MODULI SPACES OF HIGHER SPIN KLEIN SURFACES

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Abstract. We study the connected components of the space of higher spin bundles on hyperbolic Klein surfaces. A Klein surface is a generalisation of a Riemann surface to the case of non-orientable surfaces or surfaces with boundary. The category of Klein surfaces is isomorphic to the category of real algebraic curves. An m-spin bundle on a Klein surface is a complex line bundle whose m-th tensor power is the cotangent bundle. The spaces of higher spin bundles on Klein surfaces are important because of their applications in singularity theory and real algebraic geometry, in particular for the study of real forms of Gorenstein quasi-homogeneous surface singularities. In this paper we describe all connected components of the space of higher spin bundles on hyperbolic Klein surfaces in terms of their topological invariants and prove that any connected component is homeomorphic to a quotient of $\mathbb{R}^d$ by a discrete group.

Contents

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1. Introduction

A complex line bundle $e : L \to P$ on a Riemann surface P, denoted (e, P), is an *m-spin bundle* for an integer $m > 1$ if its m-th tensor power $e^{\otimes m} : L^{\otimes m} \to P$ is isomorphic to the cotangent bundle of P. The classical 2-spin structures on compact Riemann surfaces were introduced by Riemann [R] (as theta characteristics) and play an important role in mathematics. Their modern interpretation as complex line bundles and classification was given by Atiyah [Ati] and Mumford [Mum], who showed that 2-spin bundles have a topological invariant $\delta = \delta(e, P)$ in $\{0, 1\}$, the *Arf invariant*, which is determined by the parity of the dimension of the space of sections of the bundle. Moreover, the space $S^2_{g,\delta}$ of 2-spin bundles on Riemann surfaces of genus g with Arf invariant δ, i.e. the space of such pairs (e, P), is homeomorphic to a quotient of $\mathbb{R}^{6g-6}$ by a discrete group of autohomeomorphisms, see [Nat89a, Nat04].

The study of spaces of m-spin bundles for arbitrary m started more recently because of the remarkable connections between the compactified moduli space of m-spin bundles and the theory of integrable systems [Wit], and because of their applications in singularity theory [Dol, NP11, NP13]. It was shown that for odd m the space of m-spin bundles is connected, while for even m (and $g > 1$) there are two connected components, distinguished by an invariant which generalises the Arf invariant [Jar]. In all cases each connected components of the space of m-spin bundles on Riemann surfaces of genus g is homeomorphic to a quotient of $\mathbb{R}^{6g-6}$ by a discrete group of autohomeomorphisms, see [NP05, NP09]. The homology of these moduli spaces was studied further by T. Jarvis, D. Zvonkine, A. Chiodo, C. Faber, S. Shadrin, O. Randal-Williams, R. Pandharipande, A. Pixton, L. Spitz and others in [Jar, ChZ, FSZ, RW1, RW2, PPZ, SSZ].

The aim of this paper is to determine the topological structure of the space of m-spin bundles on hyperbolic Klein surfaces. A *Klein surface* is a non-orientable topological surface with a maximal atlas whose transition maps are *dianalytic*, i.e. either holomorphic or anti-holomorphic, see [AG]. Klein surfaces can be described as quotients $P/\langle\tau\rangle$, where P is a compact Riemann surface and $\tau : P \to P$ is an anti-holomorphic involution on P. The category of such pairs is isomorphic to the category of Klein surfaces via $(P, \tau) \mapsto P/\langle\tau\rangle$. Under this correspondence the fixed points of τ correspond to the boundary points of the Klein surface. In this paper a Klein surface will be understood as an isomorphism class of such pairs (P, τ). We will only consider connected compact Klein surfaces. The category of connected compact Klein surfaces is isomorphic to the category of irreducible real algebraic curves (see [AG]).

The boundary of the surface $P/\langle\tau\rangle$, if not empty, decomposes into k pairwise disjoint simple closed smooth curves. These closed curves are called *ovals* and correspond to connected components of the set of fixed points P^τ of the involution $\tau : P \to P$. On the real algebraic curve they correspond to connected components of the set of real points.

The *topological type* of the surface $P/\langle\tau\rangle$ is determined by the triple (g, k, ε), where g is the genus of P, k is the number of connected components of the boundary of $P/\langle\tau\rangle$ and $\varepsilon \in \{0, 1\}$ with $\varepsilon = 1$ if the surface is orientable and $\varepsilon = 0$ otherwise. The following conditions are satisfied: $1 \leqslant k \leqslant g+1$ and $k \equiv g+1 \bmod 2$ in the case

$\varepsilon = 1$ and $0 \leqslant k \leqslant g$ in the case $\varepsilon = 0$. These classification results were obtained by Weichold [Wei]. It was shown that the topological type completely determines the connected component of the space of Klein surfaces. Moreover, the space $M_{g,k,\varepsilon}$ of Klein surfaces of topological type (g, k, ε) is homeomorphic to the quotient of $\mathbb{R}^{3g-3}$ by a discrete subgroup of automorphisms. In addition to the invariants (g, k, ε), it is useful to consider an invariant that we will call the geometric genus of (P, τ). In the case $\varepsilon = 1$ the geometric genus $(g+1-k)/2$ is the number of handles that need to be attached to a sphere with holes to obtain a surface homeomorphic to $P/\langle\tau\rangle$. In the case $\varepsilon = 0$ the geometric genus $[(g-k)/2]$ is the half of the number of Möbius bands that need to be attached to a sphere with holes to obtain a surface homeomorphic to $P/\langle\tau\rangle$.

An *m-spin bundle on a Klein surface* (P, τ) is a pair (e, β), where $e : L \to P$ is an m-spin bundle on P and $\beta : L \to L$ is an anti-holomorphic involution on L such that $e \circ \beta = \tau \circ e$, i.e. the following diagram commutes:

$$\begin{array}{ccc} L & \xrightarrow{e} & P \\ \beta\big\downarrow & & \big\downarrow\tau \\ L & \xrightarrow{e} & P \end{array}$$

The spaces of higher spin bundles on Klein surfaces are important because of their applications in singularity theory and real algebraic geometry. We are particularly interested in applications to the classification of real forms of complex singularities. The results of this paper combined with the characterisation of Gorenstein quasi-homogeneous surfaces singularities in terms of higher spin bundles on Riemann orbifolds in [Dol, NP11, NP13] should allow a classification of real forms of this class of singularities. The first results in this direction were obtained in [Ril]. Other classes of complex singularities for which real forms have been studied are simple singularities and cusp singularities [AC, W1, W2]. Other important applications are the connections between 2-spin bundles on Klein surfaces and Abelian Yang-Mills theory on real tori [OT] and possible generalisations to m-spin bundles.

In this paper we determine the connected components of the space of m-spin bundles on Klein surfaces, i.e. equivalence classes of m-spin bundles on Klein surfaces up to topological equivalence as defined in section 3.6. We find the topological invariants that determine such an equivalence class and determine all possible values of these invariants. We also show that every equivalence class is a connected set homeomorphic to a quotient of $\mathbb{R}^n$ by a discrete group, where the dimension n and the group depend on the class. The special case $m = 2$ was studied in [Nat89b, Nat90, Nat99, Nat04].

While 2-spin bundles on a Riemann surface P can be described in terms of quadratic forms on $H_1(P, \mathbb{Z}/2\mathbb{Z})$, for higher spin bundles the situation is more complex. The main innovation of our method is to assign to every m-spin bundle on a Klein surface (P, τ) a function on the set of simple closed curves in P with values in $\mathbb{Z}/m\mathbb{Z}$, called real m-Arf function [NP16]. Thus the problem of topological classification of m-spin bundles on Klein surfaces is reduced to topological classification of real m-Arf functions. We introduce a complete set of topological invariants of real m-Arf functions. We then construct for any real m-Arf function σ a canonical generating set, i.e. a generating set of the fundamental group of P on which σ assumes values determined by the topological invariants.

We will now explain the results in more detail. Let (P,τ) be a Klein surface of type (g,k,ε). In this paper we will consider hyperbolic Klein surfaces (P,τ), i.e. we assume that the underlying Riemann surface P is hyperbolic, $g\geqslant 2$. We will also assume that the geometric genus of (P,τ) is positive, i.e. $k\leqslant g-2$ if $\varepsilon=0$ and $k\leqslant g-1$ if $\varepsilon=1$.

We show that if m is odd and there exists an m-spin bundle on the Klein surface (P,τ) then $g\equiv 1 \bmod m$. Moreover, assuming that m is odd and $g\equiv 1 \bmod m$, the space of m-spin bundles on Klein surfaces of type (g,k,ε) is not empty and is connected.

Now let m be even. Consider an m-spin bundle e on the Klein surface (P,τ). A restriction of the bundle e gives a bundle on the ovals. Let K_0 and K_1 be the sets of ovals on which the bundle is trivial and non-trivial respectively. We show that $|K_1|\cdot m/2\equiv 1-g \bmod m$.

If m is even and $\varepsilon=0$, the Arf invariant δ of the bundle e and the cardinalities $k_i=|K_i|$ for $i=0,1$ determine a (non-empty) connected component of the space of m-spin bundles on Klein surfaces of type $(g,k_0+k_1,0)$ if and only if

$$k_1\cdot\frac{m}{2}\equiv 1-g \bmod m.$$

If m is even and $\varepsilon=1$, the bundle e determines a decomposition of the set of ovals in two disjoint sets, K^0 and K^1, of *similar* ovals (for details see section 3.1). The bundle e induces m-spin bundles on connected components of $P\backslash P^\tau$. The Arf invariant $\tilde\delta$ of these induced bundles does not depend on the choice of the connected component of $P\backslash P^\tau$. This invariant $\tilde\delta$ and the cardinalities $k_i^j=|K_i\cap K^j|$ for $i,j\in\{0,1\}$ determine a connected component of the space of m-spin bundles on Klein surfaces of type $(g,k_0^0+k_0^1+k_1^0+k_1^1,1)$ if and only if

- If $g>k+1$ and $k_0^0+k_0^1\neq 0$ then $\tilde\delta=0$.
- If $g>k+1$ and $m\equiv 0 \bmod 4$ then $\tilde\delta=0$.
- If $g=k+1$ and $k_0^0+k_0^1\neq 0$ then $\tilde\delta=1$.
- If $g=k+1$ and $m\equiv 0 \bmod 4$ then $\tilde\delta=1$.
- If $g=k+1$ and $k_0^0+k_0^1=0$ and $m\equiv 2 \bmod 4$ then $\tilde\delta\in\{1,2\}$.
- $(k_1^0+k_1^1)\cdot m/2\equiv 1-g \bmod m$.

We also show that every connected component of the space of m-spin bundles on Klein surfaces of genus g is homeomorphic to a quotient of $\mathbb{R}^{3g-3}$ by a discrete subgroup of automorphisms which depends on the component (see Theorem 4.3).

The paper is organised as follows:

In section 2 we recall the classification of real m-Arf functions from [NP16]. We determine the topological invariants of real m-Arf functions in section 3. In section 4 we use these topological invariants to describe connected components of the space of m-spin bundles on Klein surfaces.

2. Higher Spin Structures on Klein Surfaces

2.1. **Higher Spin Structures.** A Riemann surface P of genus $g \geqslant 2$ can be described as a quotient $P = \mathbb{H}/\Gamma$ of the hyperbolic plane $\mathbb{H}$ by the action of a Fuchsian group Γ.

