Risk-sensitive control for a class of non-linear systems and its financial applications

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Abstract

This thesis studies the risk-sensitive control problem for a class of non-linear stochastic systems and its financial applications. The nonlinearity is of the square-root type, and is inspired by applications. The problems of optimal investment and consumption are also considered under several different assumptions on the stochastic interest rate and stochastic volatility.

At the beginning, we systematically investigate the nonlinearity of risk-sensitive control problem. It consists of quadratic and square-root terms in the state. Such an optimal control problem can be solved in an explicit closed form by the completion of squares method. As an application of the risk-sensitive control in financial mathematics, the optimal investment problem will be described in the Chapter 4. A new interest rate, which follows the stochastic process with mixed Cox-Ingersoll-Ross (CIR) model and quadratic affine term structure model (QATSM) is introduced. Such an interest rate model admits an explicit price for the zero-coupon bond.

In Chapter 5, we consider a portfolio optimization problem on an infinite time horizon. The stochastic interest rate consists not only of the quadratic terms, but also of the square-root terms. On the other hand, the double square root process is also introduced to establish the interest rate model. Under some sufficient conditions, the unique solution of the optimal investment problem is found in an explicit closed form. Furthermore, the optimal consumption problem is considered in Chapter 6 and 7. It can be solved in an explicit closed form via the methods of completion of squares and the change of measure. We provide a detailed discussion on the existence of the optimal trading strategies. Such trading strategies can be deduced for both finite and infinite time horizon cases.

Contents

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Chapter 1

Introduction

1.1 Introduction

A short literature review on the risk-sensitive control and optimal investment problems is given. We also indicate the main contributions of the thesis.

1.2 Risk-sensitive control

The risk-sensitive control problem was introduced by Jacobson [25] in 1973. He considered linear stochastic systems with additive Gaussian noise, and minimized the expectation of the *exponential* of quadratic cost. Assuming full state observation, Jacobson gave the complete solution to this problem, with the optimal control being in a linear state-feedback form. He also considered the discrete time version of this problem. For continuous-time systems with partial observation, Bensoussan and Van Schuppen [3] in 1985 obtained the complete solution, whereas Whittle [52] solved the discrete-time partial observation problem (see also [53]). A connection between the risk-sensitive control and robust control was found in 1988 by Glover and Doyle [20]. For an infinite horizon criterion and a class of nonlinear systems, the reader can refer to Fleming-McEneaney's paper [16] and James's paper [26]. A connection with dynamic games can be found in [41].

In the recent paper [12], Date and Gashi generalised the risk-sensitive control problem by introducing a more general criterion. Such a criterion has a *noise* dependent penalties on the state and control variables. A further more general risk sensitive control problem was considered in Date and Gashi [11], where the system state is extended to include certain quadratic nonlinearities in the state and control, as well as a multiplicative noise. This is an important extension of the linear risk-sensitive control that preserves the explicit closed-form solution of the problem.

1.3 Optimal investment and consumption

Investor's *optimal portfolio problem* is to choose the optimal investment and consumption strategies so as to maximize the utilities from terminal wealth and consumption. This problem has a long history beginning with the work of Markowitz [36], [37], where the *mean-variance* portfolio selection was introduced. This is a one-step discrete-time portfolio model. The continuous-time portfolio problem was introduced by Merton [38], [39]. He obtained an explicit solution to the continuous-time portfolio problem for several different utility functions and using the methods of stochastic control. Since then, there has been a great progress on this problem, with the aim of considering more general market models and utility functions. If the market coefficients are bounded, then the optimal portfolio problem is largely solved under different settings (see, for example, the textbook accounts of Korn [29], and Karatzas and Shreve [28]). However, the assumption of bounded market coefficients can be too restrictive in some models. This is usually the case when the model for the market coefficients, such as the interest rate and the volatility, are given as solutions to stochastic differential equations, which are unbounded processes in general. This means that the methods of solving the optimal portfolio problem as given in [29], [28], no longer apply in this case. Moreover, such unbounded processes include the typical models of interest rates, such as the Vasicek model or the CIR model. Hence, other methods need to be developed to deal with these situations. In [5], [6], [21], [32]), it is shown that the optimal portfolio problem under a stochastic interest rate can be interpreted as a risk-sensitive control problem. Korn and Kraft [30] gave the solution to the optimal investment problem with the Vasicek interest rate model, and Zariphopoulou [55] connected it with the consumption problem. Deelstra, Grasselli and Koehl [13]

investigated the case of an interest rate model given by a square-root process. In Date and Gashi [11], a new quadratic affine term structure model (QATSM) of interest rate is introduced.
Despite this progress, there remain many interesting market models for which the optimal portfolio problem has not been solved. In this thesis, we consider several such market models which have an unbounded interest rate, and solve the corresponding optimal control problems by the methods of risk-sensitive control.

1.4 The main contributions

- Compared with Date and Gashi's work in [11], the risk sensitive control problem is further extended in this thesis (see Chapter 3). It contains additional nonlinear components of the state, which are represented by the square-root processes. In Fei and Gashi's work [15], the scalar case of this problem is solved. A limitation of this study is that the admissibility of the proposed optimal control is only assumed rather than proved. In this chapter we provide such a proof and extend the results to multi-dimensional square-root process. The key aspect of this chapter is that the explicit closed-form solvability has been preserved. In addition, a generalised criterion of risk-sensitive control is proposed, where noise dependent penalty of state and control variables is included. The solution is obtained explicitly by the change of measure approach.

- An extension of Merton's portfolio model has been proposed in this thesis (see Chapter 4, 5, 6, 7). All the interest rates described in this thesis can be unbounded processes, which is not the case with most of the existing literature where the interest rate is assumed to be bounded (see, e.g., [29] and [28]). In [30], [55], [13], [11], the portfolio problem with a possibly unbounded interest rate is considered. In [11] the authors introduce a new interest rate model of the quadratic-affine form, and propose an extension of the Merton's optimal investment problem with exponential utility. In Chapter 4, we extend their work. A new nonlinear interest rate model, which follows a stochastic process as a combination of the square-root process and the multi-dimensional quadratic-affine term structure model, is introduced.

Furthermore, the mean rates of return $\mu(t)$ are established in a more general form. The price of a zero-coupon bond is also obtained.

- An important stochastic interest rate model is the one introduced by Longstaff [34] (also called the *double square-root process*). A special case of our results in Chapter 5 is the problem of optimal investment under such an interest rate model, and appears that this by itself is new. In that chapter, we consider a further more general interest rate model, and solve the optimal investment problem in an infinite horizon for the power and logarithmic utilities. In Chapter 6 we solve the optimal investment and consumption problem under the assumption that not only the interest rate but the volatility as well follows the Longstaff model.

- Yong in [54] pointed that some care is needed when dealing with the optimal investment problems in a market with a CIR model for the interest rate. In Chapter 4, we give a detailed discussion on the existence of the optimal trading strategies for the CIR interest rate model. Furthermore, an extension of Yong's work to multi-dimensional square-root process is given in Chapter 3.

- Chapter 7 gives an extension to Merton's optimal consumption problem in an infinite horizon with a discounted criterion [39]. This problem for the Vasicek interest rate model was considered in [45], [44], [42]. The novelty in this chapter is to use a quadratic-affine interest rate model. The problem for the logarithmic utility is solved.

1.5 Summary

The results of this thesis should be of interest to individuals working in both risk-sensitive control and mathematical finance. From the control theory point of view, a class of nonlinear stochastic systems is considered for which the risk-sensitive control problem has an explicit solution. The importance of this class of control problems is illustrated with applications to mathematical finance. From the mathematical finance point of view, the thesis introduces several interest rate models, and solves the problems of optimal investment-consumption and bond pricing. A feature that runs throughout the thesis is that explicit closed-form solutions are obtained in all cases.

Chapter 2

Preliminaries

2.1 Introduction

We review some basic results from the risk-sensitive control and mathematical finance. This includes the basic risk-sensitive control problem, the market model and the self-financing trading strategy, the arbitrage in the market, the optimal investment and consumption problem, and two nonlinear stochastic processes. We also include some useful lemmas and theorems which will be used in the later chapters.

2.2 Risk sensitive control

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a complete filtered probability space, and $w_1(\cdot)$, $w_2(\cdot)$, $w_3(\cdot)$ be three independent n-dimensional standard Brownian motions . We define $\mathcal{F}_t = \sigma\{w(s); 0 \leq s < t\}$ to be the natural filtration augmented by all $\mathbb{P}$-null sets of $\mathcal{F}$, where $w(\cdot) = [w_1(\cdot), w_2(\cdot), w_3(\cdot)]'$.

We begin with the optimal control problem introduced by Jacobson [25] in 1973.

Consider the linear stochastic control system:

$$\begin{cases} \mathrm{d}x_1(t) = [A_1 x_1(t) + B_1 u(t)]\,\mathrm{d}t + \displaystyle\sum_{j=1}^{n} C_{1j}\mathrm{d}w_{1j}(t), \\ \\ x(0) = x_0, \end{cases} \tag{2.1}$$

and the risk-sensitive cost functional:

$$\bar{J}(u(\cdot)) = \gamma \mathbb{E}\left\{ \exp\left[\frac{\gamma}{2} x'(T) S x(T) + \frac{\gamma}{2} \int_0^T \left[x'(t)\tilde{Q}x(t) + u'(t)\tilde{P}u(t) \right] \mathrm{d}t \right] \right\}. \tag{2.2}$$

Here $x(t)$ is the states of the system, and the given data are known:

$$\begin{aligned} &A_1, S \in \mathbb{R}^{n_1 \times n_1}, \quad 0 \leq \tilde{Q} \in \mathbb{R}^{n_1 \times n_1} \quad B_1 \in \mathbb{R}^{n_1 \times m}, \\ &0 < \tilde{P} \in \mathbb{R}^{m \times m} \quad 0 \neq \gamma \in \mathbb{R}, \quad C_{1j} \in \mathbb{R}^{n_1}, \quad j = 1, \cdots, n. \end{aligned}$$

The control process $u(\cdot)$ is assumed to be square integrable, i.e.

$$\mathbb{E}\left[\int_0^T u'(t)u(t)\mathrm{d}t \right] < \infty,$$

and this ensures (2.1) has a unique solution. The control problem is to find an optimal $u(t)$ that minimises (2.2) subject to (2.1). Date and Gashi in [11] extend (2.1) by introducing a more general system which has quadratic terms in $x_1(t)$ and $u(t)$:

$$\begin{cases} \mathrm{d}x_2(t) = \left[A_{12}x_1(t) + A_{22}x_2(t) + D\left(x_1(t), u(t)\right) + B_{12}u(t) \right] \mathrm{d}t \\ \qquad\qquad + \displaystyle\sum_{j=1}^{n} \left[A_{3j}x_1(t) + B_{2j}u(t) + C_{2j} \right] \mathrm{d}w_{1j}(t), \\ x_2(0) = x_{20} \end{cases} \tag{2.3}$$

where $A_{12}, A_{3j} \in \mathbb{R}^{n_2 \times n_1}, A_{22} \in \mathbb{R}^{n_2 \times n_2}, B_{12}, B_{2j} \in \mathbb{R}^{n_2 \times m}, C_{2j} \in \mathbb{R}^{n_2}$ are known constants. The vector $D\left(x_1(t), u(t)\right)$ is defined as

$$D\left(x_1(t), u(t)\right) = \begin{bmatrix} x_1'(t)Q_1x_1(t) + u'(t)R_1x_1(t) + u'(t)P_1u(t) \\ x_1'(t)Q_2x_2(t) + u'(t)R_2x_2(t) + u'(t)P_2u(t) \\ \vdots \\ x_1'(t)Q_{n_2}x_1(t) + u'(t)R_{n_2}x_1(t) + u'(t)P_{n_2}u(t) \end{bmatrix},$$

where

$$\begin{aligned}
&Q_1, Q_2, \cdots, Q_{n_2} \in \mathbb{R}^{n_1 \times n_1} \\
&R_1, R_2, \cdots, R_{n_2} \in \mathbb{R}^{m \times n_1} \\
&P_1, P_2, \cdots, P_{n_2} \in \mathbb{R}^{m \times m},
\end{aligned}$$

and $Q_j, P_j, j = 1, \cdots, n_2$ are symmetric matrixes. We define the matrices Q, R, P as

$$Q := \begin{bmatrix} Q_1 \\ Q_2 \\ \vdots \\ Q_{n_2} \end{bmatrix}, R := \begin{bmatrix} R_1 \\ R_2 \\ \vdots \\ R_{n_2} \end{bmatrix}, P := \begin{bmatrix} P_1 \\ P_2 \\ \vdots \\ P_{n_2} \end{bmatrix}.$$

They also extend the criterion (2.2) by introducing a penalty on the state $x_2(t)$:

$$\begin{aligned}
J(u(\cdot)) \quad = \quad & \gamma \mathbb{E} \left[\exp \left\{ \frac{\gamma}{2} x_1'(T) S x_1(T) + \frac{\gamma}{2} \int_0^T \left[x_1'(t) \tilde{Q} x_1(t) + u'(t) \tilde{P} u(t) \right] \mathrm{d}t \right. \right. \\
& + \frac{\gamma}{2} \int_0^T \left[L_1' x_1(t) + L_2' x_2(t) + L_u' u(t) + u'(t) \tilde{R} x_1(t) \right] \mathrm{d}t \\
& \left. \left. + \frac{\gamma}{2} S_1' x_1(T) + \frac{\gamma}{2} S_2' x_2(T) \right\} \right], \qquad (2.4)
\end{aligned}$$

The given data are:

$$\begin{aligned}
& S, \tilde{Q} \in \mathbb{R}^{n_1 \times n_1}, \quad \tilde{P} \in \mathbb{R}^{m \times m}, \quad \tilde{R} \in \mathbb{R}^{m \times n_1}, \quad L_1, S_1 \in \mathbb{R}^{n_1} \\
& L_2, S_2 \in \mathbb{R}^{n_2}, \quad L_u \in \mathbb{R}^m.
\end{aligned}$$

They solve the optimal control problem of minimising (2.4) subject to (2.1) and (2.3). The optimal control is obtained as an affine function of the state $x_1(t)$ in a explicit closed form (see Theorem 1 in Date and Gashi [11]). Further extension of the Date and Gashi's work will be described in Chapter 3 in detail.

2.3 Formulation of market model and self-financing strategies

We consider a financial market consisting of a risk-free asset, which is a bond or a bank account, and n risky assets, which are the stocks. Denote the price of a bond by $S_0(t)$ with the interest rate being $r(t)$, and the price of stock i by $S_i(t)$, $i = 1, \ldots, n$. We assume the equations of these prices to be as follows (see, e.g., section 2.1 in Korn [29]):

$$\begin{cases} \mathrm{d}S_0(t) = S_0(t)r(t)\mathtt{d}t \\ \\ S_0(0) = S_{00}, \end{cases} \tag{2.5}$$

$$\begin{cases} \mathrm{d}S_i(t) = S_i(t)\Big(\mu_i(t)\mathrm{d}t + \sum_{j=1}^{m}\sigma_{ij}(t)\mathrm{d}w_j(t)\Big) \\ \\ S_i(0) = S_{i0}, \end{cases} \tag{2.6}$$

where the vector $\mu(t) \in \mathbb{R}^{n\times 1}$, $\mu(t) = [\mu_1(t), \ldots, \mu_n(t)]'$ is the mean rate of return, and the vectors $\sigma_i(t) \in \mathbb{R}^{1\times m}$, $\sigma_i(t) = [\sigma_{i1}(t), \ldots, \sigma_{im}(t)]$ are the volatilities. Here $w(t) \in \mathbb{R}^{m\times 1}$, $w(t) = [w_1(t), \ldots, w_m(t)]'$ is an m-dimensional Brownian motion, defined on a given complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with the natural filtration

$$\mathcal{F}_t = \sigma\{w(s); 0 \le s \le t\}, \quad \mathcal{F} = \mathcal{F}_T.$$

The following definitions are given in[29]:

Definition 2.3.1. *Let $T > 0$ be fixed (the "time horizon")*
i) A trading strategy is an $\mathbb{R}^n$-valued, $\mathcal{F}_t$-adapted process $v(t)$, $t \in [0, T]$, with

$$\int_0^T |v_0(t)|\mathrm{d}t < \infty \quad a.s.,$$

$$\sum_{i=1}^{n}\sum_{j=1}^{m}\int_0^T \Big(v_i(t)S_i(t)\sigma_{ij}\Big)^2 \mathrm{d}t < \infty \quad a.s..$$

ii) Let v be a trading strategy. The process

$$y(t) := \sum_{i=0}^{n}\Big(v_i(t)S_i(t)\Big)$$

is called the wealth process ("value of the current holdings") corresponding to v. $y(0)$ is called the initial wealth.

iii) A non-negative, adapted process $c(t)$, $t \in [0, T]$, with

$$\int_0^T c(t)\mathrm{d}t < \infty \quad a.s.$$

will be called a consumption rate process.
iv) A pair (v, c) consisting of a trading strategy v and a consumption process c will be called self-financing if the wealth process $y(t)$ corresponding to v satisfies

$$y(t) = y(0) + \sum_{i=0}^{n} \int_0^t v_i(s)\mathrm{d}S_i(s) - \int_0^t c(s)\mathrm{d}s, \qquad \forall t \in [0, T].$$

We only consider the self-financing trading in this thesis. That is to say, apart from the consumption at time t, the wealth before any action at time t should be the same with the wealth after this action at time t. We first look at the discrete time example (see Karatzas et. al. [27]):

Example 2.3.1. *Let the bond and stocks with prices be $S_0(\tau)$, $S_1(\tau)$ at time τ, $\tau = 0, \ldots, n$. Let $c(\tau)$ be the consumption at time τ and $v_0(\tau)$, $v_1(\tau)$ be the trading strategy. We assume that the investor trades in a self-financing way. There exists an equation:*

$$\begin{aligned} y(\tau) &= v_0(\tau)S_0(\tau) + v_1(\tau)S_1(\tau) \\ &= v_0(\tau-1)S_0(\tau) + v_1(\tau-1)S_1(\tau) - c(\tau) \\ &= v_0(\tau-1)[S_0(\tau) - S_0(\tau-1)] + v_1(\tau-1)[S_1(\tau) - S_1(\tau-1)] \\ &\quad -c(\tau) + v_0(\tau-1)S_0(\tau-1) + v_1(\tau-1)S_1(\tau-1) \\ &\quad \vdots \\ &= y(0) + \sum_{j=1}^{\tau} \Big[v_0(j)\Delta S_0(j) + v_1(j)\Delta S_1(j)\Big] - \sum_{j=1}^{\tau} c(j), \end{aligned}$$

where $\Delta S_k(\tau) = S_k(\tau) - S_j(\tau-1), k = 0, 1.$

Then the continuous-time analogue of Example 2.3.1 is

$$\mathrm{d}y(t) = \sum_{i=0}^{n} v_i(t)\mathrm{d}S_i(t) - c(t)\mathrm{d}t. \tag{2.7}$$

And the consumption $C(t)$ is the integral of consumption rate $c(t)$,

$$C(t) = \int_0^t c(s)\mathrm{d}s.$$

We substitute (2.5) and (2.6) into (2.7), and deduce

$$\begin{aligned}\mathrm{d}y(t) &= v_0(t)S_0(t)r(t)\mathrm{d}t + \sum_{i=1}^{n} v_i(t)S_i(t)[\mu_i(t)\mathrm{d}t + \sigma_i(t)\mathrm{d}w(t)] - c(t)\mathrm{d}t \\ &= \Big\{r(t)y(t) + u'(t)[\mu(t) - r(t)\mathbf{1}] - c(t)\Big\}\mathrm{d}t + u'(t)\sigma(t)\mathrm{d}w(t).\end{aligned}$$

Here $u_i(t) = v_i(t)S_i(t)$, $i = 1, \dots, n$, $u(t) = [u_1(t), \dots, u_n(t)]'$ is the control process, which is the amount of wealth invested in the stock market; and $\mathbf{1}$ is a vector of ones,

$$\mathbf{1} = [\underbrace{1, \dots, 1}_{n}].$$

2.4 No arbitrage of the market

We first give a definition of arbitrage (see Definition 12.1.3 in Øksendal [43]).

Definition 2.4.1. *An admissible portfolio $\theta(t)$ is called an arbitrage (in the market $\{X_t\}_{t\in[0,T]}$) if the corresponding value process $V^\theta(t)$ satisfies $V^\theta(0) = 0$ and*

$$V^\theta(T) \geq 0 \ \textit{a.s. and} \ \mathbb{P}[V^\theta(T) > 0] > 0.$$

In other words, we can say that an arbitrage is a transaction which begins with zero capital and later has an increase in the value with positive probability without any risk of loss [50]. However, in our model of financial market, we only consider the situation with no arbitrage. In other words, it means having an equivalent martingale measure for the market(2.5)-(2.6). Furthermore, from I. Karatzas and S. E. Shreve [28], if there exists a market price of risk process $\phi(t)$, that satisfies the following two conditions:

$$\begin{cases} \mu(t) - r(t) = \sigma\phi(t) \\ \mathbb{E}[e^{-\int_0^T \phi(t)'dw(t) - \frac{1}{2}\int_0^T |\phi(t)|^2 dt}] = 1, \end{cases} \tag{2.8}$$

an equivalent martingale measure exists.

2.5 Optimal investment and consumption

Let $y(t)$ be the wealth process of an investor who has the initial wealth $y_0 > 0$. The problem of optimal investment and consumption is to choose some reasonable values of the control $u(t)$ and consumption rate $c(t)$ to maximize the criteria. For the finite time horizon case, the most popular criterion is

$$\mathbb{J}(y; u, c) := \mathbb{E}\left[\int_0^T U_1\Big(t, c(t)\Big)\mathrm{d}t + U_2\Big(y(T)\Big)\right],$$

where $U_1(\cdot)$ and $U_2(\cdot)$ are utility functions. In [29] and [27] this optimization problem with bounded interest rate is considered. However, in this thesis, we focus on the *unbounded interest rates,* which follow nonlinear stochastic differential equations.

For the infinite time horizon case, the following is the typical criterion:

$$\mathbb{E}\left[\int_0^\infty \mathrm{e}^{-\rho t} U\Big(c(t)\Big)\mathrm{d}t\right],$$

where ρ is a positive constant.

With different utility functions, an investor could have different attitudes towards the risk. Assuming a twice continuously differentiable utilities, we have: if $U"(x) > 0$, then the investor is risk-seeking; if $U"(x) = 0$, the investor is risk-neutral; and if $U"(x) < 0$, the investor is risk-averse. In this thesis we only consider the power utility and the logarithmic utility.

2.6 Square-root process and the multi-dimensional square-root process

2.6.1 Vasicek interest rate model

In financial mathematics, there is an interest rate model, which is called the Vasicek model [51], being utilized frequently. Let $w(t)$, $t \geq 0$, be a Brownian motion. The stochastic differential equation for the Vasicek model is

$$\mathrm{d}x(t) = \Big(\alpha - \beta x(t)\Big)\mathrm{d}t + \sigma \mathrm{d}w(t),$$

where α, β, σ are positive constants. Its solution is:

$$x(t) = \mathrm{e}^{-\beta t} x(0) + \frac{\alpha}{\beta}\left(1 - \mathrm{e}^{-\beta} t\right) + \sigma \mathrm{e}^{-\beta} t \int_0^t \mathrm{e}^{\beta s} \mathrm{d}w(s).$$

2.6.2 Cox-Ingersoll-Ross (CIR) interest rate model

In Cox, Ingersoll and Ross [10], the stochastic differential equation

$$\mathrm{d}x(t) = [\alpha - \beta x(t)]\mathrm{d}t + \sigma\sqrt{x(t)}\mathrm{d}w(t)$$

is introduced as a model of the interest rate, for some positive constants α, β, σ. A certain constraint $2\alpha \geq \sigma^2$ ensures this process has a non-negative volatility. Different from the Vasicek model, this stochastic differential equation does not have an explicit solution. However, its advantage is that it has a positive solution (given that its initial value is also positive). The price of a zero-coupon bond for this model is:

$$B(t,T) = \mathrm{e}^{f_1(t,T) - f_2(t,T) r(t)},$$

Here $f_1(t,T)$, $f_2(t,T)$ are the following funstions:

$$f_1(t,T) = \frac{2\alpha}{\sigma^2} \ln\left\{\frac{\gamma \mathrm{e}^{\beta\tau/2}}{\gamma \cosh\gamma\tau + \frac{1}{2}\beta \sinh\gamma\tau}\right\},$$

$$f_2(t,T) = \frac{\sinh\gamma\tau}{\gamma \cosh\gamma\tau + \frac{1}{2}\beta \sinh\gamma\tau},$$

$$\tau = T - t, \quad 2\gamma = (\beta^2 + 2\sigma^2)^{1/2}.$$

2.6.3 Multi-dimensional square-root process

Duffie and Kan [14] introduced a generalisation of the CIR process with the equation:

$$\mathrm{d}x(t) = [A_3 x_3(t) + B_3]\,\mathrm{d}t + \Sigma \begin{pmatrix} \sqrt{v_1(x_3)} & 0 & \cdots & 0 \\ 0 & \sqrt{v_2(x_3)} & \cdots & 0 \\ & & \ddots & \\ 0 & \cdots & 0 & \sqrt{v_n(x_3)} \end{pmatrix} \mathrm{d}w(t),$$

where $A_3 \in \mathbb{R}^{n_3 \times n_3}, B_3 \in \mathbb{R}^{n_3}, \Sigma \in \mathbb{R}^{n_3 \times n}$, and

$$v_i(x_3) = \alpha_i + \beta_i' x(t),$$

for each i, $\alpha_i \in \mathbb{R}$, $\beta_i' \in \mathbb{R}^n$. The following two conditions ensure a strictly positive volatility:

Condition 2.6.1. *For all x such that $v_i(x) = 0$, $\beta_i'(A_3 x + B_3) > \dfrac{\beta_i' \Sigma \Sigma' \beta_i}{2}$.*

Condition 2.6.2. *For all j, if $(\beta_i' \Sigma)_j \neq 0$, then $v_i = v_j$.*

For convenient use in a Chapter 3, we rewrite this state and denote it as $x_3(t)$:

$$\begin{cases} \mathrm{d}x_3(t) = [A_3 x_3(t) + B_3]\,\mathrm{d}t + \sum_{j=1}^{n} \sqrt{v_j(x_3)}\sigma_j \mathrm{d}w_{3j}(t), \\ \\ x_3(0) = x_{30}, \end{cases} \tag{2.9}$$

where $A_3 \in \mathbb{R}^{n_3 \times n_3}, B_3 \in \mathbb{R}^{n_3}$ and for each j, $j = 1, \ldots, n$,

$$v_j(x_3) = \alpha_j + \beta_j' x_3(t), \quad \alpha_j \in \mathbb{R}, \quad \beta_j' \in \mathbb{R}^n, \quad \sigma_j = \begin{bmatrix} \sigma_{1j} \\ \sigma_{2j} \\ \vdots \\ \sigma_{n_3 j} \end{bmatrix} \in \mathbb{R}^{n_3}.$$

2.7 Double square-root process

We assume $x(t)$ to be governed by the following stochastic differential equation:

$$\begin{cases} \mathrm{d}x(t) = m\mathrm{d}t + s\mathrm{d}w_r(t) \\ \\ x(0) = x_0, \end{cases} \tag{2.10}$$

where m, s are constants. And we let the interest rate $r(t) := cx^2(t)$, c is a positive constant. The differential of the interest rate is:

$$\mathrm{d}r(t) = \left[cs^2 + 2m\sqrt{c}\sqrt{r(t)}\right]\mathrm{d}t + 2s\sqrt{c}\sqrt{r(t)}\mathrm{d}w_r(t), \tag{2.11}$$

which can also be written as

$$\mathrm{d}r(t) = k_r[\theta_r - \sqrt{r(t)}]\mathrm{d}t + \sigma_r \sqrt{r(t)}\mathrm{d}w_r(t), \tag{2.12}$$

where k_r, σ_r are positive constants, and $\theta_r = \dfrac{\sigma_r^2}{4k_r}$.

The stochastic differential equation of type (2.12) was first introduced by Longstaff in 1989. Compared with the CIR process, it is designated as the *double square-root (DSR) process*, because the square-root process $\sqrt{r}$ appears twice in (2.12). Several empirical comparisons of these two models are discussed in Lonstaff [34], where this DSR model outperforms the CIR model in some situations. The closed form expression for the price of a zero-coupon bond with Longstaff interest rate is derived as follows:

$$B(t,T) = \mathrm{e}^{f_1(t,T) - f_2(t,T)r(t) - f_3(t,T)\sqrt{r(t)}},$$

where $f_1(\cdot)$, $f_2(\cdot)$, $f_3(\cdot)$ are some known explicit functions. This bond's yield is such a nonlinear case of the interest rate that the bond price is not a monotone function of current interest rate. It makes the valuation of a bond option less straightforward than usual (see Chapter 10 in [40]).

2.8 Some useful theorems and lemmas

First, we introduce two important theorems which will be used in Chapter 3 and 4 to prove the existence of admissible control.

Theorem 2.8.1. *Let $\xi_0 \in \mathbb{R}$ and let $b_0, b_1 : [0,\infty]\times\Omega \to \mathbb{R}$ and $\sigma : [0,\infty]\times\Omega \to \mathbb{R}^d$ be $\{\mathcal{F}_t\}_{t\geq 0}$-adapted processes satisfying*

$$b_0(\cdot) \in L^\infty_{\mathcal{F}}(\Omega; L^1(0,T;\mathbb{R})), \qquad b_1(\cdot) \in L^\infty_{\mathcal{F}}(0,T;\mathbb{R})$$
$$\sigma(\cdot) \in L^2_{\mathcal{F}}(0,T;L^\infty(\Omega;\mathbb{R}^d)), \qquad \forall T > 0.$$

Let $\xi(\cdot)$ be an $\{\mathcal{F}_t\}_{t\geq 0}$-adapted process satisfying

$$\begin{cases} \mathrm{d}\xi(t) \leq \Big[b_0(t) + b_1(t)\xi(t)\Big]\mathrm{d}t + \sigma(t)'\mathrm{d}w(t), & t \geq 0, \\ \xi(0) = \xi_0. \end{cases} \tag{2.13}$$

Suppose $\varphi : \mathbb{R} \to [0,\infty)$ is continuous such that for some $\gamma \in [0,2]$ and $c > 0$,

$$\lim_{x\to\infty} \frac{\varphi(x)}{x^\gamma} < c. \tag{2.14}$$

Then

$$\mathbb{E}\left[\sup_{t\in[0,T]} \mathrm{e}^{\varphi(\xi(t))}\right] < \infty \tag{2.15}$$

provided

$$\begin{aligned}
&\textit{either} \qquad \gamma \in [0,2), \\
&\textit{or} \qquad \gamma = 2, \\
&2c\left[\sup_{(t,\omega)\in[0,T]\times\Omega} \exp\left\{2\int_0^t b_1(u,\omega)\mathrm{d}u\right\}\right]\int_0^T \sup_{\omega\in\Omega} \mathrm{e}^{-2\int_0^s b_1(u,\omega)\mathrm{d}u}|\sigma(s,\omega)|^2\mathrm{d}s < 1.
\end{aligned}$$

Further,

$$\mathbb{E}\left[\mathrm{e}^{\int_0^T \varphi(\xi(t))\mathrm{d}t}\right] < \infty \tag{2.16}$$

provided

$$\begin{aligned}
&\textit{either} \qquad \gamma \in [0,2), \\
&\textit{or} \qquad \gamma = 2, \\
&2Tc\left[\sup_{(t,\omega)\in[0,T]\times\Omega} \exp\left\{2\int_0^t b_1(u,\omega)\mathrm{d}u\right\}\right]\int_0^T \sup_{\omega\in\Omega} \mathrm{e}^{-2\int_0^s b_1(u,\omega)\mathrm{d}u}|\sigma(s,\omega)|^2\mathrm{d}s < 1.
\end{aligned}$$

Theorem 2.8.2. *Let $\alpha, \beta : [0,\infty) \to (0,\infty), v : [0,\infty) \to \mathbb{R}^d$ be deterministic maps. Suppose the short interest rate $r(\cdot)$ satisfies the following SDE:*

$$\begin{cases}
\mathrm{d}r(t) = [\alpha(t) - \beta(t)r(t)]\mathrm{d}t + \sqrt{r(t)}v(t)'\mathrm{d}w(t), \\
r(0) = r_0.
\end{cases} \tag{2.17}$$

Then, for $\lambda > 0$,

$$\mathbb{E}\left[\mathrm{e}^{\lambda\int_0^T r(t)\mathrm{d}t}\right] < \infty \tag{2.18}$$

provided

$$4\alpha(t) \le |v(t)|^2, t \in [0,T], \quad \frac{\lambda T}{2}\int_0^T \mathrm{e}^{\int_0^s \beta(u)\mathrm{d}u}|v(s)|^2\mathrm{d}s < 1.$$

These two theorems are given by Yong (see Theorem 3.1 and Theorem 4.1 [54]). However, his point is only noticed by few researchers.

Another crucial theorem is stated in Liu's paper (see Lemma 2 in Appendix [33]), which is also used by Rong and Chang [7]. And his idea will be utilized in Chapter (6).

