Conjugacy for positive permutation braids

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Abstract

Positive permutation braids on n strings, which are defined to be positive n-braids where each pair of strings crosses at most once, form the elementary but non-trivial building blocks in many studies of conjugacy in the braid groups. We study conjugacy of these elementary braids which close to knots, and show that those which close to the trivial knot or to the trefoil are all conjugate. All such n-braids with the maximum possible crossing number are also shown to be conjugate.

We note that conjugacy of these braids for $n \leq 5$ depends only on the crossing number. In contrast, we exhibit two such braids on 6 strings with 9 crossings which are not conjugate but whose closures are each isotopic to the $(2, 5)$ torus knot.

Keywords: Positive permutation braids; conjugacy; cycles.

Introduction

The question of when two n-string braids are conjugate has aroused interest over many years. Algorithms for comparing braids based on refinements of Garside's algorithm, [4, 8], can be used to settle this question in individual cases. A basic complexity measure in the algorithms is the least number of permutation braids needed to present a conjugate of the given braid. The simplest general case, when this number is 1, reduces to deciding when two positive permutation braids on n strings are conjugate.

A necessary condition is that the corresponding permutations be conjugate, in other words the permutations have the same cycle type. In this investigation we shall restrict ourselves to the case where the closure of the braid is a knot, and equivalently to those permutations in S_n which are n-cycles.

While any two such permutations are conjugate, the corresponding permutation braids need not be. A sufficient condition is that the closures of the braids be isotopic *as closed braids*, in other words the closed braids must be isotopic in the solid torus which is the complement of the braid axis, [9].

We shall examine how far this condition follows from weaker necessary conditions on the braids.

Theorem 2 *Positive permutation braids on n strings which close to the unknot are all conjugate.*

Theorem 3 *Positive permutation braids on n strings which close to the trefoil are all conjugate.*

Theorem 4 *Positive permutation braids on n strings which close to the same knot are all conjugate, when $n \leq 5$.*

We also prove a general result about conjugacy of such braids which have the largest possible number of crossings in theorem 5. On the other hand we exhibit in theorem 6 two 6-string positive permutation braids which close to the (2,5) torus knot but are not conjugate. These are constructed along the lines of Murasugi and Thomas' original example of non-conjugate positive braids with isotopic closure, [11].

Further simple non-conjugacy results in theorem 7 give a range of non-conjugate positive braids closing to the trefoil, in contrast to theorem 3.

Some of our results were first noted in [6] by the second author. There has also been a recent exploration by Elrifai and Benkhalifa [3] for small values of n without restrictions on the cycle type of the permutation.

The techniques used in this paper to prove non-conjugacy are very direct; more subtle techniques, such as Fiedler's Gauss sum invariants [5], may be used in more difficult cases, or applications of the algorithm of Franco and Meneses[8]. Hall has examples coming from the realms of dynamical systems of positive permutation braids on 12 or more strings which are believed not to be conjugate to their reverse [7]. In such cases none of the techniques used here can be applied to establish non-conjugacy.

1 Permutation braids

We shall use Artin's classical description of the group B_n of braids on n strings in terms of elementary generators σ_i for $i = 1, \ldots, n-1$ with the relations:

1. $\sigma_i \sigma_j = \sigma_j \sigma_i$ for $|i - j| \geq 2$,

2. $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ for $1 \leq i \leq n-2$.

There are two simple homomorphisms from the group B_n which give initial constraints on conjugacy.

- The homomorphism $\varphi : B_n \to S_n$ defined on the generators by $\varphi(\sigma_i) = (i\ i+1)$ determines a permutation $\pi = \varphi(\beta)$ in which $\pi(j)$ gives the endpoint of the string of β which begins at j.
- The homomorphism $wr : B_n \to \mathbf{Z}$ defined by $wr(\sigma_i) = 1$, counts the *writhe* or 'algebraic crossing number' of a braid.

Two conjugate braids in B_n must then have the same writhe, since $\mathbf{Z}$ is abelian, as well as having permutations of the same cycle type.

Definition. A *positive braid* is an element of B_n which can be written as a word in positive powers of the generators $\{\sigma_i\}$, without use of the inverse elements σ_i^{-1}.

For positive braids, the writhe is simply the number of crossings in the braid.

Definition. A braid β is called a *positive permutation braid* if it is a positive braid such that no pair of strings cross more than once.

