation value of the dilaton field [32]. Moreover, table 2 reveals interesting observations and directions for future research. The first eleven models in the table contain only states that transform in non–trivial representations of an hidden non–Abelian gauge group. Thus, this class of models may give rise to the so–called crypton states that are confined into integrally charged states. We see that there is an abundance of such models. There are also numerous models with small number of crypton states that may remain asymptotically free and therefore, confined at some scale. A well known example of a model that gives rise only to crypton like states is given in [4]. The table shows the existence of a large space of models with similar characteristics. One notable difference between the vacua in this table and the one of [4], is the fact that the model in [4] uses asymmetric internal shifts, whereas the models in this table only use symmetric internal shifts. The models in the six and twelfth rows of the table, with $n_4 = n_{4'} = 2$ are interesting to study for implementation of the racetrack mechanism [32].

Turning to the other types of exotic states. The non–Abelian singlet states that are counted in the second column are fractionally charged and must decouple from the light spectrum or be sufficiently diluted. The fields counted in the first column transform as $\mathbf{5}$ and $\overline{\mathbf{5}}$ of the observable $SU(5)$ and carry 1/2 of the hypercharge compared to the standard flipped $SU(5)$ states. Such states do not arise in the flipped $SU(5)$ model studied in [4], but their colour triplet and electroweak doublet components arise generically in the standard–like heterotic–string models [7, 33]. These fields may be instrumental as intermediate matter states to resolve the conflict between heterotic–string scale unification and the low scale gauge coupling experimental data [33]. The models appearing in the 13$^{\text{th}}$ and 25$^{\text{rd}}$ rows in table 2, are interesting examples of flipped $SU(5)$ models admitting such states. The models in the 25$^{\text{rd}}$ row with $n_5 = 2$, $n_1 = 6$, $n_4 = 2$ and $n_{4'} = 2$ may accommodate both the intermediate matter thresholds and the racetrack mechanism, therefore be of particular interest.

6.4 An illustrative example

In this section, we analyse the model given in (6.2) as an illustrative of the exotic spectrum appearing in the flipped $SU(5)$ models. The twisted sectors of the model given here produce three chiral generations; one pair of heavy Higgs states; one pair of light Higgs representations. Therefore, this model may yield viable Standard Model phenomenology.

$$
(v_i|v_j) \quad = \quad
\begin{array}{c}
 \\ S \\ e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \\ b_1 \\ b_2 \\ z_1 \\ z_2 \\ \alpha
\end{array}
\begin{array}{c}
\begin{array}{cccccccccccc}
S & e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & b_1 & b_2 & z_1 & z_2 & \alpha
\end{array} \\
\left(\begin{array}{cccccccccccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 \\
1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 \\
1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\
0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & -1/2 \\
0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1/2 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1/2 \\
1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1
\end{array}\right)
\end{array}
\qquad (6.2)
$$

Additionally, the model contains the following states that transform under the hidden $SU(4)$ gauge group: six non–exotic pairs of $(\mathbf{4} + \overline{\mathbf{4}})$; one non–exotic state transforming in the vectorial $\mathbf{6}$ representation; one pair of exotic states transforming as $(\mathbf{4} + \overline{\mathbf{4}})$. The model contains the following states that transform under the hidden $SU(4)'$ gauge group: four non–exotic pairs of $(\mathbf{4} + \overline{\mathbf{4}})$; one non–exotic state transforming in the vectorial $\mathbf{6}$ representation; one pair of exotic states transforming as $(\mathbf{4}+\overline{\mathbf{4}})$. Thus, the β–functions of the $SU(4)$ and $SU(4)'$ hidden gauge groups are $\beta_4 = -4$ and $\beta_{4'} = -6$, respectively. Depending on the mass scales for the hidden sector matter states, this model may therefore provide workable example for implementing the racetrack mechanism. The model also contains one pair of exotic $(\mathbf{5}+\overline{\mathbf{5}})$ states of the observable flipped $SU(5)$ group that can be used to mitigate the gauge coupling unification problem.

n_5	n_1	n_4	n_4'	#	n_5	n_1	n_4	n_4'	#	n_5	n_1	n_4	n_4'	#
0	0	0	2	12627	2	10	3	3	9311	4	12	4	3	4889
0	0	0	4	3561	2	10	4	5	668	4	12	4	5	5720
0	0	0	6	1630	2	10	5	1	1614	4	12	4	6	965
0	0	0	8	187	2	10	5	2	4074	4	12	5	2	1479
0	0	2	0	16329	2	10	5	4	906	4	12	5	4	6105
0	0	2	2	18381	2	10	7	2	1745	4	12	6	4	608
0	0	2	4	2409	2	14	2	5	1474	4	12	8	0	153
0	0	4	0	6814	2	14	4	5	873	4	16	1	1	11395
0	0	4	2	3722	2	14	5	2	1412	5	7	2	4	1352
0	0	6	0	2338	2	14	5	4	1040	5	7	4	2	1323
0	0	8	0	356	2	18	1	1	2966	5	11	2	2	9462
0	8	2	2	1166	3	9	2	4	9505	5	11	2	5	2675
1	3	1	1	45575	3	9	3	3	2670	5	11	3	4	2491
1	3	2	8	4343	3	9	4	2	9949	5	11	4	3	2828
1	3	4	6	12465	3	13	2	2	9367	5	11	4	5	5164
1	3	6	4	12858	3	13	3	4	2562	5	11	5	2	2432
1	3	8	2	4287	3	13	4	3	2599	5	11	5	4	5074
1	11	2	4	1135	3	13	4	5	1909	5	15	2	5	4163
1	11	4	2	1336	3	13	5	4	1630	5	15	2	7	1069
2	6	0	0	13014	4	4	2	2	1231	5	15	4	5	1937
2	6	0	2	17622	4	8	1	5	1331	5	15	5	2	2605
2	6	0	4	6164	4	8	2	5	993	5	15	5	4	1170
2	6	0	6	3942	4	8	3	3	7915	5	15	7	2	670
2	6	2	0	14550	4	8	4	5	649	6	14	1	1	9970
2	6	2	2	25235	4	8	5	1	1443	6	18	0	2	2171
2	6	2	4	5864	4	8	5	2	1298	6	18	2	0	1499
2	6	3	5	10824	4	8	5	4	981	6	18	2	2	2272
2	6	4	0	5593	4	12	0	2	3255	6	18	2	4	799
2	6	4	2	4924	4	12	0	4	6986	6	18	4	2	490
2	6	4	4	2712	4	12	0	8	383	7	13	2	5	1311
2	6	4	6	1870	4	12	2	0	1795	7	13	2	7	758
2	6	5	3	10858	4	12	2	2	7064	7	13	5	2	849
2	6	6	0	2699	4	12	2	4	3139	7	13	7	2	428
2	6	6	4	2099	4	12	2	5	1269	8	12	1	1	2755
2	10	1	5	1522	4	12	3	4	5217	8	24	0	4	397
2	10	2	5	2794	4	12	4	0	3237	8	24	4	0	163
2	10	2	7	1199	4	12	4	2	2489	-	-	-	-	-

Table 2: *Number of three generation models with* ($n_{10H} \geq 1, n_{5h} \geq 1$) *against fractional charge state multiplicities in a sample of* 10^{10} *randomly selected models. Here* n_5 *is the number of* $\mathbf{5}+\overline{\mathbf{5}}$ $SU(5)$ *pairs,* n_1 *is the number of fractional* $SU(5)$ *singlet pairs,* n_4 *is the number of* $\mathbf{4}+\overline{\mathbf{4}}$ *pairs transforming under hidden* $SU(4)$ *and* n_4' *is the number of* $\mathbf{4}+\overline{\mathbf{4}}$ *pairs transforming under hidden* $SU(4)'$.

7 Conclusion

String theory continues to provide the only viable contemporary framework to explore the unification of gravity with the gauge interactions. For this reason, models with three generations must be constructed phenomenologically. Whilst examples of such fully realistic models may be a long way into the future, string theory provides an abundance of these concrete quasi–realistic examples that can be explored as toy models on the way to achieving the ultimate goal.

Here, we have continued to develop the methodology for the classification of the free fermionic heterotic–string models. The free fermionic construction [3], gave rise to some of the most realistic string models constructed to date [4, 6, 7, 8, 15, 16]. These models correspond to $\mathbb{Z}_2\times\mathbb{Z}_2$ orbifold compactifications at special points in the moduli space with discrete Wilson lines [21, 22]. The classification methodology was developed in [10] for the type II superstrings and then adopted for the classification of free fermionic heterotic–string models in [11, 12, 15]. In this method, the set of basis vectors are fixed incorporating all the possible symmetric $\mathbb{Z}_2$–shifts along the internal compactified directions, in the form of the six basis vectors e_i with adequate boundary conditions. The enumeration of the models is then obtained in terms of the GGSO projection coefficients and enables the scanning of large number of models with the aid of computers. The initial classification in [11] was with respect to chiral **16** and $\overline{\mathbf{16}}$ representations of an unbroken $SO(10)$ GUT group. Inclusion of the x–map [17] in the methodology facilitated the classification with respect to the vectorial **10** representations, which led to the discovery of spinor–vector duality [12, 13]. The classification methodology relies on writing the GGSO projections in algebraic forms, which makes the enumeration of the massless spectrum for a given configuration of GGSO phases straightforward. This classification was also extended to models in which the $SO(10)$ symmetry was broken to the Pati–Salam subgroup leading to the discovery of exophobic string vacua [15, 16].