Theorem 2.8.3. *Suppose that*

$$\frac{\partial \hat{f}}{\partial t} + \mathcal{L}\hat{f} = 0, \tag{2.19}$$

and $\hat{f}(T, X) = 1$. $\mathcal{L}$ *is the linear operator on any function* f. *Then the function* f *defined by*

$$f(t, X) = \alpha^{\frac{1}{\gamma}} \int_t^T \hat{f}(u, X)\mathrm{d}u + (1-\alpha)^{\frac{1}{\gamma}} \hat{f}(t, X) \tag{2.20}$$

satisfies

$$\frac{\partial f}{\partial t} + \mathcal{L}f + \alpha^{\frac{1}{\gamma}} = 0, \tag{2.21}$$

and $f(T, X) = (1-\alpha)^{\frac{1}{\gamma}}$.

The proof is omitted here.

We also introduce an important lemma (see Corollary C.2 in [4]).

Lemma 2.8.1. *Suppose vector* $x(t)$ *is governed by the following stochastic differential equation:*

$$\begin{cases} \mathrm{d}x(t) = [Ax(t) + a]\mathrm{d}t + D\mathrm{d}w(t), \quad t \geq 0, \\ x(0) = x_0, \end{cases}$$

where $A, D \in \mathbb{R}^{n\times n}$, $a \in \mathbb{R}^n$ *and* $w(\cdot)$ *is a n-dimensional standard Brownian motion. Then*

$$\mathbb{E}\left[\mathrm{e}^{\beta \int_0^T |x(t)|^\delta \mathrm{d}t} < \infty\right], \quad \forall \beta > 0, \delta \in [0, 2).$$

Furthermore, the above holds for $\delta = 2$, *provided the following holds for* $\beta > 0$:

$$2\beta T \int_0^T |\mathrm{e}^{At} D|^2 \mathrm{d}t < 1.$$

Chapter 3

Risk-sensitive control for a class of non-linear processes with multiplicative noise

In this chapter, we consider the risk-sensitive control problem for a class of nonlinear systems. The nonlinearity consists of quadratic and square-root terms in the state. In Fei and Gashi's work [15], the scalar case of this problem has been solved, and now we extend it further, which contains multiplicative noise. By using the completion of squares method, the solution to such an optimal control problem is obtained in an explicit closed-form. We also give some conditions on which the risk-sensitive control problem has the unique solution.

3.1 Introduction

We begin with the Date and Gashi's work [11] described in Section (2.2). In this chapter, we extend their problem further, while preserving its explicit closed form solvability. We do so by introducing further nonlinear components of the state, which are represented in (2.9). It is the multi-dimensional square root process first introduced by Duffie and Kan [14].

On the other hand, the system state $x_2(t)$ in (2.3) is further extended by introducing $x_3(t)$ term. It ensures the system $x_2(t)$ containing the quadratic term of $x_1(t)$, the control process $u(t)$, and also the square root process $x_3(t)$. These three states $x_1(t)$, $x_2(t)$ and $x_3(t)$ are not separated with each other.

Thus, on the probability space defined in Section (2.2), we formulate a new system states as follows:

$$\begin{cases} \mathrm{d}x_1(t) = [A_1 x_1(t) + B_1 u(t)]\,\mathrm{d}t + \sum_{j=1}^{n} C_{1j}\mathrm{d}w_{1j}(t), \\ \\ \mathrm{d}x_2(t) = [A_{12}x_1(t) + A_{22}x_2(t) + A_{42}x_3(t) + D\left(x_1(t), u(t)\right) + B_{12}u(t)]\,\mathrm{d}t \\ \qquad\qquad + \sum_{j=1}^{n} [A_{3j}x_1(t) + B_{2j}u(t) + C_{2j}]\,\mathrm{d}w_{1j}(t) + \sum_{j=1}^{n} \sqrt{v_j(x_3)}\sigma_j \mathrm{d}w_{2j}(t), \\ \\ \mathrm{d}x_3(t) = [A_3 x_3(t) + B_3]\,\mathrm{d}t + \sum_{j=1}^{n} \sqrt{v_j(x_3)}\sigma_j \mathrm{d}w_{3j}(t), \\ \\ x_1(0) = x_{10}, \qquad x_2(0) = x_{20}, \qquad x_3(0) = x_{30}, \end{cases} \tag{3.1}$$

where

$$A_1 \in \mathbb{R}^{n_1\times n_1}, \quad , B_1 \in \mathbb{R}^{n_1\times m}, \quad C_{1j} \in \mathbb{R}^{n_1}, \quad A_{12}, A_{3j} \in \mathbb{R}^{n_2\times n_1},$$
$$A_{22} \in \mathbb{R}^{n_2\times n_2}, \quad A_{42} \in \mathbb{R}^{n_2\times n_3}, \quad B_{12}, B_{2j} \in \mathbb{R}^{n_2\times m}, \quad C_{2j} \in \mathbb{R}^{n_2},$$
$$A_3 \in \mathbb{R}^{n_3\times n_3}, \quad B_3 \in \mathbb{R}^{n_3}, \quad \Sigma \in \mathbb{R}^{n_3\times n}$$

are known constants. The vector $D\left(x_1(t), u(t)\right)$ is defined as

$$D\left(x_1(t), u(t)\right) = \begin{bmatrix} x_1'(t)Q_1 x_1(t) + u'(t)R_1 x_1(t) + u'(t)P_1 u(t) \\ \\ x_1'(t)Q_2 x_2(t) + u'(t)R_2 x_2(t) + u'(t)P_2 u(t) \\ \\ \vdots \\ \\ x_1'(t)Q_{n_2} x_1(t) + u'(t)R_{n_2} x_1(t) + u'(t)P_{n_2} u(t) \end{bmatrix},$$

where

$$Q_1, Q_2, \cdots, Q_{n_2} \in \mathbb{R}^{n_1\times n_1}$$
$$R_1, R_2, \cdots, R_{n_2} \in \mathbb{R}^{m\times n_1}$$
$$P_1, P_2, \cdots, P_{n_2} \in \mathbb{R}^{m\times m},$$

and $Q_j, P_j, j = 1, \cdots, n_2$ are symmetric matrixes. We denote matrixes Q, R, P given by

$$Q = \begin{bmatrix} Q_1 \\ Q_2 \\ \vdots \\ Q_{n_2} \end{bmatrix}, R = \begin{bmatrix} R_1 \\ R_2 \\ \vdots \\ R_{n_2} \end{bmatrix}, P = \begin{bmatrix} P_1 \\ P_2 \\ \vdots \\ P_{n_2} \end{bmatrix}.$$

Furthermore, function $v_j(x_3)$ is represented as

$$v_j(x_3) = \alpha_j + \beta_j' x_3(t),$$

for each j, $\alpha_j \in \mathbb{R}$, $\quad \beta_j' \in \mathbb{R}^n$, and

$$\sigma_1 = \begin{bmatrix} \sigma_{11} \\ \sigma_{21} \\ \vdots \\ \sigma_{n_3 1} \end{bmatrix}, \sigma_2 = \begin{bmatrix} \sigma_{12} \\ \sigma_{22} \\ \vdots \\ \sigma_{n_3 2} \end{bmatrix}, \cdots, \sigma_n = \begin{bmatrix} \sigma_{1n} \\ \sigma_{2n} \\ \vdots \\ \sigma_{n_3 n} \end{bmatrix} \in \mathbb{R}^{n_3}.$$

Under the state systems (3.1), we extend the criterion (2.4) by introducing $x_2(t)$ and $x_3(t)$ as

$$\begin{aligned} J(u(\cdot)) \quad = \quad & \gamma \mathbb{E} \left[\exp \left\{ \frac{\gamma}{2} x_1'(T) S x_1(T) + \frac{\gamma}{2} \int_0^T \left[x_1'(t) \tilde{Q} x_1(t) + u'(t) \tilde{P} u(t) \right] \mathrm{d}t \right. \right. \\ & + \frac{\gamma}{2} \int_0^T \left[L_1' x_1(t) + L_2' x_2(t) + L_3' x_3(t) + L_u' u(t) + u'(t) \tilde{R} x_1(t) \right] \mathrm{d}t \\ & \left. \left. + \frac{\gamma}{2} S_1' x_1(T) + \frac{\gamma}{2} S_2' x_2(T) + \frac{\gamma}{2} S_3' x_3(T) \right\} \right], \end{aligned} \tag{3.2}$$

and the given data

$$\begin{aligned} & S, \tilde{Q} \in \mathbb{R}^{n_1 \times n_1}, \quad \tilde{P} \in \mathbb{R}^{m \times m}, \quad \tilde{R} \in \mathbb{R}^{m \times n_1}, \quad L_1, S_1 \in \mathbb{R}^{n_1} \\ & L_2, S_2 \in \mathbb{R}^{n_2}, \quad L_3, S_3 \in \mathbb{R}^{n_3}, \quad L_u \in \mathbb{R}^m. \end{aligned}$$

The main contribution of this chapter is the solution to the following optimal control problem:

$$\begin{cases} \min\limits_{u(\cdot) \in \mathcal{A}} J(u(\cdot)) \\ \\ s.t. (3.1) \quad holds, \end{cases} \tag{3.3}$$

where $J(u(\cdot))$ is as defined in (3.2). The set $\mathcal{A}$ is the admissible control set, which will be explained later. We obtain the solution in an explicit closed form by using the completion of squares method. This is clearly a rare example of a stochastic control problem that admits a fully explicit solution.

3.2 Risk-sensitive control

Let us introduce the processes $h(t)$ and $H(t)$ as:

$$\begin{cases} \mathrm{d}h(t) = \Big[x_1'(t)\tilde{Q}x_1(t) + u'(t)\tilde{R}x_1(t) + u'(t)\tilde{P}u(t) + L_1'x_1(t) \\ \qquad\qquad +L_2'x_2(t) + L_3'(t)x_3(t) + L_u'u(t)\Big]\,\mathrm{d}t, \\ h(0) = 0, \end{cases}$$

and

$$H(t) = h(t) + x_1'(t)G_1(t)x_1(t) + g_2'(t)x_1(t) + g_3'(t)x_2(t) + g_4'(t)x_3(t) + g_5(t).$$

where

$$\begin{aligned} &G_1(\cdot) \in L^\infty(0,T;\mathbb{R}^{n_1\times n_1}), \quad g_2(\cdot) \in L^\infty(0,T;\mathbb{R}^{n_1}), \\ &g_3(\cdot) \in L^\infty(0,T;\mathbb{R}^{n_2}), \quad g_4(\cdot) \in L^\infty(0,T;\mathbb{R}^{n_3}), \quad g_5(\cdot) \in L^\infty(0,T;\mathbb{R}). \end{aligned}$$

Here $L^\infty(\cdot)$ denotes the set of uniformly bounded functions, and $G_1(\cdot)$ is symmetric.

We further let G_1, g_2, g_3, g_4 be functions which satisfy the following Riccati and linear differential equations:

$$\begin{cases} 4\gamma\tilde{Q} + 4\gamma\dot{G}_1(t) + 8\gamma G_1(t)A_1 + 4\gamma^2\sum_{j=1}^{n} G_1(t)C_{1j}C_{1j}'G_1(t) \\ \quad +4\gamma g_3'(t)Q + \gamma^2\sum_{j=1}^{n} A_{3j}'g_3(t)g_3'(t)A_{3j} - 2K_2'(t)K_1^{-1}(t)K_2(t) = 0, \\ G_1(T) = S, \end{cases} \tag{3.4}$$

$$
\begin{cases}
L_2' + \dot{g_3}'(t) + g_3'(t)A_{22} = 0, \\
\\
g_3(T) = S_2,
\end{cases}
\tag{3.5}
$$

$$
\begin{cases}
2\gamma L_1' + 2\gamma \dot{g_2}'(t) + 2\gamma g_2'(t)A_1 + 2\gamma^2 \sum_{j=1}^{n} g_2'(t)C_{1j}C_{1j}'G_1(t) + 2\gamma g_3'(t)A_{12} \\
\quad + \gamma^2 \sum_{j=1}^{n} C_{2j}' g_3(t) g_3'(t) A_{3j} - K_3'(t)\Big[\Big(K_1^{-1}(t)\Big)' + K_1^{-1}(t)\Big]K_2(t) = 0, \\
\\
g_2(T) = S_1,
\end{cases}
\tag{3.6}
$$

$$
\begin{cases}
4L_3' + 4\dot{g_4}'(t) + 4g_4'(t)A_3 + \gamma^2 \sum_{j=1}^{n} \sigma_j'\big[g_3(t)g_3'(t) + g_4(t)g_4'(t)\big]\sigma_j\beta_j' \\
\quad + \frac{\gamma}{2} g_3'(t)A_{42} = 0, \\
g_4(T) = S_3,
\end{cases}
\tag{3.7}
$$

$$
\begin{cases}
2\gamma^2 \sum_{j=1}^{n} C_{1j}'G_1(t)C_{1j} + \gamma^2 \sum_{j=1}^{n} C_{1j}'g_2(t)g_2'(t)C_{1j} + \gamma^2 \sum_{j=1}^{n} C_{2j}'g_3(t)g_3'(t)C_{2j} \\
\quad + 4\gamma g_4'(t)B_3 + 4\gamma \dot{g_5}(t) + \gamma^2 \sum_{j=1}^{n} \alpha_j \sigma_j'\big[g_3(t)g_3'(t) + g_4(t)g_4'(t)\big]\sigma_j \\
\quad - 2K_3'(t)K_1^{-1}(t)K_3(t) = 0, \\
\\
g_5(T) = 0,
\end{cases}
\tag{3.8}
$$

where $K_1(t)$, $K_2(t)$ and $K_3(t)$ defined as following:

$$
K_1(t) \equiv \frac{\gamma}{2}\tilde{P} + \frac{\gamma}{2} g_3'(t)P + \frac{\gamma^2}{8} \sum_{j=1}^{n} B_{2j}' g_3(t) g_3'(t) B_{2j} > 0,
$$

$$K_2(t) \equiv \frac{\gamma}{2}\tilde{R} + \gamma B_1' G_1(t) + \frac{\gamma}{2} g_3'(t) R + \frac{\gamma^2}{4}\sum_{j=1}^{n} B_{2j}' g_3(t) g_3'(t) A_{3j},$$

$$K_3(t) \equiv \frac{\gamma}{2} L_u + \frac{\gamma}{2} B_1' g_2(t) + \frac{\gamma}{2} B_{12}' g_3(t) + \frac{\gamma^2}{4}\sum_{j=1}^{n} B_{2j}' g_3(t) g_3'(t) C_{2j}.$$

Here we give a numerical example which is suitable to the system and cost function.

Example 3.2.1. *We choose the value of each parameter:*

$$n_1 = 1, \quad n_2 = 1, \quad n_3 = 1, \quad m = 1, \quad n = 1, \quad \gamma = 2, \quad \tilde{Q} = 1, \quad A_1 = 1,$$
$$C_{11} = \sqrt{2}, \quad Q = 1, \quad A_{31} = 1, \quad \tilde{P} = 1, \quad P = 1, \quad B_{21} = 2, \quad \tilde{R} = 1,$$
$$B_1 = 4, \quad R = 1, \quad S = \tfrac{3}{4}, \quad L_2 = 1, \quad A_{22} = -1, \quad S_2 = 1, \quad L_1 = 1,$$
$$A_{12} = 1, \quad C_{21} = 1, \quad L_u = 1, \quad B_{12} = 1, \quad S_1 = 1, \quad L_3 = 2, \quad A_3 = 3,$$
$$\sigma_1 = 1, \quad \beta_1 = 1, \quad A_{42} = 1, \quad S_3 = -\tfrac{1}{2}, \quad B_3 = 1, \quad \alpha_1 = 1.$$

Therefore, equations (3.4), (3.5), (3.6), (3.7), (3.8) *become:*

$$\begin{cases} 8 + 8\dot{G}_1(t) + 16G_1(t) + 32G_1^2(t) + 8g_3(t) + 4g_3^2(t) - 2\dfrac{K_2^2(t)}{K_1(t)} = 0, \\[2em] G_1(T) = \dfrac{3}{4}, \end{cases}$$

$$\begin{cases} 1 + g_3(t) - g_3(t) = 0, \\[2em] g_3(T) = 1, \end{cases}$$

$$\begin{cases} 4 + 4\dot{g}_2(t) + 4g_2(t) + 16g_2(t)G_1(t) + 4g_3(t) + 4g_3^2(t) - 2\dfrac{K_3(t)K_2(t)}{K_1(t)} = 0 \\[2em] g_2(T) = 1, \end{cases}$$

$$\begin{cases} 8 + 4\dot{g}_4(t) + 12g_4(t) + 4[g_3^2(t) + g_4^2(t)] + g_3(t) = 0, \\ \\ g_4(T) = -\dfrac{1}{2}, \end{cases}$$

$$\begin{cases} 16G_1(t) + 8g_2^2(t) + 4g_3^2(t) + 8g_4(t) + 8\dot{g}_5(t) + 4(g_3^2(t) + g_4^2(t)) - 2\dfrac{K_3^2(t)}{K_1(t)} = 0, \\ \\ g_5(T) = 0, \end{cases}$$

where

$$K_1(t) = 1 + g_3(t) + 2g_3^2(t),$$

$$K_2(t) = 1 + 8G_1(t) + g_3(t) + 2g_3^2(t),$$

$$K_3(t) = 1 + 4g_2(t) + g_3(t) + 2g_3^2(t).$$

Thus $G_1(t), g_2(t), g_3(t), g_4(t)$ *can be solved as follows:*

$$\begin{cases} G_1(t) = \dfrac{3}{4}, \\ g_2(t) = -2 + 3e^{t-T}, \\ g_3(t) = 1, \\ g_4(t) = -\dfrac{3}{2} + \tan\left(-t + T + \dfrac{\pi}{4}\right), \\ g_5(t) = \dfrac{1}{2}\tan\left(-t + T + \dfrac{\pi}{4}\right) - \dfrac{1}{4}\ln\left(1 + \left(\tan\left(-t + T + \dfrac{\pi}{4}\right)\right)^2\right) + 6\,e^{t-T} \\ \qquad -\dfrac{37\,t}{8} + \dfrac{37\,T}{8} - \dfrac{13}{2} + \dfrac{1}{4}\ln(2). \end{cases}$$

3.2.1 Admissible Controls

The purpose of this section is to provide some sufficient conditions which ensure control processes $u(\cdot)$ belongs to admissible control set $\mathcal{A}$. We use the method in Date and Gashi [11], to deduce the set $\mathcal{A}$, on which, for all $u(\cdot) \in \mathcal{A}$, the following inequality holds:

$$J(u(\cdot)) = \gamma \mathbb{E}\left[\mathrm{e}^{\frac{\gamma}{2}H(T)}\right] \leq \gamma \left(\mathbb{E}\left[\mathrm{e}^{p\gamma H(T)}\right]\right)^{\frac{1}{2p}} < \infty, \qquad p > 1.$$

Let us assume the control process follow the linear case in state $x_1(t)$, which given by

$$\bar{u}(t) = \bar{K}_0 + \bar{K}_1 x_1(t),$$

where $\bar{K}_0(\cdot) \in L^\infty(0,T;\mathbb{R}^m)$ and $\bar{K}_1(\cdot) \in L^\infty(0,T;\mathbb{R}^{m\times n_1})$. Substituting $\bar{u}(t)$ into equations of $x_1(t)$ and $x_2(t)$, we have new states respectively:

$$\mathrm{d}x_1(t) = \left(\bar{A}_1 x_1(t) + \bar{B}_1\right)\mathrm{d}t + \sum_{j=1}^{n} C_{1j}\mathrm{d}w_{1j}(t), \tag{3.9}$$

$$\begin{aligned}\mathrm{d}x_2(t) = {} & \left[\bar{A}_{12}x_1(t) + A_{22}x_2(t) + A_{42}x_3(t) + \bar{D}(x_1,u)\right]\mathrm{d}t \\ & + \sum_{j=1}^{n}\left[\bar{A}_{3j}x_1(t) + \bar{C}_{2j}\right]\mathrm{d}w_{1j}(t) + \sum_{j=1}^{n}\sqrt{v_j(x_3)}\sigma_j \mathrm{d}w_{2j}(t),\end{aligned}$$

where $\bar{A}_{12} = A_{12} + B_{12}\bar{K}_1$, $\bar{A}_{3j} = A_{3j} + B_{2j}\bar{K}_1$, $\bar{C}_{2j} = B_{2j}\bar{K}_0 + C_{2j}$,

$$\bar{D}\left(x_1(t), u(t)\right) = \begin{bmatrix} x_1'(t)\bar{Q}_1 x_1(t) + \bar{R}_1 x_1(t) + \bar{P}_1 \\ x_1'(t)\bar{Q}_2 x_1(t) + \bar{R}_2 x_1(t) + \bar{P}_2 \\ \vdots \\ x_1'(t)\bar{Q}_{n_2} x_1(t) + \bar{R}_{n_2} x_1(t) + \bar{P}_{n_2} \end{bmatrix},$$

we have

$$\bar{Q}_i = Q_i + \bar{K}_1' R_i + \bar{K}_1' P_i \bar{K}_1,$$

$$\bar{R}_i = \bar{K}_0' R_i + 2\bar{K}_0' P_i \bar{K}_1,$$

$$\bar{P}_i = \bar{K}_0' P_i \bar{K}_0 + (B_{12}\bar{K}_0)_i,$$

$j = 1, \cdots, n$ and $i = 1, \cdots, n_2$. We also denote matrixes $\bar{Q}, \bar{R}, \bar{P}$ given by

$$\bar{Q} = \begin{bmatrix} \bar{Q}_1 \\ \bar{Q}_2 \\ \vdots \\ \bar{Q}_{n_2} \end{bmatrix}, \bar{R} = \begin{bmatrix} \bar{R}_1 \\ \bar{R}_2 \\ \vdots \\ \bar{R}_{n_2} \end{bmatrix}, \bar{P} = \begin{bmatrix} \bar{P}_1 \\ \bar{P}_2 \\ \vdots \\ \bar{P}_{n_2} \end{bmatrix}.$$

Using equations (3.5), we can deduce

$$\begin{aligned} & \int_0^t L_2' x_2(t) \mathrm{d}s + g_3'(t) x_2(t) \\ = \quad & g_3'(0) x_2(0) + \int_0^n \left[g_3'(s) \bar{A}_{12} x_1(s) + g_3'(s) A_{42} x_3(s) + g_3'(s) \bar{D}(x_1, u) \right] \mathrm{d}s \\ & + \sum_{j=1}^n \int_0^t \left(g_3'(s) \bar{A}_{3j} x_1(s) + g_3'(s) \bar{C}_{2j} \right) \mathrm{d}w_{1j}(s) \\ & + \sum_{j=1}^n \int_0^t \sqrt{v_j(x_3)} g_3'(s) \sigma_j \mathrm{d}w_{2j}(s), \end{aligned}$$

where the product $g_3'(t)\bar{D}(x_1, u)$ can be written as

$$g_3'(t)\bar{D}(x_1, u) = x_1'(t) g_3'(t) \bar{Q} x_1(t) + g_3'(t) \bar{R} x_1(t) + g_3'(t) \bar{P}$$

Next we find $H(t)$ under the control processes $\bar{u}(t)$

$$\begin{aligned} & H(t) \\ = \quad & g_3'(0) x_2(0) + x_1'(t) G_1(t) x_1(t) + g_2'(t) x_1(t) + g_4'(t) x_3(t) + g_5(t) \\ & + \int_0^t x_1'(s) \left\{ \tilde{Q} + \bar{K}_1' \tilde{R} + \bar{K}_1' \tilde{P} \bar{K}_1 + g_3'(s) \bar{Q} \right\} x_1(s) \mathrm{d}s \\ & + \int_0^t \left\{ \bar{K}_0' \tilde{R} + 2 \bar{K}_0' \tilde{P} \bar{K}_1 + L_1' + L_u' \bar{K}_1 + g_3'(s) \bar{A}_{12} + g_3'(s) \bar{R} \right\} x_1(s) \mathrm{d}s \\ & + \int_0^t \left\{ \bar{K}_0' \tilde{P} \bar{K}_0 + L_u' \bar{K}_0 + g_3'(s) \bar{P} \right\} \mathrm{d}s + \int_0^t L_3' x_3(s) \mathrm{d}s \end{aligned}$$

$$+\sum_{j=1}^{n}\int_0^t \sqrt{v_j(x_3)}g_3'(s)\sigma_j \mathrm{d}w_{2j}(s)$$

$$+\sum_{j=1}^{n}\int_0^t \left(g_3'(s)\bar{A}_{3j}x_1(s)+g_3'(s)\bar{C}_{2j}\right)\mathrm{d}w_{1j}(s).$$

We introduce some symmetric and differentiable function $M_1(t)$ and $M_2(t)$ and differentiable function $M_3(t)$ with the initial conditions $M_1(0)=0$, $M_2(0)=0$ and $M_3(0)=0$ respectively, also the following holds:

$$\begin{aligned}
0 \;=\; & -x_1'(t)M_1(t)x_1(t)+\int_0^t x_1'(s)\left[\dot{M}_1(s)+2M_1(s)\bar{A}_1\right]x_1(s)\mathrm{d}s \\
& -M_2'(t)x_1(t)+\int_0^t \left[2\bar{B}_1'M_1(s)+\dot{M}_2(s)+M_2'(s)\bar{A}_1\right]x_1(s)\mathrm{d}s \\
& -M_3'(t)x_3(t)+\int_0^t \left[\dot{M}_3'(s)+M_3'(s)A_3\right]x_3(s)\mathrm{d}s \\
& +\sum_{j=1}^{n}\int_0^t \left[C_{1j}'M_1(s)C_{1j}+M_2'(s)\bar{B}_1+M_3'(s)B_3\right]\mathrm{d}s \\
& +\sum_{j=1}^{n}\int_0^t \left[2C_{1j}'M_1(s)x_1(s)+M_2'(s)C_{1j}\right]\mathrm{d}w_{1j}(s) \\
& +\sum_{j=1}^{n}\int_0^t \sqrt{v_j(x_3)}M_3'(s)\sigma_j \mathrm{d}w_{2j}(s).
\end{aligned}$$

Adding this equation to the right hand side of $H(t)$, it can be obtained

$$\begin{aligned}
& H(t) \\
= \;& \int_0^t x_1'(s)\left\{\tilde{Q}+\bar{K}_1'\tilde{R}+\bar{K}_1'\tilde{P}\bar{K}_1+g_3'(s)\bar{Q}+\dot{M}_1(s)+2M_1(s)\bar{A}_1\right\}x_1(s)\mathrm{d}s \\
& +\int_0^t \Big\{\bar{K}_0'\tilde{R}+2\bar{K}_0'\tilde{P}\bar{K}_1+L_1'+L_u'\bar{K}_1+g_3'(s)\bar{A}_{12}+g_3'(s)\bar{R} \\
& \qquad +2\bar{B}_1'M_1(s)+\dot{M}_2(s)+M_2'(s)\bar{A}_1\Big\}x_1(s)\mathrm{d}s
\end{aligned}$$

$$+\int_0^t \Big\{L_3' + \dot{M}_3'(s) + M_3'(s)A_3\Big\}\, x_3(s)\mathrm{d}s$$

$$+x_1'(t)\Big[G_1(t) - M_1(t)\Big]x_1(t) + \Big[g_2'(t) - M_2'(t)\Big]x_1(t) + \Big[g_4'(t) - M_3'(t)\Big]x_3(t)$$

$$+\int_0^t \Bigg\{\bar{K}_0'\tilde{P}\bar{K}_0 + L_u'\bar{K}_0 + g_3'(s)\bar{P} + C_{1j}'M_1(s)C_{1j} + M_2'(s)\bar{B}_1 + M_3'(s)B_3\Bigg\}\mathrm{d}s$$

$$+g_3'(0)x_2(0) + g_5(t)$$

$$+\sum_{j=1}^n \int_0^t \Bigg[\Big(2C_{1j}'M_1(s) + g_3'(s)\bar{A}_{3j}\Big)x_1(s) + \Big(M_2'(s)C_{1j} + g_3'(s)\bar{C}_{2j}\Big)\Bigg]\mathrm{d}w_{1j}(s)$$

$$+\sum_{j=1}^n \int_0^t \sqrt{v_j(x_3)}\Big[M_3'(s) + g_3'(s)\Big]\sigma_j \mathrm{d}w_{2j}(s).$$

The stochastic integral part can be written as

$$\begin{aligned} &\sum_{j=1}^n \int_0^t N_{1j}(s)\mathrm{d}w_{1j}(s) + \sum_{j=1}^n \int_0^t N_{2j}(s)\mathrm{d}w_{2j}(s) \\ = \ &\sum_{j=1}^n \Bigg\{\int_0^t N_j'(s)\mathrm{d}w_j(s) - \frac{1}{2}\int_0^t N_j'(s)N_j(s)\mathrm{d}s + \frac{1}{2}\int_0^t N_j'(s)N_j(s)\mathrm{d}s\Bigg\}, \end{aligned}$$

where

$$N_j(s) = \begin{bmatrix} N_{1j}(s) \\ \\ N_{2j}(s) \end{bmatrix}, \quad w_j(s) = \begin{bmatrix} w_{1j}(s) \\ \\ w_{2j}(s) \end{bmatrix}$$

and

$$\begin{cases} N_{1j}(s) = \Big(2C_{1j}'M_1(s) + g_3'(s)\bar{A}_{3j}\Big)x_1(s) + \Big(M_2'(s)C_{1j} + g_3'(s)\bar{C}_{2j}\Big), \\ \\ N_{2j}(s) = \sqrt{v_j(x_3)}[M_3'(s) + g_3'(s)]\sigma_j. \end{cases}$$

.