Notation. We denote the set of positive braids and positive permutation braids in B_n by B_n^+ and S_n^+ respectively.

This definition of positive permutation braids was first used by Elrifai in [2, 4], where they were shown to correspond exactly to permutations. Explicitly, the homomorphism φ restricts to a bijection from the set S_n^+ of positive permutation braids to S_n. They were also identified by Elrifai with the set of *initial segments* of Garside's fundamental braid Δ_n.

It should be noted that the explicit braid word for a positive permutation braid is generally not unique. For example, the permutation (1423) can be represented in S_4^+ by braid words $\sigma_1\sigma_2\sigma_3\sigma_1\sigma_2$ and $\sigma_2\sigma_1\sigma_3\sigma_2\sigma_3$. Consequently some authors choose to label permutation braids simply by the corresponding permutation in S_n.

The number of components of the closure $\hat{\beta}$ of a braid β, constructed by identifying the initial points with the end points, is the number of cycles in the cycle type of the permutation $\varphi(\beta)$. In this paper we restrict attention to braids which close to knots, and hence we shall only look at the $(n-1)!$ permutation braids whose permutation is a single n-cycle.

2 Conjugacy results

Suppose that two braids β and γ are conjugate in B_n. Then their closures are isotopic as links in the complement of the braid axis, and so they are certainly isotopic in S^3.

Where β and γ are positive they must have the same number of crossings, because they have the same writhe. Even if they are not conjugate, two positive braids in B_n which close to isotopic knots must have the same number of crossings.

Lemma 1 *If $\beta, \gamma \in B_n^+$ and $\hat{\beta}, \hat{\gamma}$ are isotopic knots then $wr(\beta) = wr(\gamma)$.*

Proof : Suppose that the knot $\hat{\beta}$ has genus g. The closure of a non-split positive braid $\beta \in B_n$ is always a fibred link or knot. The surface found from $\hat{\beta}$ by Seifert's algorithm is a fibre surface, and has minimal genus g. Its Euler characteristic $\chi = 1 - 2g$ satisfies $1 - \chi = c - (n-1)$ where $c = wr(\beta)$ is the number of crossings in β. Hence $wr(\beta) = (n-1) + 2g = wr(\gamma)$, since $\hat{\gamma}$ is isotopic to $\hat{\beta}$ and so also has genus g. □

Consequently if a positive n-braid closes to the unknot then it must have exactly $n-1$ crossings. If it closes to the trefoil knot, which has genus 1, then it must have $n+1$ crossings. We now show that positive permutation n-braids which close to either of these knots are determined up to conjugacy.

Explicitly we have the following results.

Theorem 2 *Any positive permutation n-braid β which closes to the unknot is conjugate to $\sigma_1 \sigma_2 \cdots \sigma_{n-1}$.*

Theorem 3 *Any positive permutation n-braid β which closes to the trefoil is conjugate to $\sigma_1^3 \sigma_2 \cdots \sigma_{n-1}$.*

Proof of theorem 2:

Each generator σ_i must appear at least once in β, otherwise its closure is disconnected. Since its closure has genus 0 the braid β has $n-1$ crossings, and so each generator appears exactly once.

It is enough to manipulate the braid cyclically, as such manipulations can be realised as conjugacies. We can represent β up to conjugacy by writing the generators $\sigma_1, \ldots, \sigma_{n-1}$ in the appropriate order around a circle. Each generator appears exactly once. To prove the theorem we use the braid commutation relations to rearrange the generators in ascending order round the circle.

Assume by induction on j that the generators $\sigma_1, \ldots, \sigma_j$ occur consecutively in order. Then any generator σ_k lying on the circle between σ_j and

σ_{j+1} has $k > j+1$. These generators then commute with each of $\sigma_1, \dots, \sigma_j$, and can be moved past them to leave σ_{j+1} immediately after σ_j. The process finishes when all generators are in consecutive order. □

Proof of theorem 3:

Again represent generators on a circle. Each generator $\sigma_1, \dots, \sigma_{n-1}$ must occur at least once, otherwise the closed braid splits. Since the trefoil has genus 1 the braid has $n+1$ crossings. So either two generators σ_i and σ_j each occur twice, or one, σ_i say, occurs 3 times, and the other generators occur once only.