Definition 2.1. Let P be a compact Riemann surface. A line bundle $e : L \to P$ is an m-spin bundle (of rank 1) if the m-fold tensor power $L \otimes \cdots \otimes L$ coincides with the cotangent bundle of P. (For $m = 2$ we obtain the classical notion of a spin bundle.)

Higher spin bundles on a Riemann surface P can be described by means of associated higher Arf functions, certain functions on the space of homotopy classes of simple closed curves on P with values in $\mathbb{Z}/m\mathbb{Z}$ described by simple geometric properties.

Definition 2.2. Let Γ be a Fuchsian group that consists of hyperbolic elements. Let the corresponding Riemann surface $P = \mathbb{H}/\Gamma$ be a compact surface with finitely many holes. Let $\pi_1(P) = \pi_1(P, p)$ be the fundamental group of P with respect to a point p. We denote by $\pi_1^0(P)$ the set of all non-trivial elements of $\pi_1(P, p)$ that can be represented by simple closed curves. An *m-Arf function* is a function

$$\sigma : \pi_1^0(P) \to \mathbb{Z}/m\mathbb{Z}$$

satisfying the following conditions

1. $\sigma(bab^{-1}) = \sigma(a)$ for any elements $a, b \in \pi_1^0(P)$,
2. $\sigma(a^{-1}) = -\sigma(a)$ for any element $a \in \pi_1^0(P)$,
3. $\sigma(ab) = \sigma(a) + \sigma(b)$ for any elements a and b which can be represented by a pair of simple closed curves in P intersecting at exactly one point p with intersection number not equal to zero.
4. $\sigma(ab) = \sigma(a) + \sigma(b) - 1$ for any elements $a, b \in \pi_1^0(P)$ such that the element ab is in $\pi_1^0(P)$ and the elements a and b can be represented by a pair of simple closed curves in P intersecting at exactly one point p with intersection number equal to zero and placed in a neighbourhood of the point p as shown in Figure 1.

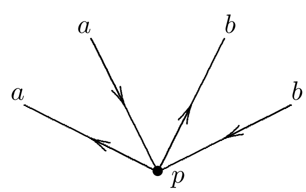

Figure 1: $\sigma(ab) = \sigma(a) + \sigma(b) - 1$

Remark. In the case $m = 2$ there is a 1-1-correspondence between the 2-Arf functions in the sense of Definition 2.2 and Arf functions in the sense of [Nat04], Chapter 1, Section 7 and [Nat91]. Namely, a function $\sigma : \pi_1^0(P) \to \mathbb{Z}/2\mathbb{Z}$ is a 2-Arf function if and only if $\omega = 1 - \sigma$ is an Arf function in the sense of [Nat04].

Higher Arf functions were introduced in [NP05, NP09], where the following result was shown:

Theorem 2.1. *There is a 1-1-correspondence between the m-spin structures and m-Arf functions on a given Riemann surface.*

We will denote an m-spin structure and its corresponding m-Arf function by the same letter.

We will use the following description of generating sets of Fuchsian groups (see [NP05, NP09, Z]):

Definition 2.3. Consider a compact Riemann surface P of genus g with n holes and the corresponding Fuchsian group Γ such that $P = \mathbb{H}/\Gamma$. Let $p \in P$. A *standard generating set* of the fundamental group $\pi_1(P, p) \cong \Gamma$ is a set

$$(a_1, b_1, \dots, a_g, b_g, c_1, \dots, c_n)$$

of elements in $\pi_1(P, p)$ that can be represented by simple closed curves on P based at p

$$(\tilde{a}_1, \tilde{b}_1, \dots, \tilde{a}_g, \tilde{b}_g, \tilde{c}_1, \dots, \tilde{c}_n)$$

with the following properties:

- The curve $\tilde{c}_i$ encloses a hole in P for $i = 1, \dots, n$.
- Any two curves only intersect at the point p.
- A neighbourhood of the point p with the curves is homeomorphic to the one shown in Figure 2.

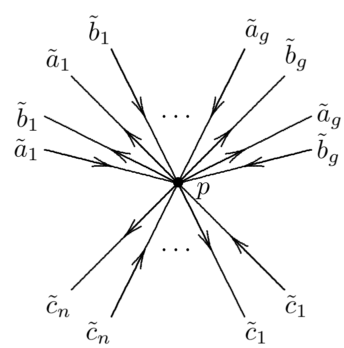

Figure 2: Standard generating set

- The system of curves cuts the surface P into $n+1$ connected components of which n are homeomorphic to an annulus and one is homeomorphic to a disc and has boundary
$$\tilde{a}_1\tilde{b}_1\tilde{a}_1^{-1}\tilde{b}_1^{-1}\dots\tilde{a}_g\tilde{b}_g\tilde{a}_g^{-1}\tilde{b}_g^{-1}\tilde{c}_1\dots\tilde{c}_n.$$

We recall the topological invariants of m-Arf functions as described in [NP05, NP09].

Definition 2.4. Let $\sigma : \pi_1^0(P) \to \mathbb{Z}/m\mathbb{Z}$ be an m-Arf function. For $g \geqslant 2$ and even m we define the *Arf invariant* $\delta = \delta(P, \sigma)$ as $\delta = 0$ if there is a standard generating set

$$\{a_i, b_i\ (i = 1, \dots, g), c_i\ (i = 1, \dots, n)\}$$

of the fundamental group $\pi_1(P)$ such that

$$\sum_{i=1}^{g}(1-\sigma(a_i))(1-\sigma(b_i)) \equiv 0 \bmod 2$$

and as $\delta = 1$ otherwise. For $g \geqslant 2$ and odd m we set $\delta = 0$. For $g \geqslant 2$ we say that an m-Arf function is *even* if $\delta = 0$ and *odd* if $\delta = 1$. For $g = 1$ we define the *Arf invariant* $\delta = \delta(P, \sigma)$ as

$$\delta = \gcd(m, \sigma(a_1), \sigma(b_1), \sigma(c_1) + 1, \ldots, \sigma(c_n) + 1),$$

where

$$\{a_1, b_1, c_i \ (i = 1, \ldots, n)\}$$

is a standard generating set of the fundamental group $\pi_1(P)$.

Remark. The Arf invariant δ is a topological invariant of an m-Arf function σ, i.e. it does not change under self-homeomorphisms of the Riemann surface P.

The following are special cases of the earlier classification results in [NP09], compare with Theorems 4.3 and 4.4 in [NP16].

Theorem 2.2. *Let P be a hyperbolic Riemann surface of genus g with n holes. Let $c_1, \ldots, c_n$ be closed curves around the holes as in Definition 2.3. Let σ be an m-Arf function on P with the Arf invariant δ. Then*

(a) If $g \geqslant 2$ and $m \equiv 1 \bmod 2$ then $\delta = 0$.
(b) If $g \geqslant 2$ and $m \equiv 0 \bmod 2$ and $\sigma(c_i) \equiv 0 \bmod 2$ for some i then $\delta = 0$.
(c) If $g = 1$ then δ is a divisor of $\gcd(m, \sigma(c_1) + 1, \ldots, \sigma(c_n) + 1)$.
(d) $\sigma(c_1) + \cdots + \sigma(c_n) \equiv (2 - 2g) - n \bmod m$.

Theorem 2.3. *Let P be a hyperbolic Riemann surface of genus g with n holes. Then for any standard generating set*

$$(a_1, b_1, \ldots, a_g, b_g, c_1, \ldots, c_n)$$

of $\pi_1(P)$ and any choice of values $\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g, \gamma_1, \ldots, \gamma_n$ in $\mathbb{Z}/m\mathbb{Z}$ with

$$\gamma_1 + \cdots + \gamma_n \equiv (2 - 2g) - n \bmod m,$$

there exists an m-Arf function σ on P such that $\sigma(a_i) = \alpha_i$, $\sigma(b_i) = \beta_i$ for $i = 1, \ldots, g$ and $\sigma(c_i) = \gamma_i$ if $i = 1, \ldots, n$. The Arf invariant δ of this m-Arf function σ satisfies the following conditions:

(a) If $g \geqslant 2$ and $m \equiv 1 \bmod 2$ then $\delta = 0$.
(b) If $g \geqslant 2$ and $m \equiv 0 \bmod 2$ and $\gamma_i \equiv 0 \bmod 2$ for some i then $\delta = 0$.
(c) If $g \geqslant 2$ and $m \equiv 0 \bmod 2$ and $\gamma_1 \equiv \cdots \equiv \gamma_n \equiv 1 \bmod 2$ then $\delta \in \{0, 1\}$ and

$$\delta \equiv \sum_{i=1}^{g}(1 - \alpha_i)(1 - \beta_i) \bmod 2.$$

(d) If $g = 1$ then $\delta = \gcd(m, \alpha_1, \beta_1, \gamma_1 + 1, \ldots, \gamma_n + 1)$.

2.2. Klein Surfaces.

Definition 2.5. A *Klein surface* is a topological surface with a maximal atlas whose transition maps are *dianalytic*, i.e. either holomorphic or anti-holomorphic. A *homomorphism* between Klein surfaces is a continuous mapping which is dianalytic in local charts.

For more information on Klein surfaces, see [AG, Nat90].

Let us consider pairs (P, τ), where P is a compact Riemann surface and $\tau : P \to P$ is an anti-holomorphic involution on P. For each such pair (P, τ) the quotient $P/\langle\tau\rangle$ is a Klein surface. Each isomorphism class of Klein surfaces contains a surface of the form $P/\langle\tau\rangle$. Moreover, two such quotients $P_1/\langle\tau_1\rangle$ and $P_2/\langle\tau_2\rangle$ are isomorphic as Klein surfaces if and only if there exists a biholomorphic map $\psi : P_1 \to P_2$ such that $\psi \circ \tau_1 = \tau_2 \circ \psi$, in which case we say that the pairs (P_1, τ_1) and (P_2, τ_2) are *isomorphic*. Hence from now on we will consider pairs (P, τ) up to isomorphism instead of Klein surfaces.

The category of such pairs (P, τ) is isomorphic to the category of real algebraic curves (see [AG]), where fixed points of τ (i.e. boundary points of the corresponding Klein surface) correspond to real points of the real algebraic curve.

For example a non-singular plane real algebraic curve given by an equation $F(x, y) = 0$ is the set of real points of such a pair (P, τ), where P is the normalisation and compactification of the surface $\{(x, y) \in \mathbb{C}^2 \mid F(x, y) = 0\}$ and τ is given by the complex conjugation, $\tau(x, y) = (\bar{x}, \bar{y})$.

Definition 2.6. Given two Klein surfaces (P_1, τ_1) and (P_2, τ_2), we say that they are *topologically equivalent* if there exists a homeomorhism $\phi : P_1 \to P_2$ such that $\phi \circ \tau_1 = \tau_2 \circ \phi$.

Let (P, τ) be a Klein surface. We say that (P, τ) is *separating* or *of type I* if the set $P \backslash P^\tau$ is not connected, otherwise we say that it is *non-separating* or *of type II*. The *topological type* of (P, τ) is the triple (g, k, ε), where g is the genus of the Riemann surface P, k is the number of connected components of the fixed point set P^τ of τ, $\varepsilon = 0$ if (P, τ) is non-separating and $\varepsilon = 1$ otherwise. In this paper we consider hyperbolic surfaces, hence $g \geqslant 2$.