Here we introduce some equations, which assumed to have a global unique solutions:

$$\begin{cases} \tilde{Q} + \bar{K}_1'\tilde{R} + \bar{K}_1'\tilde{P}\bar{K}_1 + g_3'(s)\bar{Q} + \dot{M}_1(s) + 2M_1(s)\bar{A}_1 \\ \qquad + \dfrac{\gamma p}{2}\displaystyle\sum_{j=1}^{n}\Bigg(4M_1'(s)C_{1j}C_{1j}'M_1(s) + \bar{A}_{3j}'g_3(s)g_3'(s)\bar{A}_{3j} \\ \qquad\qquad + 4M_1'(s)C_{1j}g_3'(s)\bar{A}_{3j}\Bigg) = 0, \\ M_1(0) = 0, \end{cases} \tag{3.10}$$

$$\begin{cases} \bar{K}_0'\tilde{R} + 2\bar{K}_0'\tilde{P}\bar{K}_1 + L_1' + L_u'\bar{K}_1 + g_3'(s)\bar{A}_{12} + g_3'(s)\bar{R} + 2\bar{B}_1'M_1(s) + \dot{M}_2(s) \\ \qquad + M_2'(s)\bar{A}_1 + \gamma p\displaystyle\sum_{j=1}^{n}\Bigg(2M_2'(s)C_{1j}C_{1j}'M_1(s) + \bar{C}_{2j}'g_3(s)g_3'(s)\bar{A}_{3j} \\ \qquad\qquad + 2g_3'(s)\bar{C}_{2j}C_{1j}'M_1(s) + M_2'(s)C_{1j}g_3'(s)\bar{A}_{3j}\Bigg) = 0, \\ M_2(0) = 0, \end{cases} \tag{3.11}$$

$$\begin{cases} \dfrac{\gamma p}{2}\displaystyle\sum_{j=1}^{n}\sigma_j'[M_3(s) + g_4(s)][M_3'(s) + g_4'(s)]\sigma_j\beta_j' + L_3' + \dot{M}_3'(s) + M_3'(s)A_3 = 0, \\ M_3(0) = 0. \end{cases} \tag{3.12}$$

Under these equations (3.10), (3.11) and (3.12), function $\gamma pH(t)$ becomes

$$\int_0^t \gamma p\Bigg\{\bar{K}_0'\tilde{P}\bar{K}_0 + L_u'\bar{K}_0 + g_3'(s)\bar{P} + C_{1j}'M_1(s)C_{1j} + M_2'(s)\bar{B}_1 + M_3'(s)B_3$$

$$
\begin{aligned}
&+\frac{\gamma p}{2}\sum_{j=1}^{n}\Big(C_{1j}'M_2(s)M_2'(s)C_{1j}+\bar{C}_{2j}'g_3(s)g_3'(s)\bar{C}_{2j}\\
&+2C_{1j}'M_2(s)g_3'(s)\bar{C}_{2j}+\alpha_j\sigma_j'[M_3(s)+g_3(s)][M_3'(s)+g_3'(s)]\sigma_j\Big)\Bigg\}\mathrm{d}s\\
&+\gamma p\Bigg\{x_1'(t)\Big[G_1(t)-M_1(t)\Big]x_1(t)+\Big[g_2'(t)-M_2'(t)\Big]x_1(t)+\Big[g_4'(t)-M_3'(t)\Big]x_3(t)\\
&+g_5(t)+g_3'(0)x_2(0)+g_4'(0)x_3(0)\Bigg\}\\
&+\sum_{j=1}^{n}\Bigg\{\int_0^t\gamma pN_j'(s)\mathrm{d}w_j(s)-\frac{1}{2}\int_0^t\gamma^2p^2N_j'(s)N_j(s)\mathrm{d}s\Bigg\}.
\end{aligned}
$$

Applying Hölder's inequality, the expected value of $\gamma pH(t)$ is

$$
\begin{aligned}
&\mathbb{E}\left[\mathrm{e}^{\gamma pH(t)}\right]\\
\leq\;&\mathbb{E}\Bigg[\exp\gamma pp_1\Bigg\{\int_0^t\Bigg(\bar{K}_0'\tilde{P}\bar{K}_0+L_u'\bar{K}_0+g_3'(s)\bar{P}+C_{1j}'M_1(s)C_{1j}+M_2'(s)\bar{B}_1\\
&+M_3'(s)B_3+\frac{\gamma p}{2}\sum_{j=1}^{n}\Big(C_{1j}'M_2(s)M_2'(s)C_{1j}+\bar{C}_{2j}'g_3(s)g_3'(s)\bar{C}_{2j}\\
&+2C_{1j}'M_2(s)g_3'(s)\bar{C}_{2j}+\alpha_j\sigma_j'[M_3(s)+g_3(s)][M_3'(s)+g_3'(s)]\sigma_j\Big)\Bigg)\mathrm{d}s\\
&+g_5(t)\Bigg\}\Bigg]^{\frac{1}{p_1}}\\
&\mathbb{E}\Bigg[\exp\gamma pp_2\Bigg\{g_3'(0)x_2(0)+g_4'(0)x_3(0)\Bigg\}\Bigg]^{\frac{1}{p_2}}\mathbb{E}\Bigg[\exp\gamma pp_3\Big[g_4'(t)-M_3'(t)\Big]x_3(t)\Bigg]^{\frac{1}{p_3}}\\
&\mathbb{E}\Bigg[\exp\gamma pp_4\Bigg\{x_1'(t)\Big[G_1(t)-M_1(t)\Big]x_1(t)+\Big[g_2'(t)-M_2'(t)\Big]x_1(t)\Bigg\}\Bigg]^{\frac{1}{p_4}}\\
&\mathbb{E}\Bigg[\exp\gamma pp_5\sum_{j=1}^{n}\Bigg\{\int_0^t\gamma pN_j(s)\mathrm{d}w_j(s)-\frac{1}{2}\int_0^t\gamma^2p^2N_j'(s)N_j(s)\mathrm{d}s\Bigg\}\Bigg]^{\frac{1}{p_5}}
\end{aligned}
$$

$$
\begin{aligned}
\le\ & C(t)\mathbb{E}\Bigg[\exp\gamma p p_4\Bigg\{x_1'(t)\Big[G_1(t)-M_1(t)\Big]x_1(t)+\Big[g_2'(t)-M_2'(t)\Big]x_1(t)\Bigg\}\Bigg]^{\frac{1}{p_4}} \\
& \mathbb{E}\Bigg[\exp\gamma p p_3\Big[g_4'(t)-M_3'(t)\Big]x_3(t)\Bigg]^{\frac{1}{p_3}},
\end{aligned}
$$

where

$$
\begin{aligned}
& C(t) \\
=\ & \mathbb{E}\Bigg[\exp\gamma p p_1\Bigg\{\int_0^t\Bigg(\bar{K}_0'\tilde{P}\bar{K}_0+L_u'\bar{K}_0+g_3'(s)\bar{P}+C_{1j}'M_1(s)C_{1j}+M_2'(s)\bar{B}_1 \\
& \qquad +M_3'(s)B_3+\frac{\gamma p}{2}\sum_{j=1}^{n}\Big(C_{1j}'M_2(s)M_2'(s)C_{1j}+\bar{C}_{2j}'g_3(s)g_3'(s)\bar{C}_{2j} \\
& \qquad +2C_{1j}'M_2(s)g_3'(s)\bar{C}_{2j}+\alpha_j\sigma_j'[M_3(s)+g_3(s)][M_3'(s)+g_3'(s)]\sigma_j\Big)\Bigg)\mathrm{d}s \\
& \qquad +g_5(t)\Bigg\}\Bigg]^{\frac{1}{p_1}} \\
& \mathbb{E}\Bigg[\exp\gamma p p_2\Bigg\{g_3'(0)x_2(0)+g_4'(0)x_3(0)\Bigg\}\Bigg]^{\frac{1}{p_2}} \\
<\ & \infty,
\end{aligned}
$$

and

$$
\frac{1}{p_1}+\frac{1}{p_2}+\cdots+\frac{1}{p_5}, p_1, p_2, \cdots, p_5 > 1.
$$

For the technical reason, we let

$$
\begin{aligned}
\kappa_1(t) &\equiv \gamma p p_4[G_1(t)-M_1(t)] \\
\kappa_2'(t) &\equiv \gamma p p_4[g_2'(t)-M_2'(t)] \\
\kappa_3'(t) &\equiv \gamma p p_3[g_4'(t)-M_3'(t)],
\end{aligned}
$$

and introduce some lemmas and assumptions.

Assumption 3.2.1. $\Phi(t) > 0, \forall t \in [0,T]$.

Assumption 3.2.2. $\Phi_0(t) \equiv (\Phi^{-1} - 2\kappa_1(t))^{-1} > 0, \forall t \in [0, T]$.

Lemma 3.2.1. *Let the Assumption 3.2.1, 3.2.2 hold, then there exists an inequality , which is*

$$\mathbb{E}\left[\exp\Big\{x_1'(t)\kappa_1(t)x_1(t) + \kappa_2'(t)x_1(t)\Big\}\right] < \infty.$$

Proof. From (3.9), $x_1(t)$ follows normal distribution

$$x_1(t) \sim \mathcal{N}\big(\mu(t), \Phi(t)\big)$$

Here $\mu(t)$ is the solution to the linear differential equation

$$\begin{cases} \dot{\mu}(t) - \bar{A}_1\mu(t) - \bar{B}_1 = 0, \\ \\ \mu(0) = \mu_0, \end{cases}$$

and $\Phi(t) = \Psi(t) - \mu(t)\mu'(t)$ with $\Psi(t)$ being the solution to

$$\begin{cases} \dot{\Psi}(t) - \bar{A}_1\Psi(t) - \Psi(t)\bar{A}_1' - \bar{B}_1\mu'(t) - \mu(t)\bar{B}_1' - \displaystyle\sum_{j=1}^{n} C_{1j}C_{1j}' = 0, \\ \\ \Psi(0) = \mathbb{E}[x_0x_0']. \end{cases}$$

Under Assumption 3.2.1, 3.2.2, the following holds

$$\begin{aligned} & \mathbb{E}\left[\exp\Big\{x_1'(t)\kappa_1(t)x_1(t) + \kappa_2'(t)x_1(t)\Big\}\right] \\ = & \int_{\mathbb{R}^{n_1}} \exp\Big\{x'\kappa_1(t)x + \kappa_2'(t)x\Big\}\frac{1}{(2\pi)^{n_1/2}|\Phi|^{1/2}} \\ & \qquad \times\exp\Big\{-\frac{1}{2}\big(x-\mu\big)'\Phi^{-1}\big(x-\mu\big)\Big\}\mathrm{d}x \\ = & \frac{|\Phi_0|^{1/2}}{|\Phi|^{1/2}} \\ & \times\exp\Big\{-\frac{1}{2}\big(\kappa_2'(t)+\Phi^{-1}\mu\big)'\big(\Phi^{-1}-2\kappa_1(t)\big)^{-1}\big(\kappa_2'(t)+\Phi^{-1}\mu\big) - \frac{1}{2}\mu'\Phi^{-1}\mu\Big\} \\ & \times\int_{\mathbb{R}^{n_1}} \frac{1}{(2\pi)^{n_1/2}|\Phi_0|^{1/2}} \end{aligned}$$

$$\times \exp\left\{-\frac{1}{2}\big(x-\Phi_0[\kappa_2'(t)+\Phi^{-1}\mu]\big)'\Phi_0^{-1}\big(x-\Phi_0[\kappa_2'(t)+\Phi^{-1}\mu]\big)\right\}\mathrm{d}x$$

$$\begin{aligned}
&= \frac{|\Phi_0|^{1/2}}{|\Phi|^{1/2}} \\
&\quad \times \exp\Big\{-\frac{1}{2}\big(\kappa_2'(t)+\Phi^{-1}\mu\big)'\big(\Phi^{-1}-2\kappa_1(t)\big)^{-1}\big(\kappa_2'(t)+\Phi^{-1}\mu\big)-\frac{1}{2}\mu'\Phi^{-1}\mu\Big\} \\
&< \infty.
\end{aligned}$$

□

Assumption 3.2.3. *$v_i(x_3) = \bar{\beta}_i M' x_3$, where $\bar{\beta}_i$ is a constant and M is a column vector.*

Assumption 3.2.4. *$A_3 = NM'$, where N is a column vector.*

Lemma 3.2.2. *Let the Assumption 3.2.3, 3.2.4 hold, and let $\bar{x}_3 = M'x_3$, $\bar{M} = cM$, c is a positive constant, then there exists an inequality , which is*

$$\mathbb{E}\left[\mathrm{e}^{\bar{M}'x_3(t)}\right] < \infty,$$

provided

$$M'B_3 \le \frac{1}{4}\sum_{j=1}^{n}\bar{\beta}_j\sigma_j' MM'\sigma_j,$$

$$\frac{c}{2}\left(\sum_{j=1}^{n}\bar{\beta}_j^2\sigma_j' MM'\sigma_j\right)\frac{1-\mathrm{e}^{-M'NT}}{M'N}\mathrm{e}^{M'Nt} < 1.$$

Proof. Recall stochastic differential equation,

$$\mathrm{d}x_3(t) = [A_3x_3(t)+B_3]\,\mathrm{d}t + \sum_{j=1}^{n}\sqrt{v_j(x_3)}\sigma_j\mathrm{d}w_{3j}(t),$$

we can deduce

$$\frac{\partial\sqrt{\bar{x}_3}}{\partial x_3} = \frac{1}{2}(M'x_3)^{-\frac{1}{2}}M'$$

$$\frac{\partial^2\sqrt{\bar{x}_3}}{\partial x_3^2} = -\frac{1}{4}(M'x_3)^{-\frac{3}{2}}MM'.$$

Thus

$$\begin{aligned}
\mathrm{d}(\sqrt{\bar{x}_3}) &= \left[\frac{1}{2}(M'x_3)^{-\frac{1}{2}}M'(A_3x_3+B_3) - \frac{1}{8}(M'x_3)^{-\frac{3}{2}}\sum_{j=1}^{n} v_j(x_3)\sigma_j' MM'\sigma_j\right]\mathrm{d}t \\
&\quad +\frac{1}{2}(M'x_3)^{-\frac{1}{2}}M'\sum_{j=1}^{n}\sqrt{v_j(x_3)}\sigma_j \mathrm{d}w_{3j}.
\end{aligned}$$

Due to Assumption (3.2.3) and (3.2.4), $\mathrm{d}(\sqrt{\bar{x}_3})$ can be rewritten as follows

$$\begin{aligned}
\mathrm{d}(\sqrt{\bar{x}_3}) &= \left[\frac{1}{2}(M'x_3)^{-\frac{1}{2}}M'NM'x_3 + \frac{1}{2}(M'x_3)^{-\frac{1}{2}}M'B_3\right. \\
&\quad \left. -\frac{1}{8}(M'x_3)^{-\frac{3}{2}}(M'x_3)\sum_{j=1}^{n}\bar{\beta}_j\sigma_j' MM'\sigma_j\right]\mathrm{d}t \\
&\quad +\frac{1}{2}(M'x_3)^{-\frac{1}{2}}M'(M'x_3)^{\frac{1}{2}}\sum_{j=1}^{n}\bar{\beta}_j\sigma_j \mathrm{d}w_{3j} \\
&= \left[\frac{1}{2}(M'x_3)^{\frac{1}{2}}M'N + \frac{1}{2}(M'x_3)^{-\frac{1}{2}}\left\{M'B_3 - \frac{1}{4}\sum_{j=1}^{n}\bar{\beta}_j\sigma_j' MM'\sigma_j\right\}\right]\mathrm{d}t \\
&\quad +\frac{1}{2}M'\sum_{j=1}^{n}\bar{\beta}_j\sigma_j \mathrm{d}w_{3j}.
\end{aligned}$$

When

$$M'B_3 - \frac{1}{4}\sum_{j=1}^{n}\bar{\beta}_j\sigma_j' MM'\sigma_j \le 0,$$

the following inequality holds

$$\mathrm{d}(\sqrt{\bar{x}_3}) \le \frac{1}{2}(M'x_3)^{\frac{1}{2}}M'N\mathrm{d}t + \frac{1}{2}M'\sum_{j=1}^{n}\bar{\beta}_j\sigma_j \mathrm{d}w_{3j}.$$

Let $\xi(t) = \sqrt{\bar{x}_3}$, $\varphi(\xi) = c(\xi(t))^{\gamma}$, $\gamma = 2$, and applying Lemma (2.8.1), we can prove

$$\mathbb{E}\left[\mathrm{e}^{cM'x_3(t)}\right] < \infty,$$

provided

$$\frac{c}{2}\sum_{j=1}^{n}\left(\bar{\beta}_j M'\sigma_j\right)^2 \mathrm{e}^{M'Nt}\frac{1-\mathrm{e}^{-M'NT}}{M'N} < 1.$$

□

Theorem 3.2.1. *Let the Assumptions* (3.2.1), (3.2.2), (3.2.3) *and* (3.2.4) *hold. Then the control process* $u(t) = \bar{K}_0 + \bar{K}_1 x_1(t)$ *belongs to set* $\mathcal{A}$.

Proof. We denote the vector $\kappa_3(t) = \begin{bmatrix} \kappa_{31}(t) \\ \kappa_{32}(t) \\ \vdots \\ \kappa_{3n_3(t)} \end{bmatrix}$. According to equation (3.7) and (3.11), $g_4(t)$ and $M_3(t)$ are assumed to have unique global solutions. In other words, it is obvious that $g_4(t)$ and $M_3(t)$ are bounded. Thus there exists a vector $\tilde{M}$, such that $\tilde{M} = \begin{bmatrix} \tilde{M}_1 \\ \tilde{M}_2 \\ \vdots \\ \tilde{M}_{n_3} \end{bmatrix} = \begin{bmatrix} \max \kappa_{31}(t) \\ \max \kappa_{32}(t) \\ \vdots \\ \max \kappa_{3n_3}(t) \end{bmatrix}$, which means each element in $\tilde{M}$ is the upper bound of the correspond element in $\kappa_3(t)$.

Now we take $\bar{M} = \tilde{M}$. Then it can be obtained

$$\mathbb{E}\left[\mathrm{e}^{\kappa_3'(t)x_3(t)}\right] \leq \mathbb{E}\left[\mathrm{e}^{\bar{M}'x_3(t)}\right].$$

By using Lemma 3.2.1 and Lemma 3.2.2, we can prove

$$\begin{aligned} & \mathbb{E}\left[\mathrm{e}^{\gamma p H(t)}\right] \\ \leq\ & C(t)\mathbb{E}\left[\mathrm{e}^{x_1'(t)\kappa_1 x_1(t) + \kappa_2' x_1(t)}\right]^{\frac{1}{p_4}} \mathbb{E}\left[\mathrm{e}^{\kappa_3'(t)x_3(t)}\right]^{\frac{1}{p_3}} \\ <\ & \infty. \end{aligned}$$

□

3.2.2 Solution of the Problem

Theorem 3.2.2. *Let* $G_1(\cdot)$, $g_2(\cdot)$, $g_3(\cdot)$, $g_4(\cdot)$ *and* $g_5(\cdot)$ *be solutions of differential equations* (3.4), (3.6), (3.5), (3.7) *and* (3.8) *respectively, then there exists a unique solution to problem* (3.3) *given by*

$$u^*(t) = -\frac{1}{2}K_1^{-1}(t)\left[K_2(t)x_1(t) + K_3(t)\right] \tag{3.13}$$

where $K_1(t)$, $K_2(t)$ and $K_3(t)$ defined on page 22.

The optimal cost functional is:

$$J^* = \gamma \mathbb{E}\left[\exp\left\{\frac{\gamma}{2}\left[x_{10}' G_1(0) x_{10} + g_2'(0) x_{10} + g_3'(0) x_{20} + g_4'(0) x_{30} + g_5(0)\right]\right\}\right].$$

Proof. Firstly, we introduce a new function $G(t)$, which is

$$G(t) = \mathrm{e}^{\frac{\gamma}{2} H(t)}.$$

From the definitions of $H(t)$, $h(t)$ and their initial conditions, obviously have following:

$$J(u(\cdot)) = \gamma \mathbb{E}[G(T)].$$

Applying for Itô's Lemma, the differential of $G(t)$ is

$$\begin{aligned}
& \mathrm{d}G(t) \\
= \;& G(t)\Big[\frac{\gamma}{2} x_1'(t)\tilde{Q} x_1(t) + \frac{\gamma}{2} u'(t)\tilde{R} x_1(t) + \frac{\gamma}{2} u'(t)\tilde{P} u(t) + \frac{\gamma}{2} L_1' x_1(t) + \frac{\gamma}{2} L_2' x_2(t) \\
& \quad + \frac{\gamma}{2} L_3' x_3(t) + \frac{\gamma}{2} L_u' u(t)\Big]\mathrm{d}t \\
+ \;& G(t)\Big\{\frac{\gamma^2}{8}\sum_{j=1}^{n}\Big[C_{1j}'\Big(2G_1(t) + [2G_1(t)x_1(t) + g_2(t)]\,[2G_1(t)x_1(t) + g_2(t)]'\Big)C_{1j}\Big] \\
& \quad + \frac{\gamma}{2}\big[2G_1(t)x_1(t) + g_2(t)\big]'\big[A_1 x_1(t) + B_1 u(t)\big] + \frac{\gamma}{2} x_1'(t)\dot{G}_1(t) x_1(t) \\
& \quad + \frac{\gamma}{2}\dot{g_2}'(t) x_1(t)\Big\}\mathrm{d}t \\
+ \;& G(t)\Big\{\frac{\gamma}{2} g_3'(t)\Big[A_{12}x_1(t) + A_{22}x_2(t) + A_{42}x_3(t) + D\big(x_1(t), u(t)\big) + B_{12}u(t)\Big] \\
& \quad + \frac{\gamma^2}{8}\sum_{j=1}^{n} g_3'(t)\Big(v_j(x_3)\sigma_j\sigma_j' + \big[A_{3j}x_1(t) + B_{2j}u(t) + C_{2j}\big] \\
& \qquad \times\big[A_{3j}x_1(t) + B_{2j}u(t) + C_{2j}\big]'\Big) g_3(t) + \frac{\gamma}{2}\dot{g_3}'(t) x_2(t)\Big\}\mathrm{d}t \\
+ \;& G(t)\left\{\frac{\gamma}{2}\dot{g_4}'(t) x_3(t) + \frac{\gamma}{2} g_4'(t)[A_3 x_3(t) + B_3] + \frac{\gamma^2}{8}\sum_{j=1}^{n} v_j(x_3)\sigma_j' g_4(t) g_4'(t)\sigma_j\right\}\mathrm{d}t
\end{aligned}$$

$$
\begin{aligned}
&+ \quad G(t)\frac{\gamma}{2}\dot{g_5}(t)\mathrm{d}t + \sum_{j=1}^{n} G(t)\frac{\gamma}{2}g_3'(t)\left[A_{3j}x_1(t) + B_{2j}u(t) + C_{2j}\right]\mathrm{d}w_{1j}(t) \\
&+ \quad \sum_{j=1}^{n} G(t)\frac{\gamma}{2}\left[2G_1(t)x_1(t) + g_2(t)\right]' C_{1j}\mathrm{d}w_{1j}(t) + \sum_{j=1}^{n} G(t)\frac{\gamma}{2}\sqrt{v_j(x_3)}g_4'(t)\sigma_j\mathrm{d}w_{2j}(t)
\end{aligned}
$$

We arrange the equation into a nicely organized form, and deduce the expression of $J(u(\cdot))$, which is

$$
\begin{aligned}
& J(u(\cdot)) \\
= \quad & \gamma\mathbb{E}\left[G(0)\right] \\
& +\gamma\mathbb{E}\Bigg[\int_0^T G(t)x_1'(t)\Bigg\{\frac{\gamma}{2}\tilde{Q} + \frac{\gamma}{2}\dot{G}_1(t) + \gamma G_1(t)A_1 + \frac{\gamma^2}{2}\sum_{j=1}^{n} G_1(t)C_{1j}C_{1j}'G_1(t) \\
& \qquad +\frac{\gamma}{2}g_3'(t)Q + \frac{\gamma^2}{8}\sum_{j=1}^{n} A_{3j}'g_3(t)g_3'(t)A_{3j}\Bigg\}x_1(t)\mathrm{d}t\Bigg] \\
& +\gamma\mathbb{E}\Bigg[\int_0^T G(t)\Bigg\{\frac{\gamma}{2}L_1' + \frac{\gamma}{2}\dot{g_2}'(t) + \frac{\gamma}{2}g_2'(t)A_1 + \frac{\gamma^2}{2}\sum_{j=1}^{n} g_2'(t)C_{1j}C_{1j}'G_1(t) \\
& \qquad +\frac{\gamma}{2}g_3'(t)A_{12} + \frac{\gamma^2}{4}\sum_{j=1}^{n} C_{2j}'g_3(t)g_3'(t)A_{3j}\Bigg\}x_1(t)\mathrm{d}t\Bigg] \\
& +\gamma\mathbb{E}\Bigg[\int_0^T G(t)\Bigg\{\frac{\gamma}{2}L_2' + \frac{\gamma}{2}\dot{g_3}'(t) + \frac{\gamma}{2}g_3'(t)A_{22}\Bigg\}x_2(t)\mathrm{d}t\Bigg] \\
& +\gamma\mathbb{E}\Bigg[\int_0^T G(t)\Bigg\{\frac{\gamma}{2}L_3' + \frac{\gamma}{2}\dot{g_4}'(t) + \frac{\gamma}{2}g_4'(t)A_3 + \frac{\gamma}{2}g_3'(t)A_{42} \\
& \qquad +\frac{\gamma^2}{8}\sum_{j=1}^{n} \sigma_j'\big[g_3(t)g_3'(t) + g_4(t)g_4'(t)\big]\sigma_j\beta_j'\Bigg\}x_3(t)\mathrm{d}t\Bigg] \\
& +\gamma\mathbb{E}\Bigg[\int_0^T G(t)\Bigg\{\frac{\gamma^2}{4}C_{1j}'G_1(t)C_{1j} + \frac{\gamma^2}{8}\sum_{j=1}^{n} C_{1j}'g_2(t)g_2'(t)C_{1j} + \frac{\gamma}{2}g_4'(t)B_3
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{\gamma}{2}\dot{g}_5(t)+\frac{\gamma^2}{8}\sum_{j=1}^{n}\alpha_j\sigma_j'\big[g_3(t)g_3'(t)+g_4(t)g_4'(t)\big]\sigma_j\\
&+\frac{\gamma^2}{8}\sum_{j=1}^{n}C_{2j}'g_3(t)g_3'(t)C_{2j}\Bigg\}\mathrm{d}t\Bigg]\\
+\gamma\mathbb{E}\Bigg[\int_0^T G(t)\Bigg\{&\frac{\gamma}{2}L_u'u(t)+\gamma x_1'(t)G_1(t)B_1u(t)+\frac{\gamma}{2}g_2'(t)B_1u(t)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{\gamma}{2}g_3'(t)B_{12}u(t)+\frac{\gamma^2}{4}\sum_{j=1}^{n}C_{2j}'g_3(t)g_3'(t)B_{2j}u(t)+\frac{\gamma}{2}x_1'(t)\tilde{R}u(t)\\
&+\frac{\gamma}{2}x_1'(t)\left(R'g_3(t)\right)u(t)+\frac{\gamma^2}{4}\sum_{j=1}^{n}x_1'(t)A_{3j}'g_3(t)g_3'(t)B_{2j}u(t)\\
&+\frac{\gamma}{2}u'(t)\tilde{P}u(t)+\frac{\gamma^2}{8}\sum_{j=1}^{n}u'(t)B_{2j}'g_3(t)g_3'(t)B_{2j}u(t)\\
&+\frac{\gamma}{2}u'(t)\left(g_3'(t)P\right)u(t)\Bigg\}\mathrm{d}t\Bigg].
\end{aligned}
\tag{3.14}
$$

It is observed that only the last expectation above contains the control process $u(\cdot)$. Thus we use the completion of squares method, the last can be written as:

$$
\begin{aligned}
&\gamma\mathbb{E}\Bigg[\int_0^T G(t)\Big(u'(t)K_1(t)u(t)+u'(t)K_2(t)x_1(t)+u'(t)K_3(t)\Big)\mathrm{d}t\Bigg]\\
=\;&\gamma\mathbb{E}\Bigg[\int_0^T G(t)\Bigg\{-\frac{1}{4}\Big(K_2(t)x_1(t)+K_3(t)\Big)'K_1^{-1}(t)\Big(K_2(t)x_1(t)+K_3(t)\Big)\\
&\qquad+\left[u(t)+\frac{1}{2}K_1^{-1}(t)\Big(K_2(t)x_1(t)+K_3(t)\Big)\right]'K_1(t)\\
&\qquad\times\left[u(t)+\frac{1}{2}K_1^{-1}(t)\Big(K_2(t)x_1(t)+K_3(t)\Big)\right]\Bigg\}\mathrm{d}t\Bigg],
\end{aligned}
$$

where $K_1(\cdot)$, $K_2(\cdot)$ and $K_3(\cdot)$ defined before.

The sum of the terms that are quadratic in $x_1(t)$ in (3.14) should be zero due to (3.4), indeed,

$$x_1'(t)\Bigg\{\frac{\gamma}{2}\tilde{Q}+\frac{\gamma}{2}\dot{G}_1(t)+\gamma G_1(t)A_1+\frac{\gamma^2}{2}\sum_{j=1}^{n}G_1(t)C_{1j}C_{1j}'G_1(t)+\frac{\gamma}{2}g_3'(t)Q$$
$$+\frac{\gamma^2}{8}\sum_{j=1}^{n}A_{3j}'g_3(t)g_3'(t)A_{3j}-\frac{1}{4}K_2(t)K_1^{-1}(t)K_2(t)=0\Bigg\}x_1(t)=0.$$

Similarly, the sums of the terms linear in $x_1(t)$, $x_2(t)$ and $(x_3(t)$ are also zero:

$$\Bigg\{\frac{\gamma}{2}L_1'+\frac{\gamma}{2}\dot{g_2}'(t)+\frac{\gamma}{2}g_2'(t)A_1+\frac{\gamma^2}{2}\sum_{j=1}^{n}g_2'(t)C_{1j}C_{1j}'G_1(t)+\frac{\gamma}{2}g_3'(t)A_{12}$$
$$+\frac{\gamma^2}{4}\sum_{j=1}^{n}C_{2j}'g_3(t)g_3'(t)A_{3j}-\frac{1}{4}K_3'(t)\Big[\Big(K_1^{-1}(t)\Big)'+K_1^{-1}(t)\Big]K_2(t)\Bigg\}x_1(t)=0,$$

$$\Bigg\{\frac{\gamma}{2}L_2'+\frac{\gamma}{2}\dot{g_3}'(t)+\frac{\gamma}{2}g_3'(t)A_{22}\Bigg\}x_2(t)=0,$$

$$\Bigg\{\frac{\gamma}{2}L_3'+\frac{\gamma}{2}\dot{g_4}'(t)+\frac{\gamma}{2}g_4'(t)A_3+\frac{\gamma}{2}g_3'(t)A_{42}$$
$$+\frac{\gamma^2}{8}\sum_{j=1}^{n}\sigma_j'\big[g_3(t)g_3'(t)+g_4(t)g_4'(t)\big]\sigma_j\beta_j'\Bigg\}x_3(t)=0.$$

The remaining sum of the terms that are independent of the states $x_1(t)$, $x_2(t)$, $x_3(t)$ and control $u(t)$, obviously equals to zero due to our assumption on $g_5(t)$.

Therefore, the cost function $J(u(\cdot))$ for all $u(\cdot)\in\mathcal{A}$ can be written as following:

$$\begin{aligned}
& J(u(\cdot)) \\
= \; & \gamma\mathbb{E}[G(0)] \\
& +\gamma\mathbb{E}\Bigg[\int_0^T G(t)\Bigg\{\Big[u(t)+\frac{1}{2}K_1^{-1}(t)\Big(K_2(t)x_1(t)+K_3(t)\Big)\Big]'K_1(t) \\
& \qquad\times\Big[u(t)+\frac{1}{2}K_1^{-1}(t)\Big(K_2(t)x_1(t)+K_3(t)\Big)\Big]\Bigg\}\mathrm{d}t\Bigg]
\end{aligned}$$

$$\geq \quad \gamma\mathbb{E}[G(0)].$$

It means the cost function $J(u(\cdot))$ achieves the lower bound

$$\gamma\mathbb{E}\left[\exp\left\{\frac{\gamma}{2}\left[x_{10}'G_1(0)x_{10} + g_2'(0)x_{10} + g_3'(0)x_{20} + g_4'(0)x_{30} + g_5(0)\right]\right\}\right]$$

if and only if

$$u^*(t) = -\frac{1}{2}K_1^{-1}(t)\left[K_2(t)x_1(t) + K_3(t)\right].$$

□

3.3 Generalised risk-sensitive control

Date and Gashi [12] generalised the classical risk sensitive by introducing a more general criterion which has noise dependent on the state $x_1(t)$ and control $u(t)$. In this section, we extend this problem further, which contains the square root process $x_3(t)$. Hence the generalised risk sensitive criterion is given by

$$\begin{aligned}
&\tilde{J}(u(\cdot)) \\
= \;& \gamma\mathbb{E}\Bigg[\exp\Bigg\{\frac{\gamma}{2}x_1'(T)Sx_1(T) + \frac{\gamma}{2}\int_0^T \left[x_1'(t)\tilde{Q}x_1(t) + u'(t)\tilde{P}u(t)\right]\mathrm{d}t \\
&+\frac{\gamma}{2}\int_0^T \left[L_1'x_1(t) + L_2'x_2(t) + L_3'x_3(t) + L_u'u(t) + u'(t)\tilde{R}x_1(t)\right]\mathrm{d}t \\
&+\frac{\gamma}{2}S_1'x_1(T) + \frac{\gamma}{2}S_2'x_2(T) + \frac{\gamma}{2}S_3'x_3(T) + \frac{\gamma}{2}\int_0^T \left[Q_{x_1}x_1(t) + R_u u(t)\right]'\mathrm{d}w_1(t) \\
&+\frac{\gamma}{2}\int_0^T Q_{x_3}'V(x_3)\mathrm{d}w_2(t)\Bigg\}\Bigg],
\end{aligned} \tag{3.15}$$

where $Q_{x_1} \in \mathbb{R}^{n\times n_1}$, $R_u \in \mathbb{R}^{n\times m}$, $Q_{x_3} \in \mathbb{R}^{n\times 1}$ and

$$V(x_3) = \begin{pmatrix} \sqrt{v_1(x_3)} & o & \cdots & 0 \\ 0 & \sqrt{v_2(x_3)} & \cdots & 0 \\ & & \ddots & \\ 0 & \cdots & 0 & \sqrt{v_n(x_3)} \end{pmatrix}.$$

In this section, we build the risk sensitive control problem to be

$$\begin{cases} \min\limits_{u(\cdot)\in\tilde{\mathcal{A}}} \tilde{J}(u(\cdot)) \\ \\ s.t.(3.1) holds, \end{cases} \tag{3.16}$$

where the set $\tilde{\mathcal{A}}$ is a new admissible control set. By using change of measure, we introduce the new probability measure $\tilde{\mathbb{P}}$, under which the problem (3.16) can be transformed and applied the the Theorem 3.2.2 directly.