In this paper, we extended the development of the classification methodology to the class of the free fermionic heterotic–string vacua in which the $SO(10)$ symmetry is broken to the flipped $SU(5)$ subgroup. This case presents several complications with respect to the previous ones. Whilst the earlier cases use only periodic and anti–periodic boundary conditions, the flipped $SU(5)$ class of vacua requires the use of rational boundary conditions. Another, is the variation of the basis vectors that are used to break the $SO(10)$ symmetry. With these modifications, while adaptation of the algebraic expressions is readily available, the computerised classification is substantially complicated. For this purpose, we developed two completely independent software routines, one based on JAVA and the second one being FORTRAN95 and all results presented in this paper are crossed checked using them. Our classification is limited to 1/2 rational boundary conditions, which is the case in all the quasi–realistic free fermionic models to date.

In table 3, we tabulate the number of models with sequential imposition of phe-

	Constraints	Total models in sample	Probability	Estimated number of models in class
	No Constraints	1000000000000	1	1.76×10^{13}
(1)	+ No Enhancements	762269298719	7.62×10^{-1}	1.34×10^{13}
(2)	+ Anomaly Free Flipped $SU(5)$	139544182312	1.40×10^{-1}	2.45×10^{12}
(3)	+ 3 Generations	738045321	7.38×10^{-4}	1.30×10^{10}
(4a)	+ SM Light Higgs	706396035	7.06×10^{-4}	1.24×10^{10}
(4b)	+ Flipped $SU(5)$ Heavy Higgs	46470138	4.65×10^{-5}	8.18×10^{8}
(5)	+ SM Light Higgs + & Heavy Higgs	43624911	4.36×10^{-5}	7.67×10^{8}
(6a)	+ Minimal Flipped $SU(5)$ Heavy Higgs	42310396	4.23×10^{-5}	7.44×10^{8}
(6b)	+ Minimal SM Light Higgs	25333216	2.53×10^{-5}	4.46×10^{8}
(7)	+ Minimal Flipped $SU(5)$ Heavy Higgs + & Minimal SM Light Higgs	24636896	2.46×10^{-5}	4.33×10^{8}
(8)	+ Minimal Exotic States	1218684	1.22×10^{-6}	2.14×10^{7}

Table 3: *Statistics for the flipped $SU(5)$ models with respect to phenomenological constraints. Here we note that the results of 4a and 4b have no effect on each other and this also holds for 6a and 6b.*

nomenological constraints. The total number of models in the sample is 10^{12}. We first impose that there is no enhancement of the four dimensional gauge symmetry and there are approximately 76.2% of the models that satisfy this criteria. Next we impose, that the flipped $SU(5)$ models are anomaly free with respect to the $U(1)_5$ group factor and about 14% of the total models satisfy this criterion. A further reduction, by three orders of magnitude, results from the restriction to the three chiral generations. Next, imposing the existence of both the heavy and light Higgs states to break the flipped $SU(5)$ gauge symmetry to the Standard Model gauge group and the electroweak breaking respectively, leads to a further reduction i.e. one order of magnitude. Finally, imposing the minimal number of massless exotic states results in the reduction of the number of models by a further order of magnitude. Therefore, the number of string vacua in the space of models scanned, reduces from 10^{12} to 10^6 which satisfy all the constraints that were imposed. This leaves a substantial number to accommodate further phenomenological constraints. In conclusion, we comment on the results obtained by using α_2 and α_3 in (2.3) to break the $SO(10)$ symmetry. In both cases a JAVA code was used to classify the models. The results are not that substantially different compared to the classification with α_1 and we also do not find any three generation exophobic vacua in these cases.

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A Projectors and matrix formalism

The algebraic expressions corresponding to states in the string spectrum are given by: $B^{(4,5,6)}_{pqrs}$ from (3.3), which produce light Higgs and Hidden vectorial states; $B^{(7,8,9)}_{pqrs}$ and $B^{(10,11,12)}_{pqrs}$ given in (3.4) and (3.5) respectively, which produce spinorial hidden matter states; $B^{(13,14,15)}_{pqrs}$ and $B^{(16,17,18)}_{pqrs}$ given in (3.6) and (3.7) respectively, which produce spinorial exotic states; $B^{(19,20,21)}_{pqrs}$ and $B^{(22,23,24)}_{pqrs}$ given in (3.8) and (3.9) respectively, which produce vectorial exotic states. We now enumerate these projectors with their corresponding algebraic expressions and matrix equations as follows.

A.1 Vectorial representations

The sectors in (3.3) produce vectorial states in the observable and hidden sector. These sectors are obtained from the combinations

$$B^{(4,5,6)}_{pqrs} = B^{(1,2,3)}_{pqrs} + z_1 + 2\alpha$$

The following is a list of the states produced in these sectors and the projectors that act on them:

- **States $\{\overline{\eta}^{1,2,3}\}|R\rangle$, $\{\overline{\eta}^{*1,2,3}\}|R\rangle$, $\{\overline{\psi}^{1,\dots,5}\}|R\rangle$ and $\{\overline{\psi}^{*1,\dots,5}\}|R\rangle$**

  This gives rise to the states that transform under the $SU(5)\times U(1)_5$ or $U(1)_{1/2/3}$ gauge group. The projectors are given by:

$$
\begin{aligned}
P^{(4)(\overline{\eta}^1,\overline{\psi}^{1,\dots,5})}_{pqrs} &= \frac{1}{16}\left(1-C\binom{e_1}{B^{(4)}_{pqrs}}\right)\cdot\left(1-C\binom{e_2}{B^{(4)}_{pqrs}}\right) \\
&\cdot\left(1-C\binom{z_1}{B^{(4)}_{pqrs}}\right)\cdot\left(1-C\binom{z_2}{B^{(4)}_{pqrs}}\right) \\
P^{(5)(\overline{\eta}^2,\overline{\psi}^{1,\dots,5})}_{pqrs} &= \frac{1}{16}\left(1-C\binom{e_3}{B^{(5)}_{pqrs}}\right)\cdot\left(1-C\binom{e_4}{B^{(5)}_{pqrs}}\right) \\
&\cdot\left(1-C\binom{z_1}{B^{(5)}_{pqrs}}\right)\cdot\left(1-C\binom{z_2}{B^{(5)}_{pqrs}}\right) \\
P^{(6)(\overline{\eta}^3,\overline{\psi}^{1,\dots,5})}_{pqrs} &= \frac{1}{16}\left(1-C\binom{e_5}{B^{(6)}_{pqrs}}\right)\cdot\left(1-C\binom{e_6}{B^{(6)}_{pqrs}}\right) \\
&\cdot\left(1-C\binom{z_1}{B^{(6)}_{pqrs}}\right)\cdot\left(1-C\binom{z_2}{B^{(6)}_{pqrs}}\right)
\end{aligned}
$$

  The corresponding matrix equations are given as:

$$
\begin{pmatrix}
(e_1|e_3) & (e_1|e_4) & (e_1|e_5) & (e_1|e_6) \\
(e_2|e_3) & (e_2|e_4) & (e_2|e_5) & (e_2|e_6) \\
(z_1|e_3) & (z_1|e_4) & (z_1|e_5) & (z_1|e_6) \\
(z_2|e_3) & (z_2|e_4) & (z_2|e_5) & (z_2|e_6)
\end{pmatrix}
\begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix}
=
\begin{pmatrix}
(e_1|b_1+z_1) \\
(e_2|b_1+z_1) \\
(z_1|b_1+z_1) \\
(z_2|b_1+z_1)+1
\end{pmatrix}
$$

$$
\begin{pmatrix}
(e_3|e_1) & (e_3|e_2) & (e_3|e_5) & (e_3|e_6) \\
(e_4|e_1) & (e_4|e_2) & (e_4|e_5) & (e_4|e_6) \\
(z_1|e_1) & (z_1|e_2) & (z_1|e_5) & (z_1|e_6) \\
(z_2|e_1) & (z_2|e_2) & (z_2|e_5) & (z_2|e_6)
\end{pmatrix}
\begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix}
=
\begin{pmatrix}
(e_3|b_2+z_1) \\
(e_4|b_2+z_1) \\
(z_1|b_2+z_1) \\
(z_2|b_2+z_1)+1
\end{pmatrix}
$$

$$
\begin{pmatrix}
(e_5|e_1) & (e_5|e_2) & (e_5|e_3) & (e_5|e_4) \\
(e_6|e_1) & (e_6|e_2) & (e_6|e_3) & (e_6|e_4) \\
(z_1|e_1) & (z_1|e_2) & (z_1|e_3) & (z_1|e_4) \\
(z_2|e_1) & (z_2|e_2) & (z_2|e_3) & (z_2|e_4)
\end{pmatrix}
\begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix}
=
\begin{pmatrix}
(e_5|b_1+b_2) \\
(e_6|b_1+b_2) \\
(z_1|b_1+b_2) \\
(z_2|b_1+b_2)
\end{pmatrix}
$$

