# Epistemic Quantified Boolean Logic: Expressiveness and Completeness Results

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## Abstract

We introduce epistemic quantified boolean logic (EQBL), an extension of propositional epistemic logic with quantification over propositions. We show that EQBL can express relevant properties about agents' knowledge in multi-agent contexts, such as "agent $a$ knows as much as agent $b$". We analyse the expressiveness of EQBL through a translation into monadic second-order logic, and provide completeness results w.r.t. various classes of Kripke frames. Finally, we prove that model checking EQBL is PSPACE-complete. Thus, the complexity of model checking EQBL is no harder than for (non-modal) quantified boolean logic.

## 1 Introduction

The inspiration for the present contribution comes from a series of papers [van Ditmarsch *et al.*, 2009; 2011; 2012] introducing the language of Local Properties in Modal Logic. LPML extends propositional modal logic with operators to express local properties of Kripke frames. Indeed, it is well-known that there is a tight correspondence between modal formulas and properties of Kripke frames as expressed in first-order logic [van Benthem, 1976; Blackburn *et al.*, 2001]. As an example, let $\theta(a, b, p)$ be the modal principle $K_a p \to K_b p$ and let $\Theta(a, b, x)$ represent the first-order formula $\forall y(R_b(x, y) \to R_a(x, y))$. Then $\theta$ corresponds with $\Theta$ in the following sense: $\theta(a, b, p)$ holds at a state $w$ in a Kripke frame $\mathcal{F}$ iff $\Theta(a, b, x/w)$ is true in $\mathcal{F}$, that is, every $R_b$-accessible world from $w$ is also $R_a$-accessible. Loosely speaking, given an epistemic interpretation of modal operators, to model a situation in which agent $b$ knows everything that agent $a$ knows, simply add $\theta$ to the axiom system, and assume $\Theta$ on the class of frames.

However, adding $\theta$ as an axiom to a modal logic would automatically enforce the property expressed by $\Theta$ to hold in *any* state of a frame for the logic, i.e., $\theta$ becomes *globally* true. It is impossible in 'standard' modal logic, to express something like: in state $w$, agent $b$ knows everything that $a$ knows, and agent $c$ knows *that*, but $d$ does not.

Moreover, truth in a model (as opposed to validity in a frame) of $\theta$ is a necessary but not sufficient condition for $\Theta$ to hold. That is, if $\Theta$ holds in a Kripke frame $\mathcal{F}$, then $\theta$ is true on any pointed model $(\mathcal{M}, w)$ built on $\mathcal{F}$. However, the converse does not hold in general: it may be that $\theta$ holds in $(\mathcal{M}, w)$, without $\Theta$ being true. In other words, $\theta$ may hold at $w$ because of some specifically well-chosen assignment $V$ that is part of $\mathcal{M}$.

To address the issues above, LPML considers, for every first-order property $\Theta$ of interest, a new modal operator $\boxdot_\Theta$, whose interpretation is such that the modal formula $\boxdot_\Theta(\vec{a})$ holds at state $w$ for some tuple $\vec{a}$ of agents iff $\Theta(\vec{a}, x/w)$ is true in $\mathcal{F}$. Formally, this connection is made through models of the form $\langle \mathcal{F}, I, V\rangle$, where $V$ is an assignment of propositional atoms, and $I$ is an interpretation relating $\boxdot_\Theta(\vec{a})$ to $\Theta(\vec{a}, x)$. Indeed, now structural properties are no longer determined by validities on frames, but by truths in the model.

In this paper we introduce a modal extension of quantified boolean logic to express formulas such as $\boxdot_\Theta(\vec{a})$. Specifically, for $\Theta(a, b, x) = \forall y(R_b(x, y) \to R_a(x, y))$, the formula $\boxdot_\Theta(a, b)$ is shown to be equivalent to the universal closure $\forall p(K_a p \to K_b p)$ of $\theta(a, b, p)$, which is a formula of epistemic quantified boolean logic (EQBL). Modal quantified boolean logic originates from the seminal work by Fine and Bull in the late 60's [Bull, 1969; Fine, 1970], which established axiomatisability and complexity results for monomodal languages. However, given the high complexity and some non-axiomatisability results of the formalism, modal quantified boolean logic is comparatively much less explored than its non-modal and non-quantified counterparts. Our main aim in this paper is to show that (multi-modal) EQBL provides us with the expressive power to represent and reason about epistemic properties of agents – such as those considered in [van Ditmarsch *et al.*, 2012] – at a reasonable computational cost.

**Scheme of the paper.** In Section 2 we introduce the syntax and semantics of multi-agent EQBL. Section 3 is devoted to the analysis of the expressive power of EQBL through a comparison with monadic second-order logic (MSO). In particular, we show that EQBL is strictly more expressive than LPML. In Section 4 we provide novel axiomatisations of EQBL on a number of classes of interest; while in Section 5 we establish that the model checking problem for EQBL is no harder than for (non-modal) quantified boolean logic. We conclude in Section 6 by comparing these results with related literature, and by pointing to future research.

## 2 Syntax and Semantics

**The Formal Language** We first introduce the syntax of EQBL. In what follows we fix a set $AP$ of atomic propositions and a finite set $Ag$ of indexes for agents.

**Definition 1 (EQBL)** *For $p \in AP$ and $a \in Ag$, the formulas in EQBL are defined by the following BNF:*

$$\psi \quad ::= \quad p \mid \neg\psi \mid \psi \to \psi \mid \forall p\psi \mid K_a\psi \mid C\psi$$

Our language contains epistemic formulas $K_a\phi$, for every agent name $a \in Ag$, that are read as "agent $a$ knows that $\phi$", as well as formulas $C\phi$ that intuitively mean "$\phi$ is common knowledge" in the set $Ag$ of agents [Meyer and van der Hoek, 1995]. We consider the standard abbreviations for $\wedge$, $\vee$, $\exists$, while $\sharp$ and $Q$ are placeholders for any unary operator $\neg$, $K_a$, $C$ and any quantifier $\forall$, $\exists$ respectively. Also, $M_a\phi$ is a shorthand for $\neg K_a\neg\phi$. The set *fr*$(\phi)$ of free atomic propositions in a formula $\phi$ is recursively defined as follows:

$$\begin{array}{lcl}
fr(p) & = & \{p\} \\
fr(\sharp\phi) & = & fr(\phi) \\
fr(\phi \to \phi') & = & fr(\phi) \cup fr(\phi') \\
fr(Qp\phi) & = & fr(\phi) \setminus \{p\}
\end{array}$$

Sentences in EQBL are formulas $\phi$ with an empty set of free propositions, i.e., *fr*$(\phi) = \emptyset$. The set $bnd(\phi)$ of bound propositions in $\phi$ is defined as standard as the set of all propositions $q$ appearing in the scope of any quantifier $Qq$. Hereafter we assume w.l.o.g. that for each formula $\phi$, *fr*$(\phi)$ and $bnd(\phi)$ are disjoint.