Let us introduce a stochastic process $\theta(t)$:

$$\theta(t) = \begin{bmatrix} -\frac{\gamma}{2}\Big(Q_{x_1}x_1(t) + R_u u(t)\Big) \\ \\ -\frac{\gamma}{2}V(x_3)Q_{x_3} \end{bmatrix},$$

and the process $Z(t)$:

$$\begin{cases} Z(t) = \exp\Big\{ -\int_0^t \theta'(s)\mathrm{d}w(s) - \frac{1}{2}\int_0^t \theta'(s)\theta(s)\mathrm{d}s \Big\}, \\ \\ Z(T) = Z. \end{cases}$$

Here we denote Z to be a random variable. In order to ensure that $\mathbb{E}[Z] = 1$, we give an assumption which satisfies the Novikov condition

Assumption 3.3.1.

$$\mathbb{E}\left[\mathrm{e}^{\frac{1}{2}\int_0^T \theta'(s)\theta(s)\mathrm{d}s} \right] < \infty.$$

Now we introduce a new probability measure $\tilde{\mathbb{P}}$, which is

$$\tilde{\mathbb{P}}(\alpha) = \int_\alpha Z(\omega)\mathrm{d}\mathbb{P}(\omega), \forall \alpha \in \mathcal{F}.$$

By Girsanov theorem, the standard Brownian motion $\tilde{w}(t)$ under probability measure $\tilde{\mathbb{P}}$ is defined

$$\tilde{w}(t) \equiv \begin{bmatrix} w_1(t) \\ \\ w_2(t) \end{bmatrix} + \int_0^t \begin{bmatrix} -\frac{\gamma}{2}\Big(Q_{x_1}x_1(t) + R_u u(t)\Big) \\ \\ -\frac{\gamma}{2}V(x_3)Q_{x_3} \end{bmatrix} \mathrm{d}s.$$

Under $\tilde{\mathbb{P}}$, now we deduce the criterion (3.15) as

$$
\begin{aligned}
& \tilde{J}(u(\cdot)) \\
= \; & \gamma \tilde{\mathbb{E}}\Bigg[\exp\Bigg\{\frac{\gamma}{2}x_1'(T)Sx_1(T) + \frac{\gamma}{2}\int_0^T \Big[x_1'(t)\tilde{Q}x_1(t) + u'(t)\tilde{P}u(t)\Big]\,\mathrm{d}t \\
& + \frac{\gamma}{2}\int_0^T \Big[L_1'x_1(t) + L_2'x_2(t) + L_3'x_3(t) + L_u'u(t) + u'(t)\tilde{R}x_1(t)\Big]\,\mathrm{d}t \\
& + \frac{\gamma}{2}S_1'x_1(T) + \frac{\gamma}{2}S_2'x_2(T) + \frac{\gamma}{2}S_3'x_3(T) + \frac{\gamma}{2}\int_0^T \frac{\gamma}{4}Q_{x_3}'V(x_3)V(x_3)Q_{x_3}\mathrm{d}t \\
& + \frac{\gamma}{2}\int_0^T \frac{\gamma}{4}\left[Q_{x_1}x_1(t) + R_u u(t)\right]'\left[Q_{x_1}x_1(t) + R_u u(t)\right]\mathrm{d}t\Bigg\}\Bigg] \\
= \; & \gamma \tilde{\mathbb{E}}\Bigg[\exp\Bigg\{\frac{\gamma}{2}x_1'(T)Sx_1(T) + \frac{\gamma}{2}\int_0^T \Big[x_1'(t)\hat{Q}x_1(t) + u'(t)\hat{P}u(t)\Big]\,\mathrm{d}t \\
& + \frac{\gamma}{2}\int_0^T \Big[L_1'x_1(t) + L_2'x_2(t) + \hat{L}_3'x_3(t) + L_u'u(t) + u'(t)\hat{R}x_1(t)\Big]\,\mathrm{d}t \\
& + \frac{\gamma}{2}S_1'x_1(T) + \frac{\gamma}{2}S_2'x_2(T) + \frac{\gamma}{2}S_3'x_3(T) + C\Bigg\}\Bigg], \qquad (3.17)
\end{aligned}
$$

where

$$
\begin{aligned}
& \hat{Q} = \tilde{Q} + \frac{\gamma}{4}Q_{x_1}'Q_{x_1}, \quad \hat{P} = \tilde{P} + \frac{\gamma}{2}R_u'Q_{x_1}, \quad \hat{R} = \tilde{R} + \frac{\gamma}{4}R_u'R_u \\
& \hat{L}_3' = L_3' + \frac{\gamma}{4}\sum_{j=1}^{n} Q_{x_3 j}^2\beta_j', \quad C = \frac{\gamma}{2}T\frac{\gamma}{4}\sum_{j=1}^{n} Q_{x_3 j}^2\alpha_j.
\end{aligned}
$$

It is observed that equation (3.17) has a similar form compared with equation (3.2). To ensure under the same probability measure $\tilde{\mathbb{P}}$, we need to change the measures of $x_1(t)$, $x_2(t)$ and $x_3(t)$.

$$
\begin{aligned}
& \mathrm{d}x_1(t) \\
= \; & [A_1x_1(t) + B_1u(t)]\mathrm{d}t + \sum_{j=1}^{n} C_{1j}\Big[\mathrm{d}\tilde{w}_{1j}(t) + \frac{\gamma}{2}\big(Q_{x_1 j}x_1(t) + R_{uj}u(t)\big)\mathrm{d}t\Big]
\end{aligned}
$$

$$= \quad [\tilde{A}_1 x_1(t) + \tilde{B}_1 u(t)]\mathrm{d}t + \sum_{j=1}^{n} C_{1j}\mathrm{d}\tilde{w}_{1j}(t) \tag{3.18}$$

$$\begin{aligned} & \mathrm{d}x_2(t) \\ = \quad & [A_{12}x_1(t) + A_{22}x_2(t) + A_{42}x_3(t) + D\left(x_1(t), u(t)\right) + B_{12}u(t)]\,\mathrm{d}t \\ & + \sum_{j=1}^{n} [A_{3j}x_1(t) + B_{2j}u(t) + C_{2j}]\,\mathrm{d}\tilde{w}_{2j}(t) + \sum_{j=1}^{n} \sqrt{v_j(x_3)}\sigma_j \mathrm{d}\tilde{w}_{3j}(t) \end{aligned} \tag{3.19}$$

$$\begin{aligned} & \mathrm{d}x_3(t) \\ = \quad & [A_3 x_3(t) + B_3]\,\mathrm{d}t + \sum_{j=1}^{n} \sqrt{v_j(x_3)}\sigma_j \left[\mathrm{d}\tilde{w}_{3j}(t) + \frac{\gamma}{2}\sqrt{v_j(x_3)}Q_{x_3j}\mathrm{d}t\right] \\ = \quad & \left[\tilde{A}_3 x_3(t) + \tilde{B}_3\right]\mathrm{d}t + \sum_{j=1}^{n} \sqrt{v_j(x_3)}\sigma_j \mathrm{d}\tilde{w}_{3j}(t) \end{aligned} \tag{3.20}$$

where

$$\tilde{A}_1 = A_1 + \frac{\gamma}{2}\sum_{j=1}^{n} C_{1j}Q_{x_1j}, \quad \tilde{B}_1 = B_1 + \frac{\gamma}{2}\sum_{j=1}^{n} C_{1j}R_{uj}$$

$$\tilde{A}_3 = A_3 + \frac{\gamma}{2}\sum_{j=1}^{n} Q_{x_3j}\sigma_j\beta'_j, \quad \tilde{B}_3 = B_3 + \frac{\gamma}{2}\sum_{j=1}^{n} Q_{x_3j}\alpha_j\sigma_j$$

$$Q_{x_1j} \in \mathbb{R}^{1\times n_1}, \quad R_{uj} \in \mathbb{R}^{1\times m}, \quad Q_{x_3j} \in \mathbb{R}.$$

It is notable that the control problem of minimizing (3.17) subject to (3.18), (3.19) and (3.20) is similar to the problem (3.3). Thus it can be solved by the same method. The only difference is the terminal condition $g_5(T) = C$ in equation (3.8). Therefore, under Assumption 3.3.1, these two problems have the same solutions.

Theorem 3.3.1. *Under Assumption 3.3.1, there exists a unique solution to problem* (3.16) *given by*

$$\tilde{u}^*(t) = -\frac{1}{2}K_1^{-1}(t)\left[K_2(t)x_1(t) + K_3(t)\right]$$

where $K_1(t)$, $K_2(t)$ and $K_3(t)$ defined before.

Proof. The proof is similar to that of Theorem 3.2.2. □

3.4 Summary

In this chapter, we consider the risk-sensitive control problem for a class of nonlinear systems. We extend our previous work [15] with multiplicative noise. And we generalised the classical risk sensitive control by introducing a more general criterion which has noise dependent on the state $x_1(t)$, $x_3(t)$ and control $u(t)$. Under some assumptions, the solution to such optimal control problem is obtained in an explicit form. Applications of these results to mathematical finance, such as interest rate modelling, bond pricing, and optimal investment, will be considered in later chapters.

Chapter 4

Interest rate modelling, bond pricing and optimal investment

4.1 Introduction

The optimal investment problem is an important application of the risk sensitive control. In Date and Gashi [11], they outline the results of a new interest rate model, which is a quadratic affine term structure (QATS) interest rate model. They propose a generalisation of the classical optimal investment problem with exponential utility. In this chapter, we extend this optimal investment problem by introducing a further nonlinear interest rate, which is a combination of the CIR and the multi-dimensional quadratic term structure model. The explicit solutions with logarithmic and power utilities are obtained in closed-form. Furthermore, we take into consideration Yong's result [54] about CIR model's limitation. We also derive the zero-coupon bond price for such an interest rate model.

4.2 Interest Rate Model

Recalling the systems (2.1) and (2.9), we choose the value

$$A_1 = A \in \mathbb{R}^{n_1 \times n_1}, \quad B_1 = 0, \quad C_{1j} = [D_{1j}, D_{2j}, \dots, D_{n_1 j}]' \in \mathbb{R}^{n_1},$$
$$A_3 = \alpha \in \mathbb{R}, \quad B_3 = -\beta \in \mathbb{R}, \quad \Sigma = v \in \mathbb{R}, \quad v_3(x_3) = x_3.$$

And let $w_1(t)$, $w_3(t)$ be d_1–dimensional and one-dimensional standard Brownian motions respectively. We define a new filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$

with the natural filtration

$$\mathcal{F}_t = \sigma\{w(s); 0 \leq s \leq t\}, \mathcal{F} = \mathcal{F}_T,$$

where $w(\cdot) = [w_1(\cdot), w_2(\cdot)]'$ is a $d_1 + 1$ dimensional standard Brownian motion.

The factor processes $x_1(t)$ and $x_3(t)$ are governed by the following SDEs:

$$\begin{cases} \mathrm{d}x_1(t) = Ax_1(t)\mathrm{d}t + D\mathrm{d}w_1(t), t \geq 0 \\ x_1(0) = x_{10} \end{cases} \tag{4.1}$$

$$\begin{cases} \mathrm{d}x_3(t) = [\alpha - \beta x_3(t)]\,\mathrm{d}t + \sqrt{x_3(t)}v\mathrm{d}w_3(t), t \geq 0 \\ x_3(0) = x_{30} \end{cases} \tag{4.2}$$

We propose the following model for the short interest rate $r(t)$, which has the quadratic and mixed CIR term,

$$r(t) = q_1' x_1(t) + x_1(t)' Q_2 x_1(t) + \delta x_3(t), \tag{4.3}$$

where $q_1 \in \mathbb{R}^{n_1}, Q_2 \in \mathbb{R}^{n_1 \times n_1}$and $\delta \in \mathbb{R}$. Compared this model with the known interest rate models, it has a more general case: when Q_2 and δ chosen the value zero, it becomes the Vasicek interest rate model directly; and let $q_1 = 0$, $Q_2 = 0$, it is CIR process; in particular, when δ are zero, our interest rate turns to be the traditional QATSM.

4.3 Zero-coupon Bond

The price of a zero-coupon bond can be observed in an explicit closed-form by using equation (3.2). It is denoted to be $B(t, T)$ at mature time T:

$$B(t, T) = \mathbb{E}\left[\mathrm{e}^{-\int_t^T r(\tau)\mathrm{d}\tau}\right].$$

Substituting $r(t)$ from (4.3), we obtain the expression of bond price, which is

$$B(t, T) = \mathbb{E}\left[\mathrm{e}^{-\int_t^T \left\{q_1' x_1(\tau) + x_1(\tau)' Q_2 x_1(t) + \delta x_3(\tau)\right\}\mathrm{d}\tau}\right]. \tag{4.4}$$

Beginning with time t, rather than time zero, the criterion (3.2) has the following coefficients:

$$\gamma = 1, \quad S = 0, \quad \tilde{Q} = -2Q_2, \quad \tilde{P} = 0, \quad L_1 = -2q_1, \quad L_2 = 0,$$
$$L_3 = -2\delta, \quad L_u = 0, \quad \tilde{R} = 0, \quad S_1 = 0, \quad S_2 = 0, \quad S_3 = 0.$$

Therefore, the relevant differential equations from (3.4), (3.5), (3.6), (3.7), (3.8) can be rewritten as follows:

$$\begin{cases} -2Q_2 + \dot{G}_1(t) + 2G_1(t)A + G_1(t)D'DG_1(t) = 0, \\ \\ G_1(T) = 0, \end{cases} \tag{4.5}$$

$$g_3(t) = 0, \tag{4.6}$$

$$\begin{cases} -2q_1' + \dot{g_2}'(t) + g_2'(t)A + g_2'(t)D'DG_1(t) = 0, \\ \\ g_2(T) = 0, \end{cases} \tag{4.7}$$

$$\begin{cases} -8\delta + 4\dot{g}_4(t) + 4\alpha g_4(t) + v^2 g_4^2(t) = 0, \\ \\ g_4(T) = 0, \end{cases} \tag{4.8}$$

$$\begin{cases} 2\sum_{j=1}^{n} C_{1j}'G_1(t)C_{1j} + g_2'(t)DD'g_2(t) - 4\beta g_4(t) + 4\dot{g}_5(t) = 0, \\ \\ g_5(T) = 0, \end{cases} \tag{4.9}$$

Corollary 4.3.1. *Let equations* (4.5), (4.7), (4.8) *and* (4.9) *have unique global* *solutions, then the price of zero-coupon bond at time t is*

$$B(t,T) = \mathbb{E}\left[\exp\left\{\frac{1}{2}\left[x_1'(t)G_1(t)x_1(t) + g_2'(t)x_1(t) + g_4(t)x_3(t) + g_5(t)\right]\right\}\right],$$

where $G_1(t)$, $g_2(t)$, $g_4(t)$ and $g_5(t)$ are solutions of differential equations (4.5), (4.7)*,* (4.8) *and* (4.9) *respectively.*

Proof. We can obtain the result from Theorem 3.2.2. □

This gives us a closed form of bond pricing with a new interest rate model.

4.4 Market Model

As described in Section 2.3, we consider a financial market consists of a bond, and n stocks. Denote the price of a bond to be $S_0(t)$ with the interest rate $r(t)$, and the price of stock number i to be $S_i(t)$, $i = 1, \ldots, n$. In our model, we assume these prices processes whose differentials are

$$\begin{cases} \mathrm{d}S_0(t) = S_0(t)r(t)\mathtt{d}t \\ \\ S_0(0) = S_{00}, \end{cases} \tag{4.10}$$

$$\begin{cases} \mathrm{d}S_i(t) = S_i(t)\left(\mu_i(t)\mathrm{d}t + \sum_{j}^{m}\sigma_{ij}(t)\mathrm{d}w_j(t)\right) \\ \\ S_i(0) = S_{i0}, \end{cases} \tag{4.11}$$

where the vector $\mu(t) \in \mathbb{R}^n$, $\mu(t) = [\mu_1(t), \ldots, \mu_n(t)]'$ is the mean rates of return, and the vectors $\sigma_i(t) \in \mathbb{R}^{1\times m}$, $\sigma_i(t) = [\sigma_{i1}(t), \ldots, \sigma_{im}(t)]$ is the volatility.

We use the factor processes x_1 and x_3 defined in Section refsection 4.2. According to H. Kraft [31] and T. R. Bielecki, S. Pliska and J. Yong [4], the drift process is defined: $\mu(t) = \mathbf{1}r(t) + \lambda x_1(t)$. In this chapter, we introduce a more general case of $\mu(t)$, which is $\mu(t) = \mathbf{1}r(t) + \lambda_1\tilde{x}(t) + \lambda_2 x_1(t) + \lambda_3$,

$$\tilde{x}(t) = \begin{bmatrix} \sqrt{(x_1(t)-\mu_1)'K_{11}(x_1(t)-\mu_1)+k_{21}x_3(t)+k_{31}} \\ \sqrt{(x_1(t)-\mu_1)'K_{12}(x_1(t)-\mu_1)+k_{22}x_3(t)+k_{32}} \\ \vdots \\ \sqrt{(x_1(t)-\mu_1)'K_{1n}(x_1(t)-\mu_1)+k_{2n}x_3(t)+k_{3n}} \end{bmatrix},$$

.
where

$$K_{1i} \in \mathbb{R}^{n\times n}, \quad k_{2i}, k_{3i} \in \mathbb{R}, \quad \mu_1 \in \mathbb{R}^n, \quad \lambda_1, \lambda_2 \in \mathbb{R}^{n\times n},$$
$$\lambda_3 \in \mathbb{R}^n, \quad \sigma \in \mathbb{R}^{n\times d}, \quad \mathbf{1} = [\underbrace{1,\dots,1}_{n}].$$

It should has two assumptions:

Assumption 4.4.1. $\sigma\sigma' > 0$

Assumption 4.4.2.

$$\begin{cases} {\lambda_1}'(\sigma\sigma')^{-1}\lambda_2 = 0 \\ {\lambda_3}'(\sigma\sigma')^{-1}\lambda_1 = 0 \\ {\lambda_1}'(\sigma\sigma')^{-1}\lambda_1 = \operatorname{diag}(\theta_1, \theta_2, \dots, \theta_n) \quad \textit{i.e. a diagonal matrix,} \end{cases}$$

where at least one θ_i *is zero.*

Example 4.4.1. *We give a numerical example of Assumption 4.4.1 and 4.4.2.*

$$n = 2, \quad \sigma = \begin{bmatrix} \sqrt{\frac{21}{5}} & \sqrt{\frac{4}{5}} \\ 0 & \sqrt{5} \end{bmatrix}, \quad \sigma\sigma' = \begin{bmatrix} 5 & 2 \\ 2 & 5 \end{bmatrix}, \quad \lambda_1 = \begin{bmatrix} 0 & 2 \\ 0 & 3 \end{bmatrix},$$
$$\lambda_2 = \begin{bmatrix} 11 & -11 \\ -4 & 4 \end{bmatrix}, \quad \lambda_3 = \begin{bmatrix} 11 \\ -4 \end{bmatrix}, \quad \operatorname{diag}(\theta_1, \theta_2) = \begin{bmatrix} 0 & 0 \\ 0 & \frac{41}{21} \end{bmatrix}.$$

4.4.1 No Arbitrage of the Market

According to section 2.4, if there has a market price of risk process $\phi(t)$, that satisfying the following two conditions:

$$\begin{cases} \mu(t) - r(t) = \sigma\phi(t) \\ \mathbb{E}[e^{-\int_0^T \phi(t)'dw(t) - \frac{1}{2}\int_0^T |\phi(t)|^2 dt}] = 1, \end{cases} \tag{4.12}$$

then the market has no arbitrage. Thus it is necessary to prove the existence of $\phi(t)$. According to Theorem 8.6.4 of B. ksendal [43], if the Novikov condition holds:

$$\mathbb{E}[e^{\frac{1}{2}\int_0^T |\phi(t)|^2 \mathrm{d}t}] < \infty, \tag{4.13}$$

then the second equation of (4.12) holds.

We first define $\phi(\cdot)$ to be

$$\phi(t) = \sigma^{-1}(\mu(t) - \mathbf{1}r(t)) = \sigma^{-1}(\lambda_1\tilde{x}(t) + \lambda_2 x_1(t) + \lambda_3),$$

and substitute $\phi(t)$ into

$$\mathbb{E}[e^{\frac{1}{2}\int_0^T |\phi(t)|^2 \mathrm{d}t}].$$

It can be deduced as follows:

$$\begin{aligned} &\mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^T [\lambda_1 \tilde{x(t)} + \lambda_2 x_1(t) + \lambda_3]'(\sigma\sigma')^{-1}[\lambda_1\tilde{x}(t) + \lambda_2 x_1(t) + \lambda_3]\mathrm{d}t\right)\right] \\ \leq\ & \left\{\mathbb{E}\left[\exp\left(\frac{p_1}{2}\int_0^T x_1(t)'\left[\lambda_2'(\sigma\sigma')^{-1}\lambda_2 + \sum_{i=1}^n \theta_i K_{1i}\right] x_1(t)\mathrm{d}t\right)\right]\right\}^{1/p_1} \\ & \left\{\mathbb{E}\left[\exp\left(p_2\int_0^T \left[\lambda_3'(\sigma\sigma')^{-1}\lambda_2 - \sum_{i=1}^n \theta_i\mu_1' K_{1i}\right] x_1(t)\mathrm{d}t\right)\right]\right\}^{1/p_2} \\ & \left\{\mathbb{E}\left[\exp\left(p_3\int_0^T x_2(t)\sum_{i=1}^n \theta_i k_{2i}\mathrm{d}t\right)\right]\right\}^{1/p_3} \\ & \left\{\mathbb{E}\left[\exp\left(p_4\int_0^T \left[\lambda_3'(\sigma\sigma')^{-1}\lambda_3 + \sum_{i=1}^n \theta_i(\mu_1'\mu_1 + k_{3i})\right]\mathrm{d}t\right)\right]\right\}^{1/p_4}, \end{aligned}$$

where $p_1, p_2, p_3, p_4 > 1$ and $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} + \frac{1}{p_4} = 1$.
The next work is to check the existence of following inequalities.

$$\mathbb{E}[\mathrm{e}^{\kappa \int_0^T x_1(t)' \Lambda x_1(t) \mathrm{d}t}] < \infty,$$

$$\mathbb{E}[\mathrm{e}^{\kappa \int_0^T \lambda' x_1(t) dt}] < \infty,$$

$$\mathbb{E}[\mathrm{e}^{\kappa \int_0^T x_2(t) dt}] < \infty,$$

where κ is a positive constant, and

$$\Lambda \triangleq [\lambda_2'(\sigma\sigma')^{-1}\lambda_2 + \sum_{i=1}^{n} \theta_i K_{1i}]$$
$$\lambda \triangleq [\lambda_3'(\sigma\sigma')^{-1}\lambda_2 - \sum_{i=1}^{n} \theta_i \mu_1' K_{1i}]'$$

In the Section 4.5, the proofs of these three inequalities will be illustrated.

4.4.2 Wealth processes

Suppose an investor has an initial wealth $y_0 > 0$ and takes self-finance trading strategies. The stochastic differential equation of wealth process $y(t)$ will satisfy as follows:

$$\begin{cases} \mathrm{d}y(t) = \{r(t)y(t) + u(t)'[\mu(t) - \mathbf{1}r(t)]\}\mathrm{d}t + u(t)'\sigma \mathrm{d}w(t) \\ y(0) = y_0. \end{cases} \tag{4.14}$$

We call $u(t)$ the control process of the portfolio, and define $b := [\mu(t) - \mathbf{1}r(t)]$.

4.5 Admissible Trading Strategies

The purpose of this section is to provide a set of admissible controls $\mathcal{A}[0, T]$. For any initial state y_0 and admissible control $u(\cdot) \in \mathcal{A}[0, T]$, the state equations (4.14) admits a unique solution.

Recall (4.14), if we take $u(\cdot) = 0$, which means putting all money into the bank account, then the wealth process $y(\cdot)$ is given by

$$y(t) = \exp\left\{\int_0^t r(s)\mathrm{d}s\right\} y_0,$$

Now we consider that $y(t)^\tau$ should be integrable for $\tau > 0$, which means it should request the following type of estimate:

$$\mathbb{E}\left[\mathrm{e}^{\tau\int_0^t r(s)\mathrm{d}s}\right] < \infty, \forall \tau > 0.$$

Lemma 4.5.1. *Let $x_1(\cdot), x_2(\cdot)$ be the solution of* (4.1), (4.2) *respectively, and* $r(t) = q_1' x_1(t) + x_1(t)' Q_2 x_1(t) + \delta x_2(t)$. *Then*

$$\mathbb{E}\left[\exp\left\{\tau \int_0^t r(s)\mathrm{d}s\right\}\right] < \infty, \forall \tau > 0$$

provided the following holds:

$$2\tau T^2 |\mathrm{e}^{LAL^{-1}} LD|^2 < 1, \textit{ where } L \textit{ is an invertible matrix and } Q_2 = L'L,$$

$$4\alpha \leq |v|^2, \quad \frac{\tau T}{2\beta}|v|^2\left(\mathrm{e}^{\beta T} - 1\right) < 1.$$

Proof. Substituting the interest rate model into $\mathbb{E}\left[\mathrm{e}^{\int_0^t r(s)\mathrm{d}s}\right]$, we have the following inequality:

$$\begin{aligned} & \mathbb{E}\left[\exp\left\{\int_0^t [q_1' x_1(s) + x_1(s)' Q_2 x_1(s) + \delta x_2(s)]\mathrm{d}s\right\}\right] \\ \leq & \left\{\mathbb{E}\left[\exp\left\{p_5 \int_0^t q_1' x_1(s)\mathrm{d}s\right\}\right]\right\}^{1/p_5} \\ & \left\{\mathbb{E}\left[\exp\left\{p_6 \int_0^t x_1(s)' Q_2 x_1(s)\mathrm{d}s\right\}\right]\right\}^{1/p_6} \\ & \left\{\mathbb{E}\left[\exp\left\{p_7 \int_0^t \delta x_2(s)\mathrm{d}s\right\}\right]\right\}^{1/p_7}, \end{aligned}$$

where $p_5, p_6, p_7 > 1$ and $\frac{1}{p_5} + \frac{1}{p_6} + \frac{1}{p_7} = 1$.
Since $\mathbb{E}\left[\exp\left\{p_5 \int_0^t q_1' x_1(s)\mathrm{d}s\right\}\right]$ can be written as $\mathbb{E}\left[\exp\left\{\sum_{i=1}^n p_5 q_{1i} \int_0^t x_{1i}(s)\mathrm{d}s\right\}\right]$,

again using Hölder's inequality, and by Corollary C.2 from T. R. Bielecki, S. Pliska and J. Yong's paper [4], we can prove

$$\mathbb{E}\left[\exp\left\{p_5\int_0^t q_1' x_1(s)\mathrm{d}s\right\}\right] < \infty.$$

Then, by Cholesky decomposition, there exists a invertible matrix L such that $Q_2 = L'L$. Moreover, $x_1(t)'Q_2x_1(t) = x_1(t)'L'Lx_1(t) = z(t)'z(t)$, where we denote $z(t) = Lx_1(t)$. Recall the first equation (4.1),

$$\begin{aligned}
\mathrm{d}x_1(t) &= Ax_1\mathrm{d}t + D\mathrm{d}w_1(t) \\
\mathrm{d}Lx_1(t) &= LAx_1(t)\mathrm{d}t + LD\mathrm{d}w_1(t) \\
\mathrm{d}Lx_1(t) &= LAL^{-1}Lx_1(t)\mathrm{d}t + LD\mathrm{d}w_1(t) \\
\mathrm{d}z(t) &= A_z(t)z(t)\mathrm{d}t + D_z(t)\mathrm{d}w_1(t)
\end{aligned}$$

we denote $A_z = LAL^{-1}$ and $D_z = LD$.

Again applying Corollary C.2 from T. R. Bielecki, S. Pliska and J. Yong's paper [4], we can show

$$\mathbb{E}\left[\exp\left\{p_6\int_0^t x_1(s)'Q_2x_1(s)\mathrm{d}s\right\}\right] = \mathbb{E}\left[\exp\left\{p_6\int_0^t z(s)'z(s)\mathrm{d}s\right\}\right] < \infty$$

with condition

$$2\tau T^2|\mathrm{e}^{LAL^{-1}}LD|^2 < 1, \text{where } L \text{ is an invertible matrix and } Q_2 = L'L,\ \tau > 0.$$

According to Theorem 4.1 from J. Yong [54], we can prove

$$\mathbb{E}\left[\exp\left\{p_7\int_0^t \delta x_2(t)\mathrm{d}s\right\}\right] = \mathbb{E}\left[\exp\left\{\tau\int_0^t x_2(t)\mathrm{d}s\right\}\right] < \infty$$

provided

$$4\alpha \le |v|^2, \quad \frac{\tau T}{2\beta}|v|^2\left(\mathrm{e}^{\beta T} - 1\right) < 1, \quad \tau > 0$$

Thus, it is proved. □

The next step is taking $u(\cdot) \neq 0$, and supposing $y(\cdot)$ to be the solution of (4.14). We define $\rho(t) := u(t)/y(t)$.

By Itô's formula, we have

$$\mathrm{d}[\ln y(t)] = \left\{ r(t) + \rho(t)'[\mu(t) - \mathbf{1}r(t)] - \frac{1}{2}\rho(t)'\sigma\sigma'\rho(t) \right\} \mathrm{d}t + \rho(t)'\sigma \mathrm{d}w(t).$$

This implies:

$$y(t) = y_0 \exp\left\{ \int_0^t \left[r(s) + \rho(s)'[\mu(s) - \mathbf{1}r(s)] - \frac{1}{2}\rho(s)'\sigma\sigma'\rho(s) \right] \mathrm{d}s + \int_0^t \rho(s)'\sigma \mathrm{d}w(s) \right\}.$$

Again if $y(\cdot)$ is well defined, there should exist a constant $\kappa > 0$, such that the following inequality holds:

$$\mathbb{E}\left[y(t)^\kappa\right] < \infty$$

Here we introduce an assumption:

Assumption 4.5.1.

$$\rho(\cdot) = C(\sigma\sigma')^{-1}[\mu(\cdot) - \mathbf{1}r(\cdot)], \quad \text{where } C \text{ is a constant, } C > 0$$

And we state another lemma below:

Lemma 4.5.2. *Under Assumptions* (4.4.1), (4.4.2), (4.5.1), *let* $y(\cdot)$ *to be the solution of* (4.14), $\mu(t) = \mathbf{1}r(t) + \lambda_1\tilde{x}(t) + \lambda_2 x_1(t) + \lambda_3$ *and* $r(t) = q_1' x_1(t) + x_1(t)'Q_2 x_1(t) + \delta x_2(t)$. *Then*

$$\mathbb{E}\left[y(t)^\kappa\right] < \infty, \forall \kappa > 0$$

provided the following holds:

$$2\kappa T^2 |\mathrm{e}^{LAL^{-1}} LD|^2 < 1, \text{where } L \text{ is an invertible matrix and } Q_2 = L'L,$$

$$2\kappa T^2 |\mathrm{e}^{\tilde{L}A\tilde{L}^{-1}} \tilde{L}D|^2 < 1, \text{where } \tilde{L} \text{ is an invertible matrix and } \Lambda = \tilde{L}'\tilde{L},$$

$$4\alpha \leq |v|^2, \quad \frac{\kappa T}{2\beta}|v|^2 \left(\mathrm{e}^{\beta T} - 1\right) < 1.$$

Proof. In order to prove it, we use Bielecki, Pliska and Yong's approach [4].

$$
\begin{aligned}
& \mathbb{E}\left[y(t)^\kappa\right] \\
= \ & y_0^\kappa \mathbb{E}\left[\exp\left\{\int_0^t \kappa\left(r(s)+\rho'[\mu(s)-\mathbf{1}r(s)]-\frac{1}{2}\rho'\sigma\sigma'\rho\right)\mathrm{d}s+\int_0^t \kappa\rho'\sigma\mathrm{d}w(s)\right\}\right] \\
= \ & y_0^\kappa \mathbb{E}\left[\exp\left\{\int_0^t \kappa\left(r(s)+\rho'[\mu(s)-\mathbf{1}r(s)]-\frac{p_8\kappa-1}{2}\rho'\sigma\sigma'\rho\right)\mathrm{d}s\right\}\right. \\
& \qquad \left.\exp\left\{\frac{1}{p_8}\int_0^t p_8\kappa\rho'\sigma\mathrm{d}w(s)-\frac{1}{2p_8}\int_0^t (p_8\kappa)^2\rho'\sigma\sigma'\rho\mathrm{d}s\right\}\right] \\
\leq \ & y_0^\kappa \left\{\mathbb{E}\left[\exp\left(p_9\kappa\int_0^t r(s)\mathrm{d}s\right)\right]\right\}^{1/p_9}\left\{\mathbb{E}\left[\exp\left(p_{10}\kappa\int_0^t \rho'[\mu(s)-\mathbf{1}r(s)]\mathrm{d}s\right)\right]\right\}^{1/p_{10}} \\
& \quad \left\{\mathbb{E}\left[\exp\left(p_{11}\frac{p_8(\kappa^2-\kappa)}{2}\int_0^t \rho'\sigma\sigma'\rho\mathrm{d}s\right)\right]\right\}^{1/p_{11}} \\
& \quad \left\{\mathbb{E}\left[\exp\left\{\int_0^t p_8\kappa\rho'\sigma\mathrm{d}w(s)-\frac{1}{2}\int_0^t (p_8\kappa)^2\rho'\sigma\sigma'\rho\mathrm{d}s\right\}\right]\right\}^{1/p_8}
\end{aligned}
$$

where $p_8, p_9, p_{10}, p_{11} > 1$ and $\frac{1}{p_8}+\frac{1}{p_9}+\frac{1}{p_{10}}+\frac{1}{p_{11}}=1$.