The following result of Weichold [Wei] gives a classification of Klein surfaces up to topological equivalence:

Theorem 2.4. *Two Klein surfaces are topologically equivalent if and only if they are of the same topological type. A triple (g, k, ε) is a topological type of some Klein surface if and only if either $\varepsilon = 1$, $1 \leqslant k \leqslant g + 1$, $k \equiv g + 1 \bmod 2$ or $\varepsilon = 0$, $0 \leqslant k \leqslant g$.*

Remark. The inequality $k \leqslant g + 1$ for plane real algebraic curves is known as the *Harnack inequality* [Har].

To understand the structure of a Klein surface (P, τ), we look at those closed curves which are invariant under the involution τ. There are two kinds of invariant closed curves, depending on whether the restriction of τ to the invariant closed curve is the identity or a "half-turn".

Definition 2.7. Let (P, τ) be a Klein surface. The set of fixed points of the involution τ is called the *set of real points* of (P, τ) and denoted by P^τ. The set P^τ decomposes into pairwise disjoint simple closed smooth curves, called *ovals*.

Definition 2.8. A *twist* (or *twisted oval*) is a simple closed curve in P which is invariant under the involution τ but does not contain any fixed points of τ.

Remark. A twisted oval is not an oval, however the corresponding element of $H_1(P)$ is a fixed point of the induced involution and the corresponding element of $\pi_1(P)$ is preserved up to conjugation by the induced involution.

2.3. **Symmetric Generating Sets.** Let $\mathrm{Aut}(\mathbb{H})$ be the full isometry group of the hyperbolic plane $\mathbb{H}$. Let $\mathrm{Aut}_+(\mathbb{H})$ and $\mathrm{Aut}_-(\mathbb{H})$ be the subsets of all orientation-preserving and and orientation-reversing isometries of $\mathbb{H}$. An orientation-preserving isometry is called *hyperbolic* if it has two fixed points, which lie on the boundary of $\mathbb{H}$. One of the fixed points is attracting, the other fixed point is repelling. The *axis* of a hyperbolic isometry is the geodesic between its fixed points, oriented from the repelling fixed point to the attracting fixed point.

Definition 2.9. A *real Fuchsian group* is a discrete subgroup $\hat{\Gamma}$ of $\mathrm{Aut}(\mathbb{H})$ such that the intersection $\hat{\Gamma}^+ = \hat{\Gamma} \cap \mathrm{Aut}_+(\mathbb{H})$ is a Fuchsian group consisting of hyperbolic automorphisms, $\hat{\Gamma} \neq \hat{\Gamma}^+$ and the quotient $P = \mathbb{H}/\hat{\Gamma}^+$ is a compact surface.

All Klein surfaces can be constructed from real Fuchsian groups, see [Nat04, Nat75, Nat78]. We will sketch the construction here: To a real Fuchsian group $\hat{\Gamma}$ we assign a Riemann surface $P_{\hat{\Gamma}} = \mathbb{H}/(\hat{\Gamma} \cap \mathrm{Aut}_+(\mathbb{H}))$. An anti-holomorphic involution of $P_{\hat{\Gamma}}$ is given by $\tau_{\hat{\Gamma}} = \Phi \circ g \circ \Phi^{-1}$, where g is an element of $\hat{\Gamma} \cap \mathrm{Aut}_-(\mathbb{H})$ and $\Phi : \mathbb{H} \to P_{\hat{\Gamma}}$ is the natural projection. Thus a real Fuchsian group $\hat{\Gamma}$ defines the Klein surface $[\hat{\Gamma}] = (P_{\hat{\Gamma}}, \tau_{\hat{\Gamma}})$.

We will use the following description of generating sets of real Fuchsian groups given in [Nat04, Nat75, Nat78], see also [BEGG]:

Theorem 2.5. (Generating sets of real Fuchsian groups)
For a hyperbolic isometry c, let $\bar{c}$ be the reflection whose mirror coincides with the axis of c, $\sqrt{c}$ be the hyperbolic automorphism such that $(\sqrt{c})^2 = c$ and $\tilde{c} = \bar{c}\sqrt{c}$.

1) Let $(g, k, 1)$ be a topological type of a Klein surface, i.e. $1 \leqslant k \leqslant g+1$ and $k \equiv g+1 \bmod 2$. Set $n = k$. Consider a Fuchsian group Γ such that $\mathbb{H}/\Gamma$ is a Riemann surface of genus $\tilde{g} = (g+1-n)/2$ with n holes. If $(a_1, b_1, \dots, a_{\tilde{g}}, b_{\tilde{g}}, c_1, \dots, c_n)$ is a standard generating set of Γ, then

$$(a_1, b_1, \dots, a_{\tilde{g}}, b_{\tilde{g}}, c_1, \dots, c_n, \bar{c}_1, \dots, \bar{c}_n)$$

is a generating set of a real Fuchsian group $\hat{\Gamma}$ of topological type $(g, k, 1)$. Any real Fuchsian group of topological type $(g, k, 1)$ is obtained this way.

2) Let $(g, k, 0)$ be a topological type of a Klein surface, i.e. $0 \leqslant k \leqslant g$. We choose $n \in \{k+1, \dots, g+1\}$ such that $n \equiv g+1 \bmod 2$. Consider a Fuchsian group Γ such that $\mathbb{H}/\Gamma$ is a Riemann surface of genus $\tilde{g} = (g+1-n)/2$ with n holes. If $(a_1, b_1, \dots, a_{\tilde{g}}, b_{\tilde{g}}, c_1, \dots, c_n)$ is a standard generating set of Γ, then

$$(a_1, b_1, \dots, a_{\tilde{g}}, b_{\tilde{g}}, c_1, \dots, c_n, \bar{c}_1, \dots, \bar{c}_k, \tilde{c}_{k+1}, \cdots, \tilde{c}_n)$$

is a generating set of a real Fuchsian group $\hat{\Gamma}$ of topological type $(g, k, 0)$. Any real Fuchsian group of topological type $(g, k, 0)$ is obtained this way.

3) Let $\hat{\Gamma}$ be a real Fuchsian group as in part 1 or 2 and (P,τ) be the corresponding Klein surface. We now interpret the elements $(a_1,b_1,\dots,a_{\tilde g},b_{\tilde g},c_1,\dots,c_n)$ in $\hat{\Gamma}$ as loops in P without a base point up to homotopy of free loops. We have $P^\tau = c_1\cup\cdots\cup c_k$. The curves $c_1,\dots,c_k$ correspond to ovals, the curves $c_{k+1},\dots,c_n$ correspond to twists. Let P_1 and P_2 be the connected components of the complement of the curves $c_1,\dots,c_n$ in P. Each of these components is a surface of genus $\tilde g=(g+1-n)/2$ with n holes. We have $\tau(P_1)=P_2$. We will refer to P_1 and P_2 as decomposition of (P,τ) in two halves. (Note that such a decomposition is unique if (P,τ) is separating, but is not unique if (P,τ) is non-separating since the twists $c_{k+1},\dots,c_n$ can be chosen in different ways.) Then

$$(a_1,b_1,\dots,a_{\tilde g},b_{\tilde g},c_1,\dots,c_n)$$

is a generating set of $\pi_1(P_1)$, while its image under τ gives a generating set of $\pi_1(P_2)$.

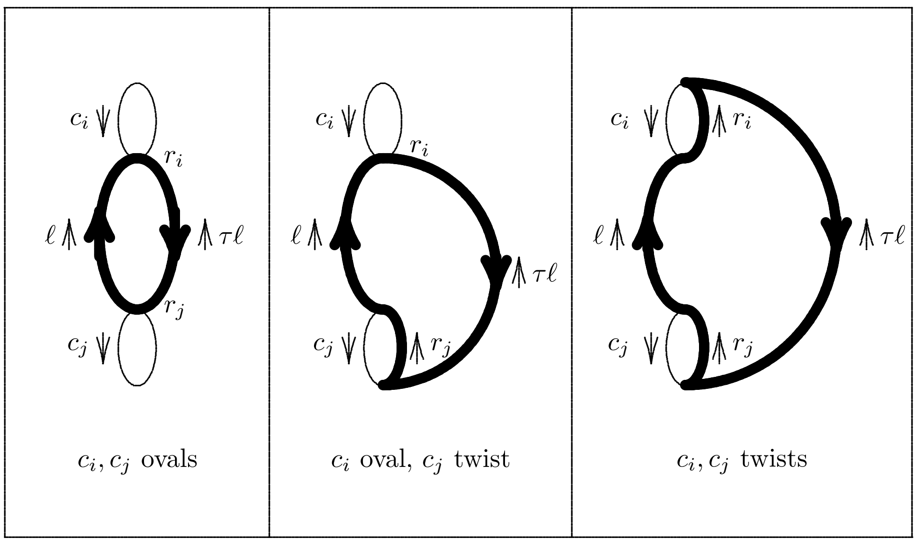

Figure 3: Bridges

Definition 2.10. Let P_1 and P_2 be a decomposition of a Klein surface (P,τ) in two halves as in Theorem 2.5. For two invariant closed curves c_i and c_j, a *bridge* between c_i and c_j is a curve of the form

$$r_i\cup(\tau\ell)^{-1}\cup r_j\cup\ell,$$

where:

- ℓ is a simple path in P_1 starting on c_j and ending on c_i.
- r_i is the path along c_i from the end point of ℓ to the end point of $\tau\ell$. (If c_i is an oval then the path r_i consists of one point.)
- r_j is the path along c_j from the starting point of $\tau\ell$ to the starting point of ℓ. (If c_j is an oval then the path r_j consists of one point.)

Figure 3 shows the shapes of the bridges for different types of invariant curves. The bridges are shown in bold. The bold arrows on the bold lines show the direction of the bridges, while the thinner arrows near the lines show the directions of the paths c_i, c_j, r_i, r_j, ℓ and $\tau\ell$.

Definition 2.11. Let (P,τ) be a Klein surface with a decomposition in two halves P_1 and P_2 as in Theorem 2.5. A *symmetric generating set* of $\pi_1(P)$ is a generating set of the form

$$(a_1, b_1, \dots, a_{\tilde g}, b_{\tilde g}, a'_1, b'_1, \dots, a'_{\tilde g}, b'_{\tilde g}, c_1, \dots, c_{n-1}, d_1, \dots, d_{n-1}),$$

where

- $(a_1, b_1, \dots, a_{\tilde g}, b_{\tilde g}, c_1, \dots, c_n)$ is a generating set of $\pi_1(P_1)$ as in Theorem 2.5,
- $a'_i = (\tau a_i)^{-1}$ and $b'_i = (\tau b_i)^{-1}$ for $i = 1, \dots, \tilde g$.
- $d_1, \dots, d_{n-1}$ are closed curves which only intersect at the base point, such that d_i is homotopic to a bridge between c_i and c_n,

Note that $\tau c_i = c_i$ and $\tau d_i = c_i^{|c_i|} d_i^{-1} c_n^{|c_n|}$, where $|c_j| = 0$ if c_j is an oval and $|c_j| = 1$ if c_j is a twist.

Remark. Note that a symmetric generating set is not a standard generating set in the sense of Definition 2.3, however it is free homotopic to a standard one.

2.4. **Real Higher Arf Functions.** In this section we recall the results from [NP16] on the classification of those higher Arf functions that correspond to m-spin structures on a Klein surface that are invariant under the anti-holomorphic involution.

Definition 2.12. A *real m-Arf function* on a Klein surface (P,τ) is an m-Arf function σ on P such that

(i) σ is compatible with τ, i.e. $\sigma(\tau c) = -\sigma(c)$ for any $c \in \pi_1^0(P)$.
(ii) σ vanishes on all twists.