- **States $\{\overline{\phi}^{1,\ldots,4}\}|R\rangle$ and $\{\overline{\phi}^{*1,\ldots,4}\}|R\rangle$**

  These states transform under the $SU(4)\times U(1)_4$ hidden gauge group. The projectors are given by:

$$
\begin{aligned}
P^{(4)(\overline{\phi}^{1,\ldots,4})}_{pqrs} &= \frac{1}{16}\left(1-C\begin{pmatrix} e_1 \\ B^{(4)}_{pqrs}\end{pmatrix}\right)\cdot\left(1-C\begin{pmatrix} e_2 \\ B^{(4)}_{pqrs}\end{pmatrix}\right)\\
&\quad\cdot\left(1+C\begin{pmatrix} z_1 \\ B^{(4)}_{pqrs}\end{pmatrix}\right)\cdot\left(1-C\begin{pmatrix} z_2 \\ B^{(4)}_{pqrs}\end{pmatrix}\right)\\
P^{(5)(\overline{\phi}^{1,\ldots,4})}_{pqrs} &= \frac{1}{16}\left(1-C\begin{pmatrix} e_3 \\ B^{(5)}_{pqrs}\end{pmatrix}\right)\cdot\left(1-C\begin{pmatrix} e_4 \\ B^{(5)}_{pqrs}\end{pmatrix}\right)\\
&\quad\cdot\left(1+C\begin{pmatrix} z_1 \\ B^{(5)}_{pqrs}\end{pmatrix}\right)\cdot\left(1-C\begin{pmatrix} z_2 \\ B^{(5)}_{pqrs}\end{pmatrix}\right)\\
P^{(6)(\overline{\phi}^{1,\ldots,4})}_{pqrs} &= \frac{1}{16}\left(1-C\begin{pmatrix} e_5 \\ B^{(6)}_{pqrs}\end{pmatrix}\right)\cdot\left(1-C\begin{pmatrix} e_6 \\ B^{(6)}_{pqrs}\end{pmatrix}\right)\\
&\quad\cdot\left(1+C\begin{pmatrix} z_1 \\ B^{(6)}_{pqrs}\end{pmatrix}\right)\cdot\left(1-C\begin{pmatrix} z_2 \\ B^{(6)}_{pqrs}\end{pmatrix}\right)
\end{aligned}
$$

  The corresponding matrix equations are given as:

$$
\begin{pmatrix}
(e_1|e_3) & (e_1|e_4) & (e_1|e_5) & (e_1|e_6)\\
(e_2|e_3) & (e_2|e_4) & (e_2|e_5) & (e_2|e_6)\\
(z_1|e_3) & (z_1|e_4) & (z_1|e_5) & (z_1|e_6)\\
(z_2|e_3) & (z_2|e_4) & (z_2|e_5) & (z_2|e_6)
\end{pmatrix}
\begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix}
=
\begin{pmatrix}
(e_1|b_1+z_1)\\
(e_2|b_1+z_1)\\
(z_1|b_1+z_1)+1\\
(z_2|b_1+z_1)+1
\end{pmatrix}
$$

$$
\begin{pmatrix}
(e_3|e_1) & (e_3|e_2) & (e_3|e_5) & (e_3|e_6)\\
(e_4|e_1) & (e_4|e_2) & (e_4|e_5) & (e_4|e_6)\\
(z_1|e_1) & (z_1|e_2) & (z_1|e_5) & (z_1|e_6)\\
(z_2|e_1) & (z_2|e_2) & (z_2|e_5) & (z_2|e_6)
\end{pmatrix}
\begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix}
=
\begin{pmatrix}
(e_3|b_2+z_1)\\
(e_4|b_2+z_1)\\
(z_1|b_2+z_1)+1\\
(z_2|b_2+z_1)+1
\end{pmatrix}
$$

$$
\begin{pmatrix}
(e_5|e_1) & (e_5|e_2) & (e_5|e_3) & (e_5|e_4)\\
(e_6|e_1) & (e_6|e_2) & (e_6|e_3) & (e_6|e_4)\\
(z_1|e_1) & (z_1|e_2) & (z_1|e_3) & (z_1|e_4)\\
(z_2|e_1) & (z_2|e_2) & (z_2|e_3) & (z_2|e_4)
\end{pmatrix}
\begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix}
=
\begin{pmatrix}
(e_5|b_1+b_2)\\
(e_6|b_1+b_2)\\
(z_1|b_1+b_2)+1\\
(z_2|b_1+b_2)
\end{pmatrix}
$$

- **State $\{\overline{\phi}^{5,\dots,8}\}|R\rangle$ and $\{\overline{\phi}^{*5,\dots,8}\}|R\rangle$**

  These states transform under the $U(1)_{hid}$ or $SO(6)$ gauge groups. The projectors on these states are given by:

$$
\begin{aligned}
P_{pqrs}^{(4)(\overline{\phi}^{5,\dots,8})} &= \frac{1}{16}\left(1 - C\begin{pmatrix} e_1 \\ B_{pqrs}^{(4)} \end{pmatrix}\right) \cdot \left(1 - C\begin{pmatrix} e_2 \\ B_{pqrs}^{(4)} \end{pmatrix}\right) \\
&\quad \cdot \left(1 - C\begin{pmatrix} z_1 \\ B_{pqrs}^{(4)} \end{pmatrix}\right) \cdot \left(1 + C\begin{pmatrix} z_2 \\ B_{pqrs}^{(4)} \end{pmatrix}\right) \\
P_{pqrs}^{(5)(\overline{\phi}^{5,\dots,8})} &= \frac{1}{16}\left(1 - C\begin{pmatrix} e_3 \\ B_{pqrs}^{(5)} \end{pmatrix}\right) \cdot \left(1 - C\begin{pmatrix} e_4 \\ B_{pqrs}^{(5)} \end{pmatrix}\right) \\
&\quad \cdot \left(1 - C\begin{pmatrix} z_1 \\ B_{pqrs}^{(5)} \end{pmatrix}\right) \cdot \left(1 + C\begin{pmatrix} z_2 \\ B_{pqrs}^{(5)} \end{pmatrix}\right) \\
P_{pqrs}^{(6)(\overline{\phi}^{5,\dots,8})} &= \frac{1}{16}\left(1 - C\begin{pmatrix} e_5 \\ B_{pqrs}^{(6)} \end{pmatrix}\right) \cdot \left(1 - C\begin{pmatrix} e_6 \\ B_{pqrs}^{(6)} \end{pmatrix}\right) \\
&\quad \cdot \left(1 - C\begin{pmatrix} z_1 \\ B_{pqrs}^{(6)} \end{pmatrix}\right) \cdot \left(1 + C\begin{pmatrix} z_2 \\ B_{pqrs}^{(6)} \end{pmatrix}\right)
\end{aligned}
$$

  The corresponding matrix equations are given as:

$$
\begin{pmatrix} (e_1|e_3) & (e_1|e_4) & (e_1|e_5) & (e_1|e_6) \\ (e_2|e_3) & (e_2|e_4) & (e_2|e_5) & (e_2|e_6) \\ (z_1|e_3) & (z_1|e_4) & (z_1|e_5) & (z_1|e_6) \\ (z_2|e_3) & (z_2|e_4) & (z_2|e_5) & (z_2|e_6) \end{pmatrix} \begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix} = \begin{pmatrix} (e_1|b_1 + z_1) \\ (e_2|b_1 + z_1) \\ (z_1|b_1 + z_1) \\ (z_2|b_1 + z_1) \end{pmatrix}
$$

$$
\begin{pmatrix} (e_3|e_1) & (e_3|e_2) & (e_3|e_5) & (e_3|e_6) \\ (e_4|e_1) & (e_4|e_2) & (e_4|e_5) & (e_4|e_6) \\ (z_1|e_1) & (z_1|e_2) & (z_1|e_5) & (z_1|e_6) \\ (z_2|e_1) & (z_2|e_2) & (z_2|e_5) & (z_2|e_6) \end{pmatrix} \begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix} = \begin{pmatrix} (e_3|b_2 + z_1) \\ (e_4|b_2 + z_1) \\ (z_1|b_2 + z_1) \\ (z_2|b_2 + z_1) \end{pmatrix}
$$

$$
\begin{pmatrix} (e_5|e_1) & (e_5|e_2) & (e_5|e_3) & (e_5|e_4) \\ (e_6|e_1) & (e_6|e_2) & (e_6|e_3) & (e_6|e_4) \\ (z_1|e_1) & (z_1|e_2) & (z_1|e_3) & (z_1|e_4) \\ (z_2|e_1) & (z_2|e_2) & (z_2|e_3) & (z_2|e_4) \end{pmatrix} \begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix} = \begin{pmatrix} (e_5|b_1 + b_2) \\ (e_6|b_1 + b_2) \\ (z_1|b_1 + b_2) \\ (z_2|b_1 + b_2) + 1 \end{pmatrix}
$$