**Definition 2 (Substitution)** *Given an atom $p \in fr(\phi)$, an EQBL formula $\psi$ is* free *for $p$ in $\phi$ iff $p$ does not appear in $\phi$ within the scope of any quantifier $Qq$ for $q$ free in $\psi$.*

If $\psi$ is free for $p \in fr(\phi)$, then we inductively define the substitution $\phi[p/\psi]$ as follows:

$$\begin{array}{lcl}
q[p/\psi] & = & \begin{cases} q & \text{for } q \text{ different from } p \\ \psi & \text{otherwise} \end{cases} \\
(\sharp\phi)[p/\psi] & = & \sharp(\phi[p/\psi]) \\
(\phi \to \phi')[p/\psi] & = & (\phi[p/\psi]) \to (\phi'[p/\psi]) \\
(\forall r\phi)[p/\psi] & = & \forall r(\phi[p/\psi]), \text{ for } r \text{ different from } p
\end{array}$$

Intuitively, $\psi$ being free for $p$ in $\phi$ means that a substitution of $p$ by $\psi$ in $\phi$ does not create any new binding. For instance, $\neg q$ is free for $p$ in $\exists t(t \to p)$ but not in $\phi = \exists q(p \leftrightarrow q)$.

**Example 1** As an example of the expressive power of EQBL, consider the specification put forward in the introduction: agent $b$ knows everything that $a$ knows, and agent $c$ knows this fact, but $d$ does not. This can be recast in EQBL as the following formula:

$$\forall p(K_a p \to K_b p) \wedge K_c \forall p(K_a p \to K_b p) \wedge \neg K_d \forall p(K_a p \to K_b p)$$

In particular, we can reason further about agent $d$'s knowledge. Indeed, agent $d$ might know that $a$ knows something ignored by $b$, without being able to explicitly point out the content of $a$'s extra knowledge. This can be recast in EQBL by the following *de dicto* formula:

$$K_d \exists p(K_a p \wedge \neg K_b p) \tag{1}$$

However, $d$ could actually know some fact, also known by $a$ but ignored by $b$, as expressed in the *de re* formula:

$$\exists p K_d(K_a p \wedge \neg K_b p) \tag{2}$$

We remark intuitively that (2) is strictly stronger than and entails (1). EQBL allows us to distinguish the two readings – *de re* and *de dicto* – of individual knowledge. ■

**Kripke Frames and Models** To interpret formulas in EQBL we consider multi-modal Kripke frames and models, defined as follows.

**Definition 3** *A Kripke frame is a tuple $\mathcal{F} = \langle W, D, R\rangle$ where*

- *$W$ is a set of* possible worlds*;*
- *$D \subseteq 2^W$ is the* domain of propositions*;*
- *$R : Ag \to 2^{W\times W}$ assigns a* binary relation *on $W$ to each agent index in $Ag$.*

As standard, for $a \in Ag$, $R_a$ is the accessibility relation between worlds in $W$. In the rest of the paper we assume that each $R_a$ is an equivalence relation (i.e., symmetric, transitive and reflexive), consistently with the epistemic reading of modal operators [Meyer and van der Hoek, 1995]. We will state explicitly when this is not the case. Also, for each agent index $a \in Ag$ and $w \in W$, we define $R_a(w) = \{w' \mid R_a(w, w')\}$. Moreover, differently from standard Kripke frames [Blackburn *et al.*, 2001], we have a set $D \subseteq 2^W$ of "admissible" propositions for the interpretation of atoms and quantifiers.

To evaluate EQBL formulas we introduce *assignments* as functions $V : AP \to D$. Also, for $U \in D$, the assignment $V^p_U$ assigns $U$ to $p$ and coincides with $V$ on all other atomic propositions. Hence, atoms can only be assigned propositions in $D \subseteq 2^W$. A *Kripke model* is then a pair $\mathcal{M} = \langle \mathcal{F}, V\rangle$. In what follows we consider particular classes of Kripke frames and models. Specifically, we say that a frame is *full* whenever $D = 2^W$; while it is *boolean* if $D$ is closed under intersection, union and complementation, i.e., it is a boolean algebra. The interest of these classes of frames, which have already appeared in the literature [Fine, 1970; Mares and Goldblatt, 2006], will become apparent later on. Finally, a model is *full* (resp. *boolean*) whenever the underlying frame is.

**Definition 4 (Semantics of EQBL)** *We define whether a Kripke model $\mathcal{M} = \langle \mathcal{F}, V\rangle$* satisfies *an EQBL formula $\varphi$ at world $w$, or $(\mathcal{M}, w) \models \varphi$, as follows, where $R^+$ denotes the transitive closure of $\bigcup_{a\in Ag} R_a$ (clauses for propositional connectives are trivial and thus omitted).*

$$\begin{array}{lll}
(\mathcal{M}, w) \models p & \textit{iff} & w \in V(p) \\
(\mathcal{M}, w) \models K_a\psi & \textit{iff} & \textit{for all } w' \in R_a(w),\ (\mathcal{M}, w') \models \psi \\
(\mathcal{M}, w) \models C\psi & \textit{iff} & \textit{for all } w' \in R^+(w),\ (\mathcal{M}, w') \models \psi \\
(\mathcal{M}, w) \models \forall p\psi & \textit{iff} & \textit{for all } U \in D,\ (\langle \mathcal{F}, V^p_U\rangle, w) \models \psi
\end{array}$$

Hence, a quantified formula $\forall p\psi$ (resp. $\exists p\psi$) is true at world $w$ iff for every (resp. some) assignment of propositions in $D$ to $p$, $\psi$ is true. Further, as standard in epistemic logic [Meyer and van der Hoek, 1995], $(\mathcal{M}, w) \models C\psi$ iff $(\mathcal{M}, w') \models \psi$ for every world $w'$ *reachable* from $w$, i.e., for every $w'$ s.t. there exists a sequence $w_0, \ldots, w_k$ of worlds and *(i)* $w_0 = w$, *(ii)* $w_k = w'$, and *(iii)* for every $i < k$, $R_a(w_i, w_{i+1})$ for some $a \in Ag$.