Theorem 2.6. *Let (P,τ) be a Klein surface. An m-spin bundle on P is invariant under τ if and only if the corresponding m-Arf function is real. The mapping that assigns to an m-spin bundle on P the corresponding m-Arf function establishes a 1-1-correspondence between m-spin bundles invariant under τ and real m-Arf functions on (P,τ).*

Let (P,τ) be a Klein surface of type (g,k,ε), $g \geqslant 2$. Let $c_1,\dots,c_n$ be invariant curves as in Theorem 2.5. Let

$$\mathcal{B} = (a_1, b_1, \dots, a_{\tilde g}, b_{\tilde g}, a'_1, b'_1, \dots, a'_{\tilde g}, b'_{\tilde g}, c_1, \dots, c_{n-1}, d_1, \dots, d_{n-1})$$

be a symmetric generating set of $\pi_1(P)$ (Definition 2.11).

Theorem 2.7. *Let m be even.*

1) Let σ be a real m-Arf function on (P,τ). Then

$$\sigma(a_i) \equiv \sigma(a'_i) \bmod m \text{ and } \sigma(b_i) \equiv \sigma(b'_i) \bmod m \quad \text{for } i = 1, \dots, \tilde g,$$
$$\sigma(c_i) \equiv 0 \bmod (m/2) \quad \text{for } i = 1, \dots, k,$$
$$\sigma(c_1) + \dots + \sigma(c_k) \equiv 1 - g \bmod m,$$
$$g \equiv 1 \bmod (m/2),$$
$$\text{and for } \varepsilon = 0 \text{ additionally } \sigma(c_i) \equiv 0 \bmod m \quad \text{for } i = k+1, \dots, n.$$

2) *Let a value set* $\mathcal{V}$ *in* $(\mathbb{Z}/m\mathbb{Z})^{4\tilde{g}+2n-2}$ *be*
$$(\alpha_1,\beta_1,\dots,\alpha_{\tilde{g}},\beta_{\tilde{g}},\alpha'_1,\beta'_1,\dots,\alpha'_{\tilde{g}},\beta'_{\tilde{g}},\gamma_1,\dots,\gamma_{n-1},\delta_1,\dots,\delta_{n-1}).$$
Assume that
$$\alpha_i=\alpha'_i \text{ and } \beta_i=\beta'_i \quad \text{for } i=1,\dots,\tilde{g},$$
$$\gamma_1,\dots,\gamma_{k-1}\in\{0,m/2\}.$$
In the case $\varepsilon=0$ *assume also that*
$$\gamma_k\in\{0,m/2\},$$
$$\gamma_1+\dots+\gamma_k\equiv 1-g \bmod m,$$
$$\gamma_{k+1}=\dots=\gamma_{n-1}=0.$$
In the case $\varepsilon=1$ *assume also that*
$$g\equiv 1 \bmod \frac{m}{2}.$$
Then there exists a real m-Arf function σ *on* (P,τ) *with*
$$\sigma(a_i)=\alpha_i,\ \sigma(b_i)=\beta_i,\ \sigma(a'_i)=\alpha'_i,\ \sigma(b'_i)=\beta'_i,\ \sigma(c_i)=\gamma_i,\ \sigma(d_i)=\delta_i.$$
For this m-Arf function we have
$$\sigma(c_n)\equiv 0 \bmod m \text{ in the case } \varepsilon=0,$$
$$\sigma(c_n)\equiv (1-g)-(\gamma_1+\dots+\gamma_{n-1}) \bmod m \text{ in the case } \varepsilon=1.$$
3) *The number of real m-Arf functions on* (P,τ) *is*
$$m^g \quad \text{in the case } \varepsilon=0, k=0,$$
$$m^g\cdot 2^{k-1} \quad \text{otherwise.}$$
4) *The Arf invariant* $\delta\in\{0,1\}$ *of a real m-Arf function* σ *on* (P,τ) *is given by*
$$\delta\equiv\sum_{i=1}^{n-1}(1-\sigma(c_i))(1-\sigma(d_i)) \bmod 2.$$
5) *Consider* $\gamma_1,\dots,\gamma_{n-1}$ *as above. Let*
$$\Sigma=\sum_{i=1}^{n-1}(1-\gamma_i)(1-\delta_i).$$
In the case $\varepsilon=1$, $m\equiv 2 \bmod 4$, $\gamma_1=\dots=\gamma_{n-1}=m/2$, *any choice of* $(\delta_1,\dots,\delta_{n-1})\in(\mathbb{Z}/m\mathbb{Z})^{k-1}$ *gives* $\Sigma\equiv 0 \bmod 2$. *In all other cases, out of the* m^{n-1} *possible choices for* $(\delta_1,\dots,\delta_{n-1})\in(\mathbb{Z}/m\mathbb{Z})^{n-1}$, *there are* $m^{n-1}/2$ *which give* $\Sigma\equiv 0 \bmod 2$ *and* $m^{n-1}/2$ *which give* $\Sigma\equiv 1 \bmod 2$.
6) *In the case* $\varepsilon=0$, *the numbers of even and odd real m-Arf functions on* (P,τ) *are both equal to*
$$\frac{m^g}{2} \quad \text{for } k=0 \quad \text{and} \quad m^g\cdot 2^{k-2} \quad \text{for } k\geqslant 1.$$
In the case $\varepsilon=1$ *and* $m\equiv 0 \bmod 4$, *the numbers of even and odd real m-Arf functions are both equal to*
$$m^g\cdot 2^{k-2}.$$

In the case $\varepsilon = 1$ and $m \equiv 2 \bmod 4$, the numbers of even and odd real m-Arf functions respectively are

$$m^g \cdot \frac{2^{k-1}+1}{2} \quad \text{and} \quad m^g \cdot \frac{2^{k-1}-1}{2}.$$

Theorem 2.8. *Let m be odd.*

1) Let σ be a real m-Arf function on (P, τ). Then

$$\begin{aligned} &\sigma(a_i) = \sigma(a_i') \text{ and } \sigma(b_i) = \sigma(b_i') \quad \text{for } i = 1, \dots, \tilde{g}, \\ &\sigma(c_1) = \cdots = \sigma(c_n) = 0, \\ &g = 1 \bmod m. \end{aligned}$$

2) Assume that

$$g = 1 \bmod m.$$

Let a value set $\mathcal{V}$ in $(\mathbb{Z}/m\mathbb{Z})^{4\tilde{g}+2n-2}$ be

$$(\alpha_1, \beta_1, \dots, \alpha_{\tilde{g}}, \beta_{\tilde{g}}, \alpha_1', \beta_1', \dots, \alpha_{\tilde{g}}', \beta_{\tilde{g}}', \gamma_1, \dots, \gamma_{n-1}, \delta_1, \dots, \delta_{n-1}).$$

Assume that

$$\begin{aligned} &\alpha_i = \alpha_i' \text{ and } \beta_i = \beta_i' \quad \text{for } i = 1, \dots, \tilde{g}, \\ &\gamma_1 = \cdots = \gamma_{n-1} = 0. \end{aligned}$$

Then there exists a real m-Arf function σ on (P, τ) with

$$\sigma(a_i) = \alpha_1,\ \sigma(b_i) = \beta_i,\ \sigma(a_i') = \alpha_i',\ \sigma(b_i') = \beta_i',\ \sigma(c_i) = \gamma_i,\ \sigma(d_i) = \delta_i.$$

For this Arf function we have $\sigma(c_n) = 0$.

3) The number of real m-Arf functions on (P, τ) is m^g.

3. Topological Types of Higher Arf Functions on Klein Surfaces

3.1. **Topological Invariants.**

Definition 3.1. Let (P, τ) be a non-separating Klein surface of type $(g, k, 0)$. Let m be even. The *topological type* of a real m-Arf function σ on (P, τ) is a tuple (g, δ, k_0, k_1), where g is the genus of P, δ is the m-Arf invariant of σ and k_j is the number of ovals of (P, τ) with value of σ equal to $j \cdot m/2$.

Real m-Arf functions with even m on separating Klein surfaces have additional topological invariants:

Definition 3.2. Let (P, τ) be a separating Klein surface of type $(g, k, 1)$. Let P_1 and P_2 be the connected components of $P \backslash P^\tau$. Let m be even. Let σ be an m-Arf function on (P, τ). We say that two ovals c_1 and c_2 are *similar* with respect to σ, $c_1 \sim c_2$, if $\sigma(\ell \cup (\tau\ell)^{-1})$ is odd, where ℓ is a simple path in P_1 connecting c_1 and c_2.

From Definition 2.2 it is clear that if $\sigma : \pi_1^0(P) \to \mathbb{Z}/m\mathbb{Z}$ is a real m-Arf function on (P, τ) and m is even, then $(\sigma \bmod 2) : \pi_1^0(P) \to \mathbb{Z}/2\mathbb{Z}$ is a real 2-Arf function on (P, τ). Note that two ovals are similar with respect to the m-Arf function σ if and only if they are similar with respect to the 2-Arf function $(\sigma \bmod 2)$, hence we obtain using [Nat04], Theorem 3.3:

Proposition 3.1. *Similarity of ovals is well-defined. Similarity is an equivalence relation on the set of all ovals with at most two equivalence classes.*

Definition 3.3. Let (P,τ) be a separating Klein surface of type $(g,k,1)$. Let P_1 and P_2 be the connected components of $P\backslash P^\tau$. Let m be even. Let us choose one similarity class of ovals. The *topological type* of a real m-Arf function σ on (P,τ) is a tuple

$$(g,\tilde\delta,k_0^0,k_1^0,k_0^1,k_1^1),$$

where g is the genus of P, $\tilde\delta$ is the m-Arf invariant of $\sigma|_{P_1}$, k_j^0 is the number of ovals in the chosen similarity class with value of σ equal to $j\cdot m/2$ and $k_j^1=k_j-k_j^0$ is the number of ovals in the other similarity class with value of σ equal to $j\cdot m/2$. (The invariants k_j^i are defined up to the swap $k_j^i\leftrightarrow k_j^{1-i}$.)

Definition 3.4. Let (P,τ) be a Klein surface of type (g,k,ε). Let m be odd. The *topological type* of a real m-Arf function σ on (P,τ) is a tuple (g,k), where g is the genus of P and k is the number of ovals of (P,τ).

Proposition 3.2. *If there exists a real m-Arf function of topological type t on a Klein surface of type (g,k,ε), $g\geqslant 2$, then t satisfies the following conditions:*

1) *Case $\varepsilon=0$, $m\equiv 0 \bmod 2$, $t=(g,\delta,k_0,k_1)$: $k_1\cdot m/2\equiv 1-g \bmod m$.*
2) *Case $\varepsilon=1$, $m\equiv 0\bmod 2$, $t=(g,\tilde\delta,k_0^0,k_1^0,k_0^1,k_1^1)$: Let $k_j=k_j^0+k_j^1$, $j=0,1$.*
   (a) *If $g>k+1$ and $m\equiv 0\bmod 4$ then $\tilde\delta=0$.*
   (b) *If $g>k+1$ and $k_0\neq 0$ then $\tilde\delta=0$.*
   (c) *If $g=k+1$ and $m\equiv 0\bmod 4$ then $\tilde\delta=1$.*
   (d) *If $g=k+1$ and $k_0\neq 0$ then $\tilde\delta=1$.*
   (e) *If $g=k+1$, $m\equiv 2\bmod 4$ and $k_0=0$ then $\tilde\delta\in\{1,2\}$.*
   (f) *$k_1\cdot m/2\equiv 1-g\bmod m$.*
3) *Case $m\equiv 1\bmod 2$, $t=(g,k)$: $g\equiv 1\bmod m$.*

Proof. Let (P,τ) be a Klein surface of type (g,k,ε), $g\geqslant 2$. Let σ be a real m-Arf function of topological type t on (P,τ). Let $c_1,\dots,c_k$ be the ovals of (P,τ).