A.2 Hidden sector representations

The sectors $B_{pqrs}^{(1,2,3)} + 2\alpha$ and $B_{pqrs}^{(1,2,3)} + z_1 + z_2 + 2\alpha$ give rise to non–exotic states that transform under the hidden gauge group. The states in these sectors descend from the **16** vectorial representation of the hidden $SO(16)$ gauge group, decomposed under the final unbroken hidden sector gauge group. The sectors

$$B_{pqrs}^{(7,8,9)} = B_{pqrs}^{(1,2,3)} + 2\alpha$$

produce states that transform under the $SU(4) \times U(1)_4$ hidden gauge group. The projectors on states arising in these sectors are given by:

$$\begin{aligned}
P_{pqrs}^{(7)} &= \frac{1}{8}\left(1 - C\begin{pmatrix} e_1 \\ B_{pqrs}^{(7)} \end{pmatrix}\right) \cdot \left(1 - C\begin{pmatrix} e_2 \\ B_{pqrs}^{(7)} \end{pmatrix}\right) \cdot \left(1 - C\begin{pmatrix} z_2 \\ B_{pqrs}^{(7)} \end{pmatrix}\right) \\
P_{pqrs}^{(8)} &= \frac{1}{8}\left(1 - C\begin{pmatrix} e_3 \\ B_{pqrs}^{(8)} \end{pmatrix}\right) \cdot \left(1 - C\begin{pmatrix} e_4 \\ B_{pqrs}^{(8)} \end{pmatrix}\right) \cdot \left(1 - C\begin{pmatrix} z_2 \\ B_{pqrs}^{(8)} \end{pmatrix}\right) \\
P_{pqrs}^{(9)} &= \frac{1}{8}\left(1 - C\begin{pmatrix} e_5 \\ B_{pqrs}^{(9)} \end{pmatrix}\right) \cdot \left(1 - C\begin{pmatrix} e_6 \\ B_{pqrs}^{(9)} \end{pmatrix}\right) \cdot \left(1 - C\begin{pmatrix} z_2 \\ B_{pqrs}^{(9)} \end{pmatrix}\right)
\end{aligned}$$

The corresponding matrix equations are given as:

$$\begin{pmatrix} (e_1|e_3) & (e_1|e_4) & (e_1|e_5) & (e_1|e_6) \\ (e_2|e_3) & (e_2|e_4) & (e_2|e_5) & (e_2|e_6) \\ (z_2|e_3) & (z_2|e_4) & (z_2|e_5) & (z_2|e_6) \end{pmatrix} \begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix} = \begin{pmatrix} (e_1|b_1) \\ (e_2|b_1) \\ (z_2|b_1) + 1 \end{pmatrix}$$

$$\begin{pmatrix} (e_3|e_1) & (e_3|e_2) & (e_3|e_5) & (e_3|e_6) \\ (e_4|e_1) & (e_4|e_2) & (e_4|e_5) & (e_4|e_6) \\ (z_2|e_1) & (z_2|e_2) & (z_2|e_5) & (z_2|e_6) \end{pmatrix} \begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix} = \begin{pmatrix} (e_3|b_2) \\ (e_4|b_2) \\ (z_2|b_2) + 1 \end{pmatrix}$$

$$\begin{pmatrix} (e_5|e_1) & (e_5|e_2) & (e_5|e_3) & (e_5|e_4) \\ (e_6|e_1) & (e_6|e_2) & (e_6|e_3) & (e_6|e_4) \\ (z_2|e_1) & (z_2|e_2) & (z_2|e_3) & (z_2|e_4) \end{pmatrix} \begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix} = \begin{pmatrix} (e_5|b_1 + b_2 + z_1) \\ (e_6|b_1 + b_2 + z_1) \\ (z_2|b_1 + b_2 + z_1) \end{pmatrix}$$

The sectors

$$B_{pqrs}^{(10,11,12)} = B_{pqrs}^{(1,2,3)} + z_1 + z_2 + 2\alpha$$

give rise to states that transform under the hidden $SO(6)$ gauge group. The projectors acting on these states are given by:

$$\begin{aligned}
P_{pqrs}^{(10)} &= \frac{1}{8}\left(1 - C\binom{e_1}{B_{pqrs}^{(10)}}\right) \cdot \left(1 - C\binom{e_2}{B_{pqrs}^{(10)}}\right) \cdot \left(1 - C\binom{z_1}{B_{pqrs}^{(10)}}\right) \\
P_{pqrs}^{(11)} &= \frac{1}{8}\left(1 - C\binom{e_3}{B_{pqrs}^{(11)}}\right) \cdot \left(1 - C\binom{e_4}{B_{pqrs}^{(11)}}\right) \cdot \left(1 - C\binom{z_1}{B_{pqrs}^{(11)}}\right) \\
P_{pqrs}^{(12)} &= \frac{1}{8}\left(1 - C\binom{e_5}{B_{pqrs}^{(12)}}\right) \cdot \left(1 - C\binom{e_6}{B_{pqrs}^{(12)}}\right) \cdot \left(1 - C\binom{z_1}{B_{pqrs}^{(12)}}\right)
\end{aligned}$$

The corresponding matrix equations are given as:

$$\begin{pmatrix} (e_1|e_3) & (e_1|e_4) & (e_1|e_5) & (e_1|e_6) \\ (e_2|e_3) & (e_2|e_4) & (e_2|e_5) & (e_2|e_6) \\ (z_1|e_3) & (z_1|e_4) & (z_1|e_5) & (z_1|e_6) \end{pmatrix} \begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix} = \begin{pmatrix} (e_1|b_1 + z_1 + z_2) \\ (e_2|b_1 + z_1 + z_2) \\ (z_1|b_1 + z_1 + z_2) \end{pmatrix}$$

$$\begin{pmatrix} (e_3|e_1) & (e_3|e_2) & (e_3|e_5) & (e_3|e_6) \\ (e_4|e_1) & (e_4|e_2) & (e_4|e_5) & (e_4|e_6) \\ (z_1|e_1) & (z_1|e_2) & (z_1|e_5) & (z_1|e_6) \end{pmatrix} \begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix} = \begin{pmatrix} (e_3|b_2 + z_1 + z_2) \\ (e_4|b_2 + z_1 + z_2) \\ (z_1|b_2 + z_1 + z_2) \end{pmatrix}$$

$$\begin{pmatrix} (e_5|e_1) & (e_5|e_2) & (e_5|e_3) & (e_5|e_4) \\ (e_6|e_1) & (e_6|e_2) & (e_6|e_3) & (e_6|e_4) \\ (z_1|e_1) & (z_1|e_2) & (z_1|e_3) & (z_1|e_4) \end{pmatrix} \begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix} = \begin{pmatrix} (e_5|b_1 + b_2 + z_2) \\ (e_6|b_1 + b_2 + z_2) \\ (z_1|b_1 + b_2 + z_2) \end{pmatrix}$$

A.3 Exotics

The exotic states are obtained from sectors containing the $SO(10)$ breaking vector α. As mentioned in section 3.3, the sectors that give rise to exotic states are classified according to their vacuum in the right–moving sector. For a given sector ξ with $\xi_R \cdot \xi_R = 6$ a right–moving oscillator of a world–sheet fermion with $\pm 1/4$ boundary conditions acting on the vacuum is needed to obtain a massless state. Sectors with $\xi_R \cdot \xi_R = 8$ produce massless states without an oscillator. The first type of sectors can therefore produce states that transform as $\mathbf{5}$ and $\overline{\mathbf{5}}$, as well as states that transform as singlets under the observable $SU(5)$ gauge group. The second type of sectors gives rise to states that transform as singlets of the observable $SU(5)$ gauge symmetry. All the exotic states transform in standard representations under the observable $SU(5)$

gauge group (including singlets) but carry exotic charge under the observable $U(1)_5$ gauge group. The sectors

$$B_{pqrs}^{(13,14,15)} = B_{pqrs}^{(1,2,3)} + z_2 + \alpha$$

produce states that transform under the $\mathbf{4}$ and $\overline{\mathbf{4}}$ of the $SO(6)$ hidden gauge group. The projectors acting on these states are given by:

$$\begin{aligned}
P_{pqrs}^{(13)} &= \frac{1}{16}\left(1 - C\begin{pmatrix} e_1 \\ B_{pqrs}^{(13)} \end{pmatrix}\right) \cdot \left(1 - C\begin{pmatrix} e_2 \\ B_{pqrs}^{(13)} \end{pmatrix}\right) \\
&\quad \left(1 + C\begin{pmatrix} z_1 \\ B_{pqrs}^{(13)} \end{pmatrix}\right) \cdot \left(1 - C\begin{pmatrix} \alpha \\ B_{pqrs}^{(13)} \end{pmatrix}\right) \\
P_{pqrs}^{(14)} &= \frac{1}{16}\left(1 - C\begin{pmatrix} e_3 \\ B_{pqrs}^{(14)} \end{pmatrix}\right) \cdot \left(1 - C\begin{pmatrix} e_4 \\ B_{pqrs}^{(14)} \end{pmatrix}\right) \\
&\quad \left(1 + C\begin{pmatrix} z_1 \\ B_{pqrs}^{(14)} \end{pmatrix}\right) \cdot \left(1 - C\begin{pmatrix} \alpha \\ B_{pqrs}^{(14)} \end{pmatrix}\right) \\
P_{pqrs}^{(15)} &= \frac{1}{16}\left(1 - C\begin{pmatrix} e_5 \\ B_{pqrs}^{(15)} \end{pmatrix}\right) \cdot \left(1 - C\begin{pmatrix} e_6 \\ B_{pqrs}^{(15)} \end{pmatrix}\right) \\
&\quad \left(1 + C\begin{pmatrix} z_1 \\ B_{pqrs}^{(15)} \end{pmatrix}\right) \cdot \left(1 - C\begin{pmatrix} \alpha \\ B_{pqrs}^{(15)} \end{pmatrix}\right)
\end{aligned}$$

The corresponding matrix equations are given as:

$$\begin{pmatrix} (e_1|e_3) & (e_1|e_4) & (e_1|e_5) & (e_1|e_6) \\ (e_2|e_3) & (e_2|e_4) & (e_2|e_5) & (e_2|e_6) \\ (z_1|e_3) & (z_1|e_4) & (z_1|e_5) & (z_1|e_6) \\ (\alpha|e_3) & (\alpha|e_4) & (\alpha|e_5) & (\alpha|e_6) \end{pmatrix} \begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix} = \begin{pmatrix} (e_1|b_1 + z_2 + \alpha) \\ (e_2|b_1 + z_2 + \alpha) \\ (z_1|b_1 + z_2 + \alpha) + 1 \\ (\alpha|b_1 + z_2 + \alpha) \end{pmatrix}$$

$$\begin{pmatrix} (e_3|e_1) & (e_3|e_2) & (e_3|e_5) & (e_3|e_6) \\ (e_4|e_1) & (e_4|e_2) & (e_4|e_5) & (e_4|e_6) \\ (z_1|e_1) & (z_1|e_2) & (z_1|e_5) & (z_1|e_6) \\ (\alpha|e_1) & (\alpha|e_2) & (\alpha|e_5) & (\alpha|e_6) \end{pmatrix} \begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix} = \begin{pmatrix} (e_3|b_2 + z_2 + \alpha) \\ (e_4|b_2 + z_2 + \alpha) \\ (z_1|b_2 + z_2 + \alpha) + 1 \\ (\alpha|b_2 + z_2 + \alpha) \end{pmatrix}$$

$$\begin{pmatrix} (e_5|e_1) & (e_5|e_2) & (e_5|e_3) & (e_5|e_4) \\ (e_6|e_1) & (e_6|e_2) & (e_6|e_3) & (e_6|e_4) \\ (z_1|e_1) & (z_1|e_2) & (z_1|e_3) & (z_1|e_4) \\ (\alpha|e_1) & (\alpha|e_2) & (\alpha|e_3) & (\alpha|e_4) \end{pmatrix} \begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix} = \begin{pmatrix} (e_5|b_3 + z_2 + \alpha) \\ (e_6|b_3 + z_2 + \alpha) \\ (z_1|b_3 + z_2 + \alpha) + 1 \\ (\alpha|b_3 + z_2 + \alpha) \end{pmatrix}$$

Similar exotic states are produced from the sectors:

$$B^{(16,17,18)}_{pqrs} = B^{(1,2,3)}_{pqrs} + z_1 + z_2 + \alpha$$

The projectors acting on these states are given by:

$$\begin{aligned}
P^{(16)}_{pqrs} = \frac{1}{16} & \left(1 - C\binom{e_1}{B^{(16)}_{pqrs}}\right) \cdot \left(1 - C\binom{e_2}{B^{(16)}_{pqrs}}\right) \\
& \left(1 + C\binom{z_1}{B^{(16)}_{pqrs}}\right) \cdot \left(1 + C\binom{\alpha}{B^{(16)}_{pqrs}}\right) \\
P^{(17)}_{pqrs} = \frac{1}{16} & \left(1 - C\binom{e_3}{B^{(17)}_{pqrs}}\right) \cdot \left(1 - C\binom{e_4}{B^{(17)}_{pqrs}}\right) \\
& \left(1 + C\binom{z_1}{B^{(17)}_{pqrs}}\right) \cdot \left(1 + C\binom{\alpha}{B^{(17)}_{pqrs}}\right) \\
P^{(18)}_{pqrs} = \frac{1}{16} & \left(1 - C\binom{e_5}{B^{(18)}_{pqrs}}\right) \cdot \left(1 - C\binom{e_6}{B^{(18)}_{pqrs}}\right) \\
& \left(1 + C\binom{z_1}{B^{(18)}_{pqrs}}\right) \cdot \left(1 + C\binom{\alpha}{B^{(18)}_{pqrs}}\right)
\end{aligned}$$

The corresponding matrix equations are given as:

$$\begin{pmatrix} (e_1|e_3) & (e_1|e_4) & (e_1|e_5) & (e_1|e_6) \\ (e_2|e_3) & (e_2|e_4) & (e_2|e_5) & (e_2|e_6) \\ (z_1|e_3) & (z_1|e_4) & (z_1|e_5) & (z_1|e_6) \\ (\alpha|e_3) & (\alpha|e_4) & (\alpha|e_5) & (\alpha|e_6) \end{pmatrix} \begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix} = \begin{pmatrix} (e_1|b_1 + z_1 + z_2 + \alpha) \\ (e_2|b_1 + z_1 + z_2 + \alpha) \\ (z_1|b_1 + z_1 + z_2 + \alpha) + 1 \\ (\alpha|b_1 + z_1 + z_2 + \alpha) + 1 \end{pmatrix}$$

$$\begin{pmatrix} (e_3|e_1) & (e_3|e_2) & (e_3|e_5) & (e_3|e_6) \\ (e_4|e_1) & (e_4|e_2) & (e_4|e_5) & (e_4|e_6) \\ (z_1|e_1) & (z_1|e_2) & (z_1|e_5) & (z_1|e_6) \\ (\alpha|e_1) & (\alpha|e_2) & (\alpha|e_5) & (\alpha|e_6) \end{pmatrix} \begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix} = \begin{pmatrix} (e_3|b_2 + z_1 + z_2 + \alpha) \\ (e_4|b_2 + z_1 + z_2 + \alpha) \\ (z_1|b_2 + z_1 + z_2 + \alpha) + 1 \\ (\alpha|b_2 + z_1 + z_2 + \alpha) + 1 \end{pmatrix}$$

$$\begin{pmatrix} (e_5|e_1) & (e_5|e_2) & (e_5|e_3) & (e_5|e_4) \\ (e_6|e_1) & (e_6|e_2) & (e_6|e_3) & (e_6|e_4) \\ (z_1|e_1) & (z_1|e_2) & (z_1|e_3) & (z_1|e_4) \\ (\alpha|e_1) & (\alpha|e_2) & (\alpha|e_3) & (\alpha|e_4) \end{pmatrix} \begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix} = \begin{pmatrix} (e_5|b_1 + b_2 + z_2 + \alpha) \\ (e_6|b_1 + b_2 + z_2 + \alpha) \\ (z_1|b_1 + b_2 + z_2 + \alpha) + 1 \\ (\alpha|b_1 + b_2 + z_2 + \alpha) + 1 \end{pmatrix}$$

The sectors

$$B^{(19,20,21)}_{pqrs} = B^{(1,2,3)}_{pqrs} + \alpha$$

produce massless states that are obtained by acting on the vacuum with a fermionic oscillator. Below we list the type of states that are produced and the projectors that act on them.