We now define various notions of truth and validity. First, we write $(\mathcal{F}, V, w) \models \phi$ as a shorthand for $(\langle \mathcal{F}, V \rangle, w) \models \phi$. Then, we say that $\phi$ is *true* at $w$, or $(\mathcal{F}, w) \models \phi$, iff $(\mathcal{F}, V, w) \models \phi$ for every assignment $V$; $\phi$ is *valid* in a frame $\mathcal{F}$, or $\mathcal{F} \models \phi$, iff $(\mathcal{F}, w) \models \phi$ for every world $w$ in $\mathcal{F}$; $\phi$ is *valid* in a class $\mathcal{K}$ of frames, or $\mathcal{K} \models \phi$, iff $\mathcal{F} \models \phi$ for every $\mathcal{F} \in \mathcal{K}$. Also, $\phi$ is *true* in a model $\mathcal{M}$, or $\mathcal{M} \models \phi$, iff $(\mathcal{M}, w) \models \phi$ for every world $w$. Finally, $\phi$ is *satisfiable* iff there exists a model $\mathcal{M}$ s.t. $(\mathcal{M}, w) \models \phi$ for some world $w$.

Hereafter we consider the class $\mathcal{K}_{all}$ of all Kripke frames, the class $\mathcal{K}_{bool}$ of all boolean frames, and the class $\mathcal{K}_{full}$ of all full frames. If we let $\mathsf{Th}(\mathcal{K}) = \{\phi \in EQBL \mid \mathcal{K} \models \phi\}$, then clearly $\mathsf{Th}(\mathcal{K}_{all}) \subseteq \mathsf{Th}(\mathcal{K}_{bool}) \subseteq \mathsf{Th}(\mathcal{K}_{full})$. We will show that these inclusions are indeed strict.

Further classes of frames could be introduced, for instance the class where every EQBL (resp. modal) formula defines a proposition. However, neither class is directly relevant for the results below and their introduction requires a non-trivial generalisation of Kripke frames [Mares and Goldblatt, 2006]. Thus, such extensions are beyond the scope of the present paper. Also, notice that the Kripke frames in Def. 3 are related to *general frames* [Blackburn *et al.*, 2001], as both assume some sort of algebraic structure on the set $D$ of proposition. Nonetheless, the use of quantification makes EQBL strictly more expressive than the theory of general frames.

**Example 2** We revisit the example of [van Ditmarsch *et al.*, 2012, Section 4.3], and consider a simple card game with three players 1, 2, and 3. The cards are identified by their colour: red ($r$), white ($w$), and blue ($b$). We assume to have 9 propositional atoms $\{r_i, w_i, b_i \mid 1 \leq i \leq 3\}$, where for instance $w_1$ denotes that agent 1 holds the white card. Also, all agents know the cards of the game, and that each agent can see his own card, but not that of the other players. The situation after which each player is dealt a card can be modeled by model $\mathcal{M}$ in Fig. 1. The state rwb in $\mathcal{M}$ denotes the situation where player 1 holds the red card, 2 holds white and 3 holds blue. In this state, we have for instance that

$$(\mathcal{M}, \mathsf{rwb}) \models r_1 \wedge K_1 r_1 \wedge \neg K_2 r_1 \wedge K_1 \neg K_2 r_1$$

i.e., if the deal is rwb, then 1 holds the red card, he knows this, but 2 does not know it, and, finally, 1 knows that 2 does not know that 1 holds the red card. In particular, we have

$$\begin{aligned}(\mathcal{M}, \mathsf{rwb}) \quad \models \quad & \exists r_1 (K_1 r_1 \wedge \neg K_2 r_1 \wedge \neg K_3 r_1) \wedge \\ & \exists w_2 (\neg K_1 w_2 \wedge K_2 w_2 \wedge \neg K_3 w_2) \wedge \\ & \exists b_3 (\neg K_1 b_3 \wedge \neg K_2 b_3 \wedge K_3 b_3) \qquad (3)\end{aligned}$$

i.e., for every player, there is a fact that the others don't know.

Now let us assume that $\mathcal{M}$ is full. Note that we also have

$$(\mathcal{M}, \mathsf{rwb}) \models \exists w_1 ((K_1 w_1 \leftrightarrow K_2 w_1) \wedge (K_2 w_1 \leftrightarrow K_3 w_1))$$

i.e., there exists an assignment for $w_1$ so that all players have the same information: in fact, there exists an assignment to atomic propositions such that all agents know the same (assign $W$ to each propositional atom)! However, the intended

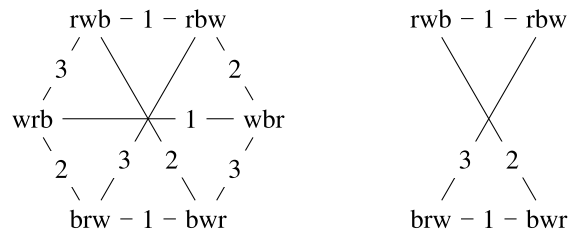


Figure 1: Two epistemic models $\mathcal{M}$ and $\mathcal{N}$

interpretation of atomic propositions should not consider such assignments. Any assignment should satisfy

$$\mathsf{deal} = C\left(\bigwedge_{i \leq 3} (r_i \oplus w_i \oplus b_i) \wedge \bigwedge_{i \neq j, p \in \{r,w,b\}} \neg (p_i \wedge p_j)\right)$$