1) Case $\varepsilon=0$, $m\equiv 0\bmod 2$, $t=(g,\delta,k_0,k_1)$: By definition of k_j, the tuple $(\sigma(c_1),\dots,\sigma(c_k))$ is a permutation of zero repeated k_0 times and $m/2$ repeated k_1 times, hence
$$\sigma(c_1)+\dots+\sigma(c_k)\equiv k_1\cdot m/2\bmod m.$$
On the other hand, according to Theorem 2.7,
$$\sigma(c_1)+\dots+\sigma(c_k)\equiv 1-g\bmod m.$$
Hence
$$k_1\cdot m/2\equiv 1-g\bmod m.$$
2) Case $\varepsilon=1$, $m\equiv 0\bmod 2$, $t=(g,\tilde\delta,k_0^0,k_1^0,k_0^1,k_1^1)$: Let P_1 and P_2 be the connected components of $P\backslash P^\tau$. Each of these components is a surface of genus $\tilde g=(g+1-k)/2$ with k holes. If σ is a real m-Arf function of topological type $(g,\tilde\delta,k_0^0,k_1^0,k_0^1,k_1^1)$ on (P,τ), then $\sigma|_{P_1}$ is an m-Arf function on a surface of genus $\tilde g$ with k holes with values on the holes equal to zero repeated k_0 times and $m/2$ repeated k_1 times. Theorem 2.2 implies that
   - If $\tilde g>1$ and $\sigma(c_i)\equiv 0\bmod 2$ for some i then $\tilde\delta=0$: Note that $\tilde g>1$ if and only if $g>k+1$. If $m\equiv 0\bmod 4$ then all $\sigma(c_i)$ are even since both 0 and $m/2$ are even, therefore $\tilde\delta=0$. If $k_0\neq 0$ then $\sigma(c_i)=0$ for some i, hence $\sigma(c_i)$ is even for some i, therefore $\tilde\delta=0$. However, if $m\equiv 2\bmod 4$ and $k_0=0$ then all $\sigma(c_i)=m/2$ are odd, hence no conclusion can be made about $\tilde\delta$. Thus we can

rewrite the condition as follows: If $g > k+1$ and ($m \equiv 0 \bmod 4$ or $k_0 \neq 0$) then $\tilde{\delta} = 0$.

- If $\tilde{g} = 1$ then $\tilde{\delta}$ is a divisor of $\gcd(m, \sigma(c_1)+1, \dots, \sigma(c_k)+1)$: Note that $\tilde{g} = 1$ if and only if $g = k+1$. If $k_0 \neq 0$ then $\sigma(c_i) = 0$ for some i, hence $\tilde{\delta}$ is a divisor of $\gcd(m, 1, \dots)$, therefore $\tilde{\delta} = 1$. If $k_0 = 0$ then $\sigma(c_i) = m/2$ for all i, hence $\tilde{\delta}$ is a divisor of $\gcd\left(m, \frac{m}{2}+1\right)$. For $m \equiv 0 \bmod 4$ we have $\gcd\left(m, \frac{m}{2}+1\right) = 1$, hence $\tilde{\delta} = 1$. For $m \equiv 2 \bmod 4$ we have $\gcd\left(m, \frac{m}{2}+1\right) = 2$, hence $\tilde{\delta} \in \{1, 2\}$. Therefore we can rewrite the condition as follows: If $g = k+1$ and ($m \equiv 0 \bmod 4$ or $k_0 \neq 0$) then $\tilde{\delta} = 1$. If $g = k+1$, $m \equiv 2 \bmod 4$ and $k_0 = 0$ then $\tilde{\delta} \in \{1, 2\}$.
- $\sigma(c_1) + \dots + \sigma(c_k) \equiv (2 - 2\tilde{g}) - k \bmod m$: Note that $\sigma(c_1) + \dots + \sigma(c_k) = k_1 \cdot m/2$ and $(2 - 2\tilde{g}) - k = 1 - g$. Hence we can rewrite the condition as follows: $k_1 \cdot m/2 \equiv 1 - g \bmod m$. (This condition also follows from Theorem 2.7.)

3) Case $m \equiv 1 \bmod 2$, $t = (g, k)$: Theorem 2.8 implies $g \equiv 1 \bmod m$.

□

Proposition 3.3. *Let (P, τ) be a Klein surface of type $(g, k, 1)$, $g \geqslant 2$, and let m be even. Let σ be an m-Arf function of type $(g, \tilde{\delta}, k_0^0, k_1^0, k_0^1, k_1^1)$ on (P, τ). Then the Arf invariant $\delta \in \{0, 1\}$ of σ is given by*

$$\delta \equiv k_0^0 \equiv k_0^1 \bmod 2 \quad \textit{if} \quad m \equiv 2 \bmod 4,$$
$$\delta \equiv k_0^0 + k_1^0 \equiv k_0^1 + k_1^1 \bmod 2 \quad \textit{if} \quad m \equiv 0 \bmod 4.$$

Proof. Consider an m-Arf function σ of type $(g, \tilde{\delta}, k_0^0, k_1^0, k_0^1, k_1^1)$ on (P, τ). Let $c_1, \dots, c_k$ be the ovals of (P, τ). We choose a symmetric generating set

$$\mathcal{B} = (a_1, b_1, \dots, a_{\tilde{g}}, b_{\tilde{g}}, a'_1, b'_1, \dots, a'_{\tilde{g}}, b'_{\tilde{g}}, c_1, \dots, c_{k-1}, d_1, \dots, d_{k-1}).$$

of $\pi_1(P)$. Set $\gamma_i = \sigma(c_i)$ for $i = 1, \dots, k$ and $\delta_i = \sigma(d_i)$ for $i = 1, \dots, k-1$. We can assume without loss of generality that the oval c_k is in the chosen similarity class (see Definition 3.2). Let $\delta_k = 1$. For $\alpha, \beta \in \{0, 1\}$ let A_α^β be the subsets of $\{1, \dots, k\}$ given by

$$A_\alpha^\beta = \{i \mid \gamma_i = \alpha \cdot m/2, \delta_i \equiv 1 - \beta \bmod 2\}.$$

Then $k \in A_0^0 \cup A_1^0$. Note that $|A_\alpha^\beta| = k_\alpha^\beta$. According to Theorem 2.7, the Arf invariant δ of σ is given by

$$\delta \equiv \sum_{i=1}^{k-1} (1 - \gamma_i)(1 - \delta_i) \bmod 2.$$

If $m \equiv 2 \bmod 4$, then

$$\sum_{i=1}^{k-1} (1 - \gamma_i)(1 - \delta_i) \equiv |A_0^1 \cap \{1, \dots, k-1\}| \equiv |A_0^1| \equiv k_0^1 \bmod 2.$$

In this case $m/2$ is odd, hence condition $k_1 \cdot m/2 \equiv 1 - g \bmod m$ can be reduced modulo 2 to $k_1 \equiv 1 - g \bmod 2$. On the other hand Theorem 2.4 implies that $k \equiv g + 1 \bmod 2$. Hence

$$k_0 = k - k_1 \equiv (g+1) - (1-g) \equiv 0 \bmod 2,$$

i.e.

$$k_0^1 = k_0 - k_0^0 \equiv k_0^0 \bmod 2.$$

If $m \equiv 0 \bmod 4$, then

$$\sum_{i=1}^{k-1}(1-\gamma_i)(1-\delta_i) \equiv |(A_0^1 \cup A_1^1) \cap \{1,\dots,k-1\}| \equiv |A_0^1 \cup A_1^1| \equiv k_0^1 + k_1^1 \bmod 2.$$

In this case $m/2$ is even, hence condition $k_1 \cdot m/2 \equiv 1-g \bmod m$ can be reduced modulo 2 to $0 \equiv 1-g \bmod 2$. On the other hand Theorem 2.4 implies that $k \equiv g+1 \bmod 2$. Hence k is even, i.e.

$$k_0^1 + k_1^1 = k - (k_0^0 + k_1^0) \equiv k_0^0 + k_1^0 \bmod 2.$$

□

3.2. **Canonical Symmetric Generating Sets.**

Definition 3.5. Let (P,τ) be a Klein surface of type (g,k,ε), $g \geqslant 2$. Let

$$(a_1, b_1, \dots, a_{\tilde{g}}, b_{\tilde{g}}, a_1', b_1', \dots, a_{\tilde{g}}', b_{\tilde{g}}', c_1, \dots, c_{n-1}, d_1, \dots, d_{n-1})$$

be a symmetric generating set of $\pi_1(P)$. Let σ be a real m-Arf function σ of topological type t on (P,τ). Let

$$\alpha_i = \sigma(a_i),\ \beta_i = \sigma(b_i),\ \alpha_i' = \sigma(a_i'),\ \beta_i' = \sigma(b_i'),\ \gamma_i = \sigma(c_i),\ \delta_i = \sigma(d_i).$$

The symmetric generating set $\mathcal{B}$ of $\pi_1(P)$ is *canonical* for the m-Arf function σ if

- Case $\varepsilon = 0$, $m \equiv 0 \bmod 2$, $t = (g, \delta, k_0, k_1)$:

  $(\alpha_1, \beta_1, \dots, \alpha_{\tilde{g}}, \beta_{\tilde{g}}) = (\alpha_1', \beta_1', \dots, \alpha_{\tilde{g}}', \beta_{\tilde{g}}') = (0,1,1,\dots,1)$ if $\tilde{g} \geqslant 2$,

  $(\alpha_1, \beta_1) = (\alpha_1', \beta_1') = (1,0)$ if $\tilde{g} = 1$,

  $\gamma_1 = \dots = \gamma_{k_0} = 0, \quad \gamma_{k_0+1} = \dots = \gamma_k = m/2, \quad \gamma_{k+1} = \dots = \gamma_{n-1} = 0,$

  $\delta_1 = \dots = \delta_{n-1} = 1 - \delta.$

- Case $\varepsilon = 1$, $m \equiv 0 \bmod 2$, $t = (g, \tilde{\delta}, k_0^0, k_1^0, k_0^1, k_1^1)$:

  $(\alpha_1, \beta_1, \dots, \alpha_{\tilde{g}}, \beta_{\tilde{g}}) = (\alpha_1', \beta_1', \dots, \alpha_{\tilde{g}}', \beta_{\tilde{g}}') = (0, 1-\tilde{\delta}, 1, \dots, 1)$ if $\tilde{g} \geqslant 2$;

  $(\alpha_1, \beta_1) = (\alpha_1', \beta_1') = (\tilde{\delta}, 0)$ if $\tilde{g} = 1$;

  $\gamma_1 = \dots = \gamma_{k_0} = 0, \quad \gamma_{k_0+1} = \dots = \gamma_{k-1} = m/2;$

  The oval c_k is in the chosen similarity class;

  $\delta_1 = \dots = \delta_{k_0^1} = 0, \quad \delta_{k_0^1+1} = \dots = \delta_{k_0} = 1,$

  $\delta_{k_0+1} = \dots = \delta_{k_0+k_1^1} = 0, \quad \delta_{k_0+k_1^1+1} = \dots = \delta_{k-1} = 1$ if $k_1 \geqslant 1$;

  $\delta_1 = \dots = \delta_{k_0^1} = 0, \quad \delta_{k_0^1+1} = \dots = \delta_{k-1} = 1$ if $k_1 = 0$.