- **States** $\{\overline{\eta}^1\}|R\rangle$ **and** $\{\overline{\psi}^{1,\ldots,5}\}|R\rangle$

  These transform as either singlets or $\mathbf{5}$ or $\overline{\mathbf{5}}$ under the observable $SU(5)$ gauge group. The projectors are given by:

$$
\begin{aligned}
P^{(19)(\overline{\eta}^1,\overline{\psi}^{1,\ldots,5})}_{pqrs} &= \frac{1}{16}\left(1 - C\binom{e_1}{B^{(19)}_{pqrs}}\right)\cdot\left(1 - C\binom{e_2}{B^{(19)}_{pqrs}}\right) \\
&\quad \cdot\left(1 + C\binom{z_1}{B^{(19)}_{pqrs}}\right)\cdot\left(1 + C\binom{z_2+\alpha}{B^{(19)}_{pqrs}}\right) \\
P^{(20)(\overline{\eta}^1,\overline{\psi}^{1,\ldots,5})}_{pqrs} &= \frac{1}{16}\left(1 - C\binom{e_3}{B^{(20)}_{pqrs}}\right)\cdot\left(1 - C\binom{e_4}{B^{(20)}_{pqrs}}\right) \\
&\quad \cdot\left(1 + C\binom{z_1}{B^{(20)}_{pqrs}}\right)\cdot\left(1 + C\binom{z_2+\alpha}{B^{(20)}_{pqrs}}\right) \\
P^{(21)(\overline{\eta}^1,\overline{\psi}^{1,\ldots,5})}_{pqrs} &= \frac{1}{16}\left(1 - C\binom{e_5}{B^{(21)}_{pqrs}}\right)\cdot\left(1 - C\binom{e_6}{B^{(21)}_{pqrs}}\right) \\
&\quad \cdot\left(1 + C\binom{z_1}{B^{(21)}_{pqrs}}\right)\cdot\left(1 + C\binom{z_2+\alpha}{B^{(21)}_{pqrs}}\right)
\end{aligned}
$$

  The corresponding matrix equations are given as:

$$
\begin{pmatrix} (e_1|e_3) & (e_1|e_4) & (e_1|e_5) & (e_1|e_6) \\ (e_2|e_3) & (e_2|e_4) & (e_2|e_5) & (e_2|e_6) \\ (z_1|e_3) & (z_1|e_4) & (z_1|e_5) & (z_1|e_6) \\ (\delta|e_3) & (\delta|e_4) & (\delta|e_5) & (\delta|e_6) \end{pmatrix}
\begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix} =
\begin{pmatrix} (e_1|b_1+\alpha) \\ (e_2|b_1+\alpha) \\ (z_1|b_1+\alpha)+1 \\ (z_2|b_1)+(\alpha|b_1+z_2+\alpha)+1 \end{pmatrix}
$$

$$
\begin{pmatrix} (e_3|e_1) & (e_3|e_2) & (e_3|e_5) & (e_3|e_6) \\ (e_4|e_1) & (e_4|e_2) & (e_4|e_5) & (e_4|e_6) \\ (z_1|e_1) & (z_1|e_2) & (z_1|e_5) & (z_1|e_6) \\ (\delta|e_1) & (\delta|e_2) & (\delta|e_5) & (\delta|e_6) \end{pmatrix}
\begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix} =
\begin{pmatrix} (e_3|b_2+\alpha) \\ (e_4|b_2+\alpha) \\ (z_1|b_2+\alpha)+1 \\ (z_2|b_2)+(\alpha|b_2+z_2+\alpha)+1 \end{pmatrix}
$$

$$
\begin{pmatrix} (e_5|e_1) & (e_5|e_2) & (e_5|e_3) & (e_5|e_4) \\ (e_6|e_1) & (e_6|e_2) & (e_6|e_3) & (e_6|e_4) \\ (z_1|e_1) & (z_1|e_2) & (z_1|e_3) & (z_1|e_4) \\ (\delta|e_1) & (\delta|e_2) & (\delta|e_3) & (\delta|e_4) \end{pmatrix}
\begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix} =
\begin{pmatrix} (e_5|b_3+\alpha) \\ (e_6|b_3+\alpha) \\ (z_1|b_3+\alpha)+1 \\ (z_2|b_3)+(\alpha|b_3+z_2+\alpha)+1 \end{pmatrix}
$$

  Where $\delta = z_2 + \alpha$

- **States** $\{\overline{\eta}^{*2,3}\}|R\rangle$

  These transform as singlets under the observable $SU(5)$ gauge group and are charged under $U(1)_{2/3}$. The projectors are given by:

$$P_{pqrs}^{(19)(\overline{\eta}^{*2,3})} = \frac{1}{16}\left(1 - C\binom{e_1}{B_{pqrs}^{(19)}}\right)\cdot\left(1 - C\binom{e_2}{B_{pqrs}^{(19)}}\right)$$
$$\cdot\left(1 + C\binom{z_1}{B_{pqrs}^{(19)}}\right)\cdot\left(1 - C\binom{z_2+\alpha}{B_{pqrs}^{(19)}}\right)$$
$$P_{pqrs}^{(20)(\overline{\eta}^{*2,3})} = \frac{1}{16}\left(1 - C\binom{e_3}{B_{pqrs}^{(20)}}\right)\cdot\left(1 - C\binom{e_4}{B_{pqrs}^{(20)}}\right)$$
$$\cdot\left(1 + C\binom{z_1}{B_{pqrs}^{(20)}}\right)\cdot\left(1 - C\binom{z_2+\alpha}{B_{pqrs}^{(20)}}\right)$$
$$P_{pqrs}^{(21)(\overline{\eta}^{*2,3})} = \frac{1}{16}\left(1 - C\binom{e_5}{B_{pqrs}^{(21)}}\right)\cdot\left(1 - C\binom{e_6}{B_{pqrs}^{(21)}}\right)$$
$$\cdot\left(1 + C\binom{z_1}{B_{pqrs}^{(21)}}\right)\cdot\left(1 - C\binom{z_2+\alpha}{B_{pqrs}^{(21)}}\right)$$

  The corresponding matrix equations are given as:

$$\begin{pmatrix} (e_1|e_3) & (e_1|e_4) & (e_1|e_5) & (e_1|e_6) \\ (e_2|e_3) & (e_2|e_4) & (e_2|e_5) & (e_2|e_6) \\ (z_1|e_3) & (z_1|e_4) & (z_1|e_5) & (z_1|e_6) \\ (\delta|e_3) & (\delta|e_4) & (\delta|e_5) & (\delta|e_6) \end{pmatrix} \begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix} = \begin{pmatrix} (e_1|b_1+\alpha) \\ (e_2|b_1+\alpha) \\ (z_1|b_1+\alpha)+1 \\ (z_2|b_1)+(\alpha|b_1+z_2+\alpha) \end{pmatrix}$$

$$\begin{pmatrix} (e_3|e_1) & (e_3|e_2) & (e_3|e_5) & (e_3|e_6) \\ (e_4|e_1) & (e_4|e_2) & (e_4|e_5) & (e_4|e_6) \\ (z_1|e_1) & (z_1|e_2) & (z_1|e_5) & (z_1|e_6) \\ (\delta|e_1) & (\delta|e_2) & (\delta|e_5) & (\delta|e_6) \end{pmatrix} \begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix} = \begin{pmatrix} (e_3|b_2+\alpha) \\ (e_4|b_2+\alpha) \\ (z_1|b_2+\alpha)+1 \\ (z_2|b_2)+(\alpha|b_2+z_2+\alpha) \end{pmatrix}$$

$$\begin{pmatrix} (e_5|e_1) & (e_5|e_2) & (e_5|e_3) & (e_5|e_4) \\ (e_6|e_1) & (e_6|e_2) & (e_6|e_3) & (e_6|e_4) \\ (z_1|e_1) & (z_1|e_2) & (z_1|e_3) & (z_1|e_4) \\ (\delta|e_1) & (\delta|e_2) & (\delta|e_3) & (\delta|e_4) \end{pmatrix} \begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix} = \begin{pmatrix} (e_5|b_3+\alpha) \\ (e_6|b_3+\alpha) \\ (z_1|b_3+\alpha)+1 \\ (z_2|b_3)+(\alpha|b_3+z_2+\alpha) \end{pmatrix}$$