From Lemma (4.5.1), we can show

$$\mathbb{E}\left[\exp\left(\kappa\int_0^t r(s)\mathrm{d}s\right)\right]<\infty.$$

with two conditions:

$$2\kappa T^2|\mathrm{e}^{LAL^{-1}}LD|^2<1, \text{where } L \text{ is an invertible matrix and } Q_2=L'L,$$

$$4\alpha\leq|v|^2, \quad \frac{\kappa T}{2\beta}|v|^2\left(\mathrm{e}^{\beta T}-1\right)<1.$$

Under Assumptions (4.4.1), (4.4.2), (4.5.1), we have

$$\mathbb{E}\left[\exp\left\{p_{10}\kappa\int_0^t \rho'[\mu(s)-\mathbf{1}r(s)]\mathrm{d}s\right\}\right]$$

$$
\begin{aligned}
&= \mathbb{E}\left[\exp\left\{p_{10}\kappa C\int_0^t [\mu(s)-\mathbf{1}r(s)]'(\sigma\sigma')^{-1}[\mu(s)-\mathbf{1}r(s)]\mathrm{d}s\right\}\right] \\
&\le \left\{\mathbb{E}\left[\exp\left(p_{12}\kappa\int_0^t x_1(s)'\left[\lambda_2'(\sigma\sigma')^{-1}\lambda_2+\sum_{i=1}^n\theta_i K_{1i}\right]x_1(s)\mathrm{d}s\right)\right]\right\}^{1/p_{12}} \\
&\quad \left\{\mathbb{E}\left[\exp\left(p_{13}\kappa\int_0^t \left[\lambda_3'(\sigma\sigma')^{-1}\lambda_2-\sum_{i=1}^n\theta_i\mu_1' K_{1i}\right]x_1(s)\mathrm{d}s\right)\right]\right\}^{1/p_{13}} \\
&\quad \left\{\mathbb{E}\left[\exp\left(p_{14}\kappa\sum_{i=1}^n\theta_i k_{2i}\int_0^t x_2(s)\mathrm{d}s\right)\right]\right\}^{1/p_{14}} \\
&\quad \left\{\mathbb{E}\left[\exp\left(p_{15}\kappa\int_0^t \left[\sum_{i=1}^n\theta_i(\mu_1'\mu_1+k_{3i})+\lambda_3'(\sigma\sigma')^{-1}\lambda_3\right]\mathrm{d}s\right)\right]\right\}^{1/p_{15}}
\end{aligned}
$$

where $p_{12}, p_{13}, p_{14}, p_{15} > 1$ and $\frac{1}{p_{12}}+\frac{1}{p_{13}}+\frac{1}{p_{14}}+\frac{1}{p_{15}}=1$.

Similar to the proof of Lemma (4.5.1), we prove three inequalities.

$$
\begin{aligned}
&\mathbb{E}\left[\exp\left(p_{12}\kappa\int_0^t x_1(s)'\left[\lambda_2'(\sigma\sigma')^{-1}\lambda_2+\sum_{i=1}^n\theta_i K_{1i}\right]x_1(s)\mathrm{d}s\right)\right] \\
&= \mathbb{E}\left[\exp\left(\kappa\int_0^t x_1(s)'\Lambda x_1(s)\mathrm{d}s\right)\right] < \infty, \qquad (4.15)
\end{aligned}
$$

provoded $2\kappa T^2|\mathrm{e}^{\tilde{L}A\tilde{L}^{-1}}\tilde{L}D|^2 < 1$, where $\tilde{L}$ is an invertible matrix and $\Lambda = \tilde{L}'\tilde{L}$.

$$
\begin{aligned}
&\mathbb{E}\left[\exp\left(p_{13}\kappa\int_0^t \left[\lambda_3'(\sigma\sigma')^{-1}\lambda_2-\sum_{i=1}^n\theta_i\mu_1' K_{1i}\right]x_1(s)\mathrm{d}s\right)\right] \\
&= \mathbb{E}\left[\exp\left(\kappa\int_0^t \lambda' x_1(s)\mathrm{d}s\right)\right] < \infty \qquad (4.16)
\end{aligned}
$$

$$\mathbb{E}\left[\exp\left(\kappa\int_0^t x_2(s)\mathrm{d}s\right)\right] < \infty, \quad \text{provided } 4\alpha \le |v|^2, \quad \frac{\kappa T}{2\beta}|v|^2\left(\mathrm{e}^{\beta T}-1\right) < 1. \tag{4.17}$$

Therefore, from inequalities (4.15), (4.16), (4.17), we proved

$$\mathbb{E}\left[\exp\left(\kappa\int_0^t \rho'[\mu(s)-\mathbf{1}r(s)]\mathrm{d}s\right)\right] < \infty$$

provided the following holds:

$$2\kappa T^2|\mathrm{e}^{\tilde{L}\Lambda\tilde{L}^{-1}}\tilde{L}D|^2 < 1, \text{where } \tilde{L} \text{ is an invertible matrix and } \Lambda = \tilde{L}'\tilde{L},$$

and

$$4\alpha \le |v|^2, \quad \frac{\kappa T}{2\beta}|v|^2\left(\mathrm{e}^{\beta T}-1\right) < 1.$$

The expectation $\mathbb{E}\left[\exp\left(p_{11}\frac{p_8(\kappa^2-\kappa)}{2}\int_0^t \rho'\sigma\sigma'\rho\mathrm{d}s\right)\right]$ can be written as

$$\mathbb{E}\left[\exp\left\{k\int_0^t (\mu(s)-\mathbf{1}r(s))'(\sigma\sigma')^{-1}(\mu(s)-\mathbf{1}r(s))\mathrm{d}s\right\}\right] < \infty,$$

where k is some constant. The proof is similar and omitted.

Using the fact

$$\mathbb{E}\left[\exp\left\{\int_0^t |p_8\kappa\rho'\sigma|\mathrm{d}w(s) - \frac{1}{2}\int_0^t |p_8\kappa\rho'\sigma|^2\mathrm{d}s\right\}\right] < 1.$$

Finally, $\mathbb{E}\left[y(t)^\kappa\right]$ can be proved to be finite with some sufficient conditions:

$$2\kappa T^2|\mathrm{e}^{LAL^{-1}}LD|^2 < 1, \text{where } L \text{ is an invertible matrix and } Q_2 = L'L,$$

$$2\kappa T^2|\mathrm{e}^{\tilde{L}A\tilde{L}^{-1}}\tilde{L}D|^2 < 1, \text{where } \tilde{L} \text{ is an invertible matrix and } \Lambda = \tilde{L}'\tilde{L},$$

$$4\alpha \le |v|^2, \quad \frac{\kappa T}{2\beta}|v|^2\left(\mathrm{e}^{\beta T}-1\right) < 1.$$

□

Then, we introduce the following set $\mathcal{A}[0,T]$:

$$\mathcal{A}[0,T]=\bigcup_{\kappa>0}\left\{u(\cdot)\mid\mathbb{E}\left[y(t)^{\kappa}\right]<\infty\right\}.$$

where the sufficient conditions of $\mathbb{E}\left[y(t)^{\kappa}\right]<\infty$ are mentioned in Lemma (4.5.2).

4.6 Solution of Problems

In this section, we define the optimal cost function follows two types of utilities, one is Logarithm Utility and the other is Power Utility. It is denoted by:

$$J(u(\cdot))=\begin{cases}\mathbb{E}\left[-\ln y(T)\right]\\ \\ \mathbb{E}\left[-y(T)^{\gamma}\right],\text{where }\gamma\in(0,1).\end{cases}\tag{4.18}$$

Then our problem is to seek a $u(\cdot)$ that minimizes $J(u(\cdot))$.

4.6.1 Logarithmic Utility

Problem 4.6.1. *For given $t\in[0,T]$, find a $u(\cdot)\in\mathcal{A}[0,T]$ such that*

$$\min_{u\in\mathcal{A}}\mathbb{E}[-\ln y(T)]$$

Theorem 4.6.1. *Under Assumption* (4.4.1)*, Problem* (4.6.1) *has a unique solution with the optimal control $u(\cdot)\in\mathcal{A}[0,T]$ given by*

$$u^{*}(t)=(\sigma\sigma')^{-1}b\tag{4.19}$$

The optimal cost is

$$\ln y_0\mathbb{E}\left[\int_0^T\left(\frac{1}{2}b'(\sigma\sigma')^{-1}b+r(s)\right)\mathrm{d}s\right].\tag{4.20}$$

Proof. By using Itô's formula and taking expectation, we have

$$\begin{aligned}&J(u(\cdot))\\ =\ &\mathbb{E}\left[-\ln y_0\right]\mathbb{E}\left[\int_0^T\left(-r(s)-u(s)'b+\frac{1}{2}u(s)'\sigma\sigma'u(s)\right)\mathrm{d}s\right]\end{aligned}$$

$$= \quad \mathbb{E}\left[-\ln y_0\right]\mathbb{E}\Bigg[\int_0^T\Bigg\{\frac{1}{2}\big[u(s)-(\sigma\sigma')^{-1}b\big]'\sigma\sigma'\big[u(s)-(\sigma\sigma')^{-1}b\big]$$

$$-\frac{1}{2}b'(\sigma\sigma')^{-1}b-r(s)\Bigg\}\mathrm{d}s\Bigg]$$

$$\geq \quad -\mathbb{E}\left[-\ln y_0\right]\mathbb{E}\left[\int_0^T\left(\frac{1}{2}b'(\sigma\sigma')^{-1}b+r(s)\right)\mathrm{d}s\right]$$

Therefore, when $u(t)=u^*(t)=(\sigma\sigma')^{-1}b$, $J(t)$ can choose the minimum value, which is

$$\mathbb{E}\left[\ln y_0\right]\mathbb{E}\left[\int_0^T\left(\frac{1}{2}b'(\sigma\sigma')^{-1}b+r(s)\right)\mathrm{d}s\right].$$

□

4.6.2 Power Utility

Recall the second equation of (4.18), we build the Problem (4.6.2).

Problem 4.6.2. *For given $t\in[0,T]$, find a $u(\cdot)\in\mathcal{A}[0,T]$ such that*

$$\min_{u\in\mathcal{A}}\mathbb{E}[-y(T)^{\gamma}],\, \textit{where } \gamma\in(0,1)$$

In order to solve Problem (4.6.2), we introduce another function $F(t,x_1,x_2,y)$ such that

$$F(t,x_1,x_2,y)=\exp\left\{g_1(t)'x_1(t)+x_1(t)'G_2(t)x_1(t)+g_3(t)x_2(t)\right\}h(t)y(t)^{\gamma} \quad (4.21)$$

with the terminal condition $g_1(T)=0, G_2(T)=0, g_3(T)=0, h(T)=-1$, where

$$g_1(\cdot)\in L^{\infty}(0,T;\mathbb{R}^n),\quad G_2(\cdot)\in L^{\infty}(0,T;\mathbb{R}^{n\times n}),\quad g_3(\cdot)\in L^{\infty}(0,T;\mathbb{R}).$$

Compared $F(t,x_t,x_2,y)$ with $-y(T)^{\gamma}$, it can be seen when $t=T$, these two functions will be the same

$$F(t,x_1,x_2,y)=-y(T)^{\gamma}.$$

We further introduce another assumption

Assumption 4.6.1. *Under Assumption* (4.4.1), (4.4.2), *functions* g_1, G_2, g_3, h *satisfy the following differential equations* (4.22),(4.23),(4.24),(4.25)*:*

$$\begin{cases} \dot{g_1}'(t) + g_1'(t)A + 2g_1'(t)DD'G_2(t) + \gamma q_1' \\ \qquad + \dfrac{\gamma}{1-\gamma}\left[\lambda_3'(\sigma\sigma')^{-1}\lambda_2 - \sum_{i=1}^{n}\theta_i\mu_1' K_{1i}\right] = 0 \\ g_1(T) = 0 \end{cases} \tag{4.22}$$

$$\begin{cases} \dot{G}_2(t) + 2G_2(t)A + 2G_2(t)DD'G_2(t) + \gamma Q_2 \\ \qquad + \dfrac{\gamma}{2(1-\gamma)}\left[\sum_{i=1}^{n}\theta_i K_{1i} + \lambda_2'(\sigma\sigma')^{-1}\lambda_2\right] = 0 \\ G_2(T) = 0 \end{cases} \tag{4.23}$$

$$\begin{cases} \dot{g_3}(t) + g_3(t)\beta + \dfrac{1}{2}g_3^2(t)v^2 + \gamma\delta + \dfrac{\gamma}{2(1-\gamma)}\sum_{i=1}^{n}\theta_i k_{2i} = 0 \\ g_3(T) = 0 \end{cases} \tag{4.24}$$

$$\begin{cases} \dot{h}(t) + h(t)g_3(t)\alpha + \dfrac{1}{2}h(t)\mathrm{tr}\left[(2G_2(t) + g_1(t)g_1'(t))DD'\right] \\ \qquad + \dfrac{\gamma h(t)}{2(1-\gamma)}\left[\sum_{i=1}^{n}\theta_i(\mu_1'\mu_1 + k_{3i}) + \lambda_3'(\sigma\sigma')^{-1}\lambda_3\right] = 0 \\ h(T) = -1. \end{cases} \tag{4.25}$$

Now we can state the main results of this section.

Theorem 4.6.2. *Under Assumption* (4.6.1), *there exist s a unique solution to*

Problem (4.6.2) *given by*

$$u^*(t) = \frac{1}{1-\gamma}(\sigma\sigma')^{-1}[\mu(t) - \mathbf{1}r(t)]y(t) \tag{4.26}$$

The optimal cost function is

$$J^*(t) \quad = \quad \mathbb{E}\left[\exp\{g_1(0)'x_{10} + x_{10}'G_2(0)x_{10} + g_3(0)x_{20}\}h(0)y_0^{\gamma}\right]. \tag{4.27}$$

with g_1, G_2, g_3, h *being solutions of differential equations* (4.22),(4.23),(4.24),(4.25) *corresponding.*

Proof. We now suppose $F(t, x_1, x_2, y) = f_1(t, x_1, x_2)f_2(t, y)$, where functions $f_1(\cdot)$ and $f_2(\cdot)$ related to $x_1(t), x_2(t)$ and $y(t)$ corresponding

$$\begin{cases} f_1(t, x_1) = \exp\{g_1(t)'x_1(t) + x_1(t)'G_2(t)x_1(t) + g_3(t)x_2(t)\} \\ \\ f_2(t, y) = h(t)y(t)^{\gamma} \end{cases}$$

By Itô's Lemma, we have the following differential equations

$$\begin{aligned} \mathrm{d}F(t) \quad = \quad & f_1(t)\Big\{\dot{h}(t)y(t)^{\gamma} + \gamma h(t)y(t)^{\gamma-1}\left\{r(t)y(t) + u(t)'\left[\mu(t) - \mathbf{1}r(t)\right]\right\} \\ & \qquad + \frac{1}{2}u(t)'\sigma\sigma'u(t)\gamma(\gamma-1)h(t)y(t)^{\gamma-2}\Big\}\,\mathrm{d}t \\ & + f_1(t)f_2(t)\Big\{\dot{g}_1(t)'x_1(t) + x_1(t)'\dot{G}_2(t)x_1(t) + g_1(t)'Ax_1(t) \\ & \qquad + 2g_1(t)'DD'G_2(t)x_1(t) + 2x_1(t)'G_2(t)DD'G_2(t)x_1(t) \\ & \qquad + 2x_1(t)'G_2Ax_1(t) + \frac{1}{2}\mathrm{tr}[D'(2G_2(t) + g_1(t)g_1(t)')D]\Big\}\,\mathrm{d}t \\ & + f_1(t)f_2(t)\left\{\dot{g}_3(t)x_2(t) + g_3(t)[\alpha - \beta x_2(t)] + \frac{1}{2}g_3(t)^2x_2(t)v^2\right\}\mathrm{d}t \\ & + f_1(t)f_2(t)g_3(t)\sqrt{x_2(t)}v\,\mathrm{d}w_2(t) + f_1(t)f_2(t)[g_1(t) + 2G_2(t)x_1(t)]'D\,\mathrm{d}w_1(t) \\ & + f_1(t)u(t)'\sigma\gamma h(t)y(t)^{\gamma-1}\mathrm{d}w(t) \end{aligned}$$

Then we substitute $f_2(t) = h(t)y(t)^{\gamma}$ into $\mathrm{d}F$, take the expectation of $\mathrm{d}F$ and arrange the equation into a nicely organized form, and deduce the expression of

$J(u(\cdot))$:

$$
\begin{aligned}
& J(u(\cdot)) \\
= \;& \mathbb{E}\left[F(0)\right] \\
& +\mathbb{E}\left[\int_0^T f_1(t)y(t)^\gamma\{h(t)\dot{g_1}(t)' + h(t)g_1(t)'A \right. \\
& \qquad\qquad \left. + 2h(t)g_1(t)'DD'G_2(t) + \gamma h(t)q_1'\}x_1(t)\mathrm{d}s\right] \\
& +\mathbb{E}\left[\int_0^T f_1(t)y(t)^\gamma x_1(t)'\{h(t)\dot{G_2}(t) + 2h(t)G_2(t)A \right. \\
& \qquad\qquad \left. + 2h(t)G_2(t)DD'G_2(t) + \gamma h(t)Q_2\}x_1(t)\mathrm{d}s\right] \\
& +\mathbb{E}\left[\int_0^T f_1(t)y(t)^\gamma x_2(t)\left\{ h(t)\dot{g_3}(t) - h(t)g_3(t)\beta \right.\right. \\
& \qquad\qquad \left.\left. +\frac{1}{2}h(t)g_3(t)^2v^2 + \gamma h(t)\delta\right\}\mathrm{d}s\right] \\
& +\mathbb{E}\left[\int_0^T f_1(t)y(t)^\gamma \left\{\dot{h}(t) + h(t)g_3(t)\alpha \right.\right. \\
& \qquad\qquad \left.\left. +\frac{1}{2}h(t)\mathrm{tr}\left[(2G_2(t) + g_1(t)g_1(t)')DD'\right]\right\}\mathrm{d}s\right] \\
& +\mathbb{E}\left[\int_0^T f_1(t)\left\{\gamma h(t)y(t)^{\gamma-1}u(t)'(\mu(t) - \mathbf{1}r(t)) \right.\right. \\
& \qquad\qquad \left.\left. +\frac{1}{2}u(t)'\sigma\sigma'u(t)\gamma(\gamma-1)h(t)y(t)^{\gamma-2}\right\}\mathrm{d}s\right]
\end{aligned}
$$

Only the last expectation above contains the control process $u(\cdot)$, and it can be written as:

$$
\mathbb{E}\left[\int_0^T f_1(s)\frac{\gamma(\gamma-1)h(s)y(s)^\gamma}{2}\left\{\rho(s)'\sigma\sigma'\rho(s) + \frac{2}{\gamma-1}\rho(s)'[\mu(s) - \mathbf{1}r(s)]\right\}\mathrm{d}s\right]
$$

$$= \mathbb{E}\left[\int_0^T f_1(s)\frac{\gamma(\gamma-1)h(s)y(s)^\gamma}{2}\left\{\left[\rho(s)+\frac{1}{\gamma-1}(\sigma\sigma')^{-1}(\mu(s)-\mathbf{1}r(s))\right]'\right.\right.$$

$$(\sigma\sigma')\left[\rho(s)+\frac{1}{\gamma-1}(\sigma\sigma')^{-1}(\mu(s)-\mathbf{1}r(s))\right]$$

$$\left.\left.-\frac{1}{(\gamma-1)^2}(\mu(s)-\mathbf{1}r(s))'(\sigma\sigma')^{-1}(\mu(s)-\mathbf{1}r(s))\right\}\mathrm{d}s\right]$$

The sum of the terms that are quadratic in $x_1(t)$ in (4.28) is zero due to equation (4.23) in Assumption (4.6.1), indeed,

$$x_1(t)'\left\{h(t)\dot{G}_2(t)+2h(t)G_2(t)A+2h(t)G_2(t)DD'G_2(t)\right.$$

$$\left.+\gamma h(t)Q_2+\frac{\gamma h(t)}{2(1-\gamma)}\left[\sum_{i=1}^n\theta_iK_{1i}+\lambda_2'(\sigma\sigma')^{-1}\lambda_2\right]\right\}x_1(t)=0$$

Similarly, the sums of the terms linear in $x_1(t)$, $x_2(t)$ are also zero:

$$\left\{h(t)\dot{g}_1(t)'+h(t)g_1(t)'A+2h(t)g_1(t)'DD'G_2(t)\right.$$

$$\left.+\gamma h(t)q_1(t)'+\frac{\gamma h(t)}{(1-\gamma)}\left[\lambda_3'(\sigma\sigma')^{-1}\lambda_2-\sum_{i=1}^n\theta_i\mu_1'K_{1i}\right]\right\}x_1(t)=0$$

$$x_2(t)\left\{h(t)\dot{g}_3(t)-h(t)g_3(t)\beta+\frac{1}{2}h(t)g_3(t)^2v^2+\gamma h(t)\delta+\frac{\gamma h(t)}{2(1-\gamma)}\sum_{i=1}^n\theta_1k_{2i}\right\}=0$$

The remaining sum of the terms that are independent of the states $x_1(t)$, $x_2(t)$ and control $u(t)$, is also zero due to our assumption on $h(t)$.

$$\dot{h}(t)+h(t)g_3(t)\alpha+\frac{1}{2}h(t)\mathrm{tr}\left[(2G_2(t)+g_1(t)g_1(t)')DD'\right]$$

$$+\frac{\gamma h(t)}{2(1-\gamma)}\left[\sum_{i=1}^{n}\theta_i(\mu_1'\mu_1+k_{3i})+\lambda_3'(\sigma\sigma')^{-1}\lambda_3\right]=0$$

Therefore, the cost function $J(u(\cdot))$ for all $u(\cdot)\in A$ can be written as follows:

$$\begin{aligned}
& J(u(\cdot)) \\
= \; & \mathbb{E}\left[\int_0^T f_1(s)\frac{\gamma(\gamma-1)h(s)y(s)^\gamma}{2}\left\{\rho(s)+\frac{1}{\gamma-1}(\sigma\sigma')^{-1}(\mu(s)-\mathbf{1}r(s))\right\}'\right. \\
& \left.(\sigma\sigma')\left\{\rho(s)+\frac{1}{\gamma-1}(\sigma\sigma')^{-1}(\mu(s)-\mathbf{1}r(s))\right\}\mathrm{d}s\right]+\mathbb{E}\left[F(0)\right] \\
\geq \; & \mathbb{E}\left[F(0)\right]
\end{aligned}$$

It means the cost function $J(u(\cdot))$ has the lower bound if and only if

$$u(t)=u^*(t)=\frac{1}{1-\gamma}(\sigma\sigma')^{-1}[\mu(t)-\mathbf{1}r(t)]y(t).$$

□

Similar to Section 3.2, we give a numerical example which is suitable to the model.

Example 4.6.1. *Let Example 4.4.1 holds, and*

$$\gamma=\frac{1}{2},\quad A=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix},\quad D=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix},\quad q_1=\begin{bmatrix}2\\ 4\end{bmatrix},\quad \mu_1=\begin{bmatrix}1\\ 2\end{bmatrix},$$

$$K_{12}=\begin{bmatrix}-21 & 21\\ 21 & -21\end{bmatrix},\quad Q_2=\begin{bmatrix}\frac{1}{8} & \frac{1}{8}\\ \frac{1}{8} & \frac{1}{8}\end{bmatrix},\quad \beta=1,\quad v=1,\quad \delta=2,$$

$$k_{22}=-\frac{21}{44},\quad \alpha=\frac{7}{4},\quad k_{32}=-16.$$

Then equations (4.22),(4.23),(4.24),(4.25) *become:*

$$\begin{cases} \dot{g}_1'(t) + g_1'(t)\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} + 2g_1'(t)\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}G_2(t) + \begin{bmatrix}1\\ 2\end{bmatrix} = 0,\\ \\ g_1(T) = 0,\end{cases}$$

$$\begin{cases} \dot{G}_2(t) + 2G_2(t)\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} + 2G_2(t)\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}G_2(t) + \begin{bmatrix}\frac{1}{16} & \frac{1}{16}\\ \frac{1}{16} & \frac{1}{16}\end{bmatrix} = 0,\\ \\ G_2(T) = 0,\end{cases}$$

$$\begin{cases} \dot{g}_3(t) + g_3(t) + \frac{1}{2}g_3^2(t) + \frac{1}{2} = 0,\\ \\ g_3(T) = 0,\end{cases}$$

$$\begin{cases} \dot{h}(t) - \frac{73}{4}h(t)g_3(t) + \frac{1}{2}h(t)\mathrm{tr}\left\{\Big(2G_2(t) + g_1(t)g_1'(t)\Big)\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right\} + h(t) = 0,\\ \\ h(T) = -1.\end{cases}$$

Thus g_1, G_2, g_3, h *can be solved as follows:*

$$g_1(t) = \begin{bmatrix} -\dfrac{\mathrm{e}^{T-t}(T-t-2) + 5T - 5t + 2}{2(T-t-2)}\\ \\ \dfrac{\mathrm{e}^{T-t}(T-t-2) - (7T - 7t - 2)}{2(T-t-2)}\end{bmatrix},$$

$$G_2(t) = \begin{bmatrix} -\dfrac{T-t}{8(T-t-2)} & -\dfrac{T-t}{8(T-t-2)} \\ \\ -\dfrac{T-t}{8(T-t-2)} & -\dfrac{T-t}{8(T-t-2)} \end{bmatrix},$$

$$g_3(t) = \frac{T-t}{T-t-2},$$

$$h(t) = -\frac{1}{2^{40}}(T-t-2)^{40}\exp\Bigg\{\frac{1}{8(T-t-2)}\Big[(e^{2(T-t)} - 4e^{T-t})(T-t-2) \\ +112(T-t)^2 - 365(T-t) - 6\Big]\Bigg\}.$$

4.7 Summary

In this chapter, we introduce a further nonlinear interest rate model, which is a combination of the CIR and the multi-dimensional quadratic term structure. The price of the zero-coupon bond with such type of interest rate model is calculated. The explicit solutions with logarithmic and power utilities are obtained in closed-form. In addition, Yong's result [54] about CIR model's limitation is carefully considered.

Chapter 5

Optimal investment with stochastic interest rate in an infinite horizon

We consider the optimal investment problem with a nonlinear stochastic interest rate in an *infinite* horizon. The nonlinearity consists not only of the quadratic but also of the square-root terms. The optimal investment is found in an explicit closed form by using the completion of squares and the change of measure methods.

5.1 Introduction

We consider the probability space defined in Section (2.2). Let $n = 1$, and define a new three-dimensional Brownian motion $w(t) = \Big[w_1(t), w_2(t), w_3(t)\Big]', t \geq 0$.

The processes x_1 and x_2 are the *factor* processes, and we assume that they are the solutions to the following equations:

$$\begin{cases} \mathrm{d}x_1(t) = [a_1 x_1(t) + a_2]\mathrm{d}t + b\mathrm{d}w_1(t), \\ \\ x_1(0) = x_{10}, \end{cases} \tag{5.1}$$

$$\begin{cases} \mathrm{d}x_2(t) = k[\theta - \sqrt{x_2(t)}]\mathrm{d}t + \sigma_2\sqrt{x_2(t)}\mathrm{d}w_2(t), \\ \\ x_2(0) = x_{20}, \end{cases} \tag{5.2}$$

where a_1, a_2, b are constants, k, σ_2 are positive constants, and $\theta = \dfrac{\sigma_2^2}{4k}$.

The state (5.1) is clearly a linear stochastic differential equation, whereas (5.2) is the Longstaff model [34]. The Longstaff model is different from the CIR model in that the square-root process $\sqrt{x_2}$ appears twice, and thus is called the *double square-root* (DSR) process. An empirical comparison of CIR and DSR models is done by Longstaff, who suggests that DSR process outperforms the CIR model in most circumstances.

We consider a financial market consisting of two assets: the bank account with price $B(t)$ and the stock with price $S(t)$. The equations of these two prices are:

$$\begin{cases} dB(t) = r(t)B(t)\mathrm{d}t, \\ \\ B(0) = B_0 > 0. \end{cases} \tag{5.3}$$

$$\begin{cases} \mathrm{d}S(t) = S(t)\Big[\mu(t)\mathrm{d}t + \sigma\mathrm{d}w_3(t)\Big], \\ \\ S(0) = S_0 > 0, \end{cases} \tag{5.4}$$

for some positive constant σ. Here $r(t)$ is the interest rate. Based on the factor processes x_1 and x_2, we assume the interest rate to be of the form

$$r(t) = q_1x_1^2(t) + q_2x_1(t) + q_3x_2(t) + q_4\sqrt{x_2(t)},$$

for some constants q_1, q_2, q_3, q_4. This interest rate model has quadratic and linear terms in x_1, which means it contains the QATSM as a special case. It also contains a linear term in x_2, and thus contains the Longstaff interest rate model as a special case. In addition, it contains a square-root term in x_2, which makes it a *new* interest rate model, in addition to being a rather general one. The motivation for introducing this additional term has mainly to do with the explicit solvability of the optimal investment problems that we consider.

In the optimal investment problems that deal with a stochastic interest rate, it is usually assumed that there is a certain relation between the drift $\mu(t)$ of the stock and the interest rate $r(t)$. In Bielecki, T.R., Pliska, S. & Yong, J., 2004, and Kraft, H., 2005, they assume that $\mu(t) = r(t) + \lambda x_1(t)$. Similarly to this, we assume the following relation to hold:

$$\mu(t) = r(t) + \lambda\sqrt{\kappa_1 x_1^2(t) + \kappa_2 x_2(t) + \kappa_3\sqrt{x_2(t)} + \kappa_4},$$

where $\lambda, \kappa_1, \kappa_2, \kappa_3, \kappa_4$ are constants.

In the above defined financial market, we consider an investor with the initial wealth of $y_0 > 0$ that can invest in the bank account and in the stock. The value of his/her self-financing portfolio is:

$$\begin{cases} \mathrm{d}y(t) = \Big\{ r(t)y(t) + u(t)\Big[\mu(t) - r(t)\Big]\Big\}\mathrm{d}t + u(t)\sigma\mathrm{d}w_3(t), \\ \\ y(0) = y_0, \end{cases} \tag{5.5}$$

where $u(t)$ denotes the amount of wealth invested in the stock.

The main objective of this chapter is to formulate and solve the optimal investment problem in an *infinite horizon*. In addition to the market model being new, our approach to solving this problem, which is based on the completion of squares and the change of measure methods, also appears to be new. The infinite horizon criteria that we use are the *average* type power and logarithmic utilities, given by, respectively:

$$J(u(\cdot)) = \lim_{T\to\infty} \frac{1}{T}\log\mathbb{E}\Big[y^{\gamma}(T)\Big], \quad \gamma \in (0,1), \tag{5.6}$$

$$J(u(\cdot)) = \lim_{T\to\infty} \frac{1}{T^2}\mathbb{E}\Big[\log y(T)\Big]. \tag{5.7}$$

The optimal investment problem is the following optimal control problem:

$$\begin{cases} \max\limits_{u(\cdot)\in\mathcal{A}} J(u(\cdot)), \\ \\ s.t. \quad (5.1), \quad (5.2), \quad (5.5), \end{cases} \tag{5.8}$$

where $\mathcal{A}$ is the set of admissible controls. We define $\mathcal{A}$ as the set of all $\mathcal{F}_t$-adapted processes $u(t)$ under which there exists of a positive and unique solution to equation (5.5), and the corresponding cost $J(u(\cdot))$ is finite.

5.2 Power utility

Let $u(\cdot) \in \mathcal{A}$, and define $\rho(t) := u(t)/y(t)$. The solution to (5.5) is

$$\begin{aligned} y(t) &= y_0 \exp\bigg\{ \int_0^t \Big[r(s) + \rho(s)\big[\mu(s) - r(s)\big] - \frac{1}{2}\rho^2(s)\sigma^2 \Big] \mathrm{d}s \bigg\} \\ &\quad \exp\bigg\{ \int_0^t \rho(s)\sigma \mathrm{d}w_3(t) \bigg\}. \end{aligned} \tag{5.9}$$

Consider the following equations:

$$\begin{aligned} \mathrm{d}\Big(g_1 x_1^2(t) + g_2 x_1(t)\Big) &= \Big[\Big(2g_1 x_1(t) + g_2\Big)\Big(a_1 x_1(t) + a_2\Big) + g_1 b^2\Big]\mathrm{d}t \\ &\quad + \Big(2g_1 x_1(t) + g_2\Big) b \mathrm{d}w_1(t), \end{aligned} \tag{5.10}$$

$$\mathrm{d}\Big(g_3 x_2(t) + g_4\sqrt{x_2(t)}\Big) = \Big[\Big(g_3 + \frac{1}{2}g_4\frac{1}{\sqrt{x_2(t)}}\Big)k\Big(\theta - \sqrt{x_2(t)}\Big) - \frac{1}{8}g_4\sigma_2^2\frac{1}{\sqrt{x_2(t)}}\Big]\mathrm{d}t$$

$$+\Big(g_3+\frac{1}{2}g_4\frac{1}{\sqrt{x_2(t)}}\Big)\sigma_2\sqrt{x_2(t)}\mathrm{d}w_2(t), \tag{5.11}$$

for some constants g_1, g_2, g_3, g_4, yet to be specified. After integration these become:

$$\begin{aligned} 0 &= -g_1x_1^2(T)-g_2x_1(T)+g_1x_{10}^2 \\ &\quad +g_2x_{10}+\int_0^T\Big[\Big(2g_1x_1(t)+g_2\Big)\Big(a_1x_1(t)+a_2\Big)+g_1b^2\Big]\mathrm{d}t \\ &\quad +\int_0^T\Big(2g_1x_1(t)+g_2\Big)b\mathrm{d}w_1(t) \end{aligned} \tag{5.12}$$

$$\begin{aligned} 0 &= -g_3x_2(T)-g_4\sqrt{x_2(T)}+g_3x_{20}+g_4\sqrt{x_{20}} \\ &\quad +\int_0^T\Big[\Big(g_3+\frac{1}{2}g_4\frac{1}{\sqrt{x_2(t)}}\Big)k\Big(\theta-\sqrt{x_2(t)}\Big)-\frac{1}{8}g_4\sigma_2^2\frac{1}{\sqrt{x_2(t)}}\Big]\mathrm{d}t \\ &\quad +\int_0^T\Big(g_3+\frac{1}{2}g_4\frac{1}{\sqrt{x_2(t)}}\Big)\sigma_2\sqrt{x_2(t)}\mathrm{d}w_2(t) \end{aligned} \tag{5.13}$$

By combining (5.9) with (5.12) and (5.13), we obtain

$$\begin{aligned} &\mathbb{E}\left[y^\gamma(T)\right] \\ &= y_0^\gamma\alpha\mathbb{E}\Bigg[\exp\Big\{\gamma\int_0^T\Big[q_1x_1^2(t)+q_2x_1(t)+q_3x_2(t)+q_4\sqrt{x_2(t)}+\rho(t)\left[\mu(t)-r(t)\right] \\ &\qquad -\frac{1}{2}\rho^2(t)\sigma^2+\Big(2g_1x_1(t)+g_2\Big)\Big(a_1x_1(t)+a_2\Big)+g_1b^2 \\ &\qquad +\Big(g_3+\frac{1}{2}g_4\frac{1}{\sqrt{x_2(t)}}\Big)k\Big(\theta-\sqrt{x_2(t)}\Big)-\frac{1}{8}g_4\sigma_2^2\frac{1}{\sqrt{x_2(t)}}\Big]\mathrm{d}t\Big\} \\ &\qquad \exp\Big\{-\gamma g_1x_1^2(T)-\gamma g_2x_1(T)-\gamma g_3x_2(T)-\gamma g_4\sqrt{x_2(T)}\Big\} \end{aligned}$$

$$\exp\left\{\gamma \int_0^T \tilde{\sigma}'(t)\mathrm{d}w(t)\right\}\Bigg], \tag{5.14}$$

where $\tilde{\sigma}(t)$ and α are defined as follows:

$$\tilde{\sigma}(t) := \begin{bmatrix} b\Big(2g_1x_1(t)+g_2\Big) \\ \\ \Big(g_3+\frac{1}{2}g_4\frac{1}{\sqrt{x_2(t)}}\Big)\sigma_2\sqrt{x_2(t)} \\ \\ \rho(t)\sigma \end{bmatrix},$$

$$\alpha := \exp\Big\{\gamma g_1 x_{10}^2 + \gamma g_2 x_{10} + \gamma g_3 x_{20} + \gamma g_4\sqrt{x_{20}}\Big\}.$$

We define the process $Z(t)$ and the random variable Z as

$$\begin{cases} Z(t) := \exp\Big\{-\displaystyle\int_0^t \gamma\tilde{\sigma}'(s)\mathrm{d}w(s) - \frac{1}{2}\int_0^t \gamma^2\tilde{\sigma}'(s)\tilde{\sigma}(s)\mathrm{d}s\Big\}, \\ \\ Z := Z(T) \end{cases}$$

Based on Z we introduce the new probability measure $\tilde{\mathbb{P}}$ as

$$\tilde{\mathbb{P}}(A) = \int_A Z(\omega)\mathrm{d}\mathbb{P}(\omega), \quad \forall A \in \mathcal{F}. \tag{5.15}$$

We define the process $\tilde{w}(t)$ as

$$\tilde{w}(t) \equiv \begin{bmatrix} \tilde{w_1}(t) \\ \\ \tilde{w_2}(t) \\ \\ \tilde{w_3}(t) \end{bmatrix} = \begin{bmatrix} w_1(t) \\ \\ w_2(t) \\ \\ w_3(t) \end{bmatrix} + \gamma\int_0^t \begin{bmatrix} \Big(2g_1x_1(t)+g_2\Big)b \\ \\ \Big(g_3+\dfrac{1}{2}g_4\dfrac{1}{\sqrt{x_2(t)}}\Big)\sigma_2\sqrt{x_2(t)} \\ \\ \rho(t)\sigma \end{bmatrix} \mathrm{d}s,$$

which is a standard Brownian motion under $\tilde{\mathbb{P}}$ by Girsanov's theorem.

We can now rewrite (5.14) as

$$
\begin{aligned}
& \mathbb{E}\left[y^{\gamma}(T)\right] \\
= \; & y_0^{\gamma}\alpha\mathbb{E}\Bigg[\exp\Bigg\{\gamma\int_0^T \Big([q_1+2a_1g_1+2\gamma b^2g_1^2]x_1^2(t) \\
& \qquad +[q_2+a_1g_2+2a_2g_1+2\gamma b^2g_1g_2]x_1(t) \\
& \qquad +[q_3+\frac{\gamma}{2}g_3^2\sigma_2^2]x_2(t)+[q_4+\frac{\gamma}{2}g_3g_4\sigma_2^2-g_3k]\sqrt{x_2(t)} \\
& \qquad +[g_1b^2+g_3k\theta+a_2g_2-\frac{1}{2}g_4k+\frac{1}{2}\gamma g_2^2b^2+\frac{\gamma}{8}g_4^2\sigma_2^2]\Big)\mathrm{d}t\Bigg\} \\
& \quad \exp\Bigg\{\gamma\int_0^T\Big(\rho(t)[\mu(t)-r(t)]-\frac{1}{2}\rho^2(t)\sigma^2+\frac{1}{2}\gamma\rho^2(t)\sigma^2\Big)\mathrm{d}t\Bigg\} \\
& \quad \exp\Bigg\{-\gamma g_1x_1^2(T)-\gamma g_2x_1(T)-\gamma g_3x_2(T)-\gamma g_4\sqrt{x_2(T)}\Bigg\} \\
& \quad \exp\Bigg\{\gamma\int_0^T\tilde{\sigma}\mathrm{d}w-\frac{1}{2}\int_0^T\gamma^2\tilde{\sigma}'\tilde{\sigma}\mathrm{d}t\Bigg\}\Bigg] \\
= \; & y_0^{\gamma}\alpha\tilde{\mathbb{E}}\Bigg[\exp\Bigg\{\gamma\int_0^T\Big(\big[q_1+2a_1g_1+2\gamma b^2g_1^2+\frac{\lambda^2\kappa_1}{2(1-\gamma)\sigma^2}\big]x_1^2(t) \\
& \qquad +\big[q_2+a_1g_2+2a_2g_1+2\gamma b^2g_1g_2\big]x_1(t) \\
& \qquad +\big[q_3+\frac{\gamma}{2}g_3^2\sigma_2^2+\frac{\lambda^2\kappa_2}{2(1-\gamma)\sigma^2}\big]x_2(t) \\
& \qquad +\big[q_4+\frac{\gamma}{2}g_3g_4\sigma_2^2-g_3k+\frac{\lambda^2\kappa_3}{2(1-\gamma)\sigma^2}\big]\sqrt{x_2(t)} \\
& \qquad +\big[2g_1b^2+g_3k\theta+a_2g_2-\frac{1}{2}g_4k+\frac{1}{2}\gamma g_2^2b^2
\end{aligned}
$$

$$+\frac{\gamma}{8}g_4^2\sigma_2^2 + \frac{\lambda^2\kappa_4}{2(1-\gamma)\sigma^2}]$$

$$-\frac{(1-\gamma)\sigma^2}{2}\left[\rho(t) - \frac{\mu(t)-r(t)}{(1-\gamma)\sigma^2}\right]^2\Big)\mathrm{d}t\Big\}$$

$$\exp\Big\{ -\gamma g_1 x_1^2(T) - \gamma g_2 x_1(T) - \gamma g_3 x_2(T) - \gamma g_4\sqrt{x_2(T)}\Big\}\Big].$$

We introduce the following assumption:

Assumption 5.2.1.

$$\begin{cases} \triangle_1 := a_1^2 - 2\gamma b^2\left(q_1 + \dfrac{\lambda^2\kappa_1}{2(1-\gamma)\sigma^2}\right) \geq 0, \\ \\ 2a_1 - \sqrt{\triangle_1} < 0, \\ \\ \dfrac{\kappa_2}{q_3} < 0. \end{cases}$$

We also define the coefficients g_1, g_2, g_3, g_4, as

$$\begin{cases} g_1 := \dfrac{-a_1 + \sqrt{\triangle_1}}{2\gamma b^2} \\ \\ g_2 := -\dfrac{q_2\gamma b^2 + a_2\sqrt{\triangle_1} - a_1 a_2}{\gamma b^2(\sqrt{\triangle_1} - a_1 + a_2)} \\ \\ g_3 := \sqrt{-\dfrac{\lambda^2\kappa_2}{\gamma(1-\gamma)\sigma^2\sigma_2^2 q_3}} \\ \\ g_4 := \dfrac{2g_3 k - \frac{\lambda^2\kappa_3}{(1-\gamma)\sigma^2} - 2q_4}{\gamma g_3\sigma_2^2}. \end{cases}$$

Due to this selection of coefficients g_1, g_2, g_3, g_4, the sum of the terms in $\mathbb{E}\left[y^{\gamma}(T)\right]$ that are quadratic in $x_1(t)$ and linear in $x_1(t)$ are zero. Similarly, the sum of the terms that are linear in $x_2(t)$ and linear in $\sqrt{x_2(t)}$ are also zero. Thus we have:

$$\begin{aligned}
&\mathbb{E}[y^{\gamma}(T)] \\
&= \; y_0^{\gamma}\alpha\tilde{\mathbb{E}}\Bigg[\exp\bigg\{\gamma\int_0^T\bigg(\Lambda_1 - \frac{(1-\gamma)\sigma^2}{2}\bigg[\rho(t) - \frac{\mu(t)-r(t)}{(1-\gamma)\sigma^2}\bigg]^2\bigg)\mathrm{d}t\bigg\} \\
&\qquad \exp\bigg\{-\gamma g_1 x_1^2(T) - \gamma g_2 x_1(T) - \gamma g_3 x_2(T) - \gamma g_4\sqrt{x_2(T)}\bigg\}\Bigg]
\end{aligned}$$

where

$$\Lambda_1 = g_1 b^2 + g_2 a_2 + g_3 k\theta - \frac{1}{2}g_4 k + \frac{1}{2}\gamma g_2^2 b^2 + \frac{\gamma}{8}g_4^2\sigma_2^2 + \frac{\lambda^2\kappa_4}{2(1-\gamma)\sigma^2}.$$

Since x_1, x_2 are independent of the control $u(t)$, we have that for all $u(\cdot) \in \mathcal{A}$ and for any fixed $T > 0$ the following upper bound holds:

$$\mathbb{E}[y^{\gamma}(T)] \leq \; y_0^{\gamma}\alpha \mathrm{e}^{\gamma\Lambda_1 T}\tilde{\mathbb{E}}\bigg[\exp\bigg\{-\gamma g_1 x_1^2(T) - \gamma g_2 x_1(T) - \gamma g_3 x_2(T) - \gamma g_4\sqrt{x_2(T)}\bigg\}\bigg].$$

This upper bound is achieved if and only if

$$\rho(t) = \rho^*(t) := \frac{\mu(t)-r(t)}{(1-\gamma)\sigma^2}, \tag{5.16}$$

in which case we have

$$\mathbb{E}[y^{\gamma}(T)] = y_0^{\gamma}\alpha \mathrm{e}^{\gamma\Lambda_1 T}\tilde{\mathbb{E}}\left[\mathrm{e}^{M_1 x_1^2(T)+M_2 x_1(T)}\right]\tilde{\mathbb{E}}\left[\mathrm{e}^{M_3 x_2(T)+M_4\sqrt{x_2(T)}}\right], \tag{5.17}$$

where $M_1 = -\gamma g_1$, $M_2 = -\gamma g_2$, $M_3 = -\gamma g_3$, $M_4 = -\gamma g_4$.

In order to show that (5.16) is indeed the optimal control, it remains to show that it belongs to the set $\mathcal{A}$. One of the requirements for the control to belong to $\mathcal{A}$ is for $J(u(\cdot))$ to be finite. From (5.17) it is clear that we need to study the behavior of the expectations of the exponential processes based on x_1 and x_2. This is our next task. We begin with the following assumption, the motivation of which becomes clear when considering the finiteness of these expectations in what follows.

Assumption 5.2.2. *We assume that* $g_i > 0$*,* $i = 1, 2, 3, 4$.

Equation (5.1) can be rewritten as:

$$\begin{aligned} \mathrm{d}x_1(t) &= \Big[(a_1 - 2\gamma b^2 g_1)x_1(t) + (a_2 - \gamma b^2 g_2)\Big]\mathrm{d}t + b\mathrm{d}\tilde{w}_1(t) \\ &= \Big[A_1 x_1(t) + B_1\Big]\mathrm{d}t + b\mathrm{d}\tilde{w}_1(t), \end{aligned} \tag{5.18}$$

where $A_1 := a_1 - 2\gamma b^2 g_1$ and $B_1 := a_2 - \gamma b^2 g_2$. Denoting by $\mu_1(t) := \tilde{\mathbb{E}}[x_1(t)]$ and $\Sigma_1(t) := \tilde{V}ar[x_1(t)]$ it is clear that $x_1(t) \sim N(\mu_1(t), \Sigma_1(t))$. The equation of $\mu_1(t)$ is

$$\begin{cases} \dot{\mu_1}(t) = A_1\mu_1(t) + B_1, \\ \mu_1(0) = \mu_{10}, \end{cases} \tag{5.19}$$

whereas for the variance we have that $\Sigma_1(t) = \Phi_1(t) - \mu_1^2(t)$, where $\Phi_1(t)$ is the solution of

$$\begin{cases} \dot{\Phi}(t) = 2A_1\Phi_1(t) + 2B_1\mu_1(t) + b^2 \\ \Phi_1(0) = x_{10}^2. \end{cases} \tag{5.20}$$

The solutions to these two equations are:

$$\mu_1(t) = \frac{A_1\mu_{10} + B_1}{A_1}\exp\{A_1 t\} - \frac{B_1}{A_1},$$

$$\Phi_1(t) = \frac{1}{2A_1}\left(2A_1 x_{10}^2 + \bar{B}_1\right)\exp\{-2\bar{A}_1\}\exp\{2\bar{A}_1 \mathrm{e}^{A_1 t}\} - \frac{\bar{B}_1}{2A_1},$$

where $\bar{A}_1 \equiv 2B_1 \dfrac{A_1\mu_{10} + B_1}{A_1}$ and $\bar{B}_1 \equiv b^2 - \dfrac{2B_1^2}{A_1}$. We thus have

$$\Sigma_1(t) = \frac{2A_1x_{10}^2 + \bar{B}_1}{2A_1}\exp\{-2\bar{A}_1\}\exp\{2\bar{A}_1\mathrm{e}^{A_1 t}\} + \frac{\bar{A}_1}{A_1}\exp\{A_1 t\} - \left(\frac{\bar{B}_1}{2A_1} + \frac{B_1^2}{A_1^2}\right).$$

Lemma 5.2.1. *The following holds*

$$\lim_{T\to\infty} \tilde{\mathbb{E}}\left[\mathrm{e}^{M_1 x_1^2(T) + M_2 x_1(T)}\right] < \infty.$$

Proof.

$$\begin{aligned} & \tilde{\mathbb{E}}\left[\mathrm{e}^{M_1 x_1^2(t) + M_2 x_1(t)}\right] \qquad (5.21) \\ = \ & \int_{\mathbb{R}} \exp\{M_1 x^2 + M_2 x\}\frac{1}{\sqrt{2\pi}\sqrt{\Sigma_1(t)}}\exp\Big\{-\frac{1}{2}[x - \mu_1(t)]^2\Sigma_1^{-1}(t)\Big\}\mathrm{d}x \\ = \ & \frac{\sqrt{\bar{\Sigma}_1(t)}}{\sqrt{\Sigma_1(t)}}\exp\left\{\frac{\left[\frac{\mu_1(t)}{\Sigma_1(t)} + M_2\right]^2}{2[\frac{1}{\Sigma_1(t)} - 2M_1]} - \frac{\mu_1^2(t)}{2\Sigma_1}\right\} \\ & \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}\sqrt{\bar{\Sigma}_1(t)}}\exp\Big\{-\frac{1}{2}[x - \bar{\mu}_1(t)]^2\bar{\Sigma}_1^{-1}(t)\Big\}\mathrm{d}x, \end{aligned}$$

where

$$\begin{cases} \bar{\mu}_1(t) \equiv \dfrac{\mu_1(t) + M_2\Sigma_1(t)}{1 - 2M_1\Sigma_1(t)}, \\ \bar{\Sigma}_1(t) \equiv \dfrac{\Sigma_1(t)}{1 - 2M_1\Sigma_1(t)}. \end{cases}$$

Since $M_1 = -\gamma g_1 < 0$, we have that $\bar{\Sigma}_1(t) > 0$, and thus

$$\int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}\sqrt{\bar{\Sigma}_1(t)}}\exp\Big\{-\frac{1}{2}[x - \bar{\mu}_1(t)]^2\bar{\Sigma}_1^{-1}(t)\Big\}\mathrm{d}x = 1,$$

which reduces (5.22) into

$$
\begin{aligned}
&\tilde{\mathbb{E}}\Big[\mathrm{e}^{M_1 x_1^2(T)+M_2 x_1(T)}\Big] \\
= \;& \sqrt{\frac{1}{1-2M_1\Sigma_1(T)}}\exp\Big\{\frac{M_2^2\Sigma_1^2(T)+2M_2\mu_1(T)\Sigma_1(T)+2M_1\mu_1^2(T)\Sigma_1(T)}{2\Sigma_1(T)[1-2M_1\Sigma_1(T)]}\Big\}
\end{aligned}
$$

Since we have assumed $A_1 < 0$, we have:

$$
\begin{cases}
\mu_1(T) \to -\dfrac{B_1}{A_1} \quad (T\to\infty), \\
\\
\Sigma_1(T) \to C_1 \quad (T \to \infty),
\end{cases}
$$

where

$$
C_1 \equiv \frac{2A_1 x_{10}^2 + \bar{B}_1}{2A_1}\mathrm{e}^{-2\bar{A}_1} - \Big(\frac{\bar{B}_1}{2A_1} + \frac{B_1^2}{A_1^2}\Big).
$$

We thus finally have

$$
\lim_{T\to\infty} \tilde{\mathbb{E}}\Big[\mathrm{e}^{M_1 x_1^2(T)+M_2 x_1(T)}\Big] = \bar{C}_1,
$$

where

$$
\bar{C}_1 = \sqrt{\frac{1}{1-2M_1C_1}}\exp\Big\{\frac{M_2^2 C_1 - 2M_2\frac{B_1}{A_1} + 2M_1\frac{B_1^2}{A_1^2}}{2(1-C_1M_1)}\Big\}.
$$

$\square$

The proof of the following result proceeds in a similar way, and we thus omit the details.

Lemma 5.2.2. *The following holds*

$$
\lim_{T\to\infty} \tilde{\mathbb{E}}\Big[\mathrm{e}^{M_3 x_2(T)+M_4\sqrt{x(T)}}\Big] < \infty.
$$

Referring to equation (5.17), and with the help of Lemma 5.2.1 and Lemma5.2.2, we have established that under the control (5.16) we have

$$
\lim_{T\to\infty} \frac{1}{T}\log \mathbb{E}\Big[y^{\gamma}(T)\Big] = \gamma\Lambda_1,
$$

Since the control $u(t) = \rho^*(t)y(t)$ is a linear function of $y(t)$, it ensures the positivity of $y(t)$. Therefore, we have proved the following main result of this chapter.

Theorem 5.2.1. *Let assumptions 5.2.1 and 5.2.2 hold. There exists a unique solution to the problem*

$$\begin{cases} \max\limits_{u(\cdot)\in\mathcal{A}} \lim\limits_{T\to\infty} \dfrac{1}{T} \log \mathbb{E}\Big[y^{\gamma}(T)\Big], \quad \gamma \in (0,1), \\ \\ s.t. \quad (5.1), \quad (5.2), \quad (5.5), \end{cases} \tag{5.22}$$

given by

$$u^*(t) = \frac{\mu(t) - r(t)}{(1-\gamma)\sigma^2} y(t).$$

The corresponding optimal cost is

$$J^* = \gamma \Lambda_1.$$

Now we give a numerical example which is suitable to Assumption 5.2.1, 5.2.2.

Example 5.2.1. *Let* $a_1 = 2, a_2 = 1, b = 5, q_1 = 4, q_2 = 1, q_3 = 1, q_4 = 1, k_1 = -5, k_2 = -1, k_3 = -1, \sigma = 3, \sigma_2 = 1, \lambda = 3.$

Thus, the coefficients g_1, g_2, g_3, g_4 *are:*

$$\begin{cases} g_1 = \dfrac{\sqrt{29} - 2}{25}, \\ \\ g_2 = \dfrac{2\sqrt{29} + 21}{25(\sqrt{29} - 1)}, \\ \\ g_3 = 2, \\ \\ g_4 = 4. \end{cases}$$

5.3 Logarithmic Utility

Here we consider the following optimal control problem:

$$\begin{cases} \max_{u(\cdot)\in\mathcal{A}} \lim_{T\to\infty} \dfrac{1}{T^2}\mathbb{E}\Big[\log y(T)\Big] \\ \\ s.t. \quad (5.1), \quad (5.2), \quad (5.5). \end{cases} \tag{5.23}$$

Remark 5.3.1. *Initially, we build the logarithmic utility with $T^{1+\epsilon}$ averaging, which is $\dfrac{1}{T^{1+\epsilon}}\mathbb{E}[\log y(T)]$. With some calculation, it is found that only $\epsilon \geq 1$, the objective can be converged in an infinite time horizon. Thus, to simplify the problem, we let $\epsilon = 1$. For this specific technical reason, the criteria is defined with the squared time averaging.*

Assumption 5.3.1. $q_3 + \frac{\lambda^2\kappa_2}{2\sigma^2} = 0$ *and* $a_1 < 0$.

We define the following coefficients:

$$\begin{cases} \Lambda_2 := 2b^2 g_1 + a_2 g_2 + k\theta g_3 - \dfrac{1}{2}k\theta g_4 + \dfrac{\lambda^2}{2\sigma^2}\kappa_4, \\ \\ g_1 := -\dfrac{\lambda^2\kappa_1 + 2\sigma^2 q_1}{4a_1\sigma^2}, \\ \\ g_2 := \dfrac{a_2\lambda^2\kappa_1 + 2\sigma^2(a_2q_1 - a_1q_2)}{2a_1^2\sigma^2}, \\ \\ g_3 := \dfrac{q_4}{k} + \dfrac{\lambda^2\kappa_3}{2\sigma^2 k}, \\ \\ g_4 := \dfrac{4b^2}{k\theta}g_1 + \dfrac{2a_2}{k\theta}g_2 + 2g_3 + \dfrac{\lambda^2\kappa_4 - 2\Lambda_2\sigma^2}{k\theta\sigma^2}. \end{cases} \tag{5.24}$$

Lemma 5.3.1. *If $a_1 < 0$, then*

$$\lim_{T\to\infty} \mathbb{E}\Big[- g_1 x_1^2(T) - g_2 x_1(T)\Big] = \hat{C}_1,$$

where $\hat{C}_1$ is some constant.

Proof. We define $\hat{\mu}_1(t) := \mathbb{E}[x_1(t)]$, which is the solution to

$$\begin{cases} \dot{\hat{\mu}}_1(t) = a_1\hat{\mu}_1(t) + a_2, \\ \\ \hat{\mu}_1(0) = \hat{\mu}_{10}. \end{cases} \tag{5.25}$$

We also define $\hat{\Phi}_1(t) := \mathbb{E}[x_1^2(t)]$, which is the solution to

$$\begin{cases} \dot{\hat{\Phi}}_1(t) = 2a_1\hat{\Phi}_1(t) + 2a_2\hat{\mu}_1(t) + b^2, \\ \\ \hat{\Phi}_1(0) = x_{10}^2. \end{cases} \tag{5.26}$$

By solving (5.25) and (5.26), we deduce

$$\begin{aligned} \mathbb{E}[x(t)] &= \frac{a_1\hat{\mu}_{10} + a_2}{a_1}\exp\{a_1 t\} - \frac{a_2}{a_1}, \\ \mathbb{E}[x^2(t)] &= \frac{1}{2a_1}\Big(2a_1x_{10}^2 + \bar{a}_2\Big)\exp\{-2\bar{a}_1\}\exp\{2\bar{a}_1\mathrm{e}^{a_1t}\} - \frac{\bar{a}_2}{2a_1}, \end{aligned}$$

where $\bar{a}_1 = 2a_2\dfrac{a_1\hat{\mu}_{10} + a_2}{a_1}$ and $\bar{a}_2 = b^2 - \dfrac{2a_2^2}{a_1}$.

Thus, due to assumption of $a_1 < 0$, when $T \to \infty$, we obtain the following results:

$$\begin{cases} \mathbb{E}[x(T)] \to -\dfrac{a_2}{a_1}, \\ \\ \mathbb{E}[x^2(T)] \to \dfrac{1}{2a_1}\Big(2a_1x_{10}^2 + \bar{a}_2\Big)\exp\{-2\bar{a}_1\} - \dfrac{\bar{a}_2}{2a_1}, \end{cases}$$

and

$$\lim_{T\to\infty} \mathbb{E}\left[- g_1 x_1^2(T) - g_2 x_1(T)\right] = \hat{C}_1,$$

where $\hat{C}_1 = -g_1 \left[\frac{1}{2a_1}\Big(2a_1 x_{10}^2 + \bar{a}_2\Big)\exp\{-2\bar{a}_1\} - \frac{\bar{a}_2}{2a_1}\right] + g_2 \frac{a_2}{a_1}$. □

From the Longstaff (1989) paper we know that for a suitable positive constant c, and constants m and s, it holds that $x_2(t) = cx^2(t)$, where $x(t)$ is the solution to the following equation

$$\begin{cases} \mathrm{d}x(t) = m\mathrm{d}t + s\mathrm{d}w_2(t) \\ \\ x(0) = x_0. \end{cases} \tag{5.27}$$

Also recall the following basic result concerning the normal random variables.

Lemma 5.3.2. *If* $X \sim N(\mu_X, \sigma_X)$, *then*

$$\mathbb{E}[|X|] = \frac{2\sigma_X}{\sqrt{2\pi}} \cosh(\mu_X/\sigma_X) + \mu_X[N(\mu_X/\sigma_X) - N(-\mu_X/\sigma_X)]. \tag{5.28}$$

Lemma 5.3.3. *The following holds:*

$$\lim_{T\to\infty} \frac{1}{T^2}\mathbb{E}\left[- g_3 x_2(T) - g_4\sqrt{x_2(T)}\right] = -g_3 cm^2.$$

Proof. It is clear that $x(t) \sim N(x_0 + mt, s^2 t)$. This in particular means that the relation between the mean and the variance of $x(t)$ tends to a constant, i.e. $\lim_{t\to\infty}(x_0 + mt)/s^2 t = m/s^2$. Note that $\mathbb{E}[x^2(t)] = s^2 t + x_0^2 + 2x_0 mt + m^2 t^2$, and $\sqrt{x_2(t)} = \sqrt{c}|x(t)|$. By making use of these facts and (5.28), we obtain

$$\frac{1}{T^2}\lim_{T\to\infty} \mathbb{E}[-g_3 x_2(T)] = -g_3 cm^2, \quad \text{and} \quad \frac{1}{T^2}\lim_{T\to\infty} \mathbb{E}[-g_4\sqrt{x_2(T)}] = 0.$$

□

Theorem 5.3.1. *If assumptions 5.3.1 holds, then there exists a unique solution to the control problem* (5.23)*, given by*

$$u^*(t) = \frac{\mu(t) - r(t)}{\sigma^2} y(t).$$

The corresponding optimal cost is $J^* = -g_3 cm^2$.

Proof. Recall equation (5.5), according to Itô's lemma, the expected value of $\log y(T)$ can be written as:

$$\mathbb{E}[\log y(T)] = \log y_0 + \mathbb{E}\left[\int_0^T \left\{r(t) + \rho(t)[\mu(t) - r(t)]\frac{1}{2}\sigma^2\rho^2(t)\right\}\mathrm{d}t\right]. \quad (5.29)$$

Similarly to the approach for the power utility, we combine (5.12), (5.13) with (5.29) to obtain:

$$\begin{aligned}
&\mathbb{E}[\log y(T)] \\
= \; & \log y_0 + g_1 x_{10}^2 + g_2 x_{10} + g_3 x_{20} + g_4\sqrt{x_{20}} \\
&+\mathbb{E}\Bigg[\int_0^T \Bigg\{ -\frac{1}{2}\sigma^2\Big(\rho(t) - \frac{\mu(t)-r(t)}{\sigma^2}\Big)^2 + \frac{[\mu(t)-r(t)]^2}{2\sigma^2} \\
&\qquad + q_1 x_1^2(t) + q_2 x_1(t) + q_3 x_2(t) + q_4\sqrt{x_2(t)} \\
&\qquad + \Big(2g_1 x_1(t) + g_2\Big)\Big(a_1 x_1(t) + a_2\Big) + g_1 b^2 \\
&\qquad + \Big(g_3 + \frac{1}{2}g_4\frac{1}{\sqrt{x_2(t)}}\Big)k\Big(\theta - \sqrt{x_2(t)}\Big) - \frac{1}{8}g_4\sigma_2^2\frac{1}{\sqrt{x_2(t)}}\Bigg\}\mathrm{d}t\Bigg] \\
&+\mathbb{E}\Bigg[-g_1 x_1^2(T) - g_2 x_1(T) - g_3 x_2(T) - g_4\sqrt{x_2(T)}\Bigg] \\
= \; & \log y_0 + g_1 x_{10}^2 + g_2 x_{10} + g_3 x_{20} + g_4\sqrt{x_{20}} \\
&+\mathbb{E}\Bigg[\int_0^T \Bigg\{ -\frac{1}{2}\sigma^2\Big(\rho(t) - \frac{\mu(t)-r(t)}{\sigma^2}\Big)^2 + \Big(q_1 + 2a_1 g_1 + \frac{\lambda^2}{2\sigma^2}\kappa_1\Big)x_1^2(t) \\
&\qquad + \Big(q_2 + a_1 g_2 + 2a_2 g_1\Big)x_1(t) + \Big(q_3 + \frac{\lambda^2}{2\sigma^2}\kappa_2\Big)x_2(t) \\
&\qquad + \Big(q_4 - kg_3 + \frac{\lambda^2}{2\sigma^2}\kappa_3\Big)\sqrt{x_2(t)} + \Lambda_2\Bigg\}\mathrm{d}t\Bigg]
\end{aligned}$$

$$+\mathbb{E}\Bigg[-g_1x_1^2(T)-g_2x_1(T)-g_3x_2(T)-g_4\sqrt{x_2(T)}\Bigg].$$

The sum of the terms that are quadratic in $x_1(t)$ and linear in $x_1(t)$ that appear in the integrand part of $\mathbb{E}[\log y(T)]$ is zero. Indeed,

$$\Big(q_1+2a_1g_1+\frac{\lambda^2}{2\sigma^2}\kappa_1\Big)x_1^2(t)+\Big(q_2+a_1g_2+2a_2g_1\Big)x_1(t)=0.$$

Similarly, the part that contains the sum of the terms that are linear in $x_2(t)$ and linear in $\sqrt{x_2(t)}$ is also zero:

$$\Big(q_3+\frac{\lambda^2}{2\sigma^2}\kappa_2\Big)x_2(t)+\Big(q_4-kg_3+\frac{\lambda^2}{2\sigma^2}\kappa_3\Big)\sqrt{x_2(t)}=0.$$

It is clear that:

$$\begin{aligned}\mathbb{E}[\log y(T)] &= \mathbb{E}\Bigg[\int_0^T\Big\{-\frac{1}{2}\sigma^2\Big(\rho(t)-\frac{\mu(t)-r(t)}{\sigma^2}\Big)^2+\Lambda_2\Big\}\mathrm{d}t\Bigg]\\ &\quad +\mathbb{E}\Bigg[-g_1x_1^2(T)-g_2x_1(T)-g_3x_2(T)-g_4\sqrt{x_2(T)}\Bigg]\\ &\quad +\log y_0+g_1x_{10}^2+g_2x_{10}+g_3x_{20}+g_4\sqrt{x_{20}}\\ &\le \Bigg\{\mathbb{E}\Bigg[-g_1x_1^2(T)-g_2x_1(T)-g_3x_2(T)-g_4\sqrt{x_2(T)}\Bigg]\\ &\quad +\log y_0+g_1x_{10}^2+g_2x_{10}+g_3x_{20}+g_4\sqrt{x_{20}}+T\Lambda_2\Bigg\}. \qquad (5.30)\end{aligned}$$

This upper bound is achieved if and only if we choose the following control $u^*(t)=\dfrac{\mu(t)-r(t)}{\sigma^2}y(t)$. It is then clear that the optimal cost is $J^*=-g_3cm^2$.- □

5.4 Summary

We have considered the problem of optimal investment in an infinite horizon. A stochastic interest rate model is introduced with two factor processes, one being linear and the other being the Longstaff model. We give the solution to the investment problem for certain infinite horizon power and logarithmic utilities. The optimal investment strategy is a linear function of the wealth.

Chapter 6

Optimal investment and consumption with a double square-root stochastic interest rate and volatility

6.1 Introduction

We consider the problem of optimal investment and consumption in a market with a stochastic interest rate and a stochastic volatility. We assume that the interest rate and the volatility follow the Longstaff model. It is shown that there exists a unique optimal trading strategy and a consumption process that maximize the logarithmic and power utility. These are obtained in an explicit closed-form by the completion of squares method.

6.2 Formulation of the problem

Considering the probability space defined in Section refsection 2.2, we choose $n = 1$, and let $w_r(\cdot) = w_1(\cdot), w_s(\cdot) = w_2(\cdot), w_\eta(\cdot) = w_3(\cdot)$.

We assume the interest rate $r(t)$ follows the model introduced by Longstaff

[34], i.e. the interest rate is a double square-root process given by

$$\begin{cases} \mathrm{d}r(t) = k_r[\theta_r - \sqrt{r(t)}]\mathrm{d}t + \sigma_r\sqrt{r(t)}\mathrm{d}w_r(t) \\ \\ r(0) = r_0 > 0, \end{cases} \tag{6.1}$$

where k_r, σ_r are positive constants, and $\theta_r = \dfrac{\sigma_r^2}{4k_r}$.

We consider a financial market consisting of two assets with equations:

$$\begin{cases} dB(t) = r(t)B(t)\mathrm{d}t, \\ \\ B(0) = B_0 > 0, \end{cases} \tag{6.2}$$

$$\begin{cases} \mathrm{d}S(t) = S(t)\Big[\mu(t)\mathrm{d}t + \sigma\sqrt{\eta(t)}\mathrm{d}w_s(t)\Big], \\ \\ S(0) = S_0 > 0, \end{cases} \tag{6.3}$$

where $\sigma > 0$ is given. Here $B(\cdot)$ is the bank account, whereas $S(\cdot)$ is the stock price. Similarly to Kraft [31], Bielecki, Pliska and Yong [5] and Chang and Rong [7], we assume that $\mu(t) = r(t) + \kappa\sqrt{\eta(t)}$, where κ is a constant.

We also assume that the volatility $\eta(t)$ satisfies the following Longstaff equation:

$$\begin{cases} d\eta(t) = k_\eta[\theta_\eta - \sqrt{\eta(t)}]\mathrm{d}t + \sigma_\eta\sqrt{\eta(t)}\mathrm{d}w_\eta(t) \\ \\ \eta(0) = \eta_0 > 0, \end{cases} \tag{6.4}$$

where k_η, σ_η are positive constants, and $\theta_\eta = \dfrac{\sigma_\eta^2}{4k_\eta}$.

In summary, the novelty in this market is the use of the Longstaff model for both the interest rate and the volatility processes. Note that the market price of risk in this case is

$$\phi(t) := \frac{\mu(t) - r(t)}{\sigma\sqrt{\eta(t)}} = \frac{1}{\sigma},$$

and thus there exists a unique risk-neutral probability measure, which ensures the absence of any arbitrage opportunities.

In this market we consider an investor with an initial wealth of $y_0 > 0$ that follows a self-financing trading strategy and is permitted to *consume*. The equation of her/his wealth $y(t)$ is thus

$$\begin{cases} \mathrm{d}y(t) = \Big\{ r(t)y(t) + u(t)\Big[\mu(t) - r(t)\Big] - c(t) \Big\}\mathrm{d}t + u(t)\sigma\sqrt{\eta(t)}\mathrm{d}w_s(t), \\ \\ y(0) = y_0, \end{cases} \tag{6.5}$$

where $u(t)$ denotes the amount of wealth invested in the stock at time t, and $c(t)$ is the consumption rate.

The main contribution of this chapter is to solve the following optimal investment and consumption problem:

$$\begin{cases} \max\limits_{u(\cdot),c(\cdot)\in\mathcal{A}} J(u(\cdot), c(\cdot)), \\ \\ s.t. \quad (6.1), \quad (6.4), \quad (6.5), \end{cases} \tag{6.6}$$

where

$$J(u(\cdot), c(\cdot)) = \mathbb{E}\left[\int_0^T \mathrm{e}^{-\beta t} U_1\Big(c(t)\Big)\mathrm{d}t + \mathrm{e}^{-\beta T} U_2\Big(y(T)\Big) \right], \tag{6.7}$$

and $U_1(\cdot)$, $U_2(\cdot)$ are either power or logarithmic utilities. Here $\beta > 0$ is a given subjective discount rate. The set $\mathcal{A}$ of admissible controls is the set of all $\mathcal{F}_t$-adapted processes $u(t)$ and $c(t)$ that ensure the existence of a unique and positive solution to (6.5) and the finiteness of the cost $J(u(\cdot), c(\cdot))$.

6.3 Admissible controls

In this section we consider the admissability of the controls $u(t)$ and $c(t)$ that are linear functions of $y(t)$. Let us thus consider the following controls:

$$u(t) \quad = \quad \frac{K[\mu(t) - r(t)]}{\sigma^2\eta(t)} y(t), \tag{6.8}$$

$$c(t) \quad = \quad \frac{y(t)}{H(t)}, \tag{6.9}$$

where K is a constant and $H(t)$ is some positive process. Provided that the wealth process $y(t)$ under these controls has sufficient integrability, then this set of controls belongs to $\mathcal{A}$. Indeed, since these controls are linear in $y(t)$ the solution to (6.5) will exist and be positive. In addition, the cost will be finite for the utilities that we consider. Thus, our task is to show that the wealth process under these controls has the following integrability:

$$\mathbb{E}[y(t)] \leq \Big(\mathbb{E}\left[y^p(t)\right]\Big)^{\frac{1}{p}} < \infty, \qquad p > 1.$$

Therefore, the problem becomes to prove the following inequality:

$$\mathbb{E}\left[y^p(t)\right] < \infty, \qquad p > 1.$$

Substituting (6.8) and (6.9) into (6.5) makes it a linear stochastic differential equation with random coefficients. The expected value of its solution raised into power p is:

$$\begin{aligned}
\mathbb{E}\left[y^p(t)\right] &= \mathbb{E}\Bigg[y_0^p \exp\left\{p \int_0^t \left[r(s) + \frac{K\kappa^2}{\sigma^2} - \frac{1}{H(s)} - \frac{1}{2}\frac{K^2\kappa^2}{\sigma^2}\right]\mathrm{d}s\right\} \\
&\qquad \times \exp\left\{p \int_0^t \frac{K\kappa}{\sigma}\mathrm{d}w_s(s)\right\}\Bigg] \\
&= \mathbb{E}\Bigg[y_0^p \exp\left\{p \int_0^t \left[r(s) - \frac{1}{H(s)} + \bar{K}\right]\mathrm{d}s\right\} \\
&\qquad \times \exp\left\{p \int_0^t \frac{K\kappa}{\sigma}\mathrm{d}w_s(s) - \frac{1}{2}\int_0^t p^2\frac{K^2\kappa^2}{\sigma^2}\mathrm{d}s\right\}\Bigg],
\end{aligned}$$

where

$$\bar{K} := \frac{K\kappa^2}{\sigma^2} - \frac{1}{2}\frac{K^2\kappa^2}{\sigma^2} + \frac{p}{2}\frac{K^2\kappa^2}{\sigma^2}.$$

Due to the positivity of $H(t)$, we have:

$$\exp\left\{p \int_0^t \left[r(s) - \frac{1}{H(s)} + \bar{K}\right]\mathrm{d}s\right\} \leq \exp\left\{p \int_0^t \left[r(s) + \bar{K}\right]\mathrm{d}s\right\}$$

By applying the Hölder's inequality, we obtain:

$$\begin{aligned}\mathbb{E}\left[y^p(t)\right] &\leq \tilde{K}(t)\mathbb{E}\Bigg[\exp\left\{pp_1\int_0^t r(s)\mathrm{d}s\right\}\Bigg]^{\frac{1}{p_1}}\\ &\quad \mathbb{E}\Bigg[\exp\left\{pp_3\int_0^t \frac{K\kappa}{\sigma}\mathrm{d}w_s(s) - \frac{1}{2}\int_0^t p^2p_3^2\frac{K^2\kappa^2}{\sigma^2}\mathrm{d}s\right\}\Bigg]^{\frac{1}{p_3}}\\ &\leq \tilde{K}(t)\mathbb{E}\Bigg[\exp\left\{pp_1\int_0^t r(s)\mathrm{d}s\right\}\Bigg]^{\frac{1}{p_1}},\end{aligned}$$

where

$$\tilde{K}(t) = \mathbb{E}\Bigg[y_0^{pp_2}\exp\left\{pp_2\int_0^t \bar{K}\mathrm{d}s\right\}\Bigg]^{\frac{1}{p_2}} < \infty, \quad \forall t \in [0,T],$$
$$\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = 1, \quad p_1, p_2, p_3 > 1.$$

Consider the following stochastic differential equation:

$$\begin{cases} \mathrm{d}x(t) = m\mathrm{d}t + s\mathrm{d}w_r(t) \\ x(0) = x_0, \end{cases} \tag{6.10}$$

where m, s are constants. From Longstaff [34], we have that $r(t) = cx^2(t)$, with c being a positive constant. The differential of $r(t)$ can be obtained as

$$\mathrm{d}r(t) = \left[cs^2 + 2m\sqrt{c}\sqrt{r(t)}\right]\mathrm{d}t + 2s\sqrt{c}\sqrt{r(t)}\mathrm{d}w_r(t),$$

which when compared with

$$\mathrm{d}r(t) = k_r[\theta_r - \sqrt{r(t)}]\mathrm{d}t + \sigma_r\sqrt{r(t)}\mathrm{d}w_r(t), \tag{6.11}$$

gives the relation between k_r, σ_r, θ_r and c, s, m.

Lemma 6.3.1. *If* $2\beta s^2 t^2 < 1$ *for some* $\beta > 0$ *and* $t \in [0,T]$, *then*

$$\mathbb{E}\Bigg[\exp\left\{\beta\int_0^t x^2(s)\mathrm{d}s\right\}\Bigg] < \infty.$$

Proof. This follows from Corollary 3.2 of Bielecki, Pliska and Yong [4], by taking $A = 0$ in their result. □

From this result it immediately follows that if $2pp_1cs^2T^2 < 1$ for some $p, p_1 > 1$, then

$$\mathbb{E}\left[\exp\left\{pp_1\int_0^T r(s)\mathrm{d}s\right\}\right]^{\frac{1}{p_1}} < \infty,$$

which concludes the proof that $\mathbb{E}\left[y^p(T)\right] < \infty$.

By substituting $2s\sqrt{c} = \sigma_r$, $\sigma_r^2 = 4\theta_r k_r$ into inequality $2pp_1cs^2T^2 < 1$, we can deduce

$$T < \frac{1}{\sqrt{2\theta_r k_r pp_1}} < \frac{1}{\sqrt{\theta_r k_r}},$$

where $p, p_1 > 1$. We thus introduce the following sufficient condition.

Condition 6.3.1. *The parameters k_r, θ_r and T are such that $T < \dfrac{1}{\sqrt{\theta_r k_r}}$.*

6.4 Power Utility

In this section we give the solution to the optimal investment and consumption problem (6.6) for the *power* utility, i. e. we consider the following criterion

$$J_{power}(u(\cdot), c(\cdot)) := \mathbb{E}\left[\int_0^T \mathrm{e}^{-\beta t}\frac{1}{\gamma}c^{\gamma}(t)\mathrm{d}t + \mathrm{e}^{-\beta T}\frac{y^{\gamma}(T)}{\gamma}\right]. \tag{6.12}$$

We introduce functions g_1, g_2, g_3, g_4, g_5, which satisfy the following differential equations:

$$\begin{cases} \dot{g}_1(t) + \dfrac{\gamma}{1-\gamma} + \dfrac{1}{2}\sigma_r^2 g_1^2(t) = 0 \\ g_1(T) = 0 \end{cases} \tag{6.13}$$

$$\begin{cases} \dot{g}_2(t) - k_r g_1(t) + \dfrac{1}{2}\sigma_r^2 g_1(t) g_2(t) = 0 \\ g_2(T) = 0 \end{cases} \tag{6.14}$$

$$\begin{cases} \dot{g}_3(t) + \frac{1}{2}\sigma_\eta^2 g_3^2(t) = 0 \\ \\ g_3(T) = 0 \end{cases} \tag{6.15}$$

$$\begin{cases} \dot{g}_4(t) - k_\eta g_3(t) + \frac{1}{2}\sigma_\eta^2 g_3(t) g_4(t) = 0 \\ \\ g_4(T) = 0 \end{cases} \tag{6.16}$$

$$\begin{cases} \dot{g}_5(t) + \frac{\sigma_r^2}{4} g_1(t) - \frac{1}{2} k_r g_2(t) + \frac{1}{8}\sigma_r^2 g_2^2 + \frac{\sigma_\eta^2}{4} g_3(t) - \frac{1}{2} k_\eta g_4(t) + \frac{1}{8}\sigma_\eta^2 g_4^2 \\ \qquad\qquad + \frac{\gamma\kappa^2}{2(1-\gamma)^2\sigma^2} - \frac{\beta}{1-\gamma} = 0 \\ \\ g_5(T) = 0 \end{cases} \tag{6.17}$$

Example 6.4.1. *Let* $\gamma = \frac{1}{2}$, *then* g_1, g_2, g_3, g_4, g_5 *can be solved as follows:*

$$g_1(t) = \frac{\sqrt{2}\tan[\frac{\sqrt{2}}{2}\sigma_r(T-t)]}{\sigma_r},$$

$$g_2(t) = \frac{\sqrt{2}k_r\sqrt{1+\tan[\frac{\sqrt{2}}{2}\sigma_r(T-t)]^2}\Big\{\sqrt{\cos[\sqrt{2}\sigma_r(T-t)]+1} - \sqrt{2}\Big\}}{\sigma_r^2},$$

$$g_3(t) = g_4(t) = 0,$$

$$
\begin{aligned}
g_5(t) = \int_t^T \Bigg\{ & \frac{1}{4\sigma_r^2 \cos\left(\frac{\sqrt{2}}{2}\sigma_r(T-s)\right)^2} \Bigg[4k_r^2 \cos\left(\frac{\sqrt{2}}{2}\sigma_r(T-s)\right) \\
& -2\sqrt{2}k_r^2 \cos\left(\frac{\sqrt{2}}{2}\sigma_r(T-s)\right)\sqrt{\cos\left(\sqrt{2}\sigma_r(T-s)\right)+1} \\
& +\sqrt{2}\sigma_r^3 \sin\left(\frac{\sqrt{2}}{2}\sigma_r(T-s)\right)\cos\left(\frac{\sqrt{2}}{2}\sigma_r(T-s)\right) \\
& +2\sqrt{2}k_r^2\sqrt{\cos\left(\sqrt{2}\sigma_r(T-s)\right)+1} + k_r^2\cos\left(\sqrt{2}\sigma_r(T-s)\right) \\
& +3k_r^2 \Bigg]\Bigg\} \mathrm{d}s - \left(\frac{k^2}{\sigma^2} - 2\beta\right)(T-t).
\end{aligned}
$$

We further introduce two new functions $h(\cdot)$ and $\hat{h}(\cdot)$, with terminal conditions $h(T) = 1$ and $\hat{h}(T) = 1$, respectively. We use the idea from Liu [33] (see Lemma 2 in the Appendix of his paper).

Lemma 6.4.1. *Suppose that*

$$
\frac{\partial \hat{h}}{\partial t} + f(\hat{h}) = 0,
$$

f is the linear operator on any function $\hat{h}$. Then the function h defined

$$
\begin{cases}
h(r,\eta,t) = \hat{h}(r,\eta,t) + \int_t^T \hat{h}(r,\eta,s)\mathrm{d}s \\
\\
\hat{h}(r,\eta,T) = 1,
\end{cases}
\tag{6.18}
$$

satisfies

$$
\frac{\partial h}{\partial t} + f(h) + 1 = 0.
$$

and $h(r,\eta,T) = 1$

Proof. It is obvious that $h(r,\eta,T) = 1$. Furthermore,

$$
\frac{\partial h}{\partial t} + f(h) \;=\; \frac{\partial\left(\int_t^T \hat{h}\mathrm{d}s\right)}{\partial t} + \int_t^T f(\hat{h})\mathrm{d}s
$$

$$= -\hat{h} - \int_t^T \frac{\partial \hat{h}}{\partial s} \mathrm{d}s = -\hat{h} - [\hat{h}(T) - \hat{h}] = -1.$$

□

The particular forms of $\hat{h}(t)$ and $h(t)$ that we need are:

$$\hat{h}(t) := \exp\Big\{ g_1(t)r(t) + g_2(t)\sqrt{r(t)} + g_3(t)\eta(t) + g_4(t)\sqrt{\eta(t)} + g_5(t) \Big\},$$

$$h(t) := \exp\Big\{ g_1(t)r(t) + g_2(t)\sqrt{r(t)} + g_3(t)\eta(t) + g_4(t)\sqrt{\eta(t)} + g_5(t) \Big\}$$
$$+ \int_t^T \exp\Big\{ g_1(s)r(s) + g_2(s)\sqrt{r(s)} + g_3(s)\eta(s) + g_4(s)\sqrt{\eta(s)} + g_5(s) \Big\} \mathrm{d}s.$$

The following is the main result of this section.

Theorem 6.4.1. *Let g_1, g_2, g_3, g_4 and g_5 be solutions of differential equations* (6.13)-(6.17) *respectively. There exists a unique solution to the problem* (6.6) *with criterion (6.12), given as:*

$$u^*(t) = \frac{\mu(t) - r(t)}{(1-\gamma)\sigma^2\eta(t)} y(t), \tag{6.19}$$

$$c^*(t) = \frac{y(t)}{h(t)}. \tag{6.20}$$

The corresponding optimal cost is

$$\mathbb{E}\left[\frac{y_0^\gamma h(0)^{1-\gamma}}{\gamma} \right].$$

Proof. Let the process $v(t)$ be defined as $v(t) := \mathrm{e}^{-\beta t} \frac{y^\gamma(t)}{\gamma} h^{1-\gamma}(r, \eta, t)$. Its differential is

$$\mathrm{d}v(t) = \mathrm{e}^{-\beta t} \frac{y^\gamma(t)}{\gamma} \Bigg\{ \frac{\partial(h^{1-\gamma})}{\partial t} + k_r[\theta_r - \sqrt{r(t)}] \frac{\partial(h^{1-\gamma})}{\partial r} + \frac{1}{2}\sigma_r^2 r(t) \frac{\partial^2(h^{1-\gamma})}{\partial r^2}$$

$$
+k_\eta[\theta_\eta - \sqrt{\eta}]\frac{\partial(h^{1-\gamma})}{\partial \eta} + \frac{1}{2}\sigma_\eta^2\eta(t)\frac{\partial^2(h^{1-\gamma})}{\partial \eta^2}\Bigg\}\mathrm{d}t
$$

$$
+\mathrm{e}^{-\beta t}h^{1-\gamma}(r,\eta,t)\Bigg\{\Big[r(t)y(t) + u(t)\Big(\mu(t) - r(t)\Big) - c(t)\Big]\frac{\partial(\frac{y^\gamma}{\gamma})}{\partial y}
$$

$$
+\frac{1}{2}u^2(t)\sigma^2\eta(t)\frac{\partial^2(\frac{y^\gamma}{\gamma})}{\partial y^2}\Bigg\}\mathrm{d}t
$$

$$
-\beta\mathrm{e}^{-\beta t}\frac{y^\gamma(t)}{\gamma}h^{1-\gamma}(r,\eta,t)\mathrm{d}t + \mathrm{e}^{-\beta t}\frac{y^\gamma(t)}{\gamma}\sigma_r\sqrt{r(t)}\frac{\partial(h^{1-\gamma})}{\partial r}\mathrm{d}w_r(t)
$$

$$
+\mathrm{e}^{-\beta t}\frac{y^\gamma(t)}{\gamma}\sigma_\eta\sqrt{\eta(t)}\frac{\partial(h^{1-\gamma})}{\partial \eta}\mathrm{d}w_\eta(t)
$$

$$
+\mathrm{e}^{-\beta t}h^{1-\gamma}(r,\eta,t)u(t)\sigma\sqrt{\eta(t)}\frac{\partial(\frac{y^\gamma}{\gamma})}{\partial y}\mathrm{d}w_s(t).
$$

After integrating both sides and taking the expectation, we obtain:

$$
\begin{aligned}
& \mathbb{E}[v(T)] \\
= \; & \mathbb{E}\Bigg[\int_0^T\Bigg\{\mathrm{e}^{-\beta t}\frac{y^\gamma(t)}{\gamma}\Bigg[k_r[\theta_r - \sqrt{r(t)}]\frac{\partial(h^{1-\gamma})}{\partial r} + \frac{1}{2}\sigma_r^2 r(t)\frac{\partial^2(h^{1-\gamma})}{\partial r^2} \\
& +\mathrm{e}^{-\beta t}\frac{y^\gamma(t)}{\gamma}\Bigg[k_\eta[\theta_\eta - \sqrt{\eta}]\frac{\partial(h^{1-\gamma})}{\partial \eta} + \frac{1}{2}\sigma_\eta^2\eta(t)\frac{\partial^2(h^{1-\gamma})}{\partial \eta^2}\Bigg] \\
& +\mathrm{e}^{-\beta t}h^{1-\gamma}(r,\eta,t)\Bigg[\Big(r(t)y(t) + u(t)[\mu(t) - r(t)] - c(t)\Big)\frac{\partial(\frac{y^\gamma}{\gamma})}{\partial y} \\
& +\frac{1}{2}u^2(t)\sigma^2\eta(t)\frac{\partial^2(\frac{y^\gamma}{\gamma})}{\partial y^2}\Bigg] + \mathrm{e}^{-\beta t}\frac{y^\gamma(t)}{\gamma}\frac{\partial(h^{1-\gamma})}{\partial t} \\
& -\beta\mathrm{e}^{-\beta t}\frac{y^\gamma(t)}{\gamma}h^{1-\gamma}(r,\eta,t)\Bigg\}\mathrm{d}t\Bigg] + \mathbb{E}[v(0)],
\end{aligned}
$$

Therefore, we can deduce $J_{power}(u(\cdot), c(\cdot))$ as:

$$
J_{power}(u(\cdot), c(\cdot))
$$

$$
\begin{aligned}
= \quad & \mathbb{E}\Bigg[\int_0^T \Bigg\{ \mathrm{e}^{-\beta t}\frac{y^\gamma(t)}{\gamma}\Bigg[k_r[\theta_r - \sqrt{r(t)}]\frac{\partial(h^{1-\gamma})}{\partial r} + \frac{1}{2}\sigma_r^2 r(t)\frac{\partial^2(h^{1-\gamma})}{\partial r^2} \\
& \quad + \mathrm{e}^{-\beta t}\frac{y^\gamma(t)}{\gamma}\Bigg[k_\eta[\theta_\eta - \sqrt{\eta}]\frac{\partial(h^{1-\gamma})}{\partial \eta} + \frac{1}{2}\sigma_\eta^2 \eta(t)\frac{\partial^2(h^{1-\gamma})}{\partial \eta^2}\Bigg] \\
& \quad + \mathrm{e}^{-\beta t}h^{1-\gamma}(r,\eta,t)\Bigg[\Big(r(t)y(t) + u(t)[\mu(t) - r(t)] - c(t)\Big)\frac{\partial(\frac{y^\gamma}{\gamma})}{\partial y} \\
& \qquad + \frac{1}{2}u^2(t)\sigma^2\eta(t)\frac{\partial^2(\frac{y^\gamma}{\gamma})}{\partial y^2}\Bigg] + \mathrm{e}^{-\beta t}\frac{y^\gamma(t)}{\gamma}\frac{\partial(h^{1-\gamma})}{\partial t} \\
& \quad - \beta \mathrm{e}^{-\beta t}\frac{y^\gamma(t)}{\gamma}h^{1-\gamma}(r,\eta,t) + \mathrm{e}^{-\beta t}\frac{c^\gamma(t)}{\gamma}\Bigg\}\mathrm{d}t\Bigg] + \mathbb{E}[v(0)] \\
= \quad & \mathbb{E}[v(0)] \\
& + \mathbb{E}\Bigg[\int_0^T \Bigg\{(1-\gamma)\mathrm{e}^{-\beta t}\frac{y^\gamma(t)}{\gamma}h^{-\gamma}(r,\eta,t)\frac{\partial(h^{1-\gamma})}{\partial t} - \beta \mathrm{e}^{-\beta t}\frac{y^\gamma(t)}{\gamma}h^{1-\gamma}(r,\eta,t) \\
& \qquad + r(t)\mathrm{e}^{-\beta t}h^{1-\gamma}(r,\eta,t)y^\gamma(t) \\
& \qquad + (1-\gamma)\mathrm{e}^{-\beta t}\frac{y^\gamma(t)}{\gamma}k_r[\theta_r - \sqrt{r(t)}]h^{-\gamma}(r,\eta,t)\frac{\partial h}{\partial r} \\
& \qquad + \frac{1}{2}(1-\gamma)\mathrm{e}^{-\beta t}\frac{y^\gamma(t)}{\gamma}\sigma_r^2 r(t)h^{-\gamma}(r,\eta,t)\frac{\partial^2 h}{\partial r^2} \\
& \qquad + (1-\gamma)\mathrm{e}^{-\beta t}\frac{y^\gamma(t)}{\gamma}k_\eta[\theta_\eta - \sqrt{\eta(t)}]h^{-\gamma}(r,\eta,t)\frac{\partial h}{\partial \eta} \\
& \qquad + \frac{1}{2}(1-\gamma)\mathrm{e}^{-\beta t}\frac{y^\gamma(t)}{\gamma}\sigma_\eta^2 \eta(t)h^{-\gamma}(r,\eta,t)\frac{\partial^2 h}{\partial \eta^2} \\
& \qquad + u(t)[\mu(t) - r(t)]\mathrm{e}^{-\beta t}y^{\gamma-1}(t)h^{1-\gamma}(r,\eta,t)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{2}(\gamma-1)u^2(t)\sigma^2\eta(t)y^{\gamma-2}(t)\mathrm{e}^{-\beta t}h^{1-\gamma}(r,\eta,t)\\
&+\mathrm{e}^{-\beta t}\frac{c^\gamma(t)}{\gamma}-\mathrm{e}^{-\beta t}c(t)y^{\gamma-1}(t)h^{1-\gamma}(r,\eta,t)\Bigg\}\mathrm{d}t\Bigg]\\
=\;& \mathbb{E}[v(0)]+\mathbb{E}\Bigg[\int_0^T\Lambda(t)\mathrm{d}t\Bigg].
\end{aligned}
$$

Now the problem is equivalent to find the optimal $u(t), c(t)$ such that

$$
\max_{u,C\in\mathcal{A}}\Lambda(t), \tag{6.21}
$$

where

$$
\begin{aligned}
&\Lambda(t)\\
=\;&(1-\gamma)\mathrm{e}^{-\beta t}\frac{y^\gamma(t)}{\gamma}h^{-\gamma}(r,\eta,t)\frac{\partial(h^{1-\gamma})}{\partial t}-\beta\mathrm{e}^{-\beta t}\frac{y^\gamma(t)}{\gamma}h^{1-\gamma}(r,\eta,t)\\
&+r(t)\mathrm{e}^{-\beta t}h^{1-\gamma}(r,\eta,t)y^\gamma(t)+(1-\gamma)\mathrm{e}^{-\beta t}\frac{y^\gamma(t)}{\gamma}k_r[\theta_r-\sqrt{r(t)}]h^{-\gamma}(r,\eta,t)\frac{\partial h}{\partial r}\\
&+\frac{1}{2}(1-\gamma)\mathrm{e}^{-\beta t}\frac{y^\gamma(t)}{\gamma}\sigma_r^2 r(t)h^{-\gamma}(r,\eta,t)\frac{\partial^2 h}{\partial r^2}\\
&+(1-\gamma)\mathrm{e}^{-\beta t}\frac{y^\gamma(t)}{\gamma}k_\eta[\theta_\eta-\sqrt{\eta(t)}]h^{-\gamma}(r,\eta,t)\frac{\partial h}{\partial \eta}\\
&+\frac{1}{2}(1-\gamma)\mathrm{e}^{-\beta t}\frac{y^\gamma(t)}{\gamma}\sigma_\eta^2\eta(t)h^{-\gamma}(r,\eta,t)\frac{\partial^2 h}{\partial \eta^2}\\
&+u(t)[\mu(t)-r(t)]\mathrm{e}^{-\beta t}y^{\gamma-1}(t)h^{1-\gamma}(r,\eta,t)\\
&+\frac{1}{2}(\gamma-1)u^2(t)\sigma^2\eta(t)y^{\gamma-2}(t)\mathrm{e}^{-\beta t}h^{1-\gamma}(r,\eta,t)\\
&+\mathrm{e}^{-\beta t}\frac{c^\gamma(t)}{\gamma}-\mathrm{e}^{-\beta t}c(t)y^{\gamma-1}(t)h^{1-\gamma}(r,\eta,t)
\end{aligned}
$$

By using the completion of squares method, It is observed that when the control $u^*(t) = \dfrac{\mu(t) - r(t)}{(1-\gamma)\sigma^2\eta(t)} y(t)$, and consumption $c^*(t) = \dfrac{y(t)}{h(t)}$, the equation $\Lambda(t)$ can be the maximum value, which is

$$\begin{aligned}
\Lambda^*(t) &= \frac{1-\gamma}{\gamma} e^{\beta t} y^{\gamma}(t) h^{-\gamma}(r,\eta,t) \Bigg\{ -\frac{\beta h(r,\eta,t)}{1-\gamma} + \frac{\partial h}{\partial t} + \frac{\gamma}{1-\gamma} r(t) h(r,\eta,t) \\
&\quad + k_r[\theta_r - \sqrt{r(t)}]\frac{\partial h}{\partial r} + \frac{1}{2}\sigma_r^2 r(t) \frac{\partial^2 h}{\partial r^2} + k_\eta[\theta_\eta - \sqrt{\eta(t)}]\frac{\partial h}{\partial \eta} + \frac{1}{2}\sigma_\eta^2 \eta(t) \frac{\partial^2 h}{\partial \eta^2} \\
&\quad + \frac{\gamma}{2(1-\gamma)^2} \frac{[\mu(t) - r(t)]^2}{\eta(t)\sigma^2} h(r,\eta,t) + 1 \Bigg\} \\
&= \frac{1-\gamma}{\gamma} e^{\beta t} y^{\gamma}(t) h^{-\gamma}(r,\eta,t) \Big\{ \frac{\partial h}{\partial t} + f(h) + 1 \Big\} \qquad (6.22)
\end{aligned}$$

where

$$\begin{aligned}
f(h) &= -\frac{\beta h}{1-\gamma} + \frac{\gamma}{1-\gamma} r(t) h + k_r[\theta_r - \sqrt{r(t)}]\frac{\partial h}{\partial r} + \frac{1}{2}\sigma_r^2 r(t) \frac{\partial^2 h}{\partial r^2} \\
&\quad + k_\eta[\theta_\eta - \sqrt{\eta(t)}]\frac{\partial h}{\partial \eta} + \frac{1}{2}\sigma_\eta^2 \eta(t) \frac{\partial^2 h}{\partial \eta^2} + \frac{\gamma}{2(1-\gamma)^2} \frac{[\mu(t) - r(t)]^2}{\eta(t)\sigma^2} h
\end{aligned}$$

Now we focus on the expression $\dfrac{\partial \hat{h}}{\partial t} + f(\hat{h})$. Substituting

$$\hat{h}(t) = \exp\{g_1(t) r(t) + g_2(t)\sqrt{r(t)} + g_3(t)\eta(t) + g_4(t)\sqrt{\eta(t)} + g_5(t)\},$$

it obviously can be written as

$$\begin{aligned}
&\hat{h}(t)\Big[\dot{g}_1(t) r(t) + \dot{g}_2(t)\sqrt{r(t)} + \dot{g}_3(t)\eta(t) + \dot{g}_4(t)\sqrt{\eta(t)} + \dot{g}_5(t)\Big] \\
&+(k_r\theta_r - k_r\sqrt{r(t)})\Big[g_1(t) + \frac{1}{2} g_2(t) r^{-\frac{1}{2}}(t)\Big]\hat{h}(t) \\
&+\frac{1}{2}\sigma_r^2 r(t)\Big[g_1^2(t) + g_1(t) g_2(t) r^{-\frac{1}{2}}(t) + \frac{1}{4} g_2^2(t) r^{-1}(t) - \frac{1}{4} g_2(t) r^{-\frac{3}{2}}(t)\Big]\hat{h}(t)
\end{aligned}$$

$$+(k_\eta\theta_\eta - k_\eta\sqrt{\eta(t)})\Big[g_3(t) + \frac{1}{2}g_4(t)\eta^{-\frac{1}{2}}(t)\Big]\hat{h}(t)$$

$$+\frac{1}{2}\sigma_\eta^2\eta(t)\Big[g_3^2(t) + g_3(t)g_4(t)\eta^{-\frac{1}{2}}(t) + \frac{1}{4}g_4^2(t)\eta^{-1}(t) - \frac{1}{4}g_4(t)\eta^{-\frac{3}{2}}(t)\Big]\hat{h}(t)$$

$$-\frac{\beta}{1-\gamma}\hat{h}(t) + \frac{\gamma}{1-\gamma}r(t)\hat{h}(t) + \frac{\gamma[\mu(t)-r(t)]^2}{2(1-\gamma)^2\sigma^2\eta(t)}\hat{h}(t) \tag{6.23}$$

Because of the values of $\mu(t)$, $k_r\theta_r$ and $k_\eta\theta_\eta$, we can arrange the equation into a nicely organized form, and deduce the whole expression, which is

$$\hat{h}(t)\Bigg\{\Big[\dot{g}_1(t) + \frac{\gamma}{1-\gamma} + \frac{1}{2}\sigma_r^2 g_1^2(t)\Big]r(t) + \Big[\dot{g}_2(t) - k_r g_1(t) + \frac{1}{2}\sigma_r^2 g_1(t)g_2(t)\Big]r^{\frac{1}{2}}(t)$$

$$+\Big[\dot{g}_3(t) + \frac{1}{2}\sigma_\eta^2 g_3^2(t)\Big]\eta(t) + \Big[\dot{g}_4(t) - k_\eta g_3(t) + \frac{1}{2}\sigma_\eta^2 g_3(t)g_4(t)\Big]\eta^{\frac{1}{2}}(t)$$

$$+\Big[\dot{g}_5(t) + \frac{\sigma_r^2}{4}g_1(t) - \frac{1}{2}k_r g_2(t) + \frac{1}{8}\sigma_r^2 g_2^2 + \frac{\sigma_\eta^2}{4}g_3(t) - \frac{1}{2}k_\eta g_4(t) + \frac{1}{8}\sigma_\eta^2 g_4^2$$
$$+\frac{\gamma\kappa^2}{2(1-\gamma)^2\sigma^2} - \frac{\beta}{1-\gamma}\Big]\Bigg\} \tag{6.24}$$

The sums of the terms in $r(t)$ and $r^{\frac{1}{2}}(t)$ in equation (6.24) should be zero due to equantion (6.13) (6.14), indeed,

$$\Big[\dot{g}_1(t) + \frac{\gamma}{1-\gamma} + \frac{1}{2}\sigma_r^2 g_1^2(t)\Big]r(t) = 0,$$

$$\Big[\dot{g}_2(t) - k_r g_1(t) + \frac{1}{2}\sigma_r^2 g_1(t)g_2(t)\Big]r^{\frac{1}{2}}(t) = 0.$$

Similarly, according to equation (6.15) (6.16), the sums of the terms in $\eta(t)$ and $\sqrt{\eta(t)}$ are also zero:

$$\Big[\dot{g}_3(t) + \frac{1}{2}\sigma_\eta^2 g_3^2(t)\Big]\eta(t) = 0,$$

$$\Big[\dot{g}_4(t) - k_\eta g_3(t) + \frac{1}{2}\sigma_\eta^2 g_3(t)g_4(t)\Big]\eta^{\frac{1}{2}}(t) = 0.$$

The remaining sum of the terms that are independent of $r(t)$, $r^{\frac{1}{2}}(t)$, $\eta(t)$ and $\eta^{\frac{1}{2}}(t)$ obviously equals to zero due to our assumption on $g_5(t)$.

Thus, it is calculated that $\dfrac{\partial \hat{h}}{\partial t} + f(\hat{h}) = 0$. Due to Lemma 6.4.1 and function $h(\cdot)$

$$\begin{cases} h(r,\eta,t) = \hat{h}(r,\eta,t) + \displaystyle\int_t^T \hat{h}(r,\eta,s)\mathrm{d}s \\ \\ \hat{h}(r,\eta,T) = 1, \end{cases}$$

then the equation $\Lambda^*(t) = \dfrac{1-\gamma}{\gamma}\mathrm{e}^{\beta t} y^{\gamma}(t) h^{-\gamma}(r,\eta,t)\Big\{\dfrac{\partial h}{\partial t} + f(h) + 1\Big\} = 0$

Therefore, the cost function $J_{power}(u(\cdot),c(\cdot))$ for all $\Big(u(\cdot),c(\cdot)\Big) \in \mathcal{A}$ can be written as:

$$J_{power}(u(\cdot),c(\cdot)) = \mathbb{E}[v(0)] + \mathbb{E}\left[\int_0^T \Lambda(t)\mathrm{d}t\right] \leq \mathbb{E}[v(0)]$$

It means the cost function $J_{power}(u(\cdot),c(\cdot))$ has the upper bound

$$\mathbb{E}\left[\frac{y_0^{\gamma}}{\gamma} h^{1-\gamma}(0)\right]$$

if and only if

$$u^*(t) = \frac{\mu(t) - r(t)}{(1-\gamma)\sigma^2\eta(t)} y(t),$$

$$c^*(t) = \frac{y(t)}{h(t)}.$$

□

6.5 Logarithmic Utility

In this section we give the solution to the optimal investment and consumption problem (6.6) for the *logarithmic* utility, i. e. we consider the following criterion

$$J_{logarithmic}(u(\cdot),c(\cdot)) = \mathbb{E}\left[\int_0^T \mathrm{e}^{-\beta t}\ln c(t)\mathrm{d}t + \mathrm{e}^{-\beta T}\ln y(T)\right]. \tag{6.25}$$

We introduce the linear differential equation

$$\begin{cases} \dot{g}_6(t) - \beta g_6(t) + 1 = 0 \\ \\ g_6(T) = 1. \end{cases} \tag{6.26}$$

From (6.26), g_6 can be solved as

$$\frac{1}{\beta} - \frac{\mathrm{e}^{\beta t}\Big(\frac{1}{\beta} - 1\Big)}{\mathrm{e}^{\beta T}}.$$

Theorem 6.5.1. *Let the Condition 6.3.1 hold. There exists a unique solution to the problem* (6.6) *with criterion (6.25), given as:*

$$\begin{aligned} u^*(t) &= \frac{\mu(t) - r(t)}{\sigma^2 \eta(t)} y(t), \\ c^*(t) &= \frac{y(t)}{g_6(t)}. \end{aligned}$$

The corresponding optimal cost is:

$$g_6(0) \ln y_0 + \mathbb{E}\left[\int_0^T \left\{ g_6(t)\mathrm{e}^{-\beta t} r(t) + g_6(t)\mathrm{e}^{-\beta t} \frac{\kappa^2}{2\sigma^2} - \mathrm{e}^{-\beta t}\Big(\ln g_6(t) - 1\Big) \right\} \mathrm{d}t \right]$$

Proof. We define $\bar{v}(t)$ as $\bar{v}(t) := g_6(t)\mathrm{e}^{-\beta t} \ln y(t)$. Its differential is:

$$\begin{aligned} \mathrm{d}\bar{v}(t) &= g_6(t)\mathrm{e}^{-\beta t} \left\{ r(t) + \frac{u(t)}{y(t)} \Big[\mu(t) - r(t)\Big] - \frac{c(t)}{y(t)} - \frac{u^2(t)}{2y^2(t)} \sigma^2 \eta(t) \right\} \mathrm{d}t \\ &\quad + g_6(t)\mathrm{e}^{-\beta t} \frac{u(t)}{y(t)} \sigma \sqrt{\eta(t)} \mathrm{d}w_s(t) + \dot{g}_6(t)\mathrm{e}^{-\beta t} \ln y(t)\mathrm{d}t - \beta \mathrm{e}^{-\beta t} g_6(t) \ln y(t)\mathrm{d}t. \end{aligned}$$

By integrating both sides an taking the expectation we can obtain $\mathbb{E}[\bar{v}(T)]$, through which we derive:

$$J_{logarithmic}\Big(u(\cdot), c(\cdot)\Big)$$

$$
\begin{aligned}
&= \mathbb{E}[\bar{v}(0)] \\
&\quad +\mathbb{E}\Bigg[\int_0^T \Bigg\{\dot{g}_6(t)\mathrm{e}^{-\beta t}\ln y(t) - \beta g_6(t)\mathrm{e}^{-\beta t}\ln y(t) + g_6(t)\mathrm{e}^{-\beta t} r(t) \\
&\qquad +g_6(t)\mathrm{e}^{-\beta t}\left[\frac{u(t)}{y(t)}\Big(\mu(t)-r(t)\Big) - \frac{1}{2}\frac{u^2(t)}{y^2(t)}\sigma^2\eta(t)\right] \\
&\qquad +\mathrm{e}^{-\beta t}\left[\ln c(t) - g_6(t)\frac{c(t)}{y(t)}\right]\Bigg\}\mathrm{d}t\Bigg].
\end{aligned}
$$

The maximum of $J_{logarithmic}\Big(u(\cdot), c(\cdot)\Big)$ is achieved for $u^*(t) = \frac{\mu(t)-r(t)}{\sigma^2\eta(t)}y(t)$ and consumption $c^*(t) = \frac{y(t)}{g_6(t)}$, which give

$$
\begin{aligned}
& J^*_{logarithmic}\Big(u(\cdot), c(\cdot)\Big) \\
&= \mathbb{E}[\bar{v}(0)] \\
&\quad +\mathbb{E}\Bigg[\int_0^T \Bigg\{\dot{g}_6(t)\mathrm{e}^{-\beta t}\ln y(t) - \beta g_6(t)\mathrm{e}^{-\beta t}\ln y(t) + \mathrm{e}^{-\beta t}\ln y(t) \\
&\qquad +g_6(t)\mathrm{e}^{-\beta t} r(t) + g_6(t)\mathrm{e}^{-\beta t}\frac{\kappa^2}{2\sigma^2} - \mathrm{e}^{-\beta t}\Big(\ln g_6(t) - 1\Big)\Bigg\}\mathrm{d}t\Bigg] \\
&= \mathbb{E}[\bar{v}(0)] + \mathbb{E}\Bigg[\int_0^T \Bigg\{g_6(t)\mathrm{e}^{-\beta t} r(t) + g_6(t)\mathrm{e}^{-\beta t}\frac{\kappa^2}{2\sigma^2} - \mathrm{e}^{-\beta t}\Big(\ln g_6(t) - 1\Big)\Bigg\}\mathrm{d}t\Bigg].
\end{aligned}
$$

□

Remark 6.5.1. *Due to Theorem 6.4.1 and 6.5.1, it is illustrated that the optimal solution contains the volatility $\eta(t)$. However, this $\eta(t)$ is from* (6.4)*, cannot be observed in practice. We need to infer volatility from other observable prices. This is the limitation of this research.*

6.6 Summary

The optimal investment and consumption problem in a finite horizon is considered in this chapter. The market model has a stochastic interest rate and a stochastic volatility. Both these processes are assumed to follow a Longstaff model. The corresponding optimal control problems are solved by a kind of completion of squares method, which is a combination of the approach of Liu [33] and the completion of squares method of stochastic control, and it appears to be new.

Chapter 7

Optimal investment and consumption in an infinite horizon

7.1 Introduction

We consider the problem of optimal investment and consumption in an infinite horizon. The optimality criterion is a *discounted* logarithmic utility from consumption, and the market is assumed to have a stochastic interest rate that follows a quadratic-affine model. The volatility of the stock is also assumed to be stochastic, and it follows the CIR model.

7.2 Formulation of the problem

Recalling the Brownian motions defined in (6.2), we let $w_r(\cdot) = w_x(\cdot)$, and consider the stochastic differential equation:

$$\begin{cases} \mathrm{d}x(t) = mx(t)\mathrm{d}t + s\mathrm{d}w_x(t) \\ x(0) = x_0, \end{cases} \tag{7.1}$$

where $m > 0$ and s are given constants. We assume that for some constants q_1 and q_2 the interest rate is defined as:

$$r(t) := q_1 x^2(t) + q_2 x(t).$$

We consider a market consisting of a bank account and a stock, the prices of which are denoted by $B(t)$ and $S(t)$, respectively, and satisfy the equations

$$\begin{cases} dB(t) = r(t)B(t)\mathrm{d}t, \\ \\ B(0) = B_0 > 0. \end{cases}$$

$$\begin{cases} \mathrm{d}S(t) = S(t)\Big[\mu(t)\mathrm{d}t + \sigma\sqrt{\eta(t)}\mathrm{d}w_s(t)\Big], \\ \\ \\ S(0) = S_0 > 0. \end{cases}$$

We assume that the volatility process $\eta(t)$ is the solution to the following equation

$$\mathrm{d}\eta(t) = k_\eta[\theta_\eta - \eta(t)]\mathrm{d}t + \sigma_\eta\sqrt{\eta(t)}\mathrm{d}w_\eta(t), \tag{7.2}$$

where $k_\eta, \theta_\eta, \sigma_\eta$ are constants.

Similarly to the work of Bielecki, T.R., Pliska, S. and & Yong, J., 2004, Kraft, H., 2005, and Chang and Rong [7], we make a certain assumption on the relation of the drift $\mu(t)$ with $r(t)$ and $\eta(t)$. In this case we assume that $\mu(t) := r(t)+k\sqrt{\eta(t)}$.

In this market, we consider an investor with an initial wealth of $y_0 > 0$ and that follows a self-financing trading strategy and is permitted to *consume*. The equation of her/his wealth $y(t)$ is

$$\mathrm{d}y(t) = \Big\{r(t)y(t) + u(t)\Big[\mu(t) - r(t)\Big] - c(t)\Big\}\mathrm{d}t + u(t)\sigma\sqrt{\eta(t)}\mathrm{d}w_s(t), \tag{7.3}$$

where $u(t)$ denotes the amount of wealth invested in the stock, whereas $c(t)$ denotes the consumption rate.

Let the *discounted* cost from consumption be defied as

$$J_T(u(\cdot), c(\cdot)) = \mathbb{E}\left[\int_0^T \mathrm{e}^{-\beta t}\mathrm{ln}c(t)\mathrm{d}t\right].$$

The objective of this chapter is solve the following *infinite horizon* optimal control problem for some $\beta > 0$.

$$\begin{cases} \max\limits_{u(\cdot),c(\cdot)\in\mathcal{A}} \lim\limits_{T\to\infty} J_T(u(\cdot),c(\cdot)), \\ \\ s.t. \quad (7.1), \quad (7.2), \quad (7.3). \end{cases} \tag{7.4}$$

Here $\mathcal{A}$ is the set of admissible controls $u(t)$ and $c(t)$ that ensure the existence of a unique and positive solution of (7.3), and that $\lim_{T\to\infty} J_T(u(\cdot),c(\cdot))$ is finite.

7.3 Solution of the problem

We introduce the following linear ordinary differential equations:

$$\begin{cases} \dot{g}(t) - \beta g(t) + 1 = 0, \\ \\ g(T) = 0, \end{cases} \tag{7.5}$$

$$\begin{cases} \dot{g}_1(t) - \beta g_1(t) + 2m g_1(t) + q_1 g(t) = 0, \\ \\ g_1(T) = 0, \end{cases} \tag{7.6}$$

$$\begin{cases} \dot{g}_2(t) - \beta g_2(t) + m g_2(t) + q_2 g(t) = 0, \\ \\ g_2(T) = 0. \end{cases} \tag{7.7}$$

By solving equations (7.5), (7.6) and (7.7),we have:

$$\begin{cases} g\,(t) = \beta^{-1} - \dfrac{e^{\beta t}}{\beta\, e^{\beta T}}, \\ \\ g(0) = \dfrac{1}{\beta} - \dfrac{1}{\beta\, e^{\beta T}}, \end{cases}$$

$$\begin{cases} g_1(t) = \dfrac{\left(-q_1 \mathrm{e}^{2\,Tm+\beta\,t}\beta + q\mathrm{e}^{\beta\,t+2\,tm}\beta - 2\,q_1 \mathrm{e}^{\beta\,t+2\,tm} m + 2\,qm\mathrm{e}^{\beta\,T+2\,tm}\right)\mathrm{e}^{-\beta\,T-2\,tm}}{2\beta\,(\beta-2\,m)\,m}, \\ \\ g_2(t) = \dfrac{\left(-q_2 \mathrm{e}^{Tm+\beta\,t}\beta + q\mathrm{e}^{\beta\,t+tm}\beta - q\mathrm{e}^{\beta\,t+tm} m + qm\mathrm{e}^{\beta\,T+tm}\right)\mathrm{e}^{-\beta\,T-tm}}{\beta\,(\beta-m)\,m}, \end{cases}$$

$$\begin{cases} g_1(0) = -\dfrac{q_1}{\beta\,(-\beta+2\,m)} + \dfrac{1}{2}\,\dfrac{q_1 \mathrm{e}^{-\beta\,T}}{\beta\,m} \\ \qquad -\dfrac{1}{\mathrm{e}^{T(\beta-2\,m)}}\left(-\dfrac{q_1}{\beta\,(-\beta+2\,m)} + \dfrac{1}{2}\,\dfrac{q\mathrm{e}^{T(\beta-2\,m)-\beta\,T+2\,Tm}}{\beta\,m}\right), \\ \\ g_2(0) = \dfrac{\left(-q_2 \mathrm{e}^{Tm}\beta + \beta\,q_2 - q_2 m + q_2 m\mathrm{e}^{\beta\,T}\right)\mathrm{e}^{-\beta\,T}}{\beta\,(\beta-m)\,m}. \end{cases}$$

We further introduce the equation:

$$\begin{aligned} & g_1(t)\mathrm{e}^{-\beta t}x^2(t) + g_2(t)\mathrm{e}^{-\beta t}x(t) \\ = \; & g_1(0)x_0^2 + g_2(0)x_0 \\ & + \int_0^t \mathrm{e}^{-\beta\tau}\Big\{ -\beta\Big[g_1(\tau)x^2(\tau) + g_2(\tau)x(\tau) + \dot{g}_1(\tau)x^2(\tau) + \dot{g}_2(\tau)x(\tau) \\ & \qquad + \Big[2g_1(\tau)x(\tau) + g_2(\tau)\Big]mx(\tau) + g_1(\tau)s^2\Big\}\mathrm{d}\tau \\ & + \int_0^t \mathrm{e}^{-\beta\tau}\Big[2g_1(\tau)x(\tau) + g_2(\tau)\Big]s\mathrm{d}w_x(\tau). \end{aligned} \tag{7.8}$$

Theorem 7.3.1. *If $\beta > 2m$, then there exists a unique solution to the problem (7.4) given by*

$$u^*(t) = \frac{\mu(t) - r(t)}{\sigma^2\eta(t)}y(t), \qquad c^*(t) = \frac{y(t)}{g(t)}.$$

The corresponding optimal cost is:

$$
\begin{aligned}
J^* &= \frac{\ln(y_0)}{\beta} + \frac{q_1}{\beta(\beta-2m)} + \frac{q_2}{\beta(\beta-m)} + \frac{q_1 s^2}{m^2\beta^2(\beta-2m)} \\
&\quad + \frac{1}{2\beta^2\sigma^2}\Big[2\beta\sigma^2\ln(\beta) + k^2 - 2\beta\sigma^2\Big].
\end{aligned}
$$

Proof. We define the process $v(t)$ as $v(t) := g(t)\mathrm{e}^{-\beta t}\ln y(t)$. Its differential is

$$
\begin{aligned}
\mathrm{d}v(t) &= g(t)\mathrm{e}^{-\beta t}\Bigg\{r(t) + \frac{u(t)}{y(t)}\Big[\mu(t) - r(t)\Big] - \frac{c(t)}{y(t)} - \frac{u^2(t)}{2y^2(t)}\sigma^2\eta(t)\Bigg\}\mathrm{d}t \\
&\quad + g(t)\mathrm{e}^{-\beta t}\frac{u(t)}{y(t)}\sigma\sqrt{\eta(t)}\mathrm{d}w_s(t) + \dot{g}(t)\mathrm{e}^{-\beta t}\ln y(t)\mathrm{d}t \\
&\quad - \beta\mathrm{e}^{-\beta t}g(t)\ln y(t)\mathrm{d}t \qquad (7.9)
\end{aligned}
$$

We can thus write the cost functional as

$$
\begin{aligned}
J_T(u(\cdot), c(\cdot)) &= \mathbb{E}\Bigg[\int_0^T \mathrm{e}^{-\beta t}\ln c(t)\mathrm{d}t + g(T)\mathrm{e}^{-\beta T}\ln y(T)\Bigg] \\
&= \mathbb{E}[v(0)] \\
&\quad + \mathbb{E}\Bigg[\int_0^T \Big\{\dot{g}(t)\mathrm{e}^{-\beta t}\ln y(t) - \beta g(t)\mathrm{e}^{-\beta t}\ln y(t) + g(t)\mathrm{e}^{-\beta t}r(t) \\
&\quad + g(t)\mathrm{e}^{-\beta t}\Big[\frac{u(t)}{y(t)}\big(\mu(t) - r(t)\big) - \frac{1}{2}\frac{u^2(t)}{y^2(t)}\sigma^2\eta(t)\Big] \\
&\quad + \mathrm{e}^{-\beta t}\Big[\ln c(t) - g(t)\frac{c(t)}{y(t)}\Big]\Big\}\mathrm{d}t\Bigg] \\
&= \mathbb{E}\Big[v(0)\Big] \\
&\quad + \mathbb{E}\Bigg[\int_0^T \Big\{\Big(\dot{g}(t) - \beta g(t) + 1\Big)\mathrm{e}^{-\beta t}\ln y(t) + g(t)\mathrm{e}^{-\beta t}r(t)
\end{aligned}
$$

$$+g(t)\mathrm{e}^{-\beta t}\sigma^2\eta(t)\Big(\frac{u(t)}{y(t)} - \frac{\mu(t)-r(t)}{\sigma^2\eta(t)}\Big)^2 + g(t)\mathrm{e}^{-\beta t}\frac{k^2}{2\sigma^2}$$

$$+\mathrm{e}^{-\beta t}\Big(\ln\frac{c(t)}{y(t)} - g(t)\frac{c(t)}{y(t)}\Big)\Big\}\mathrm{d}t\Bigg] \tag{7.10}$$

Due to equation (7.5), the sum of the terms containing $\ln y(t)$ is zero. Moreover, the following upper bound holds

$$J_T(u(\cdot), c(\cdot)) \;\; \le \;\; \mathbb{E}\Big[v(0)\Big] + \alpha_1(T) + \alpha_2(T),$$

and it is archived if and only if we choose

$$\frac{u(t)}{y(t)} = \frac{\mu(t)-r(t)}{\sigma^2\eta(t)} \Rightarrow u^*(t) = \frac{\mu(t)-r(t)}{\sigma^2\eta(t)}y(t),$$

$$\frac{c(t)}{y(t)} = \frac{1}{g(t)} \Rightarrow c^*(t) = \frac{y(t)}{g(t)}.$$

We now consider the finiteness of $\alpha_1(T)$ and $\alpha_2(T)$ as $T \to \infty$, which are given by

$$\alpha_1(T) = \mathbb{E}\Bigg[\int_0^T g(t)\mathrm{e}^{-\beta t}r(t)\mathrm{d}t\Bigg],$$

$$\alpha_2(T) = \mathbb{E}\Bigg[\int_0^T \Big\{g(t)\mathrm{e}^{-\beta t}\frac{k^2}{2\sigma^2} - \mathrm{e}^{-\beta t}\Big(\ln g(t) + 1\Big)\Big\}\mathrm{d}t\Bigg].$$

With the help of (7.8) we can write the expression for $\alpha_1(T)$ as:

$$\alpha_1(T)$$

$$= \mathbb{E}\Bigg[g_1(0)x_0^2 + g_2(0)x_0 - g_1(T)\mathrm{e}^{-\beta T}x^2(T) - g_2(T)\mathrm{e}^{-\beta T}x(T)$$

$$+\int_0^T \mathrm{e}^{-\beta t}\Big[\dot{g}_1(t)x^2(t) + \dot{g}_2(t)x(t) - \beta g_1(t)x^2(t) - \beta g_2(t)x(t)$$

$$+2mg_1(t)x^2(t) + mg_2(t)x(t) + g_1(t)s^2\Big]\mathrm{d}t$$

$$+\int_0^T g(t)\mathrm{e}^{-\beta t}\Big[q_1x^2(t) + q_2x(t)\Big]\mathrm{d}t\Bigg]$$

$$= \mathbb{E}\Bigg[g_1(0)x_0^2 + g_2(0)x_0 - g_1(T)\mathrm{e}^{-\beta T}x^2(T) - g_2(T)\mathrm{e}^{-\beta T}x(T)$$

$$+\int_0^T \mathrm{e}^{-\beta t}\Big[\Big(\dot{g}_1(t) - \beta g_1(t) + 2mg_1(t) + q_1g(t)\Big)x^2(t)$$

$$+\Big(\dot{g}_2(t) - \beta g_2(t) + mg_2(t) + q_2g(t)\Big)x(t)\Big]\mathrm{d}t$$

$$+\int_0^T s^2g_1(t)\mathrm{e}^{-\beta t}\mathrm{d}t\Bigg] \tag{7.11}$$

Due to equation (7.6), we can get rid of the terms that are quadratic in $x(t)$, since

$$\Big(\dot{g}_1(t) - \beta g_1(t) + 2mg_1(t) + q_1g(t)\Big)x^2(t) = 0.$$

Similarly, the terms that are linear in $x(t)$ also add up to zero due to equation (7.7):

$$\Big(\dot{g}_2(t) - \beta g_2(t) + mg_2(t) + q_2g(t)\Big)x(t) = 0$$

Then our aim is to consider the remaining part, which is

$$\alpha_1(T) \quad = \quad \mathbb{E}\Bigg[g_1(0)x_0^2 + g_2(0)x_0 + \int_0^T s^2g_1(t)\mathrm{e}^{-\beta t}\mathrm{d}t\Bigg]$$

When $\beta > 2m$,

$$\lim_{T\to\infty}\alpha_1(T)$$

$$= \lim_{T\to\infty}\mathbb{E}\Bigg[-\frac{q_1x_0^2}{\beta\,(-\beta+2\,m)} + \frac{1}{2}\,\frac{q_1\mathrm{e}^{-\beta\,T}x_0^2}{\beta\,m}$$

$$-\frac{x_0^2}{\mathrm{e}^{T(\beta-2\,m)}}\left(-\frac{q_1}{\beta\,(-\beta+2\,m)} + \frac{1}{2}\,\frac{q_1\mathrm{e}^{T(\beta-2\,m)-\beta\,T+2\,Tm}}{\beta\,m}\right)$$

$$+\frac{\left(-q_2\mathrm{e}^{Tm}\beta+\beta\,q_2-q_2m+q_2m\mathrm{e}^{\beta\,T}\right)\mathrm{e}^{-\beta\,T}x_0}{\beta\,\left(\beta-m\right)m}$$

$$+\frac{q_1s^2\left(-\beta^2\mathrm{e}^{2\,Tm}+2\,\beta^2mT-4\,m^2\beta\,T+4\,m^2\mathrm{e}^{\beta\,T}+\beta^2-4\,m^2\right)\mathrm{e}^{-\beta\,T}}{4m^2\left(\beta-2\,m\right)\beta^2}\Bigg]$$

$$= \frac{q_1x_0^2}{\beta(\beta-2m)}+\frac{q_2x_0}{\beta(\beta-m)}+\frac{q_1s^2}{m^2\beta^2(\beta-2m)} \tag{7.12}$$

$$\lim_{T\to\infty}\alpha_2(T)$$

$$= \lim_{T\to\infty}\mathbb{E}\Bigg[-\frac{1}{2\beta^2\sigma^2}\mathrm{e}^{-\beta T}\Bigg\{2\mathrm{e}^{\beta T}\sigma^2\beta\ln\Big(\frac{\mathrm{e}^{\beta T}-1}{\beta}\mathrm{e}^{-\beta T}\Big)-2T\beta^2\sigma^2$$

$$-2\beta\sigma^2\ln\Big(\frac{\mathrm{e}^{\beta T}-1}{\beta}\mathrm{e}^{-\beta T}\Big)+2\beta\sigma^2\mathrm{e}^{\beta T}+k^2\beta T-k^2\mathrm{e}^{\beta T}-2\beta\sigma^2+k^2\Bigg\}\Bigg]$$

$$= \frac{1}{2\beta^2\sigma^2}\Bigg[2\beta\sigma^2\ln(\beta)+k^2-2\beta\sigma^2\Bigg] \tag{7.13}$$

Therefore, we can deduce

$$\lim_{T\to\infty}J_T(u^*(\cdot),c^*(\cdot))$$

$$\leq \frac{\ln(y_0)}{\beta}+\frac{q_1x_0^2}{\beta(\beta-2m)}+\frac{q_2x_0}{\beta(\beta-m)}+\frac{q_1s^2}{m^2\beta^2(\beta-2m)}$$

$$+\frac{1}{2\beta^2\sigma^2}\Bigg[2\beta\sigma^2\ln(\beta)+k^2-2\beta\sigma^2\Bigg] \tag{7.14}$$

□

Remark 7.3.1. *Similar to Remark 6.5.1, from Theorem 7.3.1, we can find the optimal solution $u^*(t)$ contains the volatility $\eta(t)$. However, this $\eta(t)$ is from (7.2), cannot be observed in practice, and needs to be infered from other observable prices. This is the limitation of this research.*

7.4 Summary

We formulate and solve an infinite horizon optimal investment and consumption problem. The criterion has a discounting factor, the value of which must be grater than a given constant. The market coefficients are all stochastic. The interest rate follows a QATSM, the volatility follows a CIR model, whereas the drift is expressed in terms of these two coefficients. We obtained the optimal investment strategy and consumption in an explicit closed-form for the logarithmic utility. It remains an interesting open question if a similar approach can be used to solve the problem for the power utility.

Chapter 8

Conclusion

In this conclusion chapter, we summarize the main contributions of the thesis and point out some interesting open questions for future research.

8.1 Contributions

- First, based on Date and Gashi's work [11], we propose a further nonlinear system, which contains the multi-dimensional square-root process. The key contribution of Chapter 3 is that the explicit closed form solvability has been preserved. Another relevant point is the extension of Yong' work. Yong in [54] points out the limitation of using the CIR model in investment. In Chapter 3, we give a detailed discussion on the this problem which utilizes the multi-dimensional square-root process. We give some sufficient conditions to ensure the control processes belong to the admissible set. In addition, a generalised criterion of risk-sensitive control is proposed, where the noise dependents on the state and control variables. Even in this situation, the solution is obtained explicitly by the change of measure approach.

- As the applications of risk sensitive control to mathematical finance, the interest rate modelling, bond pricing, and optimal investment, are considered. Several new forms of nonlinear stochastic interest rate are introduced in this thesis. For instance, the mixed CIR and multi-dimensional quadratic term structure interest rate model is discussed in Chapter 4 and the Longstaff interest rate model is considered in Chapter 5, 6. We know the approach to obtain the price of zero-coupon bond with different kinds of stochastic interest rates. For finite time horizon case, we obtain the optimal investment strategy under a new stochastic interest rate models (see Chapter 4). In addition, the optimal solution of the portfolio which contains consumption is also given in an explict closed form (see Chapter 6). These results should be quite important to individuals who work in mathematical finance. A brief example is investment banking. The investor could know how many shares of which asset he should hold to maximise his wealth.

On the other hand, we deal with the optimal portfolio problems not only for finite but also for infinite time horizon case. Chapter 5 considers the optimal investment with nonlinear stochastic interest rate on a long time. And Merton's optimal consumption problem for the infinite time case is discussed in Chapter 7. The research about infinite time case is to consider some financial instruments such as perpetual bond or perpetual options. Under this situation, we can still obtain the optimal strategies.

8.2 Future Research

In this section, we list some open probems which can be investigated in the future.

- A numerical study to such mathematical finance problems is a good direction for research. Particularly, we can focus on how market parameters affect the dynamic behaviour of optimal strategy. In addition, due to Remark 6.5.1 7.3.1, we state the limitation of research about stochastic volatility. Therefore, how to estimated the volatility in the optimal strategy by using observed volatility, is also a quite interesting open question.

- In Chapter 3, we simplify the problem about multi-dimensional square root process by introducing Assumption 3.2.3, 3.2.4, when discussing the admissible controls in Section 3.2.1. Our aim of future research is to avoid these two strong assumptions or at least make them weaker.

- In Chapter 7, there remains an open question, which is the problem of optimal consumption with power utility on an infinite time horizon. This work is incomplete, and is illustrated as follows:

Problem 8.2.1. *Let the discounted cost from consumption be defined as*

$$J_T(u(\cdot), c(\cdot)) = \mathbb{E}\left[\int_0^T \mathrm{e}^{-\beta t}\frac{c^\gamma(t)}{\gamma}\mathrm{d}t\right].$$

The objective is solve the following infinite horizon optimal control problem for some $\beta > 0$ and $\gamma \in (0, 1)$.

$$\begin{cases} \max\limits_{u(\cdot),c(\cdot)\in\mathcal{A}} \lim\limits_{T\to\infty} J_T(u(\cdot), c(\cdot)), \\ \\ s.t. \quad (7.1), \quad (7.2), \quad (7.3). \end{cases} \tag{8.1}$$

It is believable that this problem could be solved in the future research.

Notation

$\mathbb{R}^n$	n-dimensional real Euclidean space
$\mathbb{R}^{n \times m}$	Euclidean space of real $(n \times m)$ matrices
$\text{diag}(x)$	a diagonal matrix with elements of vector x on its diagonal
$\text{tr}(X)$	the trace of a square matrix X
A'	the transpose of the vector (or matrix) A
$\dot{g}(t)$	the derivative of function $g(t)$
Ω	sample space
$\mathbb{P}$	probability measure
$\mathcal{F}_t$	filtration up to time t
T	terminal time

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