- Case $m \equiv 1 \bmod 2$, $t = (g, k)$:

  $(\alpha_1, \beta_1, \dots, \alpha_{\tilde{g}}, \beta_{\tilde{g}}) = (\alpha_1', \beta_1', \dots, \alpha_{\tilde{g}}', \beta_{\tilde{g}}') = (0,1,1,\dots,1)$ if $\tilde{g} \geqslant 2$,

  $(\alpha_1, \beta_1) = (\alpha_1', \beta_1') = (1,0)$ if $\tilde{g} = 1$,

  $\gamma_1 = \dots = \gamma_{n-1} = 0,$

  $\delta_1 = \dots = \delta_{n-1} = 0.$

Lemma 3.4. *Let (P,τ) be a Klein surface of type (g,k,ε), $g \geqslant 2$. Let the geometric genus of (P,τ) be positive, i.e. $k \leqslant g-1$ if $\varepsilon = 1$ and $k \leqslant g-2$ if $\varepsilon = 0$. In the case $\varepsilon = 1$ let $n = k$. In the case $\varepsilon = 0$ we choose $n \in \{k+1,\dots,g-1\}$ such that $n \equiv g-1 \bmod 2$. (The assumption that the geometric genus is positive implies $k+1 \leqslant g-1$, hence $\{k+1,\dots,g-1\} \neq \varnothing$.) Let $c_1,\dots,c_n$ be invariant closed* *curves as in Theorem 2.5, then bridges $d_1, \dots, d_{n-1}$ as in Definition 2.11 can be chosen in such a way that*

(i) If m is odd, then $\sigma(d_i) = 0$ for $i = 1, \dots, n-1$.
(ii) If m is even and (P, τ) is separating, then $\sigma(d_i) \in \{0, 1\}$ for $i = 1, \dots, n-1$.
(iii) If m is even and (P, τ) is non-separating, then $\sigma(d_1) = \dots = \sigma(d_{n-1}) \in \{0, 1\}$.

Proof. Let P_1 and P_2 be the connected components of the complement of the closed curves $c_1, \dots, c_n$ in P. Each of these components is a surface of genus $\tilde{g} = (g+1-n)/2$ with n holes. The assumption $n \leqslant g-1$ implies $\tilde{g} \geqslant 1$.

- Consider the real 2-Arf function $(\sigma \bmod 2) : \pi_1^0(P) \to \mathbb{Z}/2\mathbb{Z}$. If m is even and (P, τ) is non-separating, then, according to Lemma 11.2 in [Nat04], we can choose the bridges $d_1, \dots, d_{n-1}$ in such a way that
$$(\sigma \bmod 2)(d_1) = \dots = (\sigma \bmod 2)(d_{n-1}).$$
This means for the original m-Arf function σ that
$$\sigma(d_1) \equiv \dots \equiv \sigma(d_{n-1}) \bmod 2.$$
- Let Q_1 be the compact surface of genus $\tilde{g}$ with one hole obtained from P_1 after removing all bridges $d_1, \dots, d_{n-1}$. Let $\tilde{\delta}$ be the Arf invariant of $\sigma|_{Q_1}$. In the case $\tilde{g} \geqslant 2$, Lemma 5.1 in [NP09] implies that we can choose a standard generating set $(a_1, b_1, \dots, a_{\tilde{g}}, b_{\tilde{g}}, \tilde{c})$ of $\pi_1(Q_1)$ in such a way that $\sigma(a_1) = 0$. In the case $\tilde{g} = 1$, Lemma 5.2 in [NP09] implies that we can choose a standard generating set $(a_1, b_1, \tilde{c})$ of $\pi_1(Q_1)$ in such a way that $\sigma(b_1) = 0$. Thus for $\tilde{g} \geqslant 1$ there always exists a non-trivial closed curve a in P_1 with $\sigma(a) = 0$, which does not intersect any of the bridges $d_1, \dots, d_{n-1}$. If we replace d_i by $(\tau a)^{-1} d_i a$, then
$$\sigma((\tau a)^{-1} d_i a) = \sigma((\tau a)^{-1}) + \sigma(d_i) + \sigma(a) - 2.$$
Taking into account the fact that $\sigma(a) = 0$ we obtain
$$\sigma((\tau a)^{-1} d_i a) = \sigma(d_i) - 2.$$
Repeating this operation we can obtain $\sigma(d_i) = 0$ for odd m and $\sigma(d_i) \in \{0, 1\}$ for even m.
- Note that the property $\sigma(d_1) \equiv \dots \equiv \sigma(d_{n-1}) \bmod 2$ (if m is even and (P, τ) is non-separating) is preserved during this process, hence $\sigma(d_1) = \dots = \sigma(d_{n-1})$ at the end of the process.

□

Proposition 3.5. *Let (P, τ) be a Klein surface of positive geometric genus. For any real m-Arf function on (P, τ) there exists a canonical symmetric generating set of $\pi_1(P)$.*

Proof. Let (g, k, ε) be the topological type of the Klein surface (P, τ). Let σ be a real m-Arf function on (P, τ). Let $c_1, \dots, c_n$ be invariant closed curves as in Theorem 2.5.

- If $m \equiv 0 \bmod 2$ then $\sigma(c_{k+1}) = \dots = \sigma(c_n) = 0$.
- If $m \equiv 0 \bmod 2$ then $\sigma(c_1), \dots, \sigma(c_k) \in \{0, m/2\}$. We can reorder the ovals $c_1, \dots, c_k$ in such a way that
$$\sigma(c_1) = \dots = \sigma(c_{k_0}) = 0, \quad \sigma(c_{k_0+1}) = \dots = \sigma(c_k) = m/2,$$
where k_0 is the numbers of ovals of (P, τ) with the value of σ equal to 0.
- If $m \equiv 1 \bmod 2$ then $\sigma(c_1) = \dots = \sigma(c_n) = 0$.

- We can choose bridges $d_1, \dots, d_{n-1}$ with values $\sigma(d_i)$ as described in Lemma 3.4 since the assumptions of the Lemma are satisfied.
- If $\varepsilon = 1$ and $m \equiv 0 \bmod 2$, we can change the order of $c_1, \dots, c_{k_0}$ and $c_{k_0+1}, \dots, c_k$ to obtain the required values $\delta_1, \dots, \delta_{k-1}$.
- If $\varepsilon = 0$ and $m \equiv 0 \bmod 2$, there exists $\xi \in \{0, 1\}$ such that
$$\sigma(d_1) = \cdots = \sigma(d_{n-1}) = \xi.$$
According to Theorem 2.3 the Arf invariant of σ is
$$\delta \equiv \sum_{i=1}^{n-1} (1 - \sigma(c_i))(1 - \sigma(d_i)) \bmod 2.$$
Using $\sigma(d_i) = \xi$ we obtain
$$\begin{aligned}\delta &\equiv \sum_{i=1}^{n-1} (1 - \sigma(c_i))(1 - \sigma(d_i)) \\ &\equiv (1 - \xi) \cdot \sum_{i=1}^{n-1} (1 - \sigma(c_i)) \\ &\equiv (1 - \xi) \cdot \left((n-1) - \sum_{i=1}^{n-1} \sigma(c_i) \right) \\ &\equiv (1 - \xi) \cdot \left((n-1) - k_1 \cdot \frac{m}{2} \right) \bmod 2.\end{aligned}$$
Recall that $k_1 \cdot m/2 \equiv 1 - g \bmod m$ by Proposition 3.2 and $n \equiv g - 1 \bmod 2$, hence
$$(n-1) - k_1 \cdot \frac{m}{2} \equiv (g-2) - (1-g) \equiv 2g - 3 \equiv 1 \bmod 2$$
and
$$\delta \equiv (1 - \xi) \cdot \left((n-1) - k_1 \cdot \frac{m}{2} \right) \equiv 1 - \xi \bmod 2.$$
Therefore
$$\sigma(d_1) = \cdots = \sigma(d_{n-1}) = \xi = 1 - \delta.$$
- For $\tilde{g} \geqslant 2$, Lemma 5.1 in [NP09] implies that we can choose a standard generating set $(a_1, b_1, \dots, a_{\tilde{g}}, b_{\tilde{g}}, c_1, \dots, c_n)$ of $\pi_1(P_1)$ in such a way that
$$(\sigma(a_1), \sigma(b_1), \dots, \sigma(a_{\tilde{g}}), \sigma(b_{\tilde{g}})) = (0, 1 - \tilde{\delta}, 1, \dots, 1),$$
where $\tilde{\delta}$ is the Arf invariant of $\sigma|_{P_1}$. Moreover, if m is odd then $\tilde{\delta} = 0$. If m is even and $\varepsilon = 0$ then there are closed curves around holes in P_1 such that the values of σ on these closed curves are even, namely $\sigma(c_{k+1}) = \cdots = \sigma(c_n) = 0$, hence $\tilde{\delta} = 0$.
- If $\tilde{g} = 1$, Lemma 5.2 in [NP09] implies that we can choose a standard generating set $(a_1, b_1, c_1, \dots, c_n)$ of $\pi_1(P_1)$ in such a way that
$$(\sigma(a_1), \sigma(b_1)) = (\tilde{\delta}, 0),$$
where $\tilde{\delta} = \gcd(m, \sigma(a_1), \sigma(b_1), \sigma(c_1)+1, \dots, \sigma(c_n)+1)$ is the Arf invariant of $\sigma|_{P_1}$. If m is odd then $\sigma(c_1) = \cdots = \sigma(c_n) = 0$, hence $\tilde{\delta} = 1$. If $\varepsilon = 0$ then $\sigma(c_{k+1}) = \cdots = \sigma(c_n) = 0$, hence $\tilde{\delta} = 1$.

□

Proposition 3.6. *For any Klein surface (P,τ) and any symmetric generating set $\mathcal{B}$ of $\pi_1(P)$ and any tuple t that satisfies the conditions of Proposition 3.2 there exists a real m-Arf function of topological type t on (P,τ) for which $\mathcal{B}$ is canonical.*

Proof. Let $\mathcal{V}=(\alpha_i,\beta_i,\alpha_i',\beta_i',\gamma_i,\delta_i)$ satisfy the conditions in Definition 3.5.