  Where $\delta = z_2 + \alpha$

- **States** $\{\overline{\phi}^{*1,\ldots,4}\}|R\rangle$

  These transform as singlets under the observable $SU(5)$ gauge group and transform in non–trivial representation of the hidden $SU(4)\times U(1)_4$ gauge group. The projectors are given by:

$$P_{pqrs}^{(19)(\overline{\phi}^{*1,\ldots,4})} = \frac{1}{16}\left(1-C\begin{pmatrix} e_1 \\ B_{pqrs}^{(19)}\end{pmatrix}\right)\cdot\left(1-C\begin{pmatrix} e_2 \\ B_{pqrs}^{(19)}\end{pmatrix}\right)$$
$$\cdot\left(1-C\begin{pmatrix} z_1 \\ B_{pqrs}^{(19)}\end{pmatrix}\right)\cdot\left(1-C\begin{pmatrix} z_2+\alpha \\ B_{pqrs}^{(19)}\end{pmatrix}\right)$$
$$P_{pqrs}^{(20)(\overline{\phi}^{*1,\ldots,4})} = \frac{1}{16}\left(1-C\begin{pmatrix} e_3 \\ B_{pqrs}^{(20)}\end{pmatrix}\right)\cdot\left(1-C\begin{pmatrix} e_4 \\ B_{pqrs}^{(20)}\end{pmatrix}\right)$$
$$\cdot\left(1-C\begin{pmatrix} z_1 \\ B_{pqrs}^{(20)}\end{pmatrix}\right)\cdot\left(1-C\begin{pmatrix} z_2+\alpha \\ B_{pqrs}^{(20)}\end{pmatrix}\right)$$
$$P_{pqrs}^{(21)(\overline{\phi}^{*1,\ldots,4})} = \frac{1}{16}\left(1-C\begin{pmatrix} e_5 \\ B_{pqrs}^{(21)}\end{pmatrix}\right)\cdot\left(1-C\begin{pmatrix} e_6 \\ B_{pqrs}^{(21)}\end{pmatrix}\right)$$
$$\cdot\left(1-C\begin{pmatrix} z_1 \\ B_{pqrs}^{(21)}\end{pmatrix}\right)\cdot\left(1-C\begin{pmatrix} z_2+\alpha \\ B_{pqrs}^{(21)}\end{pmatrix}\right)$$

  The corresponding matrix equations are given as:

$$\begin{pmatrix} (e_1|e_3) & (e_1|e_4) & (e_1|e_5) & (e_1|e_6) \\ (e_2|e_3) & (e_2|e_4) & (e_2|e_5) & (e_2|e_6) \\ (z_1|e_3) & (z_1|e_4) & (z_1|e_5) & (z_1|e_6) \\ (\delta|e_3) & (\delta|e_4) & (\delta|e_5) & (\delta|e_6) \end{pmatrix}\begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix} = \begin{pmatrix} (e_1|b_1+\alpha) \\ (e_2|b_1+\alpha) \\ (z_1|b_1+\alpha) \\ (z_2|b_1)+(\alpha|b_1+z_2+\alpha) \end{pmatrix}$$

$$\begin{pmatrix} (e_3|e_1) & (e_3|e_2) & (e_3|e_5) & (e_3|e_6) \\ (e_4|e_1) & (e_4|e_2) & (e_4|e_5) & (e_4|e_6) \\ (z_1|e_1) & (z_1|e_2) & (z_1|e_5) & (z_1|e_6) \\ (\delta|e_1) & (\delta|e_2) & (\delta|e_5) & (\delta|e_6) \end{pmatrix}\begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix} = \begin{pmatrix} (e_3|b_2+\alpha) \\ (e_4|b_2+\alpha) \\ (z_1|b_2+\alpha) \\ (z_2|b_2)+(\alpha|b_2+z_2+\alpha) \end{pmatrix}$$

$$\begin{pmatrix} (e_5|e_1) & (e_5|e_2) & (e_5|e_3) & (e_5|e_4) \\ (e_6|e_1) & (e_6|e_2) & (e_6|e_3) & (e_6|e_4) \\ (z_1|e_1) & (z_1|e_2) & (z_1|e_3) & (z_1|e_4) \\ (\delta|e_1) & (\delta|e_2) & (\delta|e_3) & (\delta|e_4) \end{pmatrix}\begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix} = \begin{pmatrix} (e_5|b_3+\alpha) \\ (e_6|b_3+\alpha) \\ (z_1|b_3+\alpha) \\ (z_2|b_3)+(\alpha|b_3+z_2+\alpha) \end{pmatrix}$$

  Where $\delta = z_2+\alpha$

The remaining sectors

$$B_{pqrs}^{(22,23,24)} = B_{pqrs}^{(1,2,3)} + z_1 + \alpha$$

produce the following vector-like states:

- **States $\{\overline{\eta}^1\}|R\rangle$ and $\{\overline{\psi}^{1,\dots,5}\}|R\rangle$**

  These transform as either singlets or **5** or $\overline{\mathbf{5}}$ under the observable $SU(5)$ gauge group. The projectors are given by:

$$
\begin{aligned}
P^{(22)(\overline{\eta}^1,\overline{\psi}^{1,\dots,5})}_{pqrs} &= \frac{1}{16}\left(1 - C\begin{pmatrix} e_1 \\ B^{(22)}_{pqrs}\end{pmatrix}\right)\cdot\left(1 - C\begin{pmatrix} e_2 \\ B^{(22)}_{pqrs}\end{pmatrix}\right) \\
&\cdot\left(1 + C\begin{pmatrix} z_1 \\ B^{(22)}_{pqrs}\end{pmatrix}\right)\cdot\left(1 - C\begin{pmatrix} z_2+\alpha \\ B^{(22)}_{pqrs}\end{pmatrix}\right) \\
P^{(23)(\overline{\eta}^1,\overline{\psi}^{1,\dots,5})}_{pqrs} &= \frac{1}{16}\left(1 - C\begin{pmatrix} e_3 \\ B^{(23)}_{pqrs}\end{pmatrix}\right)\cdot\left(1 - C\begin{pmatrix} e_4 \\ B^{(23)}_{pqrs}\end{pmatrix}\right) \\
&\cdot\left(1 + C\begin{pmatrix} z_1 \\ B^{(23)}_{pqrs}\end{pmatrix}\right)\cdot\left(1 - C\begin{pmatrix} z_2+\alpha \\ B^{(23)}_{pqrs}\end{pmatrix}\right) \\
P^{(24)(\overline{\eta}^1,\overline{\psi}^{1,\dots,5})}_{pqrs} &= \frac{1}{16}\left(1 - C\begin{pmatrix} e_5 \\ B^{(24)}_{pqrs}\end{pmatrix}\right)\cdot\left(1 - C\begin{pmatrix} e_6 \\ B^{(24)}_{pqrs}\end{pmatrix}\right) \\
&\cdot\left(1 + C\begin{pmatrix} z_1 \\ B^{(24)}_{pqrs}\end{pmatrix}\right)\cdot\left(1 - C\begin{pmatrix} z_2+\alpha \\ B^{(24)}_{pqrs}\end{pmatrix}\right)
\end{aligned}
$$

  The corresponding matrix equations are given as:

$$
\begin{pmatrix}
(e_1|e_3) & (e_1|e_4) & (e_1|e_5) & (e_1|e_6) \\
(e_2|e_3) & (e_2|e_4) & (e_2|e_5) & (e_2|e_6) \\
(z_1|e_3) & (z_1|e_4) & (z_1|e_5) & (z_1|e_6) \\
(\delta|e_3) & (\delta|e_4) & (\delta|e_5) & (\delta|e_6)
\end{pmatrix}
\begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix}
=
\begin{pmatrix}
(e_1|b_1+z_1+\alpha) \\
(e_2|b_1+z_1+\alpha) \\
(z_1|b_1+z_1+\alpha)+1 \\
(z_2|b_1+z_1)+(\alpha|b_1+z_1+z_2+\alpha)
\end{pmatrix}
$$

$$
\begin{pmatrix}
(e_3|e_1) & (e_3|e_2) & (e_3|e_5) & (e_3|e_6) \\
(e_4|e_1) & (e_4|e_2) & (e_4|e_5) & (e_4|e_6) \\
(z_1|e_1) & (z_1|e_2) & (z_1|e_5) & (z_1|e_6) \\
(\delta|e_1) & (\delta|e_2) & (\delta|e_5) & (\delta|e_6)
\end{pmatrix}
\begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix}
=
\begin{pmatrix}
(e_3|b_2+z_1+\alpha) \\
(e_4|b_2+z_1+\alpha) \\
(z_1|b_2+z_1+\alpha)+1 \\
(z_2|b_2+z_1)+(\alpha|b_2+z_1+z_2+\alpha)
\end{pmatrix}
$$

$$
\begin{pmatrix}
(e_5|e_1) & (e_5|e_2) & (e_5|e_3) & (e_5|e_4) \\
(e_6|e_1) & (e_6|e_2) & (e_6|e_3) & (e_6|e_4) \\
(z_1|e_1) & (z_1|e_2) & (z_1|e_3) & (z_1|e_4) \\
(\delta|e_1) & (\delta|e_2) & (\delta|e_3) & (\delta|e_4)
\end{pmatrix}
\begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix}
=
\begin{pmatrix}
(e_5|b_3+z_1+\alpha) \\
(e_6|b_3+z_1+\alpha) \\
(z_1|b_3+z_1+\alpha)+1 \\
(z_2|b_1+b_2)+(\alpha|b_1+b_2+z_2+\alpha)+1
\end{pmatrix}
$$