That is, we only consider deals where each player holds a card, and no player holds two. Note that this still allows for 'un-intended' assignments, allowing for example for $\exists w_1 (w_1 \wedge \neg K_1 w_1)$ to hold. Our assignments should also satisfy the property that every player knows its own card (koc)

$$\mathsf{koc} = C \bigwedge_{i \leq 3, p \in \{r,w,b\}} (p_i \leftrightarrow K_i p_i)$$

Now, (3) can be made stronger by replacing each occurrence of $\exists p_i(\dots)$ by $\exists p_i(\mathsf{deal} \wedge \mathsf{koc} \wedge \dots)$. Note that each agent is also aware that the others know something that they don't know: which is for instance expressed by the following *de dicto* property:

$$(\mathcal{M}, \mathsf{rwb}) \models K_2 \exists w_1 (K_1 w_1 \wedge \neg K_2 w_1) \qquad (4)$$

Let us know bring some dynamics to our story (for details on Dynamic Epistemic Logic, see [van Ditmarsch *et al.*, 2007]). Suppose player 1 truthfully announces publicly that he does not own the white card. Such an announcement leads to the updated model $\mathcal{N}$ of Fig. 1, which is obtained from $\mathcal{M}$ by simply removing all states in which $w_1$ holds. Indeed, we have

$$(\mathcal{N}, \mathsf{rwb}) \models \forall r_1 (K_1 r_1 \rightarrow K_3 r_1)$$

That is, 3 knows at least what 1 knows. In fact, 3 knows strictly more than 1, since we have $(\mathcal{N}, \mathsf{rwb}) \models K_3 b_3 \wedge \neg K_1 b_3$. Moreover, not only does player 3 know more than player 1, but player 2 knows *that* as well:

$$\begin{aligned}(\mathcal{N}, \mathsf{rwb}) \quad \models \quad & K_2 \forall r_1 (K_1 r_1 \rightarrow K_3 r_1) \wedge \\ & K_2 \exists b_3 (K_3 b_3 \wedge \neg K_1 b_3) \wedge \\ & \exists b_3 K_3 (K_3 b_3 \wedge \neg K_1 b_3) \qquad (5)\end{aligned}$$

Note how (5) not only expresses that player 2 knows that 3 knows everything that 1 knows, but also that 2 knows *de dicto* that 3 knows more than 1, while 3 knows *de re* that this is the case. ■

We now show a preliminary result used in the completeness proof: an EQBL formula $\phi$ is valid iff it is valid in the class of frames where the operator $C$ is the *universal modality*, defined in every such frame $\mathcal{F}$ as $(\mathcal{F}, V, w) \models C\psi$ iff for all $w' \in W$, $(\mathcal{F}, V, w') \models \psi$. First of all, we define the submodel generated by a world $w$ as follows.

**Definition 5 (Submodel)** *Given a model $\mathcal{M} = \langle W, D, R, V\rangle$ and a world $w \in W$, the* submodel *generated by $w$ is the model $\mathcal{M}_w = \langle W_w, D_w, R_w, V_w\rangle$ s.t.*

- *$W_w$ is the set of worlds reachable from $w$, i.e., $W_w = (\bigcup_{a\in Ag} R_a)^+(w)$;*
- *$D_w = \{U_w \subseteq W_w \mid U_w = U \cap W_w$ for some $U \in D\}$;*
- *for every $a \in Ag$, $R_{w,a} = R_a \cap W_w^2$;*
- *for every $p \in AP$, $V_w(p) = V(p) \cap W_w$.*

If the frame $\mathcal{F} = \langle W, D, R\rangle$ belongs to $\mathcal{K}_{all}$ (resp. $\mathcal{K}_{bool}$, $\mathcal{K}_{full}$), then also $\mathcal{F}_w = \langle W_w, D_w, R_w\rangle \in \mathcal{K}_{all}$ (resp. $\mathcal{K}_{bool}$, $\mathcal{K}_{full}$). We can now prove the following lemma on the preservation of EQBL formulas on generated submodels.

**Lemma 1** *Let $\mathcal{M} = (\mathcal{F}, V)$ be a Kripke model and $w \in W$. For every $w' \in W_w$ and $\phi \in EQBL$,*

$$(\mathcal{F}, V, w') \models \phi \quad \textit{iff} \quad (\mathcal{F}_w, V_w, w') \models \phi$$

Now let $\mathcal{K}_{univ}$ be the class of *universal* Kripke frames, that is, every world is reachable from any other world according to $(\bigcup_{a\in Ag} R_a)^+$.

**Corollary 2** *For every class $\mathcal{K} \in \{K_{all}, \mathcal{K}_{bool}, \mathcal{K}_{full}\}$, $\mathsf{Th}(\mathcal{K}) = \mathsf{Th}(\mathcal{K} \cap \mathcal{K}_{univ})$*

By Corollary 2 we can assume w.l.o.g. that, as long as we are interested in validity, the common knowledge operator $C$ acts as a universal modality on the set $W$ of possible worlds. This fact will be used later in the completeness proof.

In what follows we analyse the formal properties of EQBL starting with its expressiveness.

## 3 Expressiveness

In this section we analyse the expressiveness of (multimodal) epistemic quantified boolean logic and define a correspondence between EQBL and monadic second-order logic (MSO), which is a language suitable to describe properties of frames. First of all, given a Kripke frame $\mathcal{F} = \langle W, D, R\rangle$ on a set $AP$ of atomic propositions, we consider an MSO alphabet containing binary predicate constants $R_C$ and $R_a$ for every agent index $a \in Ag$, and a unary predicate variable $P$ for every atom $p \in AP$. We use the same symbols for semantical and syntactical elements, as context will disambiguate. Then, MSO formulas $\psi$ are defined in BNF as follows:

$$\psi ::= P(x) \mid R_a(x,y) \mid R_C(x,y) \mid \neg\psi \mid \psi \to \psi \mid \forall x\psi \mid \forall P\psi$$

An *assignment* $\sigma$ is a function associating a world $w \in W$ to every individual variable $x$ and a set $U \in D$ to every predicate variable $P$. For $w \in W$ and $U \in D$, the variants $\sigma^x_w$ and $\sigma^P_U$ are defined similarly to EQBL.

**Definition 6 (Semantics of MSO)** *We define whether a frame $\mathcal{F}$* satisfies *an MSO formula $\varphi$ for an assignment $\sigma$, or $(\mathcal{F}, \sigma) \models \varphi$, as follows (clauses for propositional connectives are omitted):*

| | | |
|---|---|---|
| $(\mathcal{F}, \sigma) \models P(x)$ | *iff* | $\sigma(x) \in \sigma(P)$ |
| $(\mathcal{F}, \sigma) \models R_a(x,y)$ | *iff* | $R_a(\sigma(x), \sigma(y))$ |
| $(\mathcal{F}, \sigma) \models R_C(x,y)$ | *iff* | $(\bigcup_{a\in Ag} R_a)^+(\sigma(x), \sigma(y))$ |
| $(\mathcal{F}, \sigma) \models \forall x\psi$ | *iff* | *for all $w \in W$, $(\mathcal{F}, \sigma^x_w) \models \psi$* |
| $(\mathcal{F}, \sigma) \models \forall P\psi$ | *iff* | *for all $U \in D$, $(\mathcal{F}, \sigma^P_U) \models \psi$* |

We briefly remark that equality (modulo $D$) is definable in MSO as $x = y ::= \forall P(P(x) \leftrightarrow P(y)) \wedge \bigwedge_{i\in Ag\cup\{C\}} \forall z(R_i(x,z) \leftrightarrow R_i(y,z))$. Having introduced the formal machinery, we move on to define the extension to EQBL of the standard translation between modal and first-order logic. Then, $ST$ is the translation between EQBL and MSO defined as follows:

| | | |
|---|---|---|
| $ST_x(p)$ | = | $P(x)$ |
| $ST_x(\neg\phi)$ | = | $\neg ST_x(\phi)$ |
| $ST_x(\phi \to \phi')$ | = | $ST_x(\phi) \to ST_x(\phi')$ |
| $ST_x(K_a\phi)$ | = | $\forall y(R_a(x,y) \to ST_y(\phi))$ |
| $ST_x(C\phi)$ | = | $\forall y(R_C(x,y) \to ST_y(\phi))$ |
| $ST_x(\forall p\phi)$ | = | $\forall P(ST_x(\phi))$ |

Clearly, for every EQBL formula $\phi$, $ST_x(\phi)$ is an MSO formula where $x$ is the only free individual variable. If $\psi$ is a purely propositional modal formula, then $ST_x(\psi)$ is a first-order formula, as obtained via the standard translation.

The following results show that structural properties of frames can be expressed locally through EQBL formulas.

**Lemma 3** *For every model $\mathcal{M} = \langle \mathcal{F}, V\rangle$, world $w$, and formula $\psi$ in EQBL, $(\mathcal{M}, w) \models \psi$ iff $(\mathcal{F}, \sigma) \models ST_x(\psi)$, whenever $\sigma(x) = w$ and $\sigma(P_i) = V(p_i)$.*

As a consequence of Lemma 3 there is a one-to-one correspondence between EQBL formulas and their standard translations in MSO in the following sense: a Kripke frame $\mathcal{F}$ validates the universal closure $\forall\vec{p}\psi$ of an EQBL formula $\psi$ iff the MSO property $\forall x\forall\vec{P}ST_x(\psi)$ holds in $\mathcal{F}$, where $\vec{P}$ are all the unary predicates appearing in $ST_x(\psi)$. We observe that this is not the case in propositional modal logic. For instance, for the McKinsey formula $K_aM_ap \to M_aK_ap$ there is no first-order principle $\alpha$ such that $\alpha$ holds in all and only frames validating the McKinsey formula.

We now compare the expressiveness of EQBL to the extended modal logic LPML introduced in [van Ditmarsch *et al.*, 2012] to formulate local properties of models. Table 1 corresponds to Table 1 in [van Ditmarsch *et al.*, 2012]. Formulas $\theta(\vec{a}, \vec{p})$ belong to propositional epistemic logic, while $\Theta(\vec{a}, x)$ are first-order formulas with at most one free variable $x$. The formulas $\boxdot(\vec{a})$ appear in [van Ditmarsch *et al.*, 2012] precisely to describe local properties of models. The satisfaction of $\boxdot$-formulas in Kripke models is given as: $(\mathcal{F}, V, w) \models \boxdot(\vec{a})$ iff $(\mathcal{F}, \sigma) \models \Theta(\vec{a}, x)$ for $\sigma(x) = w$, that is, property $\Theta$ holds of world $w$. By the following corollary we obtain that, as regards full Kripke models, EQBL is as expressive as LPML.

**Corollary 4** *For every full model $\mathcal{M}$, world $w$, and formulas $\theta(\vec{a}, \vec{p})$ and $\boxdot(\vec{a})$ in Table 1, $(\mathcal{M}, w) \models \forall\vec{p}\theta(\vec{a}, \vec{p})$ iff $(\mathcal{M}, w) \models \boxdot(\vec{a})$.*

From Corollary 4 we conclude that EQBL can deal with all the examples of [van Ditmarsch *et al.*, 2012, Table 1] that LPML deals with. Furthermore, LPML provides a completeness proof for a language in which $\boxdot$ and $\theta_\boxdot$ are connected through some specific axiom and inference rule, and moreover $\theta_\boxdot$ is *locally good for* $\Theta$ [van Ditmarsch *et al.*, 2012, Definition 11]. In other words, for first-order properties $\Theta$ for which there is no corresponding propositional modal formula $\theta$, LPML cannot express $\Theta$ locally. We now show that EQBL

| $\theta(\vec{a},\vec{p})$ | $\Theta(\vec{a},x)$ | $\boxdot(\vec{a})$ |
|---|---|---|
| $K_a p \to K_b p$ | $\forall y(R_b(x,y) \to R_a(x,y))$ | $Sup(a,b)$ |
| $K_c p \to K_a K_b p$ | $\forall y,z(R_a(x,y) \wedge R_b(y,z) \to R_c(x,z))$ | $Trans(a,b,c)$ |
| $K_a(p \vee \neg p)$ | $\exists y R_a(x,y)$ | $Ser(a)$ |
| $K_a p \to p$ | $R_a(x,x)$ | $Refl(a)$ |
| $\neg K_a p \to K_b \neg K_c p$ | $\forall y,z(R_a(x,y) \wedge R_b(x,z) \to R_c(y,z))$ | $Eucl(a,b,c)$ |
| $\neg K_a p \to \neg K_b K_c p$ | $\forall z(R_a(x,z) \to \exists y R_b(x,y) \wedge R_c(y,z))$ | $Dens(a,b,c)$ |
| $(\neg K_a p \wedge \neg K_b q) \to \neg K_c(p \vee q)$ | $\forall y,z((R_a(x,y) \wedge R_b(x,z)) \to (y = z \wedge R_c(x,y))$ | $Func(a,b,c)$ |

Table 1: as in [12, Table 1] $\Theta(\vec{a},x)$ is a property of state $x$, and $\boxdot(\vec{a})$ is a name in the object language such that $\boxdot(\vec{a})$ holds at $w$ iff $\Theta(\vec{a},x)$ holds of $\mathcal{M}$ for $\sigma(x) = w$.

*is* sometimes able to express such $\Theta$. Consider the following EQBL formula:

$$\exists p(M_a p \wedge M_a \neg p) \tag{6}$$

It is not difficult to see that (6) holds on frames where states have at least two successors: a first-order property that is not modally definable [Blackburn *et al.*, 2001], and hence something that cannot be expressed in LPML.

Moreover, there are instances of modal properties expressible in EQBL that are not first-order definable, and hence cannot again be dealt with in LPML *locally*. Consider the following EQBL formula $\tau = \forall p(K_a p \to K_b p) \wedge \forall q(K_b K_a q \to K_b q)$. Then we can prove the following result.

**Lemma 5** *Let $\mathcal{F}$ be a full frame and $R_a^+$ the transitive closure of $R_a$. Then, $(\mathcal{F}, w) \models \tau$ iff $R_a^+(w) \subseteq R_b(w)$.*

Since being the transitive closure is not first-order definable (even though it is modally definable), this shows that EQBL is strictly more expressive than the language of [van Ditmarsch *et al.*, 2012], as the latter cannot express $\tau$ locally.

To conclude our comparison between EQBL and LPML, we observe that the $\boxdot$ operators act in fact like linguistic black boxes, bringing the metatheory language of first-order logic into the object language of modal logic. On the other hand, EQBL is more transparent, as everything is done in the object language. In addition, for the first-order conditions in [van Ditmarsch *et al.*, 2012] there must always be a suitable modal counterpart. Indeed, the axioms $\mathbf{Ax}_{\boxdot} : \boxdot(\vec{a}) \to \theta_{\boxdot}(\vec{a},\vec{p})$ in [van Ditmarsch *et al.*, 2012] make sense only as long as there is a propositional modal formula $\theta_{\boxdot}$ locally good for $\boxdot$, and this is not always the case. In what follows we will show that none of the above has to be assumed to axiomatise EQBL.

Finally, we briefly state one more result showing that EQBL is strictly more expressive than propositional modal logic. In the following lemma we consider Kripke frames and models where each $R_a$ is simply a binary relation.

**Lemma 6** *Let $\mathcal{F}$ be a full Kripke frame, $\mathcal{F} \models \exists p(K_a p \wedge \neg p)$ iff $\mathcal{F}$ is irreflexive.*

Since irreflexivity is not expressible in standard modal logic [Blackburn *et al.*, 2001], Lemma 6 illustrates the well-known fact that EQBL is more expressive than modal logic.

## 4 Completeness

In this section we present axiomatisations for the sets of validities on the classes $\mathcal{K}_{all}$, $\mathcal{K}_{bool}$, and $\mathcal{K}_{full}$ of Kripke frames built on sets $Ag$ of agent indexes and $AP$ of atomic propositions. We first present a logic for $\mathcal{K}_{all}$, where $\Box$ is a placeholder for any modal operators $C$ and $K_a$ for $a \in Ag$.

**Definition 7** *Axioms and inference rules of logic $\mathbf{K}_{all}$*

| | |
|---|---|
| **Prop** | *all instances of propositional tautologies* |
| **K** | $\Box(\phi \to \psi) \to (\Box\phi \to \Box\psi)$ |
| **T** | $\Box\phi \to \phi$ |
| **4** | $\Box\phi \to \Box\Box\phi$ |
| **5** | $\neg\Box\phi \to \Box\neg\Box\phi$ |
| **C1** | $C\phi \to K_a(\phi \wedge C\phi)$ |
| **C2** | *from* $\phi \to \bigwedge_{a \in Ag} K_a(\psi \wedge \phi)$ *infer* $\phi \to C\psi$ |
| $\mathbf{Ex}_{all}$ | $\forall p\phi \to \phi[p/q]$ |
| **BF** | $\forall p\Box\phi \to \Box\forall p\phi$ |
| **MP** | *from* $\phi \to \psi$ *and* $\phi$ *infer* $\psi$ |
| **Nec** | *from* $\phi$ *infer* $\Box\phi$ |
| **Gen** | *from* $\phi \to \psi$ *infer* $\phi \to \forall p\psi$, *for* $p$ *not free in* $\phi$ |

The logic $\mathbf{K}_{all}$ extends the standard multi-modal epistemic logic $S5_n^C$, including common knowledge, with axioms for quantification. It also includes the Barcan formula **BF**. The logic $\mathbf{K}_{bool}$ extends $\mathbf{K}_{all}$ through axiom

$\mathbf{Ex}_{bool}$: $\forall p\phi \to \phi[p/\psi]$

where $\psi$ is any boolean formula. Finally, $\mathbf{K}_{full}$ extends $\mathbf{K}_{bool}$ through axiom

$\mathbf{Ex}_{full}$: $\forall p\phi \to \phi[p/\psi]$

where $\psi$ is any EQBL formula, as well as axiom **At**:

$$\exists p(p \wedge \forall q(q \to C(p \to q)) \wedge \bigwedge_{a \in Ag} \forall r(K_a r \to C(p \to K_a r)))$$

Intuitively, the first two conjuncts of axiom **At** express that the interpretation of $p$ is an atom in $D$ (i.e., a minimal, non-empty element according to order $\subseteq$ on $D$). Moreover, by the third conjunct, for every $a \in Ag$, the worlds in the interpretation of $p$ are all connected by each $R_a$, i.e, the interpretation of $p$ is a *clique* for every $R_a$. This means that all worlds in the interpretation of $p$ are actually indistinguishable. The notions of *proof* and *theoremhood* $\vdash$ are defined as standard.

Mono-modal versions of $\mathbf{Ex}_{bool}$ and $\mathbf{Ex}_{full}$ (without common knowledge) were considered in [Fine, 1970]. Hereafter, we extend these results to a multi-agent epistemic setting, including common knowledge, and prove generalised completeness in Section 4.1.

We now state the soundness and completeness results for each logic $\mathbf{K}$ w.r.t. the corresponding class $\mathcal{K}$ of Kripke frames, starting with the former.

**Theorem 7 (Soundness)** *For every $l \in \{all, bool, full\}$, for every EQBL formula $\phi$, $\vdash_{\mathbf{K}_l} \phi$ implies $\mathcal{K}_l \models \phi$.*

As regards completeness, we show that if a formula is consistent, then we can construct a relevant model that satisfies it. For logics $\mathbf{K}_{all}$ and $\mathbf{K}_{bool}$ we need models whose underlying frame is any frame and a boolean algebra respectively. As regards $\mathbf{K}_{full}$ we need to introduce some more definitions. First of all, a frame $\mathcal{F}$ is *atomic* iff for every $w \in W$ there is an atom $U \in D$ (i.e., a minimal, non-empty $U \in D$, according to order $\subseteq$) such that $w \in U$. Moreover, the atom $U$ has to be a clique according to every $R_a$, i.e., for every $w, w' \in U$, $R_a(w, w')$. Further, $\mathcal{F}$ is *complete* iff the domain $D$ is closed under infinite unions and intersections. Notice that we use 'complete' with two different meanings: *(i)* semantical completeness of a logic, and *(ii)* algebraic completeness. The context will disambiguate. We use $at$ and $com$ as subscripts to designate the respective classes of frames. Clearly, every full frame is boolean, atomic and complete. Hence, $\mathsf{Th}(\mathcal{K}_{bool,at,com}) \subseteq \mathsf{Th}(\mathcal{K}_{full})$. The converse follows from the next well-known algebraic result, which is a consequence of Stone's representation theorem.

**Theorem 8 ([Givant and Halmos, 2009])** *Every complete atomic Boolean algebra is isomorphic to the powerset of some set.*

By Theorem 8 we can prove the following lemma.

**Lemma 9** $\mathsf{Th}(\mathcal{K}_{full}) \subseteq \mathsf{Th}(\mathcal{K}_{bool,at,com})$.

In [Fine, 1970] Fine remarks that the language of modal quantified boolean logic is not rich enough to express completeness. We prove that this is the case for EQBL as well.

**Lemma 10** $\mathsf{Th}(\mathcal{K}_{bool,at,com}) = \mathsf{Th}(\mathcal{K}_{bool,at})$

By combining Lemma 9 and 10 we obtain that $\mathsf{Th}(\mathcal{K}_{full}) = \mathsf{Th}(\mathcal{K}_{bool,at})$. Thus, for our purposes it is sufficient to prove completeness of $\mathbf{K}_{full}$ w.r.t. $\mathcal{K}_{bool,at}$.

**Theorem 11 (Completeness)** *For every $l \in \{all, bool, full\}$, for every EQBL formula $\phi$, $\mathcal{K}_l \models \phi$ implies $\vdash_{\mathbf{K}_l} \phi$.*

If $\nvdash_{\mathbf{K}_l} \phi$ then we construct a Kripke model $\mathcal{M} = \langle \mathcal{F}, V \rangle$, the *canonical model* for $\phi$, that satisfies $\neg\phi$. Moreover, $\mathcal{F}$ is shown to belong to the relevant class of Kripke frames. This implies that $\mathcal{K}_l \not\models \phi$. Hereafter we discuss the case for $l = full$. We omit the subscript $l$ whenever clear by the context.

As regards the logic $\mathbf{K}_{full}$ we remarked above that it is sufficient to prove completeness w.r.t. $\mathcal{K}_{bool,at}$. To do so, we observe that axiom **At** ensures that the canonical model contains an atom $U_w$ for every state $w \in W$, and that all worlds in these atoms are related by each $R_a$, provided that the accessibility relation $R_C = \{(w', w'') \mid \{\psi \mid C\psi \in w'\} \subseteq w''\}$ is the universal relation on $W$. To enforce the latter fact, we restrict $W$ to the set of states reachable from state $v \supseteq \{\neg\phi\}$, for a consistent $\phi$, through the relation $(\bigcup_{a \in Ag} R_a)^+$. By doing so, we guarantee that $D$ is indeed an atomic boolean algebra, that is, $\mathcal{F} \in \mathcal{K}_{bool,at}$.

We summarise the soundness and completeness results for our logics w.r.t. the relevant classes of frames.

**Theorem 12 (Soundness and Completeness)** *For every $l \in \{all, bool, full\}$, the logic $\mathbf{K}_l$ is sound and complete w.r.t. the class $\mathcal{K}_l$ of Kripke frames.*

### 4.1 Generalised Completeness

One nice feature of EQBL is that we can extend the completeness results in the previous section by considering extra axioms expressing properties of frames. Specifically, let $\mathbf{K}_l$ be any of the axiomatisations above, for $l \in \{all, bool, full\}$. Then, if we extend $\mathbf{K}_l$ with the universal closure $\forall \vec{p}\psi$ of an EQBL formula $\psi$, the resulting calculus $\mathbf{K}_l + \forall \vec{p}\psi$ is sound and complete w.r.t. the class of frames satisfying the MSO condition $\forall x \forall \vec{P} ST_x(\psi)$.

**Theorem 13** *Let $\psi$ be an EQBL formula, then the logic $\mathbf{K}_l + \forall \vec{p}\psi$ is sound and complete w.r.t. the class of $\mathcal{K}_l$ frames satisfying $\forall x \forall \vec{P} ST_x(\psi)$.*

By the result above we immediately obtain that for every formula $\theta(\vec{a}, \vec{p})$ in Table 1, $\mathbf{K}_l + \forall p\theta(\vec{a}, \vec{p})$ is sound and complete w.r.t. the class of Kripke frames satisfying $\forall x \forall \vec{P} \Theta(\vec{a}, x)$. More generally, there is a one-to-one correspondence between an EQBL axiom $\forall \vec{p}\theta$ and the MSO condition $\forall x \forall \vec{P} ST_x(\theta)$ on the corresponding class of sound and complete frames. Notice that this is not the case in propositional modal logic.

## 5 Model Checking EQBL

This section hinges on the model checking problem for EQBL, defined as follows.

**Definition 8 (Model Checking)** *Given an EQBL formula $\phi$ and a finite model $\mathcal{M}$, determine whether $\mathcal{M} \models \phi$.*

Then, we are able to prove the following complexity result.

**Theorem 14 (Model Checking)** *The model checking problem for EQBL is PSPACE-complete.*

**Algorithm 1** Computation of the satisfaction set $[[\phi]]_{\mathcal{M}}$

**switch** $(\phi)$:
**case** $\bot$:
  **return** $\emptyset$;
**case** $p$:
  **return** $V(p)$;
**case** $\neg\psi$:
  **return** $W \setminus [[\psi]]_{\mathcal{M}}$;
**case** $\psi \wedge \psi'$:
  **return** $[[\psi]]_{\mathcal{M}} \cap [[\psi']]_{\mathcal{M}}$;
**case** $K_a\psi$:
  **return** $\{w \in W \mid R_a(w) \subseteq [[\psi]]_{\mathcal{M}}\}$;
**case** $C\psi$:
  **return** $\{w \in W \mid (\bigcup_{a \in Ag} R_a)^*(w) \subseteq [[\psi]]_{\mathcal{M}}\}$;
**case** $\forall p\psi$:
  **return** $\bigcap_{U \in D}\{[[\psi]]_{\mathcal{M}'}) \mid \mathcal{M}' = \langle \mathcal{F}, V_U^p \rangle\}$;

In particular, we can prove that model checking EQBL is in PSPACE by using Algorithm 1 to compute the satisfaction set $[[\phi]]_{\mathcal{M}} = \{w \in W \mid (\mathcal{M}, w) \models \phi\}$, and then check whether $[[\phi]]_{\mathcal{M}} = W$. Algorithm 1 takes polynomial space in the size of the formula and exponential time in the size of the model.

As a result, the model checking problem for EQBL is no more computationally complex than the corresponding problem for quantified boolean logic. Thus, the enhanced expressiveness comes at no extra computational cost, when compared with quantified boolean logic. With respect to propositional epistemic logic, the complexity increases from PTIME. However, this is expected given the expressive power of propositional quantification.

## 6 Conclusions

We analysed the formal properties of multi-modal EQBL. Specifically, we provided soundness and completeness results for several classes of multi-agent Kripke frames, according to various assumptions on the domain $D$ of propositions. In particular, we showed that there is a one-to-one correspondence between an EQBL axiom $\forall \vec{p}\theta$ and the MSO condition $\forall x \forall \vec{P} ST_x(\theta)$ satisfied by the corresponding class of sound and complete frames. None of this is available in propositional modal logic. Further, we discussed the model checking problem for EQBL and showed that computationally it is no harder than the corresponding question for quantified boolean logic. In future work we plan to investigate precisely the expressive power of EQBL by exploring standard model-theoretic techniques, including (bi)simulation and unravelling. Even though some technical results are available [Kaminski and Tiomkin, 1996; ten Cate, 2006], the area is much less developed in comparison to plain propositional modal logic.

**Related Work.** The main inspiration for the present contribution comes from a series of papers on LPML, an extension of propositional modal logics to express local properties [van Ditmarsch *et al.*, 2009; 2011; 2012]. Differently from these works we do not consider *ad hoc* languages, instead we adopt the framework of modal quantified boolean logic (MQBL). MQBL has been first considered in the seminal works by Bull and Fine [Bull, 1969; Fine, 1970]. The latter proved completeness and (un)decidability results for various classes of Kripke frames. Here we extended such results by explictly considering the multi-agent, epistemic interpretation of modalities. In particular, differently from [Fine, 1970], the introduction of common knowledge is essential to mimick an universal modality that, together with axiom **At**, enforces canonical models that are atomic. More recently, MQBL and its formal properties have been considered in the literature on modal logic, even though its conceptual complexity has hindered its development. The logic is also known as second-order propositional modal logic (SOPML) for the relation with second-order logic detailed in Section 3. In [Kaminski and Tiomkin, 1996] the authors proved that the expressive power of SOPML (for modalities weaker than 4.2) is the same as MSO. While [ten Cate, 2006] provides SOPML with analogues of the van Benthem-Rosen theorem and the Goldblatt-Thomason theorem. We remark that both works hinge only on the specific class of *full* Kripke frames.

Further, there is also a rich literature on combinations of quantification and epistemic logic [Belardinelli and Lomuscio, 2009; 2012; Corsi and Orlandelli, 2013]. However, the works in this area typically consider first-order quantification and knowledge of relational facts; which is not the case in the present setting.

The present contribution is also related to general Kripke frames [Blackburn *et al.*, 2001; Mares and Goldblatt, 2006], as we also assume some algebraic structure on the set $D$ of propositions, notably for the classes of boolean and full frames. However, propositional quantification makes our framework strictly more expressive than general frames. Also, differently from the contributions above, the main motivation for the present paper comes from the application to artificial intelligence, specifically the representation of agent knowledge in multi-agent contexts. In this respect, there has not been any previous work on EQBL to our knowledge.

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