- Case $\varepsilon=0$, $m\equiv 0 \bmod 2$, $t=(g,\delta,k_0,k_1)$: We have $\gamma_1=\dots=\gamma_{k_0}=0$, $\gamma_{k_0+1}=\dots=\gamma_{k_0+k_1}=m/2$, hence
$$\gamma_1+\dots+\gamma_k=k_1\cdot m/2.$$
The tuple t satisfies the conditions of Proposition 3.2, hence
$$k_1\cdot m/2\equiv 1-g \bmod m.$$
Therefore
$$\gamma_1+\dots+\gamma_k\equiv 1-g \bmod m.$$
Other conditions of Proposition 2.7 are clearly satisfied. Hence there exists a real m-Arf function σ on P with the values $\mathcal{V}$ on $\mathcal{B}$. Let δ' be the Arf invariant of σ, then
$$\begin{aligned}\delta' &\equiv \sum_{i=1}^{n-1}(1-\gamma_i)(1-\delta_i)\equiv\sum_{i=1}^{n-1}(1-\gamma_i)(1-(1-\delta))\\ &\equiv \delta\cdot\sum_{i=1}^{n-1}(1-\gamma_i)\equiv\delta\cdot\left((n-1)-\sum_{i=1}^{n-1}\gamma_i\right)\equiv\delta\cdot\left((n-1)-k_1\cdot\frac{m}{2}\right)\bmod 2.\end{aligned}$$
Recall that $k_1\cdot m/2\equiv 1-g \bmod m$ and $n\equiv g-1 \bmod 2$, hence
$$(n-1)-k_1\cdot\frac{m}{2}\equiv(g-2)-(1-g)\equiv 2g-3\equiv 1 \bmod 2$$
and
$$\delta'\equiv\delta\cdot\left((n-1)-k_1\cdot\frac{m}{2}\right)\equiv\delta \bmod 2.$$
Hence σ is a real m-Arf function on P of type t and $\mathcal{B}$ is canonical for σ.
- Case $\varepsilon=1$, $m\equiv 0 \bmod 2$, $t=(g,\tilde\delta,k_0^0,k_1^0,k_0^1,k_1^1)$: The tuple t satisfies the conditions of Proposition 3.2, hence
$$1-g\equiv k_1\cdot\frac{m}{2}\bmod m$$
and therefore
$$1-g\equiv 0 \bmod \frac{m}{2}.$$
Other conditions of Proposition 2.7 are clearly satisfied. Hence there exists a real m-Arf function σ on P with the values $\mathcal{V}$ on $\mathcal{B}$. Let $\tilde\delta'$ be the Arf invariant of $\sigma|_{P_1}$. The m-Arf function σ is real, hence according to Proposition 3.2, we have
  - If $g>k+1$ and $m\equiv 0 \bmod 4$ then $\tilde\delta'=0$.
  - If $g>k+1$ and $k_0\neq 0$ then $\tilde\delta'=0$.
  - If $g=k+1$ and $m\equiv 0 \bmod 4$ then $\tilde\delta'=1$.
  - If $g=k+1$ and $k_0\neq 0$ then $\tilde\delta'=1$.
  - If $g=k+1$, $m\equiv 2 \bmod 4$ and $k_0=0$ then $\tilde\delta'\in\{1,2\}$.

  On the other hand $t=(g,\tilde\delta,k_0^0,k_1^0,k_0^1,k_1^1)$ satisfies the conditions of Proposition 3.2, hence
  - If $g>k+1$ and $m\equiv 0 \bmod 4$ then $\tilde\delta=0$.
  - If $g>k+1$ and $k_0\neq 0$ then $\tilde\delta=0$.

- If $g=k+1$ and $m\equiv 0 \bmod 4$ then $\tilde{\delta}=1$.
- If $g=k+1$ and $k_0\neq 0$ then $\tilde{\delta}=1$.
- If $g=k+1$, $m\equiv 2\bmod 4$ and $k_0=0$ then $\tilde{\delta}\in\{1,2\}$.

Hence if $m\equiv 0\bmod 4$ or $k_0\neq 0$ we have $\tilde{\delta}'=\tilde{\delta}$. It remains to consider the case $m\equiv 2\bmod 4$, $k_0=0$. In the case $g>k+1$, $m\equiv 2\bmod 4$, $k_0=0$, we have $\tilde{g}\geqslant 2$ and the values of the m-Arf function $\sigma|_{P_1}$ on the boundary curves $\sigma(c_i)$ are all equal to $m/2$ and hence odd. Then, according to Theorem 2.3, the Arf invariant $\tilde{\delta}'$ is given by

$$\tilde{\delta}'\equiv\sum_{i=1}^{\tilde{g}}(1-\alpha_i)(1-\beta_i)\bmod 2.$$

We have $(\alpha_1,\beta_1,\dots,\alpha_{\tilde{g}},\beta_{\tilde{g}})=(0,1-\tilde{\delta},1,\dots,1)$, hence

$$\tilde{\delta}'\equiv\sum_{i=1}^{\tilde{g}}(1-\alpha_i)(1-\beta_i)\equiv 1\cdot\tilde{\delta}+0+\dots+0\equiv\tilde{\delta}\bmod 2$$

and therefore $\tilde{\delta}'=\tilde{\delta}$. In the case $g=k+1$, $m\equiv 2\bmod 4$, $k_0=0$, we have $\tilde{g}=1$ and the values of the m-Arf function $\sigma|_{P_1}$ on the boundary curves $\sigma(c_i)$ are all equal to $m/2$. Then, according to Theorem 2.3, the Arf invariant $\tilde{\delta}'\in\{1,2\}$ is given by

$$\tilde{\delta}'=\gcd\left(m,\alpha_1,\beta_1,\frac{m}{2}+1\right).$$

We have $(\alpha_1,\beta_1)=(\tilde{\delta},0)$, hence $\gcd(\alpha_1,\beta_1)=\tilde{\delta}\in\{1,2\}$. For $m\equiv 2\bmod 4$ we have $\gcd\left(m,\frac{m}{2}+1\right)=2$. Therefore

$$\tilde{\delta}'=\gcd\left(m,\alpha_1,\beta_1,\frac{m}{2}+1\right)=\gcd(\tilde{\delta},2)=\tilde{\delta}.$$

Hence σ is a real m-Arf function on P of type t and $\mathcal{B}$ is canonical for σ.

- Case $m\equiv 1\bmod 2$, $t=(g,k)$: The tuple t satisfies the conditions of Proposition 3.2, hence $g\equiv 1\bmod m$. Other conditions of Proposition 2.8 are clearly satisfied. Hence there exists a real m-Arf function σ on P with the values $\mathcal{V}$ on $\mathcal{B}$. The topological type of σ is t and $\mathcal{B}$ is canonical for σ.

□

Proposition 3.7. *The conditions in Proposition 3.2 are necessary and sufficient for a tuple to be a topological type of a real m-Arf function.*

Proof. Proposition 3.2 shows that the conditions are necessary. Proposition 3.6 shows that the conditions are sufficient as we constructed an m-Arf function of type t for any tuple t that satisfies the conditions. □

Definition 3.6. Two m-Arf functions σ_1 and σ_2 on a Klein surface (P,τ) are *topologically equivalent* if there exists a homeomorphism $\varphi:P\to P$ such that $\varphi\circ\tau=\tau\circ\varphi$ and $\sigma_1=\sigma_2\circ\varphi_*$ for the induced automorphism φ_* of $\pi_1(P)$.

Proposition 3.8. *Let (P,τ) be a Klein surface of positive geometric genus. Two m-Arf functions on (P,τ) are topologically equivalent if and only if they have the same topological type.*

Proof. Let (g, k, ε) be the topological type of the Klein surface (P, τ). Proposition 3.5 shows that for any real m-Arf function σ of topological type t we can choose a symmetric generating set $\mathcal{B}$ (the canonical generating set for σ) with the values of σ on $\mathcal{B}$ determined completely by t. Hence any two real m-Arf functions of topological type t are topologically equivalent. □

4. Moduli Spaces

4.1. **Moduli Spaces of Klein Surfaces.** We will use the results on the moduli spaces of real Fuchsian groups and of Klein surfaces described in [Nat75, Nat78]: We consider hyperbolic Klein surfaces, i.e. we assume that the genus is $g \geqslant 2$. Let $\mathcal{M}_{g,k,\varepsilon}$ be the moduli space of Klein surfaces of topological type (g, k, ε). Let $\Gamma_{g,n}$ be the group generated by the elements

$$v = \{a_1, b_1, \dots, a_g, b_g, c_1, \dots, c_n\}$$

with a single defining relation

$$\prod_{i=1}^{g} [a_i, b_i] \prod_{i=1}^{n} c_i = 1.$$

Let $\mathrm{Aut}_+(\mathbb{H})$ be the group of all orientation-preserving isometries of $\mathbb{H}$. The *Fricke space* $\tilde{T}_{g,n}$ is the set of all monomorphisms $\psi : \Gamma_{g,n} \to \mathrm{Aut}_+(\mathbb{H})$ such that

$$\{\psi(a_1), \psi(b_1), \dots, \psi(a_g), \psi(b_g), \psi(c_1), \dots, \psi(c_n)\}$$

is a generating set of a Fuchsian group of signature (g, n). The Fricke space $\tilde{T}_{g,n}$ is homeomorphic to $\mathbb{R}^{6g-3+3n}$. The group $\mathrm{Aut}_+(\mathbb{H})$ acts on $\tilde{T}_{g,n}$ by conjugation. The *Teichmüller space* is $T_{g,n} = \tilde{T}_{g,n} / \mathrm{Aut}_+(\mathbb{H})$.

Theorem 4.1. *Let (g, k, ε) be a topological type of a Klein surface. In the case $\varepsilon = 1$ let $n = k$. In the case $\varepsilon = 0$ we choose $n \in \{k + 1, \dots, g + 1\}$ such that $n \equiv g + 1 \bmod 2$. Let $\tilde{g} = (g + 1 - n)/2$. The moduli space $\mathcal{M}_{g,k,\varepsilon}$ of Klein surfaces of topological type (g, k, ε) is the quotient of the Teichmüller space $T_{\tilde{g},n}$ by a discrete group of autohomeomorphisms $\mathrm{Mod}_{g,k,\varepsilon}$. The space $T_{\tilde{g},n}$ is homeomorphic to $\mathbb{R}^{3g-3}$.*

Theorem 4.2. *The moduli space of Klein surfaces of genus g decomposes into connected components $\mathcal{M}_{g,k,\varepsilon}$. Each connected component is homeomorphic to a quotient of $\mathbb{R}^{3g-3}$ by a discrete group action.*

4.2. **Moduli Spaces of Higher Spin Bundles on Klein Surfaces.**

Theorem 4.3. *Let (g, k, ε) be a topological type of a Klein surface. Assume that the geometric genus of such Klein surfaces is positive, i.e. $k \leqslant g - 2$ if $\varepsilon = 0$ and $k \leqslant g - 1$ if $\varepsilon = 1$. Let t be a tuple that satisfies the conditions of Proposition 3.2. The space $S(t)$ of all m-spin bundles of type t on a Klein surface of type (g, k, ε) is connected and diffeomorphic to*

$$\mathbb{R}^{3g-3} / \mathrm{Mod}_t,$$

where Mod_t is a discrete group of diffeomorphisms.