  Where $\delta = z_2 + \alpha$

- **States** $\{\overline{\eta}^{*2,3}\}|R\rangle$

  These transform as singlets under the observable $SU(5)$ gauge group and are charged under $U(1)_{2/3}$. The projectors are given by:

$$
\begin{aligned}
P^{(22)(\overline{\eta}^{*2,3})}_{pqrs} &= \frac{1}{16}\left(1 - C\binom{e_1}{B^{(22)}_{pqrs}}\right) \cdot \left(1 - C\binom{e_2}{B^{(22)}_{pqrs}}\right) \\
&\quad \cdot \left(1 + C\binom{z_1}{B^{(22)}_{pqrs}}\right) \cdot \left(1 + C\binom{z_2+\alpha}{B^{(22)}_{pqrs}}\right) \\
P^{(23)(\overline{\eta}^{*2,3})}_{pqrs} &= \frac{1}{16}\left(1 - C\binom{e_3}{B^{(23)}_{pqrs}}\right) \cdot \left(1 - C\binom{e_4}{B^{(23)}_{pqrs}}\right) \\
&\quad \cdot \left(1 + C\binom{z_1}{B^{(23)}_{pqrs}}\right) \cdot \left(1 + C\binom{z_2+\alpha}{B^{(23)}_{pqrs}}\right) \\
P^{(24)(\overline{\eta}^{*2,3})}_{pqrs} &= \frac{1}{16}\left(1 - C\binom{e_5}{B^{(24)}_{pqrs}}\right) \cdot \left(1 - C\binom{e_6}{B^{(24)}_{pqrs}}\right) \\
&\quad \cdot \left(1 + C\binom{z_1}{B^{(24)}_{pqrs}}\right) \cdot \left(1 + C\binom{z_2+\alpha}{B^{(24)}_{pqrs}}\right)
\end{aligned}
$$

  The corresponding matrix equations are given as:

$$
\begin{pmatrix}
(e_1|e_3) & (e_1|e_4) & (e_1|e_5) & (e_1|e_6) \\
(e_2|e_3) & (e_2|e_4) & (e_2|e_5) & (e_2|e_6) \\
(z_1|e_3) & (z_1|e_4) & (z_1|e_5) & (z_1|e_6) \\
(\delta|e_3) & (\delta|e_4) & (\delta|e_5) & (\delta|e_6)
\end{pmatrix}
\begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix}
=
\begin{pmatrix}
(e_1|b_1+z_1+\alpha) \\
(e_2|b_1+z_1+\alpha) \\
(z_1|b_1+z_1+\alpha)+1 \\
(z_2|b_1+z_1)+(\alpha|b_1+z_1+z_2+\alpha)+1
\end{pmatrix}
$$

$$
\begin{pmatrix}
(e_3|e_1) & (e_3|e_2) & (e_3|e_5) & (e_3|e_6) \\
(e_4|e_1) & (e_4|e_2) & (e_4|e_5) & (e_4|e_6) \\
(z_1|e_1) & (z_1|e_2) & (z_1|e_5) & (z_1|e_6) \\
(\delta|e_1) & (\delta|e_2) & (\delta|e_5) & (\delta|e_6)
\end{pmatrix}
\begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix}
=
\begin{pmatrix}
(e_3|b_2+z_1+\alpha) \\
(e_4|b_2+z_1+\alpha) \\
(z_1|b_2+z_1+\alpha)+1 \\
(z_2|b_2+z_1)+(\alpha|b_2+z_1+z_2+\alpha)+1
\end{pmatrix}
$$

$$
\begin{pmatrix}
(e_5|e_1) & (e_5|e_2) & (e_5|e_3) & (e_5|e_4) \\
(e_6|e_1) & (e_6|e_2) & (e_6|e_3) & (e_6|e_4) \\
(z_1|e_1) & (z_1|e_2) & (z_1|e_3) & (z_1|e_4) \\
(\delta|e_1) & (\delta|e_2) & (\delta|e_3) & (\delta|e_4)
\end{pmatrix}
\begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix}
=
\begin{pmatrix}
(e_5|b_3+z_1+\alpha) \\
(e_6|b_3+z_1+\alpha) \\
(z_1|b_3+z_1+\alpha)+1 \\
(z_2|b_1+b_2)+(\alpha|b_1+b_2+z_2+\alpha)
\end{pmatrix}
$$

  Where $\delta = z_2 + \alpha$

- **States** $\{\overline{\phi}^{1,\ldots,4}\}|R\rangle$

  These transform as singlets under the observable $SU(5)$ gauge group and transform in non–trivial representations of the the hidden $SU(4)\times U(1)_4$ gauge group. The projectors are given by:

$$
\begin{aligned}
P^{(22)(\overline{\phi}^{1,\ldots,4})}_{pqrs} &= \frac{1}{16}\left(1-C\begin{pmatrix} e_1 \\ B^{(22)}_{pqrs}\end{pmatrix}\right)\cdot\left(1-C\begin{pmatrix} e_2 \\ B^{(22)}_{pqrs}\end{pmatrix}\right) \\
&\quad\cdot\left(1-C\begin{pmatrix} z_1 \\ B^{(22)}_{pqrs}\end{pmatrix}\right)\cdot\left(1+C\begin{pmatrix} z_2+\alpha \\ B^{(22)}_{pqrs}\end{pmatrix}\right) \\
P^{(23)(\overline{\phi}^{1,\ldots,4})}_{pqrs} &= \frac{1}{16}\left(1-C\begin{pmatrix} e_3 \\ B^{(23)}_{pqrs}\end{pmatrix}\right)\cdot\left(1-C\begin{pmatrix} e_4 \\ B^{(23)}_{pqrs}\end{pmatrix}\right) \\
&\quad\cdot\left(1-C\begin{pmatrix} z_1 \\ B^{(23)}_{pqrs}\end{pmatrix}\right)\cdot\left(1+C\begin{pmatrix} z_2+\alpha \\ B^{(23)}_{pqrs}\end{pmatrix}\right) \\
P^{(24)(\overline{\phi}^{1,\ldots,4})}_{pqrs} &= \frac{1}{16}\left(1-C\begin{pmatrix} e_5 \\ B^{(24)}_{pqrs}\end{pmatrix}\right)\cdot\left(1-C\begin{pmatrix} e_6 \\ B^{(24)}_{pqrs}\end{pmatrix}\right) \\
&\quad\cdot\left(1-C\begin{pmatrix} z_1 \\ B^{(24)}_{pqrs}\end{pmatrix}\right)\cdot\left(1+C\begin{pmatrix} z_2+\alpha \\ B^{(24)}_{pqrs}\end{pmatrix}\right)
\end{aligned}
$$

  The corresponding matrix equations are given as:

$$
\begin{pmatrix}
(e_1|e_3) & (e_1|e_4) & (e_1|e_5) & (e_1|e_6) \\
(e_2|e_3) & (e_2|e_4) & (e_2|e_5) & (e_2|e_6) \\
(z_1|e_3) & (z_1|e_4) & (z_1|e_5) & (z_1|e_6) \\
(\delta|e_3) & (\delta|e_4) & (\delta|e_5) & (\delta|e_6)
\end{pmatrix}
\begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix}
=
\begin{pmatrix}
(e_1|b_1+z_1+\alpha) \\
(e_2|b_1+z_1+\alpha) \\
(z_1|b_1+z_1+\alpha) \\
(z_2|b_1+z_1)+(\alpha|b_1+z_1+z_2+\alpha)
\end{pmatrix}
$$

$$
\begin{pmatrix}
(e_3|e_1) & (e_3|e_2) & (e_3|e_5) & (e_3|e_6) \\
(e_4|e_1) & (e_4|e_2) & (e_4|e_5) & (e_4|e_6) \\
(z_1|e_1) & (z_1|e_2) & (z_1|e_5) & (z_1|e_6) \\
(\delta|e_1) & (\delta|e_2) & (\delta|e_5) & (\delta|e_6)
\end{pmatrix}
\begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix}
=
\begin{pmatrix}
(e_3|b_2+z_1+\alpha) \\
(e_4|b_2+z_1+\alpha) \\
(z_1|b_2+z_1+\alpha) \\
(z_2|b_2+z_1)+(\alpha|b_2+z_1+z_2+\alpha)
\end{pmatrix}
$$

$$
\begin{pmatrix}
(e_5|e_1) & (e_5|e_2) & (e_5|e_3) & (e_5|e_4) \\
(e_6|e_1) & (e_6|e_2) & (e_6|e_3) & (e_6|e_4) \\
(z_1|e_1) & (z_1|e_2) & (z_1|e_3) & (z_1|e_4) \\
(\delta|e_1) & (\delta|e_2) & (\delta|e_3) & (\delta|e_4)
\end{pmatrix}
\begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix}
=
\begin{pmatrix}
(e_5|b_3+z_1+\alpha) \\
(e_6|b_3+z_1+\alpha) \\
(z_1|b_3+z_1+\alpha) \\
(z_2|b_1+b_2)+(\alpha|b_1+b_2+z_2+\alpha)+1
\end{pmatrix}
$$

  Where $\delta = z_2+\alpha$

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