Either σ_{i-1} or σ_{i+1} must occur between two occurrences of σ_i in β, otherwise it can be rewritten with two consecutive occurrences of σ_i. This is not possible for a permutation braid, since pairs of strings cross at most once. If σ_i occurs three times then σ_{i-1} lies between one pair of occurrences of σ_i and σ_{i+1} between the other pair. We can then move all generators except σ_{i+1} past these last two occurrences of σ_i to write β with a consecutive sequence $\sigma_i\sigma_{i+1}\sigma_i$. Change this to $\sigma_{i+1}\sigma_i\sigma_{i+1}$ by the braid relation to write β with σ_i and σ_{i+1} each appearing twice.

We may thus assume that two generators σ_i and σ_j each occur twice in β, with $j > i$. If $j > i+1$ then σ_{i+1} occurs only once. We can then collect all generators σ_k with $k > i+1$ at the two ends of the braid word, and combine them at the end of the word by cycling so as to write a conjugate braid in the form AB where B is a product of generators σ_k with $k > i+1$, and includes σ_j twice, while A is a product with $k \leq i+1$, and includes σ_i twice. The closure of the braid then has three components and not one.

Hence β must contain σ_i and σ_{i+1} twice each. Furthermore their occurrences must be interleaved, otherwise we can cycle the braid and commute elements to separate it as a product of generators σ_k with $k \leq i$ and those with $k > i$, and its closure will again have three components.

We shall prove, by induction on i, that any positive braid with two interleaved occurrences of σ_i and σ_{i+1}, and single occurrences of all other generators, is conjugate to $\sigma_1^3\sigma_2 \cdots \sigma_{n-1}$.

We can assume, by cycling, that the single occurrence of σ_{i-1} does not lie between the two occurrences of σ_i. We can move all further generators except σ_{i+1} past σ_i so as to write $\sigma_i\sigma_{i+1}\sigma_i$ consecutively. The remaining occurrence of σ_{i+1} can be moved round the circle past any other generator except the single σ_{i+2}. It can then be moved one way or other round the circle to reach this block of three generators, giving either $\sigma_i\sigma_{i+1}\sigma_i\sigma_{i+1}$ or $\sigma_{i+1}\sigma_i\sigma_{i+1}\sigma_i$. The braid relation then gives a consecutive block of either $\sigma_i\sigma_i\sigma_{i+1}\sigma_i$ or $\sigma_i\sigma_{i+1}\sigma_i\sigma_i$. The single σ_{i-1} now lies between two occurrences of σ_i on the circle. Any intervening generators commute with σ_i and can be moved out to leave $\sigma_i\sigma_{i-1}\sigma_i$, which can be converted to $\sigma_{i-1}\sigma_i\sigma_{i-1}$. The cyclic braid now has two interleaving occurrences of σ_{i-1} and σ_i.

The result follows by induction on i, once we establish it for $i = 1$. In this case the argument above provides a block of either $\sigma_1\sigma_1\sigma_2\sigma_1$ or $\sigma_1\sigma_2\sigma_1\sigma_1$ on the circle. Since σ_1 commutes with all generators except σ_2 we can move the right hand occurrences of σ_1 round the circle to give a block $\sigma_1\sigma_1\sigma_1\sigma_2$. The remaining generators can then be put in ascending order as in the proof of theorem 2. □

A quick check on the possible values of the writhe for the $(n-1)!$ positive permutation braids with $n \leq 4$ which close to a knot shows that in this range conjugacy is determined simply by writhe, using theorems 2 and 3. Positive permutation braids with $n+3$ crossings arise first when $n = 5$. A direct check on the corresponding braids shows that in this case too the writhe is sufficient.

Theorem 4 *Positive permutation braids on n strings which close to a knot are conjugate if and only if they have the same number of crossings, when $n \leq 5$.*

Tables of these braids for $n = 3, 4, 5$, and the corresponding permutations, are included below.

When $n = 3$ there are just two braids which both close to the unknot.

Permutation	Braid word	Number of crossings
(123)	$\sigma_2\sigma_1$	2
(132)	$\sigma_1\sigma_2$	2

When $n = 4$ there are two conjugacy classes. The braids with writhe 3 close to the unknot, and those with writhe 5 to the trefoil.

Permutation	Braid word	Number of crossings
(1234)	$\sigma_3\sigma_2\sigma_1$	3
(1243)	$\sigma_2\sigma_1\sigma_3$	3
(1342)	$\sigma_1\sigma_3\sigma_2$	3
(1432)	$\sigma_1\sigma_2\sigma_3$	3
(1324)	$\sigma_2\sigma_1\sigma_3\sigma_2\sigma_1$	5
(1423)	$\sigma_1\sigma_2\sigma_1\sigma_3\sigma_2$	5

When $n = 5$ there are three conjugacy classes. The braids with writhe $4 = n - 1$ close to the unknot and those with writhe 6 close to the trefoil. Those with writhe $8 = n + 3$ all close to the $(2, 5)$ torus knot.

Permutation	Braid word	Number of crossings
(12345)	$\sigma_4\sigma_3\sigma_2\sigma_1$	4
(12354)	$\sigma_3\sigma_2\sigma_1\sigma_4$	4
(12453)	$\sigma_2\sigma_1\sigma_4\sigma_3$	4
(12543)	$\sigma_2\sigma_1\sigma_3\sigma_4$	4
(13452)	$\sigma_1\sigma_4\sigma_3\sigma_2$	4
(13542)	$\sigma_1\sigma_3\sigma_2\sigma_4$	4
(14532)	$\sigma_1\sigma_2\sigma_4\sigma_3$	4
(15432)	$\sigma_1\sigma_2\sigma_3\sigma_4$	4
(12435)	$\sigma_3\sigma_2\sigma_4\sigma_3\sigma_2\sigma_1$	6
(12534)	$\sigma_2\sigma_3\sigma_2\sigma_1\sigma_4\sigma_3$	6
(13245)	$\sigma_2\sigma_1\sigma_4\sigma_3\sigma_2\sigma_1$	6
(13254)	$\sigma_2\sigma_1\sigma_3\sigma_2\sigma_1\sigma_4$	6
(13524)	$\sigma_3\sigma_2\sigma_1\sigma_4\sigma_3\sigma_2$	6
(14253)	$\sigma_2\sigma_1\sigma_3\sigma_2\sigma_4\sigma_3$	6
(14352)	$\sigma_1\sigma_3\sigma_2\sigma_4\sigma_3\sigma_2$	6
(14523)	$\sigma_1\sigma_2\sigma_1\sigma_4\sigma_3\sigma_2$	6
(15342)	$\sigma_1\sigma_2\sigma_3\sigma_2\sigma_4\sigma_3$	6
(15423)	$\sigma_1\sigma_2\sigma_1\sigma_3\sigma_2\sigma_4$	6
(13425)	$\sigma_2\sigma_3\sigma_2\sigma_1\sigma_4\sigma_3\sigma_2\sigma_1$	8
(14235)	$\sigma_1\sigma_3\sigma_2\sigma_1\sigma_4\sigma_3\sigma_2\sigma_1$	8
(14325)	$\sigma_2\sigma_1\sigma_3\sigma_2\sigma_4\sigma_3\sigma_2\sigma_1$	8
(15234)	$\sigma_1\sigma_2\sigma_3\sigma_2\sigma_1\sigma_4\sigma_3\sigma_2$	8
(15243)	$\sigma_1\sigma_2\sigma_1\sigma_3\sigma_2\sigma_4\sigma_3\sigma_2$	8
(15324)	$\sigma_1\sigma_2\sigma_1\sigma_3\sigma_2\sigma_1\sigma_4\sigma_3$	8

Having looked among the closures of positive permutation braids at knots with the smallest number of crossings, in theorems 2 and 3, we now turn briefly to those with the largest possible number.

The largest number of crossings in any positive permutation braid in B_n is $\frac{1}{2}n(n-1)$, which occurs for the fundamental half-twist braid Δ_n. If the closure is to be a knot the largest number of crossings is $\frac{1}{2}n(n-1)-[\frac{1}{2}(n-1)]$.

Theorem 5 *Every positive permutation braid with $\frac{1}{2}n(n-1)-[\frac{1}{2}(n-1)]$ crossings which closes to a knot is conjugate to $\Delta_n\sigma_1^{-1}\sigma_2^{-1}\cdots\sigma_k^{-1}$ where $k = [\frac{1}{2}(n-1)]$.*

Proof : Take $n = 2k+1$ or $n = 2k+2$, so that $k = [\frac{1}{2}(n-1)]$, and let β be a positive permutation braid with $\frac{1}{2}n(n-1)-k$ crossings which closes to a knot. Then β has a complementary positive permutation braid γ in Δ_n, with $\beta\gamma = \Delta_n$. The braid γ has k crossings. Since $\beta = \Delta_n\gamma^{-1}$ closes to a knot the k crossings in γ^{-1} must be used to connect up the $k+1$ components in $\hat{\Delta}_n$. Hence the k generators in γ^{-1} must all be different. When $n = 2k+2$ the generator σ_{k+1} cannot occur, since this connects two strings which are already in the same component of $\hat{\Delta}_n$, and more generally σ_j and σ_{n-j} cannot both occur, for any j, as they both connect the same two components. In particular the generators σ_k and σ_{k+1} cannot both occur when $n = 2k+1$.

The generators in γ then belong to two mutually commuting sets, those from σ_1 up to σ_k and those from σ_{k+1} up to σ_{n-1}. Write β on a circle, with one block of generators together as Δ_n and then the k generators of γ^{-1}. Move all generators σ_j with $j > k$ to the extreme right in γ^{-1} and then round the circle to the left of Δ_n. Now move them past Δ_n, when each σ_j is converted to σ_{n-j}. Since σ_j and σ_{n-j} did not both occur in γ we get a braid $\Delta_n\alpha^{-1}$ conjugate to β in which $\sigma_1, \ldots, \sigma_k$ each occurs exactly once in α.

Following the method of theorem 2 we can arrange the k generators in α in any order up to conjugacy, once we know how to move any generator σ_j from the left to the right of α^{-1} by conjugacy. This can be done by taking it *twice* round the circle as follows. First move σ_j to the left of Δ_n, when it becomes σ_{n-j}. Then move it round the circle to the end of the word. It can then be moved left past all the remaining generators of α^{-1}, and past Δ_n once more, to become σ_j. Finally move this round the circle to the right-hand end of α^{-1}.

Consequently β is conjugate to $\Delta_n\sigma_1^{-1}\sigma_2^{-1}\cdots\sigma_k^{-1}$. □

3 Non-conjugacy results

When $n = 6$ it is possible to have two positive permutation braids with the same number of crossings which close to different knots. The permutations (124536), with braid $\sigma_3\sigma_4\sigma_3\sigma_2\sigma_5\sigma_4\sigma_3\sigma_2\sigma_1$, and (132546), with braid $\sigma_2\sigma_1\sigma_4\sigma_3\sigma_5\sigma_4\sigma_3\sigma_2\sigma_1$, close to the $(2,5)$ torus knot and the sum of two trefoils respectively, so writhe no longer determines conjugacy.

In [6], Hadji gave examples of two non-conjugate positive permutation braids with $n = 16$ each closing to the same connected sum of three knots.

In fact, non-conjugate positive permutation braids which close to the same knot show up first when $n = 6$.

Theorem 6 *The positive permutation braids in S_6^+ with permutations* (165324) *and* (152643) *have the same closure but are not conjugate.*

Proof : The braids, shown below, can be written $\beta = \sigma_1\sigma_3\sigma_5\sigma_2\sigma_4\sigma_1\sigma_3\sigma_2\sigma_1$ and $\gamma = \sigma_2\sigma_4\sigma_3\sigma_5\sigma_2\sigma_4\sigma_1\sigma_3\sigma_2$ respectively. Both of these can be reduced by Markov moves to the 4-braid $\sigma_1\sigma_3\sigma_2\sigma_1\sigma_3\sigma_2\sigma_1$, so both close to the $(2, 5)$ torus knot.

$$\beta \quad = \qquad\qquad\qquad \gamma \quad =$$

The squares of the two braids are shown here with the strings which form one component of the closure emphasised.

$$\beta^2 \quad = \qquad\qquad\qquad \gamma^2 \quad =$$

If β and γ are conjugate then so are β^2 and γ^2. Now the closure of β^2 is a link with two components, each of which turns out to be the trefoil knot, while the two components of the closure of γ^2 are trivial knots. Hence β^2 and γ^2 are not conjugate. $\square$

An alternative check can be made by calculating the 2-variable Alexander polynomial of the link consisting of the closure of β and its axis. If β is conjugate to γ this link is isotopic to the closure of γ and its axis. Its polynomial is in general the characteristic polynomial of the reduced Burau matrix of the braid, [9]. For β above, the polynomial is

$$t^9x^5 + t^7x^4 + t^5x^3 + t^4x^2 + t^2x + 1,$$

which differs, up to multiples of $\pm t^i x^j$, from the polynomial

$$t^9x^5 + t^7x^4 + (2t^5 - t^4)x^3 + (2t^4 - x^5)x^2 + t^2x + 1$$

for γ.

Other tests for conjugacy, which also rely in effect on invariants of a closed braid in a solid torus, can be used to give a contrasting result to theorem 3 about positive braids which close to the trefoil, when we do not restrict to positive permutation braids.

Theorem 7 *If $\beta \in B_n^+$ closes to the trefoil knot then β is conjugate to $\beta(i) = \sigma_1\sigma_2 \dots \sigma_{i-1}\sigma_i^3\sigma_{i+1} \dots \sigma_{n-1}$ for some i. Two such braids $\beta(i), \beta(k)$ are conjugate if and only if $k = i$ or $k = n - i$.*

Remark. When $n = 4$ the braids are examples of the construction of Murasugi and Thomas, [11]. They show that the braids $\sigma_1^p\sigma_2^q\sigma_3^r$ and $\sigma_1^p\sigma_2^r\sigma_3^q$, with p, q, r odd, which close to isotopic knots, are not conjugate when $q \neq r$. Their proof uses the exceptional homomorphism from B_4 to B_3 defined by $\sigma_1, \sigma_3 \mapsto \sigma_1, \sigma_2 \mapsto \sigma_2$, observing that the braids map to $\sigma_1^{p+r}\sigma_2^q$ and $\sigma_1^{p+q}\sigma_2^r$, which close to links with different linking numbers.

Proof of theorem 7:
1. *Conjugacy.* The only difference from the argument of theorem 3 is that one generator σ_i may occur three times, with σ_{i-1} and σ_{i+1} both lying on the circle between the same pair of occurrences of σ_i. Then all three occurrences of σ_i can be moved together and remain as a block on the circle, while the other generators are put in consecutive order, as in theorem 2. This shows that every such braid is conjugate to some $\beta(i)$. To see that the braids $\beta(i)$ and $\beta(n-i)$ are conjugate, first conjugate $\beta(i)$ by Δ_n, taking σ_i^3 to σ_{n-i}^3 and then rearrange as above.
2. *Non-conjugacy.* Any closed braid represents an element in the framed Homfly skein of closed braids in the annulus [10]. The closure of $\sigma_1\sigma_2 \dots \sigma_{k-1}$ represents an element A_k. The skein itself admits a commutative product, represented by the closures of split braids. The subspace spanned by the closure of braids in B_n has a basis consisting of monomials $A_{i_1} \dots A_{i_k}$ with $i_1 + \dots + i_k = n$. Coefficients in the skein can be taken as integer polynomials in a variable z. In the Homfly skein of braids before closure, we have $\sigma_i^3 = c(z)\sigma_i + d(z)$, for some fixed non-zero polynomials $c(z), d(z)$, so that $\beta(i) = c(z)\sigma_1 \dots \sigma_{n-1} + d(z)\sigma_1 \dots \sigma_{i-1}\sigma_{i+1} \dots \sigma_{n-1}$ in this skein. Its closure then represents $c(z)A_n + d(z)A_iA_{n-i}$ in the skein of the annulus.

If $\beta(i)$ and $\beta(k)$ are conjugate then they have the same closure in the annulus. Then

$$c(z)A_n + d(z)A_iA_{n-i} = c(z)A_n + d(z)A_kA_{n-k},$$

and hence $A_iA_{n-i} = A_kA_{n-k}$. The monomials form a basis in the skein of the annulus, so $k = i$ or $k = n - i$.

□

Remark. This same calculation can be used to show that the Conway polynomial of the closure of $\beta(i)$ and its axis differs from that of $\beta(k)$ and its axis except when $k = i$ or $n - i$.

Although we do not have a complete list of the conjugacy classes for the 120 braids in S_6^+ which close to knots, we know by theorems 2, 3 and 5 that those with $5, 7$ or 13 crossings form complete conjugacy classes. These represent the trivial knot, the trefoil and the $(3, 5)$ torus knot respectively. An inductive count shows that there are 2^{n-2} braids in S_n^+ which represent the trivial knot, and so there are 16 of these in the case $n = 6$, while a combinatorial count finds 6 braids with 13 crossings.

Among the braids with 9 crossings there is one conjugacy class with 4 braids which close to the sum of two trefoils, and at least two classes of braids which close to the $(2, 5)$ torus knot.

We can modify the result of Cromwell, [1], on recognising connected sums among positive braid closures to show that none of the braids with 11 crossings give rise to connected sums. We have not, however, checked through to separate them into conjugacy classes.

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