Proof. In the case $\varepsilon = 1$ let $n = k$. In the case $\varepsilon = 0$ we choose $n \in \{k+1, \dots, g-1\}$ such that $n \equiv g - 1 \bmod 2$. Let $\tilde{g} = (g + 1 - n)/2$. By definition, to any $\psi \in \tilde{T}_{\tilde{g},n}$ corresponds a generating set

$$V = \{\psi(a_1), \psi(b_1), \dots, \psi(a_{\tilde{g}}), \psi(b_{\tilde{g}}), \psi(c_1), \dots, \psi(c_n)\}$$

of a Fuchsian group of signature $(\tilde{g}, n)$. The generating set V together with

$$\{\overline{\psi(c_1)}, \dots, \overline{\psi(c_k)}, \widetilde{\psi(c_{k+1})}, \dots, \widetilde{\psi(c_n)}\}$$

generates a real Fuchsian group Γ_ψ. On the Klein surface $(P, \tau) = [\Gamma_\psi]$, we consider the corresponding symmetric generating set

$$\mathcal{B}_\psi = (a_1, b_1, \dots, a_{\tilde{g}}, b_{\tilde{g}}, a'_1, b'_1, \dots, a'_{\tilde{g}}, b'_{\tilde{g}}, c_1, \dots, c_{n-1}, d_1, \dots, d_{n-1}).$$

Proposition 3.6 implies that there exists a real m-Arf function $\sigma = \sigma_\psi$ of type t for which $\mathcal{B}_\psi$ is canonical. According to Theorem 2.6, an m-spin bundle $\Omega(\psi) \in S(t)$ is associated with this m-Arf function. The correspondence $\psi \mapsto \Omega(\psi)$ induces a map $\Omega : T_{\tilde{g},n} \to S(t)$. Let us prove that $\Omega(T_{\tilde{g},n}) = S(t)$. Indeed, by Theorem 4.1, the map

$$\Psi = \Phi \circ \Omega : T_{\tilde{g},n} \to S(t) \to \mathcal{M}_{g,k,\varepsilon},$$

where Φ is the natural projection, satisfies the condition

$$\Psi(T_{\tilde{g},n}) = \mathcal{M}_{g,k,\varepsilon}.$$

The fibre of the map Ψ is represented by the group $\mathrm{Mod}_{g,k,\varepsilon}$ of all self-homeomorphisms of the Klein surface (P, τ). By Proposition 3.8, this group acts transitively on the set of all real m-Arf functions of type t and hence, by Theorem 2.6, transitively on the fibres $\Phi^{-1}((P, \tau))$. Thus

$$\Omega(T_{\tilde{g},n}) = S(t) = T_{\tilde{g},n}/\,\mathrm{Mod}_t, \quad \text{where } \mathrm{Mod}_t \subset \mathrm{Mod}_{g,k,\varepsilon}$$

According to Theorem 4.1, the space $T_{\tilde{g},n}$ is diffeomorphic to $\mathbb{R}^{3g-3}$. □

4.3. **Branching Indices of Moduli Spaces.**

Theorem 4.4. *Let (g, k, ε) be a topological type of a Klein surface. Assume that the geometric genus of such Klein surfaces is positive, i.e. $k \leqslant g - 2$ if $\varepsilon = 0$ and $k \leqslant g - 1$ if $\varepsilon = 1$. Let t be a tuple that satisfies the conditions of Proposition 3.2. The space $S(t)$ of all real m-spin bundles of type t on a Klein surface of type (g, k, ε) is an $N(t)$-fold covering of $\mathcal{M}_{g,k,\varepsilon}$, where $N(t)$ is the number of real m-Arf functions on (P, τ) of topological type t. The number $N(t)$ is equal to*

1) Case $\varepsilon = 0$, $m \equiv 0 \bmod 2$, $t = (g, \delta, k_0, k_1)$:

$$N(t) = \binom{k}{k_1} \cdot \frac{m^g}{2}.$$

2) Case $\varepsilon = 1$, $m \equiv 0 \bmod 2$, $t = (g, \tilde{\delta}, k_0^0, k_1^0, k_0^1, k_1^1)$: Let

$$M = \binom{k}{k_0} \cdot \binom{k_0}{k_0^0} \cdot \binom{k_1}{k_1^0}.$$

- *Case $g > k + 1$, ($m \equiv 0 \bmod 4$ or $k_0 \neq 0$):*

$$N(t) = 2^{1-k} \cdot m^g \cdot M \quad \text{for } \tilde{\delta} = 0 \text{ and } N(t) = 0 \quad \text{for } \tilde{\delta} = 1.$$

- *Case $g > k + 1$, $m \equiv 2 \bmod 4$, $k_0 = 0$:*

$$N(t) = \left(2^{-k} + 2^{-\frac{g+k+1}{2}}\right) \cdot m^g \cdot M \quad \text{for } \tilde{\delta} = 0,$$

$$N(t) = \left(2^{-k} - 2^{-\frac{g+k+1}{2}}\right) \cdot m^g \cdot M \quad \text{for } \tilde{\delta} = 1.$$

- *Case $g = k + 1$, ($m \equiv 0 \bmod 4$ or $k_0 \neq 0$):*

$$N(t) = 2^{-(k-1)} \cdot m^{k+1} \cdot M \quad \text{for } \tilde{\delta} = 1 \text{ and } N(t) = 0 \quad \text{for } \tilde{\delta} = 2.$$

- *Case $g = k+1$, $m \equiv 2 \bmod 4$, $k_0 = 0$:*

$$N(t) = 3 \cdot 2^{-(k+1)} \cdot m^{k+1} \cdot M \quad \text{for } \tilde{\delta} = 1,$$
$$N(t) = 2^{-(k+1)} \cdot m^{k+1} \cdot M \quad \text{for } \tilde{\delta} = 2.$$

3) Case $m \equiv 1 \bmod 2$, $t = (g,k)$:

$$N(t) = m^g.$$

Proof. According to Theorem 4.3, $S(t) \cong T_{\tilde{g},n}/\operatorname{Mod}_t$, where $\operatorname{Mod}_t \subset \operatorname{Mod}_{g,k,\varepsilon}$, hence $S(t)$ is a branched covering of $\mathcal{M}_{g,k,\varepsilon} = T_{\tilde{g},n}/\operatorname{Mod}_{g,k,\varepsilon}$ and the branching index is equal to the index of the subgroup Mod_t in $\operatorname{Mod}_{g,k,\varepsilon}$, i.e. is equal to the number $N(t)$ of real m-Arf functions on (P,τ) of topological type t. Let

$$\mathcal{B} = (a_1, b_1, \dots, a_{\tilde{g}}, b_{\tilde{g}}, a'_1, b'_1, \dots, a'_{\tilde{g}}, b'_{\tilde{g}}, c_1, d_1, \dots, c_{n-1}, d_{n-1})$$

be a symmetric generating set of $\pi_1(P)$. Let $\mathcal{V} = (\alpha_i, \beta_i, \alpha'_i, \beta'_i, \gamma_i, \delta_i)$ denote the set of values of an m-Arf function on $\mathcal{B}$.

1) Case $\varepsilon = 0$, $m \equiv 0 \bmod 2$, $t = (g, \delta, k_0, k_1)$: There are $\binom{k}{k_1}$ ways to choose the values γ_i. There are $m^{2\tilde{g}}$ ways to choose $\alpha_i = \alpha'_i$ and $\beta_i = \beta'_i$. According to Theorem 2.7, out of m^{n-1} ways to choose $\delta_1, \dots, \delta_{n-1}$ there are $m^{n-1}/2$ which give $\Sigma \equiv 0 \bmod 2$ and $m^{n-1}/2$ which give $\Sigma \equiv 1 \bmod 2$. Thus the number of real m-Arf functions of type (g, δ, k_0, k_1) is

$$\binom{k}{k_1} \cdot m^{2\tilde{g}} \cdot \frac{m^{n-1}}{2} = \binom{k}{k_1} \cdot \frac{m^{2\tilde{g}+n-1}}{2} = \binom{k}{k_1} \cdot \frac{m^g}{2}.$$

2) Case $\varepsilon = 1$, $m \equiv 0 \bmod 2$, $t = (g, \tilde{\delta}, k_0^0, k_1^0, k_0^1, k_1^1)$: There are $M = \binom{k}{k_0} \cdot \binom{k_0}{k_0^0} \cdot \binom{k_1}{k_1^0}$ ways to choose the values γ_i. Furthermore having fixed the parity of δ_i, there are $(m/2)^{k-1}$ ways to choose the values of δ_i. Hence the number of such real m-Arf functions on P is equal to

$$m^{2\tilde{g}} \cdot \left(\frac{m}{2}\right)^{k-1} \cdot M = \frac{m^{2\tilde{g}+k-1}}{2^{k-1}} \cdot M = m^g \cdot 2^{1-k} \cdot M.$$

- In the case $g > k+1$, $m \equiv 2 \bmod 4$, $k_0 = 0$, the resulting invariant $\tilde{\delta}$ is given by

$$\tilde{\delta} \equiv \sum_{i=1}^{\tilde{g}} (1-\alpha_i)(1-\beta_i) \bmod 2.$$

It can be shown by induction that out of $m^{2\tilde{g}}$ ways to choose the values α_i, β_i we get the Arf invariant $\tilde{\delta} = 0$ in $2^{\tilde{g}-1}(2^{\tilde{g}}+1)(m/2)^{2\tilde{g}}$ cases and $\tilde{\delta} = 1$ in $2^{\tilde{g}-1}(2^{\tilde{g}}-1)(m/2)^{2\tilde{g}}$ cases. Hence the number $N(t)$ with $\tilde{\delta}$ equal to 0 and 1 respectively is

$$2^{\tilde{g}-1}(2^{\tilde{g}} \pm 1)\left(\frac{m}{2}\right)^{2\tilde{g}}\left(\frac{m}{2}\right)^{k-1} \cdot M.$$

We simplify

$$\begin{aligned} 2^{\tilde{g}-1}(2^{\tilde{g}} \pm 1)\left(\frac{m}{2}\right)^{2\tilde{g}}\left(\frac{m}{2}\right)^{k-1} &= (2^{2\tilde{g}-1} \pm 2^{\tilde{g}-1})\left(\frac{m}{2}\right)^{2\tilde{g}+k-1} \\ = \left(2^{g-k} \pm 2^{\frac{g-k-1}{2}}\right)\left(\frac{m}{2}\right)^g &= \left(2^{g-k} \pm 2^{\frac{g-k-1}{2}}\right)2^{-g} \cdot m^g \\ = \left(2^{-k} \pm 2^{\frac{-g-k-1}{2}}\right)m^g &= \left(2^{-k} \pm 2^{-\frac{g+k+1}{2}}\right)m^g \end{aligned}$$

to obtain $N(t)$ as stated.

- In the case $g > k+1$, ($m \equiv 0 \bmod 4$ or $k_0 \neq 0$), the Arf invariant of all m-Arf functions we construct is $\tilde{\delta} = 0$, hence $N(t)$ is as stated.
- In the case $g = k+1$, $m \equiv 2 \bmod 4$, $k_0 = 0$, the Arf invariant of the resulting m-Arf function is given by
$$\tilde{\delta} = \gcd\left(m, \alpha_1, \beta_1, \frac{m}{2} + 1\right).$$
Note that for $m \equiv 2 \bmod 4$ we have $\gcd(m, m/2+1) = 2$, hence $\tilde{\delta} = 2$ if α_1 and β_1 are both even and $\tilde{\delta} = 1$ otherwise. Out of m^2 ways to choose the values α_1, β_1 we get $\tilde{\delta} = 1$ in $3m^2/4$ cases and $\tilde{\delta} = 2$ in $m^2/4$ cases. Hence the number $N(t)$ with $\tilde{\delta}$ equal to 1 and 2 respectively is
$$\frac{2 \pm 1}{4} \cdot m^2 \left(\frac{m}{2}\right)^{k-1} \cdot M = (2 \pm 1) \cdot \left(\frac{m}{2}\right)^{k+1} \cdot M.$$
- In the case $g = k+1$, ($m \equiv 0 \bmod 4$ or $k_0 \neq 0$), the Arf invariant of all m-Arf functions we construct is $\tilde{\delta} = 1$, hence $N(t)$ is as stated.

3) Case $m \equiv 1 \bmod 2$, $t = (g, k)$: The statement follows from Theorem 2.8.